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- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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## ERNST L'VOVICH PRESMAN

(to the 85th birthday)

On February 28, 2026, Ernst L'vovich Presman, Doctor of Physical and Mathematical Sciences, Chief Researcher of the Central Economics and Mathematics Institute of the Russian Academy of Sciences, a Member of Editorial Board of "Eurasian Mathematical Journal". turned 85 years old.

Ernst L'vovich Presman was born on February 28, 1941, in Moscow. In 1958, he graduated with honors from Moscow Secondary School No. 124 and entered the Moscow Institute of Physics and Technology (MPhTI) the same year.

In 1964, under the supervision of Academician Yu. V. Prokhorov, he graduated with honors and entered graduate school at MPhTI.

After completing his graduate studies in 1967, Ernst L'vovich joined the Central Economics and Mathematics Institute of the Russian Academy of Sciences, where he remains employed till now, having risen from a junior research fellow to a chief research fellow.

During his graduate studies and for the following four years, Ernst L'vovich taught mathematics in the Department of Mathematics at MPhTI. In 1994-1995 he was a Visiting Professor at the Hosei University, Tokyo, Japan, in 1996 a Visiting Professor at the University of North Caroline at Charlotte, Department of Mathematics. From 2008 to 2015, he worked as a professor at the Department of Fundamental and Applied Mathematics at the Russian State University for the Humanities, and from 2008 to the present, he has taught mathematics at the Moscow School of Economics at the M.V. Lomonosov Moscow State University.

As a researcher on leave he visited (from one to four months) SUNY at Stony Brook (1989), University of Toronto (1990, 1995, 1997), University of Texas at Dallas (2000, 2001, 2003), Universite de Franche Comte, Besancon (2001).

Ernst L'vovich's mathematical interests spanned a wide range. His first scientific paper appeared while he was still a student and was published in the Proceedings of the Steklov Mathematical Institute of the RAS (1964). His diploma thesis was devoted to queuing theory and was published in the journal "Probability Theory and Its Applications".

In 1968, Ernst L'vovich defended his PhD dissertation, completed under the supervision of Academician Yu.V. Prokhorov. The results of his dissertation, devoted to factorization methods for solving boundary value problems in probability theory, were published in the Bulletin of the Academy of Sciences and received wide international recognition. In 1996, at the Steklov Mathematical Institute of the RAS he defended his DSc dissertation: "Investigations on stochastic optimal control" (Committee: Academician I.A.Ibragimov, Academician A.N.Shiryaev, Professor A.Yu. Veretennikov).

Throughout his career, Ernst L'vovich devoted considerable attention to limit theorems in probability theory. His work on the multivariate version of Kolmogorov's uniform limit theorem, on the approximation of various distributions to a family of accompanying laws, his joint work with UzAS Academician Formanov and later with RAS Academician I.A. Ibragimov on modifying the Lindeberg and Rotar conditions in the Central Limit Theorem, and his other works in this area are well known.

After joining CEMI, Ernst L'vovich began working on optimal stopping problems, random process control, inventory and production management models, and financial mathematics. His joint work with I.M. Sonin on the random choice problem became a classic. Together with I.M. Sonin, he published the monograph "Sequential Control with Incomplete Information: The Bayesian Approach to Many-Armed Bandit Problems", Academic Press, 1990 (a revised translation from the Russian edition by "Nauka", 1982).

First independently, and then jointly with T.A. Belkina and Yu.V. Kabanov, he obtained important results on the stochastic linear regulator. Together with A.D. Slastnikov, he studied various stochastic models of economic dynamics, worked with S. Sethi on problems of financial mathematics, and then, with S. Sethi and his students, studied stochastic manufacturing systems.

Ernst L'vovich has published over 200 scientific papers to date (see https://scholar.google.com, Profile of Ernst Presman). He has 30 co-authors and has participated in over 100 scientific conferences. Two PhD dissertations were defended under his supervision.

In 1993 - 2000 Ernst L'vovich was Executive Secretary and a member of Editorial Board of the Journal "Theory of Probability and its Applications" and in 2000 – 2013 a member of Advisory Board of the same Journal. In 2018 – 2023 he was a member of Editorial Board of the Journal B«Theory of Stochastic ProcessesB», and from 2010 till now he is a Member of Editorial Board of the "Eurasian Mathematical Journal".

He is one of the leaders of the general institute seminar CEMI B«Problems of stochastic control and stochastic models in Economics, Finance and InsuranceB».

Ernst L'vovich was Project Manager of the Russian Foundation for Basic Research "Controlled Random Processes" through 1994 – 2015.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Ernst L'vovich on the occasion of his 85th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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# CONSTRUCTION OF A DISCRETE D-OPTIMAL DESIGN FOR A LINEAR REGRESSION MODEL WITH HAAR BASIS FUNCTIONS

**A.A. Adamov, M.B. Gabbassov, A.U. Kussebay**

Communicated by E.L. Presman

**Key words:** regression model, optimal design, optimality criterion, information matrix, Haar functions.

**AMS Mathematics Subject Classification:** 62F10, 62K05, 49K10.

**Abstract.** Parametrically linear regression models are widely used in practice to describe various types of dependencies. The goal of such experiments is to estimate the unknown parameters of the model and to verify the optimality of the chosen design points according to certain criteria.

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## 1 Introduction

The aim of this paper is to construct a D-optimal experimental design if the basis functions are the Haar functions. It is known that many Monte Carlo algorithms and experimental design methods are based on selecting a certain probability distribution $\rho$ of the design points defined on a measurable space $X$ [1, 4]. In this work, it is proved that a given discrete design is D-optimal if the design points are distributed almost uniformly across the intervals of constancy of the Haar functions.

Let us consider a parametrically linear regression model, where the measurement results $y(x_j)$ at points $x_j \in X$ are represented as follows [2, 4]:

$$y_j = y\left(x_j\right) = \sum_{i=1}^{n} \theta_i \varphi_i(x_j) + \varepsilon(x_j), \quad j = 1, 2, \ldots, N, \tag{1.1}$$

where $\varphi\left(x\right) = \left\{\varphi_1\left(x\right), \varphi_2\left(x\right), \ldots, \varphi_N\left(x\right)\right\}^T$– is the basis function vector on X, in our case the Haar functions, $\theta_i$– are the unknown parameters, $\varepsilon\left(x_j\right)$– are random errors, X – is the design space. Standard assumptions are made regarding the errors [2]:

$$E\varepsilon\left(x\right) = 0, E\varepsilon^2\left(x\right) = \sigma^2\left(x\right) = const < \infty, \ E\varepsilon\left(x_i\right)\varepsilon\left(x_j\right) = 0, \quad i \neq j,$$

where $E$ is the mathematical expectation.

The regression model (1.1) can also be written in the form $Y = F\theta + \varepsilon$, where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}, \quad F = \begin{pmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \ldots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \ldots & \varphi_n(x_2) \\ \vdots & \vdots & \ldots & \vdots \\ \varphi_1(x_N) & \varphi_2(x_N) & \ldots & \varphi_n(x_N) \end{pmatrix} = \begin{pmatrix} \varphi^T(x_1) \\ \varphi^T(x_2) \\ \vdots \\ \varphi^T(x_N) \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}.$$

It is required to estimate the unknown parameters $\theta$ and their variances, as well as to verify the D-optimality of the obtained design, i.e., to find the optimal density

$$\rho^* = \underset{\rho \in P}{argmin}(det[D(\rho)]) \tag{1.2}$$

The Haar function system on the set X=[0,1] with a given (probability) measure $\mu$ is defined as follows [3].

We divide the set $X$ with measure $\mu(X) = 1$ into $2^m$ disjoint subsets $d\,(m,s)\,(1 \le s \le 2^{m-1}, m = 1,2,\ldots)$ each of the same measure $\mu$. The subsets $d\,(m,s)(1 \le s \le 2^{m-1})$ are defined by the equality:

$$d\,(m,s) = \left[\tfrac{s-1}{2^{m-1}}\ ,\ \tfrac{s}{2^{m-1}}\right),$$

where $s$ ranges from 1 to $2^{m-1}$, and $m = 1,\ 2,\ \ldots$ (of course, for $s = 2^{m-1}$ we consider $d(m,\ s)$ to be closed on the right as well). It is easy to see that for each $m$:

$$d(m,1)\bigcup d(m,2)\bigcup \ldots \bigcup d(m,2^{m-1}) = [0,1].$$

The Haar function system $\chi_{ms}(x)$ is conveniently defined in groups: the group with number $m$ contains $2^{m-1}$ functions $\chi_{ms}\,(x)$,$s = 1,2,\ldots,2^{m-1}$, defined by the following equalities:

$$\chi_{ms}(x) = \begin{cases} 2^{\frac{m-1}{2}}, & \text{at} \quad x \in d(m+1,2s-1), \\ -2^{\frac{m-1}{2}}, & \text{at} \quad x \in d(m+1,2s), \\ 0, & \text{at} \quad x \notin d(m,s). \end{cases} \tag{1.3}$$

Let $k_j$ $(j = 1,2,...,n)$ be the number of points belonging to the subset $d(m,j)$ of the set X=[0,1] and $\sum\limits_{j=1}^{n} k_j = N$.

It is known [2], that if the matrix of the system of the normal equations $F^T F$ is non-degenerate, then the least squares estimate has the form:

$$\hat{\theta} = \left(F^T F\right)^{-1} F^T Y \tag{1.4}$$

and the variance has the form:

$$D\hat{\theta} = \sigma^2 \left(F^T F\right)^{-1}, \tag{1.5}$$

where

$$F^T F = \begin{pmatrix} (\phi_1,\phi_1) & (\phi_1,\phi_2) & \cdots & (\phi_1,\phi_n) \\ (\phi_2,\phi_1) & (\phi_2,\phi_2) & \cdots & (\phi_2,\phi_n) \\ \cdots & \cdots & \cdots & \cdots \\ (\phi_n,\phi_1) & (\phi_n,\phi_2) & \cdots & (\phi_n,\phi_n) \end{pmatrix},$$

$$(\phi,\psi) = \sum_{j=1}^{N} \phi(x_j)\psi(x_j), \quad x_j \in X.$$

The matrix $M = F^T F$ is called the information matrix, and the matrix $D\hat{\theta} = \sigma^2 M^{-1}$–is called the variance matrix for model (1.1).

## 2 Cases with two and four basis functions

To simplify the analysis, let us first consider the cases of $m = 1,\ 2$, i.e., the interval $[0,1]$ is divided into 1 and 2 subsets respectively, and the regression model contains 2 and 4 unknown parameters $\theta$.

In the case of $m = 1$, the Haar functions take the following form:

$$\phi_1\,(x) = 1, \phi_2\,(x) = \begin{cases} +1, & for \quad x \in [0,\,\frac{1}{2}) \\ -1, & for \quad x \in [\frac{1}{2},1] \end{cases} \tag{2.1}$$

Let $k_1$ and $k_2$ be the number of points selected from the subsets $[0,\frac{1}{2})$ and $[\frac{1}{2},1]$. respectively, with $k_1 + k_2 = N$. The information matrix takes the following form:

$$F^T F = \begin{pmatrix} (\phi_1, \phi_1) & (\phi_1, \phi_2) \\ (\phi_2, \phi_1) & (\phi_2, \phi_2) \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & k_1 - k_2 \\ k_1 - k_2 & k_1 + k_2 \end{pmatrix}.$$

The determinant of this matrix is equal to $k_1 k_2$. Therefore, for the matrix to be non-degenerate, it is necessary and sufficient that the conditions $k_1$ and $k_2 > 0$ are satisfied.

$$\left(F^T F\right)^{-1} = \frac{1}{4} \begin{pmatrix} \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} - \frac{1}{k_2} \\ \frac{1}{k_1} - \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} \end{pmatrix}, \quad F^T Y = \begin{pmatrix} \sum_{i=1}^{k_1} y_i & + & \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \sum_{i=1}^{k_1} y_i & - & \sum_{i=k_1+1}^{k_1+k_2} y_i \end{pmatrix}.$$

Therefore, the estimate of the unknown parameters $\theta = (\theta_1, \theta_2)$ is given by:

$$\hat{\theta} = \left(F^T F\right)^{-1} F^T Y = \begin{pmatrix} \frac{1}{2k_1} \sum_{i=1}^{k_1} y_i & + & \frac{1}{2k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \frac{1}{2k_1} \sum_{i=1}^{k_1} y_i & - & \frac{1}{2k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i \end{pmatrix},$$

and their variance has the following form:

$$D\hat{\theta} = \frac{\sigma^2}{4} \begin{pmatrix} \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} - \frac{1}{k_2} \\ \frac{1}{k_1} - \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} \end{pmatrix}.$$

In the case $m = 2$ we have four Haar functions: $\phi_1(x)$, $\phi_2(x)$ from (2.1) and

$$\phi_3(x) = \begin{cases} +\sqrt{2}, & for \quad x \in \left[0, \frac{1}{4}\right), \\ -\sqrt{2}, & for \quad x \in \left[\frac{1}{4}, \frac{1}{2}\right), \\ 0, & for \quad x \in \left[\frac{1}{2}, 1\right], \end{cases} \quad \phi_4(x) = \begin{cases} +\sqrt{2}, & for \quad x \in \left[\frac{1}{2}, \frac{3}{4}\right), \\ -\sqrt{2}, & for \quad x \in \left[\frac{3}{4}, 1\right), \\ 0, & for \quad x \in \left[0, \frac{1}{2}\right]. \end{cases} \tag{2.2}$$

Let $k_1, k_2, k_3$ и $k_4$ be the number of points selected from the subsets $\left[0, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{3}{4}\right)$ and $\left[\frac{3}{4}, 1\right]$, respectively, with $k_1 + k_2 + k_3 + k_4 = N$. The matrix $F^T F$ has the following form:

$$F^T F = \begin{pmatrix} k_1 + k_2 + k_3 + k_4 & k_1 + k_2 - k_3 - k_4 & \sqrt{2}(k_1 - k_2) & \sqrt{2}(k_3 - k_4) \\ k_1 + k_2 - k_3 - k_4 & k_1 + k_2 + k_3 + k_4 & \sqrt{2}(k_1 - k_2) & -\sqrt{2}(k_3 - k_4) \\ \sqrt{2}(k_1 - k_2) & \sqrt{2}(k_1 - k_2) & 2\left(k_1 + k_2\right) & 0 \\ \sqrt{2}(k_3 - k_4) & -\sqrt{2}(k_3 - k_4) & 0 & 2\left(k_3 + k_4\right) \end{pmatrix}.$$

The determinant of this matrix is $\det\left(F^T F\right) = 256 k_1 k_2 k_3 k_4$, where $k_i > 0, i = 1, 2, 3, 4$. The invers matrix to the matrix $\left(F^T F\right)^{-1}$ and the vector $F^T Y$ have the following form:

$$\left(F^T F\right)^{-1} = \frac{1}{16} \begin{pmatrix} \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} & \frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_3} - \frac{1}{k_4} & \sqrt{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) & \sqrt{2}\left(\frac{1}{k_3} - \frac{1}{k_4}\right) \\ \frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_3} - \frac{1}{k_4} & \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} & \sqrt{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) & -\sqrt{2}\left(\frac{1}{k_3} - \frac{1}{k_4}\right) \\ \sqrt{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) & \sqrt{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) & 2\left(\frac{1}{k_1} + \frac{1}{k_2}\right) & 0 \\ \sqrt{2}\left(\frac{1}{k_3} - \frac{1}{k_4}\right) & -\sqrt{2}\left(\frac{1}{k_3} - \frac{1}{k_4}\right) & 0 & 2\left(\frac{1}{k_3} + \frac{1}{k_4}\right) \end{pmatrix},$$

$$F^T Y = \begin{pmatrix} \sum_{i=1}^{k_1+k_2+k_3+k_4} y_i \\ \sum_{i=1}^{k_1+k_2} y_i - \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3+k_4} y_i \\ \sqrt{2} \sum_{i=1}^{k_1} y_i - \sqrt{2} \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \sqrt{2} \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \sqrt{2} \sum_{i=k_1+k_2+k_3+1}^{k_1+k_2+k_3+k_4} y_i \end{pmatrix}.$$

Then, the least squares estimates of the unknown parameters $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ determined by the formula $\hat{\theta} = \left(F^T F\right)^{-1} F^T Y$ are given by:

$$\begin{pmatrix} \frac{1}{4k_1} \sum_{i=1}^{k_1} y_i + \frac{1}{4k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i + \frac{1}{4k_3} \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i + \frac{1}{4k_4} \sum_{i=k_1+k_2+k_3+1}^{N} y_i \\ \frac{1}{4k_1} \sum_{i=1}^{k_1} y_i + \frac{1}{4k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i - \frac{1}{4k_3} \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \frac{1}{4k_4} \sum_{i=k_1+k_2+k_3+1}^{N} y_i \\ \frac{\sqrt{2}}{4k_1} \sum_{i=1}^{k_1} y_i - \frac{\sqrt{2}}{4k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \frac{\sqrt{2}}{4k_3} \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \frac{\sqrt{2}}{4k_4} \sum_{i=k_1+k_2+k_3+1}^{N} y_i \end{pmatrix},$$

and the variance of these estimates by: $D\hat{\theta} = \sigma^2 \left(F^T F\right)^{-1}$.

## 3 General case

We obtain the same result using another approach, which allows us to generalize the findings to the case of an arbitrary n. To do this, we introduce a system of characteristic functions $f_1(x)\, f_2(x), f_3(x), f_4(x)$ corresponding to the segments $d(2,1) = \left[0, \frac{1}{4}\right)$, $d(2,2) = \left[\frac{1}{4}, \frac{1}{2}\right)$, $d(2,3) = \left[\frac{1}{2}, \frac{3}{4}\right)$ and $d(2,4) = \left[\frac{3}{4}, 1\right]$ defined as follows:

$$f_i(x) = \begin{cases} 1, & if \quad x \in d(2,i), \\ 0, & if \quad x \notin d(2,i). \end{cases} \tag{3.1}$$

It is then evident that the Haar functions can be expressed in terms of the characteristic functions$f_i(x), i = 1,2,3,4$, in the following form:

$$\phi_1(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x),$$

$$\phi_2(x) = f_1(x) + f_2(x) - f_3(x) - f_4(x),$$

$$\phi_3(x) = \sqrt{2}\left(f_1(x) - f_2(x)\right),$$

$$\phi_4(x) = \sqrt{2}\left(f_3(x) - f_4(x)\right)$$

or

$$\phi(x) = Lf(x), \tag{3.2}$$

where

$$L = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}, \tag{3.3}$$

$$\phi(x) = \left(\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)\right)^T, \ f(x) = \left(f_1(x), f_2(x), f_3(x), f_4(x)\right)^T.$$

Substituting (3.2) into (1.1), we obtain

$$Y = F\theta + \epsilon = \theta^T LG + \epsilon, \tag{3.4}$$

where

$$G = \begin{pmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_N) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_N) \\ f_3(x_1) & f_3(x_2) & \dots & f_3(x_N) \\ f_4(x_1) & f_4(x_2) & \dots & f_4(x_N) \end{pmatrix}. \tag{3.5}$$

From this we obtain:

$$\theta^T L = \theta^T_{(1)}, \ \theta^T = \theta^T_{(1)} L^{-1}, \ \theta = \left(L^{-1}\right)^T \theta_{(1)}. \tag{3.6}$$

If now $\hat{\theta}_{(1)}$ is the least squares estimate of the unknown parameters under an arbitrary design $\xi$ with the dispersion matrix $D\hat{\theta}_{(1)}$, then, according to Theorem 1.4 [2], we have that the estimate $\hat{\theta} = \left(L^{-1}\right)^T \theta_{(1)}$ is also a least squares estimate, and its variance is calculated using the formula:

$$D\hat{\theta} = \left(L^{-1}\right)^T D\hat{\theta}_{(1)} L^{-1}, \tag{3.7}$$

Hence,

$$\det D\hat{\theta} = (\det L)^{-2} \det D\hat{\theta}_{(1)}, \tag{3.8}$$

which coincides with the previously obtained result, because $\det L = -16$.

The latter method for obtaining least squares estimates for the system of Haar functions allows us to generalize the estimation process to the general case. Given that Haar functions are linear combinations of step (characteristic) functions of the type (3.1), the more general case where the number of basis functions exceeds four can be studied using the previously obtained results for step functions [4].

In the general case of arbitrary $m$, we have $n = 2^m$ Haar functions defined on the interval [0,1], which is divided into $2^m$ subintervals $d(m,s)\, s = 1, 2, ..., 2^m$ of equal length.

Let us consider the system of characteristic functions $f_s(x)$ defined by the equalities:

$$f_s(x) = \begin{cases} 1, & if \quad x \in \left[\frac{s-1}{2^m}, \frac{s}{2^m}\right] \\ 0, & if \quad x \notin \left[\frac{s-1}{2^m}, \frac{s}{2^m}\right] \end{cases}, \; s = 1, 2, \ldots, 2^m.$$

**Lemma 3.1.** *Let $m$ be a natural number. Then the Haar basis functions $\{\phi_s(x)\}\,(s = 1, 2, \ldots, 2^m)$ can be expressed in terms of the characteristic functions $\{f_s(x)\}\,(s = 1, 2, \ldots, 2^m)$ using formula (3.2), where the elements of the matrix $L_m = (l_{m,i,j})_{i,j=1}^{n}$ are defined by the following recursive formula:*

$$l_{m,i,j} = \begin{cases} l_{m-1,i,\left[\frac{j+1}{2}\right]}, \quad if \quad i \le 2^{m-1} \\ 2^{\frac{m-1}{2}}, \quad if \quad i > 2^{m-1}, \quad j = i - 2^{m-1} \\ -2^{\frac{m-1}{2}}, \quad if \quad i > 2^{m-1}, \quad j = i + 1 - 2^{m-1} \\ 0, \quad if \quad i > 2^{m-1}, \quad j \ne i - 2^{m-1}, \quad j \ne i + 1 - 2^{m-1} \end{cases}. \tag{3.9}$$

*Proof.* We proceed by the method of mathematical induction. For $m = 1$, the system of Haar functions $\phi_1(x), \phi_2(x)$ is defined in equation (2.1), and the matrix

$$L_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Assume that the conditions of the theorem hold for $m = k$. We will prove formula (3.9) for the case $m = k + 1$. The subintervals $d(k+1,s)s = 1, 2, ..., 2^{k+1}$ are obtained from the subintervals $d(k,s)s = 1, 2, ..., 2^k$ by dividing each interval in half. The Haar function system consists of $2^k$ Haar functions $\{\phi_s(x)\}\,(s = 1, 2, \ldots, 2^k)$ for the case $m = k$, and $2^k$ functions $\chi_{k+1,s}(x), s = 1, 2, \ldots, 2^k$ from equation (1.3). Then, the first $2^k$ rows of the matrix $L_{k+1}$ consist of the elements of the matrix $L_k$. The next $2^k$ rows are defined by the formulas in equation (1.3), in which two elements are non-zero and the rest are zero, as reflected in the last three expressions of formula (3.9). The first $2^k$ rows of the matrix $L_k$ have $2^k$ columns, but since the subintervals $d(k+1,s)s = 1, 2, ..., 2^{k+1}$ are halves of the subintervals $d(k,s)s = 1, 2, ..., 2^k$, each element must be repeated twice. □

**Lemma 3.2.** *For the matrix $L_m$the following equality holds $L_m^T L_m = 2^m E$, where $E$ is the identity matrix.*

*Proof.* The statement of the lemma means that all rows of the matrix $L_m$ are orthogonal, and the sum of the squares of the elements in each row is equal to $2^m$. The second statement is obvious. Let us prove the orthogonality of the rows, i.e., that the scalar product of any two distinct rows is zero.

The first row is orthogonal to all other rows, since the remaining rows contain an even number of non-zero elements, half of which have opposite signs.

The last $2^{m-1}$ rows are mutually orthogonal, as each row contains exactly two non-zero elements located in different columns.

The first $2^{m-1}$ rows are also mutually orthogonal, since they are obtained by duplicating the elements of the matrix $L_{m-1}$, whose rows are orthogonal by assumption.

The scalar product of any row from the first half with any row from the second half consists of two terms with opposite signs and therefore equals zero. □

Substituting (3.2) into (1.1), we obtain:

$$Y = \theta^T F + \epsilon = \theta^T L G + \epsilon = \theta_{(1)}^T G + \epsilon, \tag{3.10}$$

where

$$G = \begin{pmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_N) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_N) \\ \dots & \dots & \dots & \dots \\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_N) \end{pmatrix}. \tag{3.11}$$

$\theta = (\theta_{1,}\theta_{2,}..., \theta_n)^T$are the unknown parameters in the case of Haar functions, and $\theta_1 = \left(\theta_1^{(1)}, \theta_2^{(1)}, \dots, \theta_n^{(1)}\right)^T$ in the case of characteristic functions. From this follow equalities (3.6).

If now $\hat{\theta}_{(1)}$ is the least squares estimate of the unknown parameters under an arbitrary design $\xi$ with the dispersion matrix $D\hat{\theta}_{(1)}$, then, according to Theorem 1.4 [2], the estimate $\hat{\theta} = (L^{-1})^T\,\theta_{(1)}$ is also a least squares estimate, and its variance is calculated using formula (3.7), while the determinant is given by formula (3.8).

Since the determinant of the information matrix $M(\hat{\theta}_{(1)})$ in the case of step functions is easily computed and equals [1]

$$\det M\left(\hat{\theta}_{(1)}\right) = n^n k_1 k_2 \dots k_n, \tag{3.12}$$

then the determinant reaches its maximum value when $k_1 = k_2 = ... = k_n$.

Thus, if the number of measurement points $N$ is divisible by the number of unknown parameters $n = 2^m$, then the following theorem holds.

**Theorem 3.1.** *Let $X = [0,1], n = 2^m, \{\phi_i(x)\}\,(i = 1,2,...,n)$ be the system of Haar functions. Consider the linear regression model (1.1) $Y = F\theta + \epsilon$. If the number of measurement points $N$ is divisible by $n$, i.e., $N = ln$, then the determinant in formula (3.8) reaches its minimum value when $k_1 = k_2 = ... = k_n = l$*

**Theorem 3.2.** *Let $X = [0,1], n = 2^m, \{\phi_i(x)\}\,(i = 1,2,...,n)$ be the system of Haar functions. Consider the linear regression model (1.1) $Y = F\theta + \epsilon$. Now suppose the number of measurement points $N$ is not divisible by the number of partitions $n$, i.e., $N = ln + r$, where $0 < r < n$.*

*Then, in the optimal design, all $k_i$ differ from each other by no more than one, i.e.,*

$$k_1 = k_2 = ... = k_r = l + 1, \quad k_{r+1} = k_{r+2} = ... = k_n = l.$$

*The number of such designs is equal to $C_n^r$.*

*Proof.* Indeed, to proceed, we order the values $k_1, k_2, \dots, k_n$ and obtain a variational series

$$k_1 \le k_2 \le \dots \le k_n,$$

where

$$k_1 k_2 \dots k_n = \frac{1}{n^n} \det M(\xi).$$

Suppose that $k_n - k_1 > 1$. We construct a new design in which all $k_i$, except for $k_1$ and $k_n$, remain unchanged, while $k_n$ is replaced by $k_n - 1$ and $k_1$ is replaced by $k_1 + 1$. For this new design, the determinant equals $(k_1+1)k_2 \dots k_n - 1(k_n - 1) > k_1, k_2, \dots,\ k_n$, i.e., the original design is not optimal, and therefore our assumption is incorrect. Thus, in the optimal design, all $k_i$ differ from each other by no more than one. The number of such designs is equal to $C_n^r$. One such design is:

$$\xi = \begin{pmatrix} x_1 & x_2 & \cdots & x_r & x_{r+1} & \cdots & x_n \\ \frac{l+1}{N} & \frac{l+1}{N} & \cdots & \frac{l+1}{N} & \frac{l}{N} & \cdots & \frac{l}{N} \end{pmatrix}.$$

□

Thus, a D-optimal design has been constructed for the system of Haar functions in a linear regression model with parameters appearing linearly.

## Discussion

Thus, we have obtained an explicit form of the discrete optimal design under the $D$-optimality criterion for a linear-in-parameters regression model in the case of the system of Haar basis functions. Furthermore, we derived an explicit form of the transformation matrix $L_m$ converting step functions into Haar functions, which satisfies the conditions of Theorem 1.4 in [2]. The resulting construction extends naturally to functions defined on an arbitrary finite interval $X = [a, b]$. One may also replace the Haar system with other basis functions possessing similar structural properties. In addition, it is possible to construct optimal designs for other optimality criteria (G, MV) and to verify the assertion of the Kiefer–Wolfowitz equivalence theorem (Theorem 2.3 in [2]).

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# ON COINCIDENCE POINTS OF MAPPINGS ON COMPACT DOMAINS

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**Key words:** compact domain, fixed point, covering mapping, strict Lipschitz inequality, coincidence point, generalized coincidence point.

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**Abstract.** In the paper, we study coincidence points of two mappings defined on a compact metric space. We assume that the first mapping satisfies the covering condition with a constant $\alpha > 0$ and the second mapping satisfies the strict Lipschitz inequality with the same constant $\alpha$. We prove that under certain continuity assumptions these two mappings have a coincidence point. An analogous result on the existence of a coincidence point and a generalized coincidence point of set-valued mappings is obtained.

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## 1 Introduction

Given metric spaces $(X, \rho_X)$ and $(Y, \rho_Y)$, denote by $B_X(x_0, r)$ the closed ball centered at point $x_0 \in X$ with radius $r \geq 0$ in the space $X$. An analogous notation we will use for closed balls in the space $Y$. Let $\psi, \varphi : X \to Y$ be given mappings.

A point $\xi \in X$ is said to be a coincidence point of the mappings $\psi$ and $\varphi$ if $\psi(\xi) = \varphi(\xi)$. In [1], there was developed the coincidence point theory. It was shown, that under natural continuity and completeness assumptions, if the mapping $\psi$ satisfies a certain covering assumption and the mapping $\varphi$ is Lipschitz with a sufficiently small Lipschitz constant, then there exists a coincidence point of the mappings $\psi$ and $\varphi$. Analogous results were derived for set-valued mappings. Let us recall the concepts in use.

Given a real number $\alpha > 0$, the mapping $\psi$ is said to be $\alpha$-covering if the following inclusion takes place

$$B_Y(\psi(x_0), \alpha r) \subset \psi(B_X(x_0, r)) \quad \forall\, x_0 \in X, \quad \forall\, r \geq 0.$$

Given a real $\beta > 0$, the mapping $\varphi$ is said to be $\beta$-Lipschitz if the following inequality takes place

$$\rho_Y(\varphi(x_1), \varphi(x_2)) \leq \beta \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in X.$$

The mapping $\psi$ is said to be closed if its graph

$$\operatorname{gph} \psi := \{(x, \psi(x)) :\, x \in X\}$$

is a closed subset of the metric space $X \times Y$ endowed with the metric defined by formula

$$\rho((x_1, y_1), (x_2, y_2)) := \rho_X(x_1, x_2) + \rho_Y(y_1, y_2), \quad (x_1, y_1),\, (x_2, y_2) \in X \times Y.$$

Note that there could be defined other equivalent metrics in the spaces $X \times Y$. However, the one defined above is more convenient in the subsequent.

Recall now the coincidence point theorem from [1]. *Assume that the metric space* $(X, \rho_X)$ *is complete, the mapping* $\psi : X \to Y$ *is* $\alpha$*-covering and closed, the mapping* $\varphi : X \to Y$ *is* $\beta$*-Lipschitzian and* $\alpha > \beta > 0$*. Then for every point* $x_0 \in X$ *there exists a point* $\xi = \xi(x_0) \in X$ *such that*

$$\psi(\xi) = \varphi(\xi) \quad \text{and} \quad \rho_X(x_0, \xi) \le \frac{\rho_Y(\psi(x_0), \varphi(x_0))}{\alpha - \beta}.$$

In the particular case, in which the space $(Y, \rho_Y)$ coincides with the complete space $(X, \rho_X)$ and $\psi$ is the identity mapping, this coincidence point theorem implies the fixed point theorem by S. Banach and R. Caccioppoli (see, for example, [5, Chapter 1, §1]).

In this paper, we consider the coincidence point problem for the specific case $\alpha = \beta$.

If $\alpha = \beta$ and all assumptions of the cited coincidence point theorem are preserved except $\alpha > \beta$, then the mappings $\psi$ and $\varphi$ could have no coincidence point even under the additional assumption that the domain $X$ is compact. For example, assume that $X$ is a circle in the Euclidian plane $\mathbb{R}^2$ with the induced metric, $Y = X$, $\psi : X \to X$ is the identity mapping, $\varphi$ is a rotation mapping by a fixed angle different from $2\pi m$ and $m \in \{..., -1, 0, 1, ...\}$. Take $\alpha = \beta = 1$. Then, all the assumption of the coincidence point Theorem 1 in [1] hold except the assumption $\alpha > \beta$. At the same time, the mappings $\psi$ and $\varphi$ have no coincidence points, although the domain $X$ is compact.

Assume now that the mapping $\varphi$ satisfies the strict Lipschitz inequality with the constant $\alpha$, i.e.

$$\rho_Y(\varphi(x_1), \varphi(x_2)) < \alpha \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in X : \quad x_1 \ne x_2. \tag{1.1}$$

Then, the $\alpha$-covering mapping $\psi$ and the mapping $\varphi$ can also have no coincidence point. For example, assume that $X = Y = [1, +\infty)$ and the mappings $\psi : X \to X$ and $\varphi : X \to X$ are defined by formulae

$$\psi(x) = x, \quad \varphi(x) = x + \frac{1}{x} + c, \quad x \in [1, +\infty).$$

Here, real $c \ge 0$ is given. Take $\alpha := 1$. Then the mapping $\psi$ is $\alpha$-covering and continuous, whereas the mapping $\varphi$ satisfies strict Lipschitz inequality (1.1). However, the mappings $\psi$ and $\varphi$ have no coincidence points. Moreover,

$$|\psi(x) - \varphi(x)| > c \quad \forall\, x \in [1, +\infty). \tag{1.2}$$

Note that, both considered examples were given in monograph [8] and illustrated the absence of a fixed point of the corresponding mappings.

As is known, if the metric space $X$ is compact, $Y = X$ and the mapping $\varphi : X \to X$ is a strict contraction (i.e. inequality (1.1) holds with $\alpha = 1$), then the mapping $\varphi$ has the only coincidence point (see, for example, [5, §1.6 (A.7)] or [8, §1.6]). In other words, in the contraction mapping theorem, the contraction assumption can be weakened by replacing it with the assumption of strict contraction, and the assumption of completeness of the space $X$ can be strengthened by replacing it with the assumption of compactness of this space. For more results on fixed points of mappings satisfying various contraction assumptions see, for example, [6, 7, 9]. For the results on application to optimization see, for example, [3], [4].

In connection with the above, the following natural question arises. Is Theorem 1 in [1] true if the assumption of completeness of $X$ is replaced by the assumption of compactness of $X$, and the assumption that $\varphi$ is $\beta$-Lipschitz with $\beta < \alpha$ is replaced by the assumption that strict Lipschitz inequality (1.1) holds with the same constant $\alpha$?

Below, we will provide a positive answer to this question. In addition, we obtain sufficient conditions for the existence of coincidence points and generalized coincidence points of set-valued mappings.

## 2 Coincidence points of single-valued mappings

Let us now present sufficient conditions for the existence of coincidence points of two mappings defined on compact domain.

**Theorem 2.1.** *Let a metric space $(X, \rho_X)$ be compact. Given $\alpha > 0$, assume that a mapping $\psi : X \to Y$ is $\alpha$-covering and closed, a mapping $\varphi : X \to Y$ satisfies strict Lipschitz inequality (1.1) with the constant $\alpha$.*

*Then, there exists a coincidence point $\xi \in X$ of the mappings $\psi$ and $\varphi$, i.e. $\psi(\xi) = \varphi(\xi)$.*

*Proof.* Denote

$$\mathcal{D} := \{(x_1, x_2) \in X \times X : \psi(x_2) = \varphi(x_1)\}.$$

Let us prove that the set $\mathcal{D}$ is a closed subset of the space $X \times X$ endowed with the metric

$$\rho((x_1, x_2), (\bar{x}_1, \bar{x}_2)) := \rho_X(x_1, \bar{x}_1) + \rho_Y(x_2, \bar{x}_2), \quad (x_1, x_2), (\bar{x}_1, \bar{x}_2) \in X \times X.$$

Take an arbitrary sequence $\{(x_1^i, x_2^i)\} \subset \mathcal{D}$ convergent to a point $(\bar{x}_1, \bar{x}_2) \in X \times X$. Then, $\varphi(x_1^i) \to \varphi(\bar{x}_1)$ as $i \to +\infty$ since the mapping $\varphi$ is Lipschitz and, therefore, continuous. Since $\{(x_1^i, x_2^i)\} \subset \mathcal{D}$, the definition of the set $\mathcal{D}$ implies that $\psi(x_2^i) = \varphi(x_1^i)$ for every $i$. Therefore, $\psi(x_2^i) \to \varphi(\bar{x}_1)$ as $i \to +\infty$ as well. The closedness of the mapping $\psi$ and the relations $x_2^i \to \bar{x}_2$ and $\psi(x_2^i) \to \varphi(\bar{x}_1)$ as $i \to +\infty$ imply that the point $(\bar{x}_2, \varphi(\bar{x}_1))$ belongs to the graph of the mapping $\psi$. Therefore, the equality $\psi(\bar{x}_2) = \varphi(\bar{x}_1)$ takes place. Hence, $(\bar{x}_1, \bar{x}_2) \in \mathcal{D}$. So, it is shown that the set $\mathcal{D}$ is a closed subset of the space $X \times X$.

Let us consider the following constrained optimization problem:

$$\text{minimize } \rho_X(x_1, x_2) \quad \text{subject to the condition } (x_1, x_2) \in \mathcal{D}. \tag{2.1}$$

The set $\mathcal{D}$ is compact since this set is a closed subset of the compact space $X \times X$. Moreover, the function $(x_1, x_2) \mapsto \rho_X(x_1, x_2)$ is continuous on the entire space $X \times X$. Therefore, the Weierstrass theorem implies that there exists at least one point $(\xi_1, \xi_2) \in \mathcal{D}$ which is a solution to Problem (2.1), i.e. $\rho_X(\xi_1, \xi_2) \le \rho_X(x_1, x_2)$ for every $(x_1, x_2) \in \mathcal{D}$.

Let us prove that the minimal value $\rho_X(\xi_1, \xi_2)$ to Problem (2.1) equals zero. Assume the contrary, i.e. $\rho_X(\xi_1, \xi_2) > 0$ or equivalently $\xi_1 \ne \xi_2$. Since the mapping $\psi$ is $\alpha$-covering, there exists a point $\xi_3 \in X$ such that $\psi(\xi_3) = \varphi(\xi_2)$ and the inequality $\rho_X(\xi_2, \xi_3) \le \alpha^{-1}\rho_Y(\psi(\xi_2), \varphi(\xi_2))$ takes place. Applying this inequality we obtain

$$\rho_X(\xi_2, \xi_3) \le \alpha^{-1}\rho_Y(\psi(\xi_2), \varphi(\xi_2)) = \alpha^{-1}\rho_Y(\varphi(\xi_1), \varphi(\xi_2)) < \rho_X(\xi_1, \xi_2).$$

Here, the equality follows from the inclusion $(\xi_1, \xi_2) \in \mathcal{D}$ and the strict inequality follows from (1.1) since $\xi_1 \ne \xi_2$.

So, we have $(\xi_2, \xi_3) \in \mathcal{D}$, since $\psi(\xi_3) = \varphi(\xi_2)$. At the same time, it is shown that $\rho_X(\xi_2, \xi_3) < \rho_X(\xi_1, \xi_2)$. Therefore, the point $(\xi_1, \xi_2)$ is not a solution to Problem (2.1). The contradiction obtained proves that $\rho_X(\xi_1, \xi_2) = 0$.

Denote $\xi := \xi_1$. Since $\rho_X(\xi_1, \xi_2) = 0$, we have $\xi_2 = \xi$. Thus, the inclusion $(\xi_1, \xi_2) \in \mathcal{D}$ implies that $\psi(\xi) = \psi(\xi_1) = \varphi(\xi_1) = \varphi(\xi)$. □

Let us compare the obtained coincidence point theorem with Theorem 7.2 in [2]. In Theorem 7.2 it is assumed that the domain $X$ as well as the target space $Y$ are Banach spaces, whereas in Theorem 2.1 these spaces are metric ones and $X$ is compact. In [2], it is assumed that the $\alpha$-covering mapping $\psi$ is smooth, whereas in Theorem 2.1 $\psi$ is closed. In [2], the mapping $\varphi$ satisfies a stronger assumption than (1.1). So, these two coincidence point theorems do not follow from each other. These two theorems also differs from Theorem 1 in [1]. The key difference is that they can be applied if the mapping $\varphi$ is not $\beta$-Lipschitz with $\beta < \alpha$. However, Theorem 2.1 here as well as the result in [2] are not applicable if the metric space $X$ is neither compact nor Banach. At the same time Theorem 1 in [1] can be applied to certain mappings in this case.

## 3 Coincidence points of set-valued mappings

Let us pass to the study of the coincidence points of set-valued mappings.

For a nonempty set $M \subset Y$ and a real number $r \geq 0$ denote $B_Y(M, r) := \bigcup_{y \in M} B_Y(y, r)$. Denote by $h_Y$ the Hausdorff distance between subsets of the space $Y$, i.e.

$$h_Y(M, N) := \inf\{r > 0 : \ B_Y(M, r) \supset N, \ \ B_Y(N, r) \supset M\}$$

for nonempty closed subsets $M, N \subset Y$. Here, if the set in the right-hand side of the equality is empty, then we assume that $h_Y(M, N) := +\infty$. Below we will also use the following distance function between sets

$$\mathrm{dist}_Y(M, N) := \inf\{\rho_Y(y_1, y_2) : \ y_1 \in M, \ \ y_2 \in N\}.$$

Here $M, N \subset Y$ are arbitrary nonempty closed subsets.

Let $\Psi, \Phi : X \rightrightarrows Y$ be given set-valued mappings, i.e. the mappings that correspond to each point $x \in X$ closed nonempty subsets of the space $Y$. A point $\xi \in X$ is said to be a coincidence point of the mappings $\Psi$ and $\Phi$ if $\Psi(\xi) \cap \Phi(\xi) \neq \emptyset$. A point $\xi \in X$ is said to be a generalized coincidence point of the mappings $\Psi$ and $\Phi$ if $\mathrm{dist}(\Psi(\xi), \Phi(\xi)) = 0$.

Given a number $\alpha > 0$, recall that the set-valued mapping $\Psi$ is said to be $\alpha$-covering if

$$B_Y(\Psi(x_0), \alpha r) \subset \Psi(B_X(x_0, r)) \quad \forall\, x_0 \in X, \quad \forall\, r \geq 0.$$

The set-valued mapping $\Psi$ is said to be upper semicontinuous if for every point $x \in X$ and every sequence $\{(x^i, y^i)\} \subset \mathrm{gph}\, \Psi$ the relation $\mathrm{dist}(y^i, \Psi(x)) \to 0$ as $i \to \infty$ takes place. Here,

$$\mathrm{gph}\, \Psi := \{(x, y) \in X \times Y : \ x \in X, \ \ y \in \Psi(x)\}.$$

Let us formulate now sufficient conditions for the existence of a generalized coincidence point of two set-valued mappings.

**Theorem 3.1.** *Let a metric space $(X, \rho_X)$ be compact. Given $\alpha > 0$, assume that a set-valued mapping $\Psi : X \rightrightarrows Y$ is $\alpha$-covering and upper semicontinuous, and a set-valued mapping $\Phi : X \rightrightarrows Y$ satisfies the strict Lipschitz inequality with the constant $\alpha$, i.e.*

$$h_Y(\Phi(x_1), \Phi(x_2)) < \alpha \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in X : \quad x_1 \neq x_2. \tag{3.1}$$

*Then, there exists a generalized coincidence point $\xi \in X$ of the set-valued mappings $\Psi$ and $\Phi$, i.e.* $\mathrm{dist}_Y(\Psi(\xi), \Phi(\xi)) = 0$.

Before moving on to the proof of the main results of this section, we present and discuss auxiliary constructions. First, consider a constrained optimization problem (2.1) with the set $\mathcal{D}$ defined in a new way by the formula

$$\mathcal{D} := \{(x_1, x_2) \in X \times X : \ \Psi(x_2) \cap \Phi(x_1) \neq \emptyset\}. \tag{3.2}$$

If this problem has a solution and the minimal value for this problem equals zero, then obviously there exists a generalized coincidence point $\xi \in X$ of the set-valued mappings $\Psi$ and $\Phi$. In the particular case when the mappings $\Psi$ and $\Phi$ are single-valued, the set $\mathcal{D}$ coincides with the set $\mathcal{D}$ in the proof of Theorem 2.1. In this case, the set $\mathcal{D}$ is compact. This fact was shown in the proof of Theorem 2.1. However, if the mapping $\Phi$ is set-valued, then the set $\mathcal{D}$ is not necessarily compact. Consider the corresponding example.

**Example 1.** Put $X := [0,1]$, $Y := \{0\} \cup [1,+\infty)$,

$$\Psi(x) := 1/x, \quad x \in (0,1], \quad \Psi(0) := 0,$$

$$\Phi(x) := [1,+\infty), \quad x \in [0,1].$$

Then, the space $X$ is compact, the mapping $\Psi$ is closed and 1-covering. The mapping $\Phi$ is constant, so it satisfies strict Lipschitz inequality (3.1) with the constant $\alpha = 1$.

The points $(1/i, 1/i)$ belong to the set $\mathcal{D}$ for every $i$, since $\Psi(1/i) = i \in [1,+\infty) = \Phi(1/i)$. At the same time the limit $(0,0)$ of the sequence $(1/i,1/i)$ does not belong to the set $\mathcal{D}$, since $\Psi(0) = 0 \notin [1,+\infty) = \Phi(0)$. Therefore, the set $\mathcal{D}$ is not compact.

This example shows that the reasonings from the proof of Theorem 2.1 are not valid when the assumptions of Theorem 3.1 hold. Namely, we cannot apply the Weierstrass theorem to Problem (2.1), since the set of admissible points $\mathcal{D}$ is not necessarily compact. Thus, we cannot prove the existence of a solution to Problem (2.1). In fact, Example 2 below shows that a solution may not exist. So, we need the following auxiliary assertions.

**Lemma 3.1.** *Let a metric space $(X, \rho_X)$ be compact. Given $\alpha > 0$, assume that a set-valued mapping $\Phi : X \rightrightarrows Y$ satisfies inequality (3.1).*

*Then, for every real number $\mu > 0$ there exists a nonnegative real number $\beta = \beta(\mu) \ < \alpha$ such that*

$$h_Y(\Phi(x_1), \Phi(x_2)) \le \beta \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in X : \quad \rho_X(x_1, x_2) \ge \mu.$$

*Proof.* Take an arbitrary real number $\mu > 0$. Denote

$$\mathcal{M} := \{(x_1, x_2) : \rho_X(x_1, x_2) \ge \mu\}.$$

It is obvious that $\mathcal{M}$ is a compact subset of the space $X \times X$.

Consider the function $f : \mathcal{M} \to \mathbb{R}$ defined by formula

$$f(x_1, x_2) := \frac{h_Y(\Phi(x_1), \Phi(x_2))}{\rho_X(x_1, x_2)}, \quad (x_1, x_2) \in \mathcal{M}.$$

Since $\rho(x_1, x_2) \ge \mu > 0$ for every $(x_1, x_2) \in \mathcal{M}$, inequality (3.1) implies that this function $f$ is continuous. The Weierstrass theorem implies that the function $f$ attains its maximal value $\beta$ at some point $(\bar{x}_1, \bar{x}_2) \in \mathcal{M}$. Then $\bar{x}_1 \ne \bar{x}_2$. So, inequality (3.1) implies that $\beta < \alpha$. Moreover,

$$h_Y(\Phi(x_1), \Phi(x_2)) = f(x_1, x_2) \rho_X(x_1, x_2) \le \beta \rho_X(x_1, x_2) \quad \forall\, (x_1, x_2) \in \mathcal{M}.$$

Thus, the constructed value $\beta$ is the desired one. □

**Lemma 3.2.** *Let a metric space $(X, \rho_X)$ be compact. Given $\alpha > 0$, assume that a set-valued mapping $\Psi : X \rightrightarrows Y$ is $\alpha$-covering and a set-valued mapping $\Phi : X \rightrightarrows Y$ satisfies inequality (3.1).*

*Then the infimum in problem (2.1) equals zero, i.e.*

$$\inf\{\rho_X(x_1, x_2) : (x_1,\, x_2) \in \mathcal{D}\} = 0. \tag{3.3}$$

*Here, the set $\mathcal{D}$ is defined by formula (3.2).*

*Proof.* Denote $\mu := \inf\{\rho_X(x_1, x_2) : (x_1,\, x_2) \in \mathcal{D}\}$. It is obvious that $\mu \ge 0$. Let us prove that $\mu = 0$.

Assume the contrary: $\mu > 0$. For each $(x_1, x_2) \in \mathcal{D}$ we have $\rho_X(x_1, x_2) \ge \mu$. Then Lemma 3.1 implies that there exists a nonnegative real number $\beta < \alpha$ such that

$$h_Y(\Phi(x_1), \Phi(x_2)) \le \beta \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in \mathcal{D}.$$

Thus, there exists a real number $\theta > 1$ such that $\beta\theta < \alpha$.

Since $\mu > 0$, the definition of $\mu$ implies that there exists a point $(x_1, x_2) \in \mathcal{D}$ such that $\rho_X(x_1, x_2) < \mu\alpha/(\beta\theta)$. Since $(x_1, x_2) \in \mathcal{D}$, there exists a point $y_1 \in Y$ such that $y_1 \in \Psi(x_2) \cap \Phi(x_1)$. Thus, the definition of the Hausdorff distance implies that the inequality $\operatorname{dist}(y_1, \Phi(x_2)) < \theta h_Y(\Phi(x_1), \Phi(x_2))$ holds. So, there exists a point $y_2 \in \Phi(x_2)$ such that $\rho_Y(y_1, y_2) \leq \theta h_Y(\Phi(x_1), \Phi(x_2))$.

Since the mapping $\Psi$ is $\alpha$-covering, there exists a point $x_3 \in X$ which satisfies the relations

$$y_2 \in \Psi(x_3) \quad \text{and} \quad \rho_X(x_2, x_3) \leq \frac{1}{\alpha}\rho_Y(y_1, y_2).$$

Note that the obtained inequalities imply that

$$\rho_X(x_2, x_3) \leq \frac{1}{\alpha}\rho_Y(y_1, y_2) \leq \frac{\theta}{\alpha}h_Y(\Phi(x_1), \Phi(x_2)) \leq \frac{\beta\theta}{\alpha}\rho_X(x_1, x_2) < \mu.$$

Moreover, $(x_2, x_3) \in \mathcal{D}$, since $y_2 \in \Psi(x_3)$ and $y_2 \in \Phi(x_2)$ as is shown above. The obtained inequality $\rho_X(x_2, x_3) < \mu$ and the inclusion $(x_2, x_3) \in \mathcal{D}$ imply that the value $\mu$ is less than $\inf\{\rho_X(x_1, x_2) : (x_1,\, x_2) \in \mathcal{D}\}$. This contradicts the fact that $\mu$ is the infimum. The obtained contradiction proves equality (3.3). $\square$

Lemma 3.2 shows that infimum of $\rho_X(x_1, x_2)$ in (2.1) equals zero. The compactness assumption is essential for this assertion. Indeed, let $X = Y = [1, +\infty)$, $\psi(x) = x$, $\varphi(x) = x + 1/x + c$, $x \in [1, +\infty)$ and $c > 0$ be a given real number. Then, by virtue of (1.2), the statement of Lemma 3.2 fails even for single-valued mappings.

*Proof of Theorem 3.1.* Lemma 3.2 implies that

$$\inf\{\rho_X(x_1, x_2) : (x_1,\, x_2) \in \mathcal{D}\} = 0.$$

Here, $\mathcal{D}$ is the set defined by formula (3.2). So, there exists a sequence $\{(x_1^j, x_2^j)\} \subset \mathcal{D}$ such that $\rho_X(x_1^j, x_2^j) \to 0$ as $j \to \infty$. This sequence has a convergent subsequence, since the space $X$ is compact. Denote this subsequence by $\{(x_1^j, x_2^j)\}$ as well. Denote its limit by $(\xi_1, \xi_2) \in X \times X$. Since $\rho_X(x_1^j, x_2^j)$ tends zero as $j \to \infty$, we obtain that $\xi_1 = \xi_2$. Denote $\xi := \xi_1$. So, we have $x_1^j \to \xi$ and $x_2^j \to \xi$ as $j \to \infty$.

Since $\{(x_1^j, x_2^j)\} \subset \mathcal{D}$, we have $\Psi(x_2^j) \cap \Phi(x_1^j) \neq \emptyset$ for each $j$. Therefore, there exists a sequence $\{y_j\} \subset Y$ such that $y_j \in \Psi(x_2^j) \cap \Phi(x_1^j)$ for each $j$. Since the set-valued mapping $\Psi$ is upper semicontinuous, $y_j \in \Psi(x_2^j)$ for all $j$ and $x_2^j \to \xi$, then $\operatorname{dist}_Y(y_j, \Psi(\xi)) \to 0$ as $j \to \infty$. Since $y_j \in \Phi(x_1^j)$ for each $j$, inequality (3.1) implies that $\operatorname{dist}_Y(y_j, \Phi(\xi)) \to 0$ as $j \to \infty$.

Let us prove now that $\operatorname{dist}_Y(\Psi(\xi), \Phi(\xi)) = 0$. For each $j$ take $\psi_j \in \Psi(\xi)$, $\varphi_j \in \Phi(\xi)$ and a sequence of positive real numbers $\delta_j$ such that

$$\rho_Y(\psi_j, y_j) \leq \operatorname{dist}_Y(\Psi(\xi), y_j) + \delta_j, \quad \rho_Y(\varphi_j, y_j) \leq \operatorname{dist}_Y(\Phi(\xi), y_j) + \delta_j, \quad \delta_j \to 0+ \quad \text{as} \quad j \to \infty.$$

Applying the triangle inequality, we have

$$\rho_Y(\psi_j, \varphi_j) \leq \rho_Y(\psi_j, y_j) + \rho_Y(y_j, \varphi_j) \leq \operatorname{dist}_Y(\Psi(\xi), y_j) + \operatorname{dist}_Y(\Phi(\xi), y_j) + 2\delta_j.$$

This inequality and the obtained relations imply that $\rho_Y(\psi_j, \varphi_j) \to 0$ as $j \to \infty$. Thus, the inequality $\operatorname{dist}_Y(\Psi(\xi), \Phi(\xi)) \leq \rho_Y(\psi_j, \varphi_j)$ implies that $\operatorname{dist}_Y(\Psi(\xi), \Phi(\xi)) = 0$. $\square$

Theorem 3.1 implies the following assertion on coincidence points.

**Corollary 3.1.** *Let the assumptions of Theorem 3.1 hold. Namely,* $\alpha > 0$ *is given, a metric space* $(X, \rho_X)$ *is compact, a set-valued mapping* $\Psi : X \rightrightarrows Y$ *is* $\alpha$*-covering and upper semicontinuous, a set-valued mapping* $\Phi : X \rightrightarrows Y$ *satisfies inequality* (3.1)*. Assume additionally that for each* $x \in X$ *at least one of two sets either* $\Psi(x)$ *or* $\Phi(x)$ *is compact.*

*Then there exists a coincidence point* $\xi \in X$ *of the set-valued mappings* $\Psi$ *and* $\Phi$*, i.e.* $\Psi(\xi) \cap \Phi(\xi) \neq \emptyset$*.*

*Proof.* Theorem 3.1 implies that there exists a point $\xi \in X$ such that $\mathrm{dist}_Y(\Psi(\xi), \Phi(\xi)) = 0$. Since at least one of two sets either $\Psi(x)$ or $\Phi(x)$ is compact, it follows from this equality that $\Psi(\xi) \cap \Phi(\xi) \neq \emptyset$. $\square$

**Remark 1.** Theorem 2.1 does not follow from Corollary 3.1 of Theorem 3.1, since the closedness assumption in Theorem 2.1 is weaker than upper continuity assumption in Corollary 3.1.

In Theorem 3.1, the coincidence point may not exist. Let us demonstrate this fact by the following example which was presented in [1].

**Example 2.** Put $\Pi := \{(x, x^2/2) : x \in [0, 1]\} \subset \mathbb{R}^2$. Consider the spaces $X = [0, 1]$, $Y = ([0, 1] \times [0, 1]) \setminus \Pi$ with the metrics induced by the metric of $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Put

$$\Psi(x) := \{(x, t) : t \in [0, 1]\} \setminus \Pi, \quad \Phi(x) := \{(t, tx/2) : t \in [0, 1]\} \setminus \Pi, \quad x \in [0, 1].$$

Obviously, the metric space $X$ is compact. Put $\alpha := 1$. The set-valued mapping $\Psi : X \rightrightarrows Y$ is $\alpha$-covering and upper semicontinuous. Moreover, the set-valued mapping $\Phi : X \rightrightarrows Y$ satisfies strict Lipschitz inequality (3.1) with the constant $\alpha = 1$. So, all the assumptions of Theorem 3.1 are satisfied. Thus, $\Psi$ and $\Phi$ has a generalized coincidence point. However, $\Psi(x) \cap \Phi(x) = \emptyset$ for each $x \in X$. For more details see [1, Example 1].

Finally, note that the upper semicontinuity assumption in Theorem 3.1 cannot be replaced by the weaker closedness assumption. For the corresponding example see [1, Example 2].

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# EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THIRD-KIND LINEAR VOLTERRA INTEGRAL EQUATIONS

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**Key words:** Volterra integral equation of the third kind, existence and regularity of solutions.

**AMS Mathematics Subject Classification:** 45A05, 45D05, 34A12.

**Abstract.** In this paper, there are studied third-kind linear Volterra integral equations with smooth data and the operator of multiplying by a smooth function that degenerates at the initial point of the integration interval. A theorem of existence, uniqueness, and continuity of a solution is proved. Conditions for smoothness and the degree of smoothness of a solution are obtained. Additionally, the existence and uniqueness of the solution in the $L^p$ space are proved.

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## 1 Introduction

According to the theory of integral equations, the solution of first-kind linear Volterra integral equations exists and is unique in the space of continuous functions if the input data is smooth and the kernel $K(x,t)$ does not vanish on the diagonal $t = x$. In this case, it is possible to precisely determine the value of the desired function at the initial point of the segment, which is requried for the numerical solution of the equation. Naturally, the question arises whether similar results can be obtained for third-kind linear Volterra integral equations in the case of smooth known functions.

G.C. Evans, in [4, 5], thoroughly studied the solvability conditions for Volterra integral equations

$$a(x)u(x) + \int_0^x K(x,t)u(t)dt = f(x), \tag{1.1}$$

with the right-hand side $f(x) = a(x)g(x)$ and a singular kernel.

Since then, only a few works have been published addressing the existence of solutions for third-kind Volterra integral equations [1, 6, 8, 15, 19]. These works have investigated only special classes of third-kind equations with continuous data [1, 19] and weakly singular kernels [1, 6, 8, 15].

T. Sato [19] constructed the solution of nonlinear third-kind Volterra integral equations in the form of a power series when $a(x) = x$. T.F. Fényes [6] demonstrated the existence of a locally integrable solution to equation (1.1) with a convolution-type kernel and $a(x) = x + c, c < 0$. A.M. Nakhushev [15], using the apparatus of fractional differentiation, investigated integral equation (1.1) with a weakly singular kernel for $a(x) = x^\beta, 0 < \beta < 1$. P. Grandits [8] obtained conditions under which equation (1.1) with the special kernel $K(x,t) = 1 + \Gamma(x,t)$ has a unique continuous solution. S.S. Allaei, Z.-W. Yang, and H. Brunner [1], using the properties of cordial Volterra integral operators, proved the existence of a unique continuous solution to equation (1.1) with a continuous (or weakly singular) kernel for $a(x) = x^\beta, \beta > 0$. A multiparameter family of solutions to equation (1.1) in Banach spaces of a special type was obtained by N.A. Magnitskii [14].

G.C. Evans's method [4] has been further developed in the theory of regularization of third-kind Volterra integral equations. In the works of A. Asanov [2], M.I. Imanaliev and A. Asanov [10], and T.T. Karakeev [12], issues of regularizing the solution of equation (1.1) in the space of continuous functions, when $a(x)$ is monotonic (in the scalar case), have been investigated. T.D. Omurov [16] and S.B. Tagaeva [20] demonstrated the applicability of the regularization method in the space of summable functions for (1.1) with a non-decreasing function $a(x)$. The regularization of equation (1.1) with a weakly singular kernel is addressed in the work of S.V. Pereverzev and S.A. Prössdorf [18]. S. Iskandarov [11] studied the conditions for the uniqueness of a solution of equation (1.1) on the half-line.

In this work, we will prove the existence and uniqueness of a solution to integral equations (1.1) both in the class of continuous functions and in the space $L_p(0, b)$. We will define the smoothness order of a solution.

## 2 Resolving equation

G.C. Evans [4] proved the existence of a unique bounded solution to a second-kind Volterra integral equation with a non-integrable order kernel. The study of such equations was continued by L.I. Panov [17], who demonstrated the unique solvability of the equation in a special Banach space of continuous functions. In this section, we will consider a special class of second-kind Volterra integral equations, which can be categorized as the type of equations studied in [4, 17]. The results of this section will be used in subsequent sections.

Let $G(x), a(x), f(x)$ be known functions on the segment $[0, b]$ and a function $Q(x, t)$ be defined in the domain $D := \{(x, t) : 0 \le t \le x \le b\}$. Assume that the following condition holds:

$(A)$ $\quad a(0) = 0, \int\limits_0^x \frac{dt}{a(t)} = +\infty, a(x) > 0, \forall x \in (0, b].$

Consider the integral equation

$$v(x) = \int\limits_0^x exp\left( - \int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)} \right) \frac{G(t)}{a(t)} \left\{ \int\limits_0^t Q(t, s)v(s)ds + f(t) \right\} dt. \tag{2.1}$$

By $C^{n,0}(D)$ we denote the space of continuous functions $z(x, t)$ on $D$ that have continuous derivatives $\frac{\partial^n z}{\partial x^n}$, and functions $\frac{\partial d^i z}{\partial dx^i}(x, x), i = 0, 1, ..., n-1$, are differentiable on $[0, b]$ up to order $n-i-1$ inclusively. $C^{n,1}(D)$ is the space of continuous functions $w(x, t)$ on $D$ that have continuous derivatives $\frac{\partial w}{\partial t}, \frac{\partial^n w}{\partial x^n}$ and $\frac{\partial w}{\partial t} \in C^{n,0}(D)$.

**Theorem 2.1.** *Let functions $a(x), f(x)$, and $G(x)$ be continuous on $[0, b]$, a function $Q(x, t)$ be continuous on the domain D, and the function $a(x)$ satisfy condition (A), and $G(x) > 0$. Then, equation* (2.1) *has a unique solution $v(x) \in C[0, b]$ with $v(0) = f(0)$. If $a(x), f(x) \in C^n[0, b], G(x) \in C^{n-1}[0, b], Q(x, t) \in C^{n,0}(D)$, and the following condition holds*

$$G_m(x) = G(x) + ma'(x) \ge d_1 > 0, m = 0, 1, ..., n, \tag{2.2}$$

*then the solution $v(x) \in C^n[0, b]$.*

*Proof.* Let $Q_1 = max\{|Q(x, t)|, (x, t) \in D\}, N_0 = max\{|f(x)|, x \in [0, b]\}$. From (2.1), using Dirichlet's rule for the repeated integral, we obtain

$$v(x) = \int\limits_0^x L(x, t)v(t)dt + F(x),$$

where

$$L(x,t) = \int\limits_t^x exp\left(-\int\limits_s^x \frac{G(\tau)d\tau}{a(\tau)}\right)\frac{G(s)}{a(s)}Q(s,t)ds,$$

$$F(x) = \int\limits_0^x exp\left(-\int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)}\right)\frac{G(t)}{a(t)}f(t)dt.$$

Since, according to (A) and (2.2),

$$0 \leq exp\left(-\int\limits_0^x \frac{G(\tau)d\tau}{a(\tau)}\right) \leq exp\left(-d_1\int\limits_0^x \frac{d\tau}{a(\tau)}\right) = 0,$$

we have

$$exp\left(-\int\limits_0^x \frac{G(\tau)d\tau}{a(\tau)}\right) = 0.$$

Thus, for any $x \in (0,b]$

$$\int\limits_0^x \frac{G(\tau)d\tau}{a(\tau)} = +\infty.$$

Let $\xi = \int\limits_x^b \frac{G(\tau)d\tau}{a(\tau)}, \eta = \int\limits_t^b \frac{G(\tau)d\tau}{a(\tau)}$. At $t = x$, we have $\eta = \xi$. If t=0, then $\eta = +\infty$ . Then [4, Section 6]

$$\int\limits_0^x exp\left(\int\limits_x^b \frac{G(\tau)d\tau}{a(\tau)} - \int\limits_t^b \frac{G(\tau)d\tau}{a(\tau)}\right)\frac{G(t)}{a(t)}dt = \int\limits_\xi^\infty exp(\xi-\eta)d\eta = 1$$

and

$$\int\limits_t^x exp\left(-\int\limits_s^x \frac{G(\tau)d\tau}{a(\tau)}\right)\frac{G(s)}{a(s)}ds = \int\limits_\xi^\eta exp(\xi-\sigma)d\sigma = 1 - exp\left(-\int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)}\right),$$

where $\sigma = \eta$ at $s = t$. Due to this, the following estimates hold

$$|F(x)| \leq N_0, \quad |L(x,t)| \leq Q_1,$$

and

$$|v(x)| \leq Q_1\int\limits_0^x |v(t)|dt + N_0. \tag{2.3}$$

If a function $g(x)$ is integrable on $[0,b]$, $g(x) \geq 0$, and a function $f(x)$ is continuous on $[0,b]$, then the theory of definite integrals allows the application of the mean value theorem in a generalized form [9, p. 324]: for some $x_1 \in [0,b]$

$$\int\limits_a^b f(t)g(t)dt = f(x_1)\int\limits_a^b g(t)dt.$$

Due to this formula and condition (A) we have that for some $\bar{x} \in [0, x_0]$

$$F(x_0) = f(\bar{x}) \int\limits_0^{x_0} exp\left( - \int\limits_t^{x_0} \frac{G(\tau)d\tau}{a(\tau)} \right) \frac{G(t)}{a(t)} dt = f(\bar{x}).$$

where $x_0$ is an arbitrary point from $[0, b]$. From this, when $x_0 \to 0$ it follows that $F(0) = f(0)$. So the kernel $L(x, t)$ degenerates on the diagonal $t = x : L(x, x) = 0$.

For the function $F(x)$ and the kernel $L(x, t)$, it is easily established by standard methods that the increments $\Delta F \to 0$ as $\Delta x \to 0$ and $\Delta L \to 0$ as $\Delta x \to 0, \Delta t \to 0$.

From the above and estimate (2.3), it follows that the function $F(x)$ and the kernel $L(x, t)$ are continuous, respectively, in the regions $[0, b]$ and $D$. According to the theory of Volterra integral equations of the second kind [3, p. 5], there exists a unique solution $v(x)$ of equation (2.1) in C[0,b] and $v(0) = f(0)$.

Let $a(x), f(x) \in C^n[0, b], G(x) \in C^{n-1}[0, b], Q(x, t) \in C^{n,0}(D)$, and condition (2.2) be satisfied. From (A) and (2.2) it follows that

$$exp\left( - \int\limits_0^x \frac{G_m(\tau)d\tau}{a(\tau)} \right) = 0, m = 0, 1, ..., n. \tag{2.4}$$

By integrating by parts, due to (2.4), we get

$$\int\limits_0^x exp\left( - \int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)} \right) \frac{G(t)}{a(t)} f(t)dt = f(x) - \int\limits_0^x exp\left( - \int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)} \right) f'(t)dt$$

and

$$\int\limits_0^x exp\left( - \int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)} \right) \frac{G(t)}{a(t)} \int\limits_0^t Q(t, s)v(s)dsdt$$

$$= \int\limits_0^x Q(x, s)v(s)ds - \int\limits_0^x exp\left( - \int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)} \right) c(t)dt,$$

where $c(x) = Q(x, x)v(x) + \int\limits_0^x Q_x(x, s)v(s)ds$.

We will demonstrate the existence of a continuous derivative for the solution of equation (2.1). Since

$$exp\left( - \int\limits_t^x \frac{a'(\tau)d\tau}{a(\tau)} \right) = \frac{a(t)}{a(x)},$$

from condition (2.4) it follows that the function

$$w(x) = \int\limits_0^x exp\left( - \int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)} \right) c_0(t)dt,$$

is continuously differentiable and

$$w'(x) = c_0(x) - G(x) \int\limits_0^x exp\left( - \int\limits_t^x \frac{G_1(\tau)d\tau}{a(\tau)} \right) \frac{c_0(t)}{a(t)} dt.$$

where $c_0(x) = c(x) + f'(x)$.

Thus, the solution of equation (2.1) has a continuous derivative

$$v'(x) = c_0(x) - w'(x) = G(x)\int\limits_0^x exp\left(-\int\limits_t^x \frac{G_1(\tau)d\tau}{a(\tau)}\right)\frac{c_0(t)}{a(t)}dt. \tag{2.5}$$

Let $x_0$ be an arbitrary point from $[0,b]$. For $v'(x_0)$, using the generalized mean value theorem, we obtain

$$v'(x_0) = G(x_0)\int\limits_0^{x_0} exp\left(-\int\limits_t^{x_0} \frac{G_1(\tau)d\tau}{a(\tau)}\right)\frac{G_1(t)}{a(t)}c_1(t)dt = G(x_0)\frac{c_0(\zeta)}{G_1(\zeta)},$$

for some $0 \le \zeta \le x_0$, where $c_1(x) = \frac{c_0(x)}{G_1(x)}$. From this, when $x_0 \to 0$

$$v'(0) = G(0)\frac{c_0(0)}{G_1(0)} = G(0)\frac{Q(0,0)f(0) + f'(0)}{G_1(0)}. \tag{2.6}$$

In a similar manner, for any continuous function $c_0(x)$, one can derive the following representation:

$$w(x_0) = \frac{c_0(\zeta)}{G_1(\zeta)}a(\zeta), 0 \le \zeta \le x_0.$$

Therefore, $w(0) = 0$, since $a(0) = 0$. This fact will be used repeatedly in the following computations.

We introduce the notations

$$c_1(x) = \frac{c_0(x)}{G_1(x)}, \quad c_m(x) = \frac{c'_{m-1}(x)}{G_m(x)}, m = 2,3,...,n,$$

$$W_m(x,t) = exp\left(-\int\limits_t^x \frac{G_m(\tau)d\tau}{a(\tau)}\right), m = 0,1,...,n,$$

$$A_1(x) = \int\limits_0^x \frac{c_0(t)}{a(t)}W_1(x,t)dt, \quad A_m(x) = \int\limits_0^x \frac{c'_{m-1}(t)}{a(t)}W_m(x,t)dt, m = 2,3,...,n. \tag{2.7}$$

We write (2.5) in the form

$$v'(x) = G(x)A_1(x). \tag{2.8}$$

The function

$$A_1(x) = \int\limits_0^x c_1(t)W_1(x,t)\frac{G_1(t)}{a(t)}dt = c_1(x) - \int\limits_0^x W_1(x,t)c'_1(t)dt$$

is continuous on $[0,b]$ and $A_1(0) = c_1(0)$. According to condition (2.2), the following estimate holds

$$|A_1(x)| \le d_1^{-1}\|c_0(x)\|_\infty \int\limits_0^x W_1(x,t)\frac{G_1(t)}{a(t)}dt \le d_1^{-1}\|c_0(x)\|_\infty.$$

The function $A_1(x)$ is continuously differentiable and

$$A'_1(x) = \frac{G_1(x)}{a(x)}\int\limits_0^x W_1(x,t)c'_1(t)dt = G_1(x)\int\limits_0^x W_2(x,t)\frac{c'_1(t)}{a(t)}dt = G_1(x)A_2(x).$$

Since

$$A_2(x) = \int_0^x c_2(t) W_2(x,t) \frac{G_2(t)}{a(t)} dt = c_2(x) - \int_0^x W_2(x,t) c_2'(t) dt,$$

we get

$$A_1'(0) = G_1(0) c_2(0), \quad |A_1'(x)| \le d_1^{-1} \|G_1(x)\|_\infty \|c_1'(x)\|_\infty .$$

Then, there exists a continuous second derivative on $[0,b]$ and

$$v''(x) = G'(x) A_1(x) + G(x) A_1'(x). \tag{2.9}$$

Note that formula (2.8), for determining $v'(x)$, contains the first derivatives of $f'(x), a'(x)$ and $Q_x(x,t)$, while in formula (2.9) for $v''(x)$, the second derivatives $f''(x)$, $a''(x)$, $Q_{xx}(x,t)$ and the first derivatives of the functions $Q(x,x), v(x), G(x)$, also the derivative $Q_x(x,x)$ are present.

By differentiating $n-1$ times from (2.8), we get

$$v^{(n)}(x) = (G(x) A_1(x))^{(n-1)}.$$

We will show that $A_1(x)$ has continuous derivatives of order $n-1$. Above, it was shown that the function $A_1(x)$ has a first derivative. Let us differentiate $A_2(x)$.

$$A_2'(x) = \frac{G_2(x)}{a(x)} \int_0^x W_2(x,t) c_2'(t) dt = G_2(x) \int_0^x W_3(x,t) \frac{c_2'(t)}{a(t)} dt = G_2(x) A_3(x)$$

and

$$A_2'(0) = G_2(0) c_3(0), \quad |A_2'(x)| \le d_1^{-1} \|G_2(x)\|_\infty \|c_2'(x)\|_\infty .$$

Next, the second derivative exists:

$$A_1''(x) = (G_1(x) A_2(x)))' = G_1'(x) A_2(x) + G_1(x) A_2'(x).$$

The third-order derivative $A_1^{(3)}(x)$ is given by the formulas:

$$A_1^{(3)}(x) = (G_1(x) A_2(x)))'' = G_1''(x) A_2(x) + 2G_1'(x) A_2'(x) + G_1(x) A_2''(x),$$

$$A_2''(x) = (G_2(x) A_3(x)))' = G_2'(x) A_3(x) + G_2(x) A_3'(x).$$

Here, we need to show the differentiability of

$$A_3(x) = \int_0^x W_3(x,t) \frac{c_2'(t)}{a(t)} dt.$$

All other functions on the right-hand sides of these equations are defined and continuous on $[0,b]$. By virtue of (2.2), we have

$$A_3'(x) = G_3(x) \int_0^x W_4(x,t) \frac{c_3'(t)}{a(t)} dt = G_3(x) A_4(x)$$

and

$$A_3'(0) = G_3(0) c_4(0), \quad |A_3'(x)| \le d_1^{-1} \|G_3(x)\|_\infty \|c_3'(x)\|_\infty .$$

Using the method of induction, it is easy to establish that $A_1^{(n-1)}(x)$ is determined by the following system of formulas:

$$A_1^{(n-1)}(x) = (G_1(x)A_2(x))^{(n-2)}, A_2^{(n-2)}(x) = (G_2(x)A_3(x))^{(n-3)},$$

$$..., A_{n-1}^{(1)}(x) = G_{n-1}(x)A_n(x).$$

At each step $m$, we need to find $A'_m(x)$, for which the following formulas hold:

$$A'_m(x) = G_m(x)A_{m+1}(x), \quad A'_m(0) = G_m(0)c_{m+1}(0), m = 1, 2, ..., n-1$$

and

$$|A'_m(x)| \leq d_1^{-1} \|G_m(x)\|_\infty \|c'_m(x)\|_\infty, m = 1, 2, ..., n-1.$$

The function $c'_{n-1}(x)$ in the equality

$$A_n(x) = \int_0^x \frac{c'_{n-1}(t)}{a(t)} W_n(x,t)dt,$$

contains $a^{(n)}(x), f^{(n)}(x), v^{(n-1)}(x), Q_x^{(n)}(x,t)$ and lower-order derivatives, and

$$A_n(0) = c_n(0), \quad |A_n(x)| \leq d_1^{-1} \left\|c'_{n-1}(x)\right\|_\infty.$$

Thus, the function $A_1(x)$ is continuously differentiable $n-1$ times. Then, by Leibniz's rule it follows that the function $v(x)$ is continuously differentiable up to order $n$ inclusively. Therefore, the solution $v(x)$ of equation (2.1) belongs to $C^n[0,b]$. □

**Definition 1.** We will call equation (2.1) the resolving equation for equation (1.1) if $G(x) = K(x,x)$ and $Q(x,t) = \frac{\partial}{\partial t}\left(\frac{K(x,t)}{G(t)}\right)$.

## 3 Theorem of existence and uniqueness of regular solutions

Theorem 2.1 serves as the basis for establishing the conditions of existence, uniqueness, and continuity of the solution to the linear Volterra integral equation of the third kind (1.1), where $a(x), K(x,t)$ and $f(x)$ are known functions, and $u(x)$ is the desired function.

V. Volterra [21, pp. 97-98] demonstrated the possibility of solving Volterra integral equations of the first kind in two ways: by differentiating the equation and by integrating by parts with the introduction of a new desired function through an integral substitution. In both cases, we obtain a Volterra integral equation of the second kind. The first method has found wide application, including in the construction of numerical solutions. The second method involves finding the derivative of the new desired function and is rarely used. We are particularly interested in the second method, the generalization of which proves to be useful for investigating the existence, uniqueness, and continuity of the solution to Volterra integral equations of the third kind. Moreover, it is possible to identify the conditions that determine the order of smoothness of the solution.

For equation (1.1), condition (2.2) takes the form

$$G_m(x) = K(x,x) + ma'(x) \geq d_1 > 0, m = 0, 1, ..., n. \tag{3.1}$$

**Theorem 3.1.** *Let $a(x), f(x) \in C^1[0,b], K(x,t) \in C^{1,1}(D), f(0) = 0$ and let conditions (A), and (3.1) (for m=0,1) hold. Then equation (1.1) has a unique solution $u(x) \in C[0,b]$, which at x=0 takes the value*

$$u(0) = \frac{f'(0)}{K(0,0) + a'(0)} \tag{3.2}$$

*and the following estimate holds:*

$$\|u(x)\|_{\infty} \leq N_1 \|f(x)\|_{C^1},$$

*where* $N_1$ *is a positive constant,* $\|\cdot\|_{C^1}$ *is the norm in the space* $C^1[0,b]$*. Furthermore, if* $a(x), f(x) \in C^n[0,b], K(x,t) \in C^{n,1}(D)$ *and condition* (3.1) *holds, then* $u(x) \in C^{n-1}[0,b]$.

*Proof.* Let us rewrite equation (1.1) as

$$a(x)u(x) + \int_0^x G(t)u(t)dt = \int_0^x Q(x,t) \int_0^t G(s)u(s)dsdt + f(x), \tag{3.3}$$

where $G(x) = K(x,x), Q(x,t) = \frac{\partial}{\partial t}\left(\frac{K(x,t)}{G(t)}\right)$. We introduce a new unknown function by substituting

$$\int_0^x G(t)u(t)dt = v(x). \tag{3.4}$$

Then, from (1.1), we obtain the integro-differential equation

$$a(x)v'(x) + G(x)v(x) = \int_0^x G(x)Q(x,t)v(t)dt + G(x)f(x), \tag{3.5}$$

with the initial condition $v(0) = 0$. We reduce this zero initial value problem to the integral equation of the followig form [5, Section 11]:

$$v(x) = exp\left(\int_x^b \frac{G(\tau)}{a(\tau)}d\tau\right)\int_0^x exp\left(-\int_t^b \frac{G(\tau)}{a(\tau)}d\tau\right)\frac{G(t)}{a(t)}\left[\int_0^t Q(t,s)v(s)ds + f(t)\right]dt,$$

which can be represented in form (2.1), where $G(x)$ and $Q(x,t)$ are defined according to (3.3). Any solution of integro-differential equation (3.5) with the initial condition $v(0) = 0$ is a solution of integral equation (2.1) and vice versa, because at $x = 0$ the solution of equation (2.1) takes the value $v(0) = f(0)$ and, by the assumptions of the theorem, $f(0) = 0$. Consequently, solving the zero Cauchy problem for (3.5) is equivalent to solving resolving integral equation (2.1). From the conditions imposed on the function $K(x,t)$, it follows that $Q(x,t) \in C^{1,0}(D)$. By virtue of Theorem 2.1, resolving equation (2.1) has a unique solution $v(x) \in C^1[0,b]$. Then, from (3.4), we find the unique solution of equation (1.1)

$$u(x) = \frac{v'(x)}{G(x)}, \tag{3.6}$$

which belongs to the space $C[0,b]$. At $x = 0$, from (3.6) and (2.6), we obtain (3.2). The estimate of this theorem directly follows from (3.6), (2.3), and (2.5).

Suppose $a(x), f(x) \in C^n[0,b], K(x,t) \in C^{n,1}(D)$ and condition (3.1) is satisfied. Then, according to Theorem 2.1, the solution of resolving equation (2.1) $v(x) \in C^n[0,b]$, and from (3.6) it follows that the solution of equation (1.1) $u(x) \in C^{n-1}[0,b]$. □

Formula (3.2) is well consistent with the theory of first-kind Volterra integral equations. If $a(x) \equiv 0$, then from (1.1) we obtain a linear first-kind Volterra integral equation. When the kernel and the right-hand side of the first-kind equation are smooth and $K(x,x) > 0$, it has a unique continuous solution $u(x)$ and

$$u(0) = \frac{f'(0)}{K(0,0)},$$

which we also obtain from (3.2) when $a(x) \equiv 0$.

**Remark 1.** Under the assumptions of Teorem 3.1 estimate (2.3), it follows that the solution of equation (1.1) can be constructed by the method of successive approximations according to the rule:

$$u_{n+1}(x) = \int_0^x \frac{W_1(x,t)}{a(t)} \left\{ Q(t,t)v_{n+1}(t) + \int_0^t Q_x(t,s)v_{n+1}(s)ds + f'(t) \right\} dt,$$

$$v_{n+1}(x) = \int_0^x W_0(x,t) \frac{G(t)}{a(t)} \left\{ \int_0^t Q(t,s)v_n(s)ds + f(t) \right\} dt, n = 0, 1, ....$$

**Example 1.** Let $K(x,t) = x - t + 1/2, 0 \le t \le x \le 1$, $f(x) = 3x/2 + 7x^2/4 + x^3/6$, $a(x) = x$, $0 \le x \le 1$. Then, $G(x) = 1/2, Q(x,t) = -2$, $F(x) = x/2 + 7x^2/20 + x^3/42$.
If we choose the initial approximation $v_0(x) = F(x)$ and apply the method of successive approximations to (2.1), then

$$v_n(x) = x/2 + x^2/4 + (-1)^n(\beta_n/105 + x\gamma_n/42)x^{n+2}, n = 1, 2, ...,$$

where $\beta_1 = 1, \beta_n = 2^{n-1}\prod_{m=4}^{n+2} \frac{1}{m(2m+1)}$ $(n = 2, 3, ...)$, $\gamma_n = \frac{2\beta_n}{(n+3)(2n+7)}$ $(n = 1, 2, 3, ...)$.

Then, $v(x) = \lim_{n\to\infty} v_n(x) = x/2 + x^2/4$ and the solution of integral equation (1.1), according to (3.6), is $u(x) = 1 + x$ and $u(0) = \frac{f'(0)}{K(0,0)+a'(0)} = \frac{3/2}{1/2+1} = 1$.

## 4 Existence and uniqueness of a solution in $L^p(0, b)$

The smoothness properties of the function $a(x)$, the kernel $K(x,t)$, and the right-hand side $f(x)$ of equation (1.1) allowed us to prove the existence and uniqueness of a continuous solution to equation (1.1). If we discard some of these conditions, equation (1.1) may not have a solution in $C[0, b]$.

Let a function $f(x)$ be continuous on $[0, b]$ and $f'(x) \in L^p(0, b), p \ge 1$. Then, the fulfillment of conditions (A) and (3.1) (for m=0) does not imply that equation (1.1) has a solution in the space of continuous functions. In this case, it is reasonable to investigate the solvability of equation (1.1) in the space $L^p(0, b)$. In this case, by a solution of equation (1.1), we mean a function delonging to $L^p(0, b)$ which, when substituted into (1.1), transforms the equation into an identity in the integral sense

Let the function $a(x)$ be continuously differentiable on $[0, b]$ and satisfy condition (A), and let $K(x,t) \in C^{1,1}(D)$. We require the fulfillment of the condition

$$k_m G_0(x) + ma'(x) \ge d_{m+1}, m = 0, 1, k_1 = min(k_0, \frac{q}{2}), k_0 = 1, \tag{4.1}$$

where $d_1, d_2$ are positive constants, $1/p + 1/q = 1, p \ge 1$, and $G_0(x) = K(x,x)$.

If $f(0) = 0, f(x) \in C^\gamma[0, b], 0 < \gamma \le 1$, then equation (2.1), where $G(x)$ and $Q(x,t)$ are defined according to (3.3), is the resolving equation for equation (1.1). By virtue of the continuity $f(x)$ and $Q(x,t) = \partial/\partial t(K(x,t)/G(t))$, respectively on $[0, b]$ and in $D$, according to Theorem 2.1, resolving equation (2.1) has a unique continuous solution $v(x)$. From (3.4) and (2.1) we find the solution of equation (1.1)

$$u(x) = F_0(x) + \int_0^x W_1(x,t)\frac{Q(t,t)}{a(t)}v(t)dt + \int_0^x v(t) \int_t^x W_1(x,s)\frac{Q_x(s,t)}{a(s)}dsdt, \tag{4.2}$$

where

$$F_0(x) = \frac{f(x)}{a(x)} - \frac{1}{a(x)} \int_0^x W_0(x,t)\frac{G_0(t)}{a(t)}f(t)dt. \tag{4.3}$$

**Lemma 4.1.** *Let a function $f(x) \in C[0,b]$ and $f'(x) \in L^p(0,b)$, $p \geq 1$. If $f(0) = 0$ and conditions (A) and (4.1) are satisfied, then the following estimate holds:*

$$\|F_0(x)\|_{L^p} \leq \left(d_2\right)^{-\frac{1}{q}} \left(\frac{pd_1}{2}\right)^{-\frac{1}{p}} ||f'(x)||_{L^p}. \tag{4.4}$$

*Proof.* Integrating by parts, from (4.3) we get:

$$\|F_0(x)\|_{L^p}^p = \int_0^b \left| \frac{f(x)}{a(x)} - \frac{1}{a(x)} \int_0^x exp\left(-\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) \frac{G_0(t)}{a(t)} f(t)dt \right|^p dx$$

$$= \int_0^b \left| \frac{1}{a(x)} \int_0^x exp\left(-\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) f'(t)dt \right|^p dx.$$

According to Hölder's inequality, we have:

$$\left(\int_0^x exp\left(-\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) |f'(t)|dt\right)^p \leq \left(\int_0^x exp\left(-\frac{q}{2}\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) dt\right)^{\frac{p}{q}}$$

$$\times \int_0^x exp\left(-\frac{p}{2}\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) |f'(t)|^p dt.$$

Then

$$\|F_0(x)\|_{L^p}^p = \int_0^b \left| \frac{1}{a(x)} \int_0^x exp\left(-\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) f'(t)dt \right|^p dx$$

$$\leq \int_0^b \left(\frac{1}{a(x)}\right)^p \left(\int_0^x exp\left(-\frac{q}{2}\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) dt\right)^{\frac{p}{q}} \int_0^x exp\left(-\frac{p}{2}\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) |f'(t)|^p dtdx.$$

Since

$$\left(\frac{1}{a(x)}\right)^p \left(\int_0^x exp\left(-\frac{q}{2}\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) dt\right)^{\frac{p}{q}} = \frac{1}{a(x)} \left(\int_0^x exp\left(-\frac{q}{2}\int_t^x \frac{G_{01}(\tau)}{a(\tau)} d\tau\right) \frac{dt}{a(t)}\right)^{\frac{p}{q}},$$

where $G_{01}(x) = \frac{q}{2}G_0(x) + a'(x)$, then

$$\|F_0(x)\|_{L^p}^p \leq (d_2)^{-\frac{p}{q}} \int_0^b \frac{1}{a(x)} \int_0^x exp\left(-\frac{p}{2}\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) |f'(t)|^p dtdx.$$

Using Dirichlet's rule, we get the estimate

$$\|F_0(x)\|_{L^p}^p \leq (d_2)^{-\frac{p}{q}} \int_0^b |f'(t)|^p \int_t^b exp\left(-\frac{p}{2}\int_t^x \frac{G_0(\tau)d\tau}{a(\tau)}\right) \frac{1}{a(x)} dxdt$$

$$\leq (d_2)^{-\frac{p}{q}} \frac{2}{pd_1} \int_0^b |f'(t)|^p dt.$$

From this, estimate (4.4) follows. □

Since

$$\left|\int_0^x W_1(x,t)\frac{Q(t,t)}{a(t)}v(t)dt\right| \le Q_1 d_2^{-1}||v(x)||_\infty$$

and

$$\left|\int_0^x v(t)\int_t^x W_1(x,s)\frac{Q_x(s,t)}{a(s)}dsdt\right| \le Q_2 b d_2^{-1}||v(x)||_\infty,$$

where $Q_{i+1} = max\{|Q_x^{(i)}(x,t)|, (x,t) \in D\}, i = 0,1$, by virtue of Minkowski's inequality and estimate (4.4), from (4.2) we arrive at the following estimate:

$$\|u(x)\|_{L^p} \le \left(d_2\right)^{-\frac{1}{q}}\left(\frac{pd_1}{2}\right)^{-\frac{1}{p}}\|f'(x)\|_{L^p} + (Q_1+Q_2b)b^{\frac{1}{p}}d_2^{-1}\|v(x)\|_\infty.$$

The uniqueness of a solution to resolving equation (2.1) and this estimate guarantee the uniqueness of the solution to equation (1.1) in $L^p(0,b)$.

**Theorem 4.1.** *Let $a(x) \in C^1[0,b], K(x,t) \in C^{1,1}(D)$, and a function f(x) be continuous on $[0,b]$, with $f(0) = 0$ and $f'(x) \in L^p(0,b)$. Suppose that conditions (A) and (4.1) are satisfied. Then, there exists a unique solution to equation (1.1) in $L^p(0,b)$, and the following estimate holds:*

$$\|u(x)\|_{L^p} \le \left(d_2\right)^{-\frac{1}{q}}\left(\frac{pd_1}{2}\right)^{-\frac{1}{p}}\|f'(x)\|_{L^p} + M_2\|v(x)\|_\infty,$$

*where $M_2 = (Q_1+Q_2b)b^{\frac{1}{p}}d_2^{-1}$.*

**Example 2.** Let the functions $a(x)$ and $K(x,t)$ be the same as in Example 1, and the function $f(x)$ be given by

$$f(x) = (7/4 + 9x/10)\sqrt[3]{x^2}, 0 \le x \le 1.$$

Then,

$$F(x) = \left(\frac{3}{4} + \frac{27}{130}x\right)\sqrt[3]{x^2},$$

and $G_0(x) = \frac{1}{2}, \quad Q(x,t) = -2$. If we choose the initial approximation $v_0(x) = F(x)$ and apply the method of successive approximations to (2.1), then we get:

$$v_1(x) = \left(\frac{3}{4} - \frac{243}{65}\beta_1 x^2\right)\sqrt[3]{x^2}, ...,$$

$$v_n(x) = \left(\frac{3}{4} + (-1)^n\frac{3^{2n+3}}{65}2^{n-1}\beta_n x^{n+1}\right)\sqrt[3]{x^2}, ...,$$

where $\beta_n = \prod_{m=1}^n \frac{1}{(3m+5)(6m+13)}, (n = 1,2,3,...)$. Hence, $v(x) = \lim_{n\to\infty} v_n(x) = \frac{3}{4}\sqrt[3]{x^2}$. Then, the solution of integral equation (1.1), according to (3.6), is $u(x) = \frac{1}{\sqrt[3]{x}}$.

## 5 Conclusions

In regularization theory, as well as in the numerical solution of Volterra integral equations of the third kind, one of the main difficulties lies in the fact that the value $u(0)$ is unknown. Therefore, an approximate value $u_\delta$ is often used instead of $u(0)$ [13], which is not always easy to determine.

Theorem 3.1 for equation (1.1) with smooth data eliminates this problem since the exact value of $u(0)$ is known and is determined by formula (3.2). Moreover, Theorem 3.1 can serve as a theoretical basis for constructing a regularizing operator and numerically solving equation (1.1) not only for a monotonic function $a(x)$ [7], but also for a broader class of functions $a(x)$.

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# PSEUDOFINITE UNAR THEORY WITH ARBITRARY NUMBER OF SEMICHAINS AND ANTICHAINS

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**Key words:** approximation of theory, unar, pseudofinite theory, omega stable unar, semichain, antichain.

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**Abstract.** The paper focuses on constructing a pseudofinite theory of a unar with an arbitrary number of antichains and semichains and studying its model-theoretic properties. The question of elementary equivalence of unars with different numbers of antichains and semichains remains open. The obtained theory is shown to be omega-stable, strengthening the known result stating that all complete theories of unars are superstable. Prime models of this theory turn out to be omega-categorical, implying their smooth approximability.

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## 1 Introduction

The study of pseudofinite structures is important because it provides a way to model infinite structures that are similar to finite structures. This allows the transfer of techniques and results from finite mathematics to the study of infinite structures. For example, in model theory, pseudofinite structures are used to study the behaviour of infinite structures under different model-theoretic interpretations.

The family of (pseudofinite) theories is studied in [32, 33, 34, 27, 16], their rank characteristics and topologies in [14, 15], pseudofinite acyclic graphs in [18], and equivalence relation in [10, 19].

In the paper [3] there are considered surjective quadratic Jordan algebras, which has connections with problems of decomposition of algebraic structures, as in [11], where there is studied the effective decomposition of Abelian groups. In both cases, the issues of decomposition and model construction are important. Paper [4] is devoted to countable models of small stable theories, which has connections with the topics of [5, 12], where there is studied the difficulty of recognizing decidable theories.

Problems concerning complete theories of unars were considered in works [9, 13, 18, 22, 23, 30, 28, 8]. Yu.E. Shishmarev [30] obtained a description of uncountably and countably categorical theories of unars, in [13] L. Marcus solved the problem of the number of non-isomorphic countable models in the complete theory of unars, A.A. Ivanov [9] obtained a criterion for elementary equivalence of unars.

This paper is a continuation of the study on approximations of unar theory by finite unar theories starting from [31, 17, 8]. The characterization of pseudofinite theories of unars depends essentially on the combinatorial structure of their connected components, particularly on the distribution of semichains and antichains. While the case in which these components occur in equal numbers is comparatively tractable, the general situation with arbitrary finite or infinite numbers of semichains and antichains is substantially more involved. The interaction between these two types of components significantly influences the existence of finite approximations and, consequently, the pseudofiniteness of the corresponding theories. Therefore, obtaining necessary and sufficient conditions for pseudofiniteness in the general case constitutes a natural and important extension of previous investigations and provides a complete characterization of pseudofinite unar theories.

In [29] an algebraic description of projective, weakly-, quasi- and pseudo-projective, injective, weakly-, quasi- and pseudo-injective unars is presented.

In [31], it is proved that a coproduct of finite acts over monoid is pseudofinite.

As usual, we will use the standard terminology. In particular, $\omega$ denotes the set of all non-negative integers.

**Definition 1.** A unar is a structure $\mathcal{U} = \langle U; f\rangle$ where $f : U \to U$ is a unary operation on a set $U$. For any $u \in U$, let $f^0(u) = u$, $f^{n+1}(u) = f(f^n(u))$ for all $n \in \omega$, $f^{-1}(u) = \{w \in U | f(w) = u\}$. A unar $\mathcal{U}$ is called a *cycle* of length $n \in \omega$ if there exists $u \in U$ such that $U = \{f^i(u) | 0 \leq i < n\}$, $f^n(u) = u$, $f^i(u) \neq f^j(u)$ for all distinct $i, j \in \{0, \ldots, n-1\}$. A set $\{u_i | i \in \omega\} \subseteq U$ is called a *semichain* if $f(u_i) = u_{i+1}$ and $u_i \neq u_j$ for all distinct $i, j \in \omega$. A set $\{u_i | i \in \omega\} \subseteq U$ is called an infinite *antichain* if $f(u_{i+1}) = u_i$ and $u_i \neq u_j$ for all distinct $i, j \in \omega$. If the cardinality of the set $\{w \in U | f(w) = u\}$ is $k$, we say that $u$ is a *$k$-branching point*, or *$k$-valence point*. In the language of the graph theory we say that $u$ is a point of *degree $k$*.

In the first section basic facts are given about model-theoretic properties of unars with some explanations. In the second section there are given equivalent definitions of pseudofiniteness. The third section is devoted to approximations developed by S.V. Sudoplatov in [33]. In the fourth section, a pseudofinite theory of a unar with an arbitrary number of semichains and antichains is constructed. It is proved that this theory is omega-stable.

## 2 Preliminaries

**Definition 2.** Let $X \subseteq U$ and $u, v \in U$. We say that $u, v$ are *connected* if there are $n, m \in \omega$ such that $f^n(u) = f^m(v)$. We say that $X$ is *connected* if any two elements of $X$ are connected. A maximal connected subset of $U$ is called a *connected component* of $U$.

A connected component is defined as for graphs. Two vertices $u, v$ are connected if there is a $f$-path connecting them: $u = w_0, w_1, \ldots, w_k = v$, where either $f(w_i) = w_{i+1}$ for each $0 \leq i \leq k-1$ or $f(w_i) = w_{i-1}$ for each $1 \leq i \leq k$. We can define the distance $\rho(u, v)$ as the minimum length of a path connecting $u, v$ if these vertices lie in the same component; and if they lie in different components, then we assume $\rho(u, v) = \infty$.

The cycle (ring) of a component (graph) $\Gamma$ is the set consisting of all $u \in \Gamma$ for which there exists $n > 0$ such that $f^n(u) = u$. A component $\Gamma$ is called a component with a cycle if its cycle is not empty.

If a component does not contain a cycle of length $n > 0$, we call it a *tree*. If a unar is disconnected and each component does not contain a cycle of length $n > 0$, we call it a *forest*.

The *root of depth* $n$ of an element $u$ is the set $K_n(u) = \{w \in \mathcal{U} |$ there is $i \leq n$ such that $f^i(w) = u\}$. The *root* of $u$ is

$$K(u) = \bigcup_{i \in \omega} K_n(u).$$

A connected subset of the root $K_n(u)$ that contains $u$ is called a subroot of depth $n$ of the element $u$.

**Fact 1.** *There is at most one cycle in one component.*

If $u$ does not belong to a cycle, then $K(u)$ is a tree. Indeed, suppose for vertices $u', v' \in K(u)$ there are two paths $u' = x_0, x_1, \ldots, x_k = v'$ and $u' = x_0, y_1, \ldots, y_{m-1}, x_k = v'$. We can assume that the common vertices of these two paths are only $u'$ and $v'$. One of these paths "comes" to $u'$, let us say $f(x_1) = u'$. Then, we have $f(x_{i+1}) = f(x_i)$ for each $i \in \{0, 1, \ldots, k\}$, and the second path leads from $u'$ to $v'$. The union of these paths gives a cycle containing $u'$ and $v'$, but not containing $u$, a contradiction with $u', v' \in K(u)$.

Similarly, it can be proved that if a vertex $u$ belongs to a cycle $C$, then $(K(u) \setminus C) \cup \{u\}$ is a tree. For simplicity of further exposition, we will also denote this tree by $K(u)$ (when the vertex $u$ belongs to a cycle).

*Description of connected components of a unar.*

(1) There is a cycle in a component. Then the component is a forest of trees planted on the vertices of this cycle.

(2) There is no cycle in a component. If there is an element that does not have an $f$-preimage, then the component is a forest of trees planted on the $f$-trajectory (semichain) of this element.

(3) There is no cycle in a component. All elements have an $f$-preimage and the component is a forest of trees planted on the $f$-trajectory ($\mathbb{Z}$-chain) of any element.

**Definition 3.** A theory $T$ is said to be *limited* if there exists a natural number $N$ such that the following formula is true in $T$:

$$(\forall u)[\bigvee_{n,m=1}^{N} (f^n(u) = f^{n+m}(u))].$$

**Fact 2.** [23, 30] *A theory $T$ is $\omega$-categorical if and only if*

*i) $T$ is limited,*

*ii) if $\mathcal{U} \models T$, then there are only finitely many non-isomorphic sets of the form $\bigcup_{n<\omega} f^{-n}(u)$ in $\mathcal{U}$ or equivalently, $\mathcal{U}$ realizes finitely many 1-types.*

In components of form (2) and (3) there are vertices $u, v$ with the formula property $\rho(u, v) = n$ for any natural $n$, and there is a unique $f$-path without repetitions connecting these vertices $u, v$. Therefore, in a countably categorical unar, all components contain a cycle. The lengths of cycles are uniformly bounded, and all trees planted on the vertices of a cycle have finite height, and these heights are also uniformly bounded.

Let us prove the following: all trees of the form $K(u)$ can be assigned labels so that isomorphic trees and only they have the same labels. A label is a formula that tells about the property of the root of this tree. The total number of labels is finite, which is not surprising since up to equivalence the number of formulas (with one free variable) in a countably categorical structure is finite. A label of a tree of height $h$ will have the superscript $h$.

We will assign labels by induction on the height of the trees. There is only one tree of height zero, it is assigned the label $p_0^0$, and the formula says that the root does not have a $f$-preimage.

Let us assume that all trees of height $\leq k$ are assigned labels. For each label $p_j^i$ with $j \leq k$, there is a natural number $m_j^i$ such that if the $f$-preimage of a vertex contains more than $m_j^i$ vertices $u$ for which $K(u)$ has the label $p_j^i$, then the $f$-preimage contains infinitely many such vertices (again, due to countable categoricity).

Let a tree of the form $K(u)$ have height $k+1$. Then we study the composition of the set $f^{-1}(u)$, each vertex $v$ from this set defines a tree $K(v)$, and therefore some label $p_j^i$ with $j \leq k$. We count how many vertices from $f^{-1}(u)$ define a tree with label $p_j^i$: if there are $n \leq m_j^i$ pieces, then we compose the formula $\psi^{-1}(x)$, which says that "$f^{-1}(x)$ contains $n$ vertices with label $p_j^i$", and if there are $n > m_j^i$ pieces, then we take the formula $\psi^{-1}(x)$, which says that "$f^{-1}(x)$ contains $m_j^i+1$ vertices with label $p_j^i$". The conjunction of all formulas $\psi^{-1}(x)$ will be a label for $u$.

The distribution of labels to vertices is complete.

It is easy to prove by induction on height that if the labels of the vertices $u, v$ coincide in a countable model, then $K(u) \cong K(v)$.

Now consider two components. They are isomorphic if and only if their cycles have the same length and the labels of the vertices on the cycles are such that there is an isomorphism between the cycles that maps the vertices with the same labels. Since there are finitely many labels and the lengths of the cycles are bounded by some number, there will be finitely many non-isomorphic components in a countable model. A complete theory of a unar simply specifies how many components of a certain type of isomorphism there are in its model.

Note also that in a countably categorical unar the connectivity relation is formulaic and therefore each component is distinguished by a formula (with a parameter).

**Fact 3.** [30] *A limited theory $T$ is uncountably categorical if and only if it satisfies the following conditions:*

*i) $|f^{-1}(u)|$ is infinite for at most one $u$, and is otherwise bounded.*

*ii) If $|f^{-1}(u)|$ is finite for all $u$, then all except one type of connected component of $\mathcal{U}$ are finite.*

*iii) If there is $u \in \mathcal{U}$ such that $|f^{-1}(u)|$ is infinite, then all types of connected components of $\mathcal{U}$ are finite and all $K_n(u)$ for $f(w) = u$ are isomorphic except for finitely many $u$.*

A set of $N$-neighborhood of $V \subseteq U$ is the set

$$\{u \in U : \text{there is } v \in V \text{ such that } \bigvee_{n,m}^{N} f^n(v) = f^m(u)\}.$$

**Fact 4.** [30] *An unlimited theory $T$ is uncountably categorical if it satisfies the following conditions:*

*i) $|f^{-1}(u)|$ is bounded.*

*ii) For each $n \in \omega$ there are only finitely many connected components whose cycle consists of $n$ elements.*

*iii) There exist a finite set $V_0 \subseteq U$, a set $V \subseteq U$, $m \in \omega$, and a set $\{P_v : v \in V\}$ such that $U = V_0 \cup \bigcup_{v \in V} P_v$, $P_v$ is a subroot of depth $m$ for $v \in V$, and for $v, w \in V$, the subroots $P_v$ and $P_w$ are isomorphic and this isomorphism can be continued to an isomorphism of their $2m$-neighborhoods.*

The following statement perfectly and succinctly summarizes the last two facts above.

**Theorem 2.1.** [22] *Let $\langle M, f\rangle$ be a unar. Then $Th(\langle M, f\rangle)$ is $\omega_1$- categorical if and only if it is quasisimilar to the theory of infinite sets without any structure.*

In work of J. Ax [2] the concept of pseudofiniteness was first defined. The ground works obtained to date for pseudofinite structures directly depend on the results of J. Ax. The basic definitions of pseudofiniteness are the following:

**Definition 4.** [2] An infinite structure $\mathcal{M}$ of a fixed language $L$ is *pseudofinite* if for all $L$-sentences $\varphi$, $\mathcal{M} \models \varphi$ implies that there is a finite $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = Th(\mathcal{M})$ of the pseudofinite structure $\mathcal{M}$ is called *pseudofinite.*

Many beautiful theorems in model theory of the 1950-60s were proved using the ultraproducts. Set theorists love ultraproducts since they give rise to elementary embeddings, a staple of large cardinal theory. J. Ax in [2] connects the notion of pseudofiniteness and the construction of ultraproducts.

**Theorem 2.2.** [2] *Fix a language $L$ and an $L$-structure $\mathcal{M}$. Then the following are equivalent:*
*1) an $L$-structure $\mathcal{M}$ is pseudofinite;*
*2) $\mathcal{M} \models T_f$, where $T_f$ is the common theory of all finite $L$-structures;*
*3) $\mathcal{M}$ is elementarily equivalent to an ultraproduct of finite $L$-structures.*

In classical logic, the following property is a straightforward corollary of pseudofiniteness.

**Theorem 2.3.** *Let $\mathcal{M}$ be a pseudofinite structure and $f : M^k \to M^k$ be a definable function. Then $f$ is injective if and only if $f$ is surjective.*

Pseudofinite fields are studied in [2], more detailed information can be found in [7]. Also a survey on pseudofinite groups is given by D. Macpherson in [20, 21]. The investigation of pseudofinite rings was started in [6, 1], more complicated pseudofinite rings are studied in the series of papers [24, 25, 26].

**Definition 5.** [33] Let $\mathcal{T}$ be a family of theories and $T$ be a theory such that $T \notin \mathcal{T}$. The theory $T$ is said to be *$\mathcal{T}$-approximated,* or *approximated by the family $\mathcal{T}$*, or a *pseudo-$\mathcal{T}$-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ for which $\varphi \in T'$.

If the theory $T$ is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for $T$, and theories $T' \in \mathcal{T}$ are said to be *approximations* for $T$.

**Definition 6.** [35] *A disjoint union* $\bigsqcup_{n\in\omega} \mathcal{M}_n$ of pairwise disjoint systems $\mathcal{M}_n$ of pairwise disjoint predicate signatures $\Sigma_n, n \in \omega$, is a system of the signature $\bigcup_{n\in\omega} \Sigma_n \cup \{P_n^{(1)} | n \in \omega\}$ with the support $\bigsqcup_{n\in\omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols from $\Sigma_n$ that coincide with their interpretations in systems $\mathcal{M}_n$, $n \in \omega$.

*A disjoint union of theories* $T_n$, of pairwise disjoint predicate signatures $\Sigma_n$, respectively, $n \in \omega$, is the theory

$$\bigsqcup_{n\in\omega} T_n \rightleftharpoons Th(\bigsqcup_{n\in\omega} \mathcal{M}_n),$$

where $\mathcal{M}_n \models T_n, n \in \omega$.

Obviously, the $T_1 \sqcup T_2$ theory does not depend on choice of the disjunctive union $\mathcal{M}_1 \sqcup \mathcal{M}_2$ of models $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$.

## 3 An example of pseudofinite unar theory with arbitrary number of semichains and antichains

In this section, we give an example of a complete, pseudofinite graph theory $T$ in a language $L = \{=, f^{(1)}\}$ with equality and a unary function $f^{(1)}$.

We immediately move on to listing the axioms of our theory. We present them in groups, sometimes providing corresponding semantic consequences. The groups of axioms are numbered in the order of their appearance in our presentation. We avoid formal writing of axioms in the language of first-order logic, limiting ourselves to their semantic description.

(1) The theory says that $f$ is a unary function defined as in Definition 1. We will freely use the terms of unar theory (such as *k-branching vertice*, *distance between vertices*, *antichain*, *semichain* etc.) for the semantic description of the given axioms.

(2) For every natural number $i \geq 2$ there is an axiom $\varphi_i$, which says that in the models of the theory $T$ there are no cycles of length $i$. This means that every connected component of a unar that

is a model of $T$ will be acyclic. Moreover, for any two vertices from one connected component, the distance is defined as the length of the only chain connecting these vertices.

(3) There is one and only one vertex of valency $\geq 3$, the remaining vertices have valency 1 or 0. Moreover, for each $i \geq 3$ there is an axiom $\delta_i$, which says that if the valency of a vertex $\geq i$, then the degree of this vertex $\geq i+1$. As a consequence, in the model of $T$, there is exactly one vertex of infinite valency, all other vertices have degree 1 or 0. Note that a vertex of infinite valency is formulaically definable (as a vertex of valency $\geq 3$), so we will call it a root vertex, and in the future we will use this name to describe other axioms.

(4) For every natural number $i \geq 0$ there is an axiom $\psi_i$, which says that there is a unique vertex of valency 0, the distance from which to the root vertex is $i$.

(5) There is one axiom of the theory that says that the chain connecting two vertices of valency 0 has length $\geq 3$. For every natural number $i \geq 3$ there is an axiom $\epsilon_i$ that says that every chain of length $i$ connecting two vertices of valency 1 passes through the root vertex.

We call the component containing the root vertex the *root component*. We represent the root component as acyclic unar with a root (of infinite valence) to which branches (i.e. maximal chains descending to the root vertex) descend. If the distance from a vertex of valence 0 to the root is finite, then the branch is a finite chain. In the case where the distance is infinite, then the model contains a semichain starting from the vertex of valence 0 and an antichain descending to the root.

Note that different branches do not have common vertices because the valences of non-root vertices are $\leq 1$ and there are no finite cycles. Since each vertex of the root component is connected to the root by some finite chain, the vertices on all branches exhaust all non-root vertices of the root component.

The group of axioms (4) states that for each natural number $i \geq 1$ there is exactly one branch of length $i$. On the other hand, the model of our theory may contain an arbitrary number of infinite branches (antichains) in the root component or none at all: our axioms do not contain any restrictions on the number of infinite branches.

Now, we turn to the description of non-root components of the potential model of our theory $T$. In such components, all vertices have valence $\leq 1$.

For convenience, we introduce the following definitions. The standard models $\langle \mathbb{N}; =, succ\rangle$ and $\langle \mathbb{Z}; =, succ\rangle$, where *succ* is the successor function, will be called the $\mathbb{N}$-model and $\mathbb{Z}$-model, respectively. A graph component will be called the $\mathbb{N}$-component and $\mathbb{Z}$-component if it is isomorphic to the $\mathbb{N}$-model and $\mathbb{Z}$-model, respectively.

Suppose that the non-root component of a model of the theory $T$ contains $k \geq 2$ vertices of valence 0. We choose two vertices of valence 0 in the component; By the definition of connectivity, there is a finite chain connecting these two vertices, and obviously this chain does not contain the root vertex. We obtain a contradiction with the group of axioms (5).

Thus, the non-root component of a model of the theory $T$ can either contain exactly one vertex of valence 0, then it will obviously be an $\mathbb{N}$-component, or not contain a vertex of valence 0 at all, then it will obviously be a $\mathbb{Z}$-component, since by the axiom group (2) there are no finite cycles in a unar.

Thus, we have practically given a description of all models of the theory $T$: it consists of a root component, where for each natural number $i \geq 1$ there are exactly one branch of length $i$ and an arbitrary number of infinite branches, and an arbitrary number of $\mathbb{N}$-components and $\mathbb{Z}$-components.

**Theorem 3.1.** *Any model of theory $T$ has an elementary extension with countably many $\mathbb{N}$-components, countably many $\mathbb{Z}$-components, and countably many infinite branches in the root component.*

*Proof.* First, we will supplement our language with the following notations and axioms in accordance with them.

1. Since the root vertex $r$ has infinite valency, it means that an infinite number of branches descend to it. If we denote the branch number by $j \in \omega$, then the valency of the vertex will be $|f^{(-1)}(r)| = \omega$ , where each previous vertex $a_{1j}(a_{0j} = r)$ which is in the corresponding branch $j$ which in turn descends to $r$, is written in general form as $a_{ij}$, and the lengths of each branch are determined through the indices $i \in \omega, k \in Z$. Now, we introduce an extension of the theory $T$ by adding the following axiom

$$f(a_{ij} = a_{kj} \Leftrightarrow j \wedge i - k = 1 \wedge (i > k))$$

2. The vertex $b_{0j}$ has valency 0, and the vertex $b_{ij}$ has valency equal to 1. The corresponding axiom will have the form, for $j \in \omega, i \in \omega, k \in Z, f(b_{ij}) = b_{kj} \Leftrightarrow j \wedge i - k = 1 \wedge (i > k)$.

3. The vertex $c_{ij}$ has valency 1, and the axiom for $j \in \omega, i \in \omega, k \in Z, f(c_{ij}) = c_{kj} \Leftrightarrow j \wedge i - k = 1 \wedge (i > k)$.

It is clear that the chain ${a_{ij}}_{i\in\omega}$ is an infinite branch, the chain ${b_{ij}}_{i\in\omega}$ is the $N$-component, and the chain ${c_{ij}}_{i\in\mathbb{Z}}$ is the $Z$-component for each fixed $j \in \omega$ in any model of the extended theory. Let us assume that we are given some model $\langle U; f\rangle$ of the theory $T$. Let us further enrich our language by highlighting each element of $U$ as a constant. As a result, our language is represented as $L \cup \{a_{ij}, b_{ij}, c_{kj}, f\}_{i,j\in\omega,k\in\mathbb{Z},f\in U}$. It is enough for us that the extended theory is consistent with the elementary diagram $T_\triangle = Th(\langle U; f\rangle)$ of the model $\langle U; f\rangle$.

The latter is a consequence of the compactness theorem. Let $T_0$ be a finite set of sentences consisting of several sentences of the elementary diagram $T_\triangle$ and several axioms of the extension $T$ given above. We can assume that these axioms (extensions) mention all the constants from the set $a_{ij}, b_{ij}, c_{ij}, 0 \leq i, j \leq n \in \omega$, and only them, and $T_0$ includes all axioms describing the mapping between these constants and their properties; we will show that these constants can be interpreted in the $\langle U; f\rangle$ so that all expansion axioms included in $T_0$ will be true.

As for the finite fragment $T_\triangle$, we can assume that they only mention elements from the final branches of the root component, since other elements do not interfere with us when interpreting the constants $a_{ij}, b_{ij}, c_{ij}, 0 \leq i, j \leq n$, in the model $\langle U; f\rangle$.

First, we select all finite branches of length $\leq m$ of the root component, covering all constants from $\{f : A \to A\}$ mentioned in sentences from $T_0$, and here we interpret each constant $a_{ij}, b_{ij}, c_{kj}$ as an element $U$. Let $N = max\{m + 1, n + 1\}$. Next, we use $n$ branches with lengths $N + 1, N + 2, \ldots, N + n$ to interpret the constants $a_{ij}, 0 \leq i, j \leq n$. After this, $n$ branches with lengths $N+n+1, N+n+2, \ldots, N+2n$ are used to interpret the constants $b_{ij}, 0 \leq i, j \leq n$. Finally, branches with lengths $N + 2n + 1, N + 2n + 2, \ldots, N + 3n$ are used to interpret the constants $c_{ij}, 0 \leq i, j \leq n$. □

It is easy to see that an unar containing only finite branches in the root component is a prime model of the theory, and the model with countably many $\mathbb{N}$-components, countably many $\mathbb{Z}$-components and countably many infinite branches in the root component is a countably saturated model of the theory $T$.

**Theorem 3.2.** *The theory $T$ is $\omega$-stable.*

*Proof.* As we mentioned, let us consider a saturated model $M$ of the theory $T$. In this model, each element belongs to one of three components:

Root component: Contains a single root of infinite valency and infinite branches.

$\mathbb{N}$-components: Chains isomorphic to $(\mathbb{N}, f)$.

$\mathbb{Z}$-components: Chains isomorphic to $(\mathbb{Z}, f)$. We will show that the number of element types in the entire model is at most countable.

Let us directly count the number of types. In the root component, we have:

A unique root, which is definable by a formula. Vertices on finite branches, which are distinguished by their distance from the root. These vertices are fully determined by their distance, meaning their types depend only on an integer. Vertices on infinite branches, whose position is determined by their relative distance from the root. Due to this strict structure, the number of distinct types here does not exceed $\omega$.

In each $\mathbb{N}$-chain, the $f$ acts as: $i \mapsto i + 1$.

Elements differ only by their position in the chain, meaning the type of an element is determined by an integer (its distance from a reference point). Since a saturated model contains only countably many such elements, the number of types is at most $\omega$.

In $\mathbb{Z}$-chains, the situation is similar: each element's position is determined by its distance from a fixed point. Thus, the number of types is again at most $\omega$.

We have proved that T is $\omega$-stable since, in every saturated expansion, the number of element types remains at most countable.

□

## 4 Concluding remarks

An elementary equivalence of pseudofinite unars with $n$ semichains and $m$ antichains, where $n \neq m$, remained an interesting question. In this paper, a pseudofinite theory of a unar with an arbitrary number of antichains and semichains is constructed. It is known that any complete theory of unars is superstable. It is proved that this theory is omega-stable. A prime model of this omega-stable theory is omega categorical, hence, smoothly approximable.

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# THE VARIATIONAL APPROACH TO TIME DISCRETIZATION OF BIRKHOFF'S EQUATIONS FOR INFINITE-DIMENSIONAL SYSTEMS

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Communicated by V.I. Burenkov

**Key words:** Birkhoff's equations, infinite-dimensional systems, discretization, potentiality.

**AMS Mathematics Subject Classification:** 70H30, 47G40.

**Abstract.** Difference methods are widely used for the numerical solution of problems in mechanics and physics. When constructing discrete analogues, it is important to preserve the basic properties of the original differential problem. The main goal of this work is to discretize a system of equations of the form $C(x,t,u)\,u_t + E(x,t,u_\alpha) = 0$, based on its functional — the Hamiltonian action. Necessary and sufficient conditions for potentiality with respect to a given bilinear form are obtained. The Hamiltonian action for this system is constructed and its representation in the form of Birkhoff's equations for infinite-dimensional systems is obtained. By approximating the constructed functional by its discrete analogue, a discrete-time analogue of Birkhoff's equations is obtained based on the variational principle. Theoretical results are illustrated by an example of a wave equation with axial symmetry.

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## 1 Introduction and problem statement

Let the configuration of an infinite-dimensional potential system be defined by the vector function $u(x,t) = (u^1(x,t), u^2(x,t), \dots, u^{2n}(x,t))^{\mathbb{T}}$, $(x,t) \in Q_T = \Omega \times (0,T)$, $\Omega$ be a bounded domain in $\mathbb{R}^m$ with a piecewise-smooth boundary $\partial\Omega$.

Consider the following system of equations:

$$N(u) \equiv C(x,t,u)u_t + E(x,t,u_\alpha) = 0, \tag{1.1}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $\alpha_i$ (for $i = \overline{1,m}$) are non-negative integers, $|\alpha| = \sum_{i=1}^m \alpha_i$, $|\alpha| = \overline{0,s}$; $C(x,t,u)$ is a given matrix $[C_{ik}(x,t,u)]_{2n\times 2n}$, $E(x,t,u_\alpha) = (E_1(x,t,u_\alpha), E_2(x,t,u_\alpha), \dots, E_{2n}(x,t,u_\alpha))^{\mathbb{T}}$ is a given vector function, and $u = (u^1, \dots, u^{2n})$ is the unknown vector function. Here $u_t^i = \dfrac{\partial u^i}{\partial t}$ for $i = \overline{1,2n}$, and $u_\alpha = D_\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_m^{\alpha_m}}$.

Moreover, $C_{ik} : \overline{\Omega} \times [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ and $E_i : \overline{\Omega} \times [0,T] \times \mathbb{R}^q \to \mathbb{R}$ are given smooth functions, where $q$ is the length of the vector $\{u_\alpha\}$, $|\alpha| = \overline{0,s}$ and $\overline{\Omega} = \partial\Omega \cup \Omega$.

We will consider system of equations (1.1) on the set

$$\begin{aligned} D(N) = \Big\{ u \in U = (U^1, \dots, U^{2n})^{\mathbb{T}} : u^i \in U^i = C^{2s,1}_{x,t}(\overline{\Omega} \times [0,T]) : u^i\big|_{t=0} = \varphi_0^i(x), \\ u^i\big|_{t=T} = \varphi_1^i(x), \left.\frac{\partial^\nu u^i}{\partial n_x^\nu}\right|_{\Gamma_T} = \psi_\nu^i(x,t), i = \overline{1,2n}, |\nu| = \overline{0,s-1} \Big\}, \end{aligned} \tag{1.2}$$

where $\Gamma_T = \partial\Omega \times (0, T)$, $n_x$ is the outward normal to $\partial\Omega$; $\varphi_0^i$, $\varphi_1^i$, $\psi_\nu^i(x,t)$ are given smooth functions.

Note that (1.1) is a generalization of system of equations of the form

$$C(t,u)\dot{u}(t) + E(t,u) = 0, \tag{1.3}$$

where $\dot{u}(t) = \dfrac{du}{dt}$.

In [4] it was proved that (1.1) admits a classical variational formulation if and only if it can be represented in the form of Birkhoff's equations [1]. In this case, its Hamiltonian action [8] was constructed.

In [5] the potentiality of a discrete system obtained from equations of form (1.3) with continuous time was investigated. Necessary and sufficient conditions for potentiality with respect to a given bilinear form were provided. An algorithm for constructing the corresponding functional, the discrete analogue of the Hamiltonian action, was outlined.

The main goal of this work is to construct a discrete-time analogue of system (1.1) based on its Hamiltonian action.

## 2 Necessary and sufficient conditions for potentiality

Let $V = (V^1, V^2, \ldots, V^{2n}) : V^i = C(\overline{\Omega} \times [0,T]), i = \overline{1, 2n}$. Define a bilinear form $\langle \cdot, \cdot \rangle : V \times U \to \mathbb{R}$ as follows:

$$\langle v, g \rangle = \int\limits_0^T \int\limits_\Omega \sum_{i=1}^{2n} v^i g^i dx dt. \tag{2.1}$$

Following [2, 6], we say that problem (1.1), (1.2) admits a direct variational formulation with respect to (2.1) if there exists a differentiable Gâteaux functional $F_N : D(N) \to \mathbb{R}$ such that its differential has the form:

$$\delta F_N[u,h] = \langle N(u), h \rangle, \quad \forall u \in D(N), \forall h \in D(N'_u).$$

Here, $D(N'_u)$ is the domain of the Gâteaux derivative $N'_u$ of the operator $N$ at a point $u \in D(N)$. In this case, it is also said that the operator $N$ is potential on $D(N)$ with respect to bilinear form (2.1).

The criterion for potentiality of $N$ on the convex set $D(N)$ is the symmetry condition [2, 6]

$$\langle N'_u h, g \rangle = \langle N'_u g, h \rangle, \quad \forall u \in D(N), \forall h, g \in D(N'_u). \tag{2.2}$$

When this condition is satisfied, the desired functional $F_N$ — the Hamiltonian action — can be found using the formula

$$F_N[u] = \int_0^1 \langle N(\hat{u} + \lambda(u - \hat{u})), u - \hat{u} \rangle \, d\lambda + \text{const}, \tag{2.3}$$

where $\hat{u}$ is an arbitrary fixed element in $D(N)$.

Let us denote $N_j \equiv \sum\limits_{k=1}^{2n} C_{jk} u_t^k + E_j$, and $N(u) \equiv (N_1(u), N_2(u), \ldots, N_{2n}(u))$.

Let us find the Gâteaux derivative of the operator $N_j$

$$(N'_u h)_j = \sum_{k=1}^{2n} \sum_{i=1}^{2n} \frac{\partial C_{jk}}{\partial u^i} h^i u_t^k + \sum_{i=1}^{2n} C_{ji} h_t^i + \sum_{i=1}^{2n} \sum_{|\alpha|=0}^{s} \frac{\partial E_j}{\partial u_\alpha^i} h_\alpha^i.$$

Using condition (2.2), we get

$$\begin{aligned}\langle N_u' h, g\rangle &= \int_0^T \int_\Omega \sum_{j=1}^{2n} (N_u' h)_j . g^j dx dt \\ &= \int_0^T \int_\Omega \sum_{j=1}^{2n} \left[ \sum_{k=1}^{2n} \sum_{i=1}^{2n} \frac{\partial C_{jk}}{\partial u^i} h^i u_t^k + \sum_{i=1}^{2n} C_{ji} h_t^i + \sum_{i=1}^{2n} \sum_{|\alpha|=0}^{s} \frac{\partial E_j}{\partial u_\alpha^i} h_\alpha^i \right] g^j dx dt\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\langle N_u' g, h\rangle &= \int_0^T \int_\Omega \sum_{i=1}^{2n} (N_u' g)_i . h^i dx dt \\ &= \int_0^T \int_\Omega \sum_{i=1}^{2n} \left[ \sum_{k=1}^{2n} \sum_{j=1}^{2n} \frac{\partial C_{ik}}{\partial u^j} g^j u_t^k + \sum_{j=1}^{2n} C_{ij} g_t^j + \sum_{j=1}^{2n} \sum_{|\beta|=0}^{s} \frac{\partial E_i}{\partial u_\beta^j} g_\beta^j \right] h^i dx dt.\end{aligned}$$

Integrating by parts, from (2.4) we find

$$\begin{aligned}\langle N_u' h, g\rangle = \int_0^T \int_\Omega \sum_{j=1}^{2n} &\left[ \sum_{k=1}^{2n} \sum_{i=1}^{2n} \frac{\partial C_{jk}}{\partial u^i} h^i g^j u_t^k - \sum_{i=1}^{2n} \frac{d}{dt}\left(C_{ji} g^j\right) h^i + \right. \\ &\left. + \sum_{i=1}^{2n} \sum_{|\alpha|=0}^{s} (-1)^{|\alpha|} D_\alpha \left( \frac{\partial E_j}{\partial u_\alpha^i} g^j \right) h^i \right] dx dt.\end{aligned}$$

It should be noted that

$$\frac{d}{dt}\left(C_{ji} g^j\right) = \frac{d}{dt}\left(C_{ji}\right) g^j + C_{ji} g_t^j = \sum_{k=1}^{2n} \frac{\partial C_{ji}}{\partial u^k} u_t^k g^j + \frac{\partial C_{ji}}{\partial t} g^j + C_{ji} g_t^j,$$

$$\sum_{|\alpha|=0}^{s} (-1)^{|\alpha|} D_\alpha \left( \frac{\partial E_j}{\partial u_\alpha^i} g^j \right) = \sum_{|\alpha|,|\beta|=0}^{s} (-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left( \frac{\partial E_j}{\partial u_\alpha^i} \right) g_\beta^j,$$

where

$$\binom{\alpha}{\beta} = \begin{cases} \binom{\alpha_1}{\beta_1}\binom{\alpha_1}{\beta_1}\ldots\binom{\alpha_m}{\beta_m}, \text{if } \forall i \in \{1,2,...,m\} : \alpha_i \geq \beta_i, \\ 0, \text{if } \exists i \in \{1,2,...,m\} : \alpha_i < \beta_i, \end{cases}$$

$$\binom{\alpha_i}{\beta_i} = \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!}.$$

Taking into account this for the potentiality of operator $N$ (1.1), we obtain

$$\langle N_u'h, g\rangle - \langle N_u'g, h\rangle = \int_0^T \int_\Omega \sum_{i=1}^{2n} \left\{ \left[ \sum_{k=1}^{2n} \sum_{j=1}^{2n} \frac{\partial C_{jk}}{\partial u^i} g^j u_t^k - \sum_{j=1}^{2n} \sum_{k=1}^{2n} \frac{\partial C_{ji}}{\partial u^k} u_t^k g^j - \right.\right.$$

$$\left. - \sum_{j=1}^{2n} \frac{\partial C_{ji}}{\partial t} g^j - \sum_{j=1}^{2n} C_{ji} g_t^j + \sum_{j=1}^{2n} \sum_{|\alpha|,|\beta|=0}^{s} (-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left( \frac{\partial E_j}{\partial u_\alpha^i} \right) g_\beta^j \right] -$$

$$\left. - \left[ \sum_{k=1}^{2n} \sum_{j=1}^{2n} \frac{\partial C_{ik}}{\partial u^j} g^j u_t^k + \sum_{j=1}^{2n} C_{ij} g_t^j + \sum_{j=1}^{2n} \sum_{|\beta|=0}^{s} \frac{\partial E_i}{\partial u_\beta^j} g_\beta^j \right] \right\} h^i dxdt$$

$$= \int_0^T \int_\Omega \sum_{i=1}^{2n} \left\{ \sum_{j,k=1}^{2n} \left( \frac{\partial C_{jk}}{\partial u^i} - \frac{\partial C_{ji}}{\partial u^k} - \frac{\partial C_{ik}}{\partial u^j} \right) g^j u_t^k - \sum_{j=1}^{2n} (C_{ji} + C_{ij}) g_t^j - \sum_{j=1}^{2n} \frac{\partial C_{ji}}{\partial t} g^j + \right.$$

$$\left. + \sum_{j=1}^{2n} \sum_{|\alpha|,|\beta|=0}^{s} (-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left( \frac{\partial E_j}{\partial u_\alpha^i} \right) g_\beta^j - \sum_{j=1}^{2n} \sum_{|\beta|=0}^{s} \frac{\partial E_i}{\partial u_\beta^j} g_\beta^j \right\} h^i dxdt = 0.$$

By virtue of arbitrariness of the functions $h^i$, we come to the conditions:

$$\begin{cases} C_{ij} + C_{ji} = 0, \\ \dfrac{\partial C_{ji}}{\partial u^k} + \dfrac{\partial C_{ik}}{\partial u^j} + \dfrac{\partial C_{kj}}{\partial u^i} = 0, \\ \dfrac{\partial C_{ji}}{\partial t} = \displaystyle\sum_{|\alpha|=0}^{s} (-1)^{|\alpha|} D_\alpha \left( \frac{\partial E_j}{\partial u_\alpha^i} \right) - \frac{\partial E_i}{\partial u^j}, \\ \displaystyle\sum_{|\alpha|=1}^{s} (-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left( \frac{\partial E_j}{\partial u_\alpha^i} \right) - \frac{\partial E_i}{\partial u_\beta^j} = 0 \end{cases} \tag{2.5}$$

for $\forall (x,t) \in Q_T, \forall u \in D(N)$, where $i,j,k = \overline{1,2n}$, and $|\beta| = \overline{1,s}$.

**Theorem 2.1.** *System (1.1) is potential on $D(N)$ (1.2) with respect to bilinear form (2.1) if and only if conditions (2.5) are satisfied.*

# 3 Construction of the Hamiltonian action

If conditions (2.5) are satisfied, the desired functional $F_N$ can be constructed using formula (2.3). Another approach can be taken to this problem. Let us look for the Hamiltonian action for (1.1) in the form

$$F_N = \int_0^T \int_\Omega \left( \sum_{i=1}^{2n} R_i u_t^i - B \right) dxdt, \tag{3.1}$$

where $R_i(x,t,u)$, $B(x,t,u_\alpha)$ are the unknown smooth functions.

The Gâteaux differential of functional (3.1) is given by

$$\delta F_N [u,h] = \int_0^T \int_\Omega \left[ \sum_{i=1}^{2n} \sum_{k=1}^{2n} \frac{\partial R_i}{\partial u^k} h^k u_t^i + \sum_{i=1}^{2n} R_i h_t^i - \sum_{i=1}^{2n} \sum_{|\gamma|=0}^{s} \frac{\partial B}{\partial u_\gamma^i} h_\gamma^i \right] dxdt.$$

Integrating by parts, we obtain

$$\delta F_N[u,h] = \int\limits_0^T \int\limits_\Omega \left[ \sum_{i=1}^{2n} \sum_{k=1}^{2n} \frac{\partial R_k}{\partial u^i} h^i u_t^k - \sum_{i=1}^{2n} \frac{dR_i}{dt} h^i - \right.$$
$$\left. - \sum_{i=1}^{2n} \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left( \frac{\partial B}{\partial u_\gamma^i} \right) h^i \right] dxdt. \tag{3.2}$$

Since

$$\frac{dR_i}{dt} = \sum_{k=1}^{2n} \frac{\partial R_i}{\partial u^k} u_t^k + \frac{\partial R_i}{\partial t},$$

we can write (3.2) as

$$\delta F_N[u,h] = \int\limits_0^T \int\limits_\Omega \sum_{i=1}^{2n} \left[ \sum_{k=1}^{2n} \left( \frac{\partial R_k}{\partial u^i} - \frac{\partial R_i}{\partial u^k} \right) u_t^k - \frac{\partial R_i}{\partial t} - \right.$$
$$\left. - \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left( \frac{\partial B}{\partial u_\gamma^i} \right) \right] h^i dxdt.$$

From the definition of potentiality, we have

$$\int\limits_0^T \int\limits_\Omega \sum_{i=1}^{2n} \left[ \sum_{k=1}^{2n} \left( \frac{\partial R_k}{\partial u^i} - \frac{\partial R_i}{\partial u^k} \right) u_t^k - \frac{\partial R_i}{\partial t} - \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left( \frac{\partial B}{\partial u_\gamma^i} \right) \right] h^i dxdt =$$
$$= \int\limits_0^T \int\limits_\Omega \sum_{i=1}^{2n} \left[ \sum_{k=1}^{2n} C_{ik} u_t^k + E_i \right] h^i dxdt.$$

Since the elements $h^i$ are arbitrary, we obtain

$$\sum_{k=1}^{2n} \left( \frac{\partial R_k}{\partial u^i} - \frac{\partial R_i}{\partial u^k} \right) u_t^k - \frac{\partial R_i}{\partial t} - \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left( \frac{\partial B}{\partial u_\gamma^i} \right) = \sum_{k=1}^{2n} C_{ik} u_t^k + E_i, \tag{3.3}$$

where $i = \overline{1, 2n}$.

Comparing the left- and right-hand sides of (3.3), we find

$$\begin{cases} \dfrac{\partial R_k}{\partial u^i} - \dfrac{\partial R_i}{\partial u^k} = C_{ik}, \\ -\dfrac{\partial R_i}{\partial t} - \displaystyle\sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left( \frac{\partial B}{\partial u_\gamma^i} \right) = E_i, \end{cases} \tag{3.4}$$

where $i, k = \overline{1, 2n}$. For the first group of equations in system (3.4), we obtain the following solution [4]:

$$R_i = - \int\limits_0^1 \sum_{k=1}^{2n} \lambda C_{ik} \left( x, t, \widehat{u} + \lambda(u - \widehat{u}) \right) (u^k - \widehat{u}^k) d\lambda, \; i = \overline{1, 2n}.$$

Let $\mathfrak{B}[t,u] = \int\limits_{\Omega} B(x,t,u_\alpha)dx$; $\dfrac{\delta\mathfrak{B}}{\delta u^i}$ be the functional derivative of $\mathfrak{B}$ with respect to $u^i$, $i=\overline{1,2n}$.

Let us rewrite the second group of equations in system (3.4) in the following form

$$\sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left(\frac{\partial B}{\partial u^i_\gamma}\right) = -\frac{\partial R_i}{\partial t} - E_i.$$

or

$$\frac{\delta\mathfrak{B}}{\delta u^i} = -\frac{\partial R_i}{\partial t} - E_i, i=\overline{1,2n}.$$

Using formula (2.3), we obtain

$$\mathfrak{B}[t,u] = -\int\limits_{\Omega}\int\limits_{0}^{1}\sum_{i=1}^{2n}\left[\frac{\partial R_i}{\partial t}(x,t,\widehat{u}+\lambda(u-\widehat{u})) + E_i(x,t,\widehat{u}_\alpha+\lambda(u_\alpha-\widehat{u}_\alpha))\right]\cdot$$
$$\cdot(u^i-\widehat{u}^i)d\lambda dx + \text{const}.$$

Thus, we arrive to the following Birkhoff's equations for infinite-dimensional systems:

$$N_i \equiv \sum_{k=1}^{2n}\left(\frac{\partial R_k}{\partial u^i} - \frac{\partial R_i}{\partial u^k}\right)u^k_t - \frac{\partial R_i}{\partial t} - \frac{\delta\mathfrak{B}}{\delta u^i} = 0, i=\overline{1,2n}. \tag{3.5}$$

**Theorem 3.1.** *The extremals of functional* (3.1) *are solutions to the system of equations* (3.5).

## 4 Discretization

Let us divide the interval $[0,T]$ into $l$ equal parts with nodes $t_j = j\tau$, $j=\overline{0,l}$, where $\tau = l^{-1}T$. Let us introduce the narrowing operators [7]

$$\overline{\mathcal{T}}_r u(x,t) = \overline{u}_r = \left(u^1(x,t_1), u^2(x,t_1),\ldots,u^{2n}(x,t_1),u^1(x,t_2),u^2(x,t_2),\ldots,\right.$$
$$\left. u^{2n}(x,t_2),\ldots,u^1(x,t_{l-1}),u^2(x,t_{l-1}),\ldots,u^{2n}(x,t_{l-1})\right),$$

where $r = 2n(l-1)$. Such vectors form a linear space, which we will denote as $\overline{U}_r$. For convenience, let us denote $\widetilde{u}_j = u(x,t_j)$, $\widetilde{u}^i_j = u^i(x,t_j)$, $i=\overline{1,2n}$, $j=\overline{0,l}$.

Denote by $\overline{N}$ the operator of the discrete analogue of problem (3.5), (1.2), obtained on the basis of functional (3.1).

Let us define

$$D\left(\overline{N}\right) = \Big\{(\widetilde{u}_0,\overline{u}_r,\widetilde{u}_l) : \overline{u}_r\in\overline{U}_r, \widetilde{u}^i_0 = \varphi^i_0(x), \widetilde{u}^i_l = \varphi^i_1(x), \widetilde{u}^i_j \in C^{2s}\left(\overline{\Omega}\right),$$
$$\left.\frac{\partial^\nu \widetilde{u}^i_j}{\partial n^\nu_x}\right|_{\partial\Omega} = \psi^i_\nu(x,t_j), i=\overline{1,2n}, |\nu| = \overline{0,s-1}, j=\overline{0,l}\Big\},$$
$$D\left(\overline{N}'_u\right) = \Big\{(\widetilde{h}_0,\overline{h}_r,\widetilde{h}_l) : \overline{h}_r\in\overline{U}_r, \widetilde{h}^i_0 = 0, \widetilde{h}^i_l = 0, \widetilde{h}^i_j \in C^{2s}\left(\overline{\Omega}\right),$$
$$\left.\frac{\partial^\nu \widetilde{u}^i_j}{\partial n^\nu_x}\right|_{\partial\Omega} = 0, i=\overline{1,2n}, |\nu| = \overline{0,s-1}, j=\overline{0,l}\Big\},$$

where $\overline{N}'_u$ is the Gâteaux derivative of the operator $\overline{N}$.

Next, we approximate the integrals as follows:

$$\int_{t_j}^{t_{j+1}} \int_{\Omega} \left( \sum_{i=1}^{2n} R_i u_t^i - B \right) dx dt \approx \frac{T}{l} \int_{\Omega} \left( \sum_{i=1}^{2n} R_{i,j} \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} - B_j \right) dx,$$

where $R_{i,j} = R_i(x, t_j, \widetilde{u}_j)$, $B_j = (x, t_j, D_\gamma \widetilde{u}_j)$.

We replace functional (3.1) by the discrete Hamiltonian action:

$$\overline{F}(\overline{u}_r) = \frac{T}{l} \sum_{j=0}^{l-1} \int_{\Omega} \left( \sum_{i=1}^{2n} R_{i,j} \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} - B_j \right) dx.$$

Then, we have

$$\delta\overline{F}\left[\overline{u}_r, \overline{h}_r\right] = \frac{T}{l} \sum_{j=0}^{l-1} \int_{\Omega} \left( \sum_{i=1}^{2n} \sum_{k=1}^{2n} \frac{\partial R_{i,j}}{\partial \widetilde{u}_j^k} \widetilde{h}_j^k \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} + \sum_{k=1}^{2n} R_{k,j} \frac{\widetilde{h}_{j+1}^k - \widetilde{h}_j^k}{\tau} - \right.$$
$$\left. - \sum_{k=1}^{2n} \sum_{|\gamma|=0}^{s} \frac{\partial B_j}{\partial D_\gamma \left(\widetilde{u}_j^k\right)} D_\gamma \widetilde{h}_j^k \right) dx. \tag{4.1}$$

Since

$$\widetilde{h}_0^i = 0, \widetilde{h}_l^i = 0, \left. \frac{\partial^\nu \left(\widetilde{h}_j^i\right)}{\partial n_x^\nu} \right|_{\partial\Omega} = 0, i = \overline{1, 2n}, j = \overline{0, l}, |\nu| = \overline{0, s-1},$$

integrating by parts in (4.1), we get

$$\delta\overline{F}\left[\overline{u}_r, \overline{h}_r\right] = \frac{T}{l} \sum_{j=1}^{l-1} \int_{\Omega} \left[ \sum_{i=1}^{2n} \sum_{k=1}^{2n} \frac{\partial R_{i,j}}{\partial \widetilde{u}_j^k} \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} - \sum_{k=1}^{2n} \frac{R_{k,j} - R_{k,j-1}}{\tau} - \right.$$
$$\left. - \sum_{k=1}^{2n} \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left( \frac{\partial B_j}{\partial D_\gamma \left(\widetilde{u}_j^k\right)} \right) \right] \widetilde{h}_j^k dx.$$

From the equality $\delta F\left[\overline{u}_r, \overline{h}_r\right] = 0$, $\forall \overline{u}_r \in D\left(\overline{N}\right)$, $\forall \overline{h}_r \in D\left(\overline{N}'_u\right)$ we obtain the system of equations for discrete-time motion as follows:

$$\overline{N}_{k,j} \equiv \sum_{i=1}^{2n} \frac{\partial R_{i,j}}{\partial \widetilde{u}_j^k} \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} - \frac{R_{k,j} - R_{k,j-1}}{\tau} - \frac{\delta \mathfrak{B}_j}{\delta \widetilde{u}_j^k} = 0,$$
$$j = \overline{1, l-1}, k = \overline{1, 2n}, \tag{4.2}$$

where $\mathfrak{B}_j = \mathfrak{B}[t_j, \widetilde{u}_j]$.

**Theorem 4.1.** *Equations (4.2) are discrete-time analogues of (3.5).*

## 5 Example

Let us consider the wave equation with axial symmetry [3]:

$$w_{tt} = a^2 \left( w_{\rho\rho} + \frac{1}{\rho} w_\rho \right), t \in [0, T], \rho \in [\rho_1, \rho_2], \tag{5.1}$$

with the following boundary conditions:

$$w\big|_{t=0} = \varphi_1(\rho), w\big|_{t=T} = \varphi_2(\rho),$$
$$w\big|_{\rho=\rho_1} = \phi_1(t), w\big|_{\rho=\rho_2} = \phi_2(t),$$

where $w(t,\rho)$ is the unknown function, $\rho$ is the radial coordinate, and $a$ is a constant coefficient.

Let us denote

$$\begin{cases} w = u^1, \\ w_t = u^2. \end{cases}$$

We can represent equation (5.1) as a system of equations:

$$N(u) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_t^1 \\ u_t^2 \end{pmatrix} + \begin{pmatrix} -u^2 \\ -a^2\left(u_{\rho\rho}^1 + \dfrac{1}{\rho}u_\rho^1\right) \end{pmatrix} = 0. \tag{5.2}$$

According to Theorem 2.1, operator (5.2) is not potential. Using conditions (2.5), we can find the matrix variational multiplier as follows:

$$M = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}.$$

Then, the system

$$\widehat{N} = MN = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix} \begin{pmatrix} u_t^1 \\ u_t^2 \end{pmatrix} + \begin{pmatrix} -a^2\rho u_{\rho\rho}^1 - a^2 u_\rho^1 \\ \rho u^2 \end{pmatrix} = 0 \tag{5.3}$$

admits a representation in the form of Birkhoff's equations and we find

$$R_1 = -\frac{1}{2}\rho u^2, R_2 = \frac{1}{2}\rho u^1, B = -\frac{1}{2}a^2\rho\left(u_\rho^1\right)^2 - \frac{1}{2}\rho\left(u^2\right)^2.$$

Therefore, according to formula (3.1), the Hamiltonian action for (5.3) takes the following form:

$$F_{\widehat{N}}[u] = \int_0^1 \int_\Omega \frac{1}{2}\left[-\rho u^2 u_t^1 + \rho u^1 u_t^2 + a^2\rho\left(u_\rho^1\right)^2 + \rho\left(u^2\right)^2\right] dxdt.$$

By converting it to its discrete version, we can easily obtain the discrete version of system of equations (5.3) as follows:

$$\overline{N}_{1,j} \equiv \frac{1}{2}\rho\frac{\widetilde{u}_{j+1}^2 - \widetilde{u}_j^2}{\tau} + \frac{1}{2}\rho\frac{\widetilde{u}_j^2 - \widetilde{u}_{j-1}^2}{\tau} - a^2\rho\frac{\partial^2}{\partial\rho^2}\widetilde{u}_j^1 - a^2\frac{\partial}{\partial\rho^1}\widetilde{u}_j^1 = 0, j = \overline{1, l-1},$$
$$\overline{N}_{2,j} \equiv -\frac{1}{2}\rho\frac{\widetilde{u}_{j+1}^1 - \widetilde{u}_j^1}{\tau} - \frac{1}{2}\rho\frac{\widetilde{u}_j^1 - \widetilde{u}_{j-1}^1}{\tau} + \rho\widetilde{u}_j^2 = 0, j = \overline{1, l-1}.$$

## 6 Conclusion

Necessary and sufficient conditions for the potentiality of the system of equations of the form $C(x,t,u)u_t + E(x,t,u_\alpha) = 0$ with respect to a given bilinear form have been obtained. An algorithm for constructing the corresponding Hamiltonian action and transforming this system into the form of Birkhoff's equations for infinite-dimensional systems is presented. Based on the derived Hamiltonian action, a discrete analogue of this system of equations has been obtained. An illustrative example has been considered.

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# ON KAISER CLASS OF UNARS IN EXPANDED SIGNATURE

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**Key words:** Model Theory, Jonsson theory, semantic model, center of Jonsson theory, normal Jonsson theory, semantic Jonsson quasivariety, Jonsson spectrum, cosemanticness class, unar, Kaiser class.

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**Abstract.** The present paper is connected with studying properties of Jonsson theories of an unar. The main idea is to study a structure of the signature with one unary functional symbol by expanding it with both a new constant symbol and a unary predicate symbol. We construct the semantic Jonsson quasivariety using the semantic models of enriched Jonsson primitives of unars and consider its Jonsson spectrum, divided by cosemanticness relation onto factor-set and the obtained factor-set is divided by new equivalence relation with regard to Kaiser class. Additionally, we consider the notion of normal Jonsson theory and prove that the theory of all unars is normal, and obtain new results concerning components of unars and the Kaiser class of their theories.

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## 1 Introduction

The research concerning complete theories of unars is connected with the works of Yu.Ye. Shishmarev [31], A.A. Ivanov [11, 12], A.N. Ryaskin [27, 28], L. Marcus [16, 17]. Unary algebras as structures were studied in the works of S. Burris [5, 6], G.E. Puninskii [25], and A.W. Miller [19].

Recent papers in Eurasian Mathematical Journal also show that questions related to algebraic methods and model-theoretic structures remain active in different but connected directions. In particular, generalized reduction algorithms were considered in [2], while algebras of binary formulas for weakly circularly minimal theories with equivalence relations were studied in [14]. These works illustrate the broader context in which algebraic and logical methods interact.

Since Jonsson theories are, generally speaking, not complete theories, it is interesting to study Jonsson unars and consider their general problems in terms of Model Theory (for example using results from [26, 30, 18, 24, 3]), as well as in the framework of universal algebra, concerning the nature of unars.

The study of unars in terms of Jonsson theories started from the works of T.G. Mustafin and A.R. Yeshkeyev [38, 37]. The authors considered the properties of Jonsson universals and Jonsson primitives, as well as obtained the definition of characteristics of their semantic models. In [40] the authors expanded the signature of unars by a new constant symbol and one unary predicate, that distinguished an existentially closed Jonsson unar and obtained some properties of the enriched Jonsson universals and primitives. The main idea for the present paper was to naturally continue using this technique in terms of constructing Jonsson spectrum of semantic Jonsson quasivariety of unars.

It is important to note, that the concept of double factorization was first introduced by A.R. Yeshkeyev in [39]. However, in [39] the double factorization was used in terms of cosemanticness relation and Jonsson semantic and Jonsson syntactic similarity. In order to distinct, we will use the denotation D1F for the double factorization in the present paper - in terms of cosemanticness relation and $K_T$-equivalence.

More information on Jonsson theories and their properties one can find in [35, 13, 22, 1]. For the convenience of readers, we recall the basic terminology used in the paper. A first-order language, or signature, is understood as a collection of non-logical symbols, such as constant, function and relation symbols. A theory is a set of first-order sentences in a fixed language. A unar is a structure whose signature contains one unary functional symbol; in this paper we work in the signature $\sigma = \{f^1\}$. For standard model-theoretic terminology see, for example, [32]; for unars in the signature of one unary function see [31, 28].

## 2 Jonsson theories of unars

Throughout this paper a unar is understood as a structure of the signature $\sigma = \{f^1\}$, where $f$ is a unary functional symbol; equivalently, it is a model with one unary operation. We first recall some important notions and definitions concerning model-theoretic properties of Jonsson theories, starting with the definition of Jonsson theory itself.

**Definition 1.** [35]A theory $T$ is said to be Jonsson, if it satisfies the following conditions:
1) $T$ has at least one infinite model;
2) $T$ is $\forall\exists$-axiomatisable;
3) $T$ has joint embedding property ($JEP$);
4) $T$ has amalgamation property ($AP$).

The existence of at least one infinite model is a strong demand, this follows from the first condition. The second condition is syntactic and it coincides with property of inductiveness, i.e., the chain of any models of this theory will be a model of this theory as well. The third and fourth are very important properties of AP and JEP (more information on mentioned properties one can find in [35]) that play a big role in algebras having a syntactic property (for example, expressibility by sentences from a specific fragment of first-order logic such as equational, Horn, universal, $\forall\exists$, existential, positive-primitive, etc.). Let us recall the definition of an inductive theory.

**Definition 2.** [4] Let $L$ be a first-order language. A theory $T$ in the language $L$ is called inductive (or a $\forall\exists$-theory) if it is axiomatizable by a set of $\forall\exists$-sentences; that is, it is equivalent to a set of sentences of the form $\forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m \; \psi(x_1, \dots, x_n, y_1, \dots, y_m)$, where $\psi$ is a quantifier-free $L$-formula (contains no quantifiers) and $m, n \geq 0$.

More detailed information on inductive theories can be found in [32, 8, 33, 10].

**Definition 3.** [23] Let $T$ be a Jonsson theory. A model $\mathfrak{C}_T$ of power $2^{|\omega|}$ is said to be a semantic model of the theory $T$ if $\mathfrak{C}_T$ is a $\omega^+$-homogeneous $\omega^+$-universal model of the theory $T$.

If a Jonsson theory has several semantic models then they are all equal up to isomorphism. The following criterion is crucial.

**Theorem 2.1.** [23] *Inductive theory $T$ is Jonsson if and only if it has a semantic model $\mathfrak{C}_T$.*

It is easy to see, that if a theory is Jonsson then it has a semantic model and vice versa. The next definition of the center of a Jonsson theory was introduced by T.G. Mustafin.

**Definition 4.** [35]The elementary theory of a semantic model of a Jonsson theory $T$ is called the center of this theory. It is denoted by $T^*$, i.e. $Th(\mathfrak{C}_T) = T^*$.

It is important to note, that the center of a Jonsson theory is a complete theory by definition. However, we cannot claim that for any Jonsson theory its center is a Jonsson theory too. For example, group theory is a Jonsson theory, but its center is not a Jonsson theory. But in the case of perfect Jonsson theories, such as, for example, theories of abelian groups, this statement is true. More detailed information can be found in [35].

The present work is associated with the Jonsson universal of unars in the frame of first-order language $L$ of the signature $\sigma = \{f^1\}$, where $f$ is a unary functional symbol. Let us recall some important concepts.

**Denotation 1.** [38]Let $\nabla$ be $\Pi_1 \cup \Sigma_1$, i.e. $\nabla$ is the set of all universal or existential $L$-formulas.

Thus, $\nabla$ is the set of all formulas that have in prenex normal form either one universal quantifier or one existential quantifier.

**Definition 5.** [38] 1) If $T = T_{\forall}$, i.e. the theory is equal to its universal logical consequences, then $T$ is said to be universal;

2) If $T = T_{\nabla}$, i.e. the theory is equal to its universal or existential logical consequences, then the theory $T$ is called primitive.

The connection between two Jonsson universals with regards to their centers and semantic models is presented in the form of the following proposition.

**Proposition 2.1.** [38] *Let $T_{\forall_1}, T_{\forall_2}$ be Jonsson universals. Then the following conditions are equivalent:*

*1) $T_{\forall_1} = T_{\forall_2}$;*

*2) $\mathfrak{C}_{T_{\forall_1}} \simeq \mathfrak{C}_{T_{\forall_2}}$;*

*3) $T^*_{\forall_1} = T^*_{\forall_2}$.*

Here $\mathfrak{C}_{T_{\forall_1}}$ and $\mathfrak{C}_{T_{\forall_2}}$ are semantic models, $T^*_{\forall_1}$ and $T^*_{\forall_2}$ are the centers of Jonsson universals $T_{\forall_1}, T_{\forall_2}$ respectively.

Let $\mathfrak{A}$ be some unar, i.e. a model of signature $\sigma = \{f^1\}$, where $f$ is a unary functional symbol. Since each model of a Jonsson theory embeds into its semantic model by Definition 3, the following fact is true for Jonsson universals of unars. Jonsson universals of unars, considering Definition 5, are theories consisting of universal sentences in terms of $\sigma = \{f^1\}$.

**Lemma 2.1.** [38] *For any unar $\mathfrak{A}$ the following holds*

$$\mathfrak{A} \text{ is a model of } T_{\forall} \Leftrightarrow \mathfrak{A} \text{ embeds in } \mathfrak{C}_{T_{\forall}},$$

*where $\mathfrak{C}_{T_{\forall}}$ is a semantic model of a Jonsson universal of unars $T_{\forall}$.*

In what follows, the relation "$\mathfrak{A}$ is a model of $T_{\forall}$" will be denoted by the standard satisfaction symbol $\models$. Thus, instead of writing that a structure $\mathfrak{A}$ is a model of a theory $T_{\forall}$, we will write $\mathfrak{A} \models T_{\forall}$.

Let $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$, $n \in \omega$. Elements $a, b \in \mathfrak{A}$ are called $\mathfrak{A}$-connected in $X \subseteq \mathfrak{A}$ if there exist natural numbers $m$ and $n$ such that $f^m(a) = f^n(b)$ and $f^0(a), ..., f^m(a), f^0(b), ..., f^n(b) \in X$.

A set $X \subseteq \mathfrak{A}$ is called $\mathfrak{A}$-connected if any two elements from $X$ are $\mathfrak{A}$-connected. A subsystem $\mathfrak{B} \subseteq \mathfrak{A}$ whose universe is the maximal $\mathfrak{A}$-connected subset of the universe of $\mathfrak{A}$ is called a component in $\mathfrak{A}$.

Let us recall some important definitions.

**Denotation 2.** [38]For any $a \in \mathfrak{C}_{T_\forall}$ let

$$\chi(a) = \begin{cases} \omega, \text{if } \ f^n(a) \neq f^k(a), \text{for any } n < k < \omega \\ \langle n, m \rangle, \text{if } \ \langle n, m \rangle = min\{\langle n, m \rangle : f^n(a) = f^{n+m}(a)\}. \end{cases}$$

**Definition 6.** [38]A set $\{a_1, ..., a_m\}$ of elements of $\mathfrak{C}_{T_\forall}$ is called an $m$-cycle, if $a_i \neq a_j, f(a_i) = a_{i+1}$ for all $1 \leq i < j \leq m$ and $f(a_m) = a_1$.

**Denotation 3.** [38]For any $a \in \mathfrak{C}_{T_\forall}$ let

$$k(a) = |\{b \in \mathfrak{C}_{T_\forall} : f(b) = a\}|$$

i.e. $k(a)$ is the cardinality of the set of all preimages of $a$ by $f$.

Probably one of the most important definitions is Definition 7. It defines the characteristic of a semantic model of a Jonsson universal of unars.

**Definition 7.** [38] A tetrad $(\Omega, \nu, \mu, \varepsilon)$ is said to be the characteristic $\mathfrak{C}_{T_\forall}$ and denoted by $char(\mathfrak{C}_{T_\forall})$, if

$\Omega = \{\chi(a) : a \in \mathfrak{C}_{T_\forall}\}$,

$\nu : \omega \setminus \{0\} \to \omega \cup \{\infty\}$ such that for any $m > 0$,

$$\nu(m) = \begin{cases} k, \text{if the number of } \ m\text{-cycles in } \ \mathfrak{C}_{T_\forall} \ \text{ is finite and equals } \ k, \\ \infty, \text{otherwise;} \end{cases}$$

$\mu : \Omega \to \omega \cup \{\infty\}$ such that if $\alpha \in \Omega$ and $\alpha \in \chi(a)$, then $\mu(\alpha) = k(a)$, if $k(a) < \omega$ and $\mu(\alpha) = \infty$, if $k(a) = |\mathfrak{C}_{T_\forall}|$;

$$\varepsilon = \begin{cases} 0, \text{if } \ |\{a \in \mathfrak{C}_{T_\forall} : \chi(a) = \omega\}| = 0, \\ \infty, \text{otherwise.} \end{cases}$$

Let us give a more detailed look at Definition 7.

1) $\Omega$ is the set that consists of $\chi(a), a \in \mathfrak{C}_{T_\forall}$, i.e. $a$ is an element of the universe of a semantic model, i.e., $\chi(a) = \omega$ if $f^n(a) \neq f^k(a)$, $n$ and $k$ indicate the number of acts on element $a$ by a function $f$, for any $n < k < \omega$, both $n$ and $k$ are finite numbers, $k > n$. $\chi(a) = \langle n, m \rangle$, $\langle n, m \rangle$ is the minimal ordered pair such that $f^n(a) = f^{n+m}(a)$. Hence, $\Omega$ expresses whether there are finite or infinite cycles in the semantic model.

2) $\nu$ is a correspondence from the set $\omega$ without 0 into the set that consists of $\omega$ or $\{\infty\}$. The correspondence holds as follows: for any $m > 0$,

$$\nu(m) = \begin{cases} k, \text{if the number of } \ m\text{-cycles in } \ \mathfrak{C}_{T_\forall} \ \text{ is finite and equals } \ k, \\ \infty, \text{otherwise.} \end{cases}$$

That is, the correspondence $\nu(m)$ expresses the number of $m$-cycles, where $m$ is the number of elements in the cycle.

3) $\mu$ is a correspondence from the set $\Omega = \{\chi(a) : a \in \mathfrak{C}_{T_\forall}\}$ into the set that consists of $\omega$ or $\{\infty\}$. The correspondence holds as follows: if $\alpha \in \Omega$ and $\alpha \in \chi(a)$, then $\mu(\alpha) = k(a)$, if $k(a) < \omega$ and $\mu(\alpha) = \infty$, then $k(a) = |\mathfrak{C}_{T_\forall}|$. Since we are working in the frame of [23], $|\mathfrak{C}_{T_\forall}| \ = 2^\omega$.

$k(a)$ is the power of the set that consists of all roots of element $b \in \mathfrak{C}_{T_\forall}$ such that $f(b) = a$, $a \in \mathfrak{C}_{T_\forall}$. The properties and dependencies between the concepts of $k(a)$ and $\chi(a)$ were studied in [38].

4) $\varepsilon$ can vary between two values: 0 and $\infty$. It depends on the power of the set that consists of such elements $a \in \mathfrak{C}_{T_\forall}$, that $\chi(a) = \omega$. If no such elements exists the power of this set will be equal to 0, i.e. $\varepsilon = 0$, if there are such elements, then $\varepsilon = \infty$. $\varepsilon$ expresses the existence of cycles in the semantic model.

The following properties follow immediately from Definition 7.

**Lemma 2.2.** [38] *If* $char(\mathfrak{C}_{T_\forall}) = (\Omega, \nu, \mu, \varepsilon)$, *then*

1°. $\varnothing \neq \Omega \subseteq \{\omega\} \cup (\omega \times \omega)$;

2°. *For any* $\langle n, m\rangle \in \Omega$ *and* $0 \le k < n$ *we have* $\langle k, m\rangle \in \Omega$;

3°. $\nu(m) > 0 \Leftrightarrow (0, m) \in \Omega$;

4°. $\omega \in \Omega \Leftrightarrow \varepsilon = \infty$;

5°. $|\Omega| = \omega \Rightarrow \omega \in \Omega$;

6°. $\langle n, m\rangle \in \Omega \Rightarrow \langle n+1, m\rangle \notin \Omega \Leftrightarrow \mu(\langle n, m\rangle) = 0$;

7°. *If* $\omega \notin \Omega$ *and* $\Omega$ *is finite, there is* $m < \omega$ *such that* $\nu(m) = \infty$ *or there are* $n, m < \omega$ *such that* $\langle n, m\rangle \in \Omega$ *and* $\mu(\langle n, m\rangle) = \infty$;

8°. *If* $|\Omega| = \omega$ *we have either* $\mu(\omega) \ge k$, *if there are finite* $k, l$ *such that* $k = max\{\mu(\langle n, m\rangle) \in \Omega, n + m \ge l\}$ *or if such* $k, l$ *do not exist, then* $\mu(\omega) = \infty$.

The characteristic $\mathfrak{C}_{T_\forall}$ is considered to be an invariant of the Jonsson universal of unars. It is important to note that in [38, 37] T.G. Mustafin and A.R. Yeshkeyev obtained the result that shows that for any arbitrary characteristics there is a universal of unars.

## 3 On normal Jonsson theory of unars

For this section we need to consider the definition of a perfect Jonsson theory.

**Definition 8.** [35] A Jonsson theory $T$ is called perfect if its semantic model $\mathfrak{C}_T$ is $\omega^+$-saturated.

The example of a Jonsson theory that is not perfect is a group theory.

**Theorem 3.1.** [35] *Let* $T$ *be an arbitrary Jonsson theory. Then the following conditions are equivalent:*

*1)* $T$ *is perfect;*

*2)* $T^*$ *is the model completion of* $T$.

**Theorem 3.2.** [37] *Let* $T$ *be a Jonsson universal of unars,* $T^*$ *its center. Then*

*1)* $T^*$ *is the model completion of* $T$;

*2)* $T^*$ *allows quantifier elimination (i.e. submodel complete);*

*3)* $T^*$ *is* $\omega$*-stable.*

By aforementioned Theorems 3.1 and 3.2 the universal of unars is a perfect Jonsson theory. This fact was proved in work [37].

Let $\mathbb{T}_U$ be the theory of all unars. It is known that $\mathbb{T}_U$ is an empty theory (the set of axioms of this theory is empty), i.e. it is a set of all sentences in the language $L$ of the signature $\sigma = \{f^1\}$, where $f$ is a unary functional symbol. It was proved in [40] that the theory of all unars is a Jonsson theory. Let $\mathfrak{C}_{\mathbb{T}_U}$ be its semantic model.

According to Definition 3 of a semantic model of a Jonsson theory and Definition 7 of $char(\mathfrak{C}_{T_\forall})$, $\mathfrak{C}_{\mathbb{T}_U}$ (semantic model of theory of all unars) is a concatenation of components.

In [40] the following lemma was proved:

**Lemma 3.1.** [40] *Let* $\mathbb{T}_U$ *be the Jonsson theory of all unars, and let* $M$ *be its model. Then* $M$ *is a component of the theory* $\mathbb{T}_U$ *if and only if* $M \in E_{\mathbb{T}_U}$, *where* $E_{\mathbb{T}_U}$ *is the class of all existentially closed models of the theory* $\mathbb{T}_U$.

Hence, any component of the theory of all unars is an existentially closed model. Since the theory of all unars $\mathbb{T}_U$ is a universal theory [40], according to Theorem 3.2 $\mathbb{T}_U$ is a perfect Jonsson theory.

**Lemma 3.2.** [35] *If* $T$ *is perfect, then the class of all* $\Sigma_1$*-closed models* $E_T$ *coincides with* $Mod\ (T^*)$.

Moreover, the following statement holds.

**Theorem 3.3.** [35] *If $T$ is perfect, then $T^*$ is a Jonsson theory.*

Thus, by Lemma 3.2 and Theorem 3.3, $Th(\mathfrak{C}_{\mathbb{T}_U}) = \mathbb{T}_U^*$ is a complete perfect Jonsson theory and $Mod(\mathbb{T}_U^*) = E_{\mathbb{T}_U}$.

In order to give the definition of a normal Jonsson theory, we need to consider first two important notions of a cosemanticness relation and a Kaiser class.

**Definition 9.** [35] Let $T_1$ and $T_2$ be Jonsson theories, $C_{T_1}$ and $C_{T_2}$ be their semantic models, respectively. $T_1$ and $T_2$ are said to be cosemantic Jonsson theories (denoted by $T_1 \bowtie T_2$), if $C_{T_1} = C_{T_2}$.

**Definition 10.** Let $T$ be a Jonsson theory. The class of all models $K_T \subseteq Mod(T)$ such that $K_T = \{\mathfrak{A} \mid \mathfrak{A} \in Mod(T)$ and $Th_{\forall\exists}(\mathfrak{A})$ is a Jonsson theory$\}$ is called the Kaiser class of $T$.

In the framework of studying the theories of all unars the following statement holds:

**Lemma 3.3.** *Let $\mathbb{T}_U$ be a theory of all unars, $\mathbb{T}_U^*$ be its center. If $E_{\mathbb{T}_U}$ is its class of existentially closed models, $K_{\mathbb{T}_U}$ is its Kaiser class, and $K_{\mathbb{T}_U}^*$ is the Kaiser class of $\mathbb{T}_U^*$, then $K_{\mathbb{T}_U} \supseteq E_{\mathbb{T}_U}$ and $K_{\mathbb{T}_U}^* = E_{\mathbb{T}_U}$.*

*Proof.* The theory of all unars is a perfect Jonsson theory and $Mod(\mathbb{T}_U^*) = E_{\mathbb{T}_U}$. The statement of the lemma follows from the following theorem.

**Theorem 3.4.** [36] *Let $T$ be a Jonsson theory. Then, for any model $A \in E_T$, the theory $T^0(A) = Th_{\forall\exists}(A)$ is Jonsson.*

It is trivial, that $K_{\mathbb{T}_U}^* = E_{\mathbb{T}_U}$ by Theorem 3.4 $Mod(\mathbb{T}_U) = Inf(\mathbb{T}_U) \cup K_{\mathbb{T}_U} \cup Fin(\mathbb{T}_U)$, where $Inf(\mathbb{T}_U)$ is the set all of such infinite models $\mathfrak{A}$ that $Th_{\forall\exists}(\mathfrak{A})$ is not a Jonsson theory (such theory does not have $AP$ property), and $Fin(\mathbb{T}_U)$ is the set of all finite models of the theory $\mathbb{T}_U$. The existence of such models is guaranteed by the fact that Jonsson theories are, generally speaking, not complete theories. By Theorem 3.4 $E_{\mathbb{T}_U} \notin Inf(\mathbb{T}_U)$, and, obviously, $E_{\mathbb{T}_U} \notin Fin(\mathbb{T}_U)$, therefore, it is evident that $K_{\mathbb{T}_U} \supseteq E_{\mathbb{T}_U}$. □

Theorem 3.4 shows that a Kaiser class always exists for any Jonsson theory. Since by Definition 10 $E_T$ (the class of all existentially closed models of Jonsson theory $T$) will always be a subset of $K_T$.

**Theorem 3.5.** *Let $\mathbb{T}_U$ be the Jonsson theory of all unars, and let $\mathfrak{M}$ be its model. Then $\mathfrak{M}$ is a component of the theory $\mathbb{T}_U$ if and only if $\mathfrak{M} \in K_{\mathbb{T}_U}$, where $K_{\mathbb{T}_U}$ is the Kaiser class of the theory $\mathbb{T}_U$.*

*Proof.* The proof follows from Lemma 3.1 and Theorem 3.4. Since $K_{\mathbb{T}_U} \supseteq E_{\mathbb{T}_U}$ for the theory of all unars, it is evident that for any component $\mathfrak{M}$ of the theory $\mathbb{T}_U$: $\mathfrak{M} \in K_{\mathbb{T}_U}$ and, moreover, by the definition of Kaiser class (Definition 10), $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory. □

**Definition 11.** A Jonsson theory $T$ is called a normal theory if for any $\mathfrak{M} \in K_T : \mathfrak{M}^0 = Th_{\forall\exists}(\mathfrak{M})$, and $\mathfrak{C}_{\mathfrak{M}^0}$ is an existentially closed submodel of $\mathfrak{C}_T$ ($\mathfrak{C}_{\mathfrak{M}^0}$ is a semantic model of $\mathfrak{M}^0$, $\mathfrak{C}_T$ is a semantic model of $T$).

As an example of a normal Jonsson theory we can consider the theory of all abelian groups, more detailed explanation one can find in [39].

**Theorem 3.6.** *The theory of all unars $\mathbb{T}_U$ is a normal Jonsson theory.*

*Proof.* Let us consider the theory of all unars $\mathbb{T}_U$ and its semantic model $\mathfrak{C}_{\mathbb{T}_U}$. $\mathfrak{M} \in K_{\mathbb{T}_U}$ means that $\mathfrak{M} \in Mod(\mathbb{T}_U)$ and $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory, where $K_{\mathbb{T}_U}$ is a Kaiser class of the theory of all unars. Since by Theorem 3.2 a Jonsson universal of unars admits quantifier elimination, $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory. Since it is a Jonsson theory by Theorem 2.1 it has a semantic model. Let $\mathfrak{M}^0 = Th_{\forall\exists}(\mathfrak{M})$ and $\mathfrak{C}_{\mathfrak{M}^0}$ be a semantic model of $\mathfrak{M}^0$ by virtue of cosemanticness relation (Definition 9).

Let us consider $\mathfrak{C}_{\mathfrak{M}^0}$ and $\mathfrak{C}_{\mathbb{T}_U}$. Here, we can use the following lemma.

**Lemma 3.4.** [35] *The semantic model $C_T$ of a Jonsson theory $T$ is $T$-existentially closed.*

$\mathfrak{C}_{\mathfrak{M}^0} \in Mod(\mathbb{T}_U)$ holds since any $L$-structure is a model of the theory of all unars due to it being an empty theory. It means that since $\mathfrak{C}_{\mathfrak{M}^0} \in Mod(\mathbb{T}_U)$ and $\mathfrak{C}_{\mathbb{T}_U}$ by Lemma 3.4 is an existentially closed model, then $\exists x\varphi(x,a)$, where $\varphi(x,a)$ is a formula of the language $L$, $a \in \mathfrak{C}_{\mathfrak{M}^0}$, from the fact that $\mathfrak{C}_{\mathbb{T}_U} \models \exists x\varphi(x,a)$ follows that there exists such $a' \in \mathfrak{C}_{\mathfrak{M}^0}$ that $\mathfrak{C}_{\mathfrak{M}^0} \models \exists x\varphi(x,a')$. It is evident that by the definition of characteristic of semantic models of all unars (Definition 7) and the fact that the theory of all unars is a perfect Jonsson theory, therefore, $\mathfrak{C}_{\mathbb{T}_U}$ is an $\omega^+$-saturated model (Definition 8), this expression holds.

□

## 4 On double factorization and permissible expansion of the signature of unars

Let $K$ be a class of models of a fixed signature $\sigma$. Then we can consider Jonsson spectrum for $K$ defined as follows.

**Definition 12.** [36] The set $JSp(K)$ of Jonsson theories of the signature $\sigma$, where

$$JSp(K) = \{\Delta \mid \Delta \text{ is a Jonsson theory and } K \subseteq Mod(\Delta)\},$$

is called the Jonsson spectrum for the class $K$.

A more detailed look at the Jonsson spectrum one can find in [36]. Now, considering this set we can introduce various equivalence relations on it and obtain its factor-sets. Let us give the definition of the following binary relation.

**Definition 13.** Let $T_1$ and $T_2$ be Jonsson theories of the considered language $L$. $T_1$ and $T_2$ are called $K_T$-equivalent ($T_1 \mathrel{\tilde{\bowtie}} T_2$) if $K_{T_1} = K_{T_2}$.

We can consider the cosemanticness relation on Jonsson spectrum $JSp(K)$ and obtain a partition of $JSp(K)$ onto disjoint equivalence classes with coinciding semantic models. We get a factor-set, denoted as $JSp(K)_{/\bowtie}$. We introduce the $K_T$-equivalent relation on the factor-set and obtain its double factorization $JSp(K)/_{D1F}$.

Let $K$ be a class of quasivariety in the sense of [15] of a first-order language $L$, $L_0 \subset L$, where $L_0$ is the set of all sentences of the language $L$, i.e. $L_0 = L/L_{Fm}$, here $L_{Fm}$ is the set of all formulas with at least one free variable. We consider the elementary theory $Th(K)$ of such class $K$, by adding to $Th(K)$ such $\forall\exists$ sentences of the language $L$ that are not contained in the $Th(K)$. We can consider the set of Jonsson theories $J(Th(K))$ defined as follows.

**Denotation 4.** [36]A set $J(Th(K)) = \{\Delta \mid \Delta \text{ is a Jonsson theory}, \ \Delta = Th(K) \cup \{\varphi^i\}\}$, where $\varphi^i \in \forall\exists(L_0)$ and $\varphi^i \notin Th(K)$, $i = 0$ or $i = 1$ (i.e. it is a formula or its negation), $Th(K)$ is an elementary theory of the class of quasivariety $K$, $\forall\exists(L_0)$ is the set of all $\forall\exists$ sentences of first-order language $L$.

Obviously, every $\Delta$ has its own semantic model. Let us consider the set of all such semantic models and denote it as $J\mathbb{C}$.

**Denotation 5.** [36]The set $J\mathbb{C} = \{\mathfrak{C}_\Delta \mid \Delta \in J(Th(K)), \mathfrak{C}_\Delta$ is a semantic model of $\Delta\}$.

We will call the set $J\mathbb{C}$ semantic Jonsson quasivariety of class $K$ if its elementary theory $Th(J\mathbb{C})$ is a Jonsson theory. A thorough study on semantic Jonsson quasivarieties and their Jonsson spectra can be found in [36].

Let us consider the first-order language $L$ of the signature $\sigma = \{f^1\}$, where $f$ is a unary functional symbol and expand it by symbols of a new constant $c$ and a unary predicate $P^1$. More on expansion of a signature can be found in [9, 34, 21, 29, 20, 7].

Let $\sigma'' = \sigma \cup \sigma'$, where $\sigma = \{f^1\}$, $\sigma' = (P^1, c)$. We consider the theory $\overline{T}_\forall$ in the new expanded signature $\sigma''$ as follows [40]:

$$\overline{T}_\forall = T_\forall \cup Th_\forall(\mathfrak{C}_{T_\forall}, a)_{a \in P^1(\mathfrak{C}_{T_\forall}) \cup P^1(c)} \cup \{P^1, \subseteq\} \cup P^1(c).$$

Here, $P^1$ is a new unary predicate symbol, $\{P^1, \subseteq\}$ is an infinite set of sentences, which are expressing the fact that in $\mathfrak{C}_{T_\forall}$ the predicate $P^1$ distinguishes an existentially closed submodel of $\mathfrak{C}_{T_\forall}$, i.e. $P^1(\mathfrak{C}_{T_\forall}) = \mathfrak{M}, \mathfrak{M} \in K_{T_\forall}$, $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory, and $K_{T_\forall}$ is a Kaiser class of theory $T_\forall$. Let us denote the center of $\overline{T}_\forall$ as follows:

$$\overline{T}^*_\forall = Th(\overline{\mathfrak{C}}_{T_\forall}) = Th(\mathfrak{C}_{T_\forall}, c, a)_{c, a \in P^1(\mathfrak{C}_{T_\forall})}$$

It was proved in [40], that the theory which was constructed in such a way is Jonsson and, moreover, it is a perfect Jonsson theory.

**Definition 14.** [36] A Jonsson theory is said to be *hereditary* if, in any of its permissible expansions, it preserves its Jonssonness.

Therefore, the following theorem was proved.

**Theorem 4.1.** [40] *If a Jonsson theory of unars $T_\forall$ is a perfect Jonsson theory and $\overline{T}_\forall$ is its hereditary expansion, then $\overline{T}_\forall$ is also a perfect Jonsson theory of unars.*

Moreover, it turns out, that the theory $\overline{T}_\forall$ will also be a normal theory. The normality property of a Jonsson theory is preserved in an enriched Jonsson theory if the expansion of the signature is hereditary.

**Theorem 4.2.** *If a Jonsson theory of unars $T_\forall$ is a normal perfect Jonsson theory and $\overline{T}_\forall$ is its hereditary expansion, then $\overline{T}_\forall$ is also a normal perfect Jonsson theory of unars.*

*Proof.* By Theorem 4.1 $\overline{T}_\forall$ is a perfect Jonsson theory of unars, and $\overline{\mathfrak{C}}_{T_\forall}$ (semantic model of enriched theory of all unars in the expanded signature) is a concatenation of components. A normality property of the Jonsson theory (Definition 11) means that any semantic model $\mathfrak{C}_{\mathfrak{M}^0}$ ($\mathfrak{M}^0 = Th_{\forall\exists}(\mathfrak{M})$) is an existentially closed submodel of $\overline{\mathfrak{C}}_{T_\forall}$. Moreover, any $P^1(\mathfrak{C}_{T_\forall}) = \mathfrak{M}, \mathfrak{M} \in K_{T_\forall}$ distinguished by the unary predicate is a component by virtue of Lemma 3.1. Since $\overline{\mathfrak{C}}_{T_\forall}$ is a concatenation of components by Definition 3 and Definition 7, this property of normality will be preserved in the permissible expansion introduced above, therefore, $\overline{T}_\forall$ is a normal perfect Jonsson theory of unars. □

**Lemma 4.1.** *If $T_\forall$ is a normal perfect Jonsson universal of unars and $\overline{T}_\forall$ is its hereditary expansion, then $\overline{T}^*_\forall$ is a perfect Jonsson theory.*

*Proof.* Let us consider a Kaiser class $K_{T_\forall}$ and a class of existentially closed models $E_{T_\forall}$ of the theory $T_\forall$. By virtue of Lemma 3.3 $E_{T_\forall} \subseteq K_{T_\forall}$. Since $T_\forall$ is a perfect Jonsson theory, its center $T_\forall^*$ $(Th(\mathfrak{C}_{T_\forall}))$ is a complete perfect Jonsson theory and $E_{T_\forall} = Mod(T_\forall^*)$. Therefore, $Mod(T_\forall^*) \subseteq K_{T_\forall}$. This means that any $\mathfrak{A} \in Mod(T_\forall^*)$ is an existentially closed model and its $Th_{\forall\exists}(\mathfrak{A})$ theory is a Jonsson theory. Hence, $T_\forall^*$ is model complete by Theorem 3.2. Considering the expansion of $T_\forall^*$ as above, and the fact that $T_\forall$ admits quantifier elimination, $\overline{T}_\forall^*$ is model complete as well. Hence, $\overline{T}_\forall^*$ is a perfect Jonsson theory. □

We consider the theory $\overline{T}_\nabla$ in the new expanded signature $\sigma''$ as follows [40]:

$$\overline{T}_\nabla = T_\nabla \cup Th_\nabla(\mathfrak{C}_{T_\nabla}, a)_{a\in P^1(\mathfrak{C}_{T_\nabla})\cup P^1(c)} \cup \{P^1, \subseteq\} \cup P^1(c).$$

Here $P^1$ is a new unary predicate symbol, $\{P^1, \subseteq\}$ is an infinite set of sentences, which are expressing the fact that in the semantic model $\mathfrak{C}_{T_\nabla}$ of $T_\nabla$ the predicate $P^1$ distinguishes an existentially closed submodel of $\mathfrak{C}_{T_\nabla}$, i.e. $P^1(\mathfrak{C}_{T_\nabla}) = \mathfrak{M}, \mathfrak{M} \in K_{T_\nabla}$, $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory, $K_{T_\nabla}$ is a Kaiser class of theory $T_\nabla$. Let us denote the center of $\overline{T}_\nabla$ as follows:

$$\overline{T}_\nabla^* = Th(\overline{\mathfrak{C}}_{T_\nabla}) = Th(\mathfrak{C}_{T_\nabla}, c, a)_{c,a\in P^1(\mathfrak{C}_{T_\nabla})}$$

Let us consider the following notions in the new expanded signature $\sigma''$. Let us start from Denotation 4.

**Denotation 6.** A set $J(Th(\overline{K})) = \{\overline{\Delta} \mid \overline{\Delta}$ is a Jonsson primitive of unars, $\overline{\Delta} = Th(\overline{K}) \cup \{\varphi^i\}\}$, where $\varphi^i \in \forall\exists(\overline{L}_0)$ and $\varphi^i \notin Th(\overline{K})$, $i = 0$ or $i = 1$ (i.e. it is a formula or its negation), $Th(\overline{K})$ is an elementary theory of the class of the quasivariety of unars $\overline{K}$ in the new expanded signature $\sigma''$, $\forall\exists(\overline{L}_0)$ is the set of all $\forall\exists$ sentences of first-order language $\overline{L}$ of the signature $\sigma''$.

**Definition 15.** A set $J\overline{\mathbb{C}}_{\mathbb{T}_\nabla} = \{\overline{\mathfrak{C}}_\Delta \mid \Delta \in J(Th(\overline{K})), \overline{\mathfrak{C}}_\Delta$ is a semantic model of $\overline{\Delta}\}$ is called a semantic Jonsson quasivariety of Jonsson primitives of unars in a signature $\sigma''$ if its elementary theory $Th(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})$ is a Jonsson theory.

Let us consider a Jonsson spectrum of semantic Jonsson quasivariety of unars in the new expanded signature as follows.

**Definition 16.** The set $JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})$ of all Jonsson theories of signature $\sigma''$, where

$$JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla}) = \{\overline{T}_\nabla \mid \overline{T}_\nabla \text{ is a Jonsson primitive of unars and } J\overline{\mathbb{C}}_{\mathbb{T}_\nabla} \subseteq Mod(\overline{T}_\nabla)\},$$

is called the Jonsson spectrum for the class $J\overline{\mathbb{C}}_{\mathbb{T}_\nabla}$, where $J\overline{\mathbb{C}}_{\mathbb{T}_\nabla}$ is a semantic Jonsson quasivariety of Jonsson primitives of unars in the signature $\sigma''$.

Let us introduce the double factorization on the set $JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})$ and obtain such disjoint equivalence classes $[[\overline{T}_\nabla]] \in JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})/_{D1F}$. $[[\overline{T}_\nabla]]$ is an equivalence class of Jonsson primitives of unars in the new expanded signature, containing theories that have coinciding centers, semantic models and Kaiser classes, as well as the class of models of such theories is equal to or contains semantic Jonsson quasivariety of primitives of unars $J\overline{\mathbb{C}}_{\mathbb{T}_\nabla}$ (Definition 15).

Let $[[\overline{T}_\nabla]] \in JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})/_{D1F}$, where $[[\overline{T}_\nabla]]$ is the class described above. Let $[[\overline{T}_\nabla]]^*$ be its center, $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]}$ be its semantic model.

**Theorem 4.3.** *Let* $[[\overline{T}_\nabla]]$ *be a double factorization class (D1F), and let* $[[\overline{T}_\nabla]]^*$ *be its center. Then,* $[[\overline{T}_\nabla]]_\forall = [[\overline{T}_\nabla]]_\forall^*$.

*Proof.* It is obvious that $[[\overline{T}_\nabla]]$ is a proper subset or equals $[[\overline{T}_\nabla]]^*$, hence, $[[\overline{T}_\nabla]]_\forall$ is also a proper subset or equals $[[\overline{T}_\nabla]]^*_\forall$ (this follows from the definition of universals and primitives). Suppose the opposite: let us assume that there is such a sentence $\varphi$ in the complement of $[[\overline{T}_\nabla]]_\forall$ to $[[\overline{T}_\nabla]]^*_\forall$ that $\varphi \in [[\overline{T}_\nabla]]^*_\forall \setminus [[\overline{T}_\nabla]]_\forall$. Let $\varphi = \forall \bar{x}\, \psi(\bar{x})$ be a universal sentence. Since $[[\overline{T}_\nabla]] \supseteq [[\overline{T}_\nabla]]_\forall$ and $\varphi \notin [[\overline{T}_\nabla]]_\forall$, $[[\overline{T}_\nabla]] \vdash \varphi$ does not hold, then it follows that a theory $[[\overline{T}_\nabla]] \cup \{\neg\varphi\}$ is a consistent theory.

If the theory is consistent, then it has a model. Let $\mathfrak{A} \models [[\overline{T}_\nabla]] \cup \{\neg\varphi\}$. Then, from the fact that $\varphi$ is a universal sentence follows $\neg\varphi \equiv \exists\bar{x}\,\neg\psi(\bar{x})$ and the following is true for the model $\mathfrak{A}$: $\mathfrak{A} \models \exists\bar{x}\,\neg\psi(\bar{x})$, $\mathfrak{A} \models [[\overline{T}_\nabla]]$. Due to the $\omega^+$-universality of the model $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]}$, we can assume that $\mathfrak{A} \subseteq \overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]}$, where $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]}$ is the semantic model of class $[[\overline{T}_\nabla]]$ and $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models [[\overline{T}_\nabla]]^*$, since $[[\overline{T}_\nabla]]^*$ is $Th(\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]})$ by the definition of the center of a Jonsson theory. Let $\bar{a} \in \mathfrak{A}$ be such that $\mathfrak{A} \models \neg\psi(\bar{a})$. Since the formula $\neg\psi(\bar{x})$ contains no quantifiers, $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models \neg\psi(\bar{a})$. However, since $\varphi \in [[\overline{T}_\nabla]]^*$ and $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models [[\overline{T}_\nabla]]^*$, we have $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models \varphi$, that is, $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models \forall\bar{x}\,\psi(\bar{x})$.

This leads to a contradiction. □

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# Short communications

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### ON INTERPOLATION OF LOCAL MORREY-TYPE SPACES

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**Key words:** local Morrey-type spaces, interpolation method for anisotropic spaces, interpolation theorem.

**AMS Mathematics Subject Classification:** 42B35, 46B70, 46E30.

**Abstract.** In this paper, we study the interpolation properties of local Morrey-type spaces related to the interpolation method for anisotropic spaces. We define approximation local Morrey spaces $\overline{LM_{pr}^{\lambda q}}$ and approximation spaces $\widetilde{LM_{pr}^{\lambda q}}$, and in terms of these spaces we obtain a description of interpolation spaces for pairs of local Morrey-type spaces $\left(LM_{p_0,q_0}^{\lambda_0}, LM_{p_1,q_1}^{\lambda_1}\right)$ in the case $p_0 \neq p_1$.

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## 1 Introduction

Morrey spaces [22] and their generalizations have been widely applied in various problems of function theory and partial differential equations (see, for example, survey papers [10, 11]).

Let $0 < p \leq \infty$ and $0 \leq \lambda \leq n/p$. The Morrey spaces $M_p^\lambda$ were defined in [22] as the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^\lambda} = \sup_{x\in\mathbb{R}^n} \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty,$$

where $B(x,r)$ is the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$. If $\lambda = 0$, then $M_p^0 = L_p(\mathbb{R}^n)$, while if $\lambda = n/p$, then $M_p^{n/p} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n/p$, then $M_p^\lambda = \Theta$, where $\Theta$ is the set of all functions that are equivalent to zero on $\mathbb{R}^n$.

Interpolation of these spaces was considered in [27, 18, 25]. According to the results of [25], we have

$$\left(M_p^{\lambda_0}, M_p^{\lambda_1}\right)_{\theta,\infty} \hookrightarrow M_p^\lambda,$$

where $1 \leq p \leq \infty$, $0 \leq \lambda_0 \neq \lambda_1 \leq n/p$, $0 < \theta < 1$, $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, and the symbol $\hookrightarrow$ denotes continuous embedding. In [26, 9] it was established that this inclusion is strict, which raised the problem of giving a complete description of the interpolation spaces.

In [15] a similar problem was considered for a local variant of the Morrey spaces and for their generalizations involving an additional parameter.

**Definition 1.** Let $0 < p, q \leq \infty$ and $0 < \lambda < \infty$ if $q < \infty$ and $0 \leq \lambda < \infty$ if $q = \infty$. The local Morrey-type spaces $LM_{p,q}^\lambda$ are defined as the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ such that for $q < \infty$

$$\|f\|_{LM_{p,q}^\lambda} = \left(\int_0^\infty \left(t^{-\lambda}\|f\|_{L_p(B(0,t))}\right)^q \frac{dt}{t}\right)^{1/q} < \infty,$$

and for $q = \infty$
$$\|f\|_{LM^{\lambda}_{p,q}} = \sup_{t>0} t^{-\lambda} \|f\|_{L_p(B(0,t))} < \infty.$$

Unlike the Morrey spaces $M^{\lambda}_{p}$, the scale of local Morrey-type spaces $LM^{\lambda}_{p,q}$ remains closed under interpolation when $p_0 = p_1$. Specifically, the following statement was proved in [15].

**Theorem 1.1** ([15]). *Let $0 < p, q_0, q_1, q \leq \infty$ and $0 < \theta < 1$. Suppose, in addition, that $\lambda_0 \neq \lambda_1$ and $0 < \lambda_0, \lambda_1 < n/p$ if $p < \infty$ and at least one of the parameters $q_0, q_1$ and $q$ is finite, and $0 \leq \lambda_0, \lambda_1 \leq n/p$ if $q_0 = q_1 = q = \infty$. Then*
$$\left(LM^{\lambda_0}_{p,q_0}, LM^{\lambda_1}_{p,q_1}\right)_{\theta,q} = LM^{\lambda}_{p,q},$$
*where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$.*

Later, in works [13, 17, 12, 16, 24, 14] generalizations of this result were obtained, and their applications to the study of various problems in analysis were explored.

In [5, 6, 7, 8] further modifications and generalizations of local and global Morrey spaces were defined, based on the use of norms of general symmetric spaces $X$ and $l$ (instead of Lebesgue space norms $L_p$ and $l_q$). In these works there were investigated problems related to the interpolation of such spaces. Below, we provide some of the definitions from these works.

A Banach space $X$ of measurable functions on $\mathbb{R}^n$ is called an ideal space [21] if, for every function $f \in X$, any measurable function $g$ satisfying $|g(t)| \leq |f(t)|$ for almost all $t \in \mathbb{R}^n$ also belongs to $X$ and satisfies $\|g\|_X \leq \|f\|_X$ (the symbol $\|f\|_X$ denotes the norm of an element $f$ in the space $X$).

For $f : \mathbb{R}^n \to \mathbb{R}$ we denote by $m(f,\gamma)$ the distribution function of $f$, namely,
$$m(f,\gamma) = \mu(\{t \in \mathbb{R}^n : |f(t)| > \gamma\}),$$
where $\mu$ is the Lebesgue measure on $\mathbb{R}^n$.

An ideal space $X$ is said to be symmetric if from the condition $f \in X$, the measurability of $g$ follows, and also the validity of the inequality $m(g,\gamma) \leq m(f,\gamma)$ for all $\gamma \in \mathbb{R}_+$ that $g \in X$ and $\|g\|_X \leq \|f\|_X$.

Along with ideal function spaces, we need to define ideal sequence spaces. Let $e_i = \{\ldots, 0, 1, 0, \ldots\}$ (1 stands in the $i$th place, $i \in \mathbb{Z}$) be the standard basis in the space of two-sided sequences. We denote by $l$ an ideal sequence space consisting of sequences $x = \sum_{i=-\infty}^{\infty} x_i e_i$ with the norm $\|\cdot\|_l$. By definition, $l$ is an ideal sequence space if, for every sequence $x = \{x_i\}_{i\in\mathbb{Z}} \in l$ and every sequence $y = \{y_i\}_{i\in\mathbb{Z}}$ satisfying $|y_i| \leq |x_i|$ for all $i \in \mathbb{Z}$, we have $y \in l$ and $\|y\|_l \leq \|x\|_l$.

Let $U(0,1) \subset \mathbb{R}^n$ be such that $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$. We also assume that $U(0,1)$ is star-shaped with respect to the point 0, that is, if $t \in U(0,1)$, then $\nu t \in U(0,1)$ for $\nu \in (0,1)$.

Let $U(0,r)$ be the homothetic set to the set $U(0,1)$ with a coefficient $r > 0$. We denote by $\Upsilon$ the set of all non-negative number sequences $\tau = \{\tau_i\}$ each of which satisfies the conditions
$$\forall i : \tau_i < \tau_{i+1}, \quad \bigcup_i (\tau_i, \tau_{i+1}] = \mathbb{R}_+.$$
If $\tau_{i+1} = \infty$, we assume that $(\tau_i, \infty] = (\tau_i, \infty)$. For every sequence $\tau = \{\tau_i\}$ we construct a family of sets $U(0,\tau_i)$ and a family of disjoint annuli $R(0,\tau_{i-1},\tau_i) = U(0,\tau_i) \setminus U(0,\tau_{i-1})$.

**Definition 2.** Let an ideal space $X$ on $\mathbb{R}^n$, an ideal space $l$ of two-sided sequences with the standard basis $\{e_i\}$ and a sequence $\tau \in \Upsilon$ be given. The discrete local Morrey spaces $M^{\tau}_{l,X}$ are defined as the spaces of all functions $f \in L^{loc}_1(\mathbb{R}^n)$ such that
$$\|f\|_{M^{\tau}_{l,X}} = \left\| \sum_{i=-\infty}^{\infty} e_i \, \|f\chi(U(0,\tau_i))\|_X \right\|_l < \infty.$$

The approximation local Morrey space $\overline{M^{\tau}_{l,X}}$ is defined as the space of all functions $f \in L^{loc}_1(\mathbb{R}^n)$ such that

$$\|f\|_{\overline{M^{\tau}_{l,X}}} = \left\| \sum_{i=-\infty}^{\infty} e_i \left\| f\chi(R(0,\tau_{i-1},\tau_i)) \right\|_X \right\|_l < \infty.$$

Here $\chi(D)$ is the characteristic function of a set $D$.

In [5], conditions under which the equality $\overline{M^{\tau}_{l,X}} = M^{\tau}_{l,X}$ holds were studied. It was also shown that the approximation local Morrey space $\overline{M^{\tau}_{l,X}}$ is a retract of the space of vector-valued sequences $l(X)$.

Note that in the framework of the classical interpolation method, it is only possible to describe the interpolation result for pairs $\left(LM^{\lambda_0}_{p,q_0}, LM^{\lambda_1}_{p,q_1}\right)$ with the same parameter $p$. We are interested in the problem of describing interpolation spaces for pairs $\left(LM^{\lambda_0}_{p_0,q_0}, LM^{\lambda_1}_{p_1,q_1}\right)$ when $p_0 \neq p_1$.

## 2 Anisotropic interpolation method and spaces of vector-valued sequences

Let us consider the interpolation method for anisotropic spaces proposed by E.D. Nursultanov [23] (see also [1, 2, 19]).

Let $A_1$ be a Banach space, and $A_2$ be a functional Banach lattice (see [20]). We denote by $\mathbf{A} = (A_1, A_2)$ the space of $A_1$-valued measurable functions such that $\|f\|_{A_1} \in A_2$, with the norm $\|f\|_{\mathbf{A}} = \left\| \|f\|_{A_1} \right\|_{A_2}$. This space is called a mixed-norm space.

Let $\mathbf{A}_0 = (A_1^0, A_2^0)$, $\mathbf{A}_1 = (A_1^1, A_2^1)$ be two mixed-norm spaces, and let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in E = \{0,1\}^2$. We define the space $\mathbf{A}_\varepsilon = (A_1^{\varepsilon_1}, A_2^{\varepsilon_2})$ with the norm

$$\|a\|_{\mathbf{A}_\varepsilon} = \left\| \|a\|_{A_1^{\varepsilon_1}} \right\|_{A_2^{\varepsilon_2}}.$$

A pair of mixed-norm spaces $\mathbf{A_0} = (A_1^0, A_2^0)$, $\mathbf{A_1} = (A_1^1, A_2^1)$ is called *compatible* if there exists a linear topological Hausdorff space $\mathcal{A}$, containing the spaces $\mathbf{A}_\varepsilon$ as subspaces for all $\varepsilon \in E$.

We define a functional for $a \in \sum_{\varepsilon \in E} \mathbf{A}_\varepsilon$ as follows:

$$K(\mathbf{t}, a; \mathbf{A}) = \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} \mathbf{t}^\varepsilon \|a_\varepsilon\|_{\mathbf{A}_\varepsilon}.$$

Let $\mathbf{0} < \theta = (\theta_1, \theta_2) < \mathbf{1}$, $\mathbf{0} < \mathbf{r} = (r_1, r_2) \leq \infty$. For any rearrangement $\star = (j_1, j_2)$ of the set $\{1, 2\}$ let $\mathbf{r}^\star = (r_{j_1}, r_{j_2})$. We denote by $\mathbf{A}_{\theta\mathbf{r}^\star} = (\mathbf{A}_0, \mathbf{A}_1)_{\theta\mathbf{r}^\star}$ the linear subset of $\sum_{\varepsilon \in E} \mathbf{A}_\varepsilon$ for all elements of which the following quasi-norm (norm if $\mathbf{r} \geq \mathbf{1}$) is finite:

$$\|a\|_{\mathbf{A}_{\theta\mathbf{r}^\star}} = \left( \int_0^\infty \left( \int_0^\infty \left( t_{j_1}^{-\theta_{j_1}} t_{j_2}^{-\theta_{j_2}} K(\mathbf{t}, a; \mathbf{A}_0, \mathbf{A}_1) \right)^{r_{j_1}} \frac{dt_{j_1}}{t_{j_1}} \right)^{r_{j_2}/r_{j_1}} \frac{dt_{j_2}}{t_{j_2}} \right)^{1/r_{j_2}} < \infty.$$

**Lemma 2.1** ([23]). *Let $\mathbf{0} < \theta < \mathbf{1}$, $\mathbf{0} < \mathbf{r} \leq \infty$, $\star = (j_1, j_2)$ be some rearrangement of the set $\{1, 2\}$, $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon \in E}$ and $\mathbf{B} = \{B_\varepsilon\}_{\varepsilon \in E}$ be two compatible families of Banach spaces. If a linear operator $T : A_\varepsilon \to B_\varepsilon$ with the quasi-norm $M_\varepsilon$ for any $\varepsilon \in E$, then*

$$T : \mathbf{A}_{\theta\mathbf{r}^\star} \to \mathbf{B}_{\theta\mathbf{r}^\star},$$

*with the quasi-norm*

$$\|T\|_{\mathbf{A}_{\theta\mathbf{r}^\star} \to \mathbf{B}_{\theta\mathbf{r}^\star}} \lesssim \max_{\varepsilon \in E} M_\varepsilon.$$

*Here, $A \lesssim B$ means that there exists a constant $c > 0$ such that $A \leq cB$ for all $A$, $B$ under consideration.*

Below we give the definition of the Lorentz space $L_{pr}(\mathbb{R}^n)$, and spaces of vector-valued sequences $l_q^\sigma(L_{pr})(\mathbb{R}^n)$ and $\widetilde{L_{pr}(l_q^\sigma)}(\mathbb{R}^n)$.

Let $0 < p, r \leq \infty$ and $0 < p < \infty$ if $r < \infty$ and $0 < p \leq \infty$ if $r = \infty$. The Lorentz spaces $L_{pr}(\mathbb{R}^n)$ are defined as the spaces of all functions $f$ measurable on $\mathbb{R}^n$ such that for $r < \infty$

$$\|f\|_{L_{pr}(\mathbb{R}^n)} = \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^r \frac{dt}{t}\right)^{1/r} < \infty,$$

and for $r = \infty$

$$\|f\|_{L_{pr}(\mathbb{R}^n)} = \sup_{t>0} t^{1/p} f^*(t) < \infty,$$

where $f^*$ is the nonincreasing rearrangement of $f$.

Let $-\infty < \sigma < \infty$, $1 \leq q, p, r \leq \infty$, we define spaces $l_q^\sigma(L_{pr})(\mathbb{R}^n)$ and $\widetilde{L_{pr}(l_q^\sigma)}(\mathbb{R}^n)$, as the set of sequences $a = \{a_k(x)\}_{k=-\infty}^{\infty}$, where $a_k(x)$ are functions, measurable on $\mathbb{R}^n$, for which the following norms are respectively finite:

$$\|a\|_{l_q^\sigma(L_{pr})(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} \left(2^{\sigma k} \|a_k\|_{L_{pr}(\mathbb{R}^n)}\right)^q\right)^{1/q},$$

$$\|a\|_{\widetilde{L_{pr}(l_q^\sigma)}(\mathbb{R}^n)} = \left\|\left(\sum_{k=-\infty}^{\infty} \left(2^{\sigma k} a_k^*\right)^q\right)^{1/q}\right\|_{L_{pr}(\mathbb{R}^n)},$$

with the standard modification for $q = \infty$.

## 3 Main result

We use the scheme for constructing local Morrey spaces from Definition 2.

Let $U(0,1) \subset \mathbb{R}^n$ satisfy the conditions imposed earlier in Introduction. Set $\tau = \{\tau_i\}$, where $\tau_i = 2^i$, $i \in \mathbb{Z}$. Let $1 \leq p, r \leq \infty$ and $1 \leq p < \infty$ if $r < \infty$ and $1 \leq p \leq \infty$ if $r = \infty$, $1 \leq q \leq \infty$ and $0 < \lambda < \infty$ if $q < \infty$ and $0 \leq \lambda < \infty$ if $q = \infty$. As the space $X$, we take the Lorentz space $L_{pr}(\mathbb{R}^n)$, and as the space $l$, we take the space $l_q^{-\lambda}$. The thus-defined discrete local Morrey spaces $M_{l,X}^\tau$ and approximation local Morrey spaces $\overline{M_{l,X}^\tau}$ will be denoted by the symbols $LM_{pr}^{\lambda q}$ and $\overline{LM_{pr}^{\lambda q}}$, respectively.

**Remark 1.** From the results of [5], it follows that in this case, the equality $LM_{pr}^{\lambda q} = \overline{LM_{pr}^{\lambda q}}$ holds. It can also be shown that when $p = r$ the space $LM_{pr}^{\lambda q}$ coincides with the local Morrey-type spaces $LM_{p,q}^{\lambda}$ (with norms equivalence).

The norm in the approximation local Morrey space $\overline{LM_{pr}^{\lambda q}}$ can be written as follows

$$\|f\|_{LM_{pr}^{\lambda q}} = \left\|\left\{f\chi(R(0, 2^{i-1}, 2^i))\right\}\right\|_{l_q^{-\lambda}(L_{pr})(\mathbb{R}^n)}.$$

We also define the approximation space $\widetilde{LM_{pr}^{\lambda q}}$ as the space of all functions $f \in L_1^{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{LM_{pr}^{\lambda q}}} = \left\|\left\{f\chi(R(0, 2^{i-1}, 2^i))\right\}\right\|_{\widetilde{L_{pr}(l_q^{-\lambda})}(\mathbb{R}^n)} < \infty.$$

We have obtained the following theorem.

**Theorem 3.1.** *Let $1 \le p_0 \neq p_1 \le \infty$, $1 \le r \le \infty$ and $1 \le q_0, q_1, q \le \infty$. Suppose, in addition, that $\lambda_0 \neq \lambda_1$ and $0 < \lambda_i < n/p_i$ if $p_i < \infty$ $(i = 0, 1)$ and at least one of the parameters $q_0, q_1$ and $q$ is finite, and $0 \le \lambda_i \le n/p_i$ $(i = 0, 1)$ if $q_0 = q_1 = q = \infty$. Then, for $0 < \theta_1, \theta_2 < 1$, the following equalities hold:*

a) *if $\star_1 = (1, 2)$, then*

$$\left(LM^{\lambda_0}_{p_0,q_0}, LM^{\lambda_1}_{p_1,q_1}\right)_{(\theta_1,\theta_2),(r,q)^{\star_1}} = \overline{LM^{\lambda q}_{pr}},$$

b) *if $\star_2 = (2, 1)$, then*

$$\left(LM^{\lambda_0}_{p_0,q_0}, LM^{\lambda_1}_{p_1,q_1}\right)_{(\theta_1,\theta_2),(r,q)^{\star_2}} = \widetilde{LM^{\lambda q}_{pr}},$$

*where $\lambda = (1 - \theta_2)\lambda_0 + \theta_2\lambda_1$, $1/p = (1 - \theta_1)/p_0 + \theta_1/p_1$.*

**Remark 2.** This approach was previously used by us to study the interpolation properties of Nikol'skii-Besov and Lizorkin-Triebel type spaces by applying the interpolation method for anisotropic spaces (see [3, 4]).

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