EURASIAN MATHEMATICAL JOURNAL
volume 17 , number 1, 2026

CONTENTS

EURASIAN NATIONAL UNIVERSITY

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

EURASIAN MATHEMATICAL JOURNAL

ISSN 2077-9879
9 772077 987003

VOLUME 17, NUMBER 1 2026

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2026, Volume 17, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Republic of Kazakhstan

The Moscow Editorial Office
The Patrice Lumumba Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russian Federation

BOKAYEV NURZHAN ADILKHANOVICH

(to the 70th birthday)

January 5, 2026, marks the 70th birthday of Nurzhan Adilkhanovich Bokayev, Doctor of Physical and Mathematical Sciences (1996), Professor (2002), member of the Editorial Board of the Eurasian Mathematical Journal (2010).

Nurzhan Adilkhanovich Bokayev was born on 5 January, 1956 in the village of Urnek, Karabalyk District, Kostanay Region. He graduated in 1972, with a gold medal from the Burlin Secondary School in the district. That same year, he entered the Mathematics Department of Karaganda State University and graduated with honors in 1977. From 1978 to 1979, he served in the Soviet Army. In 1980, he completed an internship, and from 1981 to 1984, he studied in the graduate program at Lomonosov Moscow State University in the Department of Function Theory and Functional Analysis. In 1985, he defended his candidate's dissertation there under the supervision of Corresponding Member of the Academy of Sciences of the USSR D.E. Menshov and Professor V.A. Skvortsov. In 1996, he defended his doctoral dissertation, "Fourier Coefficients and Uniqueness Theorems for Series in Generalized Walsh and Haar Systems", at the Institute of Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan, speciality Mathematical Analysis (01.01.01).

After completing his postgraduate studies, he worked as a lecturer, senior lecturer, associate professor, and professor in the Department of Mathematical Analysis at E.A. Buketov Karaganda State University (1985-1999). He headed the Department of Mathematics and Mathematical Modeling (1996-1999), and was a dean of the Faculty of Mathematics at E.A. Buketov Karaganda State University (1999-2005). Since 2005, he has been a professor in the Faculty of Mechanics and Mathematics at the L.N. Gumilyov Eurasian National University. From 2009 to 2018, he was the Head of the Department of Higher Mathematics at the L.N. Gumilyov Eurasian National University, and from 2018 to the present, he has been a professor in the Department of Fundamental Mathematics.

Professor Bokaev's research focuses on problems in function theory and functional analysis, the theory of orthogonal series for generalized Walsh and Haar systems, and operator theory in various function spaces. He has proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, and constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems. He obtained conditions for functions to belong to various functional classes in terms of the Fourier coefficients of generalized Haar and Walsh systems, and embedding criteria for Nikol'skii-Besov spaces constructed on the basis of multiplicative systems. He also obtained conditions for the boundedness and compactness of the commutator of the Riesz potential in general Morrey-type spaces, and conditions for boundedness of generalizedRiesz and Bessel potentials and generalized fractional-maximal operators in rearrangement-invariant spaces.

His co-authors include Professor V.A. Skvortsov (Moscow State University, Moscow), Professors V.I. Burenkov and M.L. Goldman (Peoples' Friendship University of Russia (RUDN University). Moscow), Dr. A. Gogatishvili (Institute of Mathematics of the Czech Academy of Sciences, Prague). His doctoral students' foreign advisors include Professors W. Sickel (Friedrich-Schiller-Universitity, Jena, Germany), Massimo Lanza de Cristoforis (University of Padova, Padova, Italy), V. Ruzhansky (Ghent University, Ghent, Belgium), U. Goginava (United Arab Emirates University, Al Ain, United Arab Emirates), and E. Panakhov (Institute of Applied Mathematics at Baku State University, Baku, Azerbaijan).

Under his supervision, 15 dissertations (4 candidate's and 11 PhD) were defended. He has published over 220 scientific papers, 2 monographs and 2 textbooks.

He is a three-time recipient of the state grant "Best University Teacher" of the Republic of Kazakhstan (2006, 2010, 2024) and served as Vice President of the Mathematical Society of Turkic-Speaking Countries (2014-2023). He was awarded the "For Contribution to the Development of Science" badge (2022).

Over the last ten years, he has been and continues to be a head of more than 5 national and international funded projects.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Nurzhan Adilkhanovich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 17, Number 1 (2026), 10 – 22

HÖLDER INEQUALITY ON THE SPACE OF UPPER SEMICONTINUOUS FUNCTIONS

Sh.A. Ayupov, M.R. Eshimbetov, A.A. Zaitov

Communicated by T. Bekjan

Key words: idempotent measure; max-plus linear functional; Borel sets; upper semicontinuous functions.

AMS Mathematics Subject Classification: Primary 47B65; Secondary 28C05.

Abstract. For a compact Hausdorff space X, we consider the space $I_{\mathfrak{B}}(X)$ of all idempotent probability measures on X, which are defined as set-functions on the σ-algebra of all Borel subsets of X, and also the space $I_{USC}(X)$ of all normalized max-plus linear functionals on the linear space of all upper semicontinuous functions on X, equipped with idempotent operations. In the main result it is established that a max-plus version of the Hölder inequality holds on the space of upper semicontinuous functions.

DOI: https://doi.org/10.32523/2077-9879-2026-17-1-10-22

1 Introduction

Idempotent mathematics is a branch of mathematical sciences, rapidly developing and gaining popularity over the last four decades. An important stage of development of the subject was presented in the book "Idempotency" [7] edited by J. Gunawardena. This book arose out of the well-known international workshop that was held in Bristol, England, in October 1994. Idempotent mathematics is based on replacing the usual arithmetic operations with a new set of basic operations, i.e., on replacing numerical fields by idempotent semirings and semifields. Typical example is the so-called max-plus algebra $\mathbb{R}_{\max}$ [7], [11].

In [15] M. Zarichnyi considered categorical properties of the space of idempotent probability measures on compact Hausdorff spaces. Later the space of idempotent probability measures was investigated on the class of compact metric spaces [16]. The study of spaces of idempotent probability measures leads to similar problems for topological spaces which are wider than the class of compact Hausdorff (or metrizable) spaces, in particular, for the case of Tychonoff spaces. In this case it is natural to apply the methods proposed in works [1], [2], [6] or [13].

The results obtained in [14], [19], [17] show that in order to establish "good" properties of the space of idempotent probability measures, methods are required which are different from classical methods (i.e., from methods suitable for probability measures which have been applied in [6], [13] and others). Note that idempotent probability measures in general are not linear and for a given compact Hausdorff space X they are defined as max-plus functionals on $C(X)$ [14], while the usual probability measures on X are a positive, linear, normalized functional on $C(X)$ [3], [4].

Unlike the above mentioned papers in the present work for a compact Hausdorff space X we introduce the notion of idempotent measure as a set-function on the σ-algebra $\mathfrak{B}(X)$ of all Borel subsets of X. For a compact Hausdorff space X, we denote by $I_{\mathfrak{B}}(X)$ the space of all idempotent probability measures on X, which define as set-functions $\mu\colon \mathfrak{B}(X) \to \overline{\mathbb{R}}_+$, and by $I_{USC}(X)$ the space of all normalized max-plus linear functionals $\nu\colon USC(X) \to \mathbb{R}_{\max}$ on the space $USC(X)$ of all upper semicontinuous functions on X. Then, we present a max-plus version of the well-known Riesz Representation Theorem. Further, we obtain the main result, which states that the spaces $I_{\mathfrak{B}}(X)$ and $I_{USC}(X)$ are homeomorphic. Finally, the Hölder inequality on the space of all upper semicontinuous functions is proved.

2 Preliminaries

Let X be a compact Hausdorff space and let $\mathfrak{B}(X)$ denote the family of all Borel subsets of X. We denote $\overline{\mathbb{R}}_+ = [0, +\infty) \cup \{+\infty\} = [0, +\infty]$. The symbol $\mathfrak{A}$ denotes a directed set. Following [10], we introduce the following notion.

Definition 1. A set function $\mu\colon \mathfrak{B}(X) \to \overline{\mathbb{R}}_+$ is said to be an idempotent measure on X if it satisfies the following conditions:

1) $\mu(\varnothing) = 0$;

2) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for any $A, B \in \mathfrak{B}(X)$;

3) $\mu\left(\bigcup\limits_{\alpha\in\mathfrak{A}} A_\alpha\right) = \sup\limits_{\alpha\in\mathfrak{A}}\{\mu(A_\alpha)\}$ for every increasing net $\{A_\alpha\colon \alpha \in \mathfrak{A}\} \subset \mathfrak{B}(X)$ such that $\bigcup\limits_{\alpha\in\mathfrak{A}} A_\alpha \in \mathfrak{B}(X)$.

Remark 1. Every idempotent measure μ is increasing, i.e., for $A, B \in \mathfrak{B}(X)$ if $A \subset B$ then $\mu(A) \le \mu(B)$.

The set of all idempotent measure on X we denote by $IM(X)$. Let $\mathcal{B}$ be a base of the topology on X. For an idempotent measure $\mu \in IM(X)$ a system of sets

$$\langle \mu;\, U_1,\, \ldots,\, U_n;\, \varepsilon\rangle = \{\nu \in IM(X)\colon\ |\nu(U_i) - \mu(U_i)| < \varepsilon,\ i = 1,\, \ldots,\, n\} \tag{2.1}$$

forms [8] a base of a topology on $IM(X)$ at μ. Here $U_i \in \mathcal{B}$, $i = 1, \ldots, n$, and $\varepsilon > 0$.

If $\mu(X) = 1$, the idempotent measure μ is called an idempotent probability measure on X. We denote

$$I_{\mathfrak{B}}(X) = \{\mu \in IM(X)\colon\ \mu(X) = 1\}.$$

Let (X, μ) be an idempotent measure space such that $\mu(X) < \infty$. We adopt the convention that $\infty \cdot 0 = 0$. For $A \subset X$ its characteristic function χ_A is defined as $\chi_A(x) = 1$ at $x \in A$, and $\chi_A(x) = 0$ at $x \in X \setminus A$.

Definition 2. [10] For a function $f\colon\ X \to \overline{\mathbb{R}}_+$ we define the idempotent integral of f with respect to μ by

$$\int\limits_X^{\oplus} f d\mu = \sup_{t\in\overline{\mathbb{R}}_+}\{t \cdot \mu\{x \in X\colon f(x) \ge t\}\}.$$

For $A \subset X$, we let $\int\limits_A^{\oplus} f d\mu = \int\limits_X^{\oplus} f\chi_A d\mu$.

Lemma 2.1. [18] *For every couple A and B of Borel subsets of a compact Hausdorff space X the following equality holds*

$$\int\limits_{A\cup B}^{\oplus} f\, d\mu = \int\limits_A^{\oplus} f\, d\mu \oplus \int\limits_B^{\oplus} f\, d\mu.$$

The following two statements will be applied to establish Lemma 2.3

Lemma 2.2. [9] *For a function $f\colon X \to \overline{\mathbb{R}}_+$ we have*

$$\int\limits_X^{\oplus} f d\mu = \sup_{x\in X}\{f(x)\cdot \mu(f^{-1}(f(x)))\}.$$

Theorem 2.1. [10] *For every $c \in \overline{\mathbb{R}}_+$, any $\overline{\mathbb{R}}_+$-valued functions f, g and a net $\{f_j\}_{j\in J}$ of $\overline{\mathbb{R}}_+$-valued functions on X the following properties hold:*

1) $\int\limits_X^{\oplus} 0\, d\mu = 0;$

2) $\int\limits_X^{\oplus} 1\, d\mu = \mu(X);$

3) $\int\limits_X^{\oplus} f\, d\mu \le \int\limits_X^{\oplus} g\, d\mu$ if $f \le g;$

4) $\int\limits_X^{\oplus} (c\cdot f)\, d\mu = c\cdot \int\limits_X^{\oplus} f\, d\mu;$

5) $\int\limits_X^{\oplus} (f\oplus g)\, d\mu = \int\limits_X^{\oplus} f\, d\mu \oplus \int\limits_X^{\oplus} g\, d\mu;$

6) $\int\limits_X^{\oplus} (f+g)\, d\mu \le \int\limits_X^{\oplus} f\, d\mu + \int\limits_X^{\oplus} g\, d\mu;$

7) $\left|\int\limits_X^{\oplus} f\, d\mu - \int\limits_X^{\oplus} g\, d\mu\right| \le \int\limits_X^{\oplus} |f-g|\, d\mu$ provided the left-hand side is well defined;

8) $\int\limits_X^{\oplus} \sup_{j\in J} f_j\, d\mu = \sup_{j\in J} \int\limits_X^{\oplus} f_j\, d\mu.$

Now, for $\varphi\colon X \to \overline{\mathbb{R}}_+$ and $p > 0$ we define

$$||\varphi||_p = \left(\int\limits_X^{\oplus} \varphi^p(x) d\mu\right)^{\frac{1}{p}} \quad \text{and} \quad ||\varphi||_\infty = \sup_{\{x\in X\colon\, \mu(\{x\})>0\}} \{\varphi(x)\}.$$

Lemma 2.3. *Let φ, $\psi\colon X \to \overline{\mathbb{R}}_+$.*

1) *Let $p \in [1, +\infty]$ and $q \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\int\limits_X^{\oplus} \varphi(x)\psi(x) d\mu \le ||\varphi||_p \cdot ||\psi||_q;$$

2) *If $\mu(X) = 1$, then $||\varphi||_p \le ||\varphi||_q$, where $0 < p < q$.*

Proof. 1) From [3], [12], we have the following inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{ where } a,\, b \geq 0,\ p \geq 1,\, q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

Now, let $I_1 = \int\limits_X^{\oplus} \varphi^p(x)d\mu \neq 0$ and $I_2 = \int\limits_X^{\oplus} \psi^q(x)d\mu \neq 0$. Then,

$$\frac{\varphi}{I_1^{\frac{1}{p}}} \cdot \frac{\psi}{I_2^{\frac{1}{q}}} \leq \frac{\varphi^p}{pI_1} + \frac{\psi^q}{qI_2} \Rightarrow$$

$$\int\limits_X^{\oplus} \frac{\varphi(x)\psi(x)}{I_1^{\frac{1}{p}} I_2^{\frac{1}{q}}} d\mu \leq \int\limits_X^{\oplus} \left(\frac{\varphi^p(x)}{pI_1} + \frac{\psi^q(x)}{qI_2} \right) d\mu \leq$$

$$\leq \int\limits_X^{\oplus} \frac{\varphi^p(x)}{pI_1} d\mu + \int\limits_X^{\oplus} \frac{\psi^q(x)}{qI_2} d\mu = \frac{1}{pI_1} \int\limits_X^{\oplus} \varphi^p(x)d\mu + \frac{1}{qI_2} \int\limits_X^{\oplus} \psi^q(x)d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

Consequently, we obtain

$$\int\limits_X^{\oplus} \varphi(x)\psi(x)d\mu \leq I_1^{\frac{1}{p}} \cdot I_2^{\frac{1}{q}} = ||\varphi||_p \cdot ||\psi||_q.$$

If $I_1 = 0$ or $I_2 = 0$, then $0 \leq \int\limits_X^{\oplus} \varphi(x)\psi(x)d\mu \leq I_1^{\frac{1}{p}} \cdot I_2^{\frac{1}{q}} = 0$.

Hence, $\int\limits_X^{\oplus} \varphi(x)\psi(x)d\mu = 0$.

2) If $0 < p < q$, then $r = \frac{q}{p} > 1$. According to the first part of Lemma 2.3 and $\mu(X) = 1$, we have

$$\int\limits_X^{\oplus} (\varphi^p(x) \cdot 1)d\mu \leq \left(\int\limits_X^{\oplus} \varphi^{p \cdot r}(x)d\mu \right)^{\frac{1}{r}} \cdot \left(\int\limits_X^{\oplus} 1^q d\mu \right)^{\frac{1}{q}} =$$

$$= \left(\int\limits_X^{\oplus} \varphi^q(x)d\mu \right)^{\frac{p}{q}} \cdot (\mu(X))^{\frac{1}{q}} = \left(\int\limits_X^{\oplus} \varphi^q(x)d\mu \right)^{\frac{p}{q}} \Rightarrow$$

$$\Rightarrow \left(\int\limits_X^{\oplus} \varphi^p(x)d\mu \right)^{\frac{1}{p}} \leq \left(\int\limits_X^{\oplus} \varphi^q(x)d\mu \right)^{\frac{1}{q}}.$$

Thus, $||\varphi||_p \leq ||\varphi||_q$.

□

To obtain the main results we need the following notions and facts.

Definition 3. [10] We say that a function $f\colon X \to \overline{\mathbb{R}}_+$ is maximable (or μ-maximable), if $\int\limits_X^{\oplus} f(x)d\mu < \infty$ and, moreover, $\int\limits_X^{\oplus} f(x)\chi_{\{x \in X\colon f(x) > t\}} d\mu \to 0$ as $t \to \infty$.

Theorem 2.2. [10] *A function $f\colon X \to \overline{\mathbb{R}}_+$ is maximable if and only if there exists a monotonically increasing function $F\colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that $\frac{F(x)}{x} \to \infty$ as $x \to \infty$ and $\int\limits_X^{\oplus} (F(x) \circ f(x))d\mu < \infty$.*

Definition 4. [10] A net $\{f_\alpha\colon \alpha \in \mathfrak{A}\}$ of $\overline{\mathbb{R}}_+$-valued functions on X is said to be uniformly maximable (or μ-uniformly maximable) if

$$\limsup_{\alpha\in\mathfrak{A}} \int\limits_X^{\oplus} f_\alpha(x)\chi_{\{x\in X\colon f_\alpha(x)>t\}} d\mu \to 0 \quad \text{as} \quad t\to\infty.$$

Theorem 2.3. [10] *A net $\{f_\alpha\colon \alpha\in\mathfrak{A}\}$ is uniformly maximable if and only if the following conditions hold:*

1) $\limsup\limits_{\alpha\in\mathfrak{A}} \int\limits_X^{\oplus} f_\alpha(x)d\mu < \infty$;

2) *for every $\varepsilon > 0$ there exists $\eta > 0$ such that $\limsup\limits_{\alpha\in\mathfrak{A}} \int\limits_{A_\alpha}^{\oplus} f_\alpha(x)d\mu < \varepsilon$ for every net of sets $\{A_\alpha\colon \alpha\in\mathfrak{A}\}$ such that $\limsup\limits_{\alpha\in\mathfrak{A}}\{\mu(A_\alpha)\} < \eta$.*

Theorem 2.4. [10] *Let $\{f_\alpha\colon \alpha\in\mathfrak{A}\}$ be a net of $\overline{\mathbb{R}}_+$-valued functions of X and f be an $\overline{\mathbb{R}}_+$-valued function on X.*

1) *If $\liminf\limits_{\alpha\in\mathfrak{A}}\{f_\alpha\} \overset{\mu-a.e.}{\geq} f$, then $\liminf\limits_{\alpha\in\mathfrak{A}} \int\limits_X^{\oplus} f_\alpha(x)d\mu \geq \int\limits_X^{\oplus} f(x)d\mu$;*

2) *If $f_\alpha \overset{\mu}{\to} f$ and net $\{f_\alpha\}$ is uniformly maximable, then $\lim\limits_{\alpha\in\mathfrak{A}} \int\limits_X^{\oplus} f_\alpha(x)d\mu = \int\limits_X^{\oplus} f(x)d\mu$;*

3) *If $f_\alpha \overset{\mu-a.e.}{\uparrow} f$, then $\lim\limits_{\alpha\in\mathfrak{A}} \int\limits_X^{\oplus} f_\alpha(x)d\mu = \int\limits_X^{\oplus} f(x)d\mu$.*

Definition 5. [5] Let X be a Tychonoff space. A function $\varphi\colon X \to \mathbb{R}_{\max}$ is said to be upper semicontinuous if for every $a\in\mathbb{R}$ the set $\{x\in X\colon \varphi(x) < a\}$ is open.

3 Max-plus version of the Riesz representation theorem

We denote by $USC(X)$ the linear space of all upper semicontinuous functions defined on X which is endowed with the sup-norm $\|\varphi\| = \sup\{|\varphi(x)|\colon x\in X\}$ and denote by $USC_b(X)$ the space of all real-valued upper semicontinuous functions on X bounded above. Note that $(-\infty)_X \in USC(X)$ and $\|(-\infty)_X\| = +\infty$.

Now we present the version of the convergence theorem in the Lebesgue sense of idempotent integrals.

Theorem 3.1. *Let for an arbitrary sequence of functions $\varphi_n(x) \in USC_b(X)$, the sequence $\{e^{\varphi_n(x)}\colon n\in\mathbb{N}\}$ be uniformly maximable. If $\varphi_n \overset{\mu}{\to} \varphi$, then*

$$\lim_{n\to\infty} \int\limits_X^{\oplus} e^{\varphi_n(x)} d\mu = \int\limits_X^{\oplus} e^{\lim\limits_{n\to\infty}\varphi_n(x)} d\mu = \int\limits_X^{\oplus} e^{\varphi(x)} d\mu. \tag{3.1}$$

Proof. First, let us prove the first part of the theorem. We introduce the following notation $A_{n,t} = \{x\in X\colon e^{\varphi_n(x)} > t\}$. Clearly, $A_{n,t}\to\varnothing$ as $t\to\infty$. (Note the symbol $A_{n,t}\to\varnothing$ means the following: $A_{n,t}\supset A_{n,t'}$ for $t < t'$ and $\bigcap\limits_t A_{n,t} = \varnothing$ for each n).

Then, according to the Definitions 2 and 4

$$\sup_{n\in\mathbb{N}} \int\limits_X^{\oplus} e^{\varphi_n(x)} \chi_{A_{n,t}} d\mu = \sup_{n\in\mathbb{N}} \int\limits_{A_{n,t}}^{\oplus} e^{\varphi_n(x)} d\mu =$$
$$= \sup_{n\in\mathbb{N}}\{ \sup_{t\in[0,\infty]} \{t\cdot \mu\{x\in A_{n,t}\colon\ e^{\varphi_n(x)} \geq t\}\} =$$
$$= \sup_{n\in\mathbb{N}}\{ \sup_{t\in[0,\infty]} \{t\cdot \mu(A_{n,t})\} \to \sup_{n\in\mathbb{N}}\{\infty\cdot\mu(\varnothing)\} = 0 \quad \text{as} \quad t\to\infty.$$

So, the set $\{e^{\varphi_n(x)}\colon\ n\in\mathbb{N}\}$ is uniformly maximable.

Now we will prove equality (3.1). By the assumption of the theorem $\lim\limits_{n\to\infty}\varphi_n(x) = \varphi(x)$. Then, according to the property of continuous and measurable functions

$$\lim_{n\to\infty} e^{\varphi_n(x)} = e^{\lim\limits_{n\to\infty}\varphi_n(x)} = e^{\varphi(x)}.$$

Hence and by part 2) of Theorem 2.4, we have

$$\lim_{n\to\infty} \int\limits_X^{\oplus} e^{\varphi_n(x)} d\mu = \int\limits_X^{\oplus} e^{\lim\limits_{n\to\infty}\varphi_n(x)} d\mu = \int\limits_X^{\oplus} e^{\varphi(x)} d\mu.$$

□

Now, we introduce the notion of a max-plus-linear functional on $USC(X)$.

For each $c\in\mathbb{R}_{\max}$ we denote by c_X the constant function in $USC(X)$ defined by the formula $c_X(x) = c$ for each $x\in X$. Define on the set $USC(X)$ operations $\oplus$ and $\odot$ by $\varphi\oplus\psi = \max\{\varphi,\ \psi\}$ and $\varphi\odot\psi = \varphi+\psi$, where $\varphi,\ \psi\in USC(X)$.

Definition 6. We say that a functional $\nu\colon\ USC(X)\to\mathbb{R}_{\max}$ is max-plus-linear, if it has the following properties:

1) $\nu(\varphi\oplus\psi) = \nu(\varphi)\oplus\nu(\psi)$ for any $\varphi,\ \psi\in USC(X)$;

2) $\nu(c\odot\varphi) = c\odot\nu(\varphi)$ for every $c\in\mathbb{R}_{\max}$ and $\varphi\in USC(X)$.

The set of all max-plus linear functionals on $USC(X)$ we denote by $USC(X)^{\oplus}$. For a max-plus linear functional $\nu\in USC(X)^{\oplus}$ a system of sets

$$\langle\nu;\ \varphi_1,\ \ldots,\ \varphi_n;\ \varepsilon\rangle = \{\nu'\in USC(X)^{\oplus}\colon\ |\nu'(\varphi_i)-\nu(\varphi_i)|<\varepsilon,\ i=1,\ \ldots,\ n\}$$

forms a base of $USC(X)^{\oplus}$ at ν. Here, $\varphi_i\in USC(X)$, $i=1,\ \ldots,\ n$, and $\varepsilon>0$. The proof of the last statement can be obtained in the same way as in [15], if we accept the convention $(-\infty)-(-\infty)=0$.

In order to give the following definition we note that $\nu(0_X)=0$ if and only if $\nu(c_X)=c$ for every $c\in\mathbb{R}_{\max}$. Indeed, for every $c\in\mathbb{R}_{\max}$ we have $\nu(c_X)=\nu(c\odot 0_X)=c\odot\nu(0_X)$. Hence, we get $\nu(c_X)-c=\nu(0_X)$.

Definition 7. A max-plus linear functional $\nu\colon\ USC(X)\to\mathbb{R}_{\max}$ is said to be normalized, if

3) $\nu(c_X)=c$ for each $c\in\mathbb{R}_{\max}$.

Put

$$I_{USC}(X) = \{\nu\in USC(X)^{\oplus}\colon\ \nu(0_X)=0\}.$$

Theorem 3.2. *For each normalized max-plus linear functional* $\nu\colon USC(X)\to\mathbb{R}_{\max}$ *a set-function* $\mu_\nu\colon \mathfrak{B}(X)\to\mathbb{R}$ *defined by the rule*

$$\mu_\nu(A)=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\ \varphi\geq\chi_A\},\qquad A\in\mathfrak{B}(X),$$

is an idempotent probability measure on X.

Proof. Clearly, $\mu_\nu(\varnothing)=0$ and $\mu_\nu(X)=1$. Equality 2) in Definition 1 holds. Indeed,

$$\begin{aligned}\mu_\nu(A\cup B)&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\ \varphi\geq\chi_{A\cup B}\}=\\&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\ \varphi\geq\chi_A\oplus\chi_B\}=\\&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\ (\varphi\geq\chi_A)\wedge(\varphi\geq\chi_B)\}=\\&=\inf\{\nu(\varphi_1\oplus\varphi_2)\colon \varphi_1,\ \varphi_2\in USC(X),\ (\varphi_1\geq\chi_A)\wedge(\varphi_2\geq\chi_B)\}=\\&=\inf\{\nu(\varphi_1)\oplus\nu(\varphi_2)\colon \varphi_1,\ \varphi_2\in USC(X),\ (\varphi_1\geq\chi_A)\wedge(\varphi_2\geq\chi_B)\}=\\&=\max\{\inf\{\nu(\varphi_1)\colon \varphi_1\in USC(X),\ \varphi_1\geq\chi_A\},\\&\qquad\quad\inf\{\nu(\varphi_2)\colon \varphi_2\in USC(X),\ \varphi_2\geq\chi_B\}\}=\\&=\mu_\nu(A)\oplus\mu_\nu(B).\end{aligned}$$

We have to show that equality 3) in Definition 1 is also true.
Let $\{A_\alpha\colon \alpha\in\mathfrak{A}\}\subset\mathfrak{B}(X)$ be an increasing net such that $\bigcup\limits_{\alpha\in\mathfrak{A}}A_\alpha\in\mathfrak{B}(X)$. Then, we have

$$\begin{aligned}\mu_\nu\Big(\bigcup_{\alpha\in\mathfrak{A}}A_\alpha\Big)&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\ \varphi\geq\chi_{\bigcup\limits_{\alpha\in\mathfrak{A}}A_\alpha}\}=\\&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\ \varphi\geq\sup_{\alpha\in\mathfrak{A}}\chi_{A_\alpha}\}=\\&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\ \varphi\geq\chi_{A_\alpha},\ \alpha\in\mathfrak{A}\}=\\&=\inf\{\nu(\bigoplus_{\alpha\in\mathfrak{A}}\varphi_\alpha)\colon \varphi_\alpha\in USC(X),\ \varphi_\alpha\geq\chi_{A_\alpha},\ \alpha\in\mathfrak{A}\}=\\&=\inf\{\sup_{\alpha\in\mathfrak{A}}\nu(\varphi_\alpha)\colon \varphi_\alpha\in USC(X),\ \varphi_\alpha\geq\chi_{A_\alpha},\ \alpha\in\mathfrak{A}\}=\\&=\sup_{\alpha\in\mathfrak{A}}\{\inf\{\nu(\varphi_\alpha)\colon \varphi_\alpha\in USC(X),\ \varphi_\alpha\geq\chi_{A_\alpha}\}\}=\\&=\sup_{\alpha\in\mathfrak{A}}\{\mu_\nu(A_\alpha)\}.\end{aligned}$$

□

Theorem 3.3. *For a compact Hausdorff space* X *and for any normalized max-plus functional* $\nu\colon USC(X)\to\mathbb{R}_{\max}$ *there exists a unique idempotent probability measure* μ_ν *on* $\mathfrak{B}(X)$ *such that*

$$\nu(\varphi)=\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}e^{\varphi(x)}d\mu_\nu\right),\qquad \varphi\in USC(X).\tag{3.2}$$

Proof. We need to verify that the functional ν defined by (3.2) satisfies the conditions in Definition 1. Indeed,

1) according to Lemma 2.1 for every pair of $\varphi,\ \psi\in USC(X)$ one has

$$\begin{aligned}\nu(\varphi\oplus\psi)&=\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}e^{\varphi\oplus\psi}d\mu_\nu\right)=\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}\left(e^{\varphi}\oplus e^{\psi}\right)d\mu_\nu\right)=\\&=\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}e^{\varphi}d\mu_\nu\right)\oplus\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}e^{\psi}d\mu_\nu\right)=\nu(\varphi)\oplus\nu(\psi);\end{aligned}$$

2) for every $c \in \mathbb{R}_{\max}$ and $\varphi \in USC(X)$

$$\nu(c \odot \varphi) = \ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus} e^{c\odot\varphi}d\mu_\nu\right) = \ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus} e^{c+\varphi}d\mu_\nu\right) =$$
$$= \ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus} (e^c \cdot e^{\varphi})\, d\mu_\nu\right) = \ln\left(e^c \cdot \frac{1}{\mu_\nu(X)}\int_X^{\oplus} e^{\varphi}d\mu_\nu\right) =$$
$$= c + \ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus} e^{\varphi}d\mu_\nu\right) = c + \nu(\varphi) = c \odot \nu(\varphi).$$

Finally, $\nu(c_X) = \ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus} e^c d\mu_\nu\right) = \ln\left(\frac{1}{\mu_\nu(X)} \cdot e^c \cdot \mu_\nu(X)\right) = c.$

Uniqueness. Suppose, that for some ν there exist two different measures μ_1 and μ_2 satisfying (3.2). Then, according to Theorem 3.2, there exists a set $A \in \mathfrak{B}(X)$ such that $\mu_1(A) \neq \mu_2(A)$. Recall that one has

$$|\mu_1(A) - \mu_2(A)| =$$
$$= |\inf\{\nu(\varphi) \colon \varphi \in USC(X),\, \varphi \geq \chi_A\} - \inf\{\nu(\varphi) \colon \varphi \in USC(X),\, \varphi \geq \chi_A\}| = 0.$$

This contradiction shows that $\mu_1 = \mu_2$.

□

Clearly, Theorem 3.3 is a max-plus variant of the well-known Riesz Representation Theorem.

Considering $I_{\mathfrak{B}}(X)$ with the topology generated by the sets of form (2.1) and $I_{USC}(X)$ equipped with the pointwise convergence topology, by Theorems 3.2 and 3.3 we get

Corollary 3.1. *For every compact Hausdorff space X the topological spaces $I_{\mathfrak{B}}(X)$ and $I_{USC}(X)$ are homeomorphic.*

Now, let us consider a more general case. Let X be a Tychonoff space. Denote by βX the Stone-Čech compactification of X. We define the following set:

$$I_\tau(X) = \{\mu \in I(\beta X) \colon \ \mu(F) = 0 \ \text{for every } F \in \mathfrak{B}(\beta X), F \subset \beta X \setminus X\}.$$

Elements of $I_\tau(X)$ are called *τ-smooth idempotent probability measures* (see [8]).

For each $\mu \in I_\tau(X)$ we define the set function $\tilde{\mu} \colon \mathfrak{B}(X) \to \mathbb{R}$ on the family $\mathfrak{B}(X)$ of all Borel subsets of X by the formula

$$\tilde{\mu}(A) = \inf\{\mu(B) \colon \ B \in \mathfrak{B}(\beta X),\, B \supset A\}, \qquad A \in \mathfrak{B}(X). \tag{3.3}$$

Lemma 3.1. [8] *$\tilde{\mu}$ is an idempotent probability measure on X.*

Now, we shall extend the assertions of Theorems 3.2 and 3.3 to a wider class of topological spaces.

Theorem 3.4. *Let X be a Tychonoff space. If $\tilde{\mu}$ is τ-smooth idempotent probability measure on $\mathfrak{B}(X)$, then integration*

$$\tilde{\varphi} \mapsto \ln\left(\frac{1}{\tilde{\mu}(X)}\int_X^{\oplus} e^{\tilde{\varphi}(x)}d\tilde{\mu}\right)$$

is a normalized max-plus linear functional on the linear space $USC_b(X)$. Conversely, for any normalized max-plus linear functional $\tilde{\nu}\colon USC_b(X) \to \mathbb{R}_{\max}$ there exists a unique τ-smooth idempotent probability measure $\tilde{\mu}$ on $\mathfrak{B}(X)$ such that

$$\tilde{\nu}(\tilde{\varphi}) = \ln\left(\frac{1}{\tilde{\mu}(X)} \int_X^{\oplus} e^{\tilde{\varphi}(x)} d\tilde{\mu}\right), \qquad \tilde{\varphi} \in USC_b(X).$$

Proof. We define a max-plus linear functional ν on $USC(\beta X)$ by $\nu(\varphi) = \tilde{\nu}(\tilde{\varphi})$, where $\tilde{\varphi}$ denotes the restriction of $\varphi \in USC(\beta X)$ to X. It is obvious that ν satisfies the conditions of Definition 6. According to Theorem 3.3 there exists a unique idempotent probability measure μ on $\mathfrak{B}(\beta X)$ such that

$$\nu(\varphi) = \ln\left(\frac{1}{\mu(\beta X)} \int_{\beta X}^{\oplus} e^{\varphi(x)} d\mu\right), \qquad \varphi \in USC(\beta X).$$

Now, we prove the converse part of Theorem 3.4. By Theorem 3.2, we can write

$$\mu(B) = \inf\{\nu(\varphi)\colon \varphi \in USC(\beta X),\, \varphi \geq \chi_B\}, \qquad B \in \mathfrak{B}(\beta X).$$

Then, applying (3.3) for each $\mu \in I_\tau(X)$ we have

$$\tilde{\mu}(A) = \inf\{\mu(B)\colon\ B \in \mathfrak{B}(\beta X),\, B \supset A\}, \qquad A \in \mathfrak{B}(X).$$

According to Lemma 3.1 $\tilde{\mu}$ is an idempotent probability measure on X.

Uniqueness. Suppose, for some μ there exist two different measures $\tilde{\mu_1}$ and $\tilde{\mu_2}$. Then, there exists a set $A \in \mathfrak{B}(X)$ such that

$$\begin{aligned} 0 \neq &|\tilde{\mu_1}(A) - \tilde{\mu_2}(A)| = \\ &= |\inf\{\mu(B)\colon B \in \mathfrak{B}(\beta X),\, B \supset A\} - \inf\{\mu(B)\colon B \in \mathfrak{B}(\beta X),\, B \supset A\}| = 0. \end{aligned}$$

This contradiction implies that $\tilde{\mu_1} = \tilde{\mu_2}$.

□

4 Max-plus version of the Hölder inequality

Now, we introduce a notion of an inner product in the space $USC_b(X)$.

Theorem 4.1. *The following equality defines an inner product on the linear space $USC_b(X)$:*

$$(\varphi,\, \psi) = \ln\left(\frac{1}{\mu(X)} \int_X^{\oplus} e^{\varphi(x)\odot\psi(x)} d\mu\right) \tag{4.1}$$

for $\varphi,\, \psi \in USC_b(X)$.

Proof. By the definition of an inner product, we have to verify the following properties:

1) for every $\varphi,\, \psi \in USC_b(X)$ one has $(\varphi,\, \psi) = (\psi,\, \varphi)$ (it is obvious);

2) for every φ, ψ, $\chi \in USC_b(X)$ we have

$$(\varphi \oplus \psi,\, \chi) = \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{(\varphi(x)\oplus\psi(x))\odot\chi(x)} d\mu\right) =$$
$$= \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{(\varphi(x)\odot\chi(x))\oplus(\psi(x)\odot\chi(x))} d\mu\right) =$$
$$= \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{\varphi(x)\odot\chi(x)} d\mu\right) \oplus \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{\psi(x)\odot\chi(x)} d\mu\right) =$$
$$= (\varphi,\, \chi) \oplus (\psi,\, \chi);$$

3) for each φ, $\psi \in USC_b(X)$ and $c \in \mathbb{R}_{\max}$ we have

$$(c \odot \varphi,\, \psi) = \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{(c\odot\varphi(x))\odot\psi(x)} d\mu\right) = \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{c\odot(\varphi(x)\odot\psi(x))} d\mu\right) =$$
$$= \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{c}\cdot e^{\varphi(x)\odot\psi(x)} d\mu\right) = \ln\left(e^{c}\cdot\frac{1}{\mu(X)}\int_X^{\oplus} e^{\varphi(x)\odot\psi(x)} d\mu\right) =$$
$$= c + \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{\varphi(x)\odot\psi(x)} d\mu\right) = c + (\varphi,\, \psi) = c \odot (\varphi,\, \psi);$$

4) for any $\varphi \in USC_b(X)$ we have $e^{\varphi(x)} \geq 0$. According to the monotonicity of the idempotent integration $\int_X^{\oplus} e^{\varphi(x)} d\mu \geq 0$. Then, $(\varphi,\, \varphi) \geq \mathbf{0}$.

□

Lemma 4.1. *Inner product (4.1) is continuous.*

Proof. Let $\varphi_n \to \varphi$, $\psi_n \to \psi$ and $\lambda_n \to \lambda$ for φ_n, $\psi_n \in USC_b(X)$ and $\lambda_n \in \mathbb{R}$. Then, $\varphi_n \oplus \psi_n \to \varphi \oplus \psi$ and $\lambda_n \odot \varphi_n \to \lambda \odot \varphi$. By Theorem 3.1 $(\varphi_n,\, \psi_n) \to (\varphi,\, \psi)$. □

For $\varphi \in USC_b(X)$, we define a $\oplus$-norm of φ by $||\varphi||_{\oplus 2} = \sqrt{(|\varphi|,\, |\varphi|)}$. For $p > 2$ we put $||\varphi||_{\oplus p} = \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{p\cdot|\varphi|(x)} d\mu\right)^{\frac{1}{p}}$.

Theorem 4.2. ***(Hölder inequality)*** 1) *Let $p \in [1, +\infty]$ and $q \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have the following inequality*

$$|(\varphi,\, \psi)| \leq ||\varphi||_{\oplus p} \odot ||\psi||_{\oplus q}, \quad \textit{for every} \quad \varphi,\, \psi \in USC_b(X);$$

2) *If $\mu(X) = 1$, then $||\varphi||_{\oplus p} \leq ||\varphi||_{\oplus q}$, where $0 < p < q$.*

Proof. By the first part of Lemma 2.3 we have

$$\int\limits_X^{\oplus} e^{\varphi(x)} \cdot e^{\psi(x)} d\mu \leq \left(\int\limits_X^{\oplus} e^{p\cdot\varphi(x)} d\mu\right)^{\frac{1}{p}} \cdot \left(\int\limits_X^{\oplus} e^{q\cdot\psi(x)} d\mu\right)^{\frac{1}{q}} \Rightarrow$$

$$\frac{1}{\mu(X)}\int\limits_X^{\oplus} e^{\varphi(x)} \cdot e^{\psi(x)} d\mu \leq \left(\frac{1}{\mu(X)}\int\limits_X^{\oplus} e^{p\cdot\varphi(x)} d\mu\right)^{\frac{1}{p}} \cdot \left(\frac{1}{\mu(X)}\int\limits_X^{\oplus} e^{q\cdot\psi(x)} d\mu\right)^{\frac{1}{q}} \Rightarrow$$

$$\left|\ln\left(\frac{1}{\mu(X)}\int\limits_X^{\oplus} e^{\varphi(x)} \cdot e^{\psi(x)} d\mu\right)\right| \leq \ln\left(\frac{1}{\mu(X)}\int\limits_X^{\oplus} e^{p\cdot|\varphi|(x)} d\mu\right)^{\frac{1}{p}} \odot \ln\left(\frac{1}{\mu(X)}\int\limits_X^{\oplus} e^{q\cdot|\psi|(x)} d\mu\right)^{\frac{1}{q}}.$$

Hence, $|(\varphi, \psi)| \leq ||\varphi||_{\oplus p} \odot ||\psi||_{\oplus q}$.

According to the second part of Lemma 2.3 and from a property of logarithmic function one has $||\varphi||_{\oplus p} \leq ||\varphi||_{\oplus q}$. □

Acknowledgments

The authors thank the unknown referee for critical comments, suggestions, and useful advice. Also, the authors would like to express gratitude to the editor for the revealed shortcomings and specified remarks.

References

[1] S. Albeverio, Sh.A. Ayupov, A.A. Zaitov, *On certain properties of the spaces of order-preserving functionals.* Topology and its Applications 155 (2008), no. 16, 1792-1799. https://doi.org/10.1016/j.topol.2008.05.019

[2] Sh.A. Ayupov, A.A. Zaitov, *Functor of weakly additive τ-smooth functionals and mappings.* Ukrainian Mathematical Journal 61 (2009), 1380-1386. https://doi.org/10.1007/s11253-010-0283-0

[3] V.I. Bogachev, *Measure theory, Vo I.* Springer, 2007.

[4] V.I. Bogachev, *Measure theory, Vo II.* Springer, 2007.

[5] R. Engelking, *General topology.* – Revised and completed ed. – Berlin: Heldermann, 1989.

[6] V.V. Fedorchuk, Yu.V. Sadovnichii, *On some topological and categorical properties of measures with alternating signs.* Fundam. Prikl. Mat. 5 (1999), no. 2, 597-618.

[7] J. Gunawardena, *Idempotency.* Publ. of the Newton Institute, 1998, Vol. 11, Cambridge University Press, Cambridge, (Online publication date: May 2010).

[8] L.D.R. Kocinac, A.A. Zaitov, M.R. Eshimbetov, *On the Čech-completeness of the space of τ-smooth idempotent measures.* Axioms, 13 (2024), no. 8, 569-582.

[9] B. Makarov, A. Podkorytov, *Real analysis: measures, integrals and applications.* Springer-Verlag, London 2013.

[10] A. Puhalskii, *Large deviations and idempotent probability.* Chapman & Hall/CRC, 2001.

[11] T. Radul, *Idempotent measures: absolute retracts and soft maps.* Topological Methods in Nonlinear Analysis [online] 56 (2020), no. 1, 161-172.

[12] A. Sheldon, *Measure, integration and real analysis.* Springer Open, 2010.

[13] V.S. Varadarajan, *Measures on topological spaces.* Mat.Sb. (N.S.) 55 (1961), no. 1, 35-100.

[14] A.A. Zaitov, *On a metric on the space of idempotent probability measures.* Applied General Topology 16 (2020), no. 1, 35-51.

[15] M. Zarichnyi, *Spaces and mappings of idempotent measures.* Izvestiya: Math. 74(2010), no. 3, 45-64.

[16] M. Zarichnyi, L. Bazylevych, D. Repovs, *Spaces of idempotent measures of compact metric spaces.* Topology and its Applications 157 (2010), no. 1, 136-144.

[17] A.A. Zaitov, Kh.F. Kholturayev, *On some properties of infinite iterations of the functor of idempotent probobility measures.* Lobachevskii Journal of Mathematics 43 (2022), no. 8, 2341-2348.

[18] A.A. Zaitov, M.R. Eshimbetov, *On a max-plus variant of the Riesz representation theorem.* Bulletin of the Institute of Mathematics 6 (2023), no. 5, 73-79.

[19] A.A. Zaitov, A.Ya. Ishmetov, *Homotopy properties of the space $I_f(X)$ of idempotent probability measures.* Math. Notes 106 (2019), no. 3-4, 562-571.

Shavkat Abdullayevich Ayupov
Scientific laboratory of algebra and its applications
V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences
9 Universitet St,
100174, Tashkent, Uzbekistan

and

Department of Algebra and Functional Analysis
National University of Uzbekistan
4 Universitet St,
100174, Tashkent, Uzbekistan
E-mail: shavkat.ayupov@mathinst.uz

Muzaffar Reyimbayevich Eshimbetov
Department of Mathematics
Tashkent International University of Financial Management and Technologies
15 Amir Temur St,
100047, Tashkent, Uzbekistan
E-mail: mr.eshimbetov@gmail.com

Adilbek Atakhanovich Zaitov
Department of Research and Innovation
Tashkent University of Architecture and Civil Engineering
9 Yangi Shahar St,
100194, Tashkent, Uzbekistan
E-mail: adilbek_zaitov@mail.ru

Received: 17.08.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 17, Number 1 (2026), 23 – 32

p-NUMERICAL RADIUS INEQUALITIES FOR THE TENSOR PRODUCT OF OPERATORS

A. Frakis, F. Kittaneh, S. Soltani

Communicated by T. Bekjan

Key words: Schatten p-norm, p-numerical radius, tensor product, inequality.

AMS Mathematics Subject Classification: 15A60, 15A69, 47A12, 47A30, 47A80, 47B10.

Abstract. In this paper, we give several inequalities for the tensor product of two operators involving the p-numerical radius and the Schatten p-norms.

DOI: https://doi.org/10.32523/2077-9879-2026-17-1-23-32

1 Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the $\mathcal{C}^*$-algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, let $T = \Re(T) + i\Im(T)$ be the Cartesian decomposition of T, where $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$.

Let $\mathcal{K}(\mathcal{H})$ denote the class of all compact operators in $\mathcal{B}(\mathcal{H})$. For a compact operator T, the Schatten p-norm of T is defined by $\|T\|_p = (tr|T|^p)^{\frac{1}{p}}$, where $1 \leq p \leq \infty$ and $|T| = (T^*T)^{\frac{1}{2}}$. For $0 < p < 1$, $\|\cdot\|_p$ is a quasi-norm. The Schatten p-class in $\mathcal{B}(\mathcal{H})$, denoted by $\mathcal{B}_p(\mathcal{H})$ is defined by

$$\mathcal{B}_p(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \|T\|_p < \infty\}.$$

Note that when $p = \infty$, the Schatten p-norm of T is the operator norm $\|T\|_\infty = \|T\| = \sup_{\|x\|=1} \|Tx\|$, and when $p = 2$, it is the Hilbert–Schmidt norm $\|T\|_2 = (tr\, T^*T)^{\frac{1}{2}}$. For $1 \leq p \leq q \leq \infty$, the Schatten p-norm of T satisfies the monotonicity property

$$\|T\|_\infty \leq \|T\|_q \leq \|T\|_p \leq \|T\|_1.$$

For $0 < p \leq \infty$, we have the following relations:

$$\|T\|_{sp}^s = \||T|^s\|_p = \||T^*|^s\|_p \quad \text{for } s > 0 \tag{1.1}$$

If $T, S \in \mathcal{B}_p(\mathcal{H})$, where $0 < p \leq \infty$, then

$$\left\| \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right\|_p = \left\| \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} \right\|_p = \left(\|T\|_p^p + \|S\|_p^p \right)^{\frac{1}{p}}. \tag{1.2}$$

Moreover, if $T \in \mathcal{B}_p(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, then

$$\|TS\|_p \leq \|T\|_p \|S\| \quad \text{and} \quad \|ST\|_p \leq \|S\| \|T\|_p. \tag{1.3}$$

Let $1 \leq p \leq \infty$. The p-numerical radius of T is defined by

$$w_p(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re(e^{i\theta} T) \right\|_p = \sup_{\theta \in \mathbb{R}} \left\| \Im(e^{i\theta} T) \right\|_p.$$

It is well known [1] that the p-numerical radius of T is equivalent to the Schatten p-norm. In fact, we have

$$\frac{1}{2}\|T\|_p \leq w_p(T) \leq \|T\|_p. \tag{1.4}$$

There are many papers dealing with estimates of the p-numerical radius of a bounded linear operator on a separable Hilbert space. We refer the readers to [3], [5], [6], [7] and the references therein.

Next, we focus our attention on the tensor product of operators, which has a wide range of applications in various fields. We mention here image processing, quantum computing, semidefinite programming, operator equations, operator differential equations and other disciplines.

The tensor product of operators has many interesting properties. For example the mixed-product property, which states that for any operators $T, S, A, B \in \mathcal{B}(\mathcal{H})$, we have

$$(T \otimes S)(A \otimes B) = (TA \otimes SB).$$

Other useful properties of the tensor product, that will be used later to state our results, are as follows:

$$|T \otimes S| = |T| \otimes |S|$$

and

$$(T \otimes S)^* = T^* \otimes S^*.$$

If $T, S \geq 0$ and $r > 0$, then

$$(T \otimes S)^r = T^r \otimes S^r.$$

Another useful property of the tensor product is the Schatten p-norm equality, which says that if $T, S \in \mathcal{B}_p(\mathcal{H})$, then

$$\|T \otimes S\|_p = \|T\|_p \|S\|_p.$$

In the case in which $\dim \mathcal{H} = n \in \mathbb{N}$, we identify $\mathcal{B}(\mathcal{H})$ with the matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. The tensor product (or the Kronecker product) of $T = [t_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $S = [s_{ij}] \in \mathcal{M}_n(\mathbb{C})$ is defined to be the block matrix

$$T \otimes S = [t_{ij}S] := \begin{bmatrix} t_{11}S & \dots & t_{1n}S \\ \vdots & \ddots & \vdots \\ t_{n1}S & \dots & t_{nn}S \end{bmatrix} \in \mathcal{M}_{n^2}(\mathbb{C}).$$

Many works involving the tensor product have been published. The reader may consult [2], [4], [8], [10], [11] and references therein.

In this paper, we give several p-numerical radius and Schatten p-norm inequalities for the tensor product of operators.

2 Main results

For $X \in \mathcal{M}_n(\mathbb{C})$, the Schatten p-norm distance $\Delta_p(X)$ of the matrix X from the scalar matrices is defined as

$$\Delta_p(X) = \inf_{z \in \mathbb{C}} \|X - zI\|_p.$$

By a compactness argument, one can deduce that there exists $z_0 \in \mathbb{C}$ such that $\|X - z_0 I\|_p = \Delta_p(X)$. It can be easily verified that $\Delta_p(\cdot)$ is a seminorm on $\mathcal{M}_n(\mathbb{C})$, and it is obvious that $\Delta_p(X) \leq \|X\|_p$.

In the following two theorems, we give lower bounds for the p-numerical radius of the tensor product of matrices T and S using merely the entries of these matrices.

Let $X[\mathcal{I}|\mathcal{J}]$ denote the submatrix of X consisting of the entries which belong to the rows $i \in \mathcal{I}$ and the columns $j \in \mathcal{J}$. Let $|\mathcal{I}|$ and $|\mathcal{J}|$ denote the cardinality of $\mathcal{I}$ and $\mathcal{J}$, respectively. It is easy to see that if $\mathcal{I} \cap \mathcal{J} = \emptyset$, then $(X - zI)[\mathcal{I}|\mathcal{J}] = X[\mathcal{I}|\mathcal{J}]$ for any $z \in \mathbb{C}$.

Note that the Schatten p-norm of an $m \times n$ matrix can be defined in a natural way. It is known that $\|X_0\|_p \leq \|X\|_p$, where X_0 is any submatrix of X. It should be mentioned here that some ideas of the proofs of the first two results are inspired by [9].

Theorem 2.1. *Let $T = [t_{ij}], S = [s_{ij}] \in \mathcal{M}_n(\mathbb{C})$, and let $p \geq 1$. Then*

$$\frac{1}{2n}\left(\sum_{i,j=1}^{n} |t_{ij}|^2 \sum_{i,j=1}^{n} |s_{ij}|^2 - \frac{1}{n}\Big(\sum_{i=1}^{n} t_{ii} \sum_{i=1}^{n} s_{ii}\Big)^2\right)^{\frac{1}{2}} \leq w_p(T \otimes S).$$

Proof. It is well known that for any matrix $X \in \mathcal{M}_n(\mathbb{C})$, we have

$$\|X\|_p \geq \|X\| \geq \frac{1}{\sqrt{n}}\|X\|_2.$$

Now,

$$\begin{aligned} w_p(T \otimes S) &\geq \frac{1}{2}\|T \otimes S\|_p \\ &\geq \frac{1}{2}\inf_{z\in\mathbb{C}} \|T \otimes S - zI\|_p \\ &\geq \frac{1}{2n}\inf_{z\in\mathbb{C}} \|T \otimes S - zI\|_2 \\ &= \frac{1}{2n}\sqrt{\|T \otimes S\|_2^2 - \frac{1}{n}tr^2(T \otimes S)} \\ &= \frac{1}{2n}\sqrt{\|T\|_2^2\|S\|_2^2 - \frac{1}{n}tr^2(T)tr^2(S)}, \end{aligned}$$

as required. □

Theorem 2.2. *Let $T = [t_{ij}], S = [s_{ij}] \in \mathcal{M}_n(\mathbb{C})$, and let $p \geq 1$. Then*

$$\max_{\mathcal{I}} \frac{\min_{i\in\mathcal{I}, j\notin\mathcal{I}} |t_{ij}|}{2\sqrt{|\mathcal{I}|}} \sqrt{\sum_{i\in\mathcal{I}, j\notin\mathcal{I}} |s_{ij}|^2} \leq w_p(T \otimes S),$$

where $\emptyset \neq \mathcal{I} \subset \{1, 2, \ldots, n\}$ and $|\mathcal{I}| \leq \frac{n}{2}$.

Proof. Let $\mathcal{I}, \mathcal{J}$ be as described above with $\mathcal{I} \cap \mathcal{J} = \emptyset$. Then

$$\begin{aligned} w_p(T \otimes S) &\geq \frac{1}{2}\|T \otimes S\|_p \\ &\geq \frac{1}{2}\inf_{z\in\mathbb{C}} \|T \otimes S - zI\|_p \\ &\geq \frac{1}{2}\inf_{z\in\mathbb{C}} \|(T \otimes S - zI)[\mathcal{I}|\mathcal{J}]\|_p \\ &= \frac{1}{2}\|(T \otimes S)[\mathcal{I}|\mathcal{J}]\|_p \\ &\geq \frac{1}{2\sqrt{\min(|\mathcal{I}|, |\mathcal{J}|)}}\|(T \otimes S)[\mathcal{I}|\mathcal{J}]\|_2 \\ &= \frac{\min_{i\in\mathcal{I}, j\in\mathcal{J}} |t_{ij}|}{2\sqrt{\min(|\mathcal{I}|, |\mathcal{J}|)}}\sqrt{\sum_{i\in\mathcal{I}, j\in\mathcal{J}} |s_{ij}|^2}. \end{aligned}$$

Letting $\mathcal{J} = \{1, 2, \ldots, n\} \setminus \mathcal{I}$ and restricting $|\mathcal{I}| \leq \frac{n}{2}$, gives the desired result. □

The following theorem refines the inequality $\frac{\|T\|_p \|S\|_p}{2} \leq w_p(T \otimes S)$, which can be extracted from inequality (1.4).

Theorem 2.3. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \geq 1$. Then*

$$\frac{1}{2}\left(\|T\|_p \|S\|_p + \left|\, \|\Re(T \otimes S)\|_p - \|\Im(T \otimes S)\|_p \,\right|\right) \leq w_p(T \otimes S).$$

Proof. We have

$$\begin{aligned} w_p(T \otimes S) &\geq \max\left\{\|\Re(T \otimes S)\|_p, \|\Im(T \otimes S)\|_p\right\} \\ &= \frac{\|\Re(T \otimes S)\|_p + \|\Im(T \otimes S)\|_p}{2} + \frac{\left|\, \|\Re(T \otimes S)\|_p - \|\Im(T \otimes S)\|_p \,\right|}{2} \\ &\geq \frac{\|\Re(T \otimes S) + i\Im(T \otimes S)\|_p}{2} + \frac{\left|\|\Re(T \otimes S)\|_p - \|\Im(T \otimes S)\|_p \right|}{2} \\ &= \frac{\|T\|_p \|S\|_p}{2} + \frac{\left|\, \|\Re(T \otimes S)\|_p - \|\Im(T \otimes S)\|_p \,\right|}{2}, \end{aligned}$$

as required. □

In the following corollary, we give a necessary condition for the equality

$$w_p(T \otimes S) = \frac{\|T\|_p \|S\|_p}{2}.$$

Corollary 2.1. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \geq 1$. If $w_p(T \otimes S) = \dfrac{\|T\|_p \|S\|_p}{2}$, then*

$$\|T \otimes S + T^* \otimes S^*\|_p = \|T \otimes S - T^* \otimes S^*\|_p = \|T\|_p \|S\|_p.$$

Proof. Let $w_p(T \otimes S) = \dfrac{\|T\|_p \|S\|_p}{2}$. Then by Theorem 2.3, we have $\|\Re(T \otimes S)\|_p = \|\Im(T \otimes S)\|_p$. On the other hand, we have

$$\begin{aligned} \|\Re(T \otimes S)\|_p &\leq w_p(T \otimes S) \\ &= \frac{\|T\|_p \|S\|_p}{2} \\ &\leq \frac{\|\Re(T \otimes S)\|_p + \|\Im(T \otimes S)\|_p}{2} \\ &= \|\Re(T \otimes S)\|_p. \end{aligned}$$

Hence, we obtain the required equalities. □

In the next theorem, we provide another lower bound for $w_p(T \otimes S)$.

Theorem 2.4. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \geq 2$. Then*

$$\frac{\left\||T|^2 \otimes |S|^2 + |T^*|^2 \otimes |S^*|^2\right\|_{p/2}}{4} + \frac{\left|\, \|\Re(T \otimes S)\|_p^2 - \|\Im(T \otimes S)\|_p^2 \,\right|}{2} \leq w_p^2(T \otimes S).$$

Proof. We have

$$
\begin{aligned}
w_p^2(T\otimes S) &\geq \max\left\{\|\Re(T\otimes S)\|_p^2, \|\Im(T\otimes S)\|_p^2\right\}\\
&= \frac{\|\Re(T\otimes S)\|_p^2+\|\Im(T\otimes S)\|_p^2}{2}+\frac{\left|\|\Re(T\otimes S)\|_p^2-\|\Im(T\otimes S)\|_p^2\right|}{2}\\
&= \frac{\left\|\Re^2(T\otimes S)\right\|_{p/2}+\left\|\Im^2(T\otimes S)\right\|_{p/2}}{2}+\frac{\left|\|\Re(T\otimes S)\|_p^2-\|\Im(T\otimes S)\|_p^2\right|}{2}\\
&\geq \frac{\left\|\Re^2(T\otimes S)+\Im^2(T\otimes S)\right\|_{p/2}}{2}+\frac{\left|\|\Re(T\otimes S)\|_p^2-\|\Im(T\otimes S)\|_p^2\right|}{2}\\
&= \frac{\left\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\right\|_{p/2}}{4}+\frac{\left|\|\Re(T\otimes S)\|_p^2-\|\Im(T\otimes S)\|_p^2\right|}{2},
\end{aligned}
$$

as required. □

Corollary 2.2. *Let $T,S\in\mathcal{B}_p(\mathcal{H})$, and let $p\geq 2$. If*

$$w_p^2(T\otimes S)=\frac{\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}}{4},$$

then

$$\|e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*\|_p^2=\|e^{i\theta}T\otimes S-e^{-i\theta}T^*\otimes S^*\|_p^2=\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}$$

for all $\theta\in\mathbb{R}$.

Proof. Assume that

$$w_p^2(T\otimes S)=\frac{\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}}{4}.$$

Then, from the fact that $w_p(T\otimes S)=w_p(e^{i\theta}T\otimes S)$, where $\theta\in\mathbb{R}$, and by Theorem 2.4, we get

$$\left\|\Re(e^{i\theta}T\otimes S)\right\|_p=\left\|\Im(e^{i\theta}T\otimes S)\right\|_p. \tag{2.1}$$

Now,

$$
\begin{aligned}
4w_p^2(T\otimes S) &= \frac{\|(e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*)^2-(e^{i\theta}T\otimes S-e^{-i\theta}T^*\otimes S^*)^2\|_{p/2}}{2}\\
&\leq \frac{\|(e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*)^2\|_{p/2}+\|(e^{i\theta}T\otimes S-e^{-i\theta}T^*\otimes S^*)^2\|_{p/2}}{2}\\
&= \frac{\|e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*\|_p^2+\|e^{i\theta}T\otimes S-e^{-i\theta}T^*\otimes S^*\|_p^2}{2}\\
&= \|e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*\|_p^2 \ \text{(by equality (2.1))}\\
&\leq 4w_p^2(T\otimes S)\\
&= \||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}.
\end{aligned}
$$

Hence, we get the desired result. □

Our next lower bound for $w_p(T\otimes S)$ reads as follows.

Theorem 2.5. *Let $T,S\in\mathcal{B}_p(\mathcal{H})$, and let $p\geq 1$. Then*

$$
\begin{aligned}
w_p(T\otimes S) \geq{}& \frac{\sqrt{2}}{4}\|T\|_p\|S\|_p+\frac{\left|\|T\otimes S+T^*\otimes S^*\|_p-\|T\otimes S-T^*\otimes S^*\|_p\right|}{4}\\
&+\frac{\sqrt{2}}{8}\left|\|T\otimes S+iT^*\otimes S^*\|_p-\|T\otimes S-iT^*\otimes S^*\|_p\right|.
\end{aligned}
$$

Proof. We have

$$\begin{aligned}
w_p(T\otimes S) &\geq \max\left\{\|\Re(T\otimes S)\|_p, \|\Im(T\otimes S)\|_p\right\} \\
&= \frac{\|\Re(T\otimes S)\|_p + \|\Im(T\otimes S)\|_p}{2} + \frac{\left|\|\Re(T\otimes S)\|_p - \|\Im(T\otimes S)\|_p\right|}{2} \\
&\geq \frac{1}{2}\max\left\{\|\Re(T\otimes S)+\Im(T\otimes S)\|_p, \|\Re(T\otimes S)-\Im(T\otimes S)\|_p\right\} \\
&+ \frac{\left|\|\Re(T\otimes S)\|_p - \|\Im(T\otimes S)\|_p\right|}{2} \\
&= \frac{\|\Re(T\otimes S)+\Im(T\otimes S)\|_p + \|\Re(T\otimes S)-\Im(T\otimes S)\|_p}{4} \\
&+ \frac{\left|\|\Re(T\otimes S)+\Im(T\otimes S)\|_p - \|\Re(T\otimes S)-\Im(T\otimes S)\|_p\right|}{4} \\
&+ \frac{\left|\|\Re(T\otimes S)\|_p - \|\Im(T\otimes S)\|_p\right|}{2}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{\|(\Re(T\otimes S)+\Im(T\otimes S)) + i\left(\Re(T\otimes S)-\Im(T\otimes S)\right)\|_p}{4} \\
&+ \frac{\left|\|\Re(T\otimes S)+\Im(T\otimes S)\|_p - \|\Re(T\otimes S)-\Im(T\otimes S)\|_p\right|}{4} \\
&+ \frac{\left|\|\Re(T\otimes S)\|_p - \|\Im(T\otimes S)\|_p\right|}{2} \\
&= \frac{\|(1+i)T^*\otimes S^*\|_p}{4} + \frac{\left|\|\Re(T\otimes S)\|_p - \|\Im(T\otimes S)\|_p\right|}{2} \\
&+ \frac{\left|\|\Re(T\otimes S)+\Im(T\otimes S)\|_p - \|\Re(T\otimes S)-\Im(T\otimes S)\|_p\right|}{4} \\
&= \frac{\sqrt{2}}{4}\|T\|_p\|S\|_p + \frac{|\|T\otimes S+T^*\otimes S^*\|_p - \|T\otimes S-T^*\otimes S^*\|_p|}{4} \\
&+ \frac{\sqrt{2}}{8}\left|\|T\otimes S+iT^*\otimes S^*\|_p - \|T\otimes S-iT^*\otimes S^*\|_p\right|.
\end{aligned}$$

This completes the proof. □

In the following two theorems, we give some upper bounds for $w_p(T\otimes S)$.

Theorem 2.6. *Let $T,S\in\mathcal{B}_p(\mathcal{H})$, and let $p\geq 1$. Then*

$$\begin{aligned}
w_p(T\otimes S) &\leq \|T\|_p w_p(S) \\
&+ \min\{w_p\left(\Re T\otimes S^*\right), w_p\left(\Im T\otimes S^*\right), w_p\left(\Re T\otimes S\right), w_p\left(\Im T\otimes S\right)\}.
\end{aligned}$$

Proof. Let $\theta \in \mathbb{R}$. Then

$$\begin{aligned}\|\Re(e^{i\theta}T\otimes S)\|_p &= w_p\big(\Re(e^{i\theta}T\otimes S)\big)\\ &= w_p\Big(\frac{e^{i\theta}T\otimes S}{2}+\frac{e^{-i\theta}T^*\otimes S^*}{2}\Big)\\ &= w_p\Big(T\otimes\frac{e^{i\theta}S+e^{-i\theta}S^*}{2}+e^{-i\theta}\frac{T^*\otimes S^*-T\otimes S^*}{2}\Big)\\ &\leq w_p\big(T\otimes\Re(e^{i\theta}S)\big)+w_p(\Im T\otimes S^*)\\ &\leq \|T\|_p\|\Re(e^{i\theta}S)\|_p+w_p\left(\Im T\otimes S^*\right).\end{aligned}$$

By taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality, we get

$$w_p(T\otimes S) \;\leq\; \|T\|_p w_p(S)+w_p\left(\Im T\otimes S^*\right). \tag{2.2}$$

Replacing T by iT in inequality (2.2), yields

$$\begin{aligned}w_p(T\otimes S) \;&=\; w_p(iT\otimes S)\\ &\leq\; \|T\|_p w_p(S)+w_p\left(\Re T\otimes S^*\right).\end{aligned}\tag{2.3}$$

From inequalities (2.2) and (2.3), we obtain

$$w_p(T\otimes S)\leq \|T\|_p w_p(S)+\min\big\{w_p\left(\Re T\otimes S^*\right), w_p\left(\Im T\otimes S^*\right)\big\}.$$

The required result follows from the fact that $w_p(T\otimes S)=w_p(T^*\otimes S^*)$. □

In the next theorem, we improve the inequality $w_p(T\otimes S)\leq \|T\|_p\|S\|_p$, which can be deduced from inequality (1.4).

Theorem 2.7. *Let $T,S\in\mathcal{B}_p(\mathcal{H})$, and let $p\geq 1$. Then*

$$w_p^2(T\otimes S)\leq \frac{1}{2}\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}.$$

Proof. Let $\theta \in \mathbb{R}$. Then

$$\begin{aligned}\|\Re(e^{i\theta}T\otimes S)\|_p^2 &= \frac{1}{4}\|e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*\|_p^2\\ &= \frac{1}{4}\left\|\begin{bmatrix}e^{i\theta}I & e^{-i\theta}I\\ 0 & 0\end{bmatrix}\begin{bmatrix}T\otimes S & 0\\ T^*\otimes S^* & 0\end{bmatrix}\right\|_p^2\\ &\leq \frac{1}{4}\left\|\begin{bmatrix}e^{i\theta}I & e^{-i\theta}I\\ 0 & 0\end{bmatrix}\right\|^2\left\|\begin{bmatrix}T\otimes S & 0\\ T^*\otimes S^* & 0\end{bmatrix}\right\|_p^2\\ &\quad\text{(by inequality (1.3))}\\ &\leq \frac{1}{2}\left\|\begin{bmatrix}T^*\otimes S^* & T\otimes S\\ 0 & 0\end{bmatrix}\begin{bmatrix}T\otimes S & 0\\ T^*\otimes S^* & 0\end{bmatrix}\right\|_{p/2}\\ &\quad\text{(by inequality (1.1))}\\ &= \frac{1}{2}\left\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\right\|_{p/2}\\ &\quad\text{(by inequality (1.2)).}\end{aligned}$$

The result follows by taking the supremum in the above inequality over all $\theta\in\mathbb{R}$. □

In the rest of this work, we present certain inequalities involving the Schatten p-norms of operators.

Theorem 2.8. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \geq 2$. Then*

$$\max\left\{\|T\otimes S+S\otimes T\|_p^2, \|T\otimes S-S\otimes T\|_p^2\right\}$$
$$\geq \max\left\{\left\|T^2\otimes S^2+S^2\otimes T^2\right\|_p, \left\| \,|T|^2\otimes|S|^2+|S|^2\otimes|T|^2\right\|_{p/2}\right\}$$
$$+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}.$$

Proof. Let $\mathcal{M}=\max\left\{\|T\otimes S+S\otimes T\|_p^2, \|T\otimes S-S\otimes T\|_p^2\right\}$. Then

$$\begin{aligned}
\mathcal{M} &= \frac{\|T\otimes S+S\otimes T\|_p^2+\|T\otimes S-S\otimes T\|_p^2}{2} \\
&\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2} \\
&\geq \frac{\|(T\otimes S+S\otimes T)^2\|_p+\|(T\otimes S-S\otimes T)^2\|_p}{2} \\
&\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2} \\
&\quad\text{(by the inequality } \|X\|_p^2\geq\|X^2\|_p \text{ for any } X\in\mathcal{B}_p(\mathcal{H})) \\
&\geq \frac{\|(T\otimes S+S\otimes T)^2+(T\otimes S-S\otimes T)^2\|_p}{2} \\
&\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2} \\
&= \left\|T^2\otimes S^2+S^2\otimes T^2\right\|_p \\
&\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{M} &= \frac{\|T\otimes S+S\otimes T\|_p^2+\|T\otimes S-S\otimes T\|_p^2}{2} \\
&\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2} \\
&= \frac{\|\,|T\otimes S+S\otimes T|^2\,\|_{p/2}+\|\,|T\otimes S-S\otimes T|^2\,\|_{p/2}}{2} \\
&\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2} \\
&\geq \frac{\|\,|T\otimes S+S\otimes T|^2+|T\otimes S-S\otimes T|^2\,\|_{p/2}}{2} \\
&\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2} \\
&= \left\|\,|T|^2\otimes|S|^2+|S|^2\otimes|T|^2\right\|_{p/2} \\
&\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}.
\end{aligned}$$

Hence, we obtain the desired result. □

Theorem 2.9. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \geq 2$. Then*

$$\max\left\{\|T \otimes S + S \otimes T\|_p^2, \|T \otimes S - S \otimes T\|_p^2\right\} \leq \min\{\alpha, \beta\},$$

where

$$\alpha = \|\,|T|^2 \otimes |S|^2 + |S|^2 \otimes |T|^2\|_{p/2} + \|T^*S \otimes S^*T + S^*T \otimes T^*S\|_{p/2}$$

and

$$\beta = \|\,|T^*|^2 \otimes |S^*|^2 + |S^*|^2 \otimes |T^*|^2\|_{p/2} + \|TS^* \otimes ST^* + ST^* \otimes TS^*\|_{p/2}.$$

Proof. We have $\|T \otimes S \pm S \otimes T\|_p^2$

$$\begin{aligned}
&= \|\,|T \otimes S \pm S \otimes T|^2\,\|_{p/2} \\
&= \|\,(T \otimes S \pm S \otimes T)^*(T \otimes S \pm S \otimes T)\,\|_{p/2} \\
&= \|\,|T|^2 \otimes |S|^2 + |S|^2 \otimes |T|^2 \pm (T^*S \otimes S^*T + S^*T \otimes T^*S)\|_{p/2} \\
&\leq \|\,|T|^2 \otimes |S|^2 + |S|^2 \otimes |T|^2\|_{p/2} + \|T^*S \otimes S^*T + S^*T \otimes T^*S\|_{p/2}.
\end{aligned}$$

Using the fact that $\|T\|_p = \|T^*\|_p$, gives

$$\|T \otimes S \pm S \otimes T\|_p^2 \leq \|\,|T^*|^2 \otimes |S^*|^2 + |S^*|^2 \otimes |T^*|^2\|_{p/2} + \|TS^* \otimes ST^* + ST^* \otimes TS^*\|_{p/2}.$$

Hence, we get the desired inequality. □

Acknowledgments

The authors thank the anonymous referee for comments and suggestions.

References

[1] A. Abu-Omar, F. Kittaneh, *A generalization of the numerical radius*, Linear Algebra Appl. 569 (2019), 323–334.

[2] S. Aici, A. Frakis, F. Kittaneh, *Some numerical radius inequalities for tensor products of operators*, J. Anal. 32 (2024), 3543-3556.

[3] A. Benmakhlouf, O. Hirzallah, F. Kittaneh, *On the p-numerical radii of Hilbert space operators*, Linear Multilinear Algebra 69 (2021), no. 15, 2813–2829.

[4] P. Bhunia, K. Paul, A. Sen, *Numerical radius inequalities for tensor product of operators*, Proc. Indian Acad. Sci. (Math. Sci.) 133 (2023), Paper No. 3, 12 pp.

[5] A. Frakis, F. Kittaneh, S. Soltani, *Upper and lower bounds for the p-numerical radii of operators*, Results Math. 79 (2024), Paper No. 59, 13 pp.

[6] A. Frakis, F. Kittaneh, S. Soltani, *On the p-numerical radii of* 2×2 *operator matrices*, J. Appl. Math. Comput. 70 (2024), no. 1, 335–350.

[7] A. Frakis, F. Kittaneh, *On p-numerical radius inequalities for commutators of operators*, Quaest. Math. 48 (2025), no. 3, 417–428.

[8] D. Gueridi, F. Kittaneh, *Inequalities for the Kronecker product of matrices*, Ann, Funct. Anal. 13 (2022), Paper No. 50, 17 pp.

[9] E. Jiang, X. Zhan, *Lower bounds for the spread of a Hermitian matrix*, Linear Algebra Appl. 256 (1997), 153–163.

[10] A.N. Langville, W.J. Stewart, *The Kronecker product and stochastic automata networks*, J. Comput. Appl. Math. 167 (2004), no. 2, 429–447.

[11] M. Sababheh, H.R. Moradi, *Numerical radius of Kronecker product of matrices*, J. Appl. Anal. and Comput. 13 (2023), no. 5, 2943–2954.

Abdelkader Frakis, Soumia Soltani
Department of Mathematics
Mustapha Stambouli University
29000, Mascara, Algeria
E-mails: frakis.aek@univ-mascara.dz, soumia.soltani@univ-mascara.dz

Fuad Kittaneh
Department of Mathematics
The University of Jordan
Amman, Jordan
and
Department of Mathematics
Korea University
Seoul 02841, South Korea
E-mail: fkitt@ju.edu.jo

Received: 23.10.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 17, Number 1 (2026), 33 – 46

ADAMS THEOREM FOR THE B-RIESZ POTENTIAL IN THE TOTAL B-MORREY SPACES

V.S. Guliyev, A. Akbulut, M.N. Omarova, A. Serbetci

Communicated by E.D. Nursultanov

Key words: B-maximal operator, B-Riesz potential, total B-Morrey space, Adams theorem for B-Riesz potential.

AMS Mathematics Subject Classification: 42B20, 42B25, 42B35.

Abstract. We prove Adams theorem for the Riesz potential I_γ^α (B-Riesz potential) in the total Morrey spaces $L_{p,(\lambda,\mu),\gamma}$ (total B-Morrey spaces), associated with the Laplace-Bessel differential operator Δ_B. More precisely, we obtain necessary and sufficient conditions for the operator I_γ^α to be bounded from the total B-Morrey space $L_{p,(\lambda,\mu),\gamma}$ to the total B-Morrey space $L_{q,(\lambda,\mu),\gamma}$ and from the total B-Morrey space $L_{1,(\lambda,\mu),\gamma}$ to the weak total B-Morrey space $WL_{q,(\lambda,\mu),\gamma}$.

DOI: https://doi.org/10.32523/2077-9879-2026-17-1-33-46

1 Introduction

The classical Morrey spaces were introduced by Morrey [4] for the study of solutions of some quasi-linear elliptic partial differential equations. For more applications of the Morrey spaces to partial differential equations, the reader is referred to [4, 15, 29]. In [16] the first author introduced a variant of the Morrey spaces called the total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. The total Morrey spaces generalize the classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ so that $L_{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L_{p,\lambda}(\mathbb{R}^n)$ and the modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ so that $L_{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$. See also [1, 7, 8, 14, 17, 18, 23, 24, 27, 28].

The Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \ldots, \gamma_k > 0, 1 \le k \le n$$

is known as an important operator in the Fourier-Bessel harmonic analysis and applications. The maximal operator, potentials and related topics associated with the Laplace-Bessel differential operator Δ_B have been investigated by many researchers. See B. Muckenhoupt and E. Stein [26], I. Kipriyanov [20], K. Trimeche [33], L. Lyakhov [22], K. Stempak [32], A.D. Gadjiev and I.A. Aliev [9], I.A. Aliev and S. Bayrakci [5], V.S. Guliyev [10, 11], V.S. Guliyev and J.J. Hasanov [12], S. Bayrakci [6], V.S. Guliyev, A. Serbetci and I. Ekincioglu [13], A. Akbulut, M. Dziri and I. Ekincioglu [3], A. Serbetci and I. Ekincioglu [30], E.L. Shishkina [31] and others.

In this paper, we consider the generalized shift operator generated by the Laplace-Bessel differential operator Δ_B in terms of which the B-maximal operator and the B-Riesz potential are investigated in the total B-Morrey space. We prove Adams theorem for the B-Riesz potential I_γ^α in the total B-Morrey spaces $L_{p,(\lambda,\mu),\gamma}$, namely, we will obtain necessary and sufficient conditions for the operator I_γ^α to be bounded from one total B-Morrey space $L_{p,(\lambda,\mu),\gamma}$ to another one $L_{q,(\lambda,\mu),\gamma}$ and from the total B-Morrey space $L_{1,(\lambda,\mu),\gamma}$ to the weak total B-Morrey space $WL_{q,(\lambda,\mu),\gamma}$.

The paper is organized as follows. In Section 2 we present some definitions and auxiliary results. In Section 3 we study some embeddings for the total B-Morrey spaces. In Section 4 the boundedness of the B-maximal operator M_γ on the total B-Morrey spaces $L_{p,(\lambda,\mu),\gamma}$ is proved. The main result of the paper is Adams theorem for the B-Riesz potential I_γ^α in the total B-Morrey space $L_{p,(\lambda,\mu),\gamma}$, established in Section 5.

Finally, we make some conventions on notation. Throughout this paper, we assume that the letter C denotes a positive constant that may vary at each occurrence but is independent of the essential variables. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C depending only on the numerical parameters.

2 Notations and preliminaries

Suppose that n and k are positive integers with $1 \le k \le n$ and $\mathbb{R}^n$ is n-dimensional Euclidean space. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$, $x = (x', x'') \in \mathbb{R}^n$, $n \ge 2$. Let

$$\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n;\ x_1 > 0,\ \dots,\ x_k > 0\}$$

and

$$E(x,r) = \{y \in \mathbb{R}^n_{k,+}\ ;\ |x-y| < r\}, \quad E_r = E(0,r).$$

For a measurable set E, let $|E|_\gamma = \int_E (x')^\gamma dx$, where $\gamma = (\gamma_1, \dots, \gamma_k)$ are positive numbers and $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$, $|E_r|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int_{E_1} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Define the generalized shift operator (B-shift operator) by

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\left((x',y')_\beta, x''-y''\right)\ d\nu(\beta),$$

where $(x_i,y_i)_{\beta_i} = (x_i^2 - 2x_iy_i\cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x',y')_\beta = ((x_1,y_1)_{\beta_1}, \dots, (x_k,y_k)_{\beta_k})$, $d\nu(\beta) = \prod_{i=1}^k \sin^{\gamma_i - 1}\beta_i\, d\beta_1 \dots d\beta_k$, $1 \le k \le n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)} = \frac{2^k}{\pi^k}\omega(2k,k,\gamma).$$

We remark that the generalized shift operator T^y is closely connected with the Bessel differential operator B (if $n = k = 1$ see [21] for details, if $n > 1$, $k = 1$ see [20], if $n, k > 1$ see [22]).

Let $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ be the space of all measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\gamma}} \equiv \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx\right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} |f(x)|.$$

The B-maximal function (see [10, 11]) is defined by

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int_{E_r} T^y |f(x)| (y')^\gamma dy$$

and the B-Riesz potential (see [10, 11]) is defined by

$$I_\gamma^\alpha f(x) = \int_{\mathbb{R}^n_{k,+}} T^y \left(|x|^{\alpha-Q}\right) f(y)\,(y')^\gamma dy, \ 0<\alpha<Q,$$

where T^y is the generalized shift operator generated by the Laplace-Bessel differential operator Δ_B.

The operator M_γ was introduced by Guliyev in [10]. Moreover, the strong-$(L_{p,\gamma}, L_{p,\gamma})$, $1<p\le\infty$ and weak-$(L_{1,\gamma}, L_{1,\gamma})$ boundedness of M_γ was proved in [10] (see also [11]). Also, the strong-$(L_{p,\gamma}, L_{q,\gamma})$, $1<p<q<\infty$, $1/p-1/q=\alpha/Q$ and weak-$(L_{1,\gamma}, L_{q,\gamma})$, $1<q<\infty$, $1-1/q=\alpha/Q$ boundedness of I_γ^α was proved in [10] (see also [11]).

Theorem 2.1. [10, 11] *1. If $f\in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, then $M_\gamma f\in WL_{1,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{WL_{1,\gamma}} \le C_{1,\gamma}\|f\|_{L_{1,\gamma}},$$

for all $f\in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, where $C_{1,\gamma}>0$ depends only on γ,k and n.

2. If $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1<p\le\infty$, then $M_\gamma f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|M_\gamma f\|_{L_{p,\gamma}} \le C_{p,\gamma}\,\|f\|_{L_{p,\gamma}},$$

for all $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, where $C_{p,\gamma}>0$ depends only on p, γ, k and n.

Theorem 2.2. [10, 11] *Let $0<\alpha<Q$ and $1\le p<\frac{Q}{\alpha}$.*

1) If $1<p<\frac{Q}{\alpha}$, then the condition $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}$ is necessary and sufficient for the boundedness I_γ^α from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p=1$, then the condition $1-\frac{1}{q}=\frac{\alpha}{Q}$ is necessary and sufficient for the boundedness I_γ^α from $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

3 Some embeddings for the total B-Morrey spaces

In this section we define the total B-Morrey spaces $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, give auxiliary results and some embeddings for these spaces.

Definition 1. Let $1\le p<\infty$, $\lambda\in\mathbb{R}$, $\mu\in\mathbb{R}$, $[t]_1=\min\{1,t\}$, $t>0$. We denote by $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the classical B-Morrey spaces [12], by $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the modified B-Morrey spaces [14], and by $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ the total B-Morrey spaces: the sets of all locally integrable functions f with the finite norms

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\,t>0} \left(t^{-\lambda}\int_{E_t} T^x|f(y)|^p\,(y')^\gamma dy\right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\,t>0} \left([t]_1^{-\lambda}\int_{E_t} T^x|f(y)|^p\,(y')^\gamma dy\right)^{1/p},$$

$$\|f\|_{L_{p,(\lambda,\mu),\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\,t>0} \left([t]_1^{-\lambda}\,[1/t]_1^{-\mu}\int_{E_t} T^x|f(y)|^p\,(y')^\gamma dy\right)^{1/p},$$

respectively.

Definition 2. Let $1 \le p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey spaces $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ [12], the weak modified Morrey spaces $W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ [14] and the weak total Morrey spaces $WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ as the sets of all locally integrable functions f with the finite norms

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left(t^{-\lambda} \int_{\{y\in E_t : T^x|f(y)|^p\}} (y')^\gamma dy\right)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left([t]_1^{-\lambda} \int_{\{y\in E_t : T^x|f(y)|^p\}} (y')^\gamma dy\right)^{1/p},$$

$$\|f\|_{WL_{p,(\lambda,\mu),\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left([t]_1^{-\lambda}\, [1/t]_1^{-\mu} \int_{\{y\in E_t : T^x|f(y)|^p\}} (y')^\gamma dy\right)^{1/p},$$

respectively.

Note that

$$L_{p,(0,0),\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+}),$$
$$WL_{p,(0,0),\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\gamma}(\mathbb{R}^n_{k,+}),$$
$$L_{p,(\lambda,\lambda),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}), \quad L_{p,(\lambda,0),\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}).$$

We have $\|f\|_{WL_{p,(\lambda,\mu),\gamma}} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}}$, therefore, the following continuous embedding holds

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}).$$

Furthermore,

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}),\ \mu \le \lambda \quad \text{and} \quad \|f\|_{L_{p,\lambda,\gamma}} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}}, \tag{3.1}$$

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+}),\ \mu \le \lambda \quad \text{and} \quad \|f\|_{L_{p,\mu,\gamma}} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}} \tag{3.2}$$

and

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\gamma}(\mathbb{R}^n_{k,+}) \quad \text{and} \quad \|f\|_{L_{p,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

If $\lambda < 0$ or $\lambda > Q$, then

$$L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta,$$

where $\Theta \equiv \Theta(\mathbb{R}^n_{k,+})$ is the set of all functions equivalent to 0 on $\mathbb{R}^n_{k,+}$.

Lemma 3.1. *If $1 \le p < \infty$, $0 \le \mu \le \lambda \le Q$, then*

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{L_{p,(\lambda,\mu),\gamma}} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\}.$$

Proof. Let $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Then from (3.1) and (3.2) we have that $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$ and $\max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}}$.

Now, let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$. Then

$$\|f\|_{L_{p,(\lambda,\mu),\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+}, t>0} \left([t]_1^{-\lambda}\ [1/t]_1^{\mu} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\right)^{1/p}$$
$$= \max\Big\{ \sup_{x\in\mathbb{R}^n_{k,+}, 0<t\le 1} \left(t^{-\lambda} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\right)^{1/p},$$
$$\sup_{x\in\mathbb{R}^n_{k,+}, t>1} \left(\ t^{-\mu} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\right)^{1/p} \Big\} \le \max\Big\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\Big\}.$$

Therefore, $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and the embedding $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ is valid.

Thus, $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$ and $\max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\} = \|f\|_{L_{p,(\lambda,\mu),\gamma}}$. □

Corollary 3.1. [19, Lemma 5] *If $1 \le p < \infty$, $0 \le \lambda \le Q$, then*

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\gamma}}\right\}.$$

Lemma 3.2. *If $1 \le p < \infty$, $0 \le \mu \le \lambda \le Q$, then*

$$WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap WL_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{WL_{p,(\lambda,\mu),\gamma}} = \max\left\{\|f\|_{WL_{p,\lambda,\gamma}}, \|f\|_{WL_{p,\mu,\gamma}}\right\}.$$

Remark 1. If $1 \le p < \infty$, and $\mu < 0$ or $\lambda > Q$, then

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = \Theta(\mathbb{R}^n_{k,+}).$$

Lemma 3.3. *If $1 \le p < \infty$, $0 \le \lambda \le Q$ and $0 \le \mu \le Q$, then*

$$L_{p,(Q,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{\infty,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,(\lambda,Q),\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{L_{p,(\lambda,Q),\gamma}} \le \omega(n,k,\gamma)^{1/p}\|f\|_{L_{\infty,\gamma}} \le \|f\|_{L_{p,(Q,\mu),\gamma}}.$$

Proof. Let $f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$. Then for all $x \in \mathbb{R}^n_{k,+}$ and $0 < t \le 1$

$$\left(t^{-\lambda} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\right)^{1/p} \le \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty}, \quad 0 \le \lambda \le Q$$

and for all $x \in \mathbb{R}^n_{k,+}$ and $t \ge 1$

$$\left(t^{-Q} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\right)^{1/p} \le \omega(n,k,\gamma)^{1/p} \|f\|_{L_{\infty,\gamma}}.$$

Therefore, $f \in L_{p,(\lambda,Q),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{p,(\lambda,n),\gamma}} \le \omega(n,k,\gamma)^{1/p}\|f\|_{L_{\infty,\gamma}}.$$

Let $f \in L_{p,(Q,\mu),\gamma}(\mathbb{R}^n_{k,+})$. By the Lebesgue's Theorem we have (see [12])

$$\lim_{t\to 0} |E_t|^{-1} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy = |f(x)|^p$$

for almost all $x \in \mathbb{R}^n_{k,+}$.

Then for almost all $x \in \mathbb{R}^n_{k,+}$

$$\begin{aligned} |f(x)| &= \left(\lim_{t\to 0} |E_t|^{-1} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\right)^{1/p} \\ &\le \omega(n,k,\gamma)^{-1/p} \sup_{x\in\mathbb{R}^n_{k,+},\, 0<t\le 1} \left(t^{-Q} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\right)^{1/p} \\ &\le \omega(n,k,\gamma)^{-1/p}\|f\|_{L_{p,(Q,\mu)}}. \end{aligned}$$

Therefore, $f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{\infty,\gamma}} \le \omega(n,k,\gamma)^{-1/p} \|f\|_{L_{p,(Q,\mu),\gamma}}.$$

□

Corollary 3.2. *If* $1 \le p < \infty$, *then*

$$L_{p,Q,\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,Q,\gamma}(\mathbb{R}^n_{k,+}) = L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{L_{p,Q,\gamma}} = \|f\|_{\widetilde{L}_{p,Q,\gamma}} = \omega(n,k,\gamma)^{1/p} \|f\|_{L_{\infty,\gamma}}.$$

Lemma 3.4. *If* $0 \le \lambda < Q$, $0 \le \mu < Q$, $0 \le \alpha < Q-\lambda$ *and* $0 \le \beta < Q-\mu$, *then for* $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}$

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset_{\succ} L_{1,(Q-\alpha,Q-\beta),\gamma}(\mathbb{R}^n_{k,+})$$

and for all $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ *the following inequality*

$$\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,(\lambda,\mu),\gamma}}$$

is valid.

Proof. Let $0 < \alpha < Q$, $0 \le \lambda < Q$, $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}$. By Hölder's inequality we have

$$\begin{aligned}
\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} &= \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} [t]_1^{\alpha-Q}\ [1/t]_1^{Q-\beta} \int_{E_t} T^x|f(y)|(y')^\gamma dy \\
&\le \omega(n,k,\gamma)^{1/p'} \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left([t]_1\, t^{-1}\right)^{-Q/p'} [t]_1^{\alpha-\frac{Q-\lambda}{p}} \\
&\times [1/t]_1^{Q-\beta-\frac{\mu}{p}} \left([t]_1^{-\lambda}\ [1/t]_1^{\mu} \int_{E_t} T^x|f(y)|^p(y')^\gamma dy\right)^{1/p} \\
&\le \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,(\lambda,\mu),\gamma}} \sup_{t>0} \left([t]_1\, t^{-1}\right)^{\frac{Q-\mu}{p}-\beta}\ [t]_1^{\alpha-\frac{Q-\lambda}{p}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sup_{t>0} \left([t]_1\, t^{-1}\right)^{\frac{Q-\mu}{p}-\beta} [t]_1^{\alpha-\frac{Q-\lambda}{p}} &= \max\left\{\sup_{0<t\le1} t^{\alpha-\frac{Q-\lambda}{p}}, \sup_{t>1} t^{\beta-\frac{Q-\mu}{p}}\right\} < \infty \\
&\Longleftrightarrow \frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}.
\end{aligned}$$

Therefore, $f \in L_{1,(Q-\alpha,Q-\beta),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,(\lambda,\mu),\gamma}}.$$

□

Corollary 3.3. [19, *Lemma* 6] *If* $0 \le \lambda < Q$ *and* $0 \le \alpha < Q-\lambda$, *then for* $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset_{\succ} \widetilde{L}_{1,Q-\alpha,\gamma}(\mathbb{R}^n_{k,+})$$

and for all $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *the following inequality*

$$\|f\|_{\widetilde{L}_{1,Q-\alpha,\gamma}} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$

is valid.

4 $L_{p,(\lambda,\mu),\gamma}$-boundedness of the B-maximal operator

In this section we will prove that the B-maximal operator M_γ is bounded on the total B-Morrey spaces $L_{p,(\lambda,\mu),\gamma}$. Let us begin by recalling that B-maximal operator M_γ is bounded on the B-Morrey spaces $L_{p,\lambda,\gamma}$.

Theorem 4.1. [12]
1) If $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$, $0 \le \lambda < Q$, then $M_\gamma f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|M_\gamma f\|_{L_{p,\lambda,\gamma}} \le C_{p,\lambda,\gamma}\|f\|_{L_{p,\lambda,\gamma}},$$

for some $C_{p,\lambda,\gamma} > 0$ depending only on p, λ, γ, k and n.
2) If $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $0 \le \lambda < Q$, then $M_\gamma f \in WL_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|M_\gamma f\|_{WL_{1,\lambda,\gamma}} \le C_{1,\lambda,\gamma}\|f\|_{L_{1,\lambda,\gamma}},$$

for some $C_{1,\lambda,\gamma} > 0$ depending only on λ, γ, k and n.

Applying Lemma 3.1 and Theorem 4.1, we obtain the following $L_{p,(\lambda,\mu),\gamma}$-boundedness of the B-maximal operator M_γ in the total B-Morrey spaces.

Theorem 4.2. *1) If $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$, $0 \le \mu \le \lambda < Q$, then $M_\gamma f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{L_{p,(\lambda,\mu),\gamma}} \le C_{p,\lambda,\gamma}\|f\|_{L_{p,(\lambda,\mu),\gamma}},$$

for some $C_{p,\lambda,\gamma} > 0$ depending only on p, λ, μ, γ, k and n.
2) If $f \in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $0 \le \mu \le \lambda < Q$, then $M_\gamma f \in WL_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|M_\gamma f\|_{WL_{1,(\lambda,\mu),\gamma}} \le C_{1,\lambda,\gamma}\|f\|_{L_{1,(\lambda,\mu),\gamma}},$$

for some $C_{1,\lambda,\gamma} > 0$ depending only on λ, μ, γ, k and n.

Proof. 1) Suppose that $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$ and $0 \le \mu \le \lambda < Q$. It is obvious that (see Lemma 3.1)

$$\|M_\gamma f\|_{L_{p,(\lambda,\mu),\gamma}} = \max\left\{\|M_\gamma f\|_{L_{p,\lambda,\gamma}}, \|M_\gamma f\|_{L_{p,\mu,\gamma}}\right\}.$$

Then, by the boundedness of M_γ on $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$ and $0 \le \mu \le \lambda < Q$ (see Theorem 4.1 and Lemma 3.1), we get

$$\|M_\gamma f\|_{L_{p,(\lambda,\mu),\gamma}} = \max\left\{C_{p,\lambda}, C_{p,\mu}\right\}\|f\|_{L_{p,\lambda,\gamma}}.$$

2) Suppose that $f \in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $0 \le \mu \le \lambda < Q$. It is obvious that (see Lemma 3.1)

$$\|M_\gamma f\|_{WL_{1,(\lambda,\mu),\gamma}} = \max\left\{\|M_\gamma f\|_{WL_{1,\lambda,\gamma}}, \|M_\gamma f\|_{WL_{1,\mu,\gamma}}\right\}.$$

Then, by the weak boundedness of M_γ on $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and $0 \le \mu \le \lambda < Q$ (see Theorem 4.1 and Lemma 3.1), we get

$$\|M_\gamma f\|_{WL_{1,(\lambda,\mu),\gamma}} = \max\left\{C_{1,\lambda}, C_{1,\mu}\right\}\|f\|_{L_{1,\lambda,\gamma}}.$$

□

5 Adams theorem for the B-Riesz potential in the total B-Morrey spaces

In this section we prove Adams theorem for the B-Riesz potential I_γ^α in the total B-Morrey spaces $L_{p,(\lambda,\mu),\gamma}$. Namely, we will obtain necessary and sufficient conditions on the numerical parameters for the B-Riesz potential I_γ^α to be bounded from one total B-Morrey space $L_{p,(\lambda,\mu),\gamma}$ to another one $L_{q,(\lambda,\mu),\gamma}$ and from the total B-Morrey space $L_{1,(\lambda,\mu),\gamma}$ to the weak total B-Morrey space $WL_{q,(\lambda,\mu),\gamma}$. These statements are the main results of our article.

Theorem 5.1. *Let $1 \le p < \infty$, $0 \le \mu \le \lambda < Q$, $0 < \alpha < \frac{Q-\lambda}{p}$.*

1) If $1 < p < \frac{Q-\lambda}{\alpha}$, then the condition $\frac{\alpha}{Q-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of the operator I_γ^α from $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p = 1 < \frac{Q-\lambda}{\alpha}$, then the condition $\frac{\alpha}{Q-\mu} \le 1 - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of the operator I_γ^α from $L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$.

Proof. 1) *Sufficiency.* Let $1 < p < \frac{Q-\lambda}{\alpha}$, $\frac{\alpha}{Q-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ and $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Then

$$I_\gamma^\alpha f(x) = \Big(\int_{E_t} + \int_{\mathbb{R}^n_{k,+}\setminus E_t}\Big) T^y f(x)|y|^{\alpha-Q}(y')^\gamma dy \equiv A(x,t) + C(x,t). \tag{5.1}$$

For $A(x,t)$ we have

$$|A(x,t)| \le \int_{E_t} T^y\,|f(x)|\,|y|^{\alpha-Q}(y')^\gamma dy \le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} \int_{E_{2^{j+1}t}\setminus E_{2^j t}} T^y\,|f(x)|\,(y')^\gamma dy$$
$$\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} |E_{2^{j+1}t}|_\gamma \; M_\gamma f(x) = t^\alpha\,\omega(n,k,\gamma)\,2^Q\,M_\gamma f(x)\sum_{j=1}^{\infty} 2^{-j\alpha}.$$

Hence,

$$|A(x,t)| \le C_3 t^\alpha M_\gamma f(x) \qquad \text{with} \qquad C_3 = \frac{\omega(n,k,\gamma)2^Q}{2^\alpha - 1}. \tag{5.2}$$

For $C(x,t)$ by Hölder's inequality we have

$$|C(x,t)| \le \Big(\int\limits_{\mathbb{R}^n_{k,+}\setminus E_t} |y|^{-\beta} T^y |f(x)|^p (y')^\gamma dy\Big)^{1/p} \Big(\int\limits_{\mathbb{R}^n_{k,+}\setminus E_t} |y|^{\left(\frac{\beta}{p}+\alpha-Q\right)p'} (y')^\gamma dy\Big)^{1/p'}$$
$$= J_1 \cdot J_2.$$

Let $\lambda < \beta < Q - \alpha p$. For J_1 we get

$$J_1 = \Big(\sum_{j=0}^{\infty}\int_{E_{2^{j+1}t}\setminus E_{2^j t}} T^y |f(x)|^p\;|y|^{-\beta}\;(y')^\gamma dy\Big)^{1/p}$$
$$\le 2^{\frac{\lambda}{p}}\,[t]_1^{\frac{\lambda-\beta}{p}}\;[1/t]_1^{-\frac{\mu-\beta}{p}}\;\|f\|_{L_{p,(\lambda,\mu),\gamma}} \Big(\sum_{j=0}^{\infty} 2^{(\lambda-\beta)j}\Big)^{1/p}$$
$$= C_4\,[t]_1^{\frac{\lambda-\beta}{p}}\;[1/t]_1^{-\frac{\mu-\beta}{p}}\;\|f\|_{L_{p,(\lambda,\mu),\gamma}},$$

where $C_4 = \left(\frac{2^\beta}{2^{\beta-\lambda}-1}\right)^{1/p}$.

For J_2 we obtain

$$J_2 = \left(\int_{\mathbb{S}^{n-1}_{k,+}} (\xi')^{\gamma} d\xi \int_t^{\infty} r^{Q-1+\left(\frac{\beta}{p}+\alpha-Q\right)p'} dr \right)^{\frac{1}{p'}} = C_5 t^{\frac{\beta}{p}+\alpha-\frac{Q}{p}},$$

where $C_5 = \left(\omega(n,k,\gamma) \left(Q + \left(\frac{\beta}{p} + \alpha - Q \right) p' \right)^{-1} \right)^{1/p'}$. Then

$$|C(x,t)| \le C_6 [t]_1^{\frac{\lambda-Q}{p}+\alpha} [1/t]_1^{-\frac{\mu-Q}{p}-\alpha} \|f\|_{L_{p,(\lambda,\mu),\gamma}}, \tag{5.3}$$

where $C_6 = C_4 \cdot C_5$. Thus, from (5.2) and (5.3) we have

$$\left| I_{\gamma}^{\alpha} f(x) \right| \lesssim t^{\alpha} M_{\gamma} f(x) + [t]_1^{\alpha-\frac{Q-\lambda}{p}} [1/t]_1^{-\alpha+\frac{Q-\mu}{p}} \|f\|_{L_{p,(\lambda,\mu),\gamma}}$$

for all $t > 0$. Taking

$$t = \left(\frac{\|f\|_{L_{p,(\lambda,\mu),\gamma}}}{M_{\gamma} f(x)} \right)^{\frac{p}{Q-\mu}} \quad \text{and} \quad t = \left(\frac{\|f\|_{L_{p,(\lambda,\mu),\gamma}}}{M_{\gamma} f(x)} \right)^{\frac{p}{Q-\lambda}}$$

we have

$$\begin{aligned} \left| I_{\gamma}^{\alpha} f(x) \right| &\lesssim \min \left\{ \left(M_{\gamma} f(x) \right)^{1-\frac{\alpha p}{Q-\mu}} \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{\frac{\alpha p}{Q-\mu}}, \left(M_{\gamma} f(x) \right)^{1-\frac{\alpha p}{Q-\lambda}} \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{\frac{\alpha p}{Q-\lambda}} \right\} \\ &= \min \left\{ \left(\frac{M_{\gamma} f(x)}{\|f\|_{L_{p,(\lambda,\mu),\gamma}}} \right)^{1-\frac{\alpha p}{Q-\mu}}, \left(\frac{M_{\gamma} f(x)}{\|f\|_{L_{p,(\lambda,\mu),\gamma}}} \right)^{1-\frac{\alpha p}{Q-\lambda}} \right\} \|f\|_{L_{p,(\lambda,\mu),\gamma}}, \end{aligned}$$

then

$$\left| I_{\gamma}^{\alpha} f(x) \right| \lesssim \left(\frac{M_{\gamma} f(x)}{\|f\|_{L_{p,(\lambda,\mu),\gamma}}} \right)^{\frac{p}{q}} \|f\|_{L_{p,(\lambda,\mu),\gamma}} = \left(M_{\gamma} f(x) \right)^{\frac{p}{q}} \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{p}{q}}. \tag{5.4}$$

Hence, by Theorem 4.2 and inequality (5.4), we get

$$\begin{aligned} \|I_{\gamma}^{\alpha} f\|_{L_{q,(\lambda,\mu),\gamma}} &= \sup_{x \in \mathbb{R}^n_{k,+}, t>0} \left([t]_1^{-\lambda} [1/t]_1^{\mu} \int_{E_t} T^x |I_{\gamma}^{\alpha} f(y)|^q (y')^{\gamma} dy \right)^{1/q} \\ &\lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{p}{q}} \sup_{x \in \mathbb{R}^n_{k,+}, t>0} \left([t]_1^{-\lambda} [1/t]_1^{\mu} \int_{E_t} T^x \left(\left(M_{\gamma} f(y) \right)^p \right) (y')^{\gamma} dy \right)^{\frac{1}{q}} \\ &= \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{p}{q}} \| \left(M_{\gamma} f \right)^{\frac{p}{q}} \|_{L_{q,(\lambda,\mu),\gamma}} \\ &\lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{p}{q}} \|M_{\gamma} f\|_{L_{p,(\lambda,\mu),\gamma}}^{\frac{p}{q}} \\ &\lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}} \end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned} \|I_{\gamma}^{\alpha} f\|_{WL_{q,(\lambda,\mu),\gamma}} &= \sup_{r>0} \sup_{x \in \mathbb{R}^n_{k,+}, t>0} \left([t]_1^{-\lambda} [1/t]_1^{\mu} \int_{\{y \in E_t : T^x |I_{\gamma}^{\alpha} f(y)|^q > r\}} (y')^{\gamma} dy \right)^{1/q} \\ &\lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{1}{q}} \sup_{r>0} \sup_{x \in \mathbb{R}^n_{k,+}, t>0} \left([t]_1^{-\lambda} [1/t]_1^{\mu} \int_{\{y \in E_t : T^x \left(M_{\gamma} f(y) \right) > r\}} (y')^{\gamma} dy \right)^{\frac{1}{q}} \\ &= \|f\|_{L_{1,(\lambda,\mu),\gamma}}^{1-\frac{1}{q}} \| \left(M_{\gamma} f \right)^{\frac{1}{q}} \|_{WL_{q,(\lambda,\mu),\gamma}} \\ &\lesssim \|f\|_{L_{1,(\lambda,\mu),\gamma}}^{1-\frac{1}{q}} \|M_{\gamma} f\|_{WL_{1,(\lambda,\mu),\gamma}}^{\frac{1}{q}} \\ &\lesssim \|f\|_{L_{1,(\lambda,\mu),\gamma}} \end{aligned}$$

if $p = 1 < q < \infty$.

Therefore, for $1 < p < q < \infty$ we have $I_\gamma^\alpha f \in L_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}} \lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}},$$

also for $p = 1 < q < \infty$ we have $I_\gamma^\alpha f \in WL_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\left\|I_\gamma^\alpha f\right\|_{WL_{q,(\lambda,\mu),\gamma}} \lesssim \|f\|_{L_{1,(\lambda,\mu),\gamma}}.$$

Necessity. Let $1 < p < \frac{Q-\lambda}{\alpha}$, $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and I_γ^α be bounded from $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Define $f_t(x) =: f(tx)$. Then we have

$$\begin{aligned}\left\|f_t\right\|_{L_{p,(\lambda,\mu),\gamma}} &= t^{-\frac{Q}{p}} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda}\,[1/r]_1^{\mu} \int_{E_{tr}} T^y|f(tx)|^p\,(y')^\gamma dy\right)^{1/p} \\ &= [t]_1^{-\frac{Q-\lambda}{p}}\,[1/t]_1^{\frac{Q-\mu}{p}}\,\|f\|_{L_{p,(\lambda,\mu),\gamma}}\end{aligned}$$

and

$$I_\gamma^\alpha f_t(x) = t^{-\alpha} I_\gamma^\alpha f(tx),$$

$$\begin{aligned}\left\|I_\gamma^\alpha f_t\right\|_{L_{q,(\lambda,\mu),\gamma}} &= t^{-\alpha} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda}\,[1/r]_1^{\mu} \int_{E_r} T^{ty}\left|I_\gamma^\alpha f(tx)\right|^q (y')^\gamma dy\right)^{1/q} \\ = [t]_1^{-\alpha-\frac{Q}{q}}\,[1/t]_1^{\alpha+\frac{Q}{q}} &\sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda}\,[1/r]_1^{\mu} \int_{E_{tr}} T^y\left|I_\gamma^\alpha f(x)\right|^q (y')^\gamma dy\right)^{1/q} \\ = [t]_1^{-\alpha-\frac{Q-\lambda}{q}}\,[1/t]_1^{\alpha+\frac{Q-\mu}{q}} &\left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}}.\end{aligned}$$

By the boundedness I_γ^α from $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ we get

$$\left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}} \le C_{p,q,\lambda,\gamma}\,[t]_1^{\alpha+\frac{Q-\lambda}{q}-\frac{Q-\lambda}{p}}\,[1/t]_1^{-\alpha-\frac{Q-\mu}{q}+\frac{Q-\mu}{p}}\,\|f\|_{L_{p,(\lambda,\mu),\gamma}},$$

where $C_{p,q,\lambda,\gamma} > 0$ depends only on p,q,λ,γ,k and n.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then by letting $t \to 0$ we have $\left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}} = 0$ for all $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Similarly, if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{Q-\mu}$, then by letting $t \to \infty$ we obtain $\left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}} = 0$ for all $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Therefore, we have $\frac{\alpha}{Q-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$.

Let $p = 1 < \frac{Q-\lambda}{\alpha}$, $f \in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and I_γ^α be bounded from $L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. We have

$$\begin{aligned}\left\|f_t\right\|_{L_{1,(\lambda,\mu),\gamma}} &= t^{-Q} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda}\,[1/r]_1^{\mu} \int_{E_{tr}} T^y|f(tx)|\,(y')^\gamma dy\right) \\ &= [t]_1^{-Q+\lambda}\,[1/t]_1^{Q-\mu}\,\|f\|_{L_{1,(\lambda,\mu),\gamma}}\end{aligned}$$

and

$$\left\|I_\gamma^\alpha f_t\right\|_{WL_{q,(\lambda,\mu),\gamma}} = \sup_{r>0} r \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left([\tau]_1^{-\lambda}\,[1/\tau]_1^{\mu} \int_{\{y\in E_\tau \,:\, T^y|I_\gamma^\alpha f_t(x)|>r\}} (y')^\gamma dy\right)^{1/q}$$

$$= t^{-\alpha} \sup_{r>0} rt^{\alpha} \sup_{x\in\mathbb{R}^n_{k,+},\tau>0} \Big([\tau]_1^{-\lambda} [1/\tau]_1^{\mu} \int_{\{y\in E_\tau \ : \ T^{ty}|I^\alpha_\gamma f(tx)|>rt^\alpha\}} (y')^\gamma dy \Big)^{1/q}$$

$$= [t]_1^{-\alpha-\frac{Q}{q}} [1/t]_1^{\alpha+\frac{Q}{q}} \sup_{r>0} rt^{\alpha} \sup_{x\in\mathbb{R}^n_{k,+},\tau>0} \Big([\tau]_1^{-\lambda} [1/\tau]_1^{\mu} \int_{\{y\in E_{t\tau} \ : \ T^{y}|I^\alpha_\gamma f(x)|>rt^\alpha\}} (y')^\gamma dy \Big)^{1/q}$$

$$= [t]_1^{-\alpha-\frac{Q-\lambda}{q}} [1/t]_1^{\alpha+\frac{Q-\mu}{q}} \left\|I^\alpha_\gamma f\right\|_{WL_{q,(\lambda,\mu),\gamma}}.$$

By the boundedness I^α_γ from $L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ it follows that

$$\left\|I^\alpha_\gamma f\right\|_{WL_{q,(\lambda,\mu),\gamma}} \le C_{1,q,\lambda,\gamma} [t]_1^{\alpha+\frac{Q-\lambda}{q}-(Q-\lambda)} [1/t]_1^{-\alpha-\frac{Q-\mu}{q}+Q-\mu} \|f\|_{L_{1,(\lambda,\mu),\gamma}},$$

where $C_{1,q,\lambda,\gamma}>0$ depends only on q,λ,γ,k and n.

If $1<\frac{1}{q}+\frac{\alpha}{Q-\mu}$, then by passing to the limit as $t\to 0$ we have $\left\|I^\alpha_\gamma f\right\|_{WL_{q,(\lambda,\mu),\gamma}}=0$ for all $f\in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$.

Similarly, if $1>\frac{1}{q}+\frac{\alpha}{Q-\lambda}$, then by passing to the limit as $t\to\infty$ we obtain $\left\|I^\alpha_\gamma f\right\|_{WL_{q,(\lambda,\mu),\gamma}}=0$ for all $f\in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Therefore, we have $\frac{\alpha}{Q-\mu}\le 1-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$. □

From Theorem 5.1 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

Corollary 5.1. [12] *Let $0<\alpha<Q$, $0\le\lambda<Q-\alpha$ and $1\le p<\frac{Q-\lambda}{\alpha}$.*

1) If $1<p<\frac{Q-\lambda}{\alpha}$, then the condition $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness I^α_γ from $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p=1$, then the condition $1-\frac{1}{q}=\frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness I^α_γ from $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Corollary 5.2. [19] *Let $0<\alpha<Q$, $0\le\lambda<Q-\alpha$ and $1\le p<\frac{Q-\lambda}{\alpha}$.*

1) If $1<p<\frac{Q-\lambda}{\alpha}$, then the condition $\frac{\alpha}{Q}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness I^α_γ from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p=1<\frac{Q-\lambda}{\alpha}$, then the condition $\frac{\alpha}{Q}\le 1-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness I^α_γ from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

6 Conclusion

In this paper, necessary and sufficient conditions on the numerical parameters are found ensuring that the B-Riesz potential I^α_γ is bounded from the total B-Morrey space $L_{p,(\lambda,\mu),\gamma}$ to the total B-Morrey space $L_{q,(\lambda,\mu),\gamma}$ and from the total B-Morrey space $L_{1,(\lambda,\mu),\gamma}$ to the weak total B-Morrey space $WL_{q,(\lambda,\mu),\gamma}$.

Acknowledgments

The authors thank the referee(s) for careful reading the paper and useful comments. The research of V. Guliyev was supported by the RUDN University Strategic Academic Leadership Programme.

References

[1] G.A. Abasova, M.N. Omarova, *Commutator of anisotropic maximal function with BMO functions on total anisotropic Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics 43 (2023), no. 1, 3–15.

[2] D.R. Adams, *A note on Riesz potentials*, Duke Math. J. 42 (1975), no. 4, 765–778.

[3] A. Akbulut, M. Dziri, I. Ekincioglu, *Maximal function and fractional integral associated with Laplace-Bessel differential operators on generalized Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics 42 (2022), no. 1, 8–25.

[4] A. Akbulut, S. Celik, M.N. Omarova, *Fractional maximal operator associated with Schrödinger operator and its commutators on vanishing generalized Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics 44 (2024), no. 1, 3–19.

[5] I.A. Aliev, S. Bayrakci, *On inversion of B-elliptic potentials by the method of Balakrishnan-Rubin*, Fract. Calc. Appl. Anal. 1 (1998), no. 4, 365–384.

[6] S. Bayrakci, *Commutators of the B-maximal operator and B-maximal commutators*, Filomat 34 (2020), no. 2, 499–505.

[7] V.I. Burenkov, D.J. Joseph, *Inequalities for trigonometric polynomials in periodic Morrey spaces.*, Eurasian Math. J. 15 (2024), no. 2, 92-100.

[8] V.I. Burenkov, M.A. Senouci, *Boundedness of the generalized Riemann-Liouville operator in local Morrey-type spaces with mixed quasi-norms.* Eurasian Math. J. 16 (2025), no. 4, 23-32.

[9] A.D. Gadjiev, I.A. Aliev, *On classes of operators of potential types, generated by a generalized shift*, (in Russian) Reports of enlarged Session of the Seminars of I.N.Vekua Inst. of Applied Mathematics, Tbilisi 3 (1988), no. 2, 21–24.

[10] V.S. Guliev, *Sobolev theorems for the Riesz B-potentials*, (in Russian) Doklady Academy Nauk Russia 358 (1998), no. 4, 450–451.

[11] V.S. Guliyev, *On maximal function and fractional integral, associated with the Bessel differential operator*, Math. Ineq. Appl. 6 (2003), no. 2, 317–330.

[12] V.S. Guliyev, J.J. Hasanov, *Necessary and sufficient conditions for the boundedness of B-Riesz potential in the B-Morrey spaces*, J. Math. Anal. Appl. 347 (2008), 113–122.

[13] V.S. Guliyev, A. Serbetci, I. Ekincioglu, *Necessary and sufficient conditions for the boundedness of rough B-fractional integral operators in the Lorentz spaces*, J. Math. Anal. Appl. 336 (2007), no. 1, 425–437.

[14] V.S. Guliyev, J.J. Hasanov, Y. Zeren, *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal. 5 (2011), no. 4, 491–506.

[15] V.S. Guliyev, R.V. Guliyev, M.N. Omarova, M.A. Ragusa, *Schrödinger type operators on local generalized Morrey spaces related to certain nonnegative potentials*, Discrete and Continuous Dynamical Systems - Series B 25 (2020), no. 2, 671–690.

[16] V.S. Guliyev, *Maximal commutator and commutator of maximal function on total Morrey spaces*, J. Math. Inequal. 16 (2022), no. 4, 1509–1524.

[17] V.S. Guliyev, *Characterizations of Lipschitz functions via the commutators of maximal function in total Morrey spaces*, Math. Meth. Appl. Sci. 47 (2024), no. 11, 8669–8682.

[18] V.S. Guliyev, *Hardy-Littlewood-Sobolev inequality in total Morrey spaces on stratified Lie groups*, Appl. Comput. Math. 25 (2026), no. 4, 180-196.

[19] J.J. Hasanov, X.A. Badalov, A.E. Fleydanli, *Necessary and sufficient conditions for the boundedness of B-Riesz potential in modified B-Morrey spaces*, Journal of Contemporary Applied Mathematics 4 (2014), no. 2, 18–41.

[20] I.A. Kipriyanov, *Fourier-Bessel transformations and imbedding theorems*, Trudy Math. Inst. Steklov 89 (1967), 130–213.

[21] B.M. Levitan, *Expansion in Fourier series and integrals with Bessel functions*, (in Russian) Uspehi Matem. Nauk (N.S.) 6 (1951), no. 2(42), 102–143.

[22] L.N. Lyakhov, *Multipliers of the mixed Fourier-Bessel transform*, (in Russian) Proc. Steklov Inst. Math. 214 (1997), no. 3, 234–249.

[23] Y.Y. Mammadov, V.S. Guliyev, F.A. Muslimova, *Commutator of the maximal function in total Morrey spaces for the Dunkl operator on the real line*, Azerb. J. Math. 15 (2025), no. 2, 85-104.

[24] Y.Y. Mammadov, V.S. Guliyev, A. Akbulut, M.N. Omarova, *Commutator of the maximal function in total Morrey spaces in the Dunkl setting*, TWMS Jour. Pure Appl. Math. 16 (2026), no. 1, 1–20.

[25] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 43 (1938), 126–166.

[26] B. Muckenhoupt, E.M. Stein, *Classical expansions and their relation to conjugate harmonic functions*, Trans. Amer. Math. Soc. 118 (1965), 17–92.

[27] F.A. Muslimova, *Some embeddings into the total Morrey spaces associated with the Dunkl operator on* $\mathbb{R}^d$, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics 43 (2023), no. 1, 94–102.

[28] F.A. Muslimova, *Boundedness of the fractional maximal commutator in total Morrey-Guliyev spaces for the Dunkl operator on the real line*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics 46 (2026), no. 1, 1–8.

[29] Y. Sawano, G.Di. Fazio, D.I. Hakim, *Morrey spaces - introduction and applications to integral operators and PDE's*, Vol. I. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2020.

[30] A. Serbetci, I. Ekincioglu, *On boundedness of Riesz potential generated by generalized shift operator on Ba spaces*, Czech. Math. J. 54 (2004), no. 3, 579–589.

[31] E.L. Shishkina, *On the boundedness of hyperbolic Riesz B-potential*, Lith. Math. J. 56 (2016), no. 4, 540–551.

[32] K. Stempak, *Almost everywhere summability of Laguerre series*, Studia Math. 100 (1991), no. 2, 129–147.

[33] K. Trimeche, *Inversion of the Lions transmutation operators using generalized wavelets*, Applied and Computational Harmonic Analysis 4 (1997), 97-112.

Vagif Sabir Guliyev
Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan,
Department of Mathematics, Kirsehir Ahi Evran University, Kirsehir, Turkey,
Azerbaijan University of Architecture and Construction, Baku, Azerbaijan, and
Peoples Friendship University of Russia (RUDN University), Moscow, Russian Federation
E-mail: vagif@guliyev.com
https://orcid.org/0000-0001-7486-0298

Ali Akbulut
Department of Mathematics, Kirsehir Ahi Evran University, Kirsehir, Turkey
E-mail: aakbulut@ahievran.edu.tr
https://orcid.org/0000-0002-1435-071X

Mehriban Nazim Omarova
Department of Mathematics and Mechanics, Baku State University, Baku, Azerbaijan, and
Institute of Mathematicss, Ministry of Science and Educations of the Republic
of Azerbaijan, Baku, Azerbaijan
E-mail: mehribanomarova@yahoo.com
https://orcid.org/0009-0008-2301-3273

Ayhan Serbetci
Department of Mathematics, Ankara University, Ankara, Turkey
E-mail: Ayhan.Serbetci@ankara.edu.tr
https://orcid.org/0000-0001-6362-7044

Received: 31.08.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 17, Number 1 (2026), 47 – 57

ANISOTROPIC MORREY-TYPE SPACES AND THEIR INTERPOLATION PROPERTIES

J.G. Jumabayeva, E.D. Nursultanov

Communicated by R. Oinarov

Key words: anisotropic local Morrey-type spaces, anisotropic generalized Morrey-type spaces, embedding properties of anisotropic Morrey-type spaces, interpolation properties of anisotropic Morrey-type spaces.

AMS Mathematics Subject Classification: 46E30, 46B70, 46B20.

Abstract In this paper, there are defined the anisotropic local Morrey-type spaces $LM_{\bar{p},\bar{q}}^{\bar{\lambda}}$ and the anisotropic generalized Morrey-type spaces $M_{\bar{p},\bar{q}}^{\bar{\lambda}}$, where $\bar{p}$, $\bar{q}$, and $\bar{\lambda}$ are vectors. The spaces $LM_{\bar{p},\bar{q}}^{\bar{\lambda}}$ allow relaxation of the conditions on the parameter $\bar{\lambda}$, namely, the components of the given vector can take any real value, i.e., $-\infty < \lambda_i < \infty$, $i = \overline{1, d}$, in contrast to previously studied spaces. The embedding properties of the defined spaces are investigated. Additionally, an anisotropic interpolation method is considered, which allows the study of the interpolation properties of these spaces.

DOI: https://doi.org/10.32523/2077-9879-2026-17-1-47-57

1 Introduction

To study specific operators of analysis, such as the Riesz operator, the O'Neil operator, the convolution operator, and others, it is important to correctly choose the spaces in which various properties of these operators can be described. In recent decades, the Morrey spaces M_p^λ and their various generalizations [17, 7, 8, 1] have played an important role in analysis. At the same time, there are only few studies dedicated to the anisotropic Morrey-type spaces and local Morrey-type spaces, in which functions have different characteristics for each variable. This is because traditional methods are not always effective for such spaces. For example, the real interpolation method is not applicable to the anisotropic local Morrey spaces.

The classical Morrey space was introduced in the work of Morrey [17] in 1938 in connection with the study of the properties solutions to quasilinear elliptic differential equations.

Let $0 \le \lambda \le \frac{d}{p}$ and $0 < p \le \infty$. The Morrey space $M_p^\lambda(\mathbb{R}^d)$ is the set of all Lebesgue measurable functions $f \in L_p^{loc}(\mathbb{R}^d)$ for which the following quantity is finite:

$$\|f\|_{M_p^\lambda} \equiv \|f\|_{M_p^\lambda(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B_r(x))}.$$

Here, $B_r(x)$ is the ball centered at point x with radius $r > 0$. Note that if $\lambda = 0$, then $M_p^0(\mathbb{R}^d) = L_p(\mathbb{R}^d)$; if $\lambda = \frac{d}{p}$ and $0 < p < \infty$, then $M_p^{\frac{d}{p}}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$; and if $\lambda < 0$ or $\lambda > \frac{d}{p}$, then $M_p^\lambda = \Theta$, where Θ is the set of all functions equivalent to zero on $\mathbb{R}^d$.

The question of interpolation in classical Morrey spaces was addressed in the work of Stampacchia [21] in 1964, Campanato and Murthy [16] in 1965, as well as in the work of Peetre [20] in 1969.

Peetre's studies led to the conclusion that $\left(M_p^{\lambda_0}, M_p^{\lambda_1}\right)_{\theta,\infty} \hookrightarrow M_p^{\lambda}$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, $0 < \theta < 1$. Note that this embedding is strict. From the work of Blasco, Ruiz, and Vega [6], it follows that $\left(M_{p_0}^{\lambda_0}, M_{p_1}^{\lambda_1}\right)_{\theta,\infty} \neq M_p^{\lambda}$.

In the work of Burenkov V.I. and Guliyev H.V. [10], the local Morrey-type spaces were introduced

$$LM_{p,q,x}^{\lambda} = \left\{ f : \left(\int_0^\infty \left(t^{-\lambda} \|f\|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

where $\lambda > 0$. The interpolation properties were studied by Burenkov V.I. and Nursultanov E.D. in works [9]-[14]. In particular, it was established in [9], [12] that the scale of the local Morrey-type spaces $LM_{p,\theta}^{\lambda} \equiv LM_{p,\theta,0}^{\lambda}$ is closed under interpolation with respect to the upper parameter, that is, the following equality holds:

$$\left(LM_{p,q}^{\lambda_0}, LM_{p,q}^{\lambda_1}\right)_{\theta,\infty} = LM_{p,q}^{\lambda},$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, $0 < \theta < 1$.

In the present work, the anisotropic local Morrey-type spaces and anisotropic generalized Morrey-type spaces are defined and their properties are investigated. The apparatus for studying these spaces, namely, the anisotropic interpolation method was developed in the works of Nursultanov E.D. and Bekmaganbetov K.A. [18] and [4].

2 Anisotropic local Morrey-type spaces $LM_{\bar{p},\bar{q}}^{\bar{\lambda}}(\mathbb{T})$

In the work of Nursultanov E.D. and Suragan D. [19], the interpolation properties of the following local Morrey-type spaces were studied.

Let $k \in \mathbb{Z}$, and let G_k denote the set of all cubes in $\mathbb{R}^d$ of the form

$$[0, 2^k)^d + 2^k m, \quad m \in \mathbb{Z}^d. \tag{2.1}$$

It is obvious that

$$\mathbb{R}^d = \bigsqcup_{Q \in G_k} Q, \tag{2.2}$$

where $\bigsqcup Q$ denotes the union of mutually disjoint sets.

The set $\mathbb{G} = \bigcup_{k\in\mathbb{Z}} G_k$ is called the family of dyadic cubes in $\mathbb{R}^d$. Note that each cube $Q \in G_k$ is subdivided into 2^d cubes from G_{k-1}.

Let μ be the d-dimensional Lebesgue measure in $\mathbb{R}^d$. The family of mutually disjoint cubes $\mathbb{T} = \{Q\} \subset \mathbb{G}$ is called a local decomposition of the space $\mathbb{R}^d$ if:

1. $\mu\left(\mathbb{R}^d \setminus \bigsqcup_{Q\in G_k} Q\right) = 0$;
2. $|\mathbb{T} \cap G_k| < \infty$.

Here and in the sequel, $|A|$ denotes the number of elements in the set A.

Let $\lambda \in \mathbb{R}$, $0 < p, q \leq \infty$, and $\mathbb{T}$ be a local decomposition of $\mathbb{R}^d$. The local Morrey-type space $LM_{p,q}^{\lambda}(\mathbb{T})$ is defined as the set of all measurable functions f, for which

$$\|f\|_{LM_{p,q}^{\lambda}(\mathbb{T})} = \left(\sum_{k\in\mathbb{Z}} \left(2^{-k\lambda} \sum_{Q \in \mathbb{T}_k = \mathbb{T}\cap G_k} \|f\|_{L_p(Q)} \right)^q \right)^{\frac{1}{q}} < \infty.$$

Now we define the anisotropic local Morrey-type spaces.

Let $\overline{n} = (n_1, \dots, n_d)$ where $n_i \in \mathbb{N}$, $|\overline{n}| = n_1 + \dots + n_d$, and let $\overline{k} = (k_1, \dots, k_d)$ where $k_i \in \mathbb{Z}$. Denote $G_{\overline{k}} = \{Q = Q_1 \times \dots \times Q_d : Q_i \subset G_{k_i}, i = 1, \dots, d\}$. The family of all mutually non-intersecting cubes $\mathbb{T}_i = \{Q_i\} \subset G_{k_i}$ is called a local decomposition of the space $\mathbb{R}^{n_i}$, and the families $\mathbb{T}_1, \dots, \mathbb{T}_d$ are local decompositions of the spaces $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_d}$, respectively. The family of all mutually disjoint parallelepipeds $\mathbb{T} = \mathbb{T}_1 \times \dots \times \mathbb{T}_d = \{Q = Q_1 \times \dots \times Q_d : Q_i \subset \mathbb{T}_i, i = 1, \dots, d\}$ is called, respectively, a local decomposition of the space $\mathbb{R}^{|\overline{n}|}$.

Let $\overline{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$, vectors $\overline{p} = (p_1, \dots, p_d)$ and $\overline{q} = (q_1, \dots, q_d)$ be such that $0 < p_i, q_i \le \infty$, $i = \overline{1, d}$, and $\mathbb{T}_{\overline{k}} = \mathbb{T} \cap G_{\overline{k}}$.

For arbitrary vectors $\overline{a} = (a_1, \dots, a_d)$ and $\overline{b} = (b_1, \dots, b_d)$, let $\langle \overline{a}, \overline{b} \rangle$ denote $\langle \overline{a}, \overline{b} \rangle = a_1 b_1 + \dots + a_d b_d$.

We define the anisotropic local Morrey-type space $LM^{\overline{\lambda}}_{\overline{p},\overline{q}}(\mathbb{T})$ as the set of all measurable functions f, for which

$$\|f\|_{LM^{\overline{\lambda}}_{\overline{p},\overline{q}}(\mathbb{T})} = \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\langle \overline{k}, \overline{\lambda} \rangle} \sum_{Q \in \mathbb{T}_{\overline{k}}} \|f\|_{L_{\overline{p}}(Q)} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}} < \infty. \tag{2.3}$$

In particular, in $q_i = \infty$ for $i = \overline{1, d}$, the expressions

$$\left(\int_{\Omega_d} \cdots \left(\int_{\Omega_1} |\varphi(\overline{t})|^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \cdots \frac{dt_d}{t_d} \right)^{\frac{1}{q_d}} \text{ and } \left(\sum_{k_d \in \Omega_d} \cdots \left(\sum_{k_1 \in \Omega_1} |a_{\overline{k}}|^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}}$$

are understood as $\sup\limits_{\overline{t} \in \Omega} |\varphi(\overline{t})|$ and $\sup\limits_{\overline{k} \in \Omega} |a_{\overline{k}}|$, respectively, where $\Omega = \Omega_1 \times \dots \times \Omega_d$.

For the anisotropic local Morrey-type spaces the following lemma holds.

Lemma 2.1. *(i) For vectors $\overline{n} = (n_1, \dots, n_d)$, $\overline{p}_0 = (p_1^0, \dots, p_d^0)$, $\overline{p}_1 = (p_1^1, \dots, p_d^1)$, and $\overline{q} = (q_1, \dots, q_d)$ such that $0 < p_i^0 < p_i^1 < \infty$, $0 < q_i \le \infty$ for $i = 1, \dots, d$, we have*

$$LM^{\overline{\alpha}}_{\overline{p}_1,\overline{q}}(\mathbb{T}) \hookrightarrow LM^{\overline{\beta}}_{\overline{p}_0,\overline{q}}(\mathbb{T}),$$

where $\overline{\alpha} = (\alpha_1, \dots, \alpha_d)$ and $\overline{\beta} = (\beta_1, \dots, \beta_d)$ such that $\beta_i = \alpha_i - \frac{n_i}{p_i^1} + \frac{n_i}{p_i^0}$ for $i = 1, \dots, d$, and "$\hookrightarrow$" denotes the continuous embedding.

(ii) For vectors $\overline{p} = (p_1, \dots, p_d)$, $\overline{q}_0 = (q_1^0, \dots, q_d^0)$, $\overline{q}_1 = (q_1^1, \dots, q_d^1)$ such that $0 < q_i^0 < q_i^1 \le \infty$ for $i = 1, \dots, d$, we have

$$LM^{\overline{\lambda}}_{\overline{p},\overline{q}_0}(\mathbb{T}) \hookrightarrow LM^{\overline{\lambda}}_{\overline{p},\overline{q}_1}(\mathbb{T}).$$

Proof. Let us prove (i). Let $f \in LM^{\bar{\alpha}}_{\bar{p}_1,\bar{q}}$. Applying Hölder's inequality and taking into account that $|Q_i| = 2^{k_i n_i}$, we obtain

$$\|f\|_{LM^{\bar{\beta}}_{\bar{p}_0,\bar{q}}} \le \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\langle \bar{k}, \bar{\beta} \rangle} \sum_{Q \in \mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}_1}(Q)} \prod_{i=1}^{d} |Q_i|^{\frac{1}{p_i^0} - \frac{1}{p_i^1}} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}}$$

$$\le \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\sum\limits_{i=1}^{d} k_i (\beta_i - (\frac{n_i}{p_i^1} - \frac{n_i}{p_i^0}))} \sum_{Q \in \mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}_1}(Q)} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}} = \|f\|_{LM^{\bar{\alpha}}_{\bar{p}_1,\bar{q}}}.$$

Let us prove (ii). Let $f \in LM^{\bar{\lambda}}_{\bar{p},\bar{q}_0}$. Applying Jensen's inequality, we obtain

$$\|f\|_{LM^{\bar{\lambda}}_{\bar{p},\bar{q}_1}} = \left(\sum_{k_d\in\mathbb{Z}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \left(2^{-\langle\bar{k},\bar{\lambda}\rangle} \sum_{Q\in\mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}}(Q)}\right)^{q_1^1}\right)^{\frac{q_2^1}{q_1^1}} \cdots\right)^{\frac{1}{q_d^1}\frac{q_d^0}{q_d^0}}$$

$$\leq \left(\sum_{k_d\in\mathbb{Z}} \left(\sum_{k_{d-1}\in\mathbb{Z}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \left(2^{-\langle\bar{k},\bar{\lambda}\rangle} \sum_{Q\in\mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}}(Q)}\right)^{q_1^1}\right)^{\frac{q_2^1}{q_1^1}} \cdots\right)^{\frac{q_{d-1}^0}{q_{d-1}^1}\frac{q_d^0}{q_{d-1}^0}}\right)^{\frac{1}{q_d^0}}$$

$$\cdots$$

$$\leq \left(\sum_{k_d\in\mathbb{Z}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \left(2^{-\langle\bar{k},\bar{\lambda}\rangle} \sum_{Q\in\mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}}(Q)}\right)^{q_1^0}\right)^{\frac{q_2^0}{q_1^0}} \cdots\right)^{\frac{1}{q_d^0}} = \|f\|_{LM^{\bar{\lambda}}_{\bar{p},\bar{q}_0}}.$$

□

Note that in [11] periodic Morrey spaces were studied, while in [15] local Morrey-type spaces with mixed quasi-norms were considered.

Lemma 2.2. [Hardy's inequality] *For $\alpha > 0$, $0 < q, h \leq \infty$, and $d > 1$, the following inequalities hold:*

$$\left(\sum_{k=0}^{\infty} \left(d^{-\alpha k} \left(\sum_{r=0}^{k} |b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \leq c_{\alpha,q} \left(\sum_{k=0}^{\infty} \left(d^{-\alpha k}|b_k|\right)^q\right)^{\frac{1}{q}},$$

$$\left(\sum_{k=0}^{\infty} \left(d^{\alpha k} \left(\sum_{r=k}^{\infty} |b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \leq c_{\alpha,q} \left(\sum_{k=0}^{\infty} \left(d^{\alpha k}|b_k|\right)^q\right)^{\frac{1}{q}}$$

for some $c_{\alpha,q} > 0$, depending only on α and q.

3 Interpolation of anisotropic local Morrey-type spaces $LM^{\overline{\lambda}}_{\overline{p},\overline{q}}(\mathbb{T})$

Let $\overline{A}_0 = (A_1^0, ..., A_d^0)$, $\overline{A}_1 = (A_1^1, ..., A_d^1)$ be a pair of anisotropic spaces. $E = \{\varepsilon = (\varepsilon_1, ..., \varepsilon_d) : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1,\ i = 1, ..., d\}$ be the vertices of the d-dimensional unit cube. For an arbitrary $\varepsilon \in E$, consider the space $A_\varepsilon = (A_1^{\varepsilon_1}, ..., A_d^{\varepsilon_d})$ with the norm

$$\|a_\varepsilon\|_{A_\varepsilon} = \| \ldots \|a\|_{A_1^{\varepsilon}} \ldots \|_{A_d^{\varepsilon_d}}. \tag{3.1}$$

Let us define the method of anisotropic interpolation. Let $A = (\overline{A}_0, \overline{A}_1)$ be a compatible pair of Banach spaces [5]. For an arbitrary vector $\overline{k} \in \mathbb{Z}^d$ we denote $2^{\overline{k}} = (2^{k_1}, ..., 2^{k_d})$. Let us consider the K–functional

$$K(2^{\bar{k}}, a; \bar{A}_0, \bar{A}_1) = \inf \left\{ \sum_{\varepsilon \in E} 2^{\langle \bar{k}, \varepsilon \rangle} \|a_\varepsilon\|_{A_\varepsilon} : \ a = \sum_{\varepsilon \in E} a_\varepsilon, \ a_\varepsilon \in A_\varepsilon \right\}.$$

If vectors $\overline{q} = (q_1, ..., q_d)$, $\overline{\theta} = (\theta_1, ..., \theta_d)$ are such that $0 < q_i < \infty$, $0 < \theta_i < 1$, then

$$A_{\overline{\theta}, \overline{q}} = (\overline{A}_0, \overline{A}_1)_{\overline{\theta}, \overline{q}} = \left\{ a = \sum_{\varepsilon \in E} a_\varepsilon, \ a_\varepsilon \in A_\varepsilon : \right.$$

$$\left. \|a\|_{A_{\overline{\theta}, \overline{q}}} = \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\langle \overline{\theta}, \overline{k} \rangle} K(2^{\overline{k}}, a) \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}} < \infty \right\}$$

and for $\overline{q} = \overline{\infty}$

$$A_{\overline{\theta}, \overline{\infty}} = (\overline{A}_0, \overline{A}_1)_{\overline{\theta}, \overline{\infty}} = \left\{ a = \sum_{\varepsilon \in E} a_\varepsilon, \ a_\varepsilon \in A_\varepsilon : \ \|a\|_{A_{\overline{\theta}, \overline{\infty}}} = \sup_{\overline{k} \in \mathbb{Z}^d} 2^{-\langle \overline{\theta}, \overline{k} \rangle} K(2^{\overline{k}}, a) < \infty \right\}.$$

For some vector $\overline{b} = (b_1, ..., b_d)$, $b_i > 1$, $i = \overline{1, d}$, the K-functional takes the form

$$K(b^{\bar{k}}, a; \bar{A}_0, \bar{A}_1) = \inf \left\{ \sum_{\varepsilon \in E} \prod_{i=1}^{d} b_i^{k_i \theta_i} \|a_\varepsilon\|_{A_\varepsilon} : \ a = \sum_{\varepsilon \in E} a_\varepsilon, \ a_\varepsilon \in A_\varepsilon \right\}$$

and

$$\|a\|_{A_{\overline{\theta}, \overline{q}}} = \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(\prod_{i=1}^{d} b_i^{-k_i \theta_i} K(\overline{b}^{\overline{k}}, a) \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}}.$$

Lemma 3.1. *(see [18]) Let $\{A_\varepsilon\}_{\varepsilon \in E}$ and $\{B_\varepsilon\}_{\varepsilon \in E}$ be Banach spaces that are subspaces of some linear space. Let vectors $\overline{\theta} = (\theta_1, ..., \theta_d)$ and $\overline{r} = (r_1, ..., r_d)$ be such that $0 < \theta_i < 1$, $0 < r_i \leq \infty$. If T is a linear operator such that $T: \ A_\varepsilon \to B_\varepsilon$ with the norm M_ε for any $\varepsilon \in E$ (here the spaces A_ε and B_ε are defined by norm (3.1)), then*

$$T: \quad A_{\overline{\theta}, \overline{r}} \to B_{\overline{\theta}, \overline{r}}$$

with the norm $\|T\| \leq \max_{\varepsilon \in E} M_\varepsilon$.

Here we may mention the papers [2] and [3].

Theorem 3.1. *Let vectors $\overline{\lambda}_0 = (\lambda_1^0, ..., \lambda_d^0)$, $\overline{\lambda}_1 = (\lambda_1^1, ..., \lambda_d^1)$, $\overline{\theta} = (\theta_1, ..., \theta_d)$, $\overline{p} = (p_1, ..., p_d)$, $\overline{q}_0 = (q_1^0, ..., q_d^0)$, $\overline{q}_1 = (q_1^1, ..., q_d^1)$, $\overline{q} = (q_1, ..., q_d)$, $\overline{n} = (n_1, ..., n_d)$ be such that $0 < q_i^0, q_i^1, q_i \leq \infty$, $0 < p_i \leq \infty$, $-\infty < \lambda_i^0 < \lambda_i^1 < +\infty$, $\theta_i \in (0,1)$, $n_i \in \mathbb{N}$, $i = \overline{1, d}$ and $\mathbb{T}$ is a local partition of $\mathbb{R}^{|\overline{n}|}$. Then*

$$\left(LM_{\overline{p}, \overline{q}_0}^{\overline{\lambda}_0}(\mathbb{T}), LM_{\overline{p}, \overline{q}_1}^{\overline{\lambda}_1}(\mathbb{T}) \right)_{\overline{\theta}, \overline{q}} = LM_{\overline{p}, \overline{q}}^{\overline{\lambda}}(\mathbb{T}),$$

where the vector $\bar{\lambda} = (\lambda_1, ..., \lambda_d)$ is such that $\lambda_i = (1 - \theta_i)\lambda_i^0 + \theta_i \lambda_i^1$, $i = \overline{1, d}$.

Proof. Let $r=\min\limits_{1\le i\le d}\{q_i,q_i^0,q_i^1\}$, $\overline{r}=(r,...,r)$. Since for any $\overline{\lambda}$ with coordinates $\lambda_i\in\mathbb{R}$, $i=\overline{1,d}$

$$LM_{\overline{p},\overline{r}}^{\overline{\lambda}}(\mathbb{T})\hookrightarrow LM_{\overline{p},\overline{q}}^{\overline{\lambda}}(\mathbb{T})\hookrightarrow LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}}(\mathbb{T}),$$

it suffices to prove

$$\left(LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}(\mathbb{T}),LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}(\mathbb{T})\right)_{\overline{\theta},\overline{q}}\hookrightarrow LM_{\overline{p},\overline{q}}^{\overline{\lambda}}(\mathbb{T})\hookrightarrow\left(LM_{\overline{p},\overline{r}}^{\overline{\lambda}_0}(\mathbb{T}),LM_{\overline{p},\overline{r}}^{\overline{\lambda}_1}(\mathbb{T})\right)_{\overline{\theta},\overline{q}}. \tag{3.2}$$

Let us first prove the left relation in (3.2). Let $f\in\left(LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}(\mathbb{T}),LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}(\mathbb{T})\right)_{\overline{\theta},\overline{q}}$, $\overline{m}\in\mathbb{Z}^d$, $\varepsilon=(\varepsilon_1,...,\varepsilon_d)\in E$. Let us denote $\overline{\lambda}_\varepsilon=(\lambda_1^{\varepsilon_1},...,\lambda_d^{\varepsilon_d})$, where $\lambda_i^{\varepsilon_i}=\begin{cases}\lambda_i^0, & \varepsilon_i=0,\\ \lambda_i^1, & \varepsilon_i=1.\end{cases}$ For an arbitrary representation $f=\sum\limits_{\varepsilon\in E}f_\varepsilon$, where $f_\varepsilon\in LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_\varepsilon}(\mathbb{T})$ we obtain

$$\sum_{Q\in\mathbb{T}_{\overline{m}}}\|f\|_{L_{\overline{p}}(Q)}\le\sum_{Q\in\mathbb{T}_{\overline{m}}}\sum_{\varepsilon\in E}\|f_\varepsilon\|_{L_{\overline{p}}(Q)}$$

$$=\sum_{Q\in\mathbb{T}_{\overline{m}}}\left(2^{\langle\overline{m},\overline{\lambda}_0\rangle}\sum_{\varepsilon\in E}2^{\langle\overline{m},\overline{\lambda}_\varepsilon-\overline{\lambda}_0\rangle}2^{-\langle\overline{m},\overline{\lambda}_\varepsilon\rangle}\|f_\varepsilon\|_{L_{\overline{p}}(Q)}\right)$$

$$\le2^{\langle\overline{m},\overline{\lambda}_0\rangle}\left(\sum_{\varepsilon\in E}2^{\langle\overline{m},\overline{\lambda}_\varepsilon-\overline{\lambda}_0\rangle}\|f_\varepsilon\|_{LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_\varepsilon}(\mathbb{T})}\right).$$

Taking into account the arbitrariness of the representation $f=\sum\limits_{\varepsilon\in E}f_\varepsilon$ and the definition of the K-functional, we obtain

$$\sum_{Q\in\mathbb{T}_{\overline{m}}}\|f\|_{L_{\overline{p}}(Q)}\le2^{\langle\overline{m},\overline{\lambda}_0\rangle}K(2^{\langle\overline{m},\overline{\lambda}_\varepsilon\rangle},f;LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_\varepsilon}).$$

Let $b_i=2^{\lambda_i^1-\lambda_i^0}$, $i=\overline{1,d}$, then from the last inequality we obtain

$$\|f\|_{LM_{\overline{p},\overline{q}}^{\overline{\lambda}}(\mathbb{T})}=\left(\sum_{k_d\in\mathbb{Z}}\sum_{b_d^{k_d}\le2^{m_d}<b_d^{k_d+1}}\cdots\left(\sum_{k_1\in\mathbb{Z}}\sum_{b_1^{k_1}\le2^{m_1}<b_1^{k_1+1}}\left(2^{-\langle\overline{m},\overline{\lambda}\rangle}\sum_{Q\in\mathbb{T}_{\overline{m}}}\|f\|_{L_{\overline{p}}(Q)}\right)^{q_1}\right)^{\frac{q_2}{q_1}}\cdots\right)^{\frac{1}{q_d}}$$

$$\le\left(\sum_{k_d\in\mathbb{Z}}\sum_{b_d^{k_d}\le2^{m_d}<b_d^{k_d+1}}\cdots\left(\sum_{k_1\in\mathbb{Z}}\sum_{b_1^{k_1}\le2^{m_1}<b_1^{k_1+1}}\left(2^{-\langle\overline{m},\overline{\lambda}-\overline{\lambda}_0\rangle}K(2^{\langle\overline{m},\overline{\lambda}_\varepsilon\rangle},f)\right)^{q_1}\right)^{\frac{q_2}{q_1}}\cdots\right)^{\frac{1}{q_d}}$$

$$\le c\left(\sum_{k_d\in\mathbb{Z}}\cdots\left(\sum_{k_1\in\mathbb{Z}}\left(\prod_{i=1}^d b_i^{-\theta_ik_i}K(b^{\overline{k}},f)\right)^{q_1}\right)^{\frac{q_2}{q_1}}\cdots\right)^{\frac{1}{q_d}}$$

$$\le c\|f\|_{\left(LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}(\mathbb{T}),LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}(\mathbb{T})\right)_{\overline{\theta},\overline{q}}},$$

here $c>0$ depends only on the parameters $\overline{\lambda_0}$, $\overline{\lambda_1}$. Therefore,

$$\left(LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0},LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}\right)_{\overline{\theta},\overline{q}}\hookrightarrow LM_{\overline{p},\overline{q}}^{\overline{\lambda}}.$$

Let us prove the second embedding in (3.2). Let $\overline{k} = (k_1, ..., k_d) : \ k_i \in \mathbb{Z}, \ b_i = 2^{\lambda_i^1 - \lambda_i^0} > 1, \ i = \overline{1, d}$, $f \in LM^{\overline{\lambda}}_{\overline{p},\overline{q}}(\mathbb{T})$. For an arbitrary $\varepsilon \in E$ we will consider the following intervals:

$$\Delta_{\varepsilon_i} = \begin{cases} (-\infty; k_i] \cap \mathbb{Z}, & \text{for } \varepsilon_i = 0, \\ [k_i + 1; +\infty) \cap \mathbb{Z}, & \text{for } \varepsilon_i = 1 \end{cases}$$

and $\Delta_\varepsilon = \Delta_{\varepsilon_1} \times \cdots \times \Delta_{\varepsilon_d}$. We define the functions f_ε as follows:

$$f_\varepsilon(\overline{x}) = \begin{cases} f(\overline{x}), & \text{for } \overline{x} \in \bigsqcup\limits_{\overline{m} \in \Delta\varepsilon} \bigsqcup\limits_{Q \in \mathbb{T}_{\overline{m}}} Q, \\ 0, & \text{for } \overline{x} \notin \bigsqcup\limits_{\overline{m} \in \Delta\varepsilon} \bigsqcup\limits_{Q \in \mathbb{T}_{\overline{m}}} Q. \end{cases}$$

Then, since

$$\|f_\varepsilon\|_{LM^{\overline{\lambda}_\varepsilon}_{\overline{p},\overline{r}}} = \left(\sum_{m_d \in \Delta_{\varepsilon_d}} \cdots \left(\sum_{m_1 \in \Delta_{\varepsilon_1}} \left(2^{-\langle \overline{m}, \overline{\lambda}_\varepsilon \rangle} \sum_{Q \in \mathbb{T}_{\overline{m}}} \|f\|_{L_{\overline{p}}(Q)} \right)^{r_1} \right)^{\frac{r_2}{r_1}} \cdots \right)^{\frac{1}{r_d}},$$

we get

$$\|f\|_{(LM^{\overline{\lambda}_0}_{\overline{p},\overline{r}}, LM^{\overline{\lambda}_1}_{\overline{p},\overline{r}})_{\overline{\theta}q}} \le c \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(\prod_{i=1}^d b_i^{-k_i \theta_i} K(\overline{b}^{\overline{k}}, f) \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}}$$

$$\le c \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(\prod_{i=1}^d b_i^{-k_i \theta_i} \sum_{\varepsilon \in E} \|f_\varepsilon\|_{LM^{\overline{\lambda}_\varepsilon}_{\overline{p},\overline{r}}} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}}$$

$$\le c \sum_{\varepsilon \in E} \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(\prod_{i=1}^d b_i^{(\varepsilon_i - \theta_i) k_i} \right. \right. \right.$$

$$\left. \left. \left. \times \left(\sum_{m_d \in \Delta_{\varepsilon_d}} \cdots \left(\sum_{m_1 \in \Delta_{\varepsilon_1}} \left(2^{-\langle \overline{m}, \overline{\lambda}_\varepsilon \rangle} \sum_{Q \in \mathbb{T}_{\overline{m}}} \|f\|_{L_{\overline{p}}(Q)} \right)^{r_1} \right)^{\frac{r_2}{r_1}} \cdots \right)^{\frac{1}{r_d}} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}}.$$

Taking into account, that $b_i = 2^{\lambda_i^1 - \lambda_i^0}, \ \lambda_i = (1-\theta_i)\lambda_i^0 + \theta_i \lambda_i^1 \ i = \overline{1, d}$, and applying for each parameter $r_i, \ i = \overline{1, d}$ the generalized Minkowski inequality and Hardy's inequality (Lemma 2.2), we obtain

$$\|f\|_{\left(LM^{\overline{\lambda}_0}_{\overline{p},\overline{r}}, LM^{\overline{\lambda}_1}_{\overline{p},\overline{r}}\right)_{\overline{\theta},\overline{q}}} \le c \|f\|_{LM^{\overline{\lambda}}_{\overline{p},\overline{q}}}.$$

Here, $c > 0$ depends only on the parameters $\overline{\lambda_0}, \ \overline{\lambda_1}, \ \overline{p}, \ \overline{q}$. □

4 Anisotropic generalized Morrey-type spaces $M^{\overline{\lambda}}_{\overline{p},\overline{q}}$

Let $\overline{p} = (p_1, ..., p_d), \ \overline{q} = (q_1, ..., q_d), \ \overline{\lambda} = (\lambda_1, ..., \lambda_d)$ be such that $0 < p_i \le \infty, \ 0 < q_i \le \infty, \ 0 < \lambda_i < \infty$. We define the anisotropic generalized Morrey-type spaces $M^{\overline{\lambda}}_{\overline{p},\overline{q}}$ as the set of all Lebesgue measurable functions $f \in L^{loc}_{\overline{p}}(\mathbb{R}^{|\overline{n}|})$, for which the following norm is finite

$$\|f\|_{M^{\overline{\lambda}}_{\overline{p},\overline{q}}} = \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\langle \overline{k}, \overline{\lambda} \rangle} \sup_{Q \in G_{\overline{k}}} \|f\|_{L_{\overline{p}}(Q)} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}} < \infty,$$

where $G_{\overline{k}} = \{Q = Q_1 \times \cdots \times Q_d : \ Q_i \subset G_{k_i}, \ i = \overline{1,d}\}$.

By the anisotropic Morrey space $M_{\overline{p}}^{\overline{\lambda}}$ we mean the set of all Lebesgue measurable functions $f \in L_{\overline{p}}^{loc}(\mathbb{R}^{|\overline{n}|})$, for which

$$\|f\|_{M_{\overline{p}}^{\overline{\lambda}}} = \sup_{\overline{k}\in\mathbb{Z}^d} \left(2^{-\langle \overline{k}\overline{\lambda}\rangle} \sup_{Q\in G_{\overline{k}}} \|f\|_{L_{\overline{p}}(Q)} \right) < \infty.$$

Note, that for $\overline{q} = \overline{\infty}$

$$M_{\overline{p},\overline{\infty}}^{\overline{\lambda}} = M_{\overline{p}}^{\overline{\lambda}}. \tag{4.1}$$

Let us present some properties of the introduced spaces.

Lemma 4.1. *(i) Let vectors $\overline{n} = (n_1, ..., n_d)$, $\overline{p}_0 = (p_1^0, \dots, p_d^0)$, $\overline{p}_1 = (p_1^1, \dots, p_d^1)$ and $\overline{q} = (q_1, \dots, q_d)$ be such that $0 < p_i^0 < p_i^1 < \infty, 0 < q_i \le \infty$, $i = \overline{1,d}$. Then*

$$M_{\overline{p}_1,\overline{q}}^{\overline{\lambda}_1} \hookrightarrow M_{\overline{p}_0,\overline{q}}^{\overline{\lambda}_0},$$

where $\overline{\lambda}_0 = (\lambda_1^0, \dots \lambda_d^0)$, $\overline{\lambda}_1 = (\lambda_1^1, \dots \lambda_d^1)$ are such that $\lambda_i^0 = \lambda_i^1 - \frac{n_i}{p_i^1} + \frac{n_i}{p_i^0}$, $i = \overline{1,d}$.

(ii) Let vectors $\overline{p} = (p_1, \dots, p_d)$, $\overline{q}_0 = (q_1^0, \dots, q_d^0)$, $\overline{q}_1 = (q_1^1, \dots, q_d^1)$ such that $0 < q_i^0 < q_i^1 \le \infty$, $i = \overline{1,d}$. Then

$$M_{\overline{p},\overline{q}_0}^{\overline{\lambda}} \hookrightarrow M_{\overline{p},\overline{q}_1}^{\overline{\lambda}}.$$

The proof is similar to the proof of Lemma 2.1.

Theorem 4.1. *Let vectors $\overline{p} = (p_1, \dots, p_d)$, $\overline{q} = (q_1, \dots, q_d)$, $\overline{q}_0 = (q_1^0, \dots, q_d^0)$, $\overline{q}_1 = (q_1^1, \dots, q_d^1)$, $\overline{\lambda}_0 = (\lambda_1^0, \dots, \lambda_d^0)$, $\overline{\lambda}_1 = (\lambda_1^1, \dots, \lambda_d^1)$, $\overline{\theta} = (\theta_1, \dots, \theta_d)$ be such that $\lambda_i^0 \ne \lambda_i^1$, $0 < p_i \le \infty$, $0 < q_i, q_i^0, q_i^1 \le \infty$, $\theta_i \in (0,1)$. Then*

$$\left(M_{\overline{p},\overline{q}_0}^{\overline{\lambda}_0}, M_{\overline{p},\overline{q}_1}^{\overline{\lambda}_1}\right)_{\overline{\theta},\overline{q}} \hookrightarrow M_{\overline{p},\overline{q}}^{\overline{\lambda}}, \tag{4.2}$$

where $\overline{\lambda} = (\lambda_1, \dots, \lambda_d) : \ \lambda_i = (1-\theta_i)\lambda_i^0 + \theta_i\lambda_i^1$, $i = \overline{1,d}$.

Proof. Let $f \in \left(M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}, M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}\right)_{\overline{\theta},\overline{q}}$.

For vectors $\overline{k} = (k_1, \dots, k_d)$, $\overline{\lambda} = (\lambda_1, \dots, \lambda_d)$ and $\varepsilon \in E$ we denote $\overline{\lambda}_\varepsilon = (\lambda_1^{\varepsilon_1}, \dots, \lambda_d^{\varepsilon_d}) : \ \lambda_i^{\varepsilon_i} = \begin{cases} \lambda_i^0, & \text{for } \varepsilon_i = 0, \\ \lambda_i^1, & \text{for } \varepsilon_i = 1. \end{cases}$

Let $f = \sum\limits_{\varepsilon\in E} f_\varepsilon$ be an arbitrary representation of the function f, where $f_\varepsilon \in M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_\varepsilon}$. Then we have

$$2^{-\langle \overline{k},\overline{\lambda}\rangle} \sup_{Q\in G_{\overline{k}}} \|f\|_{L_{\overline{p}}(Q)} \le 2^{-\langle \overline{k},(\overline{\lambda}-\overline{\lambda}_0)\rangle} \sum_{\varepsilon\in E} \left(2^{\langle \overline{k},\overline{\lambda}_\varepsilon\rangle} \sup_{Q\in G_{\overline{k}}} \|f_\varepsilon\|_{L_{\overline{p}}(Q)} \right).$$

Taking into account the arbitrariness of the representation $f = \sum\limits_{\varepsilon\in E} f_\varepsilon$, we obtain

$$2^{-\langle \overline{k},\overline{\lambda}\rangle} \sup_{Q\in G_{\overline{k}}} \|f\|_{L_{\overline{p}}(Q)} \le 2^{-\langle \overline{k},(\overline{\lambda}-\overline{\lambda}_0)\rangle} K(2^{\langle \overline{k},\overline{\lambda}_\varepsilon\rangle}, f; M_{\overline{p}\overline{\infty}}^{\overline{\lambda}_\varepsilon}).$$

Hence,

$$\|f\|_{M_{\overline{p}\overline{q}}^{\overline{\lambda}}} \le \left(\sum_{k_d\in\mathbb{Z}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \left(2^{-\langle \overline{k},(\overline{\lambda}-\overline{\lambda}_0)\rangle} K(2^{\overline{k}}, f) \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}} \asymp \|f\|_{\left(M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}, M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}\right)_{\overline{\theta},\overline{q}}}.$$

Thus, we have

$$\left(M^{\overline{\lambda}_0}_{\overline{p},\overline{q}_0}, M^{\overline{\lambda}_1}_{\overline{p},\overline{q}_1}\right)_{\overline{\theta},\overline{q}} \hookrightarrow \left(M^{\overline{\lambda}_0}_{\overline{p},\overline{\infty}}, M^{\overline{\lambda}_1}_{\overline{p},\overline{\infty}}\right)_{\overline{\theta},\overline{q}} \hookrightarrow M^{\overline{\lambda}}_{\overline{p},\overline{q}}.$$

□

Acknowledgments

This work was supported by the Ministry of Science and Higher Education of the Republic of Kazakhstan, grant no. AP23488613.

References

[1] D.R. Adams, *Morrey spaces.* Birkhäuser, 2015, 124 p.

[2] A.N. Bashirova, A.K. Kalidolday, E.D. Nursultanov, *Interpolation methods for anisotropic net spaces.*, Eurasian Math. J. 15 (2024), no. 2, 33-41.

[3] K.A. Bekmaganbetov, K.Ye. Kervenev, E.D. Nursultanov, *Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric: interpolation properties, embedding theorems, trace and extension theorems.*, Eurasian Math. J. 16 (2025), no. 2.

[4] K.A. Bekmaganbetov, E.D. Nursultanov, *Interpolation of Besov $B^{\sigma q}_{p\tau}$ and Lizorkin-Triebel $F^{\sigma q}_{p\tau}$ spaces.* Anal. Math. 35 (2009), 169-188.

[5] J. Bergh, J. Löfström, *Interpolation spaces. An introduction.* Springer. Berlin, 1976.

[6] O. Blasco, A. Ruiz, L. Vega, *Non interpolation in Morrey-Campanato and block spaces.* Ann. Scuola Norm. Super. Pisa 28 (1999), no. 1., 31-40.

[7] V.I. Burenkov, *Recent progress in studing the boundedness of classical operators of real analysis in general Morrey-type spaces I.* Eurasian Math. J. 3 (2012), no. 3, 11-32.

[8] V.I. Burenkov, *Recent progress in studing the boundedness of classical operators of real analysis in general Morrey-type spaces II.* Eurasian Math. J. 3 (2013), no. 1, 21-45.

[9] V.I. Burenkov, D.K. Chigambayeva, E.D. Nursultanov, *Marcinkiewicz-type interpolation theorem and estimates for convolutions for Morrey-type spaces.* Eurasian Math. J. 9 (2018), no. 2, 82-88.

[10] V.I. Burenkov, H.V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces.* Dokl. Ross. Akad. Nauk. Matematika 391, no. 5 (2003), 591-594 (in Russian). English transl. in Russian Acad. Sci. Dokl. Math. 67 (2003).

[11] V.I. Burenkov, D.J. Joseph, *Inequalities for trigonometric polynomials in periodic Morrey spaces.*, Eurasian Math. J. 15 (2024), no. 2, 92-100.

[12] V.I. Burenkov, E.D. Nursultanov, *Description of interpolation spaces for local Morrey-type spaces.* Proc. Steklov Inst. Math. 269 (2010), 46-56.

[13] V.I. Burenkov, E.D. Nursultanov, *Interpolation theorems for nonlinear operators in General Morrey-type spaces and their applications.* Proc. Steklov Inst. Math. 312 (2021), 124-149.

[14] V.I. Burenkov, E.D. Nursultanov, *Interpolation theorems for nonlinear urysohn integral operators in general Morrey-type spaces.* Eurasian Math. J. 11 (2020), no. 4, 87-94.

[15] V.I. Burenkov, M.A. Senouci, *Boundedness of the generalized Riemann-Liouville operator in local Morrey-type spaces with mixed quasi-norms.* Eurasian Math. J. 16 (2025), no. 4, 23-32.

[16] A. Campanato, M.K.V. Murthy, *Una generalizzazione del teorema di Riesz-Thorin.* Ann. Scuola Norm. Super. Pisa 19 (1965), 87-100.

[17] C.B. Morrey, *On the solution of quasi-linear elliptic partial differential equations.* Trans. Amer. Math. Soc. 43 (1938), 126-166.

[18] E.D. Nursultanov, *Interpolation theorems for anisotropic function spaces and their applications.* Doklady Mathematics 69 (2004), no. 1, 16-19.

[19] E.D. Nursultanov, D. Suragan, *On the convolution operator in Morrey spaces.* J. Math. Anal. Appl. 515 (2022), no. 1, Article 126357.

[20] J. Peetre, *On the theory of Lp spaces.* Journal Funct. Analysis. (1969), no. 4, 71-87.

[21] G. Stampacchia, *Spaces and interpolation.* Comm. Pure Appl. Math. 17 (1964), 293-306.

Jumabayeva Jamilya Gumarovna
Department of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
13 Kazhymukan St,
Z01C0X0 Astana, Republic of Kazakhstan
E-mail: jamilya_ast@mail.ru

Nursultanov Erlan Dautbekovich
Department of Fundamental and Applied Mathematics
M.V. Lomonosov Moscow State University, Kazakhstan branch
11 Kazhymukan St,
Z01C0T6 Astana, Republic of Kazakhstan
and
Geometry Limited Liability Partnership
49 Kabanbay Batyr Ave,
Z05H0H6 Astana, Republic of Kazakhstan
E-mail: er-nurs@yandex.kz

Received: 28.08.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 17, Number 1 (2026), 58 – 76

COMBINING UNSUPERVISED DIMENSION REDUCTION WITH SUFFICIENT DIMENSION REDUCTION

Zh. Mukanov, A. Sharafudinov, R. Takhanov, A. Bekembayev

Communicated by K.T. Mynbayev

Key words: unsupervised dimension reduction, sufficient dimension reduction, complex measures, hybrid setting.

AMS Mathematics Subject Classification: 62-07, 62H25, 68T10.

Abstract. We present a new method for dimension reduction that combines unsupervised dimension reduction (UDR) with sufficient dimension reduction (SDR). In unsupervised dimension reduction the goal is to find a low-dimensional linear subspace that approximates the support of a data distribution. If data is supervised, then in sufficient dimension reduction the goal is to find a low-dimensional linear subspace, called the effective subspace, such that the projection of an input vector onto that subspace maximally captures information on correlations between an input and an output.

The objective that we suggest to minimize consists of two parts. The first one is responsible for the UDR part, it forces a low-dimensional probabilistic measure μ to approximate a distribution over inputs. The second one is responsible for the SDR part, it forces a regression function f to be consistent with supervised data. Additionally, we require the support of μ and the effective subspace of f to be equal. In this hybrid setting we solve two problems, UDR and SDR, so that the UDR term serves as a regularizer of the SDR term.

We reformulate the problem as an optimization task of finding a k-dimensional linear subspace S and a pair of complex measures (μ, μ') supported in S. Instead of optimizing over complex measures, we suggest minimizing over ordinary functions (g_1, g_2) but with an additional term R that penalizes a distortion of the common support of g_1, g_2 from a k-dimensional linear subspace. The algorithm that we develop can be formulated for functions (g_1, g_2) as well as for their inverse Fourier transforms. Eventually, we report results of numerical experiments on well-known datasets.

DOI: https://doi.org/10.32523/2077-9879-2026-17-1-58-76

1 Introduction

Unsupervised dimension reduction (UDR) is a classical problem in data science that has many non-equivalent formulations coming from different contexts, such as principal component analysis [13], factor analysis, linear multidimensional scaling [10], Fisher's linear discriminant analysis [12], canonical correlations analysis [18], sufficient dimension reduction (SDR) [14], maximum autocorrelation factors [29], slow feature analysis [43], kernel methods [27, 14, 34], methods based on autoencoders [38, 3] and more.

In UDR we are given a finite number of points in $\mathbb{R}^n$ (sampled according to some unknown distribution) and our goal is to find a "low-dimensional" affine (or linear) subspace that approximates "the support" of the distribution. As was pointed out in [35], the study field currently achieved a saturation level at which developing a unifying framework to the problem becomes highly demanding.

In SDR (sometimes called supervised dimension reduction), we are given a finite number of pairs $(\mathbf{x}_i, y_i), \mathbf{x}_i \in \mathbb{R}^n, y_i \in \mathbb{R}$, also generated according to some unknown joint distribution $p(\mathbf{x}, y)$, and our goal is to find k vectors (where $k \ll n$) $\mathbf{w}_1, \cdots, \mathbf{w}_k \in \mathbb{R}^n$ such that symbolically:

$$y \perp\!\!\!\perp \mathbf{x}|\mathbf{w}_1^T\mathbf{x}, \cdots, \mathbf{w}_k^T\mathbf{x}.$$

This means that an output y is conditionally independent of $\mathbf{x}$, given $\mathbf{w}_1^T\mathbf{x}, \cdots, \mathbf{w}_k^T\mathbf{x}$ or, the conditional distribution $p(y|\mathbf{x})$ is the same as $p(y|\mathbf{w}_1^T\mathbf{x}, \cdots, \mathbf{w}_k^T\mathbf{x})$.

Obviously, the last formulation is not precise if we do not make any assumptions about the joint distribution, or more specifically about the conditional distribution $p(y|\mathbf{x})$. Typically, it is assumed that

$$y = g(\mathbf{w}_1^T\mathbf{x}, \cdots, \mathbf{w}_k^T\mathbf{x}) + \varepsilon, \tag{1.1}$$

where ε is the Gaussian noise with $\mathbb{E}\varepsilon = 0$ and $\mathbb{E}\varepsilon^2 = \delta^2$. The function g is an unknown smooth function. Then, the function $f(\mathbf{x}) = \mathbb{E}[y|\mathbf{x}] = g(\mathbf{w}_1^T\mathbf{x}, \cdots, \mathbf{w}_k^T\mathbf{x})$ is called the regression function.

Many methods have been proposed for estimation of the parameters of model (1.1) such as: sliced inverse regression [22],[9], methods based on an analysis of gradient and Hessian of the regression function [23, 44, 26], methods based on combining local classifiers [17, 28], kernel-based methods [14] and more. In such methods for the SDR problem as the Sliced Inverse Regression [22], the Principal Hessian Direction [23], the Sliced Average Variance Estimation [6], an effective subspace is recovered from the Singular Value Decomposition applied to a certain matrix that is constructed from a training set in a straightforward way. Other methods, such as the Principal Fitted Components [7], the Likelihood Acquired Direction [8], the Kernel Dimensionality Reduction [14], are based on analytic expressions measuring the affinity of a k-dimensional subspace to the effective subspace. In the second type of methods the SDR problem is reduced to an optimization problem over the Stiefel manifold, or the Grassmanian. For other methods we refer to a tutorial on SDR methods [15]. Again, an important aspect of all these methods is that, given a fixed effective subspace, the regression function that predicts an output variable has a relatively straightforward structure and is not optimized by any additional supervised learning procedure.

The SDR problem is tightly connected with the unsupervised dimension reduction problem. In [42] it was shown how a method originally developed for SDR can be turned into a UDR method, i.e. applied to unsupervised data by simply setting an output to be equal to an input. In [11], the SDR problem, together with UDR problems, is cast as an optimization problem over the Stiefel manifold. Taking into account deep connections between UDR and SDR problems, the current study's goal is to develop an approach to a hybrid setting, i.e. when we target to find a low-dimensional linear subspace that both approximates the support of data and is close to the effective subspace (that allows to predict an output). Note that one can solve these two problems independently and obtain two solutions — the span of their union maintains some of their desirable properties (at the expense of increasing the dimension to $2k$). The goal of the paper is to develop better approaches to the problem.

In the hybrid setting our goal is to approximate the empirical inputs distribution μ_{emp} by a distribution μ, supported in some k-dimensional subspace L, and find the regression function $f = g(\mathbf{w}_1^T\mathbf{x}, \cdots, \mathbf{w}_k^T\mathbf{x})$ such that the effective subspace $\text{span}(\mathbf{w}_1, \cdots, \mathbf{w}_k)$ is equal to L. This will guarantee that after projecting the input vector $\mathbf{x}$ onto L we are changing the geometry of the dataset only slightly and, additionally, do not lose the ability to predict the output y. The objective that is minimized in this setting consists of two parts, the first considering the dependence on μ and the second considering the dependence on f. Thus, for a target space L there is a trade-off between the requirement to support the whole data and the sufficiency to predict y. Note that if the main goal is SDR, then the first part of the objective can be interpreted as a regularization part that is dedicated to avoiding over-fitting. If alternatively, the main goal is UDR, then the second part can play its role in many interesting contexts. For example, let the output y be an indicator of an outlier, i.e., $y = 1$ indicates that the input $\mathbf{x}$ is an outlier. Then, it is desirable that after projecting onto the low-dimensional subspace L we are still able to distinguish outliers from typical points.

The key observation of our analysis, stated in Theorem 3.1 of Section 3, is that a class of functions of the form $g(\mathbf{w}_1^T\mathbf{x}, \cdots, \mathbf{w}_k^T\mathbf{x})$ can be characterized as functions whose Fourier transforms are supported in a k-dimensional linear subspace. Thus, the main problem in the hybrid setting is to find a probabilistic measure μ that approximates the empirical measure μ_{emp} and the regression function f such that μ and $\mathcal{F}[f]$ are both supported in the same k-dimensional linear subspace. Instead of optimizing over generalized functions with a k-dimensional support (or, k-dimensional complex measures, in our terminology), we suggest minimizing over ordinary functions given as feed-forward neural networks but with an additional soft constraint. To force the function's support to be close to a k-dimensional subspace, in Section 4 we introduce a class of penalty functions R such that large values of R indicate a strong distortion of the support from any k-dimensional linear subspace. For a specific case of R, in Section 5 we develop an algorithm for our problem that can be formulated for functions given in the frequency coordinate form as well as in the initial coordinate form. The last section is dedicated to experiments.

2 Preliminaries

Throughout the paper we will use common terminology and notations from functional analysis. The Schwartz space of functions is denoted by $\mathcal{S}(\mathbb{R}^n)$ and the set of all tempered distributions is $\mathcal{S}'(\mathbb{R}^n)$, the dual space of $\mathcal{S}(\mathbb{R}^n)$. The Fourier and the inverse Fourier transforms are first defined by

$$\mathcal{F}[f](\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\boldsymbol{\xi}^T\mathbf{x}} d\mathbf{x},$$

$$\mathcal{F}^{-1}[f](\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}^T\mathbf{x}} d\boldsymbol{\xi},$$

and then extended to continuous bijective linear operators $\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. A Borel complex measure is a mapping $\mu : \mathcal{E} \to \mathbb{C}$ where $\mathcal{E}$ is a sigma-algebra of all Borel sets on $\mathbb{R}^n$ and μ is sigma-additive. If μ is real-valued, then μ is called a finite Borel measure. A finite Borel measure with $\mu(\mathbb{R}^n) = 1$ is called the Borel probabilistic measure. The set of all Borel probabilistic measures on $\mathbb{R}^n$ is denoted by $\mathfrak{B}(\mathbb{R}^n)$. The symbol $X_1, \cdots, X_m \sim^{iid} \mu$ denotes the fact that random variables $X_1, \cdots, X_m$ are all independent and each has a distribution function μ.

If a function $f : \mathbb{R}^n \to \mathbb{C}$ is such that $\int_{\mathbb{R}^n} f(\mathbf{x})u(\mathbf{x})d\mathbf{x} < \infty$ for any $u \in \mathcal{S}(\mathbb{R}^n)$ then it induces an operator $T_f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, where $T_f(u) = \int_{\mathbb{R}^n} f(\mathbf{x})u(\mathbf{x})d\mathbf{x}$. Analogously, a Borel complex measure μ on $\mathbb{R}^n$ defines a tempered distribution $T_\mu : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, where $T_\mu(u) = \int_{\mathbb{R}^n} u(\mathbf{x})d\mu$. For simplicity of our notation, we use f and T_f (μ and T_μ) interchangeably (from the context it will alway be clear what we mean). By $L_2(\mathbb{R}^n)$ we denote the Hilbert space of all square-integrable functions from $\mathbb{R}^n$ to $\mathbb{C}$, with the inner product $\langle u, v\rangle_{L_2(\mathbb{R}^n)} = \int u(\mathbf{x})^* v(\mathbf{x})d\mu$. The induced norm is then $\|u\|_{L_2(\mathbb{R}^n)} = \sqrt{\langle u, u\rangle}$.

A positive-definite function is a complex-valued function $f : \mathbb{R}^n \mapsto \mathbb{C}$ such that for any real numbers $\mathbf{x}_1, \cdots, \mathbf{x}_s$ the matrix $A = (a_{i,j})_{i,j=1}^n$ where $a_{i,j} = f(\mathbf{x}_i - \mathbf{x}_j)$ is positive-semidefinite.

For a matrix $A = \left[a_{ij}\right]_{1\le i,j\le n}$ the Frobenius norm is $\|A\|_F = \sqrt{\sum_{ij} |a_{ij}|^2}$.

3 Problem formulation

We formulate the hybrid dimension reduction problem as an optimization task:

$$\inf_{(\mu,f)\in\mathfrak{D}} (1-\rho)I(\mu, \mu_{\text{emp}}) + \rho J(f). \tag{3.1}$$

In the last expression $\mu \in \mathfrak{B}(\mathbb{R}^n)$ denotes a Borel probabilistic measure on $\mathbb{R}^n$ that approximates the empirical distribution over inputs, μ_{emp}, and I denotes the distance function between measures (e.g. the maximum mean discrepancy or the Wasserstein distance) and $\rho \in [0, 1]$.

The object $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth real-valued function that can be given in an arbitrary form, keeping in mind the case of f defined by a feed-forward neural network. We assume that f is a candidate for the regression function and $J(f)$ is a cost function that values how strongly f fits in this role. In practice for the regression case we use the following cost function:

$$J(f) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 I_n)} |y_i - f(\mathbf{x}_i + \boldsymbol{\epsilon})|^2,$$

where the target variable y is normalized, i.e. the estimator of outputs variance $\hat{\text{Var}}(y) = 1$. We add the last remark only to make $I(\mu, \mu_{\text{emp}})$ and $J(f)$ to be of the same scale. For the binary classification case with 0-1 outputs we use:

$$J(f) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 I_n)} H\left(y_i, \frac{e^{f(\mathbf{x}_i + \boldsymbol{\epsilon})}}{1 + e^{f(\mathbf{x}_i + \boldsymbol{\epsilon})}}\right),$$

where $H(y, p) = -y \log p - (1 - y) \log(1 - p)$. Thus, $\rho = 0$ corresponds to the pure unsupervised dimension reduction task and $\rho = 1$ corresponds to the pure sufficient dimension reduction task.

We assume that f satisfies (for k fixed in advance):

$$f(\mathbf{x}) = g(\mathbf{w}_1^T \mathbf{x}, \cdots, \mathbf{w}_k^T \mathbf{x}),$$

where g is an arbitrary function and $\mathbf{w}_1, \cdots, \mathbf{w}_k \in \mathbb{R}^n$. Thus, given an input $\mathbf{x}$, the corresponding output depends on the projection of $\mathbf{x}$ onto $\text{span}(\mathbf{w}_1, \cdots, \mathbf{w}_k)$. Thus, $\text{span}(\mathbf{w}_1, \cdots, \mathbf{w}_k)$ serves as the effective subspace. We assume that the measure μ is also supported in that subspace. Thus, we define:

$$\begin{aligned} \mathfrak{D} = \{(\mu, f) | \mu \in \mathfrak{B}(\mathbb{R}^n), \underset{\mathbb{R}^n}{\exists}\, \mathbf{w}_1, \cdots, \mathbf{w}_k\ \exists g : \mathbb{R}^k \to \mathbb{R} \\ \text{such that } \underset{\text{Borel}}{\forall} A\ \mu(A) = \mu(A \cap \text{span}(\mathbf{w}_1, \cdots, \mathbf{w}_k)) \\ \text{and } \underset{\mathbb{R}^n}{\forall}\, \mathbf{x}\ f(\mathbf{x}) = g(\mathbf{w}_1^T \mathbf{x}, \cdots, \mathbf{w}_k^T \mathbf{x})\}. \end{aligned} \tag{3.2}$$

The parameter $\rho \geq 0$ regulates how strongly we prefer the sufficiency term J over the distance till the empirical distribution.

The following theorem is the key observation behind our approach to the problem (3.1).

Theorem 3.1. *A function $k(\mathbf{x})$ can be represented as $k(\mathbf{x}) = g(\mathbf{w}_1^T \mathbf{x}, \cdots, \mathbf{w}_k^T \mathbf{x}), g \in \mathcal{S}(\mathbb{R}^k)$ if and only if there is an orthonormal basis $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\} \subseteq \mathbb{R}^n$ such that:*

$$\mathcal{F}[T_l] = r(\mathbf{a}_1^T \mathbf{x}, \cdots, \mathbf{a}_{k'}^T \mathbf{x}) \prod_{i=k'+1}^{n} \delta(\mathbf{a}_i^T \mathbf{x}), r \in \mathcal{S}(\mathbb{R}^k), \tag{3.3}$$

where $\delta(\cdot)$ is the Dirac delta-function and $\text{span}(\mathbf{a}_1, \cdots, \mathbf{a}_{k'}) = \text{span}(\mathbf{w}_1, \cdots, \mathbf{w}_k)$.

Sketch of the proof. Without loss of generality, we can assume that $\mathbf{w}_1, \cdots, \mathbf{w}_k$ are linearly independent. A rigorous proof of the theorem would require a careful checking of certain integral identities. Instead, we will present a sketch of the proof at the abstraction level common to theoretical physics papers.

($\Rightarrow$) We also can assume that $\mathbf{w}_1, \cdots, \mathbf{w}_k$ are orthonormal. Indeed, after every redefinition of g given by the rule $g(s_1, \cdots, s_k) \leftarrow g(s_1, \cdots, s_i + \alpha s_j, \cdots, s_k)$ we get the same function l if we simultaneously transform $\mathbf{w}_i$ to $\mathbf{w}_i - \alpha \mathbf{w}_j$. By making such redefinitions, we can always orthogonalize $\mathbf{w}_1, \cdots, \mathbf{w}_k$ by the Gramm-Schmidt process with a subsequent scaling of g's arguments.

Let us complete $\mathbf{w}_1, \cdots, \mathbf{w}_k$ with $\mathbf{w}_{k+1}, \cdots, \mathbf{w}_n$ to form an orthonormal basis in $\mathbb{R}^n$ and set:

$$Q = \left[\mathbf{w}_1, \cdots, \mathbf{w}_n\right] = \left[Q_1, Q_2\right], Q_1 \in \mathbb{R}^{n\times k}, Q_2 \in \mathbb{R}^{n\times(n-k)}.$$

Then, in the Fourier transform formula we make the change of variables $\mathbf{x} = Q\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix} = Q_1\mathbf{y}_1 + Q_2\mathbf{y}_2$, $\mathbf{y}_1 \in \mathbb{R}^k$, $\mathbf{y}_2 \in \mathbb{R}^{n-k}$ and get:

$$\mathcal{F}[l](\boldsymbol{\xi}) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(\mathbf{w}_1^T\mathbf{x}, \cdots, \mathbf{w}_k^T\mathbf{x}) e^{-i\boldsymbol{\xi}^T\mathbf{x}} d\mathbf{x}$$

$$= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(\mathbf{y}_1) e^{-i\boldsymbol{\xi}^T Q\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}} d\mathbf{y}_1 d\mathbf{y}_2 = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} g(\mathbf{y}_1) e^{-i(Q_1^T\boldsymbol{\xi})^T\mathbf{y}_1 - i(Q_2^T\boldsymbol{\xi})^T\mathbf{y}_2} d\mathbf{y}_1 d\mathbf{y}_2$$

$$= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^k} g(\mathbf{y}_1) e^{-i(Q_1^T\boldsymbol{\xi})^T\mathbf{y}_1} d\mathbf{y}_1 \int_{\mathbb{R}^{n-k}} e^{-i(Q_2^T\boldsymbol{\xi})^T\mathbf{y}_2} d\mathbf{y}_2 = \sqrt{2\pi}^{n-k}\mathcal{F}[g](Q_1^T\boldsymbol{\xi})\delta^{n-k}(Q_2^T\boldsymbol{\xi}),$$

where $\delta^{n-k}(s_1, \cdots, s_{n-k}) = \prod_{i=1}^{n-k}\delta(s_i)$. Here we used the equality $\int_{\mathbb{R}^{n-k}} e^{-i\mathbf{z}^T\mathbf{y}_2} d\mathbf{y}_2 = (2\pi)^{n-k}\delta^{n-k}(\mathbf{z})$. Thus, we obtain the needed representation.

($\Leftarrow$) Suppose that:

$$\mathcal{F}[l] = r(\mathbf{a}_1^T\mathbf{x}, \cdots, \mathbf{a}_{k'}^T\mathbf{x}) \prod_{i=k'+1}^{n} \delta(\mathbf{a}_i^T\mathbf{x}).$$

Using the inverse Fourier transform we get:

$$l(\boldsymbol{\xi}) = \mathcal{F}^{-1}\left[\mathcal{F}[l]\right](\boldsymbol{\xi}) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} r(\mathbf{a}_1^T\mathbf{x}, \cdots, \mathbf{a}_{k'}^T\mathbf{x}) \prod_{i=k'+1}^{n} \delta(\mathbf{a}_i^T\mathbf{x}) e^{i\mathbf{x}^T\boldsymbol{\xi}} d\mathbf{x}.$$

After the change of variables $\mathbf{x} = O\mathbf{y}$, where $O = \left[\mathbf{a}_1, \cdots, \mathbf{a}_n\right]$, we get:

$$l(\boldsymbol{\xi}) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} r(y_{1:k'}) \prod_{i=k'+1}^{n} \delta(y_i) e^{i\sum_{i=1}^n y_i \mathbf{a}_i^T\boldsymbol{\xi}} dy_{1:n}$$

$$= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} r(y_{1:k'}) e^{i\sum_{i=1}^{k'} y_i \mathbf{a}_i^T\boldsymbol{\xi}} dy_{1:k'} = \frac{1}{\sqrt{2\pi}^{n-k}}\tilde{g}(\mathbf{a}_1^T\boldsymbol{\xi}, \cdots, \mathbf{a}_k^T\boldsymbol{\xi}),$$

where $\tilde{g} = \mathcal{F}^{-1}[r]$. □

Substantively, the theorem claims that if the function's value depends only on the projection of an argument $\mathbf{x}$ onto the span$(\mathbf{w}_1, \cdots, \mathbf{w}_k)$, then frequencies of the spectrum of such function are all in the span$(\mathbf{w}_1, \cdots, \mathbf{w}_k)$.

Definition 1. Let $\mathfrak{C}(\mathbb{R}^n)$ be a set of symmetric Borel complex measures on $\mathbb{R}^n$, i.e., any $\mu \in \mathfrak{C}(\mathbb{R}^n)$ is a sigma-additive complex-valued function on the sigma-algebra of Borel subsets of $\mathbb{R}^n$ and $\mu(-A) = \mu(A)^*$ (where x^* denotes the complex conjugate of x). We will call a measure $\mu \in \mathfrak{C}(\mathbb{R}^n)$ *a k-dimensional measure* if there is a k'-dimensional linear subspace $S \subseteq \mathbb{R}^n$, $k' \leq k$ such that $\mu(A \cap S) = \mu(A)$ for any Borel set A. The minimal linear space S with the last property is called the support of μ and is denoted by $\operatorname{supp}\mu$. The set of all k-dimensional Borel complex measures is denoted by $\mathcal{G}_k$. The set of all pairs (μ_1, μ_2), where $\mu_1, \mu_2 \in \mathcal{G}_k$ and $\operatorname{supp}\mu_1 = \operatorname{supp}\mu_2$, is denoted by $\mathcal{G}_k^2$.

Thus, problem (3.1) is equivalent to the followinq equality:

$$\inf_{(\mu,\mathcal{F}^{-1}(f))\in\mathfrak{D}} (1-\rho)I(\mu,\mu_{\text{emp}}) + \rho J(\mathcal{F}^{-1}(f))$$
$$= \inf_{(\mu,\mu')\in\mathcal{G}_k^2,\mu\in\mathfrak{B}(\mathbb{R}^n)} (1-\rho)I(\mu,\mu_{\text{emp}}) + \rho J(\mathcal{F}^{-1}(\mu')).$$

Instead of minimization over tempered distributions, we will relax the property that the common support of the pair (μ, μ') is k-dimensional, reducing the problem to the following one:

$$(1-\rho)I(\mu,\mu_{\text{emp}}) + \rho J(\mathcal{F}^{-1}[\mu']) \to \min_{(\mu,\mu')\in\mathfrak{B}(\mathbb{R}^n)\times\mathfrak{C}(\mathbb{R}^n),R(\mu,\mu')\le\epsilon}, \tag{3.4}$$

where $R(\mu, \mu')$ is a penalty term that penalizes (μ, μ') if the dimension of their common support is greater than k. In the next section we describe one natural approach to construct such a penalty term R.

4 Penalty function

We need the following theorem.

Theorem 4.1. *Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a positive definite continuous function and $(\mu, \mu') \in \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{C}(\mathbb{R}^n)$ be such that*

$$\forall i,j \int_{\mathbb{R}^n\times\mathbb{R}^n} x_i\gamma(\mathbf{x}-\mathbf{y})y_j d\mu(\mathbf{x})d\mu(\mathbf{y}) < \infty,$$
$$\int_{\mathbb{R}^n\times\mathbb{R}^n} x_i\gamma(\mathbf{x}-\mathbf{y})y_j d\mu'(\mathbf{x})^* d\mu'(\mathbf{y}) < \infty.$$

The pair (μ, μ') is in $\mathcal{G}_k^2$ if and only if

$$\operatorname{rank}(\mathcal{M}) \le k,$$

where

$$\mathcal{M} = a\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{x}\gamma(\mathbf{x}-\mathbf{y})\mathbf{y}^T d\mu(\mathbf{x})d\mu(\mathbf{y})$$
$$+b\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{x}\gamma(\mathbf{x}-\mathbf{y})\mathbf{y}^T d\mu'(\mathbf{x})^* d\mu'(\mathbf{y}), a>0, b>0.$$

Proof. ($\Rightarrow$) If $(\mu, \mu') \in \mathcal{G}_k^2$, then there is a k-dimensional linear subspace $S \subseteq \mathbb{R}^n$ such that $\mu(A\cap S) = \mu(A)$, $\mu'(A \cap S) = \mu'(A)$. Let $\{\mathbf{v}_i\}_{i=1}^n$ be an orthonormal basis in $\mathbb{R}^n$ such that $\mathbf{v}_i \perp S, i > k$. Then:

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{x}\gamma(\mathbf{x}-\mathbf{y})\mathbf{y}^T d\mu(\mathbf{x})d\mu(\mathbf{y}) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \sum_{i=1}^n \mathbf{v}_i\mathbf{v}_i^T\mathbf{x}\gamma(\mathbf{x}-\mathbf{y})\mathbf{y}^T d\mu(\mathbf{x})d\mu(\mathbf{y}).$$

Since $\int_{\mathbb{R}^n} (\mathbf{v}_i^T\mathbf{x})^2\gamma(\mathbf{x}-\mathbf{y})d\mu(\mathbf{x}) = 0, i > k$, then:

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \sum_{i=1}^k \mathbf{v}_i\mathbf{v}_i^T\mathbf{x}\gamma(\mathbf{x}-\mathbf{y})\mathbf{y}^T d\mu(\mathbf{x})d\mu(\mathbf{y}) = \sum_{i=1}^k \mathbf{v}_i\mathbf{v}_i^T \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{x}\gamma(\mathbf{x}-\mathbf{y})\mathbf{y}^T d\mu(\mathbf{x})d\mu(\mathbf{y}).$$

The same can be proven for μ', therefore:

$$\mathcal{M} = \sum_{i=1}^k \mathbf{v}_i\mathbf{v}_i^T\mathcal{M}$$

and we see that $\mathrm{rank}(\mathcal{M}) \le k$.

($\Leftarrow$) If $\mathrm{rank}(\mathcal{M}) \le k$, then there exist linearly independent vectors $\{\mathbf{u}_i\}_{i=k+1}^n$ such that

$$\mathbf{u}_i^T \mathcal{M}\mathbf{u}_i = a \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{u}_i^T \mathbf{x}\gamma(\mathbf{x}-\mathbf{y})\mathbf{u}_i^T \mathbf{y} d\mu(\mathbf{x}) d\mu(\mathbf{y})$$
$$+b \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{u}_i^T \mathbf{x}\gamma(\mathbf{x}-\mathbf{y})\mathbf{u}_i^T \mathbf{y} d\mu'(\mathbf{x})^* d\mu'(\mathbf{y}) = 0.$$

From positive definiteness of γ we conclude that $\mu(\{\mathbf{x}|\mathbf{u}_i^T\mathbf{x} \neq 0\}) = 0$ and $\mu'(\{\mathbf{x}|\mathbf{u}_i^T\mathbf{x} \neq 0\}) = 0$. Therefore, $\mu(A) = \mu(A \cap \{\mathbf{x}|\mathbf{u}_i^T\mathbf{x}, i = \overline{k+1,n}\}) = 0$ and $\mu'(A) = \mu'(A \cap \{\mathbf{x}|\mathbf{u}_i^T\mathbf{x} = 0, i = \overline{k+1,n}\}) = 0$. Thus, $\mathrm{supp}\,\mu \subseteq \{\mathbf{x}|\mathbf{u}_i^T\mathbf{x} = 0, i = \overline{k+1,n}\}$ and $\mathrm{supp}\,\mu' \subseteq \{\mathbf{x}|\mathbf{u}_i^T\mathbf{x} = 0, i = \overline{k+1,n}\}$. $\square$

Let us define

$$\mathcal{M}_\nu = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{x}\gamma(\mathbf{x}-\mathbf{y})\mathbf{y}^T d\nu(\mathbf{x})^* d\nu(\mathbf{y})$$

and

$$\mathcal{M}_{(\mu,\mu')} = (1-\rho)\mathcal{M}_\mu + \rho\mathcal{M}_{\mu'}.$$

Note that $\mathcal{M}_{(\mu,\mu')}$ is a positive semidefinite matrix, and therefore, the square root $\mathcal{M}_{(\mu,\mu')}^{1/2}$ is defined. Our definition for the penalty function R is as follows:

$$R(\mu,\mu') = \min_{\mathcal{M}\in\mathbb{R}^{n\times n}:\mathrm{rank}(\mathcal{M})\le k} \|\mathcal{M}_{(\mu,\mu')}^{1/2} - \mathcal{M}\|_F^2. \tag{4.1}$$

It is natural to expect that if $R(\mu,\mu') \le \epsilon$ and $\epsilon > 0$ is small, i.e., $\mathcal{M}_{(\mu,\mu')}^{1/2}$ (together with $\mathcal{M}_{(\mu,\mu')}$) is close to some rank k matrix, then the common support of (μ,μ') is approximable by a k-dimensional linear subspace. Now, our goal is to develop an algorithm for the following problem:

$$(1-\rho)I(\mu,\mu_{\mathrm{emp}}) + \rho J(\mathcal{F}^{-1}[\mu']) + \lambda R(\mu,\mu') \to \min_{(\mu,\mu')\in\mathfrak{B}(\mathbb{R}^n)\times\mathfrak{C}(\mathbb{R}^n)}, \tag{4.2}$$

where λ is a penalty parameter that can be chosen sufficiently large to force $R(\mu,\mu')$ to be small.

4.1 Another description of the penalty

Let us now give an alternative description of the penalty $R(\mu,\mu')$ that suits better to the goal of designing an algorithm for problem (3.4).

Let γ and s be smooth real-valued positive definite functions such that:

$$\gamma(\mathbf{x}-\mathbf{x}'') = \int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')s(\mathbf{x}'-\mathbf{x}'')d\mathbf{x}'.$$

For example, $\gamma(\mathbf{x}) = \sqrt{\frac{\pi}{2}}^n e^{-\|\mathbf{x}\|^2/2}$ and $s(\mathbf{x}) = e^{-\|\mathbf{x}\|^2}$. Note that $\gamma(\mathbf{x}) = \gamma(-\mathbf{x})$ and $s(\mathbf{x}) = s(-\mathbf{x})$.

Let us define

$$S_{(\mu,\mu')} = [\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}' d\mu(\mathbf{x}') \int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}' d\mu'(\mathbf{x}')]E,$$

where

$$E = \begin{bmatrix} \sqrt{1-\rho} & 0 \\ 0 & \sqrt{\rho} \end{bmatrix}.$$

This object is an $n \times 2$ matrix whose entries are functions.

By $L_2^{n\times 2}(\mathbb{R}^n)$ we denote the space of all matrices $\left[b_{ij}(\mathbf{x})\right]_{1\le i\le n, j=1,2}$, where $b_{ij} \in L_2(\mathbb{R}^n)$. Also, $L_2^2(\mathbb{R}^n) \equiv L_2^{2\times 1}(\mathbb{R}^n)$. Note that $L_2^2(\mathbb{R}^n)$ is a linear space over complex numbers. Let us also denote by $\tilde{L}_2^2(\mathbb{R}^n)$ the real linear space that is the set $L_2^2(\mathbb{R}^n)$ considered over real numbers only, equipped with the inner product

$$\langle[\phi_1,\phi_2]^T,[\psi_1,\psi_2]^T\rangle_{\tilde{L}_2^2(\mathbb{R}^n)} = \mathrm{Re}\left\{\langle\phi_1,\psi_1\rangle_{L_2(\mathbb{R}^n)} + \langle\phi_2,\psi_2\rangle_{L_2(\mathbb{R}^n)}\right\}.$$

The set of all bounded linear operators from $\tilde{L}_2^2(\mathbb{R}^n)$ to $\mathbb{R}^n$ is denoted by $\mathcal{B}^{n\times 2}$.

It is easy to see that any $A \in L_2^{n\times 2}(\mathbb{R}^n)$ defines a bounded linear operator O_A from $\tilde{L}_2^2(\mathbb{R}^n)$ to $\mathbb{R}^n$ by the following rule:

$$\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix} \in \tilde{L}_2^2(\mathbb{R}^n) \to^{O_A} \mathrm{Re}\int_{\mathbb{R}^n} A(\mathbf{x})^* \begin{bmatrix}\phi_1(\mathbf{x})\\\phi_2(\mathbf{x})\end{bmatrix} d\mathbf{x}.$$

Moreover, it is easy to see that all bounded linear operators from $\tilde{L}_2^2(\mathbb{R}^n)$ to $\mathbb{R}^n$ can be represented in this way. Obviously, $\mathcal{B}^{n\times 2}$ is a Hilbert space, where the inner product is defined as:

$$\langle O_{A_1}, O_{A_2}\rangle_{\mathcal{B}^{n\times 2}} = \mathrm{Re}\int_{\mathbb{R}^n} \mathrm{Trace}\,(A_1(\mathbf{x})^\dagger A_2(\mathbf{x}))d\mathbf{x},$$

where for any $A(\mathbf{x}) = [A_{ij}(\mathbf{x})]_{1\le i\le n, 1\le j\le 2} \in L_2^{n\times 2}(\mathbb{R}^n)$, $A(\mathbf{x})^\dagger$ denotes the matrix $[A_{ji}(\mathbf{x})^*]_{1\le i\le 2, 1\le j\le n} \in L_2^{2\times n}(\mathbb{R}^n)$. Thus, the corresponding norm coincides with the trace norm. Recall that, for a bounded linear operator $O:\mathcal{H}_1\to\mathcal{H}_2$ between Hilbert spaces $\mathcal{H}_1,\mathcal{H}_2$, the rank of O is defined as $\dim\mathrm{Im}(O)$, where $\mathrm{Im}(O)=\{O[\phi]\,|\,\phi\in\mathcal{H}_1\}$.

Theorem 4.2. *If $\mathcal{M}_{(\mu,\mu')} < \infty$, then $S_{(\mu,\mu')} \in L_2^{n\times 2}(\mathbb{R}^n)$, $O_{S_{(\mu,\mu')}}O^\dagger_{S_{(\mu,\mu')}} = \mathcal{M}_{(\mu,\mu')}$, and* $\mathrm{rank}(O_{S_{(\mu,\mu')}}) = \mathrm{rank}(\mathcal{M}_{(\mu,\mu')})$.

Sketch of the proof. Since

$$\langle O^\dagger_{S_{(\mu,\mu')}}\mathbf{y}, \begin{bmatrix}\phi_1\\\phi_2\end{bmatrix}\rangle_{\tilde{L}_2^2(\mathbb{R}^n)} = \mathbf{y}^T(O_{S_{(\mu,\mu')}}\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix})$$
$$= \mathbf{y}^T\mathrm{Re}\int_{\mathbb{R}^n}[\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'d\mu(\mathbf{x}')\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'d\mu'(\mathbf{x}')^*]E\begin{bmatrix}\phi_1(\mathbf{x})\\\phi_2(\mathbf{x})\end{bmatrix}d\mathbf{x},$$

we obtain

$$O^\dagger_{S_{(\mu,\mu')}}\mathbf{y} = E\begin{bmatrix}\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'^T\mathbf{y}d\mu(\mathbf{x}')\\\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'^T\mathbf{y}d\mu'(\mathbf{x}')\end{bmatrix}.$$

Let us now check that $O_{S_{(\mu,\mu')}}O^\dagger_{S_{(\mu,\mu')}} = \mathcal{M}_{(\mu,\mu')}$ by direct calculation:

$$\mathbf{y}\in\mathbb{R}^n \overset{O^\dagger_{S_{(\mu,\mu')}}}{\longrightarrow} E\begin{bmatrix}\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'^T\mathbf{y}d\mu(\mathbf{x}')\\\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'^T\mathbf{y}d\mu'(\mathbf{x}')\end{bmatrix}$$
$$\overset{O_{S_{(\mu,\mu')}}}{\longrightarrow} \mathrm{Re}\int_{\mathbb{R}^n}[\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'d\mu(\mathbf{x}')\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'d\mu'(\mathbf{x}')^*]$$
$$EE\begin{bmatrix}\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'^T\mathbf{y}d\mu(\mathbf{x}')\\\int_{\mathbb{R}^n} s(\mathbf{x}-\mathbf{x}')\mathbf{x}'^T\mathbf{y}d\mu'(\mathbf{x}')\end{bmatrix}d\mathbf{x}$$
$$= (1-\rho)\int_{\mathbb{R}^n}\gamma(\mathbf{x}-\mathbf{x}')\mathbf{x}\mathbf{x}'^T\mathbf{y}d\mu(\mathbf{x})d\mu(\mathbf{x}') + \rho\int_{\mathbb{R}^n}\gamma(\mathbf{x}-\mathbf{x}')\mathbf{x}\mathbf{x}'^T\mathbf{y}d\mu'(\mathbf{x})^*d\mu'(\mathbf{x}')$$
$$= \mathcal{M}_{(\mu,\mu')}\mathbf{y}.$$

From the equality $O_{S_{(\mu,\mu')}} O^{\dagger}_{S_{(\mu,\mu')}} = \mathcal{M}_{(\mu,\mu')}$ we conclude that $\mathrm{rank}(\mathcal{M}_{(\mu,\mu')}) \le \mathrm{rank}(O_{S_{(\mu,\mu')}})$. Conversely, if $\mathbf{x}_1, \cdots, \mathbf{x}_r$ is a basis of $\mathrm{Im}(\mathcal{M}_{(\mu,\mu')})^{\perp}$, then $0 = \mathbf{x}_i^T \mathcal{M}_{(\mu,\mu')} \mathbf{x}_i = \|O^{\dagger}_{S_{(\mu,\mu')}} \mathbf{x}_i\|^2_{\tilde{L}_2^2(\mathbb{R}^n)}$, i.e. $O^{\dagger}_{S_{(\mu,\mu')}} \mathbf{x}_i = \mathbf{0}$. Therefore, for $i \in [r]$, we have

$$\langle O_{S_{(\mu,\mu')}} \phi, \mathbf{x}_i \rangle_{\tilde{L}_2^2(\mathbb{R}^n)} = \langle \phi, O^{\dagger}_{S_{(\mu,\mu')}} \mathbf{x}_i \rangle_{\tilde{L}_2^2(\mathbb{R}^n)} = 0,$$

for any ϕ, i.e., $\mathrm{Im}(O_{S_{(\mu,\mu')}}) \subseteq \{\mathbf{x}_1, \cdots, \mathbf{x}_r\}^{\perp}$. Thus, $\mathrm{rank}\,(O_{S_{(\mu,\mu')}}) \le n - r = \mathrm{rank}\,(\mathcal{M}_{(\mu,\mu')})$. □

The Eckart-Young-Mirsky theorem [19, Theorem 4.4.7] in the theory of Singular Value Decomposition (SVD) gives us that

$$R(\mu, \mu') = \min_{\mathcal{M} \in \mathbb{R}^{n \times n} : \mathrm{rank}(\mathcal{M}) \le k} \|\mathcal{M}^{1/2}_{(\mu,\mu')} - \mathcal{M}\|_F^2 = \sum_{i=k+1}^{n} \lambda_i,$$

where $\lambda_1 \ge \cdots \ge \lambda_n$ are eigenvalues of $\mathcal{M}_{(\mu,\mu')} = \mathcal{M}^{\frac{1}{2}T}_{(\mu,\mu')} \mathcal{M}^{\frac{1}{2}}_{(\mu,\mu')}$. Due to the relationship $O_{S_{(\mu,\mu')}} O^{\dagger}_{S_{(\mu,\mu')}} = \mathcal{M}_{(\mu,\mu')}$ the following statement is true.

Theorem 4.3. *We have*

$$R(\mu, \mu') = \min_{S \in L_2^{n \times 2}(\mathbb{R}^n) : \mathrm{rank}(O_S) \le k} \|S_{(\mu,\mu')} - S\|^2_{L_2^{n \times 2}(\mathbb{R}^n)},$$

and the latter minimum is attained at $P_{(\mu,\mu')} S_{(\mu,\mu')}$*, where* $P_{(\mu,\mu')} = \sum_{i=1}^{k} \mathbf{u}_i \mathbf{u}_i^T$ *is the projection operator to first* k *principal components of* $\mathcal{M}_{(\mu,\mu')}$*.*

Sketch of the proof. First, we check that all arguments of the Eckart-Young-Mirsky theorem for matrices maintain in the case of bounded linear operators from $\tilde{L}_2^2(\mathbb{R}^n)$ to $\mathbb{R}^n$. Indeed, all arguments survive, because such operators are compact and can have only a finite spectrum, since $\mathbb{R}^n$ is finite-dimensional. Let us only describe an optimal S on which $\min_{S \in L_2^{n \times 2}(\mathbb{R}^n) : \mathrm{rank}(O_S) \le k} \|S_{(\mu,\mu')} - S\|^2_{L_2^{n \times 2}(\mathbb{R}^n)}$ is attained.

Let $\mathbf{u}_1, \cdots, \mathbf{u}_n$ be orthonormal eigenvectors of $\mathcal{M}_{(\mu,\mu')} = O_{S_{(\mu,\mu')}} O^{\dagger}_{S_{(\mu,\mu')}}$ and $\lambda_1 \ge \cdots \ge \lambda_{n'} > 0$ be the corresponding non-zero eigenvalues. For $\sigma_i = \sqrt{\lambda_i}$ let us define $\mathbf{v}_i = \frac{O^{\dagger}_{S_{(\mu,\mu')}}[\mathbf{u}_i]}{\sigma_i}$. Here, $\mathbf{v}_i$ is equal to the function

$$\mathbf{v}_i(\mathbf{x}) = \frac{1}{\sigma_i} \begin{bmatrix} \sqrt{1-\rho} \int_{\mathbb{R}^n} s(\mathbf{x} - \mathbf{x}') \mathbf{u}_i^T \mathbf{x}' d\mu(\mathbf{x}') \\ \sqrt{\rho} \int_{\mathbb{R}^n} s(\mathbf{x} - \mathbf{x}') \mathbf{u}_i^T \mathbf{x}' d\mu'(\mathbf{x}') \end{bmatrix}.$$

It is easy to see that $\mathbf{v}_1, \cdots \mathbf{v}_{n'}$ is an orthonormal basis in $\mathrm{Im}\, O^{\dagger}_{S_{(\mu,\mu')}}$, and SVD for $O_{S_{(\mu,\mu')}}$ is:

$$O_{S_{(\mu,\mu')}} = \sum_{i=1}^{n'} \sigma_i \mathbf{u}_i \mathbf{v}_i^{\dagger}.$$

By the Eckart-Young-Mirsky theorem, an optimal $S = O_F$ in $\min_{S \in \mathcal{B}^{n \times 2}, \mathrm{rank} S \le k} \|O_{S_{(\mu,\mu')}} - S\|^2_{\mathcal{B}^{n \times 2}}$ is defined by a truncation of SVD for $O_{S_{(\mu,\mu')}}$ at kth term, i.e.,

$$\begin{aligned} F = \Bigg[\sqrt{1-\rho} \int_{\mathbb{R}^n} s(\mathbf{x} - \mathbf{x}') \sum_{i=1}^{k} \mathbf{u}_i \mathbf{u}_i^T \mathbf{x}' d\mu(\mathbf{x}'), \\ \sqrt{\rho} \int_{\mathbb{R}^n} s(\mathbf{x} - \mathbf{x}') \sum_{i=1}^{k} \mathbf{u}_i \mathbf{u}_i^T \mathbf{x}' d\mu'(\mathbf{x}') \Bigg] = P_{(\mu,\mu')} S_{(\mu,\mu')}, \end{aligned} \tag{4.3}$$

where $P_{(\mu,\mu')} = \sum_{i=1}^{k} \mathbf{u}_i \mathbf{u}_i^T$ is the projection operator to first k principal components of $\mathcal{M}_{(\mu,\mu')}$. From the equality $\|O_A\|_{\mathcal{B}^{n \times 2}} = \|A\|_{L_2^{n \times 2}(\mathbb{R}^n)}$ we obtain the statement of theorem. □

5 The alternating scheme

Complex measures are in the weak closure of the main functional classes, L_2 and Sobolev spaces, continuous functions, single-layer neural networks, etc. The major gain from penalty formulation (4.2) is that, instead of optimizing over complex measures whose supports have empty interior, we can vary the argument over a space of ordinary functions $\mathfrak{F}$, where $\mathfrak{F} \subset C(\mathbb{R}^n)$ can be chosen as any class of functions which is dense (with respect to the weak weak topology) in $\mathfrak{D}' = \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{C}(\mathbb{R}^n)$, for example:

$$\mathfrak{F} = \{\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} | g_1, g_2 \in C(\mathbb{R}^n), \forall \mathbf{x}\, g_1(\mathbf{x}) \geq 0, \int_{\mathbb{R}^n} g_1(\mathbf{x}) d\mathbf{x} = 1, \int_{\mathbb{R}^n} |g_2(\mathbf{x})| d\mathbf{x} < \infty\}. \tag{5.1}$$

Thus, we need to solve the following optimization problem:

$$(1-\rho) I(g_1, \mu_{\mathrm{emp}}) + \rho J(\mathcal{F}^{-1}[g_2]) + \lambda R(g_1, g_2) \to \min_{(g_1, g_2) \in \mathfrak{F}}. \tag{5.2}$$

Using Theorem 4.3 we can represent problem (5.2) in the following form:

$$\Phi(\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, S) = (1-\rho) I(g_1, \mu_{\mathrm{emp}}) + \rho J(\mathcal{F}^{-1}[g_2]) + \lambda \|S_{(g_1,g_2)} - S\|^2_{L_2^{n\times 2}(\mathbb{R}^n)}$$
$$\to \min_{\mathbf{g} \in \mathfrak{F}, S \in L_2^{n\times 2}(\mathbb{R}^n): \mathrm{rank}(O_S) \leq k}.$$

The simplest idea for an optimization is to minimize over $\mathbf{g} = (g_1, g_2) \in \mathfrak{F}$ and over $S \in L_2^{n\times 2}(\mathbb{R}^n)$: $\mathrm{rank}(O_S) \leq k$ alternatingly. The first part would be an optimization over an infinite-dimensional object, which cannot be implemented in practice. To avoid infiniteness, we will fix a proper parameterized set of functions $\mathfrak{M} \subseteq \mathfrak{F}$ (e.g., deep neural networks [36, 1, 2, 39, 40, 24]) and optimize over $\mathfrak{M}$. A general scheme of optimization is given in Algorithm 1.

Algorithm 1 Alternating scheme

1: **procedure**
2: $\quad S_0 \leftarrow 0$
3: $\quad$ **for** $t = 1, \cdots, N$ **do**
4: $\quad\quad \mathbf{g}_t \leftarrow \arg\min_{\mathbf{g} \in \mathfrak{M}} \Phi(\mathbf{g}, S_{t-1})$
5: $\quad\quad S_t \leftarrow \arg\min_{S \in L_2^{n\times 2}(\mathbb{R}^n): \mathrm{rank}(O_S) \leq k} \|S_{\mathbf{g}_t} - S\|^2_{L_2^{n\times 2}(\mathbb{R}^n)}$

Note that Step 5 of the algorithm is equivalent to minimizing $\|S_{\mathbf{g}_t} - S\|^2_{L_2^{n\times 2}(\mathbb{R}^n)}$ over $S \in L_2^{n\times 2}(\mathbb{R}^n)$: $\mathrm{rank}(O_S) \leq k$. In the previous section we have already described an optimal solution for that task (equation (4.3)):

$$S_t = P_{\mathbf{g}_t} S_{\mathbf{g}_t},$$
$$S_{\mathbf{g}_t} = \int_{\mathbb{R}^n} s(\mathbf{x} - \mathbf{x}') \mathbf{x}' \mathbf{g}_t^T(\mathbf{x}') E d\mathbf{x}'.$$

Here $P_{\mathbf{g}_t} \in \mathbb{R}^{n\times n}$ is the projection operator to the first k principal components of

$$\mathcal{M}_{\mathbf{g}_t} = \int_{\mathbb{R}^n \times \mathbb{R}^n} \gamma(\mathbf{x} - \mathbf{y}) \mathbf{x} \mathbf{y}^T \mathbf{g}_t(\mathbf{x})^\dagger E^2 \mathbf{g}_t(\mathbf{y}) d\mathbf{x} d\mathbf{y}.$$

The hardest part of that step is to estimate the matrix $\mathcal{M}_{\mathbf{g}_t}$ for a given $\mathbf{g}_t \in \mathfrak{M}$. Thus, a practical implementation of our algorithm would require $\mathfrak{M}$ to be defined in such a way that the latter integral can be calculated either analytically or numerically. At the same time, $\mathfrak{M}$ should be rich enough to approximate functions from $\mathfrak{D}$ in terms of weak topology. Thus, to summarize, $\mathfrak{M}$ should be:

- dense in $\mathfrak{D}'$.
- $\mathcal{M}_{\mathbf{g}}$ is efficiently computable.

An example of $\mathfrak{M}$ that satisfies the latter 2 conditions will be given in the next section.

5.1 Return to initial coordinates

In applications, it is desirable that an algorithm for the problem deals with the function $\mathcal{F}^{-1}[\mathbf{g}] = [\mathcal{F}^{-1}[g_1], \mathcal{F}^{-1}[g_2]]$, rather than with $[g_1, g_2]$.

The pair $[\mathcal{F}^{-1}[g_1],\ \mathcal{F}^{-1}[g_2]]$ has the following interpretation. Since g_1 is the probability density function that approximates the empirical distribution μ_{emp}, then $G_1 = \mathcal{F}^{-1}[g_1]$ is the characteristic function of g_1 (up to a constant factor). By Bochner's theorem [5] we know that the function can serve as the characteristic function of a probability distribution if and only if it is continuous, positive definite and

$$G_1(\mathbf{0}) = 1.$$

The function $G_2 = \mathcal{F}^{-1}[g_2]$ is simply the regression function in the term $\rho J(\mathcal{F}^{-1}[g_2]) = \rho J(G_2)$, i.e., G_2 is a real-valued function.

Thus, in the dual reformulation we search over pairs $[G_1,\ G_2]$ where G_1 is complex-valued, continuous, positive definite and G_2 is real-valued.

The specifics of scheme 1 is that it allows such a reformulation.

Indeed, at step 4 of the algorithm we minimize the expression:

$$\Phi(\mathbf{g}, S) = (1-\rho)I(g_1, \mu_{\mathrm{emp}}) + \rho J(\mathcal{F}^{-1}[g_2]) + \lambda \|S_{\mathbf{g}} - S_{t-1}\|^2_{L_2^{n\times 2}(\mathbb{R}^n)},$$

where $S_{t-1} = P_{\mathbf{g}_{t-1}} S_{\mathbf{g}_{t-1}}$ has been calculated on the previous iteration. According to Theorem 4.3 and formula (4.3) we can rewrite the penalty term $\lambda \|S_{\mathbf{g}} - S_{t-1}\|^2_{L_2^{n\times 2}(\mathbb{R}^n)}$ as:

$$\lambda \int_{\mathbb{R}^n\times\mathbb{R}^n} \gamma(\mathbf{x}-\mathbf{y}) E(\mathbf{g}(\mathbf{x})\mathbf{x}^T - \mathbf{g}_{t-1}(\mathbf{x})\mathbf{x}^T P^T_{\mathbf{g}_{t-1}})(\mathbf{y}\mathbf{g}(\mathbf{y})^T - P_{\mathbf{g}_{t-1}}\mathbf{y}\mathbf{g}_{t-1}(\mathbf{y})^T) E d\mathbf{x}d\mathbf{y}.$$

Let us denote $\mathbf{G} = \mathcal{F}^{-1}[\mathbf{g}]$ and $\mathbf{G}_{t-1} = \mathcal{F}^{-1}[\mathbf{g}_{t-1}]$. Using the well-known duality between x_i and $-\mathrm{i}\partial_{x_i}$, we see that

$$\mathcal{F}^{-1}[\mathbf{x}\mathbf{g}(\mathbf{x})^T] = -\mathrm{i}\partial_{\mathbf{x}}\mathbf{G}^T,$$
$$\mathcal{F}^{-1}[P_{\mathbf{g}_{t-1}}\mathbf{x}\mathbf{g}_{t-1}(\mathbf{x})^T] = -\mathrm{i}P_{\mathbf{g}_{t-1}}\partial_{\mathbf{x}}\mathbf{G}^T_{t-1}.$$

The unitarity of the inverse Fourier transform and the convolution theorem gives us that the penalty term equals

$$\lambda' \int_{\mathbb{R}^n} p(\mathbf{x}) \left\| E\frac{\partial \mathbf{G}}{\partial \mathbf{x}} - E\frac{\partial \mathbf{G}_{t-1}}{\partial \mathbf{x}} P_{t-1} \right\|^2_F d\mathbf{x}, \tag{5.3}$$

where $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]_{1\le i\le 2, 1\le j\le n}$ denotes the Jacobian of $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^2$ and $p = \mathcal{F}^{-1}[\gamma]$, $P_{t-1} = P_{\mathbf{g}_{t-1}}$.

The set dual to $\mathfrak{M}$ is defined as $\mathfrak{M}' = \{\mathcal{F}^{-1}[\mathbf{h}] | \mathbf{h} \in \mathfrak{M}\}$. The matrix $\mathcal{M}_t = \mathcal{M}_{\mathbf{g}_t}$ can also be calculated using $\mathbf{G}_t$:

$$\mathcal{M}_t = \left[\int_{\mathbb{R}^n\times\mathbb{R}^n} \gamma(\mathbf{x}-\mathbf{y}) x_i \mathbf{g}_t^\dagger(\mathbf{x}) y_j E^2 \mathbf{g}_t(\mathbf{y}) d\mathbf{x}d\mathbf{y}\right]_{n\times n}$$
$$\propto \left[\int_{\mathbb{R}^n} p(\mathbf{x}) \frac{\partial E\mathbf{G}_t}{\partial x_i}^\dagger \frac{\partial E\mathbf{G}_t}{\partial x_j} d\mathbf{x}\right]_{n\times n} = \int_{\mathbb{R}^n} p(\mathbf{x}) \frac{\partial E\mathbf{G}_t}{\partial \mathbf{x}}^\dagger \frac{\partial E\mathbf{G}_t}{\partial \mathbf{x}} d\mathbf{x}.$$

For $\mathbf{G} = [G_1,\ G_2]^T$ we define:

$$\Phi'_t(\mathbf{G}) = (1-\rho)I(\mathcal{F}[G_1], \mu_{\text{emp}}) + \rho J(G_2) + \lambda' \int_{\mathbb{R}^n} p(\mathbf{x}) \left|\left| \frac{\partial E\mathbf{G}}{\partial \mathbf{x}} - \frac{\partial E\mathbf{G}_{t-1}}{\partial \mathbf{x}} P_{t-1} \right|\right|_F^2 d\mathbf{x}.$$

Note that we can assume that G_2 and $\mathbf{G}_{t2}$ are real-valued, because the objective $\Phi'_t(\mathbf{G})$ always attains its minimum on such a pair $[G_1^*\ G_2^*]$ that G_2^* is a real-valued function.

Thus, algorithm 2 is dual to algorithm 1.

Algorithm 2 Alternating scheme with initial coordinates

procedure
  $P_0 \leftarrow 0$
  for $t = 1, \cdots, N$ **do**
    $\mathbf{G}_t \leftarrow \arg\min_{\mathbf{G}\in\mathfrak{M}'} \Phi'_{t-1}(\mathbf{G})$
    $\mathcal{M}_t \leftarrow \int_{\mathbb{R}^n} p(\mathbf{x}) \frac{\partial E\mathbf{G}_t}{\partial \mathbf{x}}^\dagger \frac{\partial E\mathbf{G}_t}{\partial \mathbf{x}} d\mathbf{x}$
    $P_t \leftarrow$ projection to first k principal components of $\mathcal{M}_t$

5.2 The description of $\mathfrak{M}'$

Let us now give an example of a set $\mathfrak{M}$ that satisfies both conditions that we imposed in Section 5. Instead of defining $\mathfrak{M}$ we will define its dual

$$\mathfrak{M}' = \left\{ \mathbf{H} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \mid h_i \in FF_i \right\},$$

where FF_2 is a set of functions defined by the standard single layer neural network:

$$\sum_{i=1}^{M} w_i \psi(\mathbf{a}_i^T \mathbf{x} + b_i),$$

where $\mathbf{a}_i \in \mathbb{R}^n, w_i, b_i \in \mathbb{R}$ are parameters and M is a hyperparameter. For ψ we only assume that it is some non-constant function whose first derivatives are continuous and bounded. In practice we use the hyperbolic tangent function for ψ.

We define FF_1 as a set of functions given in the following parameterized form:

$$\sum_{i=1}^{M'} \alpha_i e^{\mathrm{i}\omega_i^T \mathbf{x}}, \tag{5.4}$$

where $\alpha_i > 0$ and $\sum_{i=1}^{M'} \alpha_i = 1$. It is easy to see that such functions are always positive definite. In practice we use the expression $\sum_{i=1}^{M'} \alpha_i \cos(\omega_i^T \mathbf{x})$, because the empirical distribution is made symmetrical before we apply the algorithm. The number of neurons in the single-layer neural network with the cosine activation function, M', is a hyperparameter.

Let us show that $\mathfrak{M}'$ is dense in $\mathcal{F}^{-1}[\mathfrak{D}']$. It is a well-known fact that for any compact set $\Omega \subseteq \mathbb{R}^n$, single-layer neural networks can approximate any function in $C(\mathbb{R}^n)$ with an arbitrary accuracy [4]. Thus, the weak closure of FF_2 contains $\mathcal{F}^{-1}[\mathfrak{C}(\mathbb{R}^n)]$ (all functions in the last class are real-valued), i.e. FF_2 covers all interesting functions that can serve as candidates for the regression function.

Using Theorem 2 from [4], it can be shown that the conical hull of $\{e^{\mathrm{i}\omega_i^T \mathbf{x}} | \omega \in \mathbb{R}^n\}$ is dense in the cone of continuous positive definite functions, and therefore, are dense with respect to weak topology in $\mathcal{F}^{-1}[\mathfrak{B}(\mathbb{R}^n)]$. Thus, by the expression (5.4) we are able to approximate all characteristic functions.

5.3 Maximum mean discrepancy metric

All our experiments were done for the following well-known distance function [16]:

$$I(\mu, \mu_{\text{emp}}) = \int_{\mathbb{R}^n} q(\mathbf{x})|\phi_\mu(\mathbf{x}) - \phi_{\text{emp}}(\mathbf{x})|^2 d\mathbf{x}, \tag{5.5}$$

where $q(\mathbf{x}) > 0$ is a continuous function such that $\int_{\mathbb{R}^n} q(\mathbf{x})d\mathbf{x} = 1$ and ϕ_μ, ϕ_{emp} are the characteristic functions of the distributions μ, μ_{emp} correspondingly. It is easy to see that

$$\phi_{\text{emp}}(\mathbf{x}) = \mathbb{E}_{\boldsymbol{\xi}\sim\mu_{\text{emp}}} e^{\mathrm{i}\mathbf{x}^T\boldsymbol{\xi}} = \frac{1}{N}\sum_{i=1}^{N} e^{\mathrm{i}\mathbf{x}_i^T\mathbf{x}}.$$

The Maclaurin series of the characteristic function has the form:

$$\phi_\mu(\mathbf{x}) = \mathbb{E}_{\boldsymbol{\xi}\sim\mu} e^{\mathrm{i}\mathbf{x}^T\boldsymbol{\xi}} = \sum_{j=0}^{\infty} \frac{\mathbb{E}_{\boldsymbol{\xi}\sim\mu}(\mathrm{i}\mathbf{x}^T\boldsymbol{\xi})^j}{j!}.$$

Thus, by approximating $\phi_{\text{emp}}(\mathbf{x})$ in a neighborhood of the origin, defined by $q(\mathbf{x})$, our method tries to approximate all moments of μ_{emp} simultaneously.

Our experiments show that algorithm 2 converges to a better solution if we set $q(\mathbf{x}) \propto p(\mathbf{x}) = \mathcal{F}^{-1}[\gamma]$.

5.4 Practical algorithm with initial coordinates

Let us set $p(\mathbf{x}) = \frac{e^{-\frac{\|\mathbf{x}\|^2}{2\delta^2}}}{\sqrt{2\pi\delta^2}^n}$. Step 4 of algorithm 2:

$$\theta_t = \arg\min_\theta \Phi'_{t-1}(\mathbf{G}_\theta)$$

is done via the gradient descent-type algorithm (we use the Adam optimizer [20], a popular tool in AI [31, 21, 30, 33, 37, 32, 41, 25]) that needs as an oracle an unbiased estimator of the gradient at a given point θ.

In practice it is natural to estimate the gradient of $\Phi'_{t-1}(\mathbf{G}_\theta)$ as follows:

$$\nabla_\theta\Bigg(\frac{1-\rho}{n_{\text{batch}}}\sum_{i=1}^{n_{\text{batch}}} \|\mathbf{G}_{\theta 1}(\mathbf{z}_i) - \phi_{\text{emp}}(\mathbf{z}_i)\|^2 + \frac{\rho}{n_{\text{batch}}}\sum_{i=1}^{n_{\text{batch}}} \|y_i - \mathbf{G}_{\theta 2}(\mathbf{x}_i + \boldsymbol{\epsilon}_i)\|^2 + \frac{\tilde{\lambda}}{n_{\text{batch}}}\sum_{i=1}^{n_{\text{batch}}} \|\frac{\partial E\mathbf{G}_\theta}{\partial \mathbf{x}}(\mathbf{z}'_i) - \frac{\partial E\mathbf{G}_{\theta_{t-1}}}{\partial \mathbf{x}}(\mathbf{z}'_i)P_{t-1}\|^2\Bigg),$$

where $\mathbf{x}_1, \cdots, \mathbf{x}_{n_{\text{batch}}} \sim^{iid} \mu_{\text{emp}}$ $(y_1, \cdots, y_{n_{\text{batch}}}$ are the corresponding outputs), $\boldsymbol{\epsilon}_1, \cdots, \boldsymbol{\epsilon}_{n_{\text{batch}}} \sim^{iid} \frac{1}{\sqrt{2\pi\sigma^2}^2} e^{-\frac{\|\mathbf{x}\|^2}{2\sigma^2}}$ and $\mathbf{z}_1, \cdots, \mathbf{z}_{n_{\text{batch}}}, \mathbf{z}'_1, \cdots, \mathbf{z}'_{n_{\text{batch}}} \sim^{iid} p$ $(= q)$.

Since

$$\mathcal{M}_t = \int_{\mathbb{R}^n} p(\mathbf{x}) \frac{\partial E\mathbf{G}_{\theta_t}}{\partial \mathbf{x}}^\dagger \frac{\partial E\mathbf{G}_{\theta_t}}{\partial \mathbf{x}} d\mathbf{x},$$

it is natural to estimate it as:

$$\hat{\mathcal{M}}_t = \frac{1}{N_{\text{batch}}}\sum_{i=1}^{N_{\text{batch}}} \frac{\partial E\mathbf{G}_{\theta_t}}{\partial \mathbf{x}}^\dagger(\mathbf{t}_i)\frac{\partial E\mathbf{G}_{\theta_t}}{\partial \mathbf{x}}(\mathbf{t}_i),$$

where $\mathbf{t}_1, \cdots, \mathbf{t}_{N_{\text{batch}}}$ are independent identically distributed with distribution p. Thus, the pseudocode of the algorithm can be found below.

Algorithm 3 Practical algorithm with initial coordinates

$P_0 \longleftarrow \mathbf{0}, \theta_0 \longleftarrow \mathbf{0}$
for $t = 1, \cdots, T$ **do**
  while θ has not converged **do**
    Sample $\mathbf{x}_1, \cdots, \mathbf{x}_{n_{\text{batch}}} \sim^{iid} \mu_{\text{emp}}$ with the corresponding outputs $y_1, \cdots, y_{n_{\text{batch}}}$
    Sample $\boldsymbol{\epsilon}_1, \cdots, \boldsymbol{\epsilon}_{n_{\text{batch}}} \sim^{iid} \frac{1}{\sqrt{2\pi\sigma^2}^n} e^{-\frac{\|\mathbf{x}\|^2}{2\sigma^2}}$
    Sample $\mathbf{z}_1, \cdots, \mathbf{z}_{n_{\text{batch}}}, \mathbf{z}'_1, \cdots, \mathbf{z}'_{n_{\text{batch}}} \sim^{iid} p$
    $L \longleftarrow \Big(\frac{1-\rho}{n_{\text{batch}}} \sum_{i=1}^{n_{\text{batch}}} \|\mathbf{G}_{\theta 1}(\mathbf{z}_i) - \phi_{\text{emp}}(\mathbf{z}_i)\|^2 + \frac{\rho}{n_{\text{batch}}} \sum_{i=1}^{n_{\text{batch}}} \|y_i - \mathbf{G}_{\theta 2}(\mathbf{x}_i + \boldsymbol{\epsilon}_i)\|^2 + \frac{\tilde{\lambda}}{n_{\text{batch}}} \sum_{i=1}^{n_{\text{batch}}} \|\frac{\partial E\mathbf{G}_\theta}{\partial \mathbf{x}}(\mathbf{z}'_i) - \frac{\partial E\mathbf{G}_{\theta_{t-1}}}{\partial \mathbf{x}}(\mathbf{z}'_i) P_{t-1}\|^2 \Big)$
    $\theta \longleftarrow \text{Adam}(\nabla_\theta L, \theta, \alpha, \beta_1, \beta_2)$
  $\theta_t \longleftarrow \theta$
  Sample $\mathbf{t}_1, \cdots, \mathbf{t}_{N_{\text{batch}}} \sim^{iid} p$
  $\hat{\mathcal{M}}_t \longleftarrow \frac{1}{N_{\text{batch}}} \sum_{i=1}^{N_{\text{batch}}} \frac{\partial E\mathbf{G}_{\theta_t}}{\partial \mathbf{x}}(\mathbf{t}_i)^\dagger \frac{\partial E\mathbf{G}_{\theta_t}}{\partial \mathbf{x}}(\mathbf{t}_i)$
  Find $\{\mathbf{v}_i\}_1^n$ s.t. $\hat{\mathcal{M}}_t \mathbf{v}_i = \lambda_i \mathbf{v}_i$, $\lambda_1 \geq \cdots \geq \lambda_n$
  $P_t \longleftarrow \sum_{i=1}^k \mathbf{v}_i \mathbf{v}_i^T$
Output: $\mathbf{v}_1, \cdots, \mathbf{v}_k$

5.5 Experiments

We made experiments on the standard datasets: Heart, Breast Cancer, Diabetes, Boston house prices and Wine quality. First, we applied Principal Component Analysis (PCA) and Sliced Inverse Regression (SIR) algorithms to the training set and calculated the effective subspace for $k = 2$. All points were projected to that space and we obtained two-dimensional representations of input points. In the last step we applied the 10 nearest neighbors algorithm (KNN) to predict outputs of the test set (for the regression case, the KNN regression was used). The same scheme was repeated with the Kernel Dimensionality Reduction (KDR) algorithm [14] and the alternating scheme with $\rho = 1, \frac{1}{2}, 0$. We verified that $\rho = 1$ corresponds to the pure sufficient dimension reduction case, because the best prediction was achieved for this value.

We experimented with algorithm 3 setting the key parameters as[1] (the data was standardized):

$$p(\mathbf{x}) = \frac{1}{\sqrt{2\pi}^n} e^{-\frac{\|\mathbf{x}\|^2}{2}},$$
$$T = 50, \tilde{\lambda} = 10.0, \sigma = 0.8,$$
$$\alpha = 10^{-4}, \beta_1 = 0.5, \beta_2 = 0.9.$$

Hyperparameters θ of the neural network model $\mathbf{G}_\theta$ were set as: $M' = N$ and M equals either 50 or 100. The parameters of the characteristic function were initialized as $\alpha_i^0 = \frac{1}{N}$, $\omega_i^0 = \mathbf{x}_i$ so that $\sum_{i=1}^{M'} \alpha_i^0 \cos(\omega_i^0 \cdot \mathbf{x}) = \phi_{\text{emp}}(\mathbf{x})$. Depending on the dataset, $n_{\text{batch}} \approx \frac{N}{10}$ and $N_{\text{batch}} \approx 10000$. In table 1 one can see the obtained test set accuracy on the classification tasks and R^2 on the regression tasks.

From Table 1 we see that the alternating scheme for $\rho = 1$ always outperforms SIR. As expected, the worst prediction accuracy is shown for $\rho = 0$, when the sufficiency term $J(f)$ is absent. Surprisingly, results for $\rho = \frac{1}{2}$ are also comparable with SIR's, which indicates that the two basic requirements, the proximity to the empirical distribution and the sufficiency to predict an output are often not mutually exclusive. The conclusion holds only if the distance to the empirical distribution

[1]Since the role of the parameter σ is similar to that of the bandwidth in the kernel density estimation, we use Silverman's rule of thumb to set σ.

Dataset	PCA %	SIR	KDR	$\rho = 1$	$\rho = \frac{1}{2}$	$\rho = 0$
Heart (acc)	79.80	81.1	84.5	83.2	80.4	81.8
Breast (acc)	93.46	96.5	95.1	97.1	95.6	94.0
Diabetes (R^2)	25.34	43.8	38.4	44.2	39.9	23.4
Boston (R^2)	56.42	76.2	70.4	76.7	74.2	51.3
Wine (R^2)	93.91	81.7	89.9	95.2	92.7	94.1

Table 1: The cross-validated accuracies and R^2 of KNN on 2-dimensional input representations.

is the maximum mean discrepancy distance. Figure 1 shows how the 2D scatter plots of points on the effective subspace look for the values of $\rho = 1, \frac{1}{2}, 0$.

Codes that simplify the reproducibility of our results can be downloaded from GitHub repository $https://github.com/k-nic/LR\_SDR$.

6 Conclusions

Unsupervised dimension reduction and sufficient dimension reduction tasks are usually treated as optimization tasks with substantially different objectives. An approach suggested in the paper deals with the hybrid setting, i.e. the objective that we study is the sum of the term that measures the consistency with an unsupervised part of data and the term that measures correlations between a projected input and an output. We develop a new approach to the minimization of such objectives that we call the alternating scheme. The results demonstrate that it is often possible to find a low-dimensional subspace that approximates well the support of unsupervised data, and at the same time can serve as an efficient subspace for the regression.

Acknowledgments

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP27510283), PI R. Takhanov.

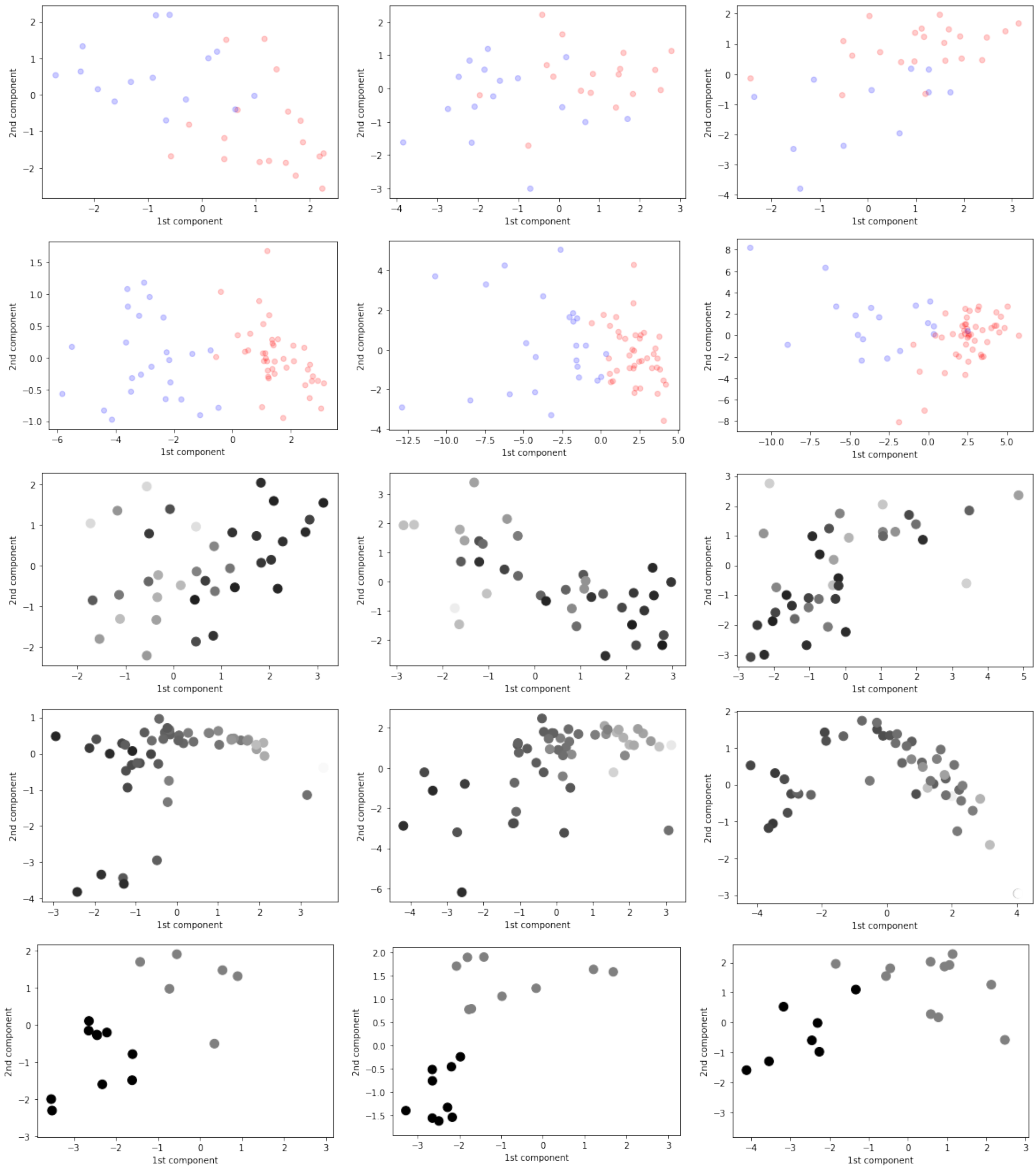

Figure 1: 2d-scatter plots of test sets for Heart, Breast Cancer, Diabetes, Boston and Wine datasets (rows) and for $\rho = 1, \frac{1}{2}, 0$ (columns). For regression tasks, the blackness of a point is proportional to a target variable's value.

References

[1] Z. Assylbekov, R. Takhanov, *Reusing weights in subword-aware neural language models.* Proceedings of the Conference of the North American Chapter of the Association for Computational Linguistics: Human Language Technologies (2018), 1413–1423.

[2] Z. Assylbekov, R. Takhanov, *Context vectors are reflections of word vectors in half the dimensions (extended abstract).* Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence (2021).

[3] M.T. Augustine, P. Patil, M. Bhushan, S. Bhartiya, *Autoencoder with ordered variance for nonlinear model identification.* Arxiv (2024), arXiv:2402.14031.

[4] A.R. Barron, *Universal approximation bounds for superpositions of a sigmoidal function.* IEEE Transactions on Information Theory 39 (1993), no. 3, 930–945.

[5] S. Bochner, *Vorlesungen über Fouriersche Integrale.* Akad. Verl.-Ges., 1932.

[6] R.D. Cook, *SAVE: a method for dimension reduction and graphics in regression.* Communications in Statistics - Theory and Methods 29 (2000), no. 9-10, 2109–2121.

[7] R.D. Cook, L. Forzani, *Principal fitted components for dimension reduction in regression.* Statistical Science 23 (2008), no. 4, 485–501.

[8] R.D. Cook, L. Forzani, *Likelihood-based sufficient dimension reduction.* Journal of the American Statistical Association 104 (2009), no. 485, 197–208.

[9] R.D. Cook, S. Weisberg, *Sliced inverse regression for dimension reduction: comment.* Journal of the American Statistical Association 86 (1991), no. 414, 328–332.

[10] T.F. Cox, M. Cox, *Multidimensional scaling.* Chapman and Hall/CRC, 2000.

[11] J.P. Cunningham, Z. Ghahramani, *Linear dimensionality reduction: survey, insights, and generalizations.* Journal of Machine Learning Research 16 (2015), 2859–2900.

[12] R.A. Fisher, *The use of multiple measurements in taxonomic problems.* Annals of Eugenics 7 (1936), no. 2, 179–188.

[13] K. Pearson, *On lines and planes of closest fit to systems of points in space.* Philosophical Magazine 2 (1901), no. 11, 559–572.

[14] K. Fukumizu, F.R. Bach, M.I. Jordan, *Dimensionality reduction for supervised learning with reproducing kernel Hilbert spaces.* Journal of Machine Learning Research 5 (2004), 73–99.

[15] B. Ghojogh, A. Ghodsi, F. Karray, M. Crowley, *Sufficient dimension reduction for high-dimensional regression and low-dimensional embedding: tutorial and survey.* Arxiv (2021), arXiv:2110.09620.

[16] A. Gretton, K. Borgwardt, M. Rasch, B. Schölkopf, A. Smola, *A kernel two-sample test.* Journal of Machine Learning Research 13 (2012), 723–773.

[17] T. Hastie, R. Tibshirani, *Discriminant analysis by Gaussian mixtures.* Journal of the Royal Statistical Society Series B 58 (1996), no. 1, 155–176.

[18] H. Hotelling, *Relations between two sets of variates.* Biometrika 28 (1936), no. 3-4, 321–377.

[19] T. Hsing, R. Eubank, *Theoretical foundations of functional data analysis.* Wiley, 2015.

[20] D.P. Kingma, J. Ba, *Adam: a method for stochastic optimization.* Proceedings of the International Conference on Learning Representations (2015).

[21] V. Kolmogorov, M. Rolinek, R. Takhanov, *Effectiveness of structural restrictions for hybrid CSPs.* Proceedings of the 26th International Symposium on Algorithms and Computation (2015), 566–577.

[22] K.-C. Li, *Sliced inverse regression for dimension reduction.* Journal of the American Statistical Association 86 (1991), no. 414, 316–327.

[23] K.-C. Li, *On principal Hessian directions for data visualization and dimension reduction.* Journal of the American Statistical Association 87 (1992), no. 420, 1025–1039.

[24] K. Mynbaev, Zh. Assylbekov, *Convergence of the partition function in the static word embedding model.* Eurasian Mathematical Journal 13 (2020), no. 4, 70–81.

[25] K. Mynbaev, *Asymptotic distribution of the OLS estimator for a mixed spatial model.* Journal of Multivariate Analysis 101 (2010), no. 3, 733–748.

[26] S. Mukherjee, D.-X. Zhou, *Learning coordinate covariances via gradients.* Journal of Machine Learning Research 7 (2006), 519–549.

[27] B. Schölkopf, A. Smola, K.-R. Müller, *Nonlinear component analysis as a kernel eigenvalue problem.* Neural Computation 10 (1998), no. 5, 1299–1319.

[28] M. Sugiyama, *Dimensionality reduction of multimodal labeled data by local Fisher discriminant analysis.* Journal of Machine Learning Research 8 (2007), 1027–1061.

[29] P. Switzer, A. Green, *Min/max autocorrelation factors for multivariate spatial imagery.* Stanford University, 1984.

[30] R. Takhanov, *Extensions of the minimum cost homomorphism problem.* Proceedings of the International Computing and Combinatorics Conference (2010), 328–337.

[31] R. Takhanov, *Hybrid VCSPs with crisp and valued conservative templates.* Proceedings of the 28th International Symposium on Algorithms and Computation (2017).

[32] R. Takhanov, *The algebraic structure of the densification and the sparsification tasks for CSPs.* Constraints 28 (2023), no. 1, 13–44.

[33] R. Takhanov, *Computing a partition function of a generalized pattern-based energy over a semiring.* Theory of Computing Systems 67 (2023), no. 4, 760–784.

[34] R. Takhanov, *On the speed of uniform convergence in Mercer's theorem.* Journal of Mathematical Analysis and Applications 518 (2023), 126718.

[35] R. Takhanov, *Reducing the dimensionality of data using tempered distributions.* Digital Signal Processing 133 (2023), 103819.

[36] R. Takhanov, *Multi-layer random features and the approximation power of neural networks.* Proceedings of the 40th Conference on Uncertainty in Artificial Intelligence (2024).

[37] R. Takhanov, *On the induced problem for fixed-template CSPs.* Proceedings of the 49th International Conference on Current Trends in Theory and Practice of Computer Science (2024), 485–499.

[38] R. Takhanov, Y.S. Abylkairov, M. Tezekbayev, *Autoencoders for a manifold learning problem with a Jacobian rank constraint.* Pattern Recognition 143 (2023), 109777.

[39] R. Takhanov, Z. Assylbekov, *Patterns versus characters in subword-aware neural language modeling.* Proceedings of the International Conference on Neural Information Processing (2017), 157–166.

[40] R. Takhanov, V. Kolmogorov, *Inference algorithms for pattern-based CRFs on sequence data.* Proceedings of the International Conference on Machine Learning (2013), 145–153.

[41] R. Takhanov, V. Kolmogorov, *Combining pattern-based CRFs and weighted context-free grammars.* Intelligent Data Analysis 26 (2022), 257–272.

[42] M. Wang, F. Sha, M.I. Jordan, *Unsupervised kernel dimension reduction.* Advances in Neural Information Processing Systems 23, 2010, 2379–2387.

[43] L. Wiskott, T.J. Sejnowski, *Slow feature analysis: unsupervised learning of invariances.* Neural Computation 14 (2002), no. 4, 715–770.

[44] Y. Xia, H. Tong, W.K. Li, L.-X. Zhu, *An adaptive estimation of dimension reduction space*. Journal of the Royal Statistical Society Series B 64 (2002), no. 3, 363–410.

Zhalgas Mukanov
Department of Fundamental Mathematics
L.N. Gumilyov Eurasian National University
13 Munaitpasov St
010008 Astana, Republic of Kazakhstan
E-mails: mukanovj@mail.ru

Anuar Sharafudinov
AILabs Technologies
159 West Broadway Ste 200
Salt Lake City, UTAH, USA
E-mail: AnuarSh@ailabs.kz

Rustem Takhanov, Arman Bekembayev
Mathematics Department
Nazarbayev University
53 Kabanbay Batyr Ave
010000 Astana, Republic of Kazakhstan
E-mails: rustem.takhanov@nu.edu.kz, arman.bekembayev@nu.edu.kz

Received: 02.08.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 17, Number 1 (2026), 77 – 90

REDUCIBILITY TO MULTIPERIODIC LINEAR SYSTEMS WITH A DIAGONAL DIFFERENTIATION OPERATOR AND ITS APPLICATION TO CONDITIONALLY PERIODIC SYSTEMS

Zh. Sartabanov

Communicated by V.I. Burenkov

Key words: reducibility, multiperiodicity, differentiation operator, periodic characteristic, cylindrical surface, Gershgorin's circles.

AMS Mathematics Subject Classification: 35B10, 35F05, 35R01, 34C46.

Abstract. In this paper, the main theorem is proved by establishing the reducibility to an equivalent multiperiodic linear system with a differentiation operator directed along the diagonal of the independent variables space. It is shown that helical lines on a circular cylindrical surface form periodic characteristics of the operator. The reducibility of a multiperiodic system is examined near a helix starting from the initial point of the phase circle, following the classical approach used for periodic systems. A monodromy matrix is introduced, which remains constant along the first integrals of the characteristic equations and possesses the properties of smoothness and multiperiodicity. The existence of localised positive eigenvalues consistent with the properties of this matrix is demonstrated. It is assumed that at the initial point of the phase circle, the monodromy matrix attains the maximal number of distinct eigenvalues. Their localisation on the cylindrical surface is established using the Gershgorin method.

DOI: https://doi.org/10.32523/2077-9879-2026-17-1-77-90

1 Introduction

The reducibility of linear systems with continuously periodic coefficients to a system with constant coefficients was given for the first time in the dissertation work of A.M. Lyapunov [19]. In modern scientific literature, the issue of the reducibility of periodic systems is primarily met with a rationale associated with Floquet's theorem [10], which pertains to the representation of their fundamental solutions.

The equivalence of the reducibility problem to the representation of fundamental solutions by Floquet was shown later in [8].

Thus, nowadays, the reducibility of periodic systems of ordinary differential equations has become well known. The interest in the reducibility of conditionally periodic systems probably began in the 1930s of the last century in connection with the application of electronic tubes in technology for generating and receiving electromagnetic oscillations with different frequencies. Work [2] is considered to be beginning of a new stage in the development of nonlinear oscillation theory, and linear oscillation theory with periodic and quasi-periodic behaviour, was at the centre of interest as a basis of studying oscillations [3].

The unresolved and important issue of the reducibility of linear conditionally periodic systems is particularly highlighted in [4]. In [15], to study this issue a method based on the transition to multiperiodic systems of partial differential equations with a special differentiation operator is proposed and the sufficiency of studying it in a small neighborhood of the diagonal of the space of independent variables is noted.

In this paper, we followed the author's recommendation of the proposed method related to the transition to systems of partial differential equations by applying Bohr's theorem [6] and used some ideas from [5, 7, 11, 12, 22].

The specificity of the characteristic equation $dt/d\tau = e$ for the operator D is as follows: *first*, although the basic system depends on this vector field, it does not depend on the original given system with the operator D, *secondly*, it can be considered an autonomous equation with a given vector field $v(t) = e$ in the phase space and, *thirdly*, the vector field $v(t) = e$ can be considered periodic with any period p and, in particular, we can put $p = \theta$.

Obviously, one and only one phase curve $t = h(\tau, \xi, \eta), \tau \in \mathbb{R}$ passes through each point of the characteristic equation's phase space (ξ, η). Hence, different phase curves of this equation do not intersect.

It is also known from the theory of autonomous systems that the solution $t = h(\tau, \xi, \eta)$ to the characteristic equation, which takes the same value $h(\tau_1, \xi, \eta) = h(\tau_2, \xi, \eta)$ at two different points $\tau = \tau_1$ and $\tau = \tau_2$, is periodic with the smallest period $\theta = \tau_2 - \tau_1 > 0$, and $\{k\theta, k \in \mathbb{Z}\}$ is the set of periods, where $\mathbb{Z}$ is a set of integers.

The characteristic equation does not have a constant solution since all components of the vector field $v(t) = e$ are positive numbers.

Note also that the initial D-system is given in the Euclidean space, therefore, following this premise in [15] the Euclidean space is also considered as the phase space of the characteristic equation. At the same time, the phase integral lines are straight lines that violate the basic specificity of the original system associated with its θ-periodicity in τ. This was a big insurmountable obstacle in the study of the system's problems related to the periodicity in τ of period θ. Therefore, the author of this method and his followers [1, 21, 26, 27, 28], along with other problems about multiperiodic oscillations, limited themselves to the study of special cases of this problem. The multiperiodic and periodic solutions of various classes of ordinary differential equations and partial differential equations were studied also in [9, 13, 14, 16, 17, 18].

It turns out that if we consider the above-mentioned specifics of the characteristic equation, this difficulty can be overcome by replacing the Euclidean phase space with a cylindrical surface [20, 23, 24, 25]. Then, as the phase integral curves of the characteristic equation of the differentiation operator, we can take a helical line at an angle of elevation $\frac{\pi}{4}$. In this case, if τ represents the shift of a point along the generatrix of the cylinder, and by t_j we mean the length of the arc of the phase circle, then the point (τ, t_j) changes along the helix in the three-dimensional Euclidean space, and t_j changes θ-periodically with respect to τ. Thus, the concept of a helical line in the three-dimensional space is introduced. Then, based on the Cartesian product of three-dimensional helical lines we have multidimensional helical lines.

Following this innovation, in this paper there is investigated the reducibility of a linear homogeneous multiperiodic system with a diagonal differentiation operator of independent variables to a similar system but with a constant on the diagonal multi-periodic matrix in terms of equivalent systems introduced in [22].

To prove the existence of the logarithm of the monodromy matrix of the original system in the ε-neighbourhood of the diagonal of independent variables we use the method of localisation of eigenvalues according to Gershgorin [11], [12], from which follows the existence of a closed line not passing through the zero of the complex plane covering the whole spectrum of the monodromy matrix. Consequently, the required logarithm is determined by the Cauchy integral formula [7].

In conclusion, the main result of the reducibility of a conditionally periodic matrix equation with an ordinary differentiation operator to a matrix equation with a constant matrix is obtained based on the transition from an equation with a differentiation operator to an equation with a diagonal matrix according to [15].

2 Problem statement

We consider the following linear system

$$\begin{aligned}&\frac{d}{d\tau}X = P(\tau,t)X, \quad \frac{dt}{d\tau} = e,\\ &P(\tau+\theta, t+k\omega) = P(\tau,t) \in C^{(0,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m),\\ &\sum_{j=0}^{m} k_j\omega_j \neq 0,\ \sum_{j=0}^{m}|k_j| \neq 0,\ k_j\in\mathbb{Z}, j=\overline{1,m},\end{aligned} \tag{2.1}$$

where $\tau \in (-\infty,+\infty) = \mathbb{R}$, $t = (t_1,\ldots,t_m) \in \mathbb{R}\times\ldots\times\mathbb{R} = \mathbb{R}^m$, $e = (1,\ldots 1)$ and $\omega = (\omega_1,\ldots,\omega_m)$ are m-vectors, $\omega_j \in \mathbb{R}, j = \overline{1,m}$, $\theta = \omega_0 = const \in \mathbb{R}$, $P(\tau,t)$ is a given n-matrix function, X is the required n-matrix function, $C^{(0,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m)$ is a class of functions of variables $(\tau,t) \in \mathbb{R}\times\mathbb{R}^m$ of smoothness $(0,e)$.

The reducibility of system (2.1) to a system

$$\frac{d}{d\tau}Y = AY, \tag{2.2}$$

with a constant matrix A is investigated, based on the linear matrix replacement

$$X = Q(\tau,t)Y,\ Q(\tau+\theta,t+\omega) = Q(\tau,t) \in C^{(1,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m),\ \det Q(\tau,t) \neq 0, \tag{2.3}$$

which is quasi-periodic along the diagonal $t = e\tau$.

The problem (2.1)–(2.3) is formulated in [4] with an indication of the characteristic equation

$$\frac{dt}{d\tau} = e, \quad e = (1,...,1), \tag{2.4}$$

which is a subsystem of system (2.1).

System (2.4) is a Cartesian product of independent scalar equations

$$\frac{dt_j}{d\tau} = 1,\ j = \overline{1,m}, \tag{2.5}$$

which are convenient for further study of vector equation (2.4).

The problem of reducibility is investigated based on the method [15], according to which, along with equation (2.1), we consider a linear multiperiodic matrix equation with a diagonal differentiation operator of the form

$$DX = P(\tau,t)X, \quad D = \frac{\partial}{\partial\tau} + \sum_{j=1}^{m}\frac{\partial}{\partial t_j}, \tag{2.6}$$

where all input data satisfies the conditions of equations (2.1) respectively.

Since [15] along the characteristics $t = \beta(\tau,\xi,\eta)$ of equation (2.4), the operator D of differentiation of the function $x(\tau,t) \in C^{(1,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m)$ passes into the ordinary operator $\frac{d}{d\tau}$ of differentiation of the function $x(\tau,\beta(\tau,\xi,\eta))$ and inversely, then for $t = e\tau = \beta(\tau,0,0)$ conditionally periodic equation (2.1) and multiperiodic equation (2.6) are equivalent.

According to Bohr's theorem [6], which states the relationship between continuous periodic functions of many variables and continuous almost periodic functions of one variable, the matrix $P(\tau,t)$ given in (2.1) and (2.6) at $t = e\tau$ becomes a conditionally periodic matrix $P(\tau,e\tau)$ with a constant basis $(\omega_0^{-1},\omega_1^{-1},...,\omega_m^{-1})$ with incommensurable frequencies $\nu_0 = \omega_0^{-1}, \nu_1 = \omega_1^{-1},...,\nu_m = \omega_m^{-1}$.

Thus, the problem of the reducibility of (2.1)–(2.3) was reduced to the problem of reducibility of multiperiodic equation (2.6) to equation (2.2) along the diagonal of independent variables $t = e\tau$ based on substitution (2.3).

3 Periodic characteristics of the diagonal differentiation operator

Assuming the variable t is one-dimensional, equation (2.5) is takes the form

$$\frac{dt}{d\tau} = 1. \tag{3.1}$$

Then if we take the straight lines

$$t = \eta + \tau - \xi$$

with the initial point $(\xi, \eta) \in \mathbb{R}^2$ as phase lines, we have rectilinear motion, which is a particular case of curvilinear motion.

If we take a circle S of length $2\pi r = \theta$ on the vOw plane as a phase line and by t we mean the length of the arc s of the circle S, and by $\varphi = \theta^{-1}\tau$ the angle of rotation of the radius vector of the arc t, then $t = t(v, w)$ is determined by its coordinates as a point (v, w) of the circle S:

$$v = rsin\left(2\pi\theta^{-1}\tau\right), \quad w = r - rcos\left(2\pi\theta^{-1}r\right) = r + rsin\left(2\pi\theta^{-1}\left(r - \frac{\theta}{4}\right)\right).$$

Obviously, $dt = d\tau$, and

$$t = \eta + \oint_{\xi}^{\tau} \sqrt{v'^2 + w'^2}d\tau \equiv \gamma(\tau, \xi, \eta)$$

represents circular motion with period θ, which is a special case of general curvilinear motion. The points t and $t + k\theta, k \in \mathbb{Z}$ are identical on the circle S.

These planar straight-line and circular motions constitute strong restrictions on the systems under consideration, namely (2.1) and (2.6).

In [15] the author limited himself to the flat rectilinear case and the problem was not investigated from a more general point of view. In the case of circular motion, systems (2.1) and (2.6) are found to be T-periodic, for which this problem is solved in [15] and [19].

The most general kind of motion is the rotational-reciprocating motion. Since, according to systems (2.1) and (2.6), rotation must have the property of periodicity, the phase integral curves of equation (2.4) must be helical in nature. This circumstance suggests the consideration of equation (3.1) on an infinite circular cylindrical surface of space $(u, v, w) \in \mathbb{R}^3$.

In this regard, we consider the characteristic equation of the operator D on a θ-circular cylindrical surface with the parametric equations

$$u = \tau, \quad v = rsin\left(\frac{t}{r}\right), \quad w = r - rcos\left(\frac{t}{r}\right) = r + rsin\left(\frac{t}{r} - \frac{\pi}{2}\right) \tag{3.2}$$

with two parameters τ and t, which are related to equation (3.1), where $2\pi r = \theta$.

According to equation (3.1), we have a β-line of the surface $\mathcal{M}$ of the form

$$t = \eta + \tau - \xi \equiv \beta(\tau, \xi, \eta), \quad (\tau, t) \in \mathcal{M} \tag{3.3}$$

with the initial data $(\xi, \eta) \in \mathcal{M}$.

Substituting (3.3) into (3.2), we obtain the equation of the phase integral curve of equation (3.1) on the surface $\mathcal{M}$ of the form

$$u = \tau, \quad v = rsin\left(\frac{\eta + \tau - \xi}{r}\right), \quad w = r + rsin\left(\frac{\eta + \tau - \xi}{r} - \frac{\pi}{2}\right) \tag{3.4}$$

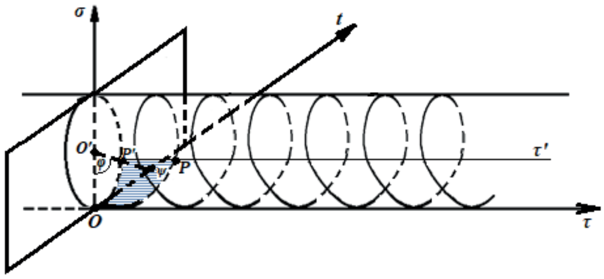

Figure 2: Projection of a circular helix onto a plane

with the parameter $\tau \in \mathbb{R}, (\xi, \eta) \in \mathcal{M}$ (see Fig. 2). The sweep function is the projection of the circular helix (see Fig. 2) on the plane τOt, where $O(0,0,0) = O$, $O'(0,0,r) = O'$, $P'(0,t,\sigma) = P'$, $P(\tau,t,\sigma) = P$, $\psi = \tan\varphi$, $[0,\psi] \subset (Ot) = \mathbb{R}$, $\varphi = \angle OO'P'$.

Equations (3.4) are parametric equations of a helix, where τ is the length of displacement of a point $P(u,v,w)$ along the cylinder generatrix, passing through this point, and $t = t(v,w)$ represents the arc length s of the phase circle corresponding to the polar angle $\varphi = \eta + \tau - \xi$ in radians. Obviously,

$$t = \eta + (\beta) \int_{\xi}^{\tau} \sqrt{v'^2(\tau) + w'^2(\tau)} d\tau \equiv \beta(\tau, \xi, \eta), (\tau, t) \in \mathcal{M}, \tag{3.5}$$

with the initial point $(\xi, \eta) \in \mathcal{M}$.

Thus, from the geometric interpretation of equation (3.1) in the plane $\mathbb{R}^2$ we have switched to its kinematic interpretation in the three-dimensional space $\mathbb{R}^3$. The independent variable τ in $\mathbb{R}^2$ is one of the coordinates of the point (τ, t), and in $\mathbb{R}^3$ space becomes a time parameter characterizing the motion of the point along a helical line drawn on a cylindrical surface.

The other dependent coordinate t of equation (3.1) on the plane $\mathbb{R}^2$, in space $\mathbb{R}^3$ characterizes the motion of a point depending on time τ (numerically equal to the displacement along the cylinder's generatrix) along the path of a kind of helical spiral located on the cylinder, moreover, the phase trajectory t is a circle S obtained by projecting a helix in the direction of the τ axis.

In general, on $\mathcal{M}$, the β-characteristics of the operator D have the following properties

$$\beta(\tau + \theta, \xi, \eta) = \beta(\tau, \xi + \theta, \eta) = \beta(\tau, \xi, \eta + \omega) - \omega = \beta(\tau, \xi, \eta) = \beta(\tau, \zeta, \beta(\zeta, \xi, \eta)). \tag{3.6}$$

It is important to have flat equivalents of helical lines (3.3)–(3.5) for quantitative analysis of system (2.6) (see Fig. 3).

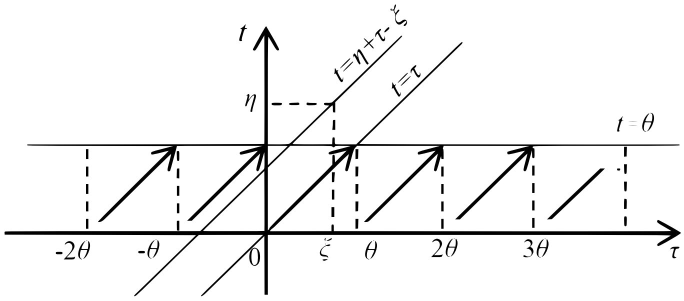

Figure 3: Cycloidal projection of helical curves

We consider the cycloidal sweep of χ-mapping of a θ-circular infinite cylindrical surface to get an idea of the equation of the helix $t = \beta(\tau, 0, 0) \equiv \beta^*(\tau)$ on the

$(\tau, t) \in \mathbb{R}^2$ plane. This surface is obtained by cutting the cylinder along its lower generatrix and unfolding it upward onto the plane by shifting the phase circle S without sliding. Then, the cylindrical surface turns into an infinite strip Π_0 with boundaries $\tau = 0$ and $\tau = \theta$.

In the χ-mapping, the areas, angles and lengths of the arcs are preserved, and the arcs of the helix are straightened and become segments. For example, the equation of the helical line $t = \beta^*(\tau)$ originating from the initial point O of space $(u, v, w) \in \mathbb{R}^3$ on the plane has the form $t = \theta\{\theta^{-1}\tau\} \equiv \}(\tau)$, where $\{\tau\}$ is the fractional part of the number τ. The graph of this jump function is well-known (Fig. 3).

The quantitative analysis related to the issue of integrating functions of variables (τ, t) along the helical lines on the plane $\mathbb{R}^2$ is carried out based on the functions $\}(\tau)$ and

$$](\tau) = \tau - \}(\tau) \equiv \theta[\theta^{-1}, \tau]$$

where $[\tau]$ is the integer part of the number τ.

Note that the fractional partial function $\}(\tau)$ is a smooth, θ-periodic on $\mathcal{M} \subset \mathbb{R}$, and the θ-step function $](\tau)$ is a function identical to zero, moreover, they are summands of the expansion of the diagonal function $t = \tau \equiv](\tau) + \}(\tau)$ on the plane $\mathbb{R}^2$. The results of the integration of functions over $\xi \in \mathbb{R}$ in the limits from τ^0 to τ are presented in terms of the first characteristic integrals of the form

$$\eta = \beta(\xi, \tau, t), \quad \tau^0 \xrightarrow{\xi} \tau, \quad (\tau, t) \in \mathcal{M}, \tag{3.7}$$

which relate to the first integrals $\eta = \delta(\xi, \tau, t)$ of the operator D by the relation

$$\beta(\xi, \tau, t)|_{(\tau,t)\in\mathcal{M}} = \delta(\xi, \tau, t) -](\xi - \tau)|_{(\tau,t)\in\mathbb{R}^2}, \quad \xi \in \mathbb{R}, \tag{3.8}$$

on the plane $(\tau, t) \in \mathbb{R}^2$.

Here we tried to consider the recommendations of [22] on a reasonable definition of the phase space of a vector field based on the specifics of the statement problem. In this case, the specifics of the problem include: 1) the vector field describes the differentiation operator of functions along its direction and, 2) on the manifold where the differentiation operator was defined, this vector field must allow periodic integral curves with a given period, although it has only linear solutions in the plane.

Information about the differentiation operator can be obtained from [5] and [15]. Based on the T-periodic characteristics (3.3)–(3.5) of equations (3.1), the characteristics

$$t_j = \beta_j(\tau, \xi, \eta_j), \quad (\tau, t_j) \in \mathcal{M}$$

of equation (3.1) with initial data $(\xi, \eta_j) \in \mathcal{M}$, $j = \overline{1, m}$ were obtained. Then, the Cartesian product of these characteristics is a vector characteristic

$$t = (\beta_1(\tau, \xi, \eta_1), ..., \beta_m(\tau, \xi, \eta_m)) \equiv \beta(\tau, \xi, \eta), \quad (\tau, t) \in \mathcal{M}^m, \tag{3.9}$$

of system (2.4) with initial data $(\xi, \eta) \in \mathcal{M}^m$, where $\eta = (\eta_1, ..., \eta_m)$.

The β-characteristics (3.9) of system (2.4) have the properties of (θ, ω)-periodicity and a group of form (3.6).

The first characteristic integrals of system (2.4) by (3.7) and (3.9) are defined in the form

$$\eta = \beta(\xi, \tau, t), (\tau, t) \in \mathcal{M}^m, \tag{3.10}$$

with the parameters $(\xi, \eta) \in \mathcal{M}^m$.

The connection of helical characteristic integrals (3.10) with the linear Euclidean characteristic integrals

$$\eta = \delta(\xi, \tau, t), \quad (\tau, t) \in \mathbb{R} \times \mathbb{R}^m = \mathbb{R}^{1+m}$$

following (3.8) and (3.9) is determined by the relation

$$\beta(\xi,\tau,t)|_{(\tau,t)\in\mathcal{M}} = \delta(\xi,\tau,t)|_{(\tau,t)\in R^{1+m}} -](\xi-\tau), \quad \xi\in\mathbb{R}, \tag{3.11}$$

Thus, the following statement is justified.

Theorem 3.1. *Characteristic equation (2.4) of the diagonal differentiation operator D on an infinite cylindrical surface $\mathcal{M}^m$ has β-characteristics (3.9) which have the properties of (θ,ω)-periodicity and group (3.6). Moreover, its corresponding first characteristic integrals (3.10) through δ-characteristics equation (2.4) in the Euclidean space $\mathbb{R}^{1+m}$ are related to relation (3.11).*

4 Reducibility of a multiperiodic linear homogeneous system with a diagonal differentiation operator

We consider a multiperiodic linear homogeneous system with a diagonal differentiation operator in matrix form (2.6). The characteristic equation (2.4) of the operator D, its β-characteristics (3.9) with properties (3.6) and the first integrals

$$\eta = \beta(\xi,\tau,t), \quad (\xi,\eta)\in\mathcal{M}^m, \quad D\beta(\xi,\tau,t)=0 \tag{4.1}$$

are defined on the cylindrical surface $\mathcal{M}^m$, where (ξ,η) is the initial point of the characteristic $t=\beta(\tau,\xi,\eta)$, which is smooth in all arguments, having the properties of (θ,ω)-periodicity and the group given in (4.1), and the first integrals $\eta=\beta(\xi,\tau,t)$ are defined based on the characteristics.

The *matricant* $X(\tau,t)$ of equation (2.6) is uniquely determined by the integral equation below and has the properties of ω-periodicity in t and smoothness in (τ,t):

$$\begin{aligned} &X(\tau,t) = E + \int_0^\tau P(\zeta,\beta(\zeta,\tau,t))X(\zeta,\beta(\zeta,\tau,t))d\zeta, \\ &X(\tau,t+\omega) = X(\tau,t)\in C^{(1,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m), \quad det X(\tau,t)\neq 0, \end{aligned} \tag{4.2}$$

where E is the unit matrix.

The *monodromy matrix* $X(\theta,\beta(0,\tau,t))$ of equation (2.6) satisfies the equation

$$X(\tau+\theta,t) = X(\tau,t)X(\theta,\beta(0,\tau,t)). \tag{4.3}$$

Obviously, both parts of equality (2.4) satisfy equation (2.6) and for $\tau=0$, by virtue of (4.1), we have the same initial condition. Hence, from the unique solvability of the initial problem for equation (2.6), we have identity (4.3).

If $\Phi(\tau,t)$ is any fundamental solution with a ω-periodic initial value of equation (2.6), then it can be represented in the form

$$\begin{aligned} &\Phi(\tau,t) = X(\tau,t)\Phi(0,\beta(0,\tau,t)), \\ &\Phi(0,t+\omega) = \Phi(0,t), \quad det\Phi(0,t)\neq 0, \end{aligned} \tag{4.4}$$

which has the following properties:

$$\begin{aligned} &\Phi(\tau+\theta,t) = \Phi(\tau,t)\cdot\Phi^{-1}(0,\beta(0,\tau,t))X(\theta,\beta(0,\tau,t))\Phi(0,\beta(0,\tau,t)), \\ &\Phi(\tau,t+\omega) = \Phi(\tau,t)\in C^{(1,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m), \end{aligned} \tag{4.5}$$

The matrix

$$U(\beta(0,\tau,t)) = \Phi^{-1}(0,\beta(0,\tau,t))X(\theta,\beta(0,\tau,t))\Phi(0,\beta(0,\tau,t)) \tag{4.6}$$

is called the *main matrix* of the solution (4.4). Next by the fundamental solution, we mean a solution with a ω-periodic in t a *nonsingular* initial matrix.

Let us show that representations (4.4) and (4.5) are valid.

Indeed, both the right and left sides of these equalities satisfy equation (2.6) and, by virtue of (4.1)–(4.3), turn into identities in $\tau = 0$. Therefore, based on the unique solvability of the initial problem for system (2.6), we have the identities for $\tau \in \mathbb{R}$. This is exactly what is needed to be proved.

Thus, the following theorem is proved.

Theorem 4.1. *Any fundamental solution* $\Phi(\tau, t)$ *of system* (2.6) *has the property*

$$\Phi(\tau + \theta, t) = \Phi(\tau, t)U(\beta(0, \tau, t)) \tag{4.7}$$

with its basic matrix $U(\beta(0, \tau, t))$ *that is defined by relation* (4.6).

Along with system (2.6), we consider another similar multiperiodic system

$$\begin{aligned} &DY = Q(\tau, t)Y, \\ &Q(\tau + \theta, t + \omega) = Q(\tau, t) \in C^{(0,e)}_{\tau,t}(\mathbb{R} \times \mathbb{R}^m) \end{aligned} \tag{4.8}$$

with the matricant $Y(\theta, t)$ having the properties of (θ, ω)-periodicity of the form

$$\begin{aligned} &Y(\tau + \theta, t) = Y(\tau, t) \cdot Y(\theta, \beta(0, \tau, t)), \\ &Y(\tau, t + \omega) = Y(\tau, t) \in C^{(1,e)}_{\tau,t}(\mathbb{R} \times \mathbb{R}^m). \end{aligned} \tag{4.9}$$

Definition 1. Systems (2.6) and (4.11) are called equivalent if there exists a matrix $T(\tau, t)$ having properties

$$T(\tau + \theta, t + \omega) = T(\tau, t) \in C^{(1,e)}_{\tau,t}(\mathbb{R} \times \mathbb{R}^m), \quad detT(\tau, t) \neq 0, \tag{4.10}$$

such that linear substitution

$$X = T(\tau, t)Y \tag{4.11}$$

transforms system (2.6) to system (4.11), where Y has properties (4.9).

Theorem 4.2. *For systems* (2.6) *and* (4.8) *to be equivalent, it is necessary and sufficient that they admit fundamental solutions of* $\Phi(\tau, t)$ *and* $\Psi(\tau, t)$ *with a common basic matrix:*

$$\begin{aligned} &U(\beta(0, \tau, t)) = V(\beta(0, \tau, t)), \\ &\Phi(\tau + \theta, t) = \Phi(\tau, t)U(\beta(0, \tau, t)), \\ &\Psi(\tau + \theta, t) = \Psi(\tau, t)V(\beta(0, \tau, t)). \end{aligned} \tag{4.12}$$

The existence of the basic matrix $V(\beta(0, \tau, t))$ *follows by Theorem* 4.1 *by virtue of* (4.7).

Proof. Let systems (2.6) and (4.8) be equivalent. Then, by virtue of (4.10) and (4.11), we have a sequence of identities of the form

$$\Phi(\tau, t) = T(\tau, t)\ \Psi(\tau, t),$$

$$\Phi(\tau + \theta, t) = \Phi(\tau, t)U(\tau, t) = T(\tau, t)\Psi(\tau + \theta, t) = T(\tau, t)\Psi(\tau, t) \cdot V(\tau, t) = \Phi(\tau, t)V(\tau, t).$$

From here we get (4.12).

Inversely, let we have (4.12). Let us prove (4.11). Dividing the second identity of (4.12) by the third one, we obtain

$$\begin{aligned} \Phi(\tau + \theta, t)\Psi^{-1}(\tau + \theta, t) &= \Phi(\tau, t)U\left(\beta(0, \tau, t)\right)V^{-1}\left(\beta(0, \tau, t)\right)\Psi^{-1}(\tau, t) \\ &= D\Phi(\tau, t)\Psi^{-1}(\tau, t). \end{aligned} \tag{4.13}$$

Further, let the following equality hold:

$$\Phi(\tau,t)\Psi^{-1}(\tau,t) = T(\tau,t). \tag{4.14}$$

As can be seen from (4.13) matrix (4.14) is (θ,ω)-periodic, and

$$\Phi(\tau,t) = T(\tau,t)\Psi(\tau,t). \tag{4.15}$$

Since $\Phi(\tau,t) = X$ and $\Psi(\tau,t) = Y$ are arbitrary fundamental solutions of equations (2.6) and (4.8), respectively, then the established relation between them (4.15) defines linear transformation (4.11). □

The (θ,ω)-periodic system (2.6) can be considered as $(2\theta,\omega)$-periodic in $(\tau,t) \in \mathbb{R}\times\mathbb{R}^m$. Then,

$$X\left(2\theta, \beta(0,\tau,t)\right) = X^2\left(\theta, \beta(0,\tau,t)\right). \tag{4.16}$$

Eigenvalues of (4.16) for fixed (τ,t) are real, and the elementary divisors corresponding to negative eigenvalues are repeated an even number of times. This is important in further studies on the logarithm of the monodromy matrix in terms of real-valued matrix functions.

The function $F\left(\beta(\xi,\tau,t)\right)$ of the first characteristic integrals $\eta = \beta(\xi,\tau,t)$ is said to be constant along the diagonal $t = e\tau$, since $\beta(\xi,\tau,e\tau) = e\xi$.

Theorem 4.3. *Equation* (2.1) *is reduced by linear substitution* (4.11) *with* (4.10) *to the equation*

$$DZ = K\left(\beta(0,\tau,t)\right)Z \tag{4.17}$$

with a diagonal constant matrix $K\left(\beta(0,\tau,t)\right)$*:*

$$K(t+\omega) = K(t) \in C_t^{(e)}\left(\mathbb{R}^m\right), \quad |t - e\tau| < \varepsilon \tag{4.18}$$

where $\varepsilon > 0$ *is a sufficiently small number.*

Proof. Under assumptions (4.1) and (4.8), equation (4.17) is (θ,ω)-periodic in (τ,t). Therefore, it is $(2\theta,\omega)$-periodic in (τ,t). For each fixed (τ,t), we have a fixed value $\eta = \beta(0,\tau,t)$ and suppose that

$$e^{2\theta K(\eta)} = X(2\theta,\eta), \tag{4.19}$$

hence, $X(2\theta,\eta) = E + \sum_{j=1}^{\infty}\frac{(2\theta)^j}{j!}K^j(\eta)$.

Since $X(2\theta,\eta) = X^2(\theta,\eta)$ according to (4.16), then the equality

$$K(\eta) = \frac{1}{2\theta}\ln X(2\theta,\eta) \tag{4.20}$$

is valid for every fixed value of the variable η.

In the case of differentiability $K\left(\beta(0,\tau,t)\right)$, the matricant of equation (4.17) has the form

$$Z = e^{\tau K(\beta(0,\tau,t))}$$

and the monodromy matrix is

$$W\left(\beta(0,\tau,t)\right) = e^{2\theta K(\beta(0,\tau,t))}. \tag{4.21}$$

By virtue of (4.18) and (4.21) we obtain the equality

$$W\left(\beta(0,\tau,t)\right) = X\left(2\theta, \beta(0,\tau,t)\right) \tag{4.22}$$

for the monodromy matrix of equation (4.17). The coincidence of the monodromy matrices (4.22) by Theorem 4.2 proves the equivalence of equations (2.6) and (4.17).

To complete the proof, it is required to show the validity of property (4.20).

Assuming, that $t = e\tau$, we have $\eta = \beta(0, \tau, e\tau)$ and the monodromy matrix of system (2.6) can be represented as

$$\begin{aligned} X\left(2\theta, \beta(0,\tau,t)\right) &= X(2\theta, 0) + \left[X\left(2\theta, \beta(0,\tau,t)\right) - X(2\theta,0)\right] \\ &= X(2\theta,0) + \Omega\left(\beta(0,\tau,t)\right), \quad \Omega(\eta+\omega) = \Omega(\eta) \in C_\eta^{(e)}(\mathbb{R}^m), \end{aligned} \tag{4.23}$$

where $\Omega(\eta) = X(2\theta, \eta) - X(2\theta, 0) \to 0$ for $\eta \to 0$.

Let the constant matrix A reduce the matrix $X(2\theta, 0)$ to the Jordan normal form

$$A^{-1}X(2\theta,0)A = J = \operatorname{diag}\left[\lambda_1 \mathrm{E}_1 + \delta_1 \mathrm{I}_1, \ldots, \lambda_l \mathrm{E}_l + \delta_l \mathrm{I}_l\right], \tag{4.24}$$

where $\lambda_1, \ldots, \lambda_l$ are real non-zero eigenvalues of the matrix $X(2\theta, 0)$, E_j are unit matrices of dimension $n_j \times n_j$, and I_j are over diagonal unit matrices, $n_1 + \ldots + n_l = n$; δ_j are sufficiently small positive constants, j=$\overline{1,l}$.

Then, by (4.24), equality (4.23) can be written in the form

$$\begin{aligned} A^{-1}X\left(2\theta, \beta(0,\tau,t)\right)A &= J + F\left(\beta(0,\tau,t)\right), \quad F(\eta) = \left[f_{ij}(\eta)\right], \\ F(\eta) = A^{-1}\Omega(\eta)A \to 0, \ \eta \to 0; &\quad F(\eta+\omega) = F(\eta) \in C_\eta^{(e)}(\mathbb{R}^m). \end{aligned} \tag{4.25}$$

According to Gershgorin's theorem [11, 12], we define circles Γ_j, in which lie the eigenvalues $\mu(\beta) = \mu_k(\beta) = \lambda_k + \nu_k(\beta)$ of the matrix $J + F(\beta)$ of form (4.25),

$$\Gamma_j : |\lambda_j + f_{jj}(\beta) - \mu(\beta)| \le \delta_j + \sum_{k=1, k\ne j}^{n} |f_{jk}(\beta)|, \ j = \overline{1,n}, \ \beta = \beta(0,\tau,t), \ |t - e\tau| < \varepsilon.$$

Obviously, $\beta(0, \tau, t) \to 0$ for $t \to e\tau$ and $f_{jk}(\eta) \to 0$ for $\eta \to 0$, and the constants $\delta_1, \ldots, \delta_i$ can be arbitrarily small. Then, the radii

$$\Delta_j(\beta) = \delta_j + \sum_{k=1, k\ne j}^{n} |f_{jk}(\beta)|$$

can be made arbitrarily small, and the centers $O_j(\beta) = \lambda_j + f_{jj}(\beta)$ can be brought closer to λ_j with any accuracy by diminishing the number $\varepsilon > 0$.

Here, inequalities $|t - e\tau| < \varepsilon$ and the concept $t \to e\tau$ should be understood in the meaning that the coordinates t_j of the vector t represent the length of the arc s_j of the circle S_Θ, and τ is the length of the shift along the generatrix of the cylinder are approximately equal to $t_j \approx \tau$ in a narrow helical strip $\mathbf{B}_\varepsilon$ of width 2ε with a central helix line $t_j = \tau$.

It can be shown that any eigenvalue of $\mu\left(\beta(0,\tau,t)\right) = \mu(\beta)$ and $\mu(\beta)$ satisfy the estimates

$$|\mu(\beta)| \ge \frac{1}{3}r, \quad |\mu(\beta) - \lambda_j| \le \frac{2}{3}r,$$

where $\min\limits_{1\le j\le l} |\lambda_j| = r > 0$,

$$\delta_j < \varepsilon, \quad |f_{jk}\left(\beta(0,\tau,t)\right)| \le \varepsilon$$

for $(\tau, t) \in \mathbf{B}_\varepsilon$ and $\varepsilon = \dfrac{r}{3n}$ for all $j, k = \overline{1,n}$.

Thus, all eigenvalue $\mu(\beta)$ of the matrix $J + F(\beta)$ will lie in one of the Gershgorin circles Γ_j, which do not intersect and do not cover the zero of the complex plane at a sufficiently small $\varepsilon > 0$.

Then, there is a closed contour γ, which includes all Gershgorin circles containing the eigenvalues of $\mu(\beta)$, but not covering the zero of the complex plane.

Consequently by [7], the logarithm $\ln[J+F(\beta)]$ is represented by the Cauchy integral formula

$$\ln[J+F(\beta)] = \frac{1}{2\pi i}\int\limits_{\gamma} (\lambda E - J - F(\beta))^{-1} \ln\lambda d\lambda, \tag{4.26}$$

and the eigenvalues of $\mu_j(\beta)$ inherit all the properties of the matrix related to (θ,ω)-periodicity and smoothness:

$$\mu_j\left(\beta(0,\tau+\theta,t+\omega)\right) = \mu_j\left(\beta(0,\tau,t)\right) \in C^{(1,e)}_{\tau,t}\left(\mathbb{R}\times\mathbb{R}^m\right), \quad 1\le j\le n. \tag{4.27}$$

Thus, from (4.26) and (4.27) follows (4.18), since by virtue of (4.25) the monodromy matrix $F(2\theta,\beta)$ is logarithmic. □

5 Reducibility of a conditionally periodic ordinary differential equation

A linear conditionally periodic matrix equation is given by (2.1). Since by [15] along the characteristics $t=\beta(\tau,\xi,\eta)$ of the equation $dt = e d\tau$, the operator D of differentiation of a function $x(\tau,t)\in C^{(1,e)}_{\tau,t}\left(\mathbb{R}\times\mathbb{R}^m\right)$ transforms into the ordinary operator $\dfrac{d}{d\tau}$ of differentiation of the function $x\left(\tau,\beta(\tau,\xi,\eta)\right)$ and inversely, it follows that for $t = e\tau = \beta(\tau,0,0)$ systems (2.6) and (2.1) are equivalent.

According to Bohr's theorem [6] on the relation between continuous periodic functions of many variables and continuous almost periodic functions of one variable, the matrix $P(\tau,t)$ given in (2.6) and (2.1) for $t=e\tau$ becomes a quasiperiodic matrix $P(\tau,e\tau)$ with constant basis $\left(\omega_0^{-1},\omega_1^{-1},\ldots,\omega_m^{-1}\right)$ with incommensurable with each other components

$$\nu_0=\omega_0^{-1}, \nu_1=\omega_1^{-1},\ldots,\nu_m=\omega_m^{-1}.$$

Thus, along the diagonal $t=e\tau$ of the independent variables $(\tau,t)\in(\mathbb{R}\times\mathbb{R}^m)$ both equation (2.6) and (2.1) can be represented as the equation

$$\begin{aligned} &\frac{d}{d\tau}Y = P(\tau,e\tau)Y, \quad Y = X(\tau,e\tau),\\ &P(\tau+\theta,e\tau+k\omega) = P(\tau,e\tau)\in C^{(0,e)}_{\tau,e\tau}(\mathbb{R}\times\mathbb{R}^m), \end{aligned} \tag{5.1}$$

where $\sum_{j=0}^m k_j\omega_j\neq 0$, $\sum_{j=0}^m |k_j|\neq 0$, $k_j\in\mathbb{Z}$, $j=\overline{0,m}$, $\omega_0=\theta$, $\omega=(\omega_0,\omega_1,\ldots,\omega_m)$.

As a result, by Theorem 4.3 applied to equation (5.1), we have the following reducibility theorem.

Theorem 5.1. *Conditionally periodic equation (5.1) is reduced by the conditionally periodic linear substitution*

$$\begin{aligned} &Y = T(\tau,e\tau)Z, \quad detT(\tau,e\tau)\neq 0,\\ &T(\tau+\theta,e\tau+\omega) = T(\tau,e\tau)\in C^{(1,e)}_{\tau,e\tau}\left(\mathbb{R}\times\mathbb{R}^m\right) \end{aligned} \tag{5.2}$$

to the equation

$$\frac{d}{d\tau}Z = AZ \tag{5.3}$$

with the constant matrix $A=K(0)$, where the transformation matrix $T(\tau,t)$ is defined by relations (4.14), (4.11) and (4.10), and the matrix $K(\eta)$ by relations (4.19).

Proof. First of all, it should be noted that Theorem 5.1 is a direct corollary of Theorem 4.3. Therefore, its justification is contained in the formulation of the theorem itself, with reference to transformation (4.11), which yields (5.2), and to equation (4.17), from which (5.3) follows. □

6 Conclusion

The general theory of almost-periodic functions of one variable and their connection with periodic functions of many variables originates from the works of P. Bohl, G. Bohr, A. Bezikovitch, V.V. Stepanov, S. Bochner, B.M. Levitan, and others.

Applied aspects of this theory in the form of differential models of oscillatory phenomena and constructive-technical processes were first investigated in the works of A.A. Andronov, N.N. Bogolyubov and N.M. Krylov, G. Bohr and O. Neugebauer, S.L. Sobolev, and others.

The problem of reducibility of linear almost periodic systems of ordinary differential equations probably appeared in these years in connection with the needs of practice and the tasks of simplifying the qualitative study of such equations as the basis of the theory of oscillations.

Research on this problem has been particularly intensified after creation of the KAM theory in 1954-1966, the creators of which are A.N. Kolmogorov, V.I. Arnold and J. Moser.

In his work [4], V.I. Arnold raised questions about the reducibility of general conditionally periodic linear equations. Further, he noted that the problem of the reducibility of linear equations with conditionally periodic coefficients naturally appears in the study of neighborhoods of invariant tori of autonomous systems that admit conditionally periodic motions.

Although this problem was formulated in terms of ordinary differential equations, it was consonant with the formulation of the reducibility problem for linear multiperiodic systems of partial differential equations by V.Kh. Kharasakhal [15].

In this study, it was established that in the Euclidean space there exists an infinite circular cylindrical surface on which the vector field of the differentiation operator with respect to the diagonal variables admits periodic characteristics [24, 25]. Using these helical characteristics, the reducibility of multiperiodic linear systems with a diagonal differentiation operator to systems with constant coefficients along the diagonal was demonstrated in a neighbourhood of the diagonal. The logarithm of the monodromy matrix was shown to exist by means of Gershgorin's localization of eigenvalues. By subsequently passing to the diagonal of independent variables, the principal result on the reducibility of conditionally periodic linear systems of ordinary differential equations was obtained in a sufficiently general situation.

Acknowledgments

This work was supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan, project no. AP19676629.

References

[1] G.M. Aitenova, Zh.A. Sartabanov, G.A. Abdikalikova, *Multiperiodic bounded oscillations in quasilinear finite-hereditary integro-differential systems convection-diffusion type.* Lobachevskii Journal of Mathematics 43 (2022), no. 8, 2046–2056. https://doi.org/10.1134/S199508022211004X

[2] A.A. Andronov, *Les cycles limites de Poincaré et la théorie des oscillations auto-entretenues.* UFN 93 (1967), no. 2, 329–331. https://doi.org/10.3367/UFNr.0093.196710h.0329

[3] A.A. Andronov, A.A. Vitt, S.E. Hajkin, *Self-oscillation theory.* Nauka, Moscow, 1981 (in Russian).

[4] V.I. Arnold, *Additional chapters of the theory of ordinary differential equations.* Nauka, Moscow, 1978 (in Russian).

[5] V.I. Arnold, *Ordinary differential equations.* Nauka, Moscow, 1971 (in Russian).

[6] H. Bohr, *Almost periodic functions.* Chelsea Publishing Company, New York, 1951.

[7] Yu.L. Daletsky, M.G. Krein, *Stability of solutions of differential equations in Banach space.* Nauka, Moscow, 1970.

[8] N.P. Erugin, *Reducible systems.* Proceedings of the Steklov Institute of Mathematics 13 (1946), 3–96 (in Russian).

[9] M. Fečkan, M. Pospisil, *On equations with generalized periodic right-hand side.* Ukrainian Mathematical Journal 70 (2018), no. 2., 288–318. https://doi.org/10.1007/s11253-018-1501-4

[10] G. Floquet, *Sur Les equations differentielles linearls a coefficients periodiques.* Annales scientifiques de l'École Normale Supérieure 12 (1883), 47–88 (in French).

[11] F.R. Gantmacher, *Theory of matrices.* Fizmatlit, Moscow, 2010 (in Russian).

[12] S. Gershgorin, *Über die Abgrenzung der eigenwerte einer Matrix.* Mathematics of the USSR-Izvestiya 6 (1931), 749–754 (in German).

[13] G.A. Grigorian, *On the reducibility systems of two linear first-order ordinary differential equations.* Monatshefte für Mathematics 194 (2021), no. 3, 513–522.

[14] P. Jebelean, C. Şerban, *Multiple periodic solutions for odd perturbations of the discrete relativistic operator.* Mathematics 10 (2022), no. 9, 1595. https://doi.org/10.3390/math10091595

[15] V.Kh. Kharasakhal, *Almost-periodic solutions of ordinary differential equations.* Nauka, Alma-Ata, 1970 (in Russian).

[16] K. Li, *Multiple periodic solutions for asymptotically linear Duffing equations with resonance.* Journal of Mathematical Analysis and Application 378 (2011), no. 2, 657–666. https://doi.org/10.1016/j.jmaa.2011.01.049

[17] C. Liu, S. Wang, *Multiple periodic solutions of second order parameter-dependent equations via rotation numbers.* AIMS Mathematics 10 (2023), no. 8, 25195–25219.

[18] K. Liu, M. Fečkan, J. Wang, *A class of (ω, T)-periodic solutions for impulsive evolution equations of Sobolev type.* Bulletin of the Iranian Mathematical Society 48 (2022), no. 5, 2743–2763. http://dx.doi.org/10.1007/s41980-021-00666-9

[19] A.M. Lyapunov, *The general problem of stability of motion.* GITTL, Moscow-Leningrad, 1980 (in Russian).

[20] B. Omarova, Zh. Sartabanov, A. Zhumagaziyev, G. Aitenova, *Multiperiodic solutions of linear systems with a differentiation operator in the directions of the diagonal of the space of independent variables.* Carpathian Journal of Mathematics 41 (2025), no. 4, 981–998. https://doi.org/10.37193/CJM.2025.04.09

[21] B.Zh. Omarova, Zh.A. Sartabanov, *On multiperiodic solutions of perturbed nonlinear autonomous systems with the differentiation operator on a vector field.* Eurasian Mathematical Journal 12 (2021), no. 1, 68–81. https://doi.org/10.32523/2077-9879-2021-12-1-68-81

[22] L.S. Pontryagin, *Ordinary differential equations.* Nauka, Moscow, 1965 (in Russian).

[23] Zh. Sartabanov, G.A. Abdikalikova, G.M. Aitenova, B.Zh. Omarova, A.Kh. Zhumagaziyev, *Implementation of the small parameter method for studying multiperiodic solutions of systems with a diagonal differentiation operator.* Lobachevskii Journal of Mathematics 45 (2024), no. 12, 6594–6609. https://doi.org/10.1134/S1995080224607197 Erratum: Lobachevskii Journal of Mathematics 46 (2025), no. 4, 1929. https://doi.org/10.1134/S1995080225040018

[24] Zh. Sartabanov, B. Omarova, G. Aitenova, A. Zhumagaziyev, *Integrating multiperiodic functions along the periodic characteristics of the diagonal differentiation operator.* Journal of Mathematics, Mechanics and Computer Science 120 (2023), no. 4, 52–68.

[25] Zh.A. Sartabanov, *Frequency of characteristics the diagonal differentiation operator.* Bulletin of the Abai KazNPU, Physical and Mathematical Sciences 82 (2023), no. 2, 40–53 (in Russian).

[26] D.U. Umbetzhanov, *Almost multiperiodic solutions of partial differential equations.* Nauka, Alma-Ata, 1979 (in Russian).

[27] D.U. Umbetzhanov, *Almost periodic solutions of evolutionary equations.* Nauka, Alma-Ata, 1990 (in Russian).

[28] A.Kh. Zhumagaziyev, Zh.A. Sartabanov, Ya.T. Sultanaev, *On a new method for investigation of multiperiodic solutions of quasilinear strictly hyperbolic system.* Azerbaijan Journal of Mathematics 12 (2022), no. 1, 32–48.

Zhaishylyk Sartabanov
Department of Mathematics
K. Zhubanov Aktobe Regional University
36 A. Moldagulova Ave,
030000 Aktobe, Republic of Kazakhstan
E-mail: sartabanov42@mail.ru

Received: 30.10.2025

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 17, Number 1 (2026), 91 – 99

GENERALIZATIONS OF HARDY-TYPE INTEGRAL INEQUALITIES FOR QUASIMONOTONE FUNCTIONS IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

M. Sofrani, A. Senouci

Communicated by V.I. Burenkov

Key words: inequalities, quasi-monotone function, Hardy operators, variable exponent.

AMS Mathematics Subject Classification: 35J20, 35J25.

Abstract. In 1992 V.I. Burenkov proved some Hardy's inequalities with sharp constants in Lebesgue spaces for monotone functions for $0 < p < 1$. Later R.A. Bandaliev established analogous estimates in weighted variable exponent Lebesgue spaces for monotone functions for $0 < p(x) \leq q(x) < 1$. In 2020 A. Senouci and A. Zanou generalized the results of R.A. Bandaliev for quasi-monotone functions. The aim of this paper is to obtain some generalizations of the previous results cited above for weighted Hardy operators by introducing a parameter $\alpha \in \mathbb{R}$. Moreover, by using the quasi-norms $\|f\|^{BT}_{L_{p(x)}(\Omega)}$ introduced by V.I. Burenkov and T.V. Tararykova, we obtain an improvement of constants in our previous estimates.

DOI: https://doi.org/10.32523/2077-9879-2026-17-1-91-99

1 Introduction

For the first time the variable exponent Lebesgue space appeared in the literature already in the thirties of the last century, being introduced by W. Orlicz. At the beginning these spaces had theoretical interest. Later, at the end of the last century, their first use beyond the function spaces theory itself, was in variational problems and studies of $p(x)$-Laplacian (see Zhikov [11], [12]) which in its turn gave an essential impulse for the development of this theory. The extensive investigation of these spaces was also widely stimulated by applications to various problems of applied mathematics, e.g., in modelling of electrorheological fluids [8].

The variable exponent Lebesgue spaces $L^{p(x)}$ for $p(x) \geq 1$ appeared in the literature for the first time in [7]. Further developement of this theory was connected with the theory of modular functions.

Many investigations are devoted to the problem of boundedness of the Hardy operator in the Lebesgue spaces $L^{p(x)}$ for $p(x) \geq 1$ (see for example [1]). However, investigations of the Hardy inequality in the variable exponent Lebesgue spaces $L^{p(x)}$ for $0 < p(x) < 1$ are much less known.

It is well known that for L_p-spaces with $0 < p < 1$, the Hardy inequalities are not satisfied for arbitrary non-negative measurable functions, but are satisfied for non-negative monotone functions (for more details see [5]). The aim of this work is to obtain weighted inequalities for the Hardy operators acting from one weighted variable exponent Lebesgue space $L_{p(x),w_1(x)}(0,\infty)$ to another weighted variable exponent Lebesgue space $L_{q(x),w_2(x)}(0,\infty)$ for $0 < p(x) \leq q(x) < 1$, for functions defined on $(0,\infty)$ and satisfying conditions of the quasi-monotonicity. Some results obtained in [10] are generalized. Moreover, by using the quasi-norms $\|f\|^{BT}_{L_{p(x),\omega(x)}}$ introduced by V.I. Burenkov and T.V. Tararykova (for more details see [4]) and a new parameter α, we establish some weighted inequalities for the same operators with improved constants.

2 Preliminaries

In this section, we state definitions, lemmas, corollaries and theorems that are useful in the proofs of main results. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, x_2, \ldots, x_n)$, Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. Suppose that p is a Lebesgue measurable function on Ω such that $0 < p(x) \leq \infty \, \forall \, x \in \Omega$, and ω is a weight function, that is a positive Lebesgue measurable function on Ω.

Definition 1. Let p be a Lebesgue measurable function, $0 < p(x) < \infty$ for all $x \in \Omega$. By $L_{p(x),\omega(x)}(\Omega)$ we denote the set of all Lebesgue measurable functions f on Ω such that

$$\rho_{p(x),\omega(x)}(f) = \int_\Omega (|f(x)|\omega(x))^{p(x)} dx < \infty. \tag{2.1}$$

Note that the expression

$$\|f\|_{L_{p(x),\omega(x)}(\Omega)} = \inf\{\lambda > 0; \int_\Omega \Big(\frac{|f(x)|\omega(x)}{\lambda}\Big)^{p(x)} dx \leq 1\} \tag{2.2}$$

defines a quasi-norm on $L_{p(x),\omega(x)}(\Omega)$. $L_{p(x),\omega(x)}(\Omega)$ is a quasi-Banach space equipped with this quasi-norm (see [9]).

In Definition 1, the case $p(x) = \infty$ is not considered. For $\omega(x) = 1$ definitions, including this case, were considered by O. Kovachik, J. Rakosnik (quasi-norm $\|f\|^{KR}_{L_{p(x)}(\Omega)}$) and V.I. Burenkov, T.V. Tararykova (quasi-norm $\|f\|^{BT}_{L_{p(x)}(\Omega)}$) (see [6] and [4], respectively). We shall use the quasi-norm based on the definition given in [4].

Definition 2. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$, $p(x) : \Omega \to (0, \infty]$, $f : \Omega \to \mathbb{C}$, Lebesgue measurable functions on Ω. Following [4], we say that $f \in L^{BT}_{p(x),\omega(x)}(\Omega)$ if

$$\|f\|^{BT}_{L_{p(x),\omega(x)}(\Omega)} = \inf\left\{\lambda > 0; \int_\Omega \left(\frac{|f(x)|\omega(x)}{\lambda}\right)^{p(x)} dx \leq 1\right\} < \infty. \tag{2.3}$$

If $\frac{|f(x)|\omega(x)}{\lambda} < 1$ and $p(x) = \infty$, then it is assumed that $\left(\frac{|f(x)|\omega(x)}{\lambda}\right)^{p(x)} = 0$.

If $\frac{|f(x)|\omega(x)}{\lambda} > 1$ and $p(x) = \infty$, then it is assumed that $\left(\frac{|f(x)|\omega(x)}{\lambda}\right)^{p(x)} = \infty$.

Remark 1. Note that $L^{BT}_{p(x)}(\Omega)$ is a quasi-normed space with the quasi-norm $\|f\|^{BT}_{L_{p(x),\omega(x)}(\Omega)}$ (norm if $p(x) \geq 1$).

Clearly, if $p(x) < \infty \; \forall x \in \Omega$, then $\|f\|^{BT}_{L_{p(x),\omega(x)}(\Omega)} = \|f\|_{L_{p(x),\omega(x)}(\Omega)}$.

The following statement is known (see [1]).

Lemma 2.1. *Let $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ be measurable sets, p be a Lebesgue measurable function on Ω_1 and q be a Lebesgue measurable function on Ω_2, $1 \leq \underline{p} \leq p(x) \leq q(y) \leq \overline{q} < \infty$ for all $x \in \Omega_1 \subset \mathbb{R}^n$ and $y \in \Omega_2 \subset \mathbb{R}^m$. If $p \in C(\Omega_1)$, $q \in C(\Omega_2)$, then the inequality*

$$\Big\|\|f\|_{L_{p(x)}(\Omega_1)}\Big\|_{L_{q(x)}(\Omega_2)} \leq C_{p,q}\Big\|\|f\|_{L_{q(x)}(\Omega_2)}\Big\|_{L_{p(x)}(\Omega_1)} \tag{2.4}$$

is valid, where

$$C_{p,q} = \Big(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + \frac{\overline{p}}{\underline{q}} + \frac{\underline{p}}{\overline{q}}\Big)(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty), \tag{2.5}$$

$$\underline{p} = \operatorname{ess}\inf_{\Omega_1} q(x),\ \overline{p} = \operatorname{ess}\sup_{\Omega_1} q(x),\ \underline{q} = \operatorname{ess}\inf_{\Omega_2} q(x),\ \overline{q} = \operatorname{ess}\sup_{\Omega_2} q(x),$$

$$\Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2;\ p(x) = q(y)\}, \qquad \Delta_2 = (\Omega_1 \times \Omega_2)\backslash \Delta_1,$$

$C(\Omega_1)$, $C(\Omega_2)$ *are the spaces of continuous functions in* Ω_1, Ω_2 *and* $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *is any Lebesgue measurable function such that* $\left\| \|f\|_{L_{q(x)}(\Omega_2)} \right\|_{L_{p(x)}(\Omega_1)} < \infty$.

The following definition was introduced in [3].

Definition 3. We say that a non-negative function f is quasimonotone on $]0,\infty[$, if for some real number α, $x^\alpha f(x)$ is a decreasing or an increasing function of x. More precisely, given $\beta \in \mathbb{R}$, we say that $f \in Q_\beta$ if only if $x^{-\beta} f(x)$ is non-increasing and $f \in Q^\beta$ if only if $x^{-\beta} f(x)$ is non-decreasing.

The following theorems were proved in [10].

Theorem 2.1. *Let* p, q *be Lebesgue measurable functions on* $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, *for* $x \in (0,\infty)$, $\beta > -1$. *Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0,\infty)$.

1) If $f \in Q_\beta$, *then the inequality*

$$\|Hf\|_{L_{q(x),\omega_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\overline{p}'}} C_{p,q} d_p \left\| \frac{t^{\frac{1}{\overline{p}'}} \| \frac{\omega_2(x)}{x} \|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{2.6}$$

holds.

2) $f \in Q^\beta$, *then the inequality*

$$\|Hf\|_{L_{q(x),\omega_2}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\overline{p}'}} C_{p,q} d_p \left\| \frac{\| [y^{-\beta}(x^{\beta+1} - y^{\beta+1})]^{\frac{1}{\overline{p}'}} \frac{\omega_2(x)}{x} \|_{L_{q(x)}(y,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{2.7}$$

holds, where

$$(Hf)(x) = \frac{1}{x} \int_0^x f(y)dy,$$

$$C_{p,q} = \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \left(\frac{\overline{p}}{\underline{q}} + \frac{\underline{p}}{\overline{q}} \right) \right) \left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)} \right),$$

$S_1 = \{x \in (0,\infty) :\ p(x) = \underline{p}\}$, $S_2 = (0,\infty)\backslash S_1$ *and*

$$d_p = \left(1 + \frac{\overline{p} - \underline{p}}{\overline{p}} + \|\chi_{S_1}\|_{L_\infty(0,\infty)} \right)^{\frac{1}{\underline{p}}}.$$

Theorem 2.2. *Let* p, q *Lebesgue measurable functions on* $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, *for* $x \in (0,\infty)$ *and* $\beta = -1$. *Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0,\infty)$.

1) *If* $f \in Q_{-1}$, *then the inequality*

$$\|H^* f\|_{L_{q(x),\omega_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}} C_{p,q} d_p \left\| \frac{t^{\frac{1}{\overline{p}'}} (\ln \frac{t}{x})^{\frac{1}{\overline{p}'}} \| \frac{\omega_2(x)}{x} \|_{L_{q(x)}(0,t)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{2.8}$$

holds.

2) *If* $f \in Q^{-1}$, *then the inequality*

$$\|Hf\|_{L_{q(x),\omega_2(x)}(0,\infty)} \leq p^{\frac{1}{p}} C_{p,q} d_p \left\| \frac{t^{\frac{1}{p'}} (\ln \frac{t}{x})^{\frac{1}{p'}} \| \frac{\omega_2(x)}{x} \|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{2.9}$$

holds.

In (2.8) - (2.9)

$$(Hf)(x) = \frac{1}{x}\int_0^x f(t)dt, \; (H^* f)(x) = \frac{1}{x}\int_x^\infty f(t)dt$$

and $C_{p,q}$, d_p *are the constants in Theorem* 2.1.

The following proposition was proved in [3].

Proposition 2.1. (*a*) *Let* $-\infty < \beta < \infty$, $f \in Q_\beta$, $0 \leq a < b < \infty$ *for* $\beta > -1$ *and* $0 < a < b \leq \infty$ *for* $\beta \leq -1$.

If $0 < p \leq 1$ *and* $\beta \neq -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \leq p|\beta+1|^{1-p} \int_a^b \left(\frac{|y^{\beta+1} - a^{\beta+1}|}{y^\beta}\right)^{p-1} f^p(y)dy. \tag{2.10}$$

If $0 < p \leq 1$ *and* $\beta = -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \leq p \int_a^b \left(y \ln\frac{y}{a}\right)^{p-1} f^p(y)dy. \tag{2.11}$$

The inequalities hold in the reversed direction if $1 \leq p < \infty$.

(*b*) *Let* $-\infty < \beta < \infty$, $f \in Q^\beta$ *and* $0 < a < b \leq \infty$ *for* $\beta < -1$ *and* $0 \leq a < b < \infty$ *for* $\beta \geq -1$.

If $0 < p \leq 1$ *and* $\beta \neq -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \leq p|\beta+1|^{1-p} \int_a^b \left(\frac{|y^{\beta+1} - b^{\beta+1}|}{y^\beta}\right)^{p-1} f^p(y)dy. \tag{2.12}$$

If $0 < p \leq 1$ *and* $\beta = -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \leq p \int_a^b \left(y \ln\frac{b}{y}\right)^{p-1} f^p(y)dy. \tag{2.13}$$

The inequalities hold in the reversed direction if $1 \leq p < \infty$.

(*c*) *The constants in these inequalities are the best possible in all cases.*

We note the following special cases of Proposition 2.1 which are useful in the proofs of main results. If we set $a = 0$ and $b = x$ in (2.10) and assume that f is nonnegative and $\omega \in Q^\alpha$, then we get for $0 < x < \infty$

$$\left(\int_0^x f(y)y^{-\alpha}\omega(y)dy\right)^p \leq \left(x^{-\alpha}\omega(x)\right)^p \left(\int_0^x f(y)dy\right)^p$$
$$\leq p(\beta+1)^{1-p}\omega^p(x)x^{-\alpha p} \int_0^x y^{p-1} f^p(y)dy.$$

Corollary 2.1. *Let* $0 < p \leq 1$, $0 < x < \infty$.

(*a*) *If* $\beta > -1$, $f \in Q_\beta$, $\omega \in Q^\alpha$, $\alpha \in \mathbb{R}$, *then*

$$\left(\int_0^x f(y)y^{-\alpha}\omega(y)dy\right)^p \leq p(\beta+1)^{1-p}\omega^p(x)x^{-\alpha p} \int_0^x y^{p-1} f^p(y)dy. \tag{2.14}$$

(*b*) *If* $\beta < -1$, $f \in Q^{\beta}$, $\omega \in Q_{\alpha}$, $\alpha \in \mathbb{R}$, *then*

$$\left(\int_x^{\infty} f(y) y^{-\alpha}\omega(y)dy\right)^p \leq p|\beta+1|^{1-p}\omega^p(x)x^{-\alpha p}\int_x^{\infty} y^{p-1}f^p(y)dy. \tag{2.15}$$

(*c*) *If* $\beta > -1$, $f \in Q^{\beta}$, $\omega \in Q^{\alpha}$, $\alpha \in \mathbb{R}$, *then*

$$\left(\int_0^{x} f(y) y^{-\alpha}\omega(y)dy\right)^p \leq p(\beta+1)^{1-p}\omega^p(x)x^{-\alpha p}\int_0^{x} \left[y^{-\beta}\left(x^{\beta+1}-y^{\beta+1}\right)\right]^{p-1}f^p(y)dy. \tag{2.16}$$

The estimates (2.15) and (2.16) are proved similarly by putting $a = x$, $b = \infty$ and $a = 0$, $b = x$ in (2.12).

If we take $a = x$, $b = \infty$ and $a = 0$, $b = x$ in (2.11) and (2.13) respectively, we obtain the following corollary.

Corollary 2.2. *Let* $0 < p \leq 1$, $\beta = -1$, $\alpha \in \mathbb{R}$, $0 < x < \infty$.

1) *If* $\omega \in Q_{\alpha}$, *then*

$$\left(\int_x^{\infty} f(y) y^{-\alpha}\omega(y)dy\right)^p \leq p\omega^p(x)x^{-\alpha p}\int_x^{\infty} \left(y\, \ln\, \frac{y}{x}\right)^{p-1}f^p(y)dy. \tag{2.17}$$

2) *If* $\omega \in Q^{\alpha}$, *then*

$$\left(\int_0^{x} f(y) y^{-\alpha}\omega(y)dy\right)^p \leq p\omega^p(x)x^{-\alpha p}\int_0^{x} \left(y\, \ln\, \frac{x}{y}\right)^{p-1}f^p(y)dy. \tag{2.18}$$

The following theorem was proved in [4].

Theorem 2.3. *Let* $\Omega \subset \mathbb{R}^n$ *be a Lebesgue measurable set;* $p, q : \Omega \to (0, \infty]$, $f : \Omega \to \mathbb{C}$, *Lebesgue measurable functions, such that*

1) *for all* $x \in \Omega$ $0 < p(x) \leq q(x) \leq \infty$;

2) $r(x) = \frac{p(x)q(x)}{q(x)-p(x)}$, *if* $p(x) < q(x) < \infty$, $r(x) = p(x)$, *if* $p(x) < q(x) = \infty$, *and* $r((x) = \infty$, *if* $p(x) = q(x)$;

3) $m = \operatorname*{ess\,inf}_{x\in\Omega} \frac{p(x)}{q(x)}$, $M = \operatorname*{ess\,sup}_{x\in\Omega} \frac{p(x)}{q(x)}$, $\underline{p} = \operatorname*{ess\,inf}_{x\in\Omega} p(x)$.

If $\underline{p} > 0$, *then*

$$\|fg\|^{BT}_{L_{p(x)}(\Omega)} \leq (1+M-m)^{\frac{1}{\underline{p}}}\|f\|^{BT}_{L_{q(x)}(\Omega)}\|g\|^{BT}_{L_{r(x)}(\Omega)} \tag{2.19}$$

for all $f \in L^{BT}_{q(x)}(\Omega)$ *and* $g \in L^{BT}_{r(x)}(\Omega)$.

Remark 2. The constant $(1+M-m)^{\frac{1}{\underline{p}}}$ in inequality (2.19) is an improvement of the constant in Corollary 2.1 of [2], with $A = M, B = 1 - m$.

If the functions p, q satisfy the conditions of Theorem 2.3, f is replaced by $f\omega_2$ and $g = \frac{\omega_1}{\omega_2}$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.3. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is a Lebesgue measurable set, then the inequality (2.19) takes the form*

$$\|f\|^{BT}_{L_{p(x),\omega_1(x)}(\Omega)} \leq (1+M-m)^{\frac{1}{\underline{p}}}\left\|\frac{\omega_1(x)}{\omega_2(x)}\right\|^{BT}_{L_{r(x)}(\Omega)}\|f\|^{BT}_{L_{q(x),\omega_2(x)}(\Omega)}, \tag{2.20}$$

for every $f \in L^{BT}_{q(x),\omega_2(x)}(\Omega)$ *and* $\frac{\omega_1}{\omega_2} \in L^{BT}_{r(x)}(\Omega)$.

3 Main results

Let ω be a weight function on $(0,\infty)$ and $\alpha \in \mathbb{R}$.

Consider the weighted Hardy operators

$$H_{\omega,\alpha} = \frac{1}{W(x)} \int_0^x f(y) y^{-\alpha} \omega(y) dy, \ \alpha \in \mathbb{R},$$

$$(H^*_{\omega,\alpha} f)(x) = \frac{1}{W(x)} \int_x^\infty f(y) y^{-\alpha} \omega(y) dy,$$

where $W(x) = \int_0^x y^{-\alpha}\omega(y)dy, \ y > 0$ and f is a non-negative Lebesgue measurable function on $(0,\infty)$.

Note that for $\omega(y) \equiv 1$, and $\alpha = 0$, the $H_{\omega,\alpha}$ and $H^*_{\omega,\alpha}$ are the usual Hardy operators H and H^*.

Theorem 3.1. *Let p, q be Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta > -1$, $f \in Q_\beta$, $\alpha \in \mathbb{R}$, and $\omega \in Q^\alpha$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then, the inequality

$$\|H_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$
$$\leq \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\underline{p}'}} C_{p,q} (1+M-m)^{\frac{1}{\underline{p}}} \left\| \frac{y^{\frac{1}{\underline{p}'}} \|\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\|_{L_{q(x)}(y,\infty)}}{\omega_1(x)} \right\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.1}$$

holds, where $C_{p,q}$, M and m are the constants in Theorems 2.1, 2.3 *respectively.*

Proof. Taking in account Remark 2.1 and by applying Corollary 2.1 and inequality (2.14) with $p = \underline{p}$, we get

$$\|H_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)} = \|\omega_2(x) H_{\omega,\alpha} f\|_{L_{q(x)}(0,\infty)} = \left\| \frac{\omega_2(x)}{W(x)} \int_0^x f(y) y^{-\alpha} w dy \right\|_{L_{q(x)}(0,\infty)}$$
$$\leq \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big(\int_0^x f^{\underline{p}}(y) y^{\underline{p}-1} dy \Big)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)}.$$

Let

$$K_1 = \left\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big(\int_0^x f^{\underline{p}}(y) y^{\underline{p}-1} dy \Big)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)},$$

then,

$$K_1 = \left\| \Big(\int_0^\infty f^{\underline{p}}(y) \chi_{(0,x)}(y) \Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big]^{\underline{p}} y^{\underline{p}-1} dy \Big)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)}$$
$$= \left\| \Big(\int_0^\infty f^{\underline{p}}(y) \chi_{(\omega,x)}(y) \Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big]^{\underline{p}} y^{\underline{p}-1} dy \Big) \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}$$
$$= \left\| \left\| f^{\underline{p}}(y) \chi_{(0,x)}(y) \Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big]^{\underline{p}} y^{\underline{p}-1} \right\|_{L_1(0,\infty)} \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}.$$

Now, one can use Lemma 2.1 and get that

$$K_1 \leq C_{p,q} \Big(\int_0^\infty \left\| [f^{\underline{p}}(y)] \chi_{(0,x)}(y) \Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big]^{\underline{p}} y^{\underline{p}-1} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dy \Big)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q}\Big(\int_0^\infty f^{\underline{p}}(y) y^{\underline{p}-1} \Big\| \chi_{(0,x)}(y) \Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\Big]^{\underline{p}} \Big\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dy\Big)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q}\Big(\int_0^\infty f^{\underline{p}}(y) y^{\underline{p}-1} \Big\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big\|^{\underline{p}}_{L_{q(x)}(y,\infty)} dy\Big)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q}\Big\| f(y) y^{\frac{1}{\underline{p}'}} \Big\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big\|_{L_{q(x)}(y,\infty)} \Big\|_{L_{\underline{p}}(0,\infty)}.$$

Let

$$K_2 = \Big\| f(y) y^{\frac{1}{\underline{p}'}} \Big\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big\|_{L_{q(x)}(y,\infty)} \Big\|_{L_{\underline{p}}(0,\infty)}.$$

Finally, applying Corollary 2.3, we have

$$K_2 \le (1+M-m)^{\frac{1}{\underline{p}}} \Big\| \frac{y^{\frac{1}{\underline{p}'}} \| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \|_{L_{q(x)}(y,\infty)}}{\omega_1(x)} \Big\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)},$$

consequently,

$$\|H_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\underline{p}'}} C_{p,q} (1+M-m)^{\frac{1}{\underline{p}}} \Big\| \frac{y^{\frac{1}{\underline{p}'}} \| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \|_{L_{q(x)}(y,\infty)}}{\omega_1(x)} \Big\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)}.$$

□

Remark 3. Note that $(1+M-m)^{\frac{1}{\underline{p}}} < d_p$. Since, under the assumptions of Theorem 3.1, $\|f\|^{BT}_{L_{p(x)}(\Omega)} = \|f\|_{L_{p(x)}(\Omega)}$, if we take $\alpha = 0$, $\omega(x) = 1$ in inequality (3.1), we get inequality (2.6) of Theorem 2.1 with an improved constant. Moreover, by putting $\beta = 0$, we get Theorem 3.1 of [2].

By applying Corollary 2.1, inequality (2.15), the following theorem is proved in a similar way.

Theorem 3.2. *Let p, q be Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta < -1$, $f \in Q^\beta$, $\alpha \in \mathbb{R}$, and $\omega \in Q_\alpha$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$. Then, the inequality*

$$\|H^*_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} |\beta+1|^{-\frac{1}{\underline{p}'}} C_{p,q} (1+M-m)^{\frac{1}{\underline{p}}} \Big\| \frac{y^{\frac{1}{\underline{p}'}} \| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \|_{L_{q(x)}(0,y)}}{\omega_1(x)} \Big\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.2}$$

holds, where $C_{p,q}$ and M, m are the constants in Theorem 3.1.

By using Corollary 2.1, inequality (2.16), the following theorem is proved similarly.

Theorem 3.3. *Let p, q Lebesgue be measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta > -1$, $f \in Q^\beta$, $\alpha \in \mathbb{R}$, and $\omega \in Q^\alpha$. Suppose that ω_1, ω_2 are weight functions defined on $(0,\infty)$.*

Then, the inequality

$$\|H_{\omega,\alpha} f\|_{L_{q(x),\omega_2}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\underline{p}'}} C_{p,q} (1+M-m)^{\frac{1}{\underline{p}}}$$

$$\times \Big\| \frac{\| [y^{-\beta}(x^{\beta+1} - y^{\beta+1})]^{\frac{1}{\underline{p}'}} \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \|_{L_{q(x)}(y,\infty)}}{\omega_1(x)} \Big\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.3}$$

holds, where $C_{p,q}$ and M, m are the constants in Theorem 3.1.

Remark 4. Since, under the assumptions of Theorem 3.3, $\|f\|^{BT}_{L_{p(x)}(\Omega)} = \|f\|_{L_{p(x)}(\Omega)}$, if we take $\alpha = 0$, $\omega(x) = 1$ in inequality (3.3), we get inequality (2.7) of Theorem 2.1, with an improved constant. Moreover, by putting $\beta = 0$, we get Theorem 3.2 of [2].

Remark 5. For constant $p(x) = q(x) = p$ and $\omega_1(x) = \omega_2(x) = x^\alpha$ and $\omega(x) = 1$, inequalities (3.1) and (3.3) with sharp constants, were proved in [3] and if $\beta = 0$ earlier in [5].

Now we consider the case $\beta = -1$.

Theorem 3.4. *Let p, q be Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ for $x \in (0,\infty)$, $\beta = -1$, and $\alpha \in \mathbb{R}$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

1) If $f \in Q_{-1}$ and $\omega \in Q_\alpha$, then the inequality

$$\|H^*_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} C_{p,q}(1+M-m)^{\frac{1}{\underline{p}}} \left\| \frac{t^{\frac{1}{p'}} (\ln \frac{t}{x})^{\frac{1}{p'}} \|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(0,t)}}{\omega_1(x)} \right\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.4}$$

holds.

2) If $f \in Q^{-1}$ and $\omega \in Q^\alpha$, then the inequality

$$\|H_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} C_{p,q}(1+M-m)^{\frac{1}{\underline{p}}} \left\| \frac{t^{\frac{1}{p'}} (\ln \frac{x}{t})^{\frac{1}{p'}} \|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.5}$$

holds.

Proof. 1. By using inequality (2.17) with $p = \underline{p}$, we obtain

$$\|H^*_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)} = \|\omega_2(x) H^*_{\omega,\alpha} f\|_{L_{q(x)}(0,\infty)} = \left\| \frac{\omega_2(x)}{W(x)} \int_x^\infty f(y) y^{-\alpha} \omega dy \right\|_{L_{q(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} \left\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \left(\int_x^\infty \left(y \ln \frac{y}{x} \right)^{p-1} f^p dy \right)^{\frac{1}{p}} \right\|_{L_{q(x)}(0,\infty)}.$$

We put

$$J_1 = \left\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \left(\int_x^\infty \left(y \ln \frac{y}{x} \right)^{p-1} f^p dy \right)^{\frac{1}{p}} \right\|_{L_{q(x)}(0,\infty)}.$$

The rest is similar to the proof of Theorem 3.1.

2. We apply inequality (2.18) with $p = \underline{p}$ and the rest of the proof is similar to that of Theorem 3.1. □

Remark 6. Since, under the assumptions of Theorem 3.4, $\|f\|^{BT}_{L_{p(x)}(\Omega)} = \|f\|_{L_{p(x)}(\Omega)}$, if we set $\alpha = 0$, $\omega(x) = 1$ in inequalities (3.4) and (3.5), we obtain inequalities (2.8) and (2.9), respectively, of Theorem 2.2 with improved constants.

Acknowledgments

The authors are very grateful to Professor V.I. Burenkov for interesting discussions of this paper.

References

[1] R.A. Bandaliev, *On an inequality in Lebesgue space with mixed norm and with variable summability.* Matem. Zametki 84 (2008), no. 3, 323–333 (in Russian). English translation in Math. Notes, 84 (2008), no. 3, 303–313.

[2] R.A. Bandaliev, *On Hardy-type inequalities in weighted variable exponent Lebesgue spaces for 0<p<1.* Eurasian Math. Journal 4 (2013), no. 4, 5–16.

[3] J. Bergh, V.I. Burenkov, L.-E. Persson, *Best constants in reversed Hardy's inequalities for quasimonotone functions.* Acta Sci. Math. (Szeged) 59 (1994), 223–241.

[4] V.I. Burenkov, T.V. Tararykova, *About Hölder's inequality in Lebesgue spaces with variable summability.* Contemporary Mathematics. Fundamental Directions 67 (2021), no. 3, 472-482 (in Russian).

[5] V.I. Burenkov, *On the best constants in Hardy's inequalities for 0<p<1 for monotone functions.* Trudy Matem. Inst. Steklov 194 (1992), 58–62 (in Russian). English translation in Proc. Steklov Inst. Math. 194 (1993), no. 4, 59–63.

[6] O. Kovachik, J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$.* Czechoslovak Math. J. 41 (1991), no. 4, 592–618.

[7] W. Orlicz, *Über konjugierte Exponentenfolgen.* Stud. Math. 3 (1931), 200–212.

[8] M. Rúžička, *Electrorheological fluids : modeling and mathematical theory.* Lecture Notes in Mathematics. 1748, Springer, Berlin (2000).

[9] S.G. Samko, *Differentiation and integration of variable order and the variable exponent Lebesgue spaces*, Proc. Inter. Conf. "Operator theory for complex and hypercomplex analysis", Mexico, 1994, Contemp. Math. 212 (1998), 203-219.

[10] A. Senouci, A. Zanou, *Some integral inequalities for quasi-monotone functions in weighted variable exponent Lebesgue spaces with* $0 < p(x) < 1$. Eurasian Math. Journal 4 (2020), no. 4, 58–65.

[11] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory.* Izv. Akad. Nauk SSSR Ser. Matem. 50 (1986), 675–710, (in Russian). English translation in Math. USSR, Izv. 29 (1987), 33–66.

[12] V.V. Zhikov, *On some variational problems.* Russian J. Math. Phys. 5 (1997), no. 1, 105–116.

Abdelkader Senouci, Mohammed Sofrani
Department of Mathematics
Laboratory of Informatics and Mathematics
University of Tiaret,
Zaaroura 14000, Algeria
e-mails: kamer295@yahoo.fr, aissamalik@yahoo.fr

Received: 24.09.2024

EURASIAN MATHEMATICAL JOURNAL

2026 – Том 17, № 1 – Астана: ЕНУ. – 100 с.

Подписано в печать 30.03.2025 г. Тираж – 40 экз.

Адрес редакции: 010008, Астана, ул. Кажымукан, 13,
Евразийский национальный университет имени Л.Н. Гумилева,
корпус № 3, каб. 306а.
Тел.: +7-7172-709500, добавочный 33312.

Дизайн: К. Булан

Отпечатано в типографии ЕНУ имени Л.Н. Гумилева

Зарегистрировано
Министерством информации и общественного развития Республики Казахстан.
Свидетельство о постановке на переучет печатного издания
№ KZ30VPY00032663 от 19.02.2021 г.
(Дата и номер первичной постановки на учет: № 10330 – Ж от 25.09.2009 г.)