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APPROXIMATION BY T MEANS WITH RESPECT TO VILENKIN SYSTEM IN LEBESGUE SPACES

N. Anakidze, N. Areshidze, L.-E. Persson, G. Tephnadze

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Key words: Vilenkin group, Vilenkin system, T means, Nörlund means, Fejér means, approximation, Lebesgue spaces, inequalities.

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Abstract. In this paper we present and prove some new results concerning approximation properties of T means with respect to the Vilenkin system in Lebesgue spaces for any $1 \leq p < \infty$. As applications, we obtain extensions of some known approximation inequalities.

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1 Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers and $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m =: (m_0, m_1, \cdots)$ be a sequence of positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \cdots, m_k - 1\}$$

the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the group Z_{m_k} with the product of the discrete topologies of Z_{m_k}'s.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{k\in\mathbb{N}} m_k < +\infty$, then we call G_m a bounded Vilenkin group. If the sequence $\{m_k\}_{k\geq 0}$ is unbounded, then G_m is said to be an unbounded Vilenkin group. In this paper we consider only bounded Vilenkin groups.

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots) \qquad (\, x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m, namely

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \ (x \in G_m, \ n \in \mathbb{N}).$$

For brevity, we also define $I_n := I_n(0)$.

Next, we define a generalized number system based on m in the following way:

$$M_0 =: 1, \quad M_{k+1} =: m_k M_k \quad (k \in \mathbb{N})$$

Then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n=\sum_{j=0}^{\infty} n_j M_j, \quad \text{where} \quad n_j \in Z_{m_j} \quad (j \in \mathbb{N})$$

and only a finite number of n_j's differ from zero. Let

$$|n| =: \max\{j \in \mathbb{N}, n_j \neq 0\}.$$

Moreover, Vilenkin (see [33, 34, 35]) investigated the group G_m and introduced the Vilenkin system $\{\psi_j\}_{j=0}^{\infty}$ as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

where $r_k(x)$ are the generalized Rademacher functions defined by

$$r_k(x) := \exp(2\pi i x_k/m_k), \quad (k \in \mathbb{N}).$$

If $m_k = 2$ for any $k \in \mathbb{N}$, then the Vilenkin group coincides with the dyadic group, which will be denoted by G_2 and Vilenkin systems include as a special case the Walsh system.

The norms (or quasi-norms) $\|f\|_p$, $0<p<\infty$, of the Lebesgue spaces $L^p(G_m)$ are defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu.$$

The Vilenkin system is orthonormal and complete in $L^2(G_m)$ (see e.g. [2] and [27]).

If $f \in L^1(G_m)$, we can define the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\begin{aligned}
\widehat{f}(k) &:= \int_{G_m} f\overline{\psi}_k d\mu, \quad (k \in \mathbb{N}), \\
S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, \quad (n \in \mathbb{N}_+, \ S_0 f := 0), \\
\sigma_n f &:= \frac{1}{n}\sum_{k=1}^{n} S_k f, \quad (n \in \mathbb{N}_+). \\
D_n &:= \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+). \\
K_n &:= \frac{1}{n}\sum_{k=1}^{n} D_k, \quad (n \in \mathbb{N}_+).
\end{aligned}$$

Recall that (see e.g. [2] and [25]),

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{1.1}$$

$$\begin{aligned}
D_{M_n-j}(x) &= D_{M_n}(x) - \overline{\psi}_{M_n-1}(-x)D_j(-x) \\
&= D_{M_n}(x) - \psi_{M_n-1}(x)\overline{D_j}(x), \quad 0 \leq j < M_n.
\end{aligned} \tag{1.2}$$

$$n\left|K_n\right| \leq 2R^2 \sum_{l=0}^{|n|} M_l \left|K_{M_l}\right|, \tag{1.3}$$

and

$$\int_{G_m} K_n(x)d\mu(x) = 1, \qquad \sup_{n\in\mathbb{N}} \int_{G_m} \left|K_n(x)\right| d\mu(x) \leq R^5. \tag{1.4}$$

where $R := \sup_{k\in\mathbb{N}} m_k$. Moreover, if $n > t$, $t, n \in \mathbb{N}$, then

$$K_{M_n}(x) = \begin{cases} \frac{M_t}{1-r_t(x)}, & x \in I_t \backslash I_{t+1}, \quad x - x_t e_t \in I_n, \\ \frac{M_n+1}{2}, & x \in I_n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

The n-th Nörlund mean t_n and T mean T_n of $f \in L^1(G_m)$ are defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f$$

and

$$T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f,$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k.$$

Here $\{q_k, \ k \geq 0\}$ is a sequence of nonnegative numbers, where $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty. \tag{1.6}$$

Then, a T mean generated by $\{q_k, \ k \geq 0\}$ is regular if and only if condition (1.6) is satisfied (see [25]).

It is evident that

$$T_n f(x) = \int_{G_m} f(t) F_n(x-y) \, d\mu(y),$$

where

$$F_n := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k D_k, \tag{1.7}$$

which are called the kernels of the T means.

By applying the Abel transformation, we get the following two useful identities:

$$Q_n := \sum_{k=0}^{n-1} q_k \cdot 1 = \sum_{k=0}^{n-2} (q_k - q_{k+1})k + q_{n-1}(n-1) \tag{1.8}$$

and

$$T_n f = \frac{1}{Q_n} \left(\sum_{k=0}^{n-2} (q_k - q_{k+1}) k \sigma_k f + q_{n-1}(n-1)\sigma_{n-1} f \right). \tag{1.9}$$

2 Historical overview

It is well-known (see e.g. [15], [25] and [39]) that, for any $1 \leq p \leq \infty$ and $f \in L^p(G_m)$, there exists $C_p > 0$, depending only on p, such that

$$\|\sigma_n f\|_p \leq C_p \|f\|_p .$$

Moreover, Skvortsov [30] (see also [1]) proved that if $1 \leq p \leq \infty$, $M_N \leq n < M_{N+1}$, $f \in L^p(G_m)$ and $n \in \mathbb{N}$, then

$$\|\sigma_n f - f\|_p \leq 2R^5 \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right), \tag{2.1}$$

where $R := \sup_{k\in\mathbb{N}} m_k$ and $\omega_p(1/M_k, f)$ is the modulus of continuity of $L^p(G_m)$ functions, $1 \leq p < \infty$ functions defined by

$$\omega_p(1/M_k, f) = \sup_{|u|<1/M_k} \|f(\cdot - u) - f(\cdot)\|_p, \qquad k \in \mathbb{N},$$

where $-$ is the inverse operation of the sum $+$ defined on G_m and the modulus $|u|$ of $u \in G_m$ is defined by

$$|u| = \sum_{i=0}^{\infty} \frac{u_i}{M_{i+1}}.$$

It follows that if $f \in Lip\,(\alpha, p)$, i.e.,

$$Lip(\alpha, p) := \{f \in L^p(G_m) : \omega_p(1/M_k, f) = O(1/M_k^{\alpha}) \quad \text{as} \quad k \to 0\},$$

then

$$|\sigma_n f - f\|_p = \begin{cases} O(1/M_N), & \text{if} \quad \alpha > 1, \\ O(N/M_N), & \text{if} \quad \alpha = 1, \\ O(1/M_n^{\alpha}), & \text{if} \quad \alpha < 1. \end{cases}$$

Moreover, (see e.g. [25]) if $1 \leq p < \infty$, $f \in L^p(G_m)$ and

$$\|\sigma_{M_n} f - f\|_p = o\left(1/M_n\right), \text{ as } n \to \infty,$$

then f is a constant function.

For the maximal operators of Vilenkin-Fejér means σ^*, defined by

$$\sigma^* f = \sup_{n\in\mathbb{N}} |\sigma_n f|$$

the weak-$(1,1)$ type inequality

$$\|\sigma^* f\|_{weak-L_1} \leq C \|f\|_1 , \qquad \left(f \in L^1(G_m)\right)$$

can be found in Schipp [26] for Walsh series and in Pál, Simon [24] and Weisz [36] for bounded Vilenkin series. The boundedness of the maximal operators of Vilenkin-Féjer means of the one- and two-dimensional cases can be found in Fridli [10], Gát [12], Goginava [14], Nagy and Tephnadze [22, 23], Simon [28, 29] and Weisz [37].

Convergence and summability of Nörlund means with respect to Vilenkin systems were studied by Areshidze and Tephnadze [3], Blahota and Nagy [4], Blahota, Persson and Tephnadze [7] (see also [5, 6]), Blyumin [8], Efimov [9], Fridli, Manchanda and Siddiqi [11], Goginava [13], Jastrebova [16], Nagy [20, 21], Memic [17], Tsutserova [31] and Zhantlesov [38].

Móricz and Siddiqi [19] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of $L^p(G_2)$ functions. In particular, they proved that if $f \in L^p(G_2)$, $1 \leq p \leq \infty$, $n = 2^j + k$, $1 \leq k \leq 2^j$ $(n \in \mathbb{N}_+)$ and $(q_k, k \in \mathbb{N})$ is a sequence of non-negative numbers, such that

$$\frac{n^{\gamma-1}}{Q_n^{\gamma}} \sum_{k=0}^{n-1} q_k^{\gamma} = O(1), \quad \text{for some} \quad 1 < \gamma \leq 2,$$

then there exists $C_p > 0$, depending only on p, such that

$$\|t_n f - f\|_p \leq \frac{C_p}{Q_n} \sum_{i=0}^{j-1} 2^i q_{n-2^i} \omega_p \left(\frac{1}{2^i}, f\right) + C_p \omega_p \left(\frac{1}{2^j}, f\right),$$

if the sequence $(q_k, k \in \mathbb{N})$ is non-decreasing, while

$$\|t_n f - f\|_p \leq \frac{C_p}{Q_n} \sum_{i=0}^{j-1} \left(Q_{n-2^i+1} - Q_{n-2^{i+1}+1}\right) \omega_p \left(\frac{1}{2^i}, f\right) + C_p \omega_p \left(\frac{1}{2^j}, f\right),$$

if the sequence $(q_k, k \in \mathbb{N})$ is non-increasing.

Tutberidze [32] (see also [25]) proved that if T_n are T means generated by either a non-increasing sequence $\{q_k, k \in \mathbb{N}\}$ or a non-decreasing sequence $\{q_k, k \in \mathbb{N}\}$ satisfying the condition

$$\frac{q_0}{Q_k} = O\left(\frac{1}{k}\right), \quad \text{as} \quad k \to \infty,$$

then there exists an absolute constant C, such that

$$\|T^* f\|_{weak-L_1} \leq C \|f\|_1, \qquad \left(f \in L^1(G_m)\right)$$

holds. From these results it follows that if $f \in L^p(G_m)$, where $1 \leq p < \infty$ and either the sequence $\{q_k, k \in \mathbb{N}\}$ is non-increasing, or $\{q_k, k \in \mathbb{N}\}$ is a sequence of non-decreasing numbers, such that the condition

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty, \tag{2.2}$$

is satisfied, then

$$\lim_{n \to \infty} \|T_n f - f\|_p = 0.$$

For the Walsh system in [18] Móricz and Rhoades proved that if $f \in L^p(G_2)$, where $1 \leq p < \infty$, and T_n are regular T means generated by a non-increasing sequence $\{q_k, \ k \in \mathbb{N}\}$, then, for any $2^N \leq n < 2^{N+1}$, we have the following approximation inequality:

$$\|T_n f - f\|_p \leq \frac{C_p}{Q_n} \sum_{s=0}^{N-1} 2^s q_{2^s} \omega_p \left(1/2^s, f\right) + C_p \omega_p \left(1/2^N, f\right). \tag{2.3}$$

In the case in which the sequence $\{q_k, \ k \in \mathbb{N}\}$ is non-decreasing and satisfying the condition

$$\frac{q_{k-1}}{Q_k} = O\left(\frac{1}{k}\right), \quad \text{as} \quad k \to \infty, \tag{2.4}$$

the following inequality holds:

$$\|T_n f - f\|_p \leq C_p \sum_{j=0}^{N-1} 2^{j-N} \omega_p \left(1/2^j, f\right) + C_p \omega_p \left(1/2^N, f\right). \tag{2.5}$$

In this paper we use a new approach and generalize inequalities in (2.3) and (2.5) for T means with respect to the Vilenkin system (see Theorems 1 and 2). We also prove a new inequality for the subsequences $\{T_{M_n}\}$ means if the sequence $\{q_k,\ k \in \mathbb{N}\}$ is non-decreasing (see Theorem 3).

3 The main results

Our first main result reads:

Theorem 3.1. *Let $f \in L^p(G_m)$, where $1 \leq p < \infty$ and T_n are T means generated by a non-increasing sequence $\{q_k,\ k \in \mathbb{N}\}$. Then, for any $n, N \in \mathbb{N}$, $M_N \leq n < M_{N+1}$, we have the following inequality:*

$$\|T_n f - f\|_p \leq \frac{6R^6}{Q_n} \sum_{j=0}^{N-1} M_j q_{M_j} \omega_p \left(1/M_j, f\right) + 4R^6 \omega_p \left(1/M_N, f\right). \tag{3.1}$$

Next we state and prove a similar inequality for non-decreasing sequences but under some restrictions.

Theorem 3.2. *Let $f \in L^p(G_m)$, where $1 \leq p < \infty$ and T_n are regular T means generated by a non-decreasing sequence $\{q_k,\ k \in \mathbb{N}\}$. Then, for any $n, N \in \mathbb{N}$, $M_N \leq n < M_{N+1}$, we have the following inequality:*

$$\|T_n f - f\|_p \leq \frac{6R^6 q_{n-1}}{Q_n} \sum_{j=0}^{N-1} M_j \omega_p \left(1/M_j, f\right) + \frac{4R^6 q_{n-1} M_N}{Q_n} \omega_p \left(1/M_N, f\right). \tag{3.2}$$

If, in addition, the sequence $\{q_k,\ k \in \mathbb{N}\}$ satisfies condition (2.2), then the inequality

$$\|T_n f - f\|_p \leq C_p \sum_{j=0}^{N} \frac{M_j}{M_N} \omega_p \left(1/M_j, f\right) \tag{3.3}$$

holds for $C_p > 0$, depending only on p.

Finally, we state and prove the third main result for non-decreasing sequences, in which we prove a more precise result than that in (3.3) and without restriction (2.2), but only for subsequences.

Theorem 3.3. *Let $f \in L^p(G_m)$, where $1 \leq p < \infty$ and T_k are regular T means generated by a non-decreasing sequence $\{q_k,\ k \in \mathbb{N}\}$. Then, for any $n \in \mathbb{N}$, the following inequality holds:*

$$\begin{aligned} \|T_{M_n} f - f\|_p &\leq R^2 \sum_{j=0}^{n-1} \frac{M_j}{M_n} \omega_p \left(1/M_j, f\right) \\ &+ \frac{2R^4}{q_0} \sum_{j=0}^{n-1} \frac{(n-j) q_{M_n - M_j} M_j}{M_n} \omega_p \left(1/M_j, f\right) + \omega_p \left(1/M_n, f\right). \end{aligned} \tag{3.4}$$

We also point out the following generalizations of some results in [18] (in that paper, only the Walsh system was considered):

Corollary 3.1. *Let $\{q_k,\ k \geq 0\}$ be a sequence of non-negative and non-increasing numbers, while in case when the sequence is non-decreasing it is assumed that also condition (2.2) is satisfied. If $f \in Lip(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p < \infty$, then*

$$\|T_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad 0 < \alpha < 1, \\ O(n^{-1}\log n), & \text{if} \quad \alpha = 1, \\ O(n^{-1}), & \text{if} \quad \alpha > 1, \end{cases}$$

Corollary 3.2. *Let $\{q_k,\ k \geq 0\}$ be a sequence of non-negative and non-increasing numbers such that*

$$q_k \sim k^{-\beta} \quad \text{for some} \quad 0 < \beta \leq 1$$

is satisfied.

If $f \in Lip(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p < \infty$, then

$$\|T_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad \alpha + \beta < 1, \\ O(n^{-(1-\beta)}\log n + n^{-\alpha}), & \text{if} \quad \alpha + \beta = 1, \\ O(n^{-(1-\beta)}), & \text{if} \quad \alpha + \beta > 1,\ \beta > 1, \\ O((\log n)^{-1}), & \text{if} \quad \beta = 1. \end{cases}$$

Corollary 3.3. *Let $\{q_k,\ k \geq 0\}$ be a sequence of non-negative and non-increasing numbers such that the equivalence*

$$q_k \sim (\log k)^{-\beta} \quad \text{for some} \quad \beta > 0$$

is satisfied.

If $f \in Lip(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p < \infty$, then

$$\|T_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad 0 < \alpha < 1,\ \beta > 0, \\ O(n^{-1}\log n), & \text{if} \quad \alpha = 1,\ 0 < \beta < 1, \\ O(n^{-1}\log n \log\log n), & \text{if} \quad \alpha = \beta = 1, \\ O(n^{-1}(\log n)^{\beta}), & \text{if} \quad \alpha > 1,\ \beta > 0. \end{cases}$$

Corollary 3.4. *Let $f \in L^p(G_m)$, where $1 \leq p < \infty$ and $\{q_k,\ k \geq 0\}$ is a sequence of non-negative and non-increasing numbers, while in case when the sequence is non-decreasing it is also assumed that condition (2.2) is satisfied. Then,*

$$\lim_{n\to\infty} \|T_n f - f\|_p = 0.$$

4 Proofs

Proof of Theorem 1. Let $M_N \leq n < M_{N+1}$. Since T_n are regular T means generated by a sequence of non-increasing numbers $\{q_k : k \in \mathbb{N}\}$, we can combine (1.8) and (1.9) and conclude that

$$\begin{aligned} \|T_n f - f\|_p &\leq \frac{1}{Q_n}\left(\sum_{j=0}^{n-2}(q_j - q_{j+1})j\|\sigma_j f - f\|_p + q_{n-1}(n-1)\|\sigma_{n-1} f - f\|_p\right) \\ &:= I + II. \end{aligned} \tag{4.1}$$

Moreover,

$$\begin{aligned} I &= \frac{1}{Q_n}\sum_{j=1}^{M_N-1}(q_j - q_{j+1})\, j\|\sigma_j f - f\|_p + \frac{1}{Q_n}\sum_{j=M_N}^{n-1}(q_j - q_{j+1})\, j\|\sigma_j f - f\|_p \\ &:= I_1 + I_2. \end{aligned} \tag{4.2}$$

Now we estimate both terms separately. By applying estimate (2.1) for I_1 we obtain that

$$\begin{aligned} I_1 &\leq \frac{2R^5}{Q_n}\sum_{k=0}^{N-1}\sum_{j=M_k}^{M_{k+1}-1}(q_j - q_{j+1})\, j \sum_{s=0}^{k}\frac{M_s}{M_k}\omega_p(1/M_s, f) \\ &\leq \frac{2R^6}{Q_n}\sum_{k=0}^{N-1} M_k \sum_{j=M_k}^{M_{k+1}-1}(q_j - q_{j+1})\sum_{s=0}^{k}\frac{M_s}{M_k}\omega_p(1/M_s, f) \\ &\leq \frac{2R^6}{Q_n}\sum_{k=0}^{N-1}\left(q_{M_k} - q_{M_{k+1}}\right)\sum_{s=0}^{k} M_s\omega_p(1/M_s, f) \\ &\leq \frac{2R^6}{Q_n}\sum_{s=0}^{N-1} M_s\omega_p(1/M_s, f)\sum_{k=s}^{N-1}\left(q_{M_k} - q_{M_{k+1}}\right) \\ &\leq \frac{2R^6}{Q_n}\sum_{s=0}^{N-1} M_s q_{M_s}\omega_p(1/M_s, f). \end{aligned} \tag{4.3}$$

Moreover,

$$\begin{aligned} I_2 &\leq \frac{2R^5}{Q_n}\sum_{j=M_N}^{n-1}(q_j - q_{j+1})\, j \sum_{s=0}^{N}\frac{M_s}{M_N}\omega_p(1/M_s, f) \\ &\leq \frac{2R^6 M_N}{Q_n}\sum_{j=M_N}^{n-1}(q_j - q_{j+1})\sum_{s=0}^{N}\frac{M_s}{M_N}\omega_p(1/M_s, f) \\ &\leq \frac{2R^6 q_{M_N}}{Q_n}\sum_{s=0}^{N} M_s\omega_p(1/M_s, f) \\ &\leq \frac{2R^6}{Q_n}\sum_{s=0}^{N} M_s q_{M_s}\omega_p(1/M_s, f) \\ &\leq \frac{2R^6}{Q_n}\sum_{s=0}^{N-1} M_s q_{M_s}\omega_p(1/M_s, f) + 2R^6\omega_p(1/M_s, f). \end{aligned} \tag{4.4}$$

For II we have that

$$\begin{aligned} II &\leq \frac{2R^5 M_{N+1} q_{n-1}}{Q_n} \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right) \\ &\leq \frac{2R^6}{Q_n} \sum_{s=0}^{N-1} M_s q_{M_s} \omega_p \left(1/M_s, f\right) + 2R^6 \omega_p \left(1/M_N, f\right). \end{aligned} \tag{4.5}$$

The proof of (3.1) is complete by just combining (4.1)-(4.5). □

Proof of Theorem 2. Let $M_N \leq n < M_{N+1}$. Since T_n are regular T means, generated by a sequence of non-decreasing numbers $\{q_k : k \in \mathbb{N}\}$, by combining (1.8) and (1.9), we find that

$$\begin{aligned} \|T_n f - f\|_p &\leq \frac{1}{Q_n} \left(\sum_{j=1}^{n-1} \left(q_{j+1} - q_j\right) j \|\sigma_j f - f\|_p + q_{n-1}(n-1) \|\sigma_n f - f\|_p \right) \\ &:= I + II. \end{aligned} \tag{4.6}$$

Furthermore,

$$\begin{aligned} I &= \frac{1}{Q_n} \sum_{j=1}^{M_N - 1} \left(q_{j+1} - q_j\right) j \|\sigma_j f - f\|_p + \frac{1}{Q_n} \sum_{j=M_N}^{n-1} \left(q_{j+1} - q_j\right) j \|\sigma_j f - f\|_p \\ &:= I_1 + I_2. \end{aligned} \tag{4.7}$$

Analogously to (4.3) we get that

$$\begin{aligned} I_1 &\leq \frac{2R^6}{Q_n} \sum_{k=0}^{N-1} \left(q_{M_{k+1}} - q_{M_k}\right) \sum_{s=0}^{k} M_s \omega_p \left(1/M_s, f\right) \\ &\leq \frac{2R^6}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) \sum_{k=s}^{N-1} \left(q_{M_{k+1}} - q_{M_k}\right) \\ &= \frac{2R^6}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) \left(q_{M_N} - q_{M_s}\right) \\ &\leq \frac{2R^6 q_{M_N}}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) \\ &\leq \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right). \end{aligned} \tag{4.8}$$

In a similar way as in (4.4) we find that

$$\begin{aligned} I_2 &\leq \frac{2R^5}{Q_n} \sum_{j=1}^{n-1} \left(q_{j+1} - q_j\right) j \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right) \\ &= \frac{2R^5}{Q_n} \left((n-1) q_{n-1} - Q_n\right) \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right) \\ &\leq \frac{2R^5 M_{N+1} q_{n-1}}{Q_n M_N} \sum_{s=0}^{N} M_s \omega_p \left(1/M_s, f\right) \end{aligned}$$

$$\leq \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N} M_s \omega_p \left(1/M_s, f\right)$$

$$\leq \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) + \frac{2R^6 q_{n-1} M_N}{Q_n} \left(1/M_N, f\right). \tag{4.9}$$

For II we have that

$$\begin{aligned} II &\leq \frac{2R^5 q_{n-1} M_{N+1}}{Q_n} \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right) \\ &\leq \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N} M_s \omega_p \left(1/M_s, f\right) \\ &= \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) + \frac{2R^6 q_{n-1} M_N}{Q_n} \left(1/M_N, f\right). \end{aligned} \tag{4.10}$$

By combining (4.6)-(4.10) we find that (3.2) holds. Moreover, by using condition (2.2) we obtain estimate (3.3), so the proof is complete. □

Proof of Theorem 3. According to (1.2) we find that

$$T_{M_n} f = D_{M_n} * f - \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k \left(\left(\psi_{M_n-1} \overline{D_k}\right) * f \right).$$

Hence, by using the Abel transformation we get that

$$\begin{aligned} T_{M_n} f &= D_{M_n} * f \\ &- \frac{1}{Q_{M_n}} \sum_{j=0}^{M_n-2} \left(q_{M_n-j} - q_{M_n-j-1}\right) j ((\psi_{M_n-1} \overline{K_j}) * f) \\ &- \frac{1}{Q_{M_n}} q_{M_n-1} (M_n - 1)(\psi_{M_n-1} \overline{K}_{M_n-1} * f) \\ &= D_{M_n} * f \\ &- \frac{1}{Q_{M_n}} \sum_{j=0}^{M_n-2} \left(q_{M_n-j} - q_{M_n-j-1}\right) j ((\psi_{M_n-1} \overline{K_j}) * f) \\ &- \frac{1}{Q_{M_n}} q_{M_n-1} M_n (\psi_{M_n-1} \overline{K}_{M_n} * f) \\ &+ \frac{q_{M_n-1}}{Q_{M_n}} (\psi_{M_n-1} \overline{D}_{M_n} * f), \end{aligned}$$

so that

$$\begin{aligned} T_{M_n} f(x) - f(x) = \int_{G_m} (f(x-t) - f(x)) D_{M_n}(t) dt \\ - \frac{1}{Q_{M_n}} \sum_{j=0}^{M_n-2} \left(q_{M_n-j} - q_{M_n-j-1}\right) j \int_{G_m} \left(f(x-t) - f(x)\right) \psi_{M_n-1}(t) \overline{K}_j(t) dt \\ - \frac{1}{Q_{M_n}} q_{M_n-1} M_n \int_{G_m} (f(x-t) - f(x)) \psi_{M_n-1}(t) \overline{K}_{M_n}(t) dt \end{aligned}$$

$$+\frac{q_{M_n-1}}{Q_{M_n}}\int_{G_m}(f(x-t)-f(x))\psi_{M_n-1}(t)\overline{D}_{M_n}(t)dt$$

$$=: I+II+III+IV. \tag{4.11}$$

By combining generalized Minkowski's inequality and (1.1) we find that

$$\|I\|_p\leq\int_{I_n}\|f(x-t)-f(x))\|_pD_{M_n}(t)dt\leq\omega_p\left(1/M_n,f\right) \tag{4.12}$$

and

$$\|IV\|_p\leq\int_{I_n}\|f(x-t)-f(x))\|_pD_{M_n}(t)dt\leq\omega_p\left(1/M_n,f\right). \tag{4.13}$$

Moreover, since

$$M_nq_{M_n-1}\leq Q_{M_n},\quad \text{for any}\quad n\in\mathbb{N},$$

we can use (1.5) and generalized Minkowski's inequality to find that

$$\begin{aligned}
\|III\|_p &\leq \int_{G_m}\left\|f\left(x-t\right)-f\left(x\right)\right\|_p\left|\overline{K}_{M_n}\left(t\right)\right|d\mu(t)\\
&= \int_{I_n}\left\|f\left(x-t\right)-f\left(x\right)\right\|_p\left|\overline{K}_{M_n}\left(t\right)\right|d\mu(t)\\
&+ \sum_{s=0}^{n-1}\sum_{n_s=1}^{m_s-1}\int_{I_n(n_se_s)}\left\|f\left(x-t\right)-f\left(x\right)\right\|_p\left|\overline{K}_{M_n}\left(t\right)\right|d\mu(t)\\
&\leq \int_{I_n}\left\|f\left(x-t\right)-f\left(x\right)\right\|_p\frac{M_n+1}{2}d\mu(t)\\
&+ \sum_{s=0}^{n-1}M_{s+1}\sum_{n_s=1}^{m_s-1}\int_{I_n(n_se_s)}\left\|f\left(x-t\right)-f\left(x\right)\right\|_pd\mu(t)\\
&\leq \omega_p\left(1/M_n,f\right)\int_{I_n}\frac{M_n+1}{2}d\mu(t)\\
&+ \sum_{s=0}^{n-1}M_{s+1}\sum_{n_s=1}^{m_s-1}\int_{I_n(n_se_s)}\omega_p\left(1/M_s,f\right)d\mu(t)\\
&\leq \omega_p\left(1/M_n,f\right)+R^2\sum_{s=0}^{n-1}\frac{M_s}{M_n}\omega_p\left(1/M_s,f\right).
\end{aligned} \tag{4.14}$$

From this inequality and the estimates in (4.14) it follows also that

$$M_n\int_{G_m}\left\|f\left(x-t\right)-f\left(x\right)\right\|_p\left|\overline{K}_{M_n}(t)\right|d\mu(t)\leq R^2\sum_{s=0}^{n}M_s\omega_p\left(1/M_s,f\right).$$

Let $M_k\leq j<M_{k+1}$. By applying (1.3) and the last estimate we find that

$$j\int_{G_m}\left\|f\left(x-t\right)-f\left(x\right)\right\|_p\left|\overline{K}_j(t)\right|d\mu(t)\leq 2R^4\sum_{l=0}^{k}\sum_{s=0}^{l}M_s\omega_p\left(1/M_s,f\right).$$

Hence, by also using (1.3) we obtain that

$$\begin{aligned}
&\|II\|_p \\
\leq\ & \frac{1}{Q_{M_n}}\sum_{j=0}^{M_n-1}\left(q_{M_n-j}-q_{M_n-j-1}\right) j\int_{G_m}\|f(x-t)-f(x)\|_p|\overline{K}_j(t)|d\mu(t) \\
\leq\ & \frac{1}{Q_{M_n}}\sum_{k=0}^{n-1}\sum_{j=M_k}^{M_{k+1}-1}\left(q_{M_n-j}-q_{M_n-j-1}\right) j\int_{G_m}\|f(x-t)-f(x)\|_p|\overline{K}_j(t)|d\mu(t) \\
\leq\ & \frac{2R^4}{Q_{M_n}}\sum_{k=0}^{n-1}\sum_{j=M_k}^{M_{k+1}-1}\left(q_{M_n-j}-q_{M_n-j-1}\right)\sum_{l=0}^{k}\sum_{s=0}^{l}M_s\omega_p\left(1/M_s,f\right) \\
\leq\ & \frac{2R^4}{Q_{M_n}}\sum_{k=0}^{n-1}\left(q_{M_n-M_k}-q_{M_n-M_{k+1}}\right)\sum_{l=0}^{k}\sum_{s=0}^{l}M_s\omega_p\left(1/M_s,f\right) \\
\leq\ & \frac{2R^4}{Q_{M_n}}\sum_{l=0}^{n-1}\sum_{k=l}^{n-1}\left(q_{M_n-M_k}-q_{M_n-M_{k+1}}\right)\sum_{s=0}^{l}M_s\omega_p\left(1/M_s,f\right) \\
\leq\ & \frac{2R^4}{Q_{M_n}}\sum_{l=0}^{n-1}q_{M_n-M_l}\sum_{s=0}^{l}M_s\omega_p\left(1/M_s,f\right) \\
\leq\ & \frac{2R^4}{Q_{M_n}}\sum_{s=0}^{n-1}M_s\omega_p\left(1/M_s,f\right)\sum_{l=s}^{n-1}q_{M_n-M_l} \\
\leq\ & \frac{2R^4}{Q_{M_n}}\sum_{s=0}^{n-1}M_s\omega_p\left(1/M_s,f\right)q_{M_n-M_s}(n-s) \\
\leq\ & 2R^4\sum_{s=0}^{n-1}\frac{(n-s)M_s}{M_n}\frac{q_{M_n-M_s}}{q_0}\omega_p\left(1/M_s,f\right). \qquad (4.15)
\end{aligned}$$

Finally, by combining (4.11)-(4.15) and using Minkowski's inequality we obtain (3.4), so the proof is complete. □

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BOUNDEDNESS OF THE GENERALIZED RIEMANN-LIOUVILLE OPERATOR IN LOCAL MORREY-TYPE SPACES WITH MIXED QUASI-NORMS

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Abstract. The objective of this paper is to establish sufficient conditions for the boundedness of the generalized Riemann-Liouville operator in local Morrey-type spaces with mixed quasi-norms on a parallelepiped and to obtain sharp estimates of the norm of this operator with respect to the lengths of the edges of this parallelepiped.

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1 Introduction

First, we recall the definition of local Morrey-type spaces.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue-measurable set, $x_0 \in \overline{\Omega}$, $0 < p,\ \theta \le \infty$, $\lambda \ge 0$. Then $f \in LM^{\lambda}_{p\theta,x_0}(\Omega)$ if f is Lebesgue-measurable on Ω and

$$\|f\|_{LM^{\lambda}_{p\theta,x_0}(\Omega)} = \|r^{-\lambda-\frac{1}{\theta}}\|f\|_{L_p(\Omega\bigcap B(x_0,r))}\|_{L_\theta(0,\infty)} < \infty,$$

where $B(x_0,r)$ is the ball centered at x_0 of radius r.

If $\lambda = 0$, then, clearly, $LM^{0}_{p\infty,x_0}(\Omega) = L_p(\Omega)$. If $\lambda = 0$ and $\theta < \infty$, then $LM^{0}_{p\theta,x_0}(\Omega)$ consists only of functions f equivalent to 0 on Ω, because for any $\rho > 0$

$$\|f\|_{LM^{0}_{p\theta,x_0}(\Omega)} \ge \|\|f\|_{L_p(\Omega\bigcap B(x_0,r))}\|_{L_\theta(\rho,\infty)} \ge \|1\|_{L_\theta(\rho,\infty)}\|f\|_{L_p(\Omega\bigcap B(x_0,\rho))}.$$

For $\Omega = \mathbb{R}^n$, $x_0 = 0$ this definition was first introduced in [1], [2].

The boundedness of various operators acting from one local Morrey-type spaces $LM^{\lambda}_{p\theta,x_0}(\Omega)$ to $LM^{\mu}_{p\sigma,x_0}(\Omega)$ was investigated in a number of papers. See, for example, [1], [2].

In this paper, we consider the following two variants of the local Morrey-type spaces with mixed quasi-norms.

We shall use the following notation for vectors $x \in \mathbb{R}^n : x \equiv \overrightarrow{x} = (x_1, ..., x_n)$ and $\overleftarrow{x} = (x_n, ..., x_1)$.

Definition 2. Let $p = (p_1, ..., p_n)$, $\theta = (\theta_1, ..., \theta_n)$, $\lambda = (\lambda_1, ..., \lambda_n)$, $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ $-\infty < a_i < b_i \le \infty$, $0 < p_i.\ \theta_i \le \infty$, $0 \le \lambda_i < \infty$, $i = 1, ..., n$, $Q(a,b) = \{x \in \mathbb{R}^n, a_i < x_i < b_i, i = 1, ..., n\}$.

Let $LM^{\lambda}_{p\theta,a}(Q(a,b))$, $\overleftarrow{LM}^{\lambda}_{p\theta,a}(Q(a,b)) \equiv LM^{\overleftarrow{\lambda}}_{\overleftarrow{p}\,\overleftarrow{\theta},\overleftarrow{a}}(Q(a,b))$ be the spaces of all Lebesgue-measurable functions on $Q(a,b)$ for which the following quasi-norms are finite

$$\|f\|_{LM^{\lambda}_{p\theta,a}\left(Q(a,b)\right)} = \left\| ... \|f(x_1,...,x_n)\|_{LM^{\lambda_1}_{p_1\theta_1,a_1,x_1}((a_1,b_1))} ... \right\|_{LM^{\lambda_n}_{p_n\theta_n,a_n,x_n}((a_n,b_n))}$$

$$= \left\| r_n^{-\lambda_n - \frac{1}{\theta_n}} \right\| ... \left\| r_1^{-\lambda_1-\frac{1}{\theta_1}} \left\| f(x_1,...,x_n) \right\|_{L_{p_1,x_1}((a_1,b_1)\bigcap(a_1-r_1,a_1+r_1))} \right\|_{L_{\theta_1}(0,\infty)}$$

$$... \Big\|_{L_{p_n,x_n}((a_n,b_n)\bigcap(a_n-r_n,a_n+r_n))} \Big\|_{L_{\theta_n}(0,\infty)} \tag{1.1}$$

and

$$\|f\|_{\overleftarrow{LM}_{p\theta,a}\left(Q(a,b)\right)} = \left\| ... \|f(x_1,...,x_n)\|_{LM^{\lambda_n}_{p_n\theta_n,a_n,x_n}((a_n,b_n))} ... \right\|_{LM^{\lambda_1}_{p_1\theta_1,a_1,x_1}((a_1,b_1))}$$

$$= \left\| r_1^{-\lambda_1-\frac{1}{\theta_1}} \right\| ... \left\| r_n^{-\lambda_n-\frac{1}{\theta_n}} \|f(x_1,...,x_n)\|_{L_{p_n,x_n}((a_n,b_n)\bigcap(a_n-r_n,a_n+r_n))} \right\|_{L_{\theta_n}(0,\infty)}$$

$$... \Big\|_{L_{p_1,x_1}((a_1,b_1)\bigcap(a_1-r_1,a_1+r_1))} \Big\|_{L_{\theta_1}(0,\infty)}, \tag{1.2}$$

respectively.

If $a = 0$, then, for brevity, we denote the corresponding spaces by $\|f\|_{LM^{\lambda}_{p\theta}(Q(0,b)}$ and $\|f\|_{\overleftarrow{LM}^{\lambda}_{p\theta}(Q(0,b))}$.

Definition 3. [5] Let $f \in L_1^{loc}(\mathbb{R}^n)$, $\alpha = (\alpha_1,...,\alpha_n)$, $0 < \alpha_i < 1$, $k = (k_1,...,k_n)$, $k_i \geq 0$, $a = (a_1,...,a_n)$, $x = (x_1,...,x_n)$, $0 \leq a_i < x_i < \infty$, $i = 1,...,n$. The generalized Riemann-Liouville fractional integral operator $I^{\alpha,k}_{a^+}$ of order α is defined by the following equality:

$$\left(I^{\alpha,k}_{a_+} f\right)(x)$$

$$= \prod_{i=1}^{n} \frac{(k_i+1)^{1-\alpha_i}}{\Gamma(\alpha_i)} \int_{a_n}^{x_n} ... \int_{a_1}^{x_1} \prod_{i=1}^{n} \left[(x_i^{k_i+1} - t_i^{k_i+1})^{\alpha_i-1} t_i^{k_i}\right] f(t_1,...,t_n) dt_1...dt_n, \tag{1.3}$$

where Γ is the Euler Gamma-function.

2 One-dimensional case

Lemma 2.1. *Let $0 < y < \infty$, $0 < p, \theta \leq \infty$, $0 < \lambda < \infty$ for $\theta < \infty$, $0 \leq \lambda < \infty$ for $\theta = \infty$. Then* [1]

$$\|f\|_{L_p(0,y)} \leq (\lambda\theta)^{\frac{1}{\theta}} y^{\lambda} \|f\|_{LM^{\lambda}_{p\theta}(0,y)} \tag{2.1}$$

Proof. It suffices to note that for $\lambda > 0$

$$\|f\|_{LM^{\lambda}_{p\theta}(0,y)} = \left(\int_0^{\infty} r^{-\lambda\theta-1} \|f\|^{\theta}_{L_p((0,y)\bigcap(-r,r))} dr \right)^{\frac{1}{\theta}}$$

$$\geq \left(\int_y^{\infty} r^{-\lambda\theta-1} \|f\|^{\theta}_{L_p((0,y)\bigcap(0,r))} dr \right)^{\frac{1}{\theta}} = \|f\|_{L_p(0,y)} \left(\int_y^{\infty} r^{-\lambda\theta-1} dr \right)^{\frac{1}{\theta}} = (\lambda\theta)^{-\frac{1}{\theta}} y^{-\lambda} \|f\|_{L_p(0,y)}$$

[1] If $\theta = \infty$, then here and in the sequel it is assumed that $(\lambda\theta)^{\frac{1}{\theta}} = 1$ for all $0 \leq \lambda < \infty$.

if $\theta < \infty$ and

$$\|f\|_{LM^{\lambda}_{p\infty}(0,y)} = \sup_{0<r<\infty} r^{-\lambda}\|f\|_{L_p((0,y)\bigcap(-r,r))} \leq y^{-\lambda}\|f\|_{L_p((0,y))}$$

if $\theta = \infty$.

The case $\lambda = 0, \theta = \infty$ is trivial, since $LM^{\lambda}_{p\infty}(0,y) = L_p(0,y)$.

□

Lemma 2.2. *Let $0 \leq a < b < \infty$, $1 < p \leq \infty$, $0 < q \leq \infty$, $\frac{1}{p} < \alpha < 1$, $k \geq 0$, $0 < \theta, \sigma \leq \infty$, $0 < \lambda < \infty$ if $\theta < \infty$, $0 \leq \lambda < \infty$ if $\theta = \infty$, $0 < \mu < \infty$ if $\sigma < \infty$, $0 \leq \mu < \infty$ if $\sigma = \infty$.*

Then there exists $C_1 > 0$ such that

$$\|I^{\alpha,k}_{a+}f\|_{LM^{\mu}_{q\sigma,a}(a,b)} \leq C_1(b-a)^{\nu}\|f\|_{LM^{\lambda}_{p\theta,a}(a,b)} \tag{2.2}$$

for all finite intervals (a,b) and for all $f \in LM^{\lambda}_{p\theta,a}(a,b)$, where

$$\nu = \lambda + \frac{1}{q} - \frac{1}{p} + (k+1)\alpha - \mu \tag{2.3}$$

under the assumption $\nu > 0$ if $\sigma < \infty$ and $\nu \geq 0$ if $\sigma = \infty$.

Moreover, ν cannot be replaced by any other number.

Proof. It suffices to consider the case in which $a = 0$, $0 < b < \infty$.

Step 1. In [5] (see inequality (2.2)) it is proved that there exists $K_1 > 0$ such that

$$\left|\left(I^{\alpha,k}_{0+}f\right)(x)\right| \leq K_1 x^{(k+1)\alpha-\frac{1}{p}}\|f\|_{L_p(0,x)}$$

for any $0 < x < b$ and for any $f \in L_p(0,x)$. By using (2.1), we obtain

$$\left|\left(I^{\alpha,k}_{0+}f\right)(x)\right| \leq K_1 x^{(k+1)\alpha-\frac{1}{p}+\lambda}(\lambda\theta)^{\frac{1}{\theta}}\|f\|_{LM^{\lambda}_{p\theta}(0,x)}$$

and

$$\left\|I^{\alpha,k}_{0+}f\right\|_{L_q((-r,r)\cap(0,b))} \leq K_1(\lambda\theta)^{\frac{1}{\theta}}\left\|x^{(k+1)\alpha-\frac{1}{p}+\lambda}\right\|_{L_q((-r,r)\cap(0,b))}\|f\|_{LM^{\lambda}_{p\theta}(0,b)}.$$

Note that

$$\|I^{\alpha,k}_{0+}f\|_{LM^{\mu}_{q\sigma}(0,\infty)} = \left\|r^{-\mu-\frac{1}{\sigma}}\left\|I^{\alpha,k}_{0+}f\right\|_{L_q((-r,r)\cap(0,b))}\right\|_{L_\sigma(0,\infty)}$$

$$= \left(\left\|r^{-\mu-\frac{1}{\sigma}}\left\|I^{\alpha,k}_{0+}f\right\|_{L_q((-r,r)\cap(0,b))}\right\|^{\sigma}_{L_\sigma(0,b)} + \left\|r^{-\mu-\frac{1}{\sigma}}\left\|I^{\alpha,k}_{0+}f\right\|_{L_q((-r,r)\cap(0,b))}\right\|^{\sigma}_{L_\sigma(b,\infty)}\right)^{\frac{1}{\sigma}} \tag{2.4}$$

Moreover, since $(k+1)\alpha - \frac{1}{p} + \lambda > 0$, it follows that

$$r^{-\mu}\left\|x^{(k+1)\alpha-\frac{1}{p}+\lambda}\right\|_{L_q((-r,r)\cap(0,b))} = \begin{cases} \left(((k+1)\alpha-\frac{1}{p}+\lambda)q+1\right)^{-\frac{1}{q}} r^{\nu}, & r \leq b, \\ \left(((k+1)\alpha-\frac{1}{p}+\lambda)q+1\right)^{-\frac{1}{q}} b^{\nu+\mu}r^{-\mu}, & r > b. \end{cases}$$

$$\leq \begin{cases} r^{\nu}, & r \leq b, \\ b^{\nu+\mu}r^{-\mu}, & r > b. \end{cases}$$

Therefore, if $r \leq b$, then, since $\nu > 0$ for $\theta < \infty$ and $\nu \geq 0$ for $\theta = \infty$,

$$\left\|I_{0+}^{\alpha,k}f\right\|_{L_q((-r,r)\cap(0,b))} = \left\|r^{-\mu-\frac{1}{\theta}}\left\|I_{0+}^{\alpha,k}f\right\|_{L_q((-r,r)\cap(0,b))}\right\|_{L_\sigma(0,b)}$$

$$\leq K_1(\lambda\theta)^{\frac{1}{\theta}}\left\|r^{\nu-\frac{1}{\sigma}}\right\|_{L_\sigma(0,b)}\|f\|_{LM_{p\theta}^{\lambda}(0,b)}$$

$$= K_1(\lambda\theta)^{\frac{1}{\theta}}(\nu\sigma)^{-\frac{1}{\sigma}}b^{\nu}\|f\|_{LM_{p\theta}^{\lambda}(0,b)}. \tag{2.5}$$

If $r > b$, then, since $\mu > 0$ for $\sigma < \infty$ and $\mu \geq 0$ for $\sigma = \infty$,

$$\left\|r^{-\mu-\frac{1}{\sigma}}\left\|I_{0+}^{\alpha,k}f\right\|_{L_q((-r,r)\cap(0,b))}\right\|_{L_\sigma(b,\infty)}$$

$$\leq K_1(\lambda\sigma)^{\frac{1}{\sigma}}b^{(k+1)\alpha-\frac{1}{p}+\lambda+\frac{1}{q}}\left\|r^{-\mu-\frac{1}{\sigma}}\right\|_{L_\sigma(b,\infty)}\|f\|_{LM_{p\theta}^{\lambda}(0,b)}$$

$$= K_1(\lambda\sigma)^{\frac{1}{\sigma}}(\mu\sigma)^{-\frac{1}{\sigma}}b^{\nu}\|f\|_{LM_{p\theta}^{\lambda}(0,b)}. \tag{2.6}$$

So, by (2.4), (2.5), (2.6) it follows that

$$\|I_{0+}^{\alpha,k}f\|_{LM_{q\sigma}^{\mu}(0,\infty)} \leq C_1\|f\|_{LM_{p\theta}^{\lambda}(0,b)},$$

where

$$C_1 = K_1(\lambda\theta)^{\frac{1}{\theta}}\sigma^{-\frac{1}{\sigma}}\left(\frac{1}{\nu}+\frac{1}{\mu}\right)^{\frac{1}{\sigma}}.$$

Step 2. Suppose that for some $K_2(b) > 0$

$$\left\|I_{0+}^{\alpha,k}f\right\|_{LM_{q\sigma}^{\mu}(0,b)} \leq K_2(b)\,\|f\|_{LM_{p\theta}^{\lambda}(0,b)}\,. \tag{2.7}$$

for all $f \in LM_{p\theta}^{\lambda}(0,b)$.

Let $f = \chi_{(\frac{b}{2},b)}$, then

$$\|\chi_{(\frac{b}{2},b)}\|_{L_p((\frac{b}{2},b)\bigcap(-r,r))} = 0, \text{ if } r \leq \frac{b}{2},$$

and

$$\|\chi_{(\frac{b}{2},b)}\|_{L_p((\frac{b}{2},b)\bigcap(-r,r))} \leq \|1\|_{L_p(\frac{b}{2},b)} = \left(\frac{b}{2}\right)^{\frac{1}{p}}, \text{ if } r > \frac{b}{2}.$$

Moreover,

$$\|f\|_{LM_{p\theta}^{\lambda}(0,b)} = \left\|\chi_{(\frac{b}{2},b)}\right\|_{LM_{p\theta}^{\lambda}(0,b)} = \left\|r^{-\lambda-\frac{1}{\theta}}\|\chi_{(\frac{b}{2},b)}\|_{L_p((\frac{b}{2},b)\bigcap(-r,r))}\right\|_{L_\theta(\frac{b}{2},\infty)}$$

$$\leq \left(\frac{b}{2}\right)^{\frac{1}{p}}\|r^{-\lambda-\frac{1}{\theta}}\|_{L_\theta(\frac{b}{2},\infty)} = K_3\,b^{\frac{1}{p}-\lambda},$$

where $K_3 = 2^{\lambda-\frac{1}{p}}(\lambda\theta)^{-\frac{1}{\theta}}$.

Next,

$$\left\|I_{0+}^{\alpha,k}f\right\|_{LM_{q\sigma}^{\mu}(\frac{b}{2},b)} = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)}\left\|\int_0^x(x^{k+1}-t^{k+1})^{\alpha-1}t^k\chi_{(\frac{b}{2},b)}(t)dt\right\|_{LM_{q\sigma}^{\mu}(\frac{b}{2},b)}$$

$$(x^{k+1}-t^{k+1}=z) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)}\left\|\int_0^{x^{k+1}}z^{\alpha-1}\frac{dz}{k+1}\right\|_{LM_{p\theta}^{\mu}(\frac{b}{2},b)} = K_4\left\|x^{\alpha(k+1)}\right\|_{LM_{q\sigma}^{\mu}(\frac{b}{2},b)},$$

where

$$K_4 = ((k+1)^\alpha \Gamma(\alpha+1))^{-1}.$$

Note that

$$\|x^{\alpha(k+1)}\|_{LM^{\mu}_{q\sigma}(\frac{b}{2},b)} \geq \left\| r^{-\frac{1}{\sigma}-\mu} \|x^{\alpha(k+1)}\|_{L_q((-r,r)\bigcap(\frac{b}{2},b))} \right\|_{L_\sigma(b,\infty)}$$
$$= \left\| r^{-\frac{1}{\sigma}-\mu} \|x^{\alpha(k+1)}\|_{L_q(\frac{b}{2},b)} \right\|_{L_\sigma(b,\infty)}$$
$$= \|x^{\alpha(k+1)}\|_{L_q(\frac{b}{2},b)} \|r^{-\frac{1}{\sigma}-\mu}\|_{L_\sigma(b,\infty)} = K_5\, b^{\alpha(k+1)+\frac{1}{q}-\mu},$$

where

$$K_5 = \left(\frac{1-2^{-\alpha(k+1)q-1}}{\alpha(k+1)q+1}\right)^{\frac{1}{q}} (\mu\sigma)^{-\frac{1}{\sigma}}.$$

Hence, we have

$$\left\| I^{\alpha,k}_{0+} f \right\|_{LM^{\mu}_{q\sigma}(\frac{b}{2},b)} \geq K_6\, b^{\alpha(k+1)+\frac{1}{q}-\mu}, \tag{2.8}$$

where $K_6 = K_5 K_4$.

By (2.7) and (2.8), we get

$$K_6\, b^{\alpha(k+1)+\frac{1}{q}-\mu} \leq K_2(b)\, K_3\, b^{\frac{1}{p}-\lambda},$$

so,

$$K_2(b) \geq \frac{K_6}{K_3} b^{\alpha(k+1)+\frac{1}{q}+\lambda-\frac{1}{p}-\mu} = \frac{K_6}{K_3} b^\nu$$

for all $b > 0$.

If (2.2) holds with $\tau \neq \nu$ replacing ν, then $\frac{K_6}{K_3} b^\nu \leq K_2(b) \leq C_1 b^\tau$ for all $b > 0$, which is impossible. □

Corollary 2.1. *If in Lemma 2.2 $\nu = 0$, then inequality (2.2) takes the form*

$$\left\| I^{\alpha,k}_{a+} f \right\|_{LM^{\mu}_{q\sigma,a}(a,b)} \leq C_1 \|f\|_{LM^{\lambda}_{p\theta,a}(a,b)}$$

for all $0 \leq a < b \leq \infty$ and for all $f \in LM^{\lambda}_{p\theta,a}(a,b)$, where

$$\mu = \lambda + (k+1)\alpha + \frac{1}{q} - \frac{1}{p}.$$

Remark 1. For $\sigma = \theta = \infty$ the statements of this section were proved in [6].

3 Multidimensional case

We start with proving a statement, in which we apply the generalized Minkowski's inequality for the Lebesgue spaces: let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ be Lebesgue-measurable sets $0 < p \leq q \leq \infty$, and $f : E \times F \to \mathbb{C}$ be a Lebesgue-measurable function. Then

$$\left\| \|f(x,y)\|_{L_{p,x}(E)} \right\|_{L_{q,y}(F)} \leq \left\| \|f(x,y)\|_{L_{q,y}(F)} \right\|_{L_{p,x}(E)}. \tag{3.1}$$

Lemma 3.1. *(Generalized Minkowski's inequality for local Morrey-type spaces.) Let* $-\infty \le a < b \le \infty$, $-\infty \le c \le d \le \infty$,

$$0 < \max\{p, \theta\} \le \min\{q, \sigma\} \le \infty,$$

$0 < \lambda$, $\mu < \infty$, *and* $f : (a,b) \times (c,d) \to \mathbb{C}$ *be a Lebesgue-measurable function.*

Then

$$\left\| \|f(x,y)\|_{LM^{\lambda}_{p\theta,a,x}(a,b)} \right\|_{LM^{\mu}_{q\sigma,c,y}(c,d)} \le \left\| \|f(x,y)\|_{LM^{\mu}_{q\sigma,c,y}(c,d)} \right\|_{LM^{\lambda}_{p\theta,a,x}(a,b)}. \tag{3.2}$$

Proof. By applying inequality (3.1) several times, we get

$$\left\| \|f(x,y)\|_{LM^{\lambda}_{p\theta,a,x}(a,b)} \right\|_{LM^{\mu}_{q\sigma,c,y}(c,d)} \le (\theta \le q)$$

$$\le \left\| \rho^{-\mu-\frac{1}{\sigma}} \left\| \left\| r^{-\lambda-\frac{1}{\theta}} \|f(x,y)\|_{L_{p,x}(a,b)\bigcap(a-r,a+r)} \right\|_{L_{q,y}((c,d)\bigcap(c-\rho,c+\rho))} \right\|_{L_\theta(0,\infty)} \right\|_{L_\sigma(0,\infty)}$$

$$\le(p\le q)\le \left\| \rho^{-\mu-\frac{1}{\sigma}} \left\| \left\| r^{-\lambda-\frac{1}{\theta}} \|f(x,y)\|_{L_{q,y}((c,d)\bigcap(c-\rho,c+\rho))} \right\|_{L_{p,x}(a,b)\bigcap(a-r,a+r)} \right\|_{L_\theta(0,\infty)} \right\|_{L_\sigma(0,\infty)}$$

$$\le(\theta\le\sigma)\le \left\| \rho^{-\mu-\frac{1}{\sigma}} \left\| \left\| r^{-\lambda-\frac{1}{\theta}} \|f(x,y)\|_{L_{q,y}((c,d)\bigcap(c-\rho,c+\rho))} \right\|_{L_{p,x}(a,b)\bigcap(a-r,a+r)} \right\|_{L_\sigma(0,\infty)} \right\|_{L_\theta(0,\infty)}$$

$$\le(p\le\sigma)\le \left\| r^{-\lambda-\frac{1}{\theta}} \left\| \left\| \rho^{-\mu-\frac{1}{\sigma}} \|f(x,y)\|_{L_{q,y}((c,d)\bigcap(c-\rho,c+\rho))} \right\|_{L_\sigma(0,\infty)} \right\|_{L_{p,x}(a,b)\bigcap(a-r,a+r)} \right\|_{L_\theta(0,\infty)}$$

$$= \left\| \|f(x,y)\|_{LM^{\mu}_{q\sigma,c,y}(c,d)} \right\|_{LM^{\lambda}_{p\theta,a,x}(a,b)}.$$

□

Theorem 3.1. *Let* $n \in \mathbb{N}, a, b, \alpha, k, p, q, \lambda, \mu, \theta, \sigma \in \mathbb{R}^n$,

$$0 \le a_i < b_i < \infty, 1 < p_i \le \infty, 0 < \theta_i, q_i, \sigma_i \le \infty, k_i \ge 0, \frac{1}{p_i} < \alpha_i < 1, i = 1, ..., n;$$

$$0 < \lambda_i < \infty \text{ if } \theta_i < \infty,\ 0 \le \lambda_i < \infty \text{ if } \theta_i = \infty,\ i = 1, ..., n.;$$

$$0 < \mu_i < \infty \text{ if } \sigma_i < \infty,\ 0 \le \mu_i < \infty \text{ if } \sigma_i = \infty,\ i = 1, ..., n.$$

Furthermore, let

$$\max\{p_i, \theta_i\} \le \min\{q_j, \sigma_j\}, i = 2, ..., n,\ j = 1, ..., i-1. \tag{3.3}$$

Then there exists $C_2 > 0$ *such that*

$$\left\| I_a^{\alpha,k} f \right\|_{\overleftarrow{LM}^{\mu}_{q\sigma,a}(Q(a,b)} \le C_2 \prod_{i=1}^{n} (b_i - a_i)^{\nu_i} \|f\|_{LM^{\lambda}_{p\theta,a}(Q(a,b))} \tag{3.4}$$

for all $f \in LM^{\lambda}_{p\theta,a}(Q(a,b)$, *where*

$$\nu_i = \lambda_i + \frac{1}{q_i} + \alpha_i(k_i+1) - \frac{1}{p_i} - \mu_i > 0,\ i = 1, ..., n, \tag{3.5}$$

under the assumptions $\nu_i > 0$ *if* $\sigma_i < \infty$ *and* $\nu_i \ge 0$ *if* $\sigma_i = \infty$, $i = 1, ...n$.

Moreover, each ν_i, $i = 1, ...; n$, *cannot be replaced by any other number.*

Proof. Without loss of generality, we assure that $a = 0$.

Step 1. A typical case is $n = 3$, which we assume. In this case by Definition 2, (1.3) and (1.2) we have

$$A \equiv \left\| I_{0+}^{\alpha,k} f \right\|_{\overleftarrow{LM}^{\mu}_{q\sigma}(Q(0,b))} = \left\| I_{0+}^{\alpha,k} f \right\|_{LM^{\overleftarrow{\mu}}_{\overleftarrow{q}\overleftarrow{\sigma}}(Q(0,b))}$$

$$= \left\| I_{0+}^{(\alpha_1,\alpha_2,\alpha_3),(k_1,k_2,k_3)} f \right\|_{LM^{(\mu_3,\mu_2,\mu_1)}_{(q_3,q_2 q_1)(\sigma_3,\sigma_2,\sigma_1)}(0,b_3)\times(0,b_2)\times(0,b_1)}$$

$$= \left\| \left\| \left\| I_{0+}^{\alpha_3,k_3} \left(I_{0+}^{\alpha_2,k_2} \left(I_{0+}^{\alpha_1,k_1} f \right) \right) \right\|_{LM^{\mu_3}_{q_3\sigma_3}(0,b_3)} \right\|_{LM^{\mu_2}_{q_2\sigma_2}(0,b_2)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)} .$$

By Lemma 2.2 with

$$\alpha = \alpha_3, k = k_3;\ \mu = \mu_3, q = q_3, \sigma = \sigma_3;\ \lambda = \lambda_3, p = p_3, \theta = \theta_3;\ a = 0, b = b_3$$

there exists $K_3 > 0$ such that

$$A \le K_3 b_3^{\nu_3} \left\| \left\| \left\| I_{0+}^{\alpha_2,k_2} \left(I_{0+}^{\alpha_1,k_1} f \right) \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)} \right\|_{LM^{\mu_2}_{q_2\sigma_2}(0,b_2)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)} .$$

By applying assumption (3.3) and inequality (2.2) first with

$$\max\{p_3, \theta_3\} \le \min\{q_2, \sigma_2\} \text{ and } \lambda = \lambda_3, p = p_3, \theta = \theta_3;\ \mu = \mu_2, q = q_2, \sigma = \sigma_2;$$

then with

$$\max\{p_3, \theta_3\} \le \min\{q_1, \sigma_1\} \text{ and } \lambda = \lambda_3, p = p_3, \theta = \theta_3;\ \mu = \mu_1, q = q_1, \sigma = \sigma_1;$$

we get

$$A \le K_3 b_3^{\nu_3} \left\| \left\| \left\| I_{0+}^{\alpha_2,k_2} \left(I_{0+}^{\alpha_1,k_1} f \right) \right\|_{LM^{\mu_2}_{q_2\sigma_2}(0,b_2)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)}$$

$$\le K_3 b_3^{\nu_3} \left\| \left\| \left\| I_{0+}^{\alpha_2,k_2} \left(I_{0+}^{\alpha_1,k_1} f \right) \right\|_{LM^{\mu_2}_{q_2\sigma_2}(0,b_2)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)} .$$

By Lemma 2.2 with

$$\alpha = \alpha_2, k = k_2;\ \mu = \mu_2, q = q_2, \sigma = \sigma_2;\ \lambda = \lambda_2, p = p_2, \theta = \theta_2;\ a = 0, b = b_2,$$

there exists $K_2 > 0$ such that

$$A \le K_2 K_3 b_2^{\nu_2} b_3^{\nu_3} \left\| \left\| \left\| I_{0+}^{\alpha_1,k_1} f \right\|_{LM^{\lambda_2}_{p_2\theta_2}(0,b_2)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)} .$$

By applying assumption (3.3) and inequality (3.1) with

$$\max\{p_2, \theta_2\} \le \min\{q_1, \sigma_1\} \text{ and } \lambda = \lambda_2, p = p_2, \theta = \theta_2;\ \mu = \mu_1, q = q_1, \sigma = \sigma_1;$$

we get

$$A \le K_2 K_3 b_2^{\nu_2} b_3^{\nu_3} = \left\| \left\| \left\| I_{0+}^{\alpha_1,k_1} f \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)} \right\|_{LM^{\lambda_2}_{p_2\theta_2}(0,b_2)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)} .$$

Finally, by Lemma 2.2 with

$$\alpha = \alpha_1, k = k_1;\ \mu = \mu_1, q = q_1, \sigma = \sigma_1;\ \lambda = \lambda_1, p = p_1, \theta = \theta_1;\ a = 0, b = b_1$$

there exists $K_1 > 0$ such that

$$A \leq K_1 K_2 K_3 b_1^{\nu_1} b_2^{\nu_2} b_3^{\nu_3} = \left\| \left\| \|f\|_{LM^{\lambda_1}_{p_1\theta_1}(0,b_1)} \right\|_{LM^{\lambda_1}_{p_2\theta_2}(0,b_2)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)}.$$

Also,

$$\left\| \left\| \|f\|_{LM^{\lambda_1}_{p_1\theta_1}(0,b_1)} \right\|_{LM^{\lambda_2}_{p_2\theta_2}(0,b_2)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)} = \|f\|_{LM^{\lambda}_{p\theta}(Q(0,b))}.$$

Therefore,

$$\left\| I^{\alpha,k}_{0+} f \right\|_{\overleftarrow{LM}^{\mu}_{q\sigma}(Q(0,b))} \leq K_1 K_2 K_3 b_1^{\nu_1} b_2^{\nu_2} b_3^{\nu_3} \|f\|_{LM^{\lambda}_{p\theta}(Q(0,b))}.$$

Step 2. Suppose that the operator $I^{\alpha,k}_{0+}$ is bounded from $LM^{\lambda}_{p,\theta}(Q(0,b))$ to $\overleftarrow{LM}^{\mu}_{q\theta}(Q(0,b))$, so, there exists $K_4(b) > 0$ such that

$$\left\| I^{\alpha,k}_{0+} f \right\|_{\overleftarrow{LM}^{\mu}_{q,\theta}(Q(0,b))} \leq K_4(b) \|f\|_{LM^{\lambda}_{p\,\theta}(Q(0,b)}. \tag{3.6}$$

Let $f(x_1, x_2) = \chi_1(x_1)\chi_2(x_2)\chi_3(x_3)$, where $\chi_i(x_i) = \chi_{(\frac{b_i}{2}, b_i)}(x_i), i = 1, 2, 3$. Then

$$\|f\|_{LM^{\lambda}_{p\,\theta}(Q(0,b))} = \|\chi_1\|_{LM^{\lambda_1}_{p_1\theta_1,x_1}(0,b_1)} \|\chi_2\|_{LM^{\lambda_2}_{p_2\theta_2,x_2}(0,b_2)} \|\chi_3\|_{LM^{\lambda_3}_{p_3\theta_3,x_3}(0,b_)},$$

where, as in Lemma 2.2,

$$\left\| \chi_{(\frac{b_i}{2},b_i)} \right\|_{LM^{\lambda_i}_{p_i\theta_i,x_i}(\frac{b_i}{2},b_i)} \leq 2^{\lambda_1 - \frac{1}{p_1}} (\lambda_1\theta_1)^{-\frac{1}{\theta_1}} b_1^{\frac{1}{p_1} - \lambda_1},\ i = 1, 2, 3.$$

Therefore, we obtain the following upper estimate:

$$\|f\|_{LM^{\lambda}_{p\theta}(Q(0,b))} \leq K_5 \prod_{i=1}^{3} b_i^{\frac{1}{p_i} - \lambda_i}, \tag{3.7}$$

where

$$K_5 = \prod_{i=1}^{3} 2^{\lambda_i - \frac{1}{p_i}} (\lambda_i\theta_i)^{-\frac{i}{\theta_i}}.$$

By applying inequality (2.8) to the equality

$$\left\| I^{\alpha,k}_{0+} f \right\|_{\overleftarrow{LM}^{\mu}_{q\,\sigma}(Q(0,b))} = \prod_{i=1}^{3} \left\| I^{\alpha_i,k_i}_{0+} \chi_i \right\|_{LM^{\mu_i}_{q_i\sigma_i}\left(\frac{b_i}{2},b_i\right)},$$

we also get the lower estimate:

$$\left\| I^{\alpha,k}_{0+} f \right\|_{\overleftarrow{LM}^{\mu}_{q\sigma}(Q(0,b))} \geq K_6 \prod_{i=1}^{3} b_i^{\alpha_i(k_i+1) + \frac{1}{q_i} - \mu_i}, \tag{3.8}$$

where

$$K_6 = \prod_{i=1}^{3}(k_i+1)^{-\alpha_i}(\Gamma(\alpha_i+1)^{-1}(1-2^{-(\alpha_i(k_i+1)q_i+1)})^{\frac{1}{q_i}}(\alpha_i(k_i+1)q_i+1)^{-\frac{1}{q_i}}(\mu_i\sigma_i)^{-\frac{1}{\sigma_i}}.$$

By using inequalities (3.6), (3.7) and (3.8), we get

$$K_6\prod_{i=1}^{3} b_1^{\alpha_i(k_i+1)+\frac{1}{q_i}-\mu_i} \leq K_4(b)K_5\prod_{i=1}^{3} b_i^{\frac{1}{p_i}-\lambda_i}.$$

So,

$$K_4(b) \geq \frac{K_6}{K_5} b_1^{\nu_1} b_2^{\nu_2} b_3^{\nu_3}.$$

If for some $i \in \{1,2,3\}$, say for $i=1$, inequality (3.4) holds with $\tau \neq \nu_1$ replacing ν_1, then

$$\frac{K_6}{K_5} b_1^{\nu_1} \leq K_4(b,1,1) \leq C_1 b_1^{\tau_1}$$

for all $b_1 > 0$, which is impossible.

Thus, we obtain the required statements for $n=3$. The case $n>3$ is considered similarly. □

Remark 2. For $\sigma = \theta = \infty$ Theorem 2.1 is an anisotropic version of Theorem 2.1 of [6].

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INFINITELY MANY PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS INVOLVING PIECEWISE ALTERNATELY ADVANCED AND RETARDED ARGUMENT

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Abstract. The manuscript introduces an innovative framework for establishing the existence of infinitely many nontrivial periodic solutions within a class of differential equations characterized by a piecewise alternately advanced and retarded argument. It comprehensively delineates the essential criteria required for the existence of these solutions and provides detailed procedures for their determination. Additionally, the study incorporates illustrative examples, including cases with infinitely many solutions, to demonstrate the effectiveness and applicability of the proposed approach.

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1 Introduction

A differential equation with a piecewise constant argument which is alternately advanced and retarded (DEPCA) is expressed as:

$$x'(t) = f\left(t, x(t), x\left(p\left[\frac{t+l}{p}\right]\right)\right), \tag{1.1}$$

where $[\cdot]$ denotes the greatest integer function, f is a continuous function defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, and $p\left[\frac{t+l}{p}\right]$ is a piecewise constant function defined by

$$p\left[\tfrac{t+l}{p}\right] = kp \quad \text{for} \quad t \in [kp-l, (k+1)p-l), \quad k \in \mathbb{Z},$$

with p and l being positive constants satisfying $p > l$.

The deviation argument of DEPCA (1.1), defined as

$$\wp(t) := t - p\left[\tfrac{t+l}{p}\right]$$

is negative within the interval $[kp-l, kp)$ and positive in $[kp, (k+1)p-l)$. This alternating sign behavior classifies DEPCA (1.1) as a differential equation of alternately advanced and retarded type.

DEPCAs represent a hybrid class of equations that combine characteristics of both discrete dynamics and continuous systems. These equations are particularly relevant in modeling applications in biomedical sciences and physical processes, as discussed in [6, 24] and further elaborated in [16, 18, 19, 20, 27]. Moreover, extensive research has been conducted to investigate various properties of DEPCAs. See, for example, [2, 3, 4, 5].

Recent studies [1, 12, 14, 15, 17, 22, 26, 28, 29, 30, 31] have analyzed specific formulations of DEPCAs.

Additionally, the works in [7, 8, 9, 10, 11] simplified the problem of n-periodic solution solvability into a system of n linear equations. Utilizing foundational principles of linear algebra, these studies systematically identified the conditions required for the existence of n-periodic solutions and provided explicit methodologies for deriving these solutions.

In 2024, M.I. Muminov and T.A. Radjabov [11] investigated the conditions under which 2-periodic solutions exist for first-order differential equations with piecewise constant delay:

$$T'(t) = a(t)T(t) + b(t)T([t]) + f(t).$$

They introduced a systematic method to identify 2-periodic solutions, clearly defined the requisite existence criteria, and provided explicit solution formulas.

Subsequently, K.-S. Chiu and F. Cordova [23] explored the conditions for the existence of 4-periodic solutions for first-order DEPCA (1.1) with the parameters $p = 2$ and $l = 1$.

To the best of our knowledge, only the two studies [11] and [23] have investigated the existence of infinitely many periodic solutions for DEPCA. However, neither has formulated detailed criteria for identifying such solutions in differential equations with a general piecewise alternately advanced and retarded argument.

This paper investigates a non-homogeneous differential equation with piecewise alternately advanced and retarded argument, given by:

$$y'(t) = a(t)y(t) + b(t)y\left(p\left[\frac{t+l}{p}\right]\right) + g(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where the functions $a(t)$, $b(t)$, and $g(t)$ are continuous and non-zero on $\mathbb{R}$. The general framework of this problem was previously analyzed in [13, 21], where the authors derived conditions ensuring the existence of a solution and demonstrated a Gronwall-type integral inequality as a practical application.

In this paper, we focus on establishing the conditions necessary for the existence of $2p$-periodic solutions to the initial value problem. An illustrative example is provided to demonstrate a case where an infinite number of $2p$-periodic solutions exist, thereby offering new perspectives that complement earlier studies on uniqueness for homogeneous cases.

2 Alternately advanced and retarded differential equation

A solution of DEPCA (1.2) is defined as follows. A function y is considered to be a solution of DEPCA (1.2) on $\mathbb{R}$ if it satisfies the following criteria:

(i) y is continuous on $\mathbb{R}$,

(ii) the derivative $y'(t)$ exists at each $t \in \mathbb{R}$, except possibly at points $t = kp - l$ for $k \in \mathbb{Z}$, where one-sided derivatives are required to exist,

(iii) y satisfies DEPCA (1.2) within each interval $\big(kp - l, (k+1)p - l\big)$ for $k \in \mathbb{Z}$, and the equation is satisfied for the right-hand derivative at $t = pk - l$ for $k \in \mathbb{Z}$.

To determine a solution of DEPCA (1.2), we follow the integration methodology described in [25]. By integrating equation (1.2), the solution can be expressed as:

$$y(t) = e^{\int_p^t a(s)ds}y(p) + \int_p^t b(s)y\left(p\left[\frac{s+l}{p}\right]\right)e^{\int_s^t a(r)dr}ds$$
$$+ \int_p^t g(s)e^{\int_s^t a(r)dr}ds, \quad t \in [p-l, 2p-l).$$

We define

$$\lambda(t,s) := e^{\int_s^t a(r)dr} + \int_s^t e^{\int_u^t a(r)dr} b(u)du,$$

$$\Psi(t,s) = \frac{\lambda\left(t, p\left[\frac{t+l}{p}\right]\right)}{\lambda\left(s, p\left[\frac{s+l}{p}\right]\right)} \prod_{j=\left[\frac{s+l}{p}\right]+1}^{\left[\frac{t+l}{p}\right]} \frac{\lambda(pj-l, p(j-1))}{\lambda(pj-l, pj)}, \quad t \geq s,$$

$$G(t,s) = \int_s^t e^{\int_u^t a(\kappa)d\kappa} g(u)du,$$

where $t, s \in [p-l, \infty)$, $\lambda(pj-l, pj) \neq 0$, $j \in \mathbb{N}$.

The following theorem provides a representation formula for the solution to DEPCA (1.2) for $t > 0$. The proof is similar to proofs of Theorem 2.1 in [13] and Theorem 2.2 in [21].

Theorem 2.1. *If $\lambda(pj-l, pj) \neq 0$ for $j \in \mathbb{N}$, then $y(t)$ represents the unique solution to DEPCA (1.2) for $t \geq p-l$ if and only if $y(t)$ can be represented as*

$$\begin{aligned} y(t) =& \Psi(t,\tau) y(\tau) + \int_\tau^{p[(\tau+l)/p]} \Psi(t,\tau) G(\tau,s)ds \\ &+ \sum_{k=[(\tau+l)/p]}^{[(t+l)/p]-1} \int_{kp}^{(k+1)p} \Psi\Big(t, (k+1)p-l\Big) G\Big((k+1)p-l, s\Big)ds \\ &+ G\Big(t, p\left[(t+l)/p\right]\Big). \end{aligned} \tag{2.1}$$

Proof. First, we demonstrate that the function $y(t)$ defined in (2.1) satisfies DEPCA (1.2). This can be readily verified using the relation $\frac{d\lambda(t,s)}{dt} = a(t)\lambda(t,s) + b(t)$, where s is fixed. Using the notation introduced earlier, we proceed as follows:

$$\begin{aligned} \frac{d\Psi(t,s)}{dt} &= \frac{\lambda'\left(t, p\left[\frac{t+l}{p}\right]\right)}{\left(s, p\left[\frac{s+l}{p}\right]\right)} \prod_{j=\left[\frac{s+l}{p}\right]+1}^{\left[\frac{t+l}{p}\right]} \frac{\lambda(pj-l, p(j-1))}{\lambda(pj-l, pj)} \\ &= \frac{a(t)\lambda\left(t, p\left[\frac{t+l}{p}\right]\right) + b(t)}{\lambda\left(s, p\left[\frac{s+l}{p}\right]\right)} \prod_{j=\left[\frac{s+l}{p}\right]+1}^{\left[\frac{t+l}{p}\right]} \frac{\lambda(pj-l, p(j-1))}{\lambda(pj-l, pj)} \\ &= a(t)\Psi(t,s) + b(t) \left(\frac{\lambda\left(p\left[\frac{t+l}{p}\right], p\left[\frac{t+l}{p}\right]\right)}{\lambda\left(s, p\left[\frac{s+l}{p}\right]\right)} \right) \prod_{j=\left[\frac{s+l}{p}\right]+1}^{\left[\frac{t+l}{p}\right]} \frac{\lambda(pj-l, p(j-1))}{\lambda(pj-l, pj)} \\ &= a(t)\Psi(t,s) + b(t)\Psi\left(p\left[\frac{t+l}{p}\right], s\right), \quad s < t. \end{aligned}$$

Conversely, suppose that $y_i(t)$ is a solution to DEPCA (1.2) on the interval $ip-l \leq t < (i+1)p-l$. Then, it satisfies

$$y_i'(t) = a(t)y_i(t) + b(t)y_i(ip) + g(t).$$

By defining $G(t,u) = \int_u^t e^{\int_s^t a(\kappa)d\kappa} g(s)ds$, the solution of this equation on $I_i = [ip-l, (i+1)p-l)$ is expressed as:

$$\begin{aligned} y_i(t) &= \left(e^{\int_{ip}^t a(s)ds} + \int_{ip}^t e^{\int_s^t a(\kappa)d\kappa} b(s)ds \right) y_i(ip) + \int_{ip}^t e^{\int_s^t a(\kappa)d\kappa} g(s)ds \\ &= \lambda(t, ip)\, y_i(ip) + G(t, ip). \end{aligned} \tag{2.2}$$

From (2.2), substituting $t = ip - l$ and taking the limit as $t \to (i+1)p - l^-$, we obtain

$$y_i(ip) = \left(\frac{y_i(ip-l) - G(ip-l, ip)}{\lambda\,(ip-l, ip)} \right). \tag{2.3}$$

Thus, based on (2.3), we deduce:

$$y_i((i+1)p-l) = \left(\frac{\lambda((i+1)p-l, ip)}{\lambda(ip-l, ip)} \right) \Big(y_i(ip-l) - G(ip-l, ip) \Big) + G((i+1)p-l, ip).$$

Similarly,

$$\begin{aligned} y_{i-1}(ip-l) = &\left(\frac{\lambda\,(ip-l,(i-1)p)}{\lambda\,(ip-l,(i-1)p)} \right) \Big(y_{i-1}((i-1)p-l) - G((i-1)p-l,(i-1)p) \Big) \\ &+ G(ip-l,(i-1)p), \qquad i \geq \left[\frac{\tau+l}{p} \right] + 2, \end{aligned}$$

and as $t \to p\left(\left[\frac{\tau+l}{p} \right] + 1 \right) - l^-$, we have

$$\begin{aligned} y\left(p\left(\left[\frac{\tau+l}{p} \right] + 1 \right) - l \right) = &\left(\frac{\lambda\left(p\left(\left[\frac{\tau+l}{p} \right] + 1 \right) - l, p\left[\frac{\tau+l}{p} \right] \right)}{\lambda\left(\tau, p\left[\frac{\tau+l}{p} \right] \right)} \right) \left(y(\tau) - G\left(\tau, p\left[\frac{\tau+l}{p} \right] \right) \right) \\ &+ G\left(p\left(\left[\frac{\tau+l}{p} \right] + 1 \right) - l, p\left[\frac{\tau+l}{p} \right] \right). \end{aligned}$$

Applying the two previous relations, we obtain

$$\begin{aligned} y_i((i+1)p-l) = &\left(\frac{\lambda\left((i+1)p-l, ip\right)}{\lambda\left(\tau, p\left[\frac{\tau+l}{p} \right] \right)} \right) \left(\prod_{j=\left[\frac{\tau+l}{p} \right]+1}^{i} \frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l, jp)} \right) y(\tau) \\ &+ \sum_{k=\left[\frac{\tau+l}{p} \right]+1}^{i} \Bigg\{ \left(\frac{\lambda\left((i+1)p-l, ip\right)}{\lambda\left((k+1)p-l,(k+1)p\right)} \right) \left(\prod_{j=k+2}^{i} \frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l, jp)} \right) \times \\ &\qquad \left(\frac{\lambda\left((k+1)p-l, kp\right)}{\lambda\left(kp-l, kp\right)} \left(-G(kp-l, kp)\right) + G((k+1)p-l, kp) \right) \Bigg\} \\ &+ \left(\frac{\lambda\left((i+1)p-l, ip\right)}{\lambda\left((k+1)p-l,(k+1)p\right)} \right) \left(\prod_{j=\left[\frac{\tau+l}{p} \right]+2}^{i} \frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l, jp)} \right) \times \\ &\left(\frac{\lambda\left(p\left(\left[\frac{\tau+l}{p} \right] + 1 \right) - l, p\left[\frac{\tau+l}{p} \right] \right)}{\lambda\left(\tau, p\left[\frac{\tau+l}{p} \right] \right)} \left(-G\left(\tau, p\left[\frac{\tau+l}{p} \right] \right) \right) + G\left(p\left(\left[\frac{\tau+l}{p} \right] + 1 \right) - l, p\left[\frac{\tau+l}{p} \right] \right) \right). \end{aligned}$$

Using the notation Ψ, we have for all $i \geq \left[\frac{\tau+l}{p}\right]+1$:

$$y_i((i+1)p-l) = \Psi\left((i+1)p-l,\tau\right)y(\tau) + \sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i} \Big[\Psi\left((i+1)p-l,kp-l\right)\cdot\left(-G(kp-l,kp)\right)$$
$$+\Psi\left((i+1)p-l,(k+1)p-l\right)\cdot G((k+1)p-l,kp)\Big]$$
$$+\Psi\left((i+1)p-l,\tau\right)\cdot\left(-G\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right)$$
$$+\Psi\left((i+1)p-l,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)\cdot G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right).$$

In particular,

$$\begin{aligned} y_i(ip-l) = \Psi\left(ip-l,\tau\right)y(\tau) + \sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i-1} \Big[\Psi\left(ip-l,kp-l\right)\left(-G(kp-l,kp)\right) \\ +\Psi\left(ip-l,(k+1)p-l\right)\cdot G((k+1)p-l,kp)\Big] \\ +\Psi\left(ip-l,\tau\right)\cdot\left(-\Psi\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right) \\ +\Psi\left(ip-l,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right). \end{aligned} \tag{2.4}$$

From equations (2.2), (2.3), and (2.4), it follows that:

$$y_i(t) = \lambda\left(t,ip\right)y_i(ip) + G(t,ip) = \frac{\lambda\left(t,ip\right)}{\lambda\left(ip-l,ip\right)}\cdot\Psi\left(ip-l,\tau\right)y(\tau)$$
$$+\frac{\lambda\left(t,ip\right)}{\lambda\left(ip-l,ip\right)}\cdot\Big[\Psi\left(ip-l,\tau\right)\cdot\left(-G\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right)$$
$$+\Psi\left(ip-l,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)\cdot G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right)\Big]$$
$$+\frac{\lambda\left(t,ip\right)}{\lambda\left(ip-l,ip\right)}\cdot\sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i-1}\Big[\Psi\left(ip-l,kp-l\right)\cdot\left(-G(kp-l,kp)\right)$$
$$+\Psi\left(ip-l,(k+1)p-l\right)\cdot G((k+1)p-l,kp)\Big]$$
$$-\frac{\lambda\left(t,ip\right)}{\lambda\left(ip-l,ip\right)}G(ip-l,ip) + G(t,ip)$$
$$= \Psi\left(t,\tau\right)y(\tau) + \Psi\left(t,\tau\right)\left(-G\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right)$$
$$+\Psi(t,(i(\tau)+1)p-l)G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right)$$
$$+\sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i-1}\Big[\Psi\left(t,kp-l\right)\left(-G(kp-l,kp)\right)+\Psi\left(t,(k+1)p-l\right)G((k+1)p-l,kp)\Big]$$

$$
\begin{aligned}
&-\frac{\lambda(t,ip)}{\lambda(ip-l,ip)}G(ip-l,ip)+G(t,ip)\\
&=\Psi(t,\tau)\,y(\tau)+G(t,ip)+\Psi(t,\tau)\left(-G\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right)\\
&+\Psi\left(t,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right)\\
+\sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i-1}&\Big[\Psi(t,kp-l)\left(-G(kp-l,kp)\right)+\Psi(t,(k+1)p-l)\,G((k+1)p-l,kp)\Big]\\
&-\frac{\lambda(t,ip)}{\lambda(ip-l,ip)}G(ip-l,ip)\\
&=\Psi(t,\tau)\,y(\tau)+\int_{\tau}^{i(\tau)p}\Psi(t,\tau)\,e^{\int_s^{\tau}a(\kappa)d\kappa}g(s)ds\\
&+\int_{p\left[\frac{\tau+l}{p}\right]}^{p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l}\Psi\left(t,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)e^{\int_s^{p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l}a(\kappa)d\kappa}g(s)ds\\
&+\sum\nolimits_{k=i(\tau)+1}^{k=i}\left(\int_{kp-l}^{kp}\Psi(t,kp-l)\,e^{\int_s^{kp-l}a(\kappa)d\kappa}g(s)ds\right)\\
&+\sum\nolimits_{k=i(\tau)+1}^{k=i-1}\left(\int_{kp}^{(k+1)p-l}\Psi(t,(k+1)p-l)\,e^{\int_s^{(k+1)p-l}a(\kappa)d\kappa}g(s)ds\right)\\
&+\int_{ip}^{t}e^{\int_s^{t}a(\kappa)d\kappa}g(s)ds\\
&=\Psi(t,\tau)\,y(\tau)+\int_{\tau}^{p\left[\frac{\tau+l}{p}\right]}\Psi(t,\tau)\,e^{\int_s^{\tau}a(\kappa)d\kappa}g(s)ds\\
&+\sum\nolimits_{k=\left[\frac{\tau+l}{p}\right]}^{k=\left[\frac{t+l}{p}\right]-1}\left(\int_{kp}^{(k+1)\cdot p}\Psi(t,(k+1)\cdot p-l)\,e^{\int_s^{(k+1)\cdot p-l}a(\kappa)d\kappa}g(s)ds\right)\\
&+\int_{ip}^{t}e^{\int_s^{t}a(\kappa)d\kappa}g(s)ds.
\end{aligned}
$$

From this, equality (2.1) follows.

If $g(t)=0$ in (2.1), we obtain the solution to linear DEPCA (1.2), given by $y(t)=\Psi(t,\tau)\,y(\tau)$. □

3 Existence of infinitely many periodic solutions

In this section, we propose an approach to identify $2p$-periodic solutions to DEPCA (1.2), assuming that the functions $a(\cdot)$, $b(\cdot)$, and $g(\cdot)$ are continuous on the interval $[p-l,\infty)$ and exhibit $2p$-periodic characteristics.

Assuming that $y(t)$ satisfies DEPCA (1.2) within the interval $kp-l\le t<(k+1)p-l$, integrating DEPCA (1.2) yields the solution of the following form:

$$
\begin{aligned}
y(t)&=e^{\int_{kp}^{t}a(s)ds}y(kp)+\int_{kp}^{t}b(s)y\left(p\left[\frac{s+l}{p}\right]\right)e^{\int_s^t a(r)dr}ds+\int_{kp}^{t}g(s)e^{\int_s^t a(r)dr}ds\\
&=\left(e^{\int_{kp}^{t}a(s)ds}+\int_{kp}^{t}b(s)e^{\int_s^t a(r)dr}ds\right)y(kp)+\int_{kp}^{t}g(s)e^{\int_s^t a(r)dr}ds.
\end{aligned}
$$

Using the notations $\lambda(t,s)$, and $G(t,s)$, the solution $y(t)$ is expressed as:

$$y(t) = \lambda(t,kp)y(kp) + G(t,kp), \quad kp - l \le t < (k+1)p - l.$$

From the above, by setting $t = kp - l$, we derive:

$$y(kp) = \frac{y(kp-l) - G(kp-l,kp)}{\lambda(kp-l,kp)}.$$

Substituting this result back, we obtain:

$$y(t) = \frac{\lambda(t,kp)}{\lambda(kp-l,kp)}\Big(y(kp-l) - G(kp-l,kp)\Big) + G(t,kp).$$

Assuming that $y(t)$ is $2p$-periodic on the interval $[p-l,\infty)$, the function $y(t)$ on $[p-l,3p-l)$ can be represented as:

$$y(t) = \begin{cases} \frac{\lambda(t,p)}{\lambda(p-l,p)}\Big(y(p-l) - G(p-l,p)\Big) + G(t,p), & t \in [p-l,2p-l), \\ \frac{\lambda(t,2p)}{\lambda(2p-l,2p)}\Big(y(2p-l) - G(2p-l,2p)\Big) + G(t,2p), & t \in [2p-l,3p-l). \end{cases} \tag{3.1}$$

This formulation highlights that the expressions on the right-hand side of (3.1) rely exclusively on the values of the unknowns $y_{p-l} = y(p-l)$ and $y_{2p-l} = y(2p-l)$. By leveraging the continuity of $y(\cdot)$, these unknowns can be defined as follows:

(i) $y_{2p-l} = y(2p-l) = \lim_{t\to 2p-l^-} y(t)$, where $t \in [p-l,2p-l)$,

(ii) $y_{3p-l} = y(3p-l) = \lim_{t\to 3p-l^-} y(t)$, where $t \in [2p-l,3p-l)$.

Given the continuity and periodicity of $y(\cdot)$, it follows that $y(p-l) = y(3p-l)$. To determine $y_{p-l} = y_{3p-l}$ from (3.1), we obtain the following system of equations:

$$\begin{cases} \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} y(p-l) - y(2p-l) = \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} G(p-l,p) - G(2p-l,p), \\ y(p-l) - \frac{\lambda(3p-l,4)}{\lambda(2p-l,2p)} y(2p-l) = -\frac{\lambda(3p-l,2p)}{\lambda(2p-l,2p)} G(2p-l,2p) + G(3p-l,2p). \end{cases} \tag{3.2}$$

Let Δ denote the determinant of the matrix $\mathcal{M}$, where

$$\mathcal{M} = \begin{pmatrix} \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} & -1 \\ 1 & -\frac{\lambda(3p-l,2p)}{\lambda(2p-l,2p)} \end{pmatrix}.$$

Using this, we establish the following theorem regarding the existence of $2p$-periodic solutions to DEPCA (1.2).

Theorem 3.1. *Let $a(\cdot)$, $b(\cdot)$, and $g(\cdot)$ be $2p$-periodic continuous functions. The following results hold.*

(a) If $\Delta \neq 0$, DEPCA (1.2) has a unique $2p$-periodic solution, as given in (3.1), where (y_{p-l}, y_{2p-l}) represents the sole solution of the system (3.2).

(b) If $\Delta = 0$ and $G(p-l,p) = G(2p-l,p) = G(2p-l,2p) = G(3p-l,2p) = 0$, DEPCA (1.2) admits an infinite number of $2p$-periodic solutions, as described below:

$$y(t) = \begin{cases} \alpha \frac{\lambda(t,p)}{\lambda(p-l,p)} \Big(y(p-l) - G(p-l,p) \Big) + G(t,p), & t \in [p-l, 2p-l), \\ \alpha \frac{\lambda(t,2p)}{\lambda(2p-l,2p)} \Big(y(2p-l) - G(2p-l,2p) \Big) + G(t,2p), & t \in [2p-l, 3p-l). \end{cases}$$

Here, (y_{p-l}, y_{2p-l}) represents an eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0, and α denotes a real-valued scalar.

(c) If $\Delta = 0$ and the rank $\mathcal{M}$ is less than the rank of the augmented matrix $(\mathcal{M}|b)$, where

$$b = \begin{pmatrix} \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} G(p-l,p) - G(2p-l,p), \\ -\frac{\lambda(3p-l,2p)}{\lambda(2p-l,2p)} G(2p-l,2p) + G(3p-l,2p) \end{pmatrix},$$

then DEPCA (1.2) does not possess a $2p$-periodic solution.

Proof. (a) Assume $y(t)$ is a $2p$-periodic solution to DEPCA (1.2). This solution can be characterized by (3.1), where (y_{p-l}, y_{2p-l}) satisfies the system (3.2). The solvability of linear system (3.2) requires that $\Delta \neq 0$. Consequently, $\Delta \neq 0$ must hold. Conversely, when $\Delta \neq 0$, DEPCA (3.2) admits a unique solution (y_{p-l}, y_{2p-l}). Furthermore, it can be demonstrated that the function $y(\cdot)$, defined in (3.1), represents the periodic solution to DEPCA (1.2).

(b) The function G assumes a value of zero at the points $(p-l,p)$, $(2p-l,p)$, $(2p-l,2p)$, and $(3p-l,2p)$. Consequently, equation (3.2) simplifies into a homogeneous form. A non-trivial solution to this equation exists if and only if $\Delta = 0$.

The pair of non-zero solutions (y_{p-l}, y_{2p-l}) serves as an eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0. Thus, $(\alpha y_{p-l}, \alpha y_{2p-l})$ represents a non-trivial solution to equation (3.2), where α denotes an arbitrary non-zero scalar. Accordingly, the $2p$-periodic function is expressed as:

$$y(t) = \begin{cases} \alpha \frac{\lambda(t,p)}{\lambda(p-l,p)} \Big(y(p-l) - G(p-l,p) \Big) + G(t,p), & t \in [p-l, 2p-l), \\ \alpha \frac{\lambda(t,2p)}{\lambda(2p-l,2p)} \Big(y(2p-l) - G(2p-l,2p) \Big) + G(t,2p), & t \in [2p-l, 3p-l). \end{cases}$$

This function satisfies DEPCA (1.2), where α can take any value.

(c) If $\Delta = 0$ and the rank of $\mathcal{M}$ is strictly less than the rank of the augmented matrix $(\mathcal{M} \,|\, b)$, where

$$b = \begin{pmatrix} \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} G(p-l,p) - G(2p-l,p) \\ -\frac{\lambda(3p-l,2p)}{\lambda(2p-l,2p)} G(2p-l,2p) + G(3p-l,2p) \end{pmatrix},$$

then equation (3.2) does not admit a solution. As a result, DEPCA (1.2) cannot have a $2p$-periodic solution.

□

4 Illustrative example

In this section, we provide a pertinent example to demonstrate the practical application of our theoretical framework. Specifically, we consider the following scalar differential equation involving a piecewise alternately advanced and retarded argument

$$y'(t) = \sin(2\pi t)\, y\left(3\left[\frac{t+1}{3}\right]\right) + g(t), \quad t \geq 2. \tag{4.1}$$

DEPCA (4.1) represents a particular case of DEPCA (1.2), specified by the parameters $p = 2$, $l = 1$, $a = 0$, $b(t) = \sin(2\pi t)$ with

$$g(t) = \begin{cases} \sin\left(\frac{t-2.5-3k}{\pi}\right), & t \in [2+3k, 3+3k), \\ -\left(\frac{\sin(0.5\cdot\pi^{-1})}{\sin(\pi^{-1})}\right)\sin\left(\frac{t-4-3k}{\pi}\right), & t \in [3+3k, 5+3k), \end{cases}$$

for $k \in \mathbb{N}_0$.

It can be readily verified that $G(2,3) = G(5,3) = G(5,6) = G(8,6) = 0$. The matrix associated with the linear system of equations involving the variables y_2 and y_5 is given as follows:

$$\mathcal{M} = \begin{pmatrix} \frac{\lambda(5,3)}{\lambda(2,3)} & -1 \\ 1 & -\frac{\lambda(8,6)}{\lambda(5,6)} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The determinant $\mathcal{M}$ is zero and vector $(1,1)$ serves as an eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0. According to Theorem 3.1(b), the solution to DEPCA is given by

$$y_\alpha(t) = \begin{cases} \alpha\lambda(t,3)y(2) + \hat{g}_1(t), & t \in [2,3), \\ \alpha\lambda(t,3)y(2) - \hat{g}_2(t), & t \in [3,5), \\ \alpha\lambda(t,6)y(5) + \hat{g}_3(t), & t \in [5,6), \\ \alpha\lambda(t,6)y(5) - \hat{g}_4(t), & t \in [6,8), \end{cases}$$

where

$$\hat{g}_1(t) = \pi\left(-\cos\left(\frac{t-2.5}{\pi}\right) + \cos\left(\frac{-0.5}{\pi}\right)\right),$$

$$\hat{g}_2(t) = \left(\frac{\sin(0.5\cdot\pi^{-1})\pi}{\sin(\pi^{-1})}\right)\left(-\cos\left(\frac{t-4}{\pi}\right) + \cos\left(\frac{-1}{\pi}\right)\right),$$

$$\hat{g}_3(t) = \pi\left(-\cos\left(\frac{t-5.5}{\pi}\right) + \cos\left(\frac{-0.5}{\pi}\right)\right),$$

$$\hat{g}_4(t) = \left(\frac{\sin(0.5\cdot\pi^{-1})\pi}{\sin(\pi^{-1})}\right)\left(-\cos\left(\frac{t-7}{\pi}\right) + \cos\left(\frac{-1}{\pi}\right)\right).$$

This solution is 6-periodic for any non-zero value of α.

The graphs of $y_\alpha(t)$ for $\alpha = 0.7$ and $\alpha = -0.5$ are presented in Figure 1 and Figure 2, respectively.

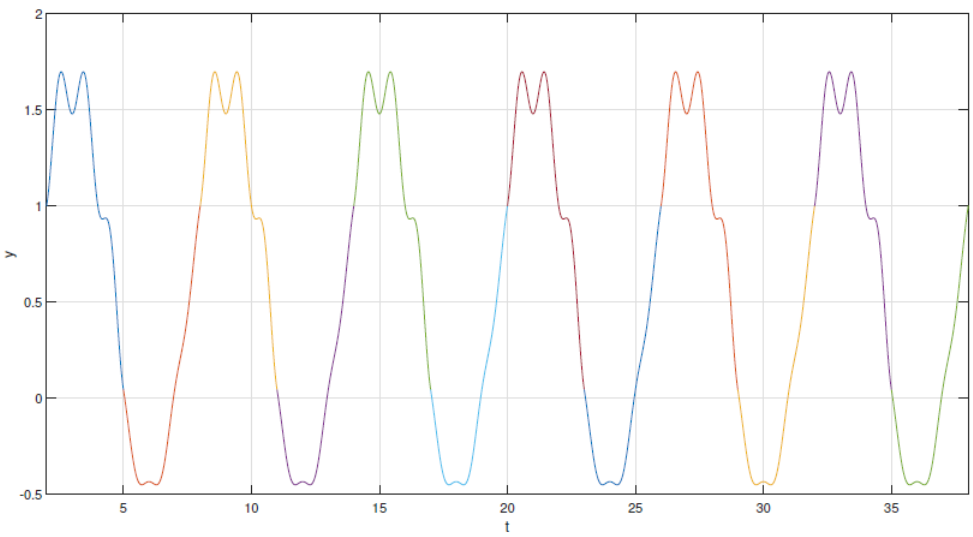

Fig. 1. 6-periodic solution to DEPCA (4.1) if $\alpha = 0.7$.

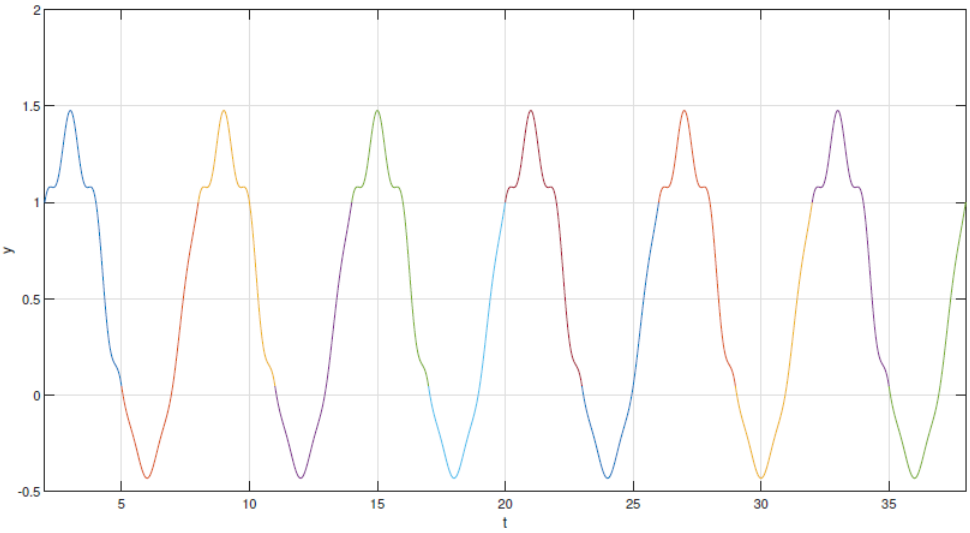

Fig. 2. 6-periodic solution to DEPCA (4.1) if $\alpha = -0.5$.

It is worth highlighting that the parameters of the equation in this example conform to the conditions outlined in the main results of [12]. Moreover, Example 4.1 extends and enhances the conclusions of Theorem 4.4 in [12], which establishes the uniqueness of the solution to the DEPCA (1.2).

5 Conclusion

This article examines the presence of infinitely many periodic solutions to first-order differential equations characterized by piecewise alternately advanced and retarded argument. Several theorems have been developed to establish both the existence and uniqueness of solutions to DEPCAs of this nature. Drawing inspiration from the methodologies in [11, 23], we have identified sufficient conditions ensuring the existence of infinitely many periodic solutions under the appropriate assumptions. Additionally, a range of numerical examples and simulations are provided to demonstrate the practical relevance and applicability of the theoretical results.

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PROPAGATION OF NONSMOOTH WAVES ALONG A STAR GRAPH WITH FIXED BOUNDARY VERTICES

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Key words: star graph, d'Alembert formula, eigenvalues problem, matching conditions

AMS Mathematics Subject Classification: 34A55, 34B05, 58C40.

Abstract. The paper studies the spread of waves along the star graph. The continuation of the initial data from the graph edges for the entire numerical axis allows to represent an analogue of the d'Alembert formula for waves on the star graph. At the same time, the continuation of the initial data is closely related to the continuation of the system of its eigenfunctions of the Sturm-Liouville problem originally defined on the star graph. The continuation of the eigenfunctions defined on the star graph is based on the continuation of the initial data of the mixed problem for the wave equation. The indicated continuation of the initial data of the mixed problem was proposed by B.M. Levitan.

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1 Statement of the main result

Let $\Gamma = \{V, E\}$ be a star graph. Here V is a set of vertices and E is a set of edges. The vertices are numerated by integer numbers from 0 to $m+1$. The interior vertex corresponds to the number $m+1$. The directed edges are denoted by $e_1, ..., e_{m+1}$, with j corresponding to the vertex of edge e_j. The length of the edge e_j is denoted by b_j.

On each edge e_j of the star graph Γ we study the initial boundary value problem (IBVP) for the wave equation with the individual continuous potential $q_j(x_j)$

$$\frac{\partial^2 \theta_j(x_j,t)}{\partial t^2} = \frac{\partial^2 \theta_j(x_j,t)}{\partial x_j^2} - q_j(x_j)\theta_j(x_j,t),\ t>0,\ x_j \in e_j \tag{1.1}$$

with the initial conditions

$$\theta_j(x_j,0) = a_j(x_j), \quad \frac{\partial \theta_j(x_j,0)}{\partial t} = b_j(x_j), \tag{1.2}$$

matching conditions

$$\theta_1(0,t) = \theta_2(0,t) = ... = \theta_m(0,t) = \theta_{m+1}(b_{m+1},t), \tag{1.3}$$

$$\frac{\partial \theta_{m+1}(b_{m+1},t)}{\partial x_{m+1}} = \sum_{j=1}^{m} \frac{\partial \theta_j(0,t)}{\partial x_j} \tag{1.4}$$

at the interior vertices of the star graph, and the Dirichlet boundary conditions

$$\theta_1(b_1,t) = \theta_2(b_2,t) = ... = \theta_m(b_m,t) = \theta_{m+1}(0,t) \tag{1.5}$$

at the boundary vertices. So, one IBVP (1.3) - (1.5) corresponds to the each edge e_j of the star graph Γ.

The goal of this work is to establish an analogue of the d'Alembert formula for the mixed problem (1.3) - (1.5). In the simplest case of a graph, representing a sequential connection of four intervals, such a formula is presented in [1].

We invoke some constructions from the monograph [1] for the statement of the main result. We extend the continuous function $q_j(x_j)$ from $(0, b_j)$ with $j = 1, ..., m+1$ continuously to the entire real axis. So, we consider the following Goursat problem

$$\frac{\partial^2 w_j(x_j,t,s)}{\partial t^2} = \frac{\partial^2 w_j(x_j,t,s)}{\partial s^2} - q_j(s) w_j(x_j,t,s), \quad t > 0, \quad s \in \mathbb{R},$$

$$w_j(x_j, t, t + x_j) = -\frac{1}{2} \int_0^t q_j(\tau + x_j) d\tau,$$

$$w_j(x_j, t, x_j - t) = -\frac{1}{2} \int_0^t q_j(x_j - \tau) d\tau$$

for $j = 1, ..., m+1$. The Goursat problem has a unique solution $w_j(x_j, t, s)$.

Now we exploit an assumption that ensures the simplicity of the spectrum for eigenvalue problem (2.1) - (2.4).

Assumption 1.1. *The lengths $b_1, ..., b_{m+1}$ of the edges $e_1, ..., e_{m+1}$ and the potentials $q_j(x_j)$, $j = 1, ..., m+1$ are chosen such that the spectra of the Sturm-Liouville operators L_j do not intersect pairwise. The operator L_j is defined by the differential expression:*
$\frac{d^2}{dx^2} - q_j(x_j)$ *on the domain* $D(L_j) = \{y(x_j) \in W_2^2[0, b_j] : y(b_j) = y(0) = 0\}$.

We note that such a choice is always possible.

We state the main result of the work. We establish an existence and uniqueness of the solution to problem (1.1) - (1.5) on the star graph in the following theorem. Moreover, we present an analogue of the d'Alembert formula for the solution to the initial boundary value problem for the wave equation on the star graph.

Theorem 1.1. *Suppose that the potentials $q_j(x_j)$ for $j = 1, ..., m+1$ represent a set of real continuous functions. Let Assumption 1.1 also be satisfied. We assume that the initial functions $a_j(x_j), b_j(x_j)$, $j = 1, ..., m+1$ are twice continuously differentiable on the corresponding edges and satisfy conditions* (1.2).

Then, the solution to problem (1.1) - (1.5) *exists and is unique. Moreover, for the solution, an analogue of the d'Alembert formula holds and for $b_j(x_j) \equiv 0$, $j = 1, ..., m+1$ the solution can be represented as follows*

$$\theta_j(x_j, t) = \frac{1}{2}\tilde{a}_j(x_j + t) + \frac{1}{2}\tilde{a}_j(x_j - t) + \frac{1}{2} \int_{x_j - t}^{x_j + t} w_j(x_j, t, s) \tilde{a}_j(s) ds.$$

Here $\tilde{a}_j(s)$ is the special continuation of the function $\tilde{a}_j(x_j)$ from the interval $(0, b_j)$ to the entire real axis. The details of this continuation are given in the proof of Theorem 1.1.

We note that $q_j(x_j) \equiv 0$ implies that $w_j(x_j, t, s) \equiv 0$.

2 Proof of Theorem 1.1

It is sufficient to prove Theorem 1.1 for $b_j(x_j) \equiv 0$ with $j = 0, 1, ..., m+1$. To state and prove Theorem 1.1 for nontrivial $b_j(x_j)$, we invoke the standard procedure from [3]. Now we state the following eigenvalue problem

$$l_j(y_j) \equiv -y_j''(x_j) + q_j(x_j)y_j(x_j) = \lambda y_j(x_j),\ x_j \in e_j \tag{2.1}$$

with the following Dirichlet boundary conditions

$$y_1(b_1) = y_2(b_2) = ... = y_m(b_m) = y_{m+1}(0) = 0 \tag{2.2}$$

at the boundary vertices of the star graph and the following matching conditions

$$y_1(0) = y_2(0) = ... = y_m(0) = y_{m+1}(b_{m+1}) \tag{2.3}$$

and

$$y'_{m+1}(b_{m+1}) = \sum_{j=1}^{m} y'_j(0) \tag{2.4}$$

at the interior vertices of the star graph. The spectral properties of eigenvalue problem (2.1) - (2.4) are essential to prove Theorem 1.1.

We state some notations and auxiliary facts related to eigenvalue problem (2.1)-(2.4) in Appendix 1. In particular, Appendix 1 proves that if Assumption 1.1 holds, then all the eigenvalues of problem (2.1) - (2.4) are simple, even if the potentials $q_j(x_j)$ with $j = 1, ..., m+1$ are complex-valued continuous functions. If the potentials $q_j(x_j)$ are real continuous functions, then all the eigenvalues are real, and the corresponding system of eigenfunctions forms an orthogonal basis in the space $L_2(\Gamma)$.

We denote by λ_k, where $k \in \mathbb{N}$, the sequence of eigenvalues of problem (2.1) - (2.4). Hence, $\lambda_k \neq \lambda_s$ if $k \neq s$ by Lemma 3.1 from Appendix 1. The corresponding system of eigenfunctions can be written as follows

$$\vec{\Phi}(\vec{x}, \lambda_k) = \{(\varphi_1(x_1, \lambda_k), ..., \varphi_{m+1}(x_{m+1}, \lambda_k))^T,\ k \geq 1\}.$$

Now we expand the initial function

$$\vec{A}(\vec{x}) = (a_1(x_1), ..., a_{m+1}(x_{m+1}))^T$$

in terms of the system of eigenfunctions of problem (2.1) - (2.4) to the following series

$$a_j(x_j) = \sum_{k=1}^{\infty} c_{jk}\ \ \varphi_j(x_j, \lambda_k), \tag{2.5}$$

where

$$c_{jk} = \frac{\int\limits_0^{b_j} a_j(x_j)\overline{\varphi_j^+(x_j, \overline{\lambda_k})}dx_j}{\int\limits_0^{b_j} \varphi_j(x_j, \lambda_k)\overline{\varphi_j^+(x_j, \overline{\lambda_k})}dx_j}.$$

Since the smooth function $\vec{A}(\vec{x})$ satisfies all conditions (2.2), (2.3) and (2.4), series (2.5) converges absolutely and uniformly on the graph Γ.

We seek the solution of problem (1.1) - (1.5) in the following form

$$\theta_j(x_j) = \sum_{k=1}^{\infty} d_{jk}(t)\,\varphi_j(x_j,\lambda_k). \tag{2.6}$$

In a standard way, we find that

$$d_{jk}(t) = c_{jk}\cos\sqrt{\lambda_k}t.$$

In the next lemma we give the representation of the solution of problem (1.1) - (1.5) in terms of the special extension of the eigenfunctions.

Lemma 2.1. *The following formula*

$$\cos\sqrt{\lambda_k}t\,\varphi_j(x_j,\lambda_k) = \tilde{\varphi}_j(x_j+t,\lambda_k) + \tilde{\varphi}_j(x_j-t,\lambda_k) + \int\limits_{x_j-t}^{x_j+t} w_j(x_j,t,s)\tilde{\varphi}_j(s,\lambda_k)ds$$

holds for $t>0$ and $x_j \in e_j$, where $\tilde{\varphi}_j(x_j,\lambda_k)$ is the special extension of the function $\varphi_j(x_j,\lambda_k)$ from the interval e_j to the entire real axis.

Proof. In fact, the potential $q_j(x_j)$ is defined only on the edge e_j. We extend the potential $q_j(x_j)$ to the entire real axis, preserving its class but otherwise arbitrarily. The function $\varphi_j(x_j,\lambda_k)$ ($j = 1,...,m+1$) is the solution of the homogeneous equation

$$-\varphi_j''(x_j) + q_j(x_j)\varphi_j(x_j) = \lambda_k\varphi_j(x_j),\ \ x_j \in e_j\ (j=1,...,m+1)$$

with Cauchy conditions at $x_j = b_j$

$$\varphi_j(b_j,\lambda_k) = 0,\ \ \varphi_j'(b_j,\lambda_k) = 1. \tag{2.7}$$

Let $\vartheta_j(x,t) = 2\cos\sqrt{\lambda_k}t\varphi_j(x,\lambda_k)$ for $x \in e_j$, $t>0$. We note that $\vartheta_j(x,t)$ is the solution of the mixed problem

$$\frac{\partial^2\vartheta_j(x,t)}{\partial t^2} = \frac{\partial^2\vartheta_j(x,t)}{\partial x^2} - q_j(x_j)\vartheta_j(x,t), \tag{2.8}$$

$$\vartheta_j(x,0) = 2\varphi_j(x,\lambda_k),\ \ \frac{\partial\vartheta_j(x,0)}{\partial t} = 0, \tag{2.9}$$

$$\vartheta_j(0,t) = 2\cos\sqrt{\lambda_k}t\varphi_j(0,\lambda_k),\ \ \vartheta_j(b_j,t) = 0. \tag{2.10}$$

Invoking the results of the monograph [2], we write the solution to the mixed problem (2.8), (2.9), (2.10) as follows

$$\vartheta_j(x,t) = \tilde{\varphi}_j(x+t,\lambda_k) + \tilde{\varphi}_j(x-t,\lambda_k) + \int\limits_{x_j-t}^{x_j+t} w_j(x,t,s)\tilde{\varphi}_j(s,\lambda_k)ds. \tag{2.11}$$

Here, $w_j(x,t,s)$ can be uniquely constructed from the function $q_j(x)$, $x \in \mathbb{R}$, as the solution of the Goursat problem

$$\frac{\partial^2 w_j}{\partial t^2} = \frac{\partial^2 w_j}{\partial s^2} - q_j(s)w_j,$$

$$w_j(x,t,x+t) = -\frac{1}{2}\int\limits_0^t q_j(x+\tau)d\tau,$$

$$w_j(x,t,x-t) = -\frac{1}{2}\int\limits_0^t q_j(x-\tau)d\tau.$$

Let us clarify how the continuation of the function $\tilde{\varphi}_j(x,\lambda_k)$ initially defined as $\varphi_j(x,\lambda_k)$ on $x \in e_j = (0,b_j)$ is extended to the entire real axis.

Substitution of representation (2.11) into the first of conditions (2.10) leads us to the following equality

$$2\cos\sqrt{\lambda_k}t\varphi_j(0,\lambda_k) = \varphi_j(t,\lambda_k) + \int\limits_0^t w_j(0,t,s)\varphi_j(s,\lambda_k)ds$$
$$+\tilde{\varphi}_j(-t,\lambda_k) + \int\limits_{-t}^0 w_j(0,t,s)\tilde{\varphi}_j(s,\lambda_k)ds.$$

For $0 \le t \le b_j$, this equality implies that

$$\tilde{\varphi}_j(-t,\lambda_k) + \int\limits_{-t}^0 w_j(0,t,s)\tilde{\varphi}_j(s,\lambda_k)ds = F_j(t), \tag{2.12}$$

where

$$F_j(t) = 2\cos\sqrt{\lambda_k}t\varphi_j(0,\lambda_k) - \varphi_j(t,\lambda_k) - \int\limits_0^t w_j(0,t,s)\varphi_j(s,\lambda_k)ds.$$

In [2] it is shown that such an extension of the function $\varphi_j(x_j,\lambda_k)$ belongs to the space $\mathbb{C}^2[-b_j,b_j]$.

Applying the second of boundary condition (2.9), we extend the function $\varphi_j(x_j,\lambda_k)$ to the interval $(b_j,2b_j)$. Therefore, the substitution of (2.10) into the second of conditions (2.9) gives the following integral equation

$$\tilde{\varphi}_j(b_j+t,\lambda_k) + \int\limits_0^{b_j+t} w_j(b_j,t,s)\tilde{\varphi}_j(s,\lambda_k)ds = R_j(t) \tag{2.13}$$

for $0 \le t \le b_j$, where

$$R_j(t) = -\varphi_j(b_j-t,\lambda_k) + \int\limits_{b_j-t}^0 w_j(b_j,t,s)\varphi_j(s,\lambda_k)ds\,(j=1,...,m+1).$$

The extension of $\varphi_j(x_j,\lambda_k)$ from the interval $(0,b_j)$ to the interval $(b_j,2b_j)$ belongs to $\mathbb{C}^2(0,2b_j)$. Integral equations (2.11) and (2.12) allow the function $\varphi_j(x_j,\lambda_k)$ to be extended to the entire real axis. □

Representation (2.6), relation (2.5) and Lemma 2.1 allow us to express the desired solution as follows

$$\theta_j(x_j,t) = \tilde{a}_j(x_j+t) + \tilde{a}_j(x_j-t) + \int\limits_{x_j-t}^{x_j+t} w_j(x_j,t,s)\tilde{a}_j(s)ds$$

for $j=1,...,m+1$, where $\tilde{a}_j(\xi_j)$ is the extension of $a_j(x_j)$ from the interval $(0,b_j)$ to the entire real axis, realized by the formula

$$\tilde{a}_j(\xi_j) = \sum_{k=1}^{\infty} c_{jk}\tilde{\varphi}(\xi_k,\lambda_k), \quad \xi_j \in (-\infty,\infty).$$

3 Appendix 1. Spectral properties of problem (2.1) - (2.4)

In this appendix, we state the spectral properties of problem (2.1) - (2.4) that are necessary to prove Theorem 1.1. In the absence of potentials $q_1(x_1) \equiv 0, ..., q_{m+1}(x_{m+1}) \equiv 0$, the spectral properties can be found in the work of N.P. Bondarenko [4]. In the case of real potentials $q_1, ..., q_{m+1}$, problem (2.1) - (2.4) is self-adjoint in the function space $L_2(\Gamma)$. Therefore, its spectrum consists of real eigenvalues and the corresponding system of eigenfunctions forms an orthogonal basis in the space $L_2(\Gamma)$. We establish conditions under which problem (2.1) - (2.4) has only simple eigenvalues. In our considerations, the potentials $q_1(\cdot) \equiv 0, ..., q_{m+1}(\cdot) \equiv 0$ may represent complex-valued continuous functions.

We denote by $\varphi_j(x_j, \lambda)$ the solution of the homogeneous equation $l_j(\varphi_j) = \lambda\varphi_j$ with $x_j \in e_j = (0, b_j)$ that satisfy the Cauchy conditions

$$\varphi_j(b_j, \lambda) = 0,\ \varphi_j'(b_j, \lambda) = 1$$

at $x_j = b_j$ for $j = 1, ..., m$. Assumption 1.1 implies that the values $\varphi_1(0, \lambda), ..., \varphi_m(0, \lambda)$ do not vanish for all complex values of the spectral parameter λ.

We also introduce the system of functions $\psi_1(x_j, \lambda) = B_j\varphi_1(x_j, \lambda)$, $x_j \in e_j$. The numbers $B_1, ..., B_m$ are chosen so that condition (2.3) is satisfied

$$B_j = B_{m+1}\prod_{\substack{i=1\\ i\neq j}}^{m}\varphi_j(0, \lambda),$$

where B_{m+1} is a common constant for all indices $j = 1, 2, ..., m$. The value of B_{m+1} may depend on the spectral parameter λ, but does not depend on $x_{m+1} \in e_{m+1}$.
Now we denote by $\varphi_{m+1}(x_{m+1}, \lambda)$ the solution of the homogeneous equation $l_{m+1}(\varphi_{m+1}) = \lambda\varphi_{m+1}(x_{m+1}, \lambda)$ for $x_{m+1} \in (0, b_{m+1})$ which at the point $x_{m+1} = b_{m+1}$ satisfies the following conditions

$$\varphi_{m+1}(b_{m+1}, \lambda) = \prod_{i=1}^{m}\varphi_j(0, \lambda),$$

$$\varphi'_{m+1}(b_{m+1}, \lambda) = \sum_{i=1}^{m}\prod_{\substack{s=1\\ s\neq i}}^{m}\varphi_s(0, \lambda)\varphi_i'(0, x).$$

Therefore, we can write the following relation

$$\varphi_{m+1}(x_{m+1}, \lambda) = \varphi_{m+1}(b_{m+1}, \lambda)c_{m+1}(x_{m+1}, \lambda) + \varphi'_{m+1}(b_{m+1}, \lambda)s_{m+1}(x_{m+1}, \lambda)$$

for all $x_{m+1} \in (0, b_{m+1})$. Here, $c_{m+1}(x_{m+1}, \lambda)$ and $s_{m+1}(x_{m+1}, \lambda)$ form a fundamental system of solutions of the homogeneous equation $l_{m+1}(y_{m+1}) = \lambda y_{m+1}(x_{m+1}, \lambda)$, $x_{m+1} \in e_{m+1}$ with the Cauchy conditions

$$s_{m+1}(b_{m+1}, \lambda) = {c'}_{m+1}(b_{m+1}, \lambda) = 0$$

and

$${s'}_{m+1}(b_{m+1}, \lambda) = c_{m+1}(b_{m+1}, \lambda) = 1$$

at $x_{m+1} = b_{m+1}$.

We note that the function $\varphi_{m+1}(x_{m+1}, \lambda)$ is an entire function of the spectral parameter λ for all fixed $x_{m+1} \in [0, b_{m+1}]$. The characteristic determinant of the eigenvalue problem (2.1) - (2.4) has the following form

$$\Delta(\lambda) \equiv \varphi_{m+1}(0, \lambda),\ \ \lambda \in \mathbb{C}.$$

Lemma 3.1. *The characteristic determinant $\Delta(\lambda)$ has only simple zeros.*

Proof. Let $\lambda, \mu \in \mathbb{C}$. We denote by $l_j^+(y) = -y''(x_j) + \overline{q}_j(x_j)y_j(x_j),\ x_j \in e_j$ the formally adjoint differential expression to $l_j(\cdot)$.

We consider the following difference

$$R = \sum_{j=1}^{m+1}\Big(\int_0^{b_j} l_j(\varphi_j(x_j,\lambda))\overline{\varphi_j^+(x_j,\overline{\mu})}dx_j - \int_0^{b_j} \varphi_j(x_j,\lambda)\overline{l_j^+(\varphi_j^+(x_j,\overline{\mu}))}dx_j\Big),$$

where the functions $\varphi_j^+(x_j,\overline{\mu})$, $j = 1,...,m+1$, are the solutions of the homogeneous equations $l_j^+(\varphi_j^+(x_j,\overline{\mu})) = \overline{\mu}\varphi_j^+(x_j,\overline{\mu})$, and their construction is analogous to the construction of $\varphi_j(x,\lambda)$, $j = 1,...,m+1$. The characteristic determinant of the adjoint problem is $\Delta^+(\overline{\mu}) = \varphi_{m+1}^+(0,\overline{\mu})$. Applying the Lagrange formula [5] and Cauchy conditions (2.7) to the difference R, we obtain

$$R = \sum_{j=1}^{m+1}\Big(\frac{d}{dx_j}\varphi_j(x_j,\lambda)\overline{\varphi_j^+(x_j,\overline{\mu})} - \varphi_j(x_j,\lambda)\overline{\frac{d\varphi_j^+(x_j,\overline{\mu})}{dx_j}}\Big)\Big|_{x_j=0}^{x_j=b_j-0}$$

$$= \frac{d\varphi_{m+1}(b_{m+1},\lambda)}{dx_{m+1}}\overline{\varphi_{m+1}^+(b_{m+1},\overline{\mu})} - \varphi_{m+1}(b_{m+1},\lambda)\overline{\frac{d\varphi_{m+1}^+(b_{m+1},\overline{\mu})}{dx_{m+1}}}$$

$$-\sum_{j=1}^{m}\frac{d\varphi_j(0,\lambda)}{dx_j}\overline{\varphi_{m+1}^+(b_{m+1},\overline{\mu})} + \varphi_{m+1}(b_{m+1},\lambda)\sum_{j=1}^{m}\overline{\frac{d\varphi_j^+(0,\overline{\mu})}{dx_j}}$$

$$-\frac{d\varphi_{m+1}(0,\lambda)}{dx_{m+1}}\overline{\varphi_{m+1}^+(0,\overline{\mu})} + \varphi_{m+1}(0,\lambda)\overline{\frac{d\varphi_{m+1}^+(0,\overline{\mu})}{dx_{m+1}}}$$

$$= \Delta(\lambda)\overline{\frac{d\varphi_{m+1}^+(0,\overline{\mu})}{dx_{m+1}}} - \frac{d\varphi_{m+1}(0,\lambda)}{dx_{m+1}}\overline{\Delta^+(\overline{\mu})}.$$

On the other hand, we have the following equality

$$R = (\lambda-\mu)\sum_{j=1}^{m+1}\int_0^{b_j}\varphi(x_j,\lambda)\overline{\varphi_j^+(x_j,\overline{\mu})}dx_j.$$

As a result, we obtain the identity

$$\sum_{j=1}^{m+1}\int_0^{b_j}\varphi(x_j,\lambda)\overline{\varphi_j^+(x_j,\overline{\mu})}dx_j =$$

$$= \frac{1}{\lambda-\mu}\Big(\Delta(\lambda)\overline{\frac{d\varphi_{m+1}^+(0,\overline{\mu})}{dx_{m+1}}} - \frac{d\varphi_{m+1}(0,\lambda)}{dx_{m+1}}\overline{\Delta^+(\overline{\mu})}\Big).$$

Let $\lambda = \lambda_0$ be an arbitrary eigenvalue of the problem (2.1) - (2.4). Then $\Delta(\lambda_0) = 0$. Thus, the following equality

$$\sum_{j=1}^{m+1}\int_0^{b_j}\varphi(x_j,\lambda)\overline{\varphi_j^+(x_j,\overline{\mu})}dx_j = -\frac{1}{\lambda_0-\mu}\overline{\Delta^+(\overline{\mu})}\frac{d\varphi_{m+1}(0,\lambda_0)}{dx_{m+1}}$$

holds. Now, let $\mu \to \lambda_0$. This equality implies the limiting relation

$$\sum_{j=1}^{m+1} \int_0^{b_j} \varphi(x_j, \lambda)\overline{\varphi_j^+(x_j, \overline{\mu})} dx_j = \frac{\overline{d\Delta^+(\overline{\lambda_0})}}{d\mu} \cdot \frac{d\varphi_{m+1}(0, \lambda_0)}{dx_{m+1}}.$$

The results of monograph [6] imply that $\overline{\lambda_0}$ is an eigenvalue of the adjoint problem, and $\varphi_j^+(x_j, \overline{\lambda_0})$ is the eigenfunction corresponding to the eigenvalue $\overline{\lambda_0}$. Moreover, $\sum_{j=1}^{m+1} \int_0^{b_j} \varphi(x_j, \lambda_0)\overline{\varphi_j^+(x_j, \overline{\mu})} dx_j \neq 0$. Hence, we get $\frac{\overline{d\Delta^+(\overline{\lambda_0})}}{d\mu} \cdot \frac{d\varphi_{m+1}(0,\lambda_0)}{dx_{m+1}} \neq 0$. We observe that $\frac{d\varphi_{m+1}(0,\lambda_0)}{dx_{m+1}} \neq 0$, since $\varphi_{m+1}(b_{m+1}, \lambda_0) = 0$. Otherwise, this would contradict Assumption 1.1. Therefore, it follows that

$$\frac{\overline{d\Delta^+(\overline{\lambda}_0)}}{d\mu} \neq 0.$$

Hence, the eigenvalue $\overline{\lambda}_0$ of the adjoint problem is simple. Consequently, the eigenvalue of original problem (2.1) - (2.4) must also be simple. □

Corollary 3.1. *If λ_0 is an eigenvalue of problem (2.1) - (2.4), then the corresponding eigenfunction has the follwing form*

$$\vec{\Phi}(\lambda_0) = (\varphi_1(x_1, \lambda_0), \varphi_2(x_2, \lambda_0), ..., \varphi_{m+1}(x_{m+1}, \lambda_0))$$

and none of the components of $\vec{\Phi}(\lambda_0)$ can be identically zero.

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CONTINUOUS EXTENSION TO THE BOUNDARY OF A DOMAIN OF THE LOGARITHMIC DOUBLE LAYER POTENTIAL

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Abstract. For the real part of the Cauchy-type integral that is known to be the logarithmic potential of the double layer, a necessary and sufficient condition for the continuous extension to the Ahlfors-regular boundary is established. Sufficient conditions involving subclasses of Ahlfors-regular curves are also considered. Illustrative examples are presented.

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1 Introduction

Let γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$, and let D^+ and D^- be the interior and exterior domains bounded by γ, respectively.

The classical theory of the logarithmic double layer potential

$$\frac{1}{2\pi}\int\limits_{\gamma} g(t)\frac{\partial}{\partial \mathbf{n}_t}\left(\ln\frac{1}{|t-z|}\right) ds_t \qquad \forall\, z\in D^{\pm} \tag{1.1}$$

is developed in the case in which the integration curve γ is a Lyapunov curve (see, for example, J. Plemelj [23]). Here $\mathbf{n}_t$ and s_t denote the unit vector of the outward normal to the curve γ at a point $t\in\gamma$ and an arc coordinate of this point, respectively, and the integral density $g:\gamma\to\mathbb{R}$ takes values in the set of real numbers $\mathbb{R}$.

J. Radon [26] established the continuous extension of the logarithmic double layer potential from the domains D^+ and D^- to the boundary γ in the case in which γ is a curve of bounded rotation, i.e., a curve for which the angle between the tangent to the curve and a fixed direction is a function of bounded variation. It is known that the class of Lyapunov curves and the class of Radon curves of bounded rotation are different, i.e., each of them contains curves that do not belong to the other class (see, for example, I.I. Danilyuk [4, p. 26]).

J. Král [19] established a necessary and sufficient condition for the curve γ, under which the logarithmic double layer potential is continuously extended from the domains D^+ and D^- to the boundary γ for all continuous functions $g:\gamma\to\mathbb{R}$.

The logarithmic double layer potential (1.1) is the real part of the Cauchy-type integral (see, for example, F.D. Gakhov [8], N.I. Muskhelishvili [20])

$$\widetilde{g}(z) := \frac{1}{2\pi i}\int\limits_{\gamma}\frac{g(t)}{t-z}\,dt \qquad \forall\, z\in D^{\pm}\,. \tag{1.2}$$

The theory of the boundary properties of the integral in (1.2) is presented in the monographs by F.D. Gakhov [8] and N.I. Muskhelishvili [20] under the classical assumptions about the smoothness of the integration curve and the Hölder density of the integral. In the papers of N.A. Davydov [6], V.V. Salaev [27], T.S. Salimov [28], E.M. Dyn'kin [7], O.F. Gerus [9, 11], the theory of the Cauchy-type integral and the Cauchy singular integral is developed on an arbitrary rectifiable Jordan curve in classes that are more general than the Hölder class of the integral density, which are defined, as a rule, in terms of the modulus of continuity of the function g.

In the paper of O.F. Gerus and M. Shapiro [12], an analog of the Davydov theorem [6] is proved for an appropriate Cauchy-type integral along an arbitrary rectifiable Jordan curve in $\mathbb{R}^2$, which takes values in the algebra of quaternions. This result is applied in the paper of O.F. Gerus and M. Shapiro [13] to establish sufficient conditions for the continuous extension to the boundary of a domain of metaharmonic potentials, a partial case of which is logarithmic double layer potential (1.1).

At the same time, the results mentioned above about the continuous extension of the logarithmic double layer potential to the boundary of a domain, which are contained in papers [23, 26, 19], are valid for arbitrary continuous functions g. It is linked to the fact that the real part of the Schwartz integral, i.e. the Poisson integral, is continuously extended to the boundary of the unit disk for an arbitrary continuous integral density, while the continuous extension of the imaginary part of the Schwartz integral requires additional assumptions about the integral density.

The purpose of this paper is to establish general results about the continuous extension of the real part of the Cauchy-type integral with a real-valued integral density, which are usable for the cases in which the classical results of papers [23, 26, 19] as well as the corresponding result of paper [13] are not applicable, generally speaking.

2 Preliminary information

In what follows, a closed rectifiable Jordan curve γ satisfies the condition (see V.V. Salaev [27])

$$\theta(\varepsilon) := \sup_{\xi\in\gamma} \theta_\xi(\varepsilon) = O(\varepsilon), \quad \varepsilon \to 0\,, \tag{2.1}$$

where $\theta_\xi(\varepsilon) := \operatorname{meas}\gamma_\varepsilon(\xi)$, $\gamma_\varepsilon(\xi) := \{t \in \gamma : |t-\xi| \le \varepsilon\}$ and meas denotes the linear Lebesgue measure on γ. Curves satisfying condition (2.1) are important in solving various problems (see, for example, V.V. Salaev [27], L. Ahlfors [1], G. David [5], C. Pommerenke [24], A. Böttcher and Y.I. Karlovich [2]). Such curves are often called *regular* (see, for example, [5]) or *Ahlfors-regular* (see, for example, [24]), or *Carleson curves* (see, for example, [2]).

It is well known that a closed rectifiable Jordan curve γ has a tangent at almost all points $t \in \gamma$. For such a point $t \in \gamma$ we denote by ϑ_t the angle between the tangent to the curve γ at this point and the direction of the real axis. J. Radon [26] called γ a *curve of bounded rotation* if the angle ϑ_t is a function of bounded variation on γ.

This implies that for a curve γ of bounded rotation, the angle ϑ_t can have at most a countable set of discontinuity points, and there are one-sided tangents at each point of the curve γ. Moreover, a curve of bounded rotation can have only a finite set of cusp points and at most a countable set of corner points. At the same time, every curve of bounded rotation satisfies condition (2.1).

This follows, for example, from the fact that the Cauchy singular integral operator is bounded in Lebesgue spaces on any curve γ of bounded rotation (see I.I. Danilyuk [4], I.I. Daniljuk and V.Yu. Šelepov [3], È. G. Gordadze [15]), and a necessary condition for this is condition (2.1) on the curve (see V.A. Paatashvili and G.A. Khuskivadze [21]).

A curve γ is called a *Lyapunov curve* if the angle ϑ_t satisfies the Hölder condition:

$$|\vartheta_{t_1} - \vartheta_{t_2}| \le c\,|t_1 - t_2|^\alpha \qquad \forall\, t_1, t_2 \in \gamma,$$

where $\alpha \in (0, 1]$ and the constant c does not depend on t_1 and t_2. It is clear that the Lyapunov curve is a smooth curve and also satisfies condition (2.1). There are Lyapunov curves that are not the Radon curves of bounded rotation (see, for example, I.I. Danilyuk [4, p. 26]).

J. Král [19] proved that logarithmic double layer potential (1.1) is extended continuously from the domains D^+ and D^- to the boundary γ for all continuous functions $g : \gamma \to \mathbb{R}$ if and only if the curve γ satisfies the condition

$$\sup_{\xi\in\gamma} \int\limits_0^{2\pi} \mu_\gamma(\xi, \phi)\, d\phi < \infty, \tag{2.2}$$

where $\mu_\gamma(\xi, \phi)$ is the number of intersection points of the curve γ with the ray $\{z = \xi + re^{i\phi} : r > 0\}$. It will be shown below that each curve γ, which satisfies condition (2.2), also satisfies condition (2.1), i.e, it is an Ahlfors-regular curve.

Note that not every smooth curve satisfies condition (2.2). In particular, an example of such a curve will be given below.

For a function $f\colon E \to \mathbb{C}$ continuous on a set $E \subset \mathbb{C}$, we shall use its modulus of continuity

$$\omega_E(f, \varepsilon) := \sup_{t_1,t_2\in E\,:\,|t_1-t_2|\le\varepsilon} |f(t_1) - f(t_2)|\,.$$

Quite often, the conditions for a given domain are formulated in terms of a mapping of the unit disk onto this domain (see, for example, I.I. Priwalow [25, §§ 12–17 of Chapter III]).

Consider a conformal mapping $\sigma : U \to D^+$ of the unit disk U onto the domain D^+. It is well known that the mapping σ is continuously extended to the boundary ∂U and defines a homeomorphism between the unit circle ∂U and the curve γ.

In paper [16], when solving boundary value problems for monogenic hypercomplex functions associated with a biharmonic equation, the following result on the continuous extension of logarithmic double layer potential (1.1) has actually been established, although it was not formulated as a theorem.

Theorem 2.1. *Let $\sigma : U \to D^+$ be a conformal mapping of the unit disk U onto the domain D^+, and let the continuous extension of σ to the circle ∂U have the nonvanishing continuous contour derivative σ' on ∂U, and let its modulus of continuity satisfy the Dini condition*

$$\int\limits_0^1 \frac{\omega_{\partial U}(\sigma', \eta)}{\eta}\, d\eta < \infty. \tag{2.3}$$

Then, for each continuous function $g\colon \gamma \to \mathbb{R}$, integral (1.1) has a continuous extension from the domain D^+ to the boundary γ.

Theorem 2.1 generalizes the corresponding result of the classical theory of the logarithmic double layer potential on the Lyapunov curves (see, for example, J. Plemelj [23]), because in the case in which γ is a Lyapunov curve, condition (2.3) is satisfied owing to the Kellogg theorem (see, for example, G.M. Goluzin [14]). Condition (2.3) is also satisfied in the more general case in which the modulus of continuity of the angle ϑ_t satisfies the condition

$$\int\limits_0^1 \frac{\omega_\gamma(\vartheta_t, \eta)}{\eta}\, \ln\frac{2}{\eta}\, d\eta < \infty\,. \tag{2.4}$$

It follows from the estimate of the modulus of continuity of the function σ' presented in Theorem 2 in the paper of J.L. Heronimus [17] (see also S.E. Warschawski [30]).

If the modulus of continuity of the function $g : \gamma \to \mathbb{R}$ satisfies the Dini condition

$$\int_0^1 \frac{\omega_\gamma(g,\eta)}{\eta}\, d\eta < \infty, \tag{2.5}$$

then the reduced singular Cauchy integral

$$\int_\gamma \frac{g(t)-g(\xi)}{t-\xi}\, dt := \lim_{\delta\to 0^+} \int_{\gamma\setminus\gamma_\delta(\xi)} \frac{g(t)-g(\xi)}{t-\xi}\, dt \qquad \forall\, \xi \in \gamma \tag{2.6}$$

exists (see O.F. Gerus [9], and also V.V. Salaev [27], where the Dini condition of form (2.5) is given in terms of the regularized modulus of continuity using the Stechkin construction).

From this and from the result of N.A. Davydov [6], it follows that Cauchy-type integral (1.2) has the limiting values $\widetilde{g}^{\,\pm}(\xi)$ at every point $\xi \in \gamma$ from the domains $D^\pm$, which are expressed by the Sokhotski–Plemelj formulas:

$$\widetilde{g}^{\,+}(\xi) = g(\xi) + \frac{1}{2\pi i}\int_\gamma \frac{g(t)-g(\xi)}{t-\xi}\, dt\,, \tag{2.7}$$

$$\widetilde{g}^{\,-}(\xi) = \frac{1}{2\pi i}\int_\gamma \frac{g(t)-g(\xi)}{t-\xi}\, dt\,. \tag{2.8}$$

Let $d := \max\limits_{t_1,t_2\in\gamma} |t_1 - t_2|$ be the diameter of the curve γ.

To single out the real part of integral (2.6), we define a branch $\arg(z-\xi)$ continuous on $\gamma\setminus\{\xi\}$ of the multivalued function $\mathrm{Arg}\,(z-\xi)$ in the following way. For each positive $\delta < d/2$, we select that connected component $\gamma_{\xi,\delta}$ of the set $\gamma_\delta(\xi)$ which contains the point ξ, and we take such a point $\xi_1 \in \gamma_{\xi,\delta}$ at which there is a tangent to γ and which does not precede the point ξ under the given orientation of the curve γ. It is obvious that in the case in which there is a tangent to γ at the point ξ, we can set $\xi_1 = \xi$. Let us cut the complex plane along the curve $\Gamma_{\xi,\delta} := \gamma[\xi,\xi_1]\cup\Gamma[\xi_1,\infty]$, where $\gamma[\xi,\xi_1]$ is the arc of γ with the initial point ξ and the end point ξ_1, and $\Gamma[\xi_1,\infty]$ is a smooth curve that connects the points ξ_1 and ∞ and lies completely (except for its ends ξ_1 and ∞) in the domain D^-. Now, let us single out a branch $\arg_\delta(z-\xi)$ of the multivalued function $\mathrm{Arg}\,(z-\xi)$, which is continuous outside the cut $\Gamma_{\xi,\delta}$ with the normalization condition $\arg_\delta(z_0-\xi) = \phi_0$, where $z_0 \in D^+$ and ϕ_0 is one of the values of the function $\mathrm{Arg}\,(z-\xi)$ at $z = z_0$. We shall use the fixed values z_0 and ϕ_0 for all positive $\delta < d/2$. As a result, we have the obvious equality

$$\arg_{\delta_1}(t-\xi) = \arg_\delta(t-\xi) \qquad \forall\, \delta_1,\delta : 0<\delta_1<\delta<d/2 \quad \forall\, t\in\gamma\setminus\gamma_\delta(\xi)$$

that implies the existence of the following limit:

$$\arg(t-\xi) := \lim_{\delta\to 0^+} \arg_\delta(t-\xi) \qquad \forall\, t \in \gamma\setminus\{\xi\}\,.$$

Thus, under the assumption that the function $g : \gamma \to \mathbb{R}$ satisfies the condition (2.5), from equality (2.6) we get the equality

$$\mathrm{Re}\left(\frac{1}{2\pi i}\int_\gamma \frac{g(t)-g(\xi)}{t-\xi}\, dt\right) = \frac{1}{2\pi}\lim_{\delta\to 0^+}\int_{\gamma\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \qquad \forall\, \xi\in\gamma\,,$$

where in the right-hand side of the equality the integral is the Stieltjes integral.

Let us accept by definition

$$\int_{\gamma} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) := \lim_{\delta\to 0^+} \int_{\gamma\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \qquad \forall\, \xi\in\gamma\,. \tag{2.9}$$

Finally, denoting

$$\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi) := \lim_{z\to\xi,\, z\in D^{\pm}} \mathrm{Re}\,\widetilde{g}(z) \qquad \forall\, \xi\in\gamma,$$

as a corollary of formulas (2.7) and (2.8), for all $\xi\in\gamma$, we obtain the equalities

$$\big(\mathrm{Re}\,\widetilde{g}\big)^{+}(\xi) = g(\xi) + \frac{1}{2\pi}\int_{\gamma} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\,, \tag{2.10}$$

$$\big(\mathrm{Re}\,\widetilde{g}\big)^{-}(\xi) = \frac{1}{2\pi}\int_{\gamma} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\,. \tag{2.11}$$

Below, we investigate the fulfillment of equalities (2.10) and (2.11), not assuming, generally speaking, neither the fulfillment of condition (2.5) for the function $g:\gamma\to\mathbb{R}$ nor the fulfillment of condition (2.2) for the curve γ.

3 A necessary and sufficient condition for the continuous extension of the real part of the Cauchy-type integral to the boundary of the domain bounded by an Ahlfors-regular curve

The following statement is true.

Theorem 3.1. *Let a closed Jordan curve γ be Ahlfors-regular and let a function $g\colon\gamma\to\mathbb{R}$ be continuous on γ. The function $\mathrm{Re}\,\widetilde{g}(z)$ has a continuous extension to the boundary γ from the domain D^+ or D^- if and only if the following condition is satisfied:*

$$\sup_{\xi\in\gamma}\ \sup_{\delta\in(0,\varepsilon)} \left| \int_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \right| \to 0\,, \qquad \varepsilon\to 0\,. \tag{3.1}$$

In the case in which condition (3.1) is satisfied, the limiting values $\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi)$ are represented by formulas (2.10) and (2.11) for all $\xi\in\gamma$.

Proof. Sufficiency. Obviously, if condition (3.1) is satisfied, the limit exists in equality (2.9).

Let us prove equality (2.10). Let $\xi\in\gamma$, $z\in D^+$ and $\varepsilon := |z-\xi| < d/8$. Denote $\varepsilon_1 := \min\limits_{t\in\gamma}|t-z|$. Let us choose the point $\xi_z\in\gamma$ closest to the point z.

We use the following representation of the difference:

$$\begin{aligned}
&\mathrm{Re}\,\widetilde{g}(z) - g(\xi) - \frac{1}{2\pi}\int_{\gamma} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\\
&\quad = \mathrm{Re}\left(\frac{1}{2\pi i}\int_{\gamma} \frac{g(t)-g(\xi_z)}{t-z}\, dt\right) - \frac{1}{2\pi}\int_{\gamma} \big(g(t)-g(\xi_z)\big)\, d\arg(t-\xi_z) + g(\xi_z) - g(\xi)\\
&\qquad + \frac{1}{2\pi}\int_{\gamma} \big(g(t)-g(\xi_z)\big)\, d\arg(t-\xi_z) - \frac{1}{2\pi}\int_{\gamma} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\,.
\end{aligned}$$

Consider the difference

$$\operatorname{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma}\frac{g(t)-g(\xi_z)}{t-z}\,dt\right)-\frac{1}{2\pi}\int\limits_{\gamma}\big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z)$$
$$=\operatorname{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma_{2\varepsilon_1}(\xi_z)}\frac{g(t)-g(\xi_z)}{t-z}\,dt\right)-\frac{1}{2\pi}\int\limits_{2\varepsilon_1(\xi_z)}\big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z)$$
$$+\operatorname{Re}\left(\frac{z-\xi_z}{2\pi i}\int\limits_{\gamma\setminus\gamma_{2\varepsilon_1}(\xi_z)}\frac{g(t)-g(\xi_z)}{(t-z)(t-\xi_z)}\,dt\right)=:I_1-I_2+I_3\,.$$

Taking into account condition (2.1), we obtain the relation

$$|I_1|\le\frac{1}{2\pi}\int\limits_{\gamma_{2\varepsilon_1}(\xi_z)}\frac{|g(t)-g(\xi_z)|}{|t-z|}\,|dt|\le\frac{\omega_\gamma(g,2\varepsilon_1)}{2\pi\varepsilon_1}\,\theta_{\xi_z}(2\varepsilon_1)\le c\,\omega_\gamma(g,2\varepsilon_1)\to0,\qquad\varepsilon_1\to0,$$

where the constant c depends only on the curve γ.

Condition (3.1) implies the relation

$$|I_2|\to0,\qquad\varepsilon_1\to0.$$

To estimate the integral I_3, we use Proposition 7.2 in [22] (see also the proof of Theorem 1 in the paper of O.F. Gerus [10]) and condition (2.1) so that we have

$$|I_3|\le\frac{|z-\xi_z|}{\pi}\int\limits_{\gamma\setminus\gamma_{2\varepsilon_1}(\xi_z)}\frac{|g(t)-g(\xi_z)|}{|t-\xi_z|^2}\,|dt|\le\frac{\varepsilon_1}{\pi}\int\limits_{[2\varepsilon_1,d]}\frac{\omega_\gamma(g,\eta)}{\eta^2}\,d\theta_{\xi_z}(\eta)$$
$$\le\frac{2\varepsilon_1}{3\pi}\int\limits_{\varepsilon_1}^{d}\frac{\theta_{\xi_z}(2\eta)\omega_\gamma(g,2\eta)}{\eta^3}\,d\eta\le c\,\varepsilon_1\int\limits_{\varepsilon_1}^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}\,d\eta\to0,\qquad\varepsilon_1\to0,$$

where the constant c depends only on the curve γ.

Now, consider the difference

$$\frac{1}{2\pi}\int\limits_{\gamma}\big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z)-\frac{1}{2\pi}\int\limits_{\gamma}\big(g(t)-g(\xi)\big)\,d\arg(t-\xi)$$
$$=\frac{1}{2\pi}\int\limits_{\gamma_\varepsilon(\xi_z)}\big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z)+\operatorname{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma_{4\varepsilon}(\xi)\setminus\gamma_\varepsilon(\xi_z)}\frac{g(t)-g(\xi_z)}{t-\xi_z}\,dt\right)$$
$$-\frac{1}{2\pi}\int\limits_{\gamma_{4\varepsilon}(\xi)}\big(g(t)-g(\xi)\big)\,d\arg(t-\xi)+\operatorname{Re}\left(\frac{\xi_z-\xi}{2\pi i}\int\limits_{\gamma\setminus\gamma_{4\varepsilon}(\xi)}\frac{g(t)-g(\xi)}{(t-\xi)(t-\xi_z)}\,dt\right)$$
$$+\operatorname{Re}\left(\frac{g(\xi)-g(\xi_z)}{2\pi i}\int\limits_{\gamma\setminus\gamma_{4\varepsilon}(\xi)}\frac{dt}{t-\xi_z}\right)=:J_1+J_2-J_3+J_4+J_5\,.$$

Condition (3.1) implies the relations

$$|J_1|\to0\quad\text{and}\quad|J_3|\to0,\qquad\varepsilon\to0.$$

The integrals J_2 and J_4 are estimated similarly to the integrals I_1 and I_3, respectively. As a result, we have the relations

$$|J_2| \to 0 \quad \text{and} \quad |J_4| \to 0, \qquad \varepsilon \to 0.$$

In addition, the following relations are satisfied (see the proof of Theorem 1 in the paper of V.V. Salaev [27]):

$$|J_5| \le \frac{|g(\xi) - g(\xi_z)|}{2\pi} \left| \int\limits_{\gamma \setminus \gamma_{4\varepsilon}(\xi)} \frac{dt}{t - \xi_z} \right| \le 2\,\omega_\gamma(g, 2\varepsilon) \to 0, \qquad \varepsilon \to 0.$$

An obvious corollary of the given relations is equality (2.10). Equality (2.11) is similarly established.

Necessity. Since the curve γ satisfies condition (2.1), the singular Cauchy integral operator is bounded in the Lebesgue spaces L_p for $p > 1$ on γ (see Theorem 1 in the paper of G. David [5]). At the same time, Cauchy-type integral (1.2) belongs to the Smirnov classes E_p (see, for example, I.I. Priwalow [25]) for $p > 1$ in the domains D^+ and D^-. In addition, its angular boundary values $\widetilde{g}^{\,\pm}_{\text{ang}}(\xi)$ from $D^\pm$ exist for almost all points $\xi \in \gamma$, and the following equality holds almost everywhere on γ:

$$g(\xi) = \widetilde{g}^{\,+}_{\text{ang}}(\xi) - \widetilde{g}^{\,-}_{\text{ang}}(\xi).$$

We denote by $\widetilde{g}^{\,\pm}$ the function that is defined by equality (1.2) in the domain $D^\pm$ and is extended almost everywhere on γ by means of the values $\widetilde{g}^{\,\pm}_{\text{ang}}$. We denote also the real part of this function by $\operatorname{Re}\widetilde{g}^{\,\pm}$.

Note that the values of the functions $\operatorname{Re}\widetilde{g}^{\,+}$ and $\operatorname{Re}\widetilde{g}^{\,-}$ are expressed by equalities of form (2.10) and (2.11) for almost all points $\xi \in \gamma$. It obviously follows that in the case in which the function $\operatorname{Re}\widetilde{g}(z)$ is continuously extended to the boundary γ from one of the domains D^+ or D^-, this function is also continuously extended to γ from the other domain.

For $\xi \in \gamma$ and $0 < \delta < \varepsilon < d$, consider the open sets $D^\pm_{\delta,\varepsilon}(\xi) := \{z \in D^\pm : \delta < |z - \xi| < \varepsilon\}$ and their boundaries $\partial D^\pm_{\delta,\varepsilon}(\xi)$, the orientation of which is induced by the orientation of γ. Denote $\Gamma^\pm_\delta := \{t \in \partial D^\pm_{\delta,\varepsilon}(\xi) \setminus \gamma : |z - \xi| = \delta\}$, $\Gamma^\pm_\varepsilon := \{t \in \partial D^\pm_{\delta,\varepsilon}(\xi) \setminus \gamma : |z - \xi| = \varepsilon\}$.

We have the equalities:

$$\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t) - g(\xi)\big)\, d\arg(t - \xi) = \operatorname{Im}\left(\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \frac{g(t) - g(\xi)}{t - \xi}\, dt \right)$$

$$= \operatorname{Im}\left(\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \frac{\widetilde{g}^{\,+}(t) - \widetilde{g}^{\,-}(t) - \operatorname{Re}\widetilde{g}^{\,+}(\xi) + \operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t - \xi}\, dt \right)$$

$$= \operatorname{Im}\left(\int\limits_{\partial D^+_{\delta,\varepsilon}(\xi)} \frac{\widetilde{g}^{\,+}(t) - \operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t - \xi}\, dt - \int\limits_{\Gamma^+_\delta} \frac{\widetilde{g}^{\,+}(t) - \operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t - \xi}\, dt - \int\limits_{\Gamma^+_\varepsilon} \frac{\widetilde{g}^{\,+}(t) - \operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t - \xi}\, dt \right)$$

$$-\operatorname{Im}\left(\int\limits_{\partial D^-_{\delta,\varepsilon}(\xi)} \frac{\widetilde{g}^{\,-}(t) - \operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t - \xi}\, dt - \int\limits_{\Gamma^-_\delta} \frac{\widetilde{g}^{\,-}(t) - \operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t - \xi}\, dt - \int\limits_{\Gamma^-_\varepsilon} \frac{\widetilde{g}^{\,-}(t) - \operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t - \xi}\, dt \right).$$

Further, taking into account that the integrals of functions from the Smirnov classes along the closed curves $\partial D^{\pm}_{\delta,\varepsilon}(\xi)$ are equal to zero, we have

$$\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)$$
$$= -\operatorname{Im}\bigg(\int\limits_{\Gamma_\delta^+}\frac{\widetilde{g}^{\,+}(t)-\operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t-\xi}\,dt+\int\limits_{\Gamma_\varepsilon^+}\frac{\widetilde{g}^{\,+}(t)-\operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t-\xi}\,dt\bigg)$$
$$+\operatorname{Im}\bigg(\int\limits_{\Gamma_\delta^-}\frac{\widetilde{g}^{\,-}(t)-\operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t-\xi}\,dt+\int\limits_{\Gamma_\varepsilon^-}\frac{\widetilde{g}^{\,-}(t)-\operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t-\xi}\,dt\bigg)$$
$$= -\operatorname{Im}\bigg(\int\limits_{\Gamma_\delta^+}\frac{\operatorname{Re}\widetilde{g}^{\,+}(t)-\operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t-\xi}\,dt+\int\limits_{\Gamma_\varepsilon^+}\frac{\operatorname{Re}\widetilde{g}^{\,+}(t)-\operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t-\xi}\,dt\bigg)$$
$$+\operatorname{Im}\bigg(\int\limits_{\Gamma_\delta^-}\frac{\operatorname{Re}\widetilde{g}^{\,-}(t)-\operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t-\xi}\,dt+\int\limits_{\Gamma_\varepsilon^-}\frac{\operatorname{Re}\widetilde{g}^{\,-}(t)-\operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t-\xi}\,dt\bigg).$$

Since the function $\operatorname{Re}\widetilde{g}(z)$ is continuously extended to the boundary γ from $D^{\pm}$ and vanishes at infinity, the function $\operatorname{Re}\widetilde{g}^{\,\pm}$ is uniformly continuous in the closure $\overline{D^{\pm}}$ of the domain $D^{\pm}$. Therefore, we obtain the following estimates:

$$\bigg|\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\bigg|$$
$$\le \int\limits_{\Gamma_\delta^+}\frac{\big|\operatorname{Re}\widetilde{g}^{\,+}(t)-\operatorname{Re}\widetilde{g}^{\,+}(\xi)\big|}{|t-\xi|}\,|dt|+\int\limits_{\Gamma_\varepsilon^+}\frac{\big|\operatorname{Re}\widetilde{g}^{\,+}(t)-\operatorname{Re}\widetilde{g}^{\,+}(\xi)\big|}{|t-\xi|}\,|dt|$$
$$+\int\limits_{\Gamma_\delta^-}\frac{\big|\operatorname{Re}\widetilde{g}^{\,-}(t)-\operatorname{Re}\widetilde{g}^{\,-}(\xi)\big|}{|t-\xi|}\,|dt|+\int\limits_{\Gamma_\varepsilon^-}\frac{\big|\operatorname{Re}\widetilde{g}^{\,-}(t)-\operatorname{Re}\widetilde{g}^{\,-}(\xi)\big|}{|t-\xi|}\,|dt|$$
$$\le 4\pi\,\omega_{\overline{D^+}}(\operatorname{Re}\widetilde{g}^{\,+},\varepsilon)+4\pi\,\omega_{\overline{D^-}}(\operatorname{Re}\widetilde{g}^{\,-},\varepsilon),$$

which imply condition (3.1). □

Theorem 3.1 is similar in a certain sense to the corresponding theorem for the Cauchy-type integral, which is proved by A.O. Tokov [29].

Let us note that in the case in which condition (3.1) is satisfied for a function $g\colon \gamma\to\mathbb{R}$ given on an Ahlfors-regular curve γ, a similar condition with the Stieltjes integral

$$\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \frac{g(t)-g(\xi)}{|t-\xi|}\,d|t-\xi|$$

may not be satisfied if the function g does not satisfy Dini condition (2.5). In this case, the function $\operatorname{Im}\widetilde{g}(z)$ has no continuous extension to the boundary γ from the domains D^+ and D^-. Indeed, the necessary condition established by A.O. Tokov [29] for the continuous extension of the Cauchy-type integral to the boundary γ is not satisfied.

4 Some properties of Ahlfors-regular curves

Note that for each $\xi \in \gamma$ and each $\delta > 0$, the function $\arg(t-\xi)$ has a bounded variation on the set $\gamma \setminus \gamma_\delta(\xi)$. However, in general, the function $\arg(t-\xi)$ can be a function of unbounded variation on γ, because, in particular, it can be unbounded in a neighborhood of the point ξ.

Consider the class of curves γ, for which the function $\arg(t-\xi)$ has a bounded variation $V_\gamma[\arg(t-\xi)]$ on $\gamma \setminus \{\xi\}$ for all $\xi \in \gamma$ and, moreover, satisfies the condition

$$\sup_{\xi\in\gamma} V_\gamma[\arg(t-\xi)] < \infty\,. \tag{4.1}$$

It is obvious that a curve satisfying the condition (4.1) has one-sided tangents at each point $\xi \in \gamma$.

Note that condition (4.1) is equivalent to condition (2.2), which follows from the Banach indicatrix theorem (see J. Král [19, Lemma 1.2]). Thus, the class of curves satisfying condition (4.1) includes curves from the corresponding classical results of J. Plemelj [23] and J. Radon [26] and from Theorem 2.1.

Curves satisfying the condition (4.1) will be called the *Král curves.*

Proposition 4.1. Every Král curve is an Ahlfors-regular curve.

Proof. Let γ be a Král curve and $\xi \in \gamma$. Let us first show that for an arbitrary $\varepsilon > 0$, the variation of the function $|t-\xi|$ on the set $\gamma_\varepsilon(\xi)$ satisfies the inequality

$$V_{\gamma_\varepsilon(\xi)}[|t-\xi|] \le c\,\varepsilon, \tag{4.2}$$

where the constant c does not depend on ξ and ε.

In the case in which we consider a fixed point $z \in \mathbb{C}\setminus\gamma$ and a variable $t \in \gamma$, we understand $\arg(t-z)$ as an arbitrary branch of the multivalued function $\mathrm{Arg}\,(t-z)$.

We use the representation $\gamma_\varepsilon(\xi) = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

$$\gamma_1 := \gamma_\varepsilon(\xi) \cap \big\{t = \xi + r\,e^{i\phi} : r > 0,\ \phi \in (-\pi/4, \pi/4) \cup (3\pi/4, 5\pi/4)\big\},$$

$$\gamma_2 := \gamma_\varepsilon(\xi) \cap \big\{t = \xi + r\,e^{i\phi} : r > 0,\ \phi \in (-3\pi/4, -\pi/4) \cup (\pi/4, 3\pi/4)\big\},$$

$$\gamma_3 := \gamma_\varepsilon(\xi) \cap \big\{t = \xi + r\,e^{i\phi} : r \ge 0,\ \phi \in \{-\pi/4, \pi/4, -3\pi/4, 3\pi/4\}\big\}.$$

For the variation of the function $|t-\xi|$ on the set γ_1, the following inequality holds (see J. Král [18, Theorem 2.10]):

$$\frac{V_{\gamma_1}[|t-\xi|]}{\varepsilon} \le c_0\Big(V_{\gamma_1}[\arg(t-\xi)] + V_{\gamma_1}[\arg(t-\xi-\varepsilon)]\Big),$$

where $c_0 = 6/\sin^2(\pi/4) = 12$. Moreover, since the curve γ satisfies condition (4.1), by virtue of Theorem 1.11 from the paper J. Král [19], the following condition is also satisfied:

$$\sup_{z\in\mathbb{C}} V_\gamma[\arg(t-z)] < \infty\,.$$

As a result, under condition (4.1) for the curve γ, we obtain the inequality

$$V_{\gamma_1}[|t-\xi|] \le c_1\,\varepsilon, \tag{4.3}$$

where the constant c_1 does not depend on ξ and ε.

In a similar way, for the variation of the function $|t-\xi|$ on the set γ_2, we obtain the inequalities

$$V_{\gamma_2}[|t-\xi|] \le c_0\Big(V_{\gamma_2}[\arg(t-\xi)] + V_{\gamma_2}[\arg(t-\xi-i\varepsilon)]\Big)\,\varepsilon \le c_1\,\varepsilon. \tag{4.4}$$

In addition, it is obvious that

$$V_{\gamma_3}[|t-\xi|] \le 4\,\varepsilon. \tag{4.5}$$

Inequalities (4.3) – (4.5) imply inequality (4.2), where $c = 2c_1 + 4$.

Now, taking into account condition (4.1) and inequality (4.2), for arbitrary $\xi \in \gamma$ and $\varepsilon > 0$, we obtain the relations

$$\theta_\xi(\varepsilon) = \int\limits_{\gamma_\varepsilon(\xi)} |dt| \le \int\limits_{\gamma_\varepsilon(\xi)} \big|d|t-\xi|\big| + \int\limits_{\gamma_\varepsilon(\xi)} |t-\xi||d\arg(t-\xi)|$$
$$\le V_{\gamma_\varepsilon(\xi)}[|t-\xi|] + \varepsilon\, V_\gamma[\arg(t-\xi)] \le c\,\varepsilon,$$

where the constant c does not depend on ξ and ε. Thus, the curve γ satisfies condition (2.1), i.e., it is an Ahlfors-regular curve. □

Among the Král curves there are curves that are not the Radon curves of bounded rotation, as the following example shows:

Example 1. Consider the curve

$$\gamma = \Big\{z = e^{i\phi} : \phi \in [0,\pi]\Big\} \cup [-1,0] \cup \bigcup_{n=1}^{\infty} [2^{-2n+1}, 2^{-2n+2}]$$
$$\cup \bigcup_{n=1}^{\infty} \Big\{z = 2^{-n} e^{i\phi} : \phi \in [0, 2^{-n}]\Big\} \cup \bigcup_{n=1}^{\infty} \Big\{z = re^{ir} : r \in [2^{-2n}, 2^{-2n+1}]\Big\}.$$

It is clear that $V_\gamma[\vartheta_t] = \infty$, but at the same time, the following relations are fulfilled:

$$V_\gamma[\arg t] \le \pi + \sum_{n=1}^{\infty} 2^{-n} + \frac{1}{2} = \pi + \frac{3}{2}\,,$$

$$V_\gamma[\arg(t-\xi)] = V_{\gamma_{|\xi|/2}(0)}[\arg(t-\xi)] + V_{\gamma_{|\xi|}(\xi)\setminus\gamma_{|\xi|/2}(0)}[\arg(t-\xi)] + V_{\gamma\setminus\gamma_{|\xi|}(\xi)\setminus\gamma_{|\xi|/2}(0)}[\arg(t-\xi)]$$
$$\le V_{\gamma_{|\xi|/2}(0)}[\arg t] + 2\pi + 2V_{\gamma\setminus\gamma_{|\xi|}(\xi)\setminus\gamma_{|\xi|/2}(0)}[\arg t] \le 2V_\gamma[\arg t] + 2\pi \qquad \forall\, \xi \in \gamma : 0 < |\xi| < 1,$$

$$V_\gamma[\arg(t-\xi)] \le V_{\gamma_{1/2}(\xi)}[\arg(t-\xi)] + \int\limits_{\gamma\setminus\gamma_{1/2}(\xi)} \frac{|dt|}{|t-\xi|} \le \pi + 2\,\mathrm{mes}\,\gamma \qquad \forall\, \xi \in \gamma : |\xi| = 1.$$

Thus, γ is a Král curve that is not the Radon curve of bounded rotation.

For points $\xi_1, \xi_2 \in \gamma$, we denote by $\gamma[\xi_1, \xi_2]$ the arc of the curve γ with the initial point ξ_1 and the end point ξ_2 at the orientation of this arc, which is induced by the orientation of the curve γ.

The following statement defines a class of smooth curves satisfying condition (4.1).

Proposition 4.2. If for a closed smooth Jordan curve γ the angle ϑ_t satisfies the condition

$$\int\limits_0^1 \frac{\omega_\gamma(\vartheta_t, \eta)}{\eta}\, d\eta < \infty, \tag{4.6}$$

then γ is a Král curve.

Proof. Let $\xi \in \gamma$. It is known (see, for example, N.I. Muskhelishvili [20]) that there exists $r_0 > 0$, which does not depend on ξ, such that each circle of radius $r \le r_0$ centered at the point ξ intersects γ in only two points.

We denote by t_- and t_+ the points of intersection of the circle $\{z \in \mathbb{C} : |z-\xi| = r_0\}$ and the curve γ, and with the given orientation γ the point t_- precedes the point ξ and the point t_+ follows it. Considering one of the arcs either $\gamma[t_-,\xi]$ or $\gamma[\xi,t_+]$, we will denote it $\widetilde{\gamma}$.

The arc $\widetilde{\gamma}$ allows the parameterization $t = \xi + r\, e^{i(\phi(r)+\phi_0)}$, $r \in [0,r_0]$, where ϕ_0 is a real constant and $\phi(r) \to 0$ as $r \to 0$. Denote $\widetilde{x}(r) := r\cos\phi(r)$, $\widetilde{y}(r) := r\sin\phi(r)$.

For all $t \in \widetilde{\gamma} \setminus \{\xi\}$, the following equalities hold:

$$d\arg(t-\xi) = d\phi(r) = d\,\mathrm{arctg}\,\frac{\widetilde{y}(r)}{\widetilde{x}(r)} = \frac{1}{1+\left(\frac{\widetilde{y}(r)}{\widetilde{x}(r)}\right)^2}\left(\frac{\widetilde{y}\,'(r)}{\widetilde{x}(r)} - \frac{\widetilde{y}(r)\widetilde{x}\,'(r)}{\big(\widetilde{x}(r)\big)^2}\right) dr$$
$$= \widetilde{x}\,'(r)\,\cos^2\phi(r)\,\frac{\frac{\widetilde{y}\,'(r)}{\widetilde{x}\,'(r)} - \mathrm{tg}\,\phi(r)}{\widetilde{x}(r)}\,dr = \widetilde{x}\,'(r)\,\cos\phi(r)\,\frac{\mathrm{tg}\,(\vartheta_t-\vartheta_\xi) - \mathrm{tg}\,\phi(r)}{r}\,dr\,.$$

Note that for a smooth arc $\widetilde{\gamma}$, for each $r \in (0,r_0]$ there exists $r_* \in [0,r]$ such that for $t_* = \xi + r_*\, e^{i(\phi(r_*)+\phi_0)}$ the following relations are fulfilled:

$$|\phi(r)| = |\vartheta_{t_*} - \vartheta_\xi| \le \omega_\gamma(\vartheta_t, r). \tag{4.7}$$

Without loss of generality, we assume r_0 to be small enough to satisfy the inequality $\omega_\gamma(\vartheta_t,r_0) < 1$. Then we get the estimate

$$\int\limits_{\widetilde{\gamma}} |d\arg(t-\xi)| \le c \int\limits_0^{r_0} \frac{\omega_\gamma(\vartheta_t,r)}{r}\,dr < \infty,$$

where the constant c depends on r_0, but does not depend on ξ.

Finally, using the obtained estimate, we estimate the variation

$$V_\gamma[\arg(t-\xi)] \le \int\limits_{\gamma[t_1,\xi]} |d\arg(t-\xi)| + \int\limits_{\gamma[\xi,t_2]} |d\arg(t-\xi)| + \int\limits_{\gamma\setminus\gamma_{r_0}(\xi)} \frac{|dt|}{|t-\xi|}$$
$$\le 2c \int\limits_0^{r_0} \frac{\omega_\gamma(\vartheta_t,r)}{r}\,dr + \frac{\mathrm{mes}\,\gamma}{r_0}\,,$$

which yields the fulfillment of condition (4.1) for the curve γ. □

It is obvious that condition (4.6) is a weaker constraint on the curve γ compared to condition (2.4).

Relation (4.7) implies the estimate

$$\omega_{\widetilde{\gamma}}\big(\arg(t-\xi),\eta\big) \le \omega_\gamma(\vartheta_t,\eta) \qquad \forall\,\eta \in [0,r_0],$$

where the arc $\widetilde{\gamma}$ is defined in the proof of Proposition 4.2. Therefore, if the modulus of continuity of the angle ϑ_t satisfies Dini condition (4.6), then the condition of the same form is also satisfied for the modulus of continuity $\omega_{\widetilde{\gamma}}\big(\arg(t-\xi),\eta\big)$ at all points $\xi \in \gamma$:

$$\int\limits_0^1 \frac{\omega_{\widetilde{\gamma}}\big(\arg(t-\xi),\eta\big)}{\eta}\,d\eta < \infty. \tag{4.8}$$

Let us show that the class of smooth Král curves differs from the class of smooth curves γ that satisfy the conditions of form (4.8) at all points $\xi \in \gamma$. First, we give an example of a smooth curve γ that is a Král curve, but condition (4.8) is not satisfied at a point $\xi \in \gamma$.

Example 2. Consider the smooth arc

$$\widetilde{\gamma} = \Big\{ t(r) = r \exp\Big(-i \, \frac{1}{\ln r} \Big) : r \in (0, r_0] \Big\} ,$$

where r_0 is the smallest positive root of the equation $\operatorname{Re} t'(r) = 0$. It is obvious that the one-sided tangent to the arc $\widetilde{\gamma}$ at the beginning point $t_0 = 0$ is the positive semi-axis of the real axis. At the end point $t(r_0)$, the arc $\widetilde{\gamma}$ has the one-sided tangent parallel to the imaginary axis of the complex plane.

Let Γ be such an arc of the ellipse that includes the points $z = x + iy$ satisfying the equation

$$\frac{x^2}{(\operatorname{Re} t(r_0))^2} + \frac{(y - \operatorname{Im} t(r_0))^2}{(\operatorname{Im} t(r_0))^2} = 1,$$

which is smoothly glued to the arc $\widetilde{\gamma}$ at the points 0 and $t(r_0)$. Then $\gamma = \widetilde{\gamma} \cup \Gamma$ is a closed smooth Jordan curve.

It is obvious that the curve γ satisfies condition (4.1) because $V_\gamma[\arg(t - \xi)] = \pi$ for all $\xi \in \gamma$. At the same time, condition (4.8) is not satisfied at the point $\xi = 0$ owing to the fact that

$$\int\limits_0^1 \frac{\omega_{\widetilde{\gamma}}\big(\arg t, \eta\big)}{\eta} \, d\eta \geq - \int\limits_0^{r_0} \frac{1}{\eta \ln \eta} \, d\eta = \infty.$$

Now we give an example of a smooth curve γ for which the conditions of form (4.8) are satisfied at all points $\xi \in \gamma$, but it is not a Král curve.

Example 3. Consider the smooth arc

$$\Gamma_1 = \Big\{ t(r) = r \exp\Big(-i \, \frac{r}{\ln r} \, \cos \frac{\pi}{r} \Big) : r \in (0, 1/2] \Big\} .$$

Let Γ_2 be such an arc of the ellipse that includes the points $z = x + iy$ satisfying the equation

$$\frac{x^2}{a^2} + \frac{(y - b)^2}{b^2} = 1$$

with fully defined positive a and b, which is smoothly glued to the arc Γ_1 at the points 0 and $t(1/2)$. Then $\gamma = \Gamma_1 \cup \Gamma_2$ is a closed smooth Jordan curve.

For each point $\xi \in \gamma$, consider the arcs $\gamma[t_-, \xi]$ and $\gamma[\xi, t_+]$ defined in the proof of Proposition 4.2, and denote them by γ_ξ^- and γ_ξ^+, respectively. Considering the function $\arg(t - \xi)$ on the arc $\gamma_\xi^\pm$, we redefine it at the point $t = \xi$ by the limiting value

$$\lim_{t \to \xi, t \in \gamma_\xi^\pm} \arg(t - \xi).$$

As a result, for each $\xi \in \gamma$, the function $\arg(t - \xi)$ satisfies the Hölder condition on each of the arcs γ_ξ^- and γ_ξ^+:

$$|\arg(t_1 - \xi) - \arg(t_2 - \xi)| \leq c \, |t_1 - t_2|^\alpha \qquad \forall \, t_1, t_2 \in \gamma_\xi^\pm$$

for all $\alpha \in (0, 1/2]$, where the constant c does not depend on t_1 and t_2. Therefore, the conditions of form (4.8) are satisfied at all points of $\xi \in \gamma$.

At the same time,

$$V_\gamma[\arg t] \geq V_{\Gamma_1}[\arg t] \geq \sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty,$$

i.e., condition (4.1) is not satisfied for the curve γ.

Example 3 also shows that condition (4.6) on the angle ϑ_t in Proposition 4.2 can not be replaced by a similar condition of form (4.8) on the function $\arg(t-\xi)$.

5 Sufficient conditions for the continuous extension of the real part of the Cauchy-type integral to the boundary of domain with unbounded variation of the function $\arg(t-\xi)$

We shall now consider curves for which condition (4.1) is not satisfied, generally speaking.

In what follows, we use the following characteristic of the function $f\colon E \to \mathbb{C}$ continuous on the set $E \subset \mathbb{C}$ (see O.F. Gerus [11]):

$$\Omega_{E,f}(a,b) := \sup_{a\le\eta\le b} \frac{\omega_E(f,\eta)}{\eta} \qquad \text{for} \quad 0<a\le b.$$

The function $\Omega_{E,f}(a,b)$ does not increase monotonically with respect to the variable a and does not decrease monotonically with respect to the variable b. In addition, the function $a\,\Omega_{E,f}(a,b)$ does not decrease monotonically with respect to the variable a.

Denote $E^{R,\psi_1,\psi_2}(\xi) := \{z=\xi+re^{i\phi} : R/2<r<R, \psi_1<\phi<\psi_2\}$.

Let us describe a certain finite set of Jordan arcs placed in the closure of domain $E^{R,0,\psi}(0)$. For this purpose, we consider two sets of points $\{\tau_j\}_{j=1}^n$ and $\{\eta_j\}_{j=1}^n$ located on the rectilinear parts of the boundary of domain $E^{R,0,\psi}(0)$ such that

$$R\ge\tau_1\ge\tau_2\ge\cdots\ge\tau_n\ge R/2,$$
$$\eta_j=|\eta_j|\,e^{i\psi} \quad \text{and} \quad R\ge|\eta_1|\ge|\eta_2|\ge\cdots\ge|\eta_n|\ge R/2.$$

Let $\Gamma := \bigcup_{j=1}^n \Gamma_j$, where Γ_j is a Jordan arc with ends at the points τ_j and η_j. Moreover, the arcs Γ_j, $j=1,2,\ldots,n$, excluding the ends, lie in the domain $E^{R,0,\psi}(0)$ and pairwise do not intersect within this domain. In addition, if the arc Γ_j is oriented from the point τ_j to the point η_j, then the next arc Γ_{j+1} is oriented in the opposite direction from the point η_{j+1} to the point τ_{j+1}, and vice versa, if the arc Γ_j is oriented from the point η_j to the point τ_j, then the arc Γ_{j+1} is oriented from the point τ_{j+1} to the point η_{j+1}.

Consider the auxiliary statements.

Lemma 5.1. *If a function $f:\Gamma\to\mathbb{R}$ is continuous on Γ, then*

$$\left|\int_\Gamma f(t)\,d\arg t\right| \le \left(R\,\Omega_{\Gamma,f}\Big(\frac{R}{n},R\Big)+\max_{t\in\Gamma}|f(t)|\right)\psi+\frac{2\,\omega_\Gamma(f,\lambda)\,\mathrm{mes}\,\Gamma}{R},$$

where $\lambda := \max_j \mathrm{mes}\,\Gamma_j$ and $\arg t$ is any branch of the multivalued function $\mathrm{Arg}\,z$, which is continuous on Γ.

Proof. As in the paper T.S. Salimov [28], we use the representation

$$\int_\Gamma f(t)\,d\arg t = \sum_{j=1}^n \int_{\Gamma_j} \big(f(t)-f(\tau_j)\big)\,d\arg t + \sum_{j=1}^n f(\tau_j)\int_{\Gamma_j} d\arg t$$

and the estimates

$$\left|\sum_{j=1}^n \int_{\Gamma_j}\big(f(t)-f(\tau_j)\big)\,d\arg t\right| \le \sum_{j=1}^n \left|\mathrm{Im}\int_{\Gamma_j}\frac{f(t)-f(\tau_j)}{t}\,dt\right| \le$$

$$\sum_{j=1}^n\int_{\Gamma_j}\frac{|f(t)-f(\tau_j)|}{|t|}\,|dt| \le \sum_{j=1}^n \frac{2\,\omega_{\Gamma_j}(f,\mathrm{mes}\,\Gamma_j)\,\mathrm{mes}\,\Gamma_j}{R} \le \frac{2\,\omega_\Gamma(f,\lambda)\,\mathrm{mes}\,\Gamma}{R},$$

$$\left|\sum_{j=1}^{n} f(\tau_j) \int\limits_{\Gamma_j} d\arg t\right| \le \left(\sum_{j=1}^{n_0} |f(\tau_{2j-1}) - f(\tau_{2j})| + 2q_0|f(\tau_n)|\right)\psi,$$

where n_0 is the integer part of the number $n/2$ and q_0 is the fractional part of the number $n/2$.

Next, taking into account the estimates (see Lemma 1 in the paper O.F. Gerus [11])

$$\sum_{j=1}^{n_0} |f(\tau_{2j-1}) - f(\tau_{2j})| \le \sum_{j=1}^{n_0} \omega_\gamma(f, |\tau_{2j-1} - \tau_{2j}|) \le R\,\Omega_{\Gamma,f}\Big(\frac{R}{2n_0}, R\Big)$$

and the inequality $2n_0 \le n$, we have

$$\left|\sum_{j=1}^{n} f(\tau_j) \int\limits_{\Gamma_j} d\arg t\right| \le \left(R\,\Omega_{\Gamma,f}\Big(\frac{R}{n}, R\Big) + \max_{t\in\Gamma} |f(t)|\right)\psi.$$

The given estimates imply the statement of the lemma. □

For a given closed rectifiable Jordan curve γ, we shall consider its intersections with the domains $E^{R,\psi_1,\psi_2}(\xi)$, where $\xi \in \gamma$. We shall denote these intersections by $\gamma_{R,\psi_1,\psi_2}(\xi)$. By $n_\gamma(\xi, R, \psi_1, \psi_2)$ we denote the number of connected components of the set $\gamma_{R,\psi_1,\psi_2}(\xi)$, the ends of which lie on the different segments $\{z = \xi + re^{i\psi_1} : R/2 \le r \le R\}$ and $\{z = \xi + re^{i\psi_2} : R/2 \le r \le R\}$. We can say that the number $n_\gamma(\xi, R, \psi_1, \psi_2)$ expresses the number of complete oscillations of the function $\arg(t - \xi)$ in the domain $E^{R,\psi_1,\psi_2}(\xi)$. Since the curve γ is rectifiable, the number $n_\gamma(\xi, R, \psi_1, \psi_2)$ is finite, but with fixed ξ and R it can tend to infinity when $\psi_2 \to \psi_1$.

Consider the case in which there exist $\xi \in \gamma$ and $R \in (0, d]$ such that

$$k_\gamma(\xi, R) := \max\Big\{1, \sup_{0\le\psi_1<\psi_2<2\pi} n_\gamma(\xi, R, \psi_1, \psi_2)\Big\} < \infty. \tag{5.1}$$

Denote by $\varphi_\gamma(\xi, R)$ the Lebesgue measure (given on the segment $[0, 2\pi]$) of the set of those $\phi \in [0, 2\pi]$ for which the rays $\{z = \xi + re^{i\phi} : r > 0\}$ have a nonempty intersection with the set $\gamma_R(\xi) \setminus \gamma_{R/2}(\xi)$.

Lemma 5.2. *Let a closed Jordan curve γ be Ahlfors-regular and satisfy condition (5.1) and let a function $g\colon \gamma \to \mathbb{R}$ be continuous on γ. Then the following estimate holds:*

$$\left|\int\limits_{\gamma_R(\xi)\setminus\gamma_{R/2}(\xi)} \big(g(t) - g(\xi)\big)\, d\arg(t - \xi)\right| \le 6\,R\,\varphi_\gamma(\xi, R)\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi, R)}\,,\, R\Big). \tag{5.2}$$

Proof. For the proof, we apply the method developed by T.S. Salimov [28] for estimating the modulus of continuity of the Cauchy singular integral on an arbitrary closed rectifiable Jordan curve and adapted by O.F. Gerus [11] for the purpose of using the modulus of continuity of the integral density instead of the regularized (by means of the Stechkin construction) modulus of continuity, which is used in the paper [28].

The set $\breve{\gamma} := \gamma_R(\xi) \setminus \gamma_{R/2}(\xi)$ is the union of no more than a countable collection of connected components of the set $\breve{\gamma} \setminus \{z \in \gamma : |z - \xi| = R\}$, which is open in the topology of the curve γ, and the closed set $\{z \in \gamma : |z - \xi| = R\}$ for which

$$\mathrm{mes}\,\{z \in \gamma : |z - \xi| = R\} \le \varphi_\gamma(\xi, R)R.$$

Therefore, there exists a finite union $\widehat{\gamma}$ of the specified connected components such that

$$\mathrm{mes}\,(\breve{\gamma} \setminus \widehat{\gamma}) \le 2\,\varphi_\gamma(\xi, R)R.$$

We have the equality

$$\int\limits_{\gamma_R(\xi)\setminus\gamma_{R/2}(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)$$

$$= \int\limits_{\widehat{\gamma}} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi) + \int\limits_{\breve{\gamma}\setminus\widehat{\gamma}} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi) \quad (5.3)$$

and the estimate

$$\Bigg|\int\limits_{\breve{\gamma}\setminus\widehat{\gamma}} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\Bigg| = \Bigg|\operatorname{Im}\int\limits_{\breve{\gamma}\setminus\widehat{\gamma}} \frac{g(t)-g(\xi)}{t-\xi}\,dt\Bigg|$$

$$\le \int\limits_{\breve{\gamma}\setminus\widehat{\gamma}} \frac{\big|g(t)-g(\xi)\big|}{|t-\xi|}\,|dt| \le \frac{\omega_\gamma(g,R)\,\mathrm{mes}\,(\breve{\gamma}\setminus\widehat{\gamma})}{R/2}$$

$$\le 4\,\varphi_\gamma(\xi,R)\,\omega_\gamma(g,R) \le 4\,R\,\varphi_\gamma(\xi,R)\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)}\,,\,R\Big). \quad (5.4)$$

To estimate the first integral in the right-hand side of equality (5.3), we partition the ring $\{z\in\mathbb{C} : R/2<|z-\xi|<R\}$ by the rays $\{z=\xi+re^{i\phi_m} : r>0\}$, $m=1,2,\ldots,k$, for $0=\phi_0<\phi_1<\cdots<\phi_k=2\pi$, so that the ends of all connected components of all nonempty sets $\widehat{\gamma}_{R,\phi_{m-1},\phi_m}(\xi):=\widehat{\gamma}\cap E^{R,\phi_{m-1},\phi_m}(\xi)$, $m=1,2,\ldots,k$, lay on the indicated rays. It is always possible due to the finite number of connected components of the set $\widehat{\gamma}$. In this case, there is the representation $\widehat{\gamma}_{R,\phi_{m-1},\phi_m}(\xi)=\widehat{\gamma}_{m,1}\cup\widehat{\gamma}_{m,2}$, where $\widehat{\gamma}_{m,1}:=\bigcup_j\widehat{\gamma}_{m,1,j}$ is the union of connected components $\widehat{\gamma}_{m,1,j}$ of the set $\widehat{\gamma}_{R,\phi_{m-1},\phi_m}(\xi)$ with the ends on one of the specified rays, and $\widehat{\gamma}_{m,2}:=\bigcup_j\widehat{\gamma}_{m,2,j}$ is the union of a finite number of connected components $\widehat{\gamma}_{m,2,j}$ of the set $\widehat{\gamma}_{R,\phi_{m-1},\phi_m}(\xi)$ with the ends on different rays.

We have the following equality for these sets:

$$\widehat{\gamma}=\bigcup_{m=1}^{k}\widehat{\gamma}_{m,1}\cup\bigcup_{m=1}^{k}\widehat{\gamma}_{m,2}\cup\bigcup_{m=1}^{k}\widehat{\gamma}_{m,3},$$

where $\widehat{\gamma}_{m,3}:=\widehat{\gamma}\cap\{z=\xi+re^{i\phi_m} : R/2\le r\le R\}$, which implies the following equality for the corresponding integrals:

$$\int\limits_{\widehat{\gamma}} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi) = \sum_{m=1}^{k}\Bigg(\int\limits_{\widehat{\gamma}_{m,1}}+\int\limits_{\widehat{\gamma}_{m,2}}+\int\limits_{\widehat{\gamma}_{m,3}}\Bigg)\big(g(t)-g(\xi)\big)\,d\arg(t-\xi)$$

$$= \sum_{m=1}^{k}\Bigg(\int\limits_{\widehat{\gamma}_{m,1}}+\int\limits_{\widehat{\gamma}_{m,2}}\Bigg)\big(g(t)-g(\xi)\big)\,d\arg(t-\xi), \quad (5.5)$$

because the integrals over the sets $\widehat{\gamma}_{m,3}$ are equal to zero.

Denote $\lambda:=\max\limits_{l=1,2}\max\limits_{m,j}\mathrm{mes}\,\widehat{\gamma}_{m,l,j}$.

Estimating the integrals over the nonempty sets $\widehat{\gamma}_{m,1}$, we denote one of the ends of the arc $\widehat{\gamma}_{m,1,j}$

by $\tau_{m,j}$, and as a result, we get

$$\Big|\int\limits_{\widehat{\gamma}_{m,1}} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\Big| = \Big|\sum_j \int\limits_{\widehat{\gamma}_{m,1,j}} \big(g(t)-g(\tau_{m,j})\big)\, d\arg(t-\xi)\Big|$$

$$= \Big|\sum_j \operatorname{Im} \int\limits_{\widehat{\gamma}_{m,1,j}} \frac{g(t)-g(\tau_{m,j})}{t-\xi}\, dt\Big| \le \sum_j \int\limits_{\widehat{\gamma}_{m,1,j}} \frac{\big|g(t)-g(\tau_{m,j})\big|}{|t-\xi|}\, |dt|$$

$$\le \sum_j \frac{\omega_\gamma(g,\lambda)\,\mathrm{mes}\,\widehat{\gamma}_{m,1,j}}{R/2} \le \frac{2\,\omega_\gamma(g,\lambda)\,\mathrm{mes}\,\widehat{\gamma}_{m,1}}{R}. \quad (5.6)$$

The integrals over the nonempty sets $\widehat{\gamma}_{m,2}$ are estimated by applying Lemma 5.1, for the application of which it is necessary to establish an obvious correspondence between the parameters of the sets Γ and $\widehat{\gamma}_{m,2}$. As a result, we have

$$\Big|\int\limits_{\widehat{\gamma}_{m,2}} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\Big|$$

$$\le (\phi_m-\phi_{m-1})\Big(R\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)},R\Big)+\omega_\gamma(g,R)\Big)+\frac{2\,\omega_\gamma(g,\lambda)\,\mathrm{mes}\,\widehat{\gamma}_{m,2}}{R}$$

$$\le 2\,(\phi_m-\phi_{m-1})\,R\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)},R\Big)+\frac{2\,\omega_\gamma(g,\lambda)\,\mathrm{mes}\,\widehat{\gamma}_{m,2}}{R}. \quad (5.7)$$

Taking into account equalities (5.3), (5.5) and estimates (5.4), (5.6) and (5.7), as well as condition (2.1) on the curve γ, we obtain the estimate

$$\Big|\int\limits_{\gamma_R(\xi)\setminus\gamma_{R/2}(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\Big| \le 4\,R\,\varphi_\gamma(\xi,R)\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)}\,,\,R\Big)$$

$$+\,2\sum_{m=1}^{k}\Big((\phi_m-\phi_{m-1})\,R\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)},R\Big)+\frac{\omega_\gamma(g,\lambda)\,\big(\mathrm{mes}\,\widehat{\gamma}_{m,1}+\mathrm{mes}\,\widehat{\gamma}_{m,2}\big)}{R}\Big)$$

$$\le 6\,R\,\varphi_\gamma(\xi,R)\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)}\,,\,R\Big)+c\,\omega_\gamma(g,\lambda),$$

where the constant c does not depend on ξ and R.

Now, as a result of a refinement of the partition of the ring $\{z\in\mathbb{C} : R/2<|z-\xi|<R\}$, if $k\to\infty$, hence $\lambda\to 0$, from the last estimate we obtain estimate (5.2). □

Lemma 5.3. *Let a closed Jordan curve γ be Ahlfors-regular and let a function $g\colon\gamma\to\mathbb{R}$ be continuous on γ. Let for $\xi\in\gamma$ condition (5.1) be satisfied for all $R\in[\delta,2\varepsilon]$, where $0<\delta<\varepsilon\le d/2$. Then the following estimate holds:*

$$\Big|\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\Big|$$

$$\le c\Big(\int\limits_{\delta}^{2\varepsilon} \widehat{\varphi}_\gamma(\xi,\eta)\,\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)}\,,\,\eta\Big)\,d\eta+\omega_\gamma(g,\varepsilon)\Big), \quad (5.8)$$

where $\widehat{\varphi}_\gamma(\xi,R):=\sup\limits_{r\in[R/2,R]}\varphi_\gamma(\xi,r)$, $\widehat{k}_\gamma(\xi,R):=\sup\limits_{r\in[R/2,R]}k_\gamma(\xi,r)$, *and the constant $c>0$ does not depend on ξ, δ and ε.*

Proof. Let $\delta \in [\varepsilon/2^n, \varepsilon/2^{n-1})$ for some natural n. Using estimate (5.2), we obtain

$$\Bigg|\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\Bigg| \le \sum_{m=0}^{n-2}\Bigg|\int\limits_{\gamma_{\varepsilon/2^m}(\xi)\setminus\gamma_{\varepsilon/2^{m+1}}(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\Bigg|$$
$$+\Bigg|\int\limits_{\gamma_{\varepsilon/2^{n-1}}(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\Bigg|$$
$$\le 6\sum_{m=0}^{n-2}\frac{\varepsilon}{2^m}\,\varphi_\gamma(\xi,\varepsilon/2^m)\,\Omega_{\gamma,g}\Big(\frac{\varepsilon}{2^m\,k_\gamma(\xi,\varepsilon/2^m)},\,\frac{\varepsilon}{2^m}\Big)$$
$$+\Bigg|\operatorname{Im}\int\limits_{\gamma_{\varepsilon/2^{n-1}}(\xi)\setminus\gamma_\delta(\xi)}\frac{g(t)-g(\xi)}{t-\xi}\,dt\Bigg| =: I(\xi,\delta,\varepsilon)\,.$$

Next, we continue the estimation, using the monotonicity properties of the function $\Omega_{\gamma,g}$ and performing the transition to the functions $\widehat{\varphi}_\gamma$ and $\widehat{k}_\gamma$:

$$I(\xi,\delta,\varepsilon) \le \frac{6}{\ln 2}\sum_{m=0}^{n-2}\frac{\varepsilon}{2^m}\,\varphi_\gamma(\xi,\varepsilon/2^m)\,\Omega_{\gamma,g}\Big(\frac{\varepsilon}{2^m\,k_\gamma(\xi,\varepsilon/2^m)},\,\frac{\varepsilon}{2^m}\Big)\int\limits_{\varepsilon/2^m}^{\varepsilon/2^{m-1}}\frac{d\eta}{\eta}$$
$$+\int\limits_{\gamma_{\varepsilon/2^{n-1}}(\xi)\setminus\gamma_\delta(\xi)}\frac{\big|g(t)-g(\xi)\big|}{|t-\xi|}\,|dt|$$
$$\le \frac{6}{\ln 2}\sum_{m=0}^{n-2}\int\limits_{\varepsilon/2^m}^{\varepsilon/2^{m-1}}\widehat{\varphi}_\gamma(\xi,\eta)\,\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)},\,\eta\Big)\,d\eta + \frac{\omega_\gamma(g,\varepsilon/2^{n-1})\,\theta_\xi(\varepsilon/2^{n-1})}{\varepsilon/2^n}\,.$$

As a result, taking into account condition (2.1) on the curve γ, we get estimate (5.8). □

The following statement is true.

Theorem 5.1. *Let a closed Jordan curve γ be Ahlfors-regular and let a function $g\colon \gamma \to \mathbb{R}$ be continuous on γ. Consider a partition $\gamma = \gamma^1 \cup \gamma^2$, for which there exists $R_0 \in (0, d]$ such that:*

(a) for all $\xi \in \gamma^1$ and all $R \in (0, R_0]$, the curve γ satisfies condition (5.1) and, in addition, the following condition is satisfied:

$$\sup_{\xi\in\gamma^1}\int\limits_0^{R_0}\widehat{\varphi}_\gamma(\xi,\eta)\,\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)},\,\eta\Big)\,d\eta < \infty\,; \tag{5.9}$$

(b) for each $\xi \in \gamma^2$ there exists $r(\xi) \in (0, R_0]$ such that the curve γ satisfies condition (5.1) for all $R \in (r(\xi), R_0]$, the function $\arg(t-\xi)$ has bounded variation on the set $\gamma_{r(\xi)}(\xi) \setminus \{\xi\}$ and, in addition, the following condition is satisfied:

$$\sup_{\xi\in\gamma^2}\Bigg(V_{\gamma_{r(\xi)}(\xi)}[\arg(t-\xi)] + \int\limits_{r(\xi)}^{R_0}\widehat{\varphi}_\gamma(\xi,\eta)\,\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)},\,\eta\Big)\,d\eta\Bigg) < \infty\,. \tag{5.10}$$

Then the function $\operatorname{Re}\widetilde{g}(z)$ has a continuous extension to the boundary γ from the domains D^+ and D^-, and the limiting values $\big(\operatorname{Re}\widetilde{g}\big)^\pm(\xi)$ are represented by formulas (2.10) and (2.11) for all $\xi \in \gamma$.

Proof. Let us show that under the assumptions of the theorem, condition (3.1) is also satisfied.

Let $\varepsilon \in (0, R_0/2]$ and $\delta \in (0, \varepsilon)$. Then estimate (5.8) holds for all $\xi \in \gamma^1$, and for each $\xi \in \gamma^2$, taking into account Lemma 5.3, we obtain the estimate

$$\Bigg| \int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \Bigg|$$

$$\le \Bigg| \int\limits_{\gamma_{r(\xi)}(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \Bigg| + \Bigg| \int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_{r(\xi)}(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \Bigg|$$

$$\le \omega_\gamma(g,\varepsilon)\, V_{\gamma_{r(\xi)}(\xi)}[\arg(t-\xi)] + c\Bigg(\int\limits_{r(\xi)}^{2\varepsilon} \widehat{\varphi}_\gamma(\xi,\eta)\, \Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)}, \eta\Big)\, d\eta + \omega_\gamma(g,\varepsilon)\Bigg),$$

where the constant $c > 0$ does not depend on ξ, δ and ε.

Under conditions (5.9) and (5.10), the given estimates yield condition (3.1).

Now, to complete the proof, it remains to apply Theorem 3.1. □

Corollary 5.1. *The statement of Theorem 5.1 remains valid if conditions (5.9), (5.10) are replaced by the conditions*

$$\sup_{\xi\in\gamma^1} \int\limits_0^{R_0} \frac{\widehat{\varphi}_\gamma(\xi,\eta)\, \widehat{k}_\gamma(\xi,\eta)\, \omega_\gamma(g,\eta)}{\eta}\, d\eta < \infty\,, \tag{5.11}$$

$$\sup_{\xi\in\gamma^2} \Bigg(V_{\gamma_{r(\xi)}(\xi)}[\arg(t-\xi)] + \int\limits_{r(\xi)}^{R_0} \frac{\widehat{\varphi}_\gamma(\xi,\eta)\, \widehat{k}_\gamma(\xi,\eta)\, \omega_\gamma(g,\eta)}{\eta}\, d\eta \Bigg) < \infty\,, \tag{5.12}$$

respectively.

It is clear that Corollary 5.1 follows from Theorem 5.1 and the inequality

$$\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)}, \eta\Big) \le \frac{\widehat{k}_\gamma(\xi,\eta)\, \omega_\gamma(g,\eta)}{\eta} \qquad \forall\, \eta \in (0, R_0].$$

Note that in the papers of T.S. Salimov [28], E.M. Dyn'kin [7] and O.F. Gerus [11], the singular Cauchy integral (2.6) on an arbitrary closed rectifiable Jordan curve is considered under certain conditions on the integral density, which are reduced to a condition of form (2.5) in the case of a curve satisfying condition (2.1). Under such conditions on the curve and the integral density, Cauchy-type integral (1.2) is continuously extended to the boundary γ from the domains D^+ and D^-, and the limiting values $\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi)$ are expressed by formulas (2.10) and (2.11) for all $\xi \in \gamma$.

At the same time, under additional assumptions of type (5.1) about the curve γ, Theorem 5.1 and Corollary 5.1 allow to construct examples (see the next example) of the curves γ that do not satisfy condition (4.1) and the functions g that do not satisfy condition (2.5) and a similar condition (see N.A. Davydov [6]) used in Theorem 2 in the paper of O.F. Gerus and M. Shapiro [13], but the limiting values $\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi)$ of logarithmic double layer potential (1.1) exist at all points $\xi \in \gamma$ and are expressed by formulas (2.10) and (2.11).

Example 4. Consider the curve

$$\gamma = \Big\{ z = e^{i\phi} : \phi \in [0,\pi] \Big\} \cup [-1, 0] \cup \bigcup_{n=1}^{\infty} [2^{-2n+1}, 2^{-2n+2}]$$
$$\cup \bigcup_{n=1}^{\infty} \Big\{ z = 2^{-n} e^{i\phi} : \phi \in [0, 1/n] \Big\} \cup \bigcup_{n=1}^{\infty} \Big\{ z = r e^{-i\frac{\ln 2}{\ln r}} : r \in [2^{-2n}, 2^{-2n+1}] \Big\}$$

and the function

$$g(t) = \begin{cases} -1/\big(\ln|t|-1\big) & \text{for} \quad t\in\gamma\setminus\{0\},\\ 0 & \text{for} \quad t=0 \end{cases}$$

that does not satisfy Dini condition (2.5) as well as a similar condition used in Theorem 2 in paper [13] because

$$\min\left\{\int_0^1 \frac{\omega_\gamma(g,\eta)}{\eta}\,d\eta,\ \int_\gamma \frac{|g(t)-g(0)|}{|t-0|}\,|dt|\right\} \ge \int_{-1}^0 \frac{|g(t)-g(0)|}{|t-0|}\,dt$$

$$= -\int_{-1}^0 \frac{dt}{|t|\,(\ln|t|-1)} = \infty\,.$$

For $0<\varepsilon\le 1/2$, denoting by n_0 the smallest natural number n that satisfies the inequality $2^{-n}\le\varepsilon$, we obtain the estimate

$$\theta(\varepsilon)=\theta_0(\varepsilon)\le 2\varepsilon+\sum_{n=n_0}^{\infty}\frac{1}{n\,2^n}+\int_0^{\varepsilon}\sqrt{1+\frac{\ln^2 2}{\ln^4 r}}\,dr\le 2\varepsilon+2\varepsilon+\frac{\sqrt{1+\ln^2 2}}{\ln 2}\,\varepsilon$$

$$\le\left(4+\frac{\sqrt{1+\ln^2 2}}{\ln 2}\right)\varepsilon\,, \qquad (5.13)$$

which proves the validity of condition (2.1) for the curve γ. Thus, γ is an Ahlfors-regular curve.

At the same time,

$$V_\gamma[\arg t]\ge\sum_{n=1}^{\infty}\frac{1}{n}=\infty$$

and condition (4.1) is not satisfied for the curve γ, i.e., γ is not a Král curve.

Let us show that the curve γ and the function g satisfy the conditions of Corolary 5.1. There is the partition $\gamma=\gamma^1\cup\gamma^2$, where $\gamma^1=\{0\}$ and $\gamma^2=\gamma\setminus\{0\}$. Let $R_0=1/2$ and $r(\xi)=\min\{2|\xi|,1/2\}$ for all $\xi\in\gamma^2$. Then for all $\xi\in\gamma^2$, we have the estimate

$$V_{\gamma_{r(\xi)}(\xi)}[\arg(t-\xi)]\le V_{\gamma_{r(\xi)/4}(\xi)}[\arg(t-\xi)]+\int_{\gamma_{r(\xi)}(\xi)\setminus\gamma_{r(\xi)/4}(\xi)}\frac{|dt|}{|t-\xi|}\le\pi+\frac{\theta\big(r(\xi)\big)}{r(\xi)/4}\le\pi+4c,$$

where $c=4+(\ln 2)^{-1}\sqrt{1+\ln^2 2}$ as it is follows from estimate (5.13).

It is obvious that $\widehat{k}_\gamma(0,\eta)\le 2$ and $\widehat{\varphi}_\gamma(0,\eta)<-1/\ln\eta$ for all $\eta\in(0,R_0]$.

In addition, $\widehat{k}_\gamma(\xi,\eta)\le 2$ for all $\xi\in\gamma^2$ and all $\eta\in(r(\xi),R_0]$.

Finally, taking into account the relations $|t|\le|t-\xi|+|\xi|\le 3\eta/2$, which hold for all $\xi\in\gamma^2$ and all $t\in\gamma$ such that $|t-\xi|=\eta\in(r(\xi),R_0]$, for the specified ξ and η, we obtain the inequality $\widehat{\varphi}_\gamma(\xi,\eta)<-3/\ln(3\eta/2)$.

Now, the validity of conditions (5.11) and (5.12) follows from the estimates

$$\sup_{\xi\in\gamma}\int_{\beta(\xi)}^{1/2}\frac{\widehat{\varphi}_\gamma(\xi,\eta)\,\widehat{k}_\gamma(\xi,\eta)\,\omega_\gamma(g,\eta)}{\eta}\,d\eta<6\int_0^{1/2}\frac{d\eta}{\eta\,\ln(3\eta/2)(\ln\eta-1)}<\infty\,,$$

where $\beta(\xi)=0$ for $\xi=0\in\gamma^1$ and $\beta(\xi)=r(\xi)$ for $\xi\in\gamma^2$.

Thus, all conditions of Corollary 5.1 are satisfied for the given curve γ and the given function g.

As a result, we can state that the limiting values $\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi)$ of logarithmic double layer potential (1.1) exist at all points $\xi\in\gamma$ and are expressed by formulas (2.10) and (2.11).

The results of this paper have been announced at a preprint of Arxiv (arXiv:2405.01482v1 [math.CV]).

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SONIN'S INVENTORY MODEL WITH A LONG-RUN AVERAGE COST FUNCTIONAL

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Abstract. We present an inventory model where a manufacturer (firm) uses for "production" a "commodity" (resource), which is consumed with the unit intensity. The price of the commodity follows a stochastic process, modelled by a continuous time Markov chain with a finite number of states and known transition rates. The firm can buy this commodity at the current price or use "stored" one. The storage cost is proportional to the storage level. The goal of the firm is to minimize the long-run average cost functional. We prove the existence of a canonical triple with an optimal threshold strategy, present an algorithm for constructing optimal thresholds and the optimal value of the functional, and discuss issues of uniqueness.

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1 Introduction

There are various mathematical inventory models (see, for example, Arrow et al., 1951, [1]; Bather, 1966, [2]; Rubalskiy, 1972, [15]; Browne, Zipkin 1991, [4]; Bayer et al., 2010, [3]; Bulinskaya, Sokolova, 2015, [5]. These models allow us to find the optimal strategy for purchasing a product that minimizes the total cost of purchasing, registration and delivery of the order, storage of goods, as well as losses from its shortage.

In this paper, we will consider the following inventory problem.

There is a manufacturer (firm) who needs to consume an intermediate product (commodity) with the unit intensity for production. If the price of a commodity is constant, then it is possible to purchase the commodity with the same intensity, and thereby production will be ensured. If the price changes over time, then at a low price it is reasonable to buy a one-time quantity in order not to overpay at a high price. But if you make a one-time purchase, then the purchased product must be stored, and you have to pay for this. An example of such a commodity is cotton, which, on one hand, is still one of the most important world products, and on the other hand, is characterized by significant price fluctuations, see for example Darekar and Reddy, 2017, [6].

I.M. Sonin proposed to consider the situation in which the price depends on the value of a Markov chain with continuous time, a finite number of n states and known transition intensities. In this case, it is reasonable to create a warehouse and make purchases and create a stock in accordance with the price.

It is assumed that purchases at the current price can be made both in large quantities and by continuously increasing (without decreasing) the quantity of the purchased goods.

It is assumed that the storage fee is proportional to the amount of goods in stock, the cost of ordering does not depend on the size of the order, the order is processed instantly.

The question is how to organize the work of the warehouse in order to minimize the expected costs of storage and purchase of goods, while the costs can be considered both discounted and marginal averages per unit of time.

This problem was studied first in the thesis of Hill, 2004, [8], and then in Hill, Sonin, 2006,[9], Katehakis, Sonin, 2013, [10] for the case $n = 2$ and special cases for $n = 3$. In these works, the marginal average expected costs per unit of time were considered and it was assumed that, in the general case, the strategy for the optimal organization of the work of the warehouse is of a threshold nature, i. e. :

for each state i of a Markov chain, there is a threshold a_i such that

if the inventory level x is less than or equal to a_i, then it is necessary to make a one-time purchase up to the level of this threshold, i.e. buy goods in the quantity $a_i - x$, and then, until the next jump of the Markov process (the moment of price change), it is necessary to purchase goods with the unit intensity so that the inventory level is equal to the threshold value a_i,

if $x > a_i$, then purchases should not be carried out until the next jump of the Markov process or the moment when the value of the inventory level, decreasing with the unit intensity becomes equal to a_i.

In Presman, Sonin, 2023, [14], using the methods developed in Presman, Sonin, 1982, [11], Presman, Sethi and Zhang, 1995, [12], Presman, Sethi, 2006, [13], a complete solution of this problem was given for the case of discounted costs. The existence of an optimal threshold strategy was proved, an algorithm for constructing optimal thresholds and the optimal value of the functional was formulated, the problem of the uniqueness of the optimal control was investigated.

We first describe the main ideas and features of the results and proofs of [14]. First, it turned out that in problems with Markov chains with continuous time it is convenient to write the optimality equation (the Bellman equation) in the form of choosing the optimal control until the moment of the first jump, i.e., consider an imbedded Markov chain. After that, from considerations of the convexity of the value function, it is easy to show that there is an optimal threshold control. Secondly, it turned out that in such problems it is convenient to pass from studying the value function to studying its derivative. Thirdly, it turned out that instead of the smooth gluing condition (continuity of the derivative of the value function), which arises in problems with diffusion processes, the condition of twice smooth gluing appears (continuity of the second derivative). Fourth, as a rule, the optimal control is unique, but, for some relations between the parameters, in which, in addition, the corresponding transition intensities are equal to zero, for some states the optimal control is unique, and for others, any optimal control is of quasi-threshold character, i.e., there is a whole interval of optimal thresholds (from zero to maximum), and if in the corresponding state the inventory level does not exceed the maximal optimal threshold, then any control that does not go beyond the interval of optimal thresholds is optimal.

In this paper, we study the case with the long-run average cost functional under the assumption that the chain is regular, i.e., from any state it is possible with a positive probability to get to any other (not necessarily after the first jump). A passage to the limit is carried out with the discounting parameter tending to zero. As a result, an analogue of the canonical triple, known in the theory of controlled Markov chains with discrete time and a finite number of states, arises. It is shown that, as in the case of discounting, there is an optimal threshold strategy, an algorithm for constructing optimal thresholds is given, and issues of uniqueness are considered.

2 Problem formulation and results for discounted costs

Let a *right-continuous* Markov process $\{m(t)\}_{0\le t<\infty}$, $m(0)=i$, with a finite number of states n (a Markov chain in continuous time) and an intensity matrix (infinitesimal operator)

$$\Lambda = (\lambda_{i,j}), \quad \text{where} \quad \lambda_{i,j} \ge 0,\ \lambda_{i,i} = -\sum_{j\ne i}\lambda_{i,j} = -\lambda_i < 0. \qquad i,j \in N = \{1,\cdots,n\},$$

are given, i.e., if at some point in time the process is in state i, then in a short time $\triangle$ with probability $\lambda_{i,j}\triangle + o(\triangle)$ it goes to state j, and with probability $1-\lambda_i\triangle+o(\triangle)$ it remains in state i.

Let $\mathcal{F}=\{\mathcal{F}_t\}_{0\le t<\infty}$ be the filtration generated by it, i.e., $\mathcal{F}_t$ contains all information about the Markov process up to and including the time t. Control u is

an $\mathcal{F}$-adapted **left continuous** nondecreasing function $u(t), u(0)=0$, whose value at time t corresponds to the total purchases of goods up to time t inclusive.

In other words, the total purchases up to time t can only depend on the behavior of the Markov process up to time t and cannot take into account whether the Markov process jumped at time t.

The moments of jumps and the sizes of jumps at these moments correspond to the moments and sizes of one-time purchases.

Since the consumption of a good occurs at the unit intensity, the equation

$$x^u(t) = x - t + u(t) \tag{2.1}$$

determines the inventory level at time t with an initial level of $x\ge 0$.

Controls for which $x^u(t)\ge 0$ for all $t\ge 0$ and the values of the functionals are finite will be called admissible. Denote by $\mathcal{U}(x)$ the set of all admissible controls for the initial point x.

For $u\in\mathcal{U}(x)$, we consider the functionals

$$V_i^{\rho,u}(x) = E_{x,i}\left\{\int_0^\infty e^{-\rho t}cx^u(t)dt + \int_0^\infty e^{-\rho t}P_{m(t)}du(t)\right\} \text{ for } \rho>0, \tag{2.2}$$

$$V_i^{0,u}(x) = \limsup_{T\to\infty}\frac{1}{T}E_{x,i}\left\{\int_0^T cx^u(t)dt + \int_0^T P_{m(t)}du(t)\right\}, \tag{2.3}$$

where c is the cost of storage of a unit of goods per unit of time, P_i is the price of a product under condition that the Markov process is in the state $i\in N$, $E_{x,i}\{\cdot\}$ – mathematical expectation when the initial state of the Markov process is equal to i and the initial inventory level is x. Without loss of generality, it is assumed that $P_i > P_{i+1}$ for $1\le i\le n-1$.

The goal is to find for $\rho\ge 0$ the value

$$V_i^\rho(x) = \inf_{u\in\mathcal{U}(x)} V_i^{\rho,u}(x),\ i\in N, \tag{2.4}$$

and determine the optimal control. The function $V_i^\rho(x)$ is called the value function. The name of the value function is also used for the column vector $V^\rho(x)$ with coordinates $V_i^\rho(x),\ i\in N$.

We will say that a control is *a_i-threshold in state* $i\in N$ if it satisfies the following conditions.

Let $m(t_0)=i,\ x(t_0)=x$, be satisfied at some time $t_0\ge 0$ and let $t_0+\tau$ be the moment of the first jump after t_0 of the process $m(t)$. To simplify the notation, we assume that $t_0=0$.

If $x \le a_i$, then a one-time purchase of $a_i - x$ is immediately made to bring the inventory level to the value of a_i, and after that, up to the moment τ, purchases are made with the unit intensity to keep the inventory at the level a_i, and hence $x^u(t) = a_i$ for $0 < t \le \tau$, i, e. $u(t) = u(0) + a_i - x + t$ for $0 < t \le \tau$;

if $x > a_i$, then at first no purchases are made (in this case, the inventory level decreases with the unit intensity), and this happens until the minimum from the moment τ and the moment when the inventory level becomes equal to a_i, and then, if $\tau > a_i - x$, until τ purchases are made with unit intensity to keep inventory level at a_i, so $u(t) = u(0) + \max[0, t - (x - a_i)]$ for $0 < t \le \tau$, which means $x^u(t) = x - t > a_i$ for $0 < t \le \min[x - a_i, \tau]$, and $x^u(t) = a_i$ for $x - a_i < t \le \tau$.

Consider a vector a with coordinates $a_1, \dots, a_n$. We call a threshold strategy a control that is a_i-threshold for every $i \in N$.

As already mentioned, [14] gave a complete solution of this problem for the case $\rho > 0$. Let us first present the results obtained there.

Let P be an n-dimensional column vector with coordinates P_i, $i \in N$; I be an n-dimensional column vector with coordinates equal to 1; E be an $n \times n$ diagonal matrix such that all diagonal element are equal to 1; $V^\rho(x)$ be the column vector with coordinates $V_i^\rho(x)$, $i \in N$; $\mathbf{0}$ be the N-dimensional column vector with zero coordinates.

We define the vector $b^\rho = (\Lambda - \rho E)(-P) + cI$, so that

$$b_i^\rho = c + (\lambda_i + \rho)P_i - \sum_{j \ne i} \lambda_{i,j} P_j = c + \rho P_i + \sum_{j \ne i} \lambda_{i,j}(P_i - P_j), \quad i \in N. \tag{2.5}$$

It turned out that for any state $i \in N$ there exists an optimal threshold control, and the equality of the optimal threshold to zero or its positiveness depends on the sign of b_i^ρ.

Let us put

$$N_+^\rho = \{i : \ b_i > 0\}, \quad N_0^\rho = \{i : \ b_i = 0\}, \quad N_-^\rho = \{i : \ b_i < 0\}. \tag{2.6}$$

Remark 1. The set N_+^ρ is non-empty because $P_1 > P_i$ for all $j \ne i$, and from the second equality in (2.5) we get that $b_1 > 0$.

Below, if it is clear which $\rho > 0$ we are talking about, we will omit the superscript ρ for all considered quantities.

The following theorem was proved in [14].

Theorem 2.1. 1) *There is a vector a^* such that the a^*-threshold strategy is optimal in the class of all admissible controls in the problem of minimization of functional (2.2).*

2) *The vector function $V(x)$ is convex and its derivative $U(x) = \dfrac{dV(x)}{dx}$ is a unique* **continuously differentiable** *solution of the equation*

$$\begin{aligned} \frac{dU(x)}{dx} = \ & \max[b(x), \mathbf{0}], \ U(0) = -P, \quad \text{where} \\ & b(x) = (\Lambda - \rho E)\, U(x) + cI = (\Lambda - \rho E)\left(U(x) - \frac{c}{\rho} I\right), \end{aligned} \tag{2.7}$$

(hereinafter for any vector d the notation $\max[d, \mathbf{0}]$ *denotes taking the coordinate-wise maximum), moreover*

$$b(0) = b, \quad b_i(x) < 0 \text{ for } 0 \le x < a_i^*, \quad b_i(x) \ge 0 \text{ for } x \ge a_i^*, \quad i \in N, \tag{2.8}$$

which means

$$a_i^* > 0 \text{ if and only if } i \in N_-. \tag{2.9}$$

Let us put $a^m = \max_i\{a_i^*\}$. *It follows from* (2.7) *and* (2.8) *that for* $x \geq a^m$

$$U(x)-\frac{c}{\rho}I=e^{(\Lambda-\rho E)(x-a^m)}\left(U(a^m)-\frac{c}{\rho}I\right)=e^{(\Lambda-\rho E)(x-a^m)}U(a^m)-\frac{c}{\rho}e^{-\rho(x-a^m)}I\,. \tag{2.10}$$

The function $V(x)$ *itself is determined from the relation*

$$V(x)=\begin{cases} c\dfrac{\rho x-1}{\rho^2}I-\displaystyle\int_x^\infty\left(U(y)-\frac{c}{\rho}I\right)dy \\ \qquad =c\dfrac{e^{-\rho(x-a^m)}+\rho x-1}{\rho^2}I+e^{(\Lambda-\rho E)(x-a^m)}(\Lambda-\rho E)^{-1}U(a^m) \text{ for } x\geq a^m, \\ -\displaystyle\int_x^{a^m}U(y)dy+V(a^m)=-\int_x^{a^m}U(y)dy+c\frac{a^m}{\rho}I+(\Lambda-\rho E)^{-1}U(a^m) \text{ for } 0\leq x<a^m. \end{cases} \tag{2.11}$$

For $a^m > 0$, in [14] an algorithm was formulated for successive construction of the vector-function $U(x)$ and the vector a^* on successive intervals between thresholds, starting from the interval $[a^{(1)}, a^{(2)}]$, where $a^{(1)} = 0$ and $a^{(2)}$ is the minimal positive threshold. On this interval, the vector-function $U(x)$ was first constructed, and then the threshold $a^{(2)}$ and the set $I^{(2)} = \{i:\ a_i^* = a^{(2)}\}$. Then, if $a^m > a^{(2)}$, then the same is done for the interval $[a^{(2)}, a^{(3)}]$, where $a^{(3)}$ is the minimal threshold of those thresholds that are greater than $a^{(2)}$, etc., up to $a^{(r)}$, where the number r was determined from the condition $a^{(r)} = a^m$.

For a more precise formulation, we need some notation. For any set of thresholds a, which is convenient for us to consider as a column vector with coordinates a_i, consider:

the number r corresponding to the number of different threshold values;

increasing numbers $a^{(1)}(a), \ldots, a^{(r)}(a)$ corresponding to different threshold values;

$$\begin{aligned} &\text{sets } I^{(l)}(a),\ 1 \leq l \leq r(a), \text{ where } I^{(l)}(a)=\{i:\ a_i=a^{(l)}(a)\}; \\ &\text{sets } N_+^{(l)}(a) = \{i:\ a_i \leq a^{(l)}(a)\}=\bigcup_{k=1}^{l} I^{(k)}(a), \text{ while } N_+^{(r)}(a)=N; \\ &\text{sets } N_-^{(l)}(a)=N\setminus N_+^{(l)}(a)=\{i:\ a_i>a^{(l)}(a)\}. \end{aligned} \tag{2.12}$$

For $a = a^*$ we will omit the dependence on a and simply write r; $a^{(l)}, I^{(l)}, N_+^{(l)}, N_-^{(l)}$, $1 \leq l \leq r$. In addition, in further notation, if it is clear which l we are talking about, we will not write the superscript "(l)".

Let us introduce the following notation:

$$\Lambda_+=(\lambda_{i,j})_{i,j\in N_+^{(l)}}\,,\quad \Lambda_-=(\lambda_{i,j})_{i,j\in N_-^{(l)}}\,,\quad \Lambda_\pm=(\lambda_{i,j})_{i\in N_+^{(l)},j\in N_-^{(l)}}\,,$$

$$\Lambda_\mp=(\lambda_{i,j})_{i\in N_-^{(l)},j\in N_+^{(l)}}\,,\quad A=\Lambda-\rho E\,,\quad A_+=\Lambda_+-\rho E_+\,,\quad A_-=\Lambda_--\rho E_-\,,$$

for any vector $d = (d_i)_{i\in N}$ we set $d_+ = (d_i)_{i\in N_+^{(l)}}$, $d_- = (d_i)_{i\in N_-^{(l)}}$. Here the subscript "$+$" (respectively "$-$") defines a vector with coordinates from $N_+^{(l)}$ (respectively $N_-^{(l)}$) and transitions from $N_+^{(l)}$ to $N_+^{(l)}$, (respectively, from $N_-^{(l)}$ to $N_-^{(l)}$). The subscript "$\pm$" (respectively "$\mp$") defines transitions from $N_+^{(l)}$ to $N_-^{(l)}$ (respectively from $N_-^{(l)}$ to $N_+^{(l)}$).

Remark 2. From the fact that A_+ corresponds to a Markov chain with the state set $N_+^{(l)}$ and the killing rate in state i equal to $\rho_i = \rho + \sum_{j\in N_-^{(l)}} \lambda_{1,j}$, the existence of the inverse matrix $(A_+)^{-1}$ follows.

Given l, $1 \le l < r$, for $0 \le x < \infty$ we define the column vector-function $F(x)$ as follows: $F(x) = U(x)$ for $0 \le x \le a^{(l)}$, and for $y \ge 0$

$$\begin{aligned} F_-(a^{(l)} + y) &= \; -P_- \\ F_+(a^{(l)} + y) &= \; -(A_+)^{-1}\left[cI_+ - \Lambda_\pm P_-\right] + (A_+)^{-1} e^{A_+ x} b_+(a^{(l)}). \end{aligned} \tag{2.13}$$

Consider also the vector-function

$$f(x) = cI + (\Lambda - \rho E)\, F(x), \;\; 0 \le x < \infty, \tag{2.14}$$

so that $f(x) = b(x)$ for $0 \le x \le a^{(l)}$.

It follows from relation (2.8) that $f_i(a^{(l)}) \ge 0$ for $i \in N_+^{(l)}$, $f_i(a^{(l)}) < 0$ for $i \in N_-^{(l)}$. Consider now the function

$$f^{\max}(x) = \max_{i \in N_-^{(l)}} f_i(x), \qquad x \ge a^{(l)}. \tag{2.15}$$

Proposition 2.1. Algorithm for constructing the vector-function $U(x)$ and the vector a^*. *Assume that for some $l < r$ we have constructed $a^{(i)}$ and $I^{(i)}$ for $1 \le i \le l$, and also $U(x)$ for $0 \le x \le a^{(l)}$, satisfying (2.7). If $l < r$, then*

$$U(x) = F(x) \;\text{ for }\; a^{(l)} \le x \le a^{(l+1)}, \tag{2.16}$$

$$a^{(l+1)} = \inf\{x : \; f^{\max}(x) > 0\}, \quad I^{(l+1)} = \{i : \; i \in N_-^{(l)}, \;\; f_i(a^{(l+1)}) = 0\}, \tag{2.17}$$

where $F(x)$, $f(x)$ and $f^{\max}(x)$ are defined in (2.13), (2.14), (2.15).

Let us pass to the study of the uniqueness of the optimal control. Let $\bar a_i = \inf\{x : \, b_i(x) > 0\}$. From (2.8) it follows that $\bar a_i \ge a_i^*$. It turns out that if $\bar a_i = a_i^*$, then the optimal control in state i is unique, and if $\bar a_i > a_i^*$, then $a_i^* = 0$, and in the state i any threshold control with a threshold not exceeding $\bar a_i$ is optimal. Moreover, in this state, the control is optimal if and only if, in this state, it prescribes not to make purchases at an inventory level greater than $\bar a_i$, and if the inventory level does not exceed $\bar a_i$, then it prescribes to make such purchases so that the trajectory of the inventory level does not go beyond the segment $[0, \bar a_i]$.

We say that a control in state $i \in J$ is a_i-quasi-threshold if it satisfies the following conditions.

Let $m(t_0) = i$, $x(t_0) = x$, be satisfied at some time $t_0 \ge 0$ and let $t_0 + \tau$ be the moment of the first jump after t_0 of the process $m(t)$.

If $x > a_i$ and $t_0 < t < t_0 + \min(x - a_i, \tau)$, then, as for threshold control, $u(t) = u(t_0)$. If $x > a_i$ and $x - a_i < t < t_0 + \tau$, or $x \le a_i$ and $t_0 < t < t_0 + \tau$, then the control u is such that $0 \le x^u(t) \le a_i$.

Let $\tilde N \subset N$. The set of $(a, \tilde N)$-quasi-threshold strategies is the set of all admissible controls that satisfy the following properties: in states $i \notin \tilde N$ they are a_i-threshold, and in states $i \in \tilde N$ they are a_i-quasi-threshold.

Denote by N_0^- the set of those states from N_0 from which one can get to states belonging to N_+ only by visiting states from N_-. The following theorem was proved in [14].

Theorem 2.2. *If $N_0^- = \emptyset$, then $\bar a_i = a_i^*$ for all $i \in N$ and in the optimization problem (2.2) the optimal control is unique and is given by a^*-threshold strategy, and if $N_0^- \ne \emptyset$, then from $i \notin N_0^-$ it follows that $\bar a_i = a_i^*$, from $i \in N_0^-$ it follows that $\bar a_i > a_i^* = 0$, $b_i(x) = 0$ for $0 \le x \le \bar a_i$ and the control is optimal if and only if it belongs to the set of $(\bar a, N_0^-)$-quasi-threshold strategies. Wherein:*

a) *$U_i(x) = -P_i$ for $0 \le x \le \bar a_i$ and $b_i(x) > 0$ for $x > \bar a_i$ (and hence, by virtue of the first equality in (2.7), $U_i(x)$ strictly increases for $x > \bar a_i$), $i \in N$.*

b) *if $i \in N_0^{(1)}$, then there exists l, $2 \le l \le r$ such that $\bar a_i = a^{(l)} > 0$, and if $\bar a_i > a^{(l)}$, then $\lambda_{i,j} = 0$ for any such j, that $\bar a_j \le a^{(l)}$.*

For $1 \le l \le r$ we set:

$$I_0^{(l)} = \{i : i \in N_0,\, \bar{a}_i = a^{(l)}\},\ N_0^{(l-1)} = \{i : i \in N_0,\, \bar{a}_i \ge a^{(l)}\}, \quad N_+^{(l)} = \{i : \bar{a}_i \le a^{(l)}\}.$$

It follows from here that $N_0^{(0)} = N_0 = \sum_{l=1}^{r} I_0^{(l)}$, $N_0^{(l)} = N_0^{(l-1)} \setminus I_0^{(l)}$. It follows from Theorem 2.2 that $N_0^{(1)} = N_0^-$.

In paper [14] the following algorithm for the sequential construction of sets $I_0^{(l)}$ was given. This algorithm is related to the structure of the zero elements of the matrix Λ.

At first it was shown how the set $I_0^{(1)}$ is constructed. First, one includes in it all those elements $i \in N_0$ for which there exists $j \in N_+$ such that $\lambda_{i,j} > 0$, then all those elements $i \in N_0$ for which there exists $\lambda_{i,j} > 0$ for j included in the previous step, and so on.

Proposition 2.2. Algorithm for constructing sets $I_0^{(l)}$. *Let the sets $I_0^{(i)}, 1 \le i \le l$ be constructed for some $l < r$ (this was done above for $l = 1$). From statement* b) *of* Theorem 3.2, *it follows that for any $i \in N_0^{(l)}$ it is true that: $\lambda_{i,j} = 0$ for any $j \in N_+^{(l)}$. To construct $I_0^{(l+1)}$, we first include in it those elements $i \in N_0^{(l)}$ for which there exists $j \in I^{(l+1)}$, such that $\lambda_{i,j} > 0$. If this set is empty, then the set $I_0^{(l+1)}$ is also empty. Otherwise, to those included in $I_0^{(l+1)}$ at the first stage, we add those elements i from the elements remaining in $N_0^{(l)}$ for which there exists j included in the first step such that $\lambda_{i,j} > 0$. If this set is empty, then the construction of the set $I_0^{(l+1)}$ is complete. If not, then we repeat the procedure, and so on. As a result, the set $I_0^{(l+1)}$ will be constructed.*

Remark 3. It is worth paying attention to the fact that, although in the formula (2.13) the vector $F_+(x)$ is written as an exponent, nevertheless, if $\bar{a}_i > a^{(l)}$, then $i \in N_+^{(l)}$ and $F_i(x) \equiv P_i$. It is possible to reformulate Proposition 2.2 in such a way that all coordinates of the new vector $F_+(x)$ are represented as constants minus decreasing functions, which are linear combinations of decreasing exponentials, perhaps multiplied by sines or cosines (in the case of complex eigenvalues of the new matrix A_+), and, perhaps, multiplied by polynomials (in the case of multiples of eigenvalues). To do this, $N_+^{(l)}$ should be determined not according to (2.12), but according to the formula: $N_+^{(1)} = N_+ \cup I_0^{(1)}$, $N_+^{(l)} = N_+^{(l-1)} \cup I_-^{(l)} \cup I_0^{(l)}$ for $1 < l < r-1$. In the sequel, we will talk about the algorithm in this formulation.

The aim of this paper is to consider the case $\rho = 0$, i.e., studying functional (2.3). This is carried out by passing to the limit as the discount coefficient tends to zero. Therefore, in what follows we return to the use of the superscript ρ.

3 Main results

3.1 Main theorems

If the chain $m(t)$ is regular, then it is natural to assume that in the problem of minimization of functional (2.3) (a long-run average cost) the optimal value of the functional $V_i^0(x)$ does not depend on i and x and is equal to some number, which we will denote V^*. In this case, it is natural to consider the problem of finding

$$G_i(x) = \inf_{u \in \mathcal{U}(x)} E_{x,i} \left\{ \int_0^\infty \left[(cx^u(t) - V^*)dt + P_{m(t)}du(t) \right] \right\}, \ i \in N. \tag{3.1}$$

If such a function exists, then, by virtue of the Bellman optimality principle, for any stopping time $\mathcal{T}$ (with respect to the process $m(t)$) with a finite mathematical expectation, the following relation holds:

$$G_i(x) = \inf_{u\in\mathcal{U}(x)} E_{x,i}\left\{\int_0^{\mathcal{T}} \left[(cx^u(t) - V^*)dt + P_{m(t)}du(t)\right] + G_{m(\mathcal{T})}(x^u(\mathcal{T}))\right\}, \ i \in N. \tag{3.2}$$

A vector-function $G(x)$ that for any Markov moment with a finite mathematical expectation satisfies relation (3.2) is naturally called an invariant function for the problem of minimization of functional (2.3).

Let there exist a strategy u^* (i.e. a set of controls that assigns to each $i \in N$ and $x \geq 0$ a control $u^*_{x,i}(t) \in \mathcal{U}(x)$) on which the minimum of functional (3.1) is achieved. Let $x^*(t)$ be the trajectory corresponding to the control $u^*_{x,i}(t)$. If $E_{x,i}G_{m(T)}(x^*(T))$ is bounded with respect to T for each x and i, then

$$G_i(x) = E_{x,i}\left\{\int_0^{\mathcal{T}} \left((cx^*(t) - V^*)dt + P_{m(t)}du^*_{x,i}(t)\right) + G_{m(\mathcal{T})}(x^*(\mathcal{T}))\right\}. \tag{3.3}$$

In this case $(V^*;\ u^*;\ G(x))$ is called the canonical triplet for the problem of minimization of functional (2.4) by analogy with the corresponding concept in control problems for discrete-time Markov chains (see [7], Chapter 7).

Theorem 3.1. *Let a chain be regular and $c > 0$. Then:*

a) *there exist a vector $a^{0,*}$, a number V^*, and a vector function $W(x)$ such that*

$$W(x) = \lim_{\rho\to 0} W^\rho(x), \text{ where } W^\rho(x) = V^\rho(x) - \frac{V^*}{\rho}I, \qquad a^{0,*} = \lim_{\rho\to 0} a^{\rho,*}, \tag{3.4}$$

and V^*, $a^{0,*}$-threshold strategy and $W(x)$ form a canonical triplet for minimization problem (2.3),

b) *the function $U^0(x) = \dfrac{dW(x)}{dx}$ for $0 \leq x \leq a^{0,m} = \max_i\{a_i^{0,*}\}$ is a unique continuously differentiable solution of the equation*

$$\frac{dU^0(x)}{dx} = \max\left(\Lambda U^0(x) + cI, \mathbf{0}\right), \quad U^0(0) = -P, \text{ for } 0 \leq x \leq a^{0,m}, \tag{3.5}$$

here $V^* = ca^{0,m} - \bar{\Lambda}U^0(a^{0,m})$, where the row vector $\bar{\Lambda}$ denotes the invariant distribution of the Markov chain.

c) *$a_i^{0,*} > 0$ if $i \in N^0_-$ and $a_i^{0,*} = 0$ otherwise,*

d) *the solution of equation (3.5) and the numbers $a_i^{0,*}$ are constructed in accordance with the algorithm from Proposition 2.1.*

Remark 4. To find V^* in case $c > 0$ it is not necessary to find $W(x)$ for $x > a^{0,m}$, but it suffices to solve equation (3.5). Nevertheless, below, in (4.12) we give an explicit expression for $W(x)$ for $x > a^{0,m}$.

For $c = 0$, the situation differs from the case $c > 0$. We show that if $c = 0$ and $\rho \to 0$, then $V^{\rho,*} \to P_n$ and the solution of problem (3.2) is obtained by passing to the limit as $\rho \to 0$. It is clear that for $c = 0$ and $V^* = P_n$ to solve problem (3.2), it is necessary for $i \neq n$ to minimize the expected costs until the hitting time of state n. We show that when solving the latter problem, the optimal control is a threshold strategy for which the thresholds coincide with $a_i^{0,*} = \lim_{\rho\to 0} a_i^{\rho,*}, i \neq n$. At the same time, it turns out that if for $i = n$ one uses the y-threshold control and after the first jump uses the obtained optimal strategy, then the larger y the smaller the value of functional is (it is due to the fact, that $a_n^{\rho,*} \to \infty$ as $\rho \to 0$). Thus, when solving problem (3.2), there exists an invariant function, but for $i = n$ there is no optimal control, while in problem (3.1) for any control, the value of the functional is infinity.

Before we formulate and prove these facts, we introduce the following notation. Denote $N_{+n} = N \setminus \{n\}$. By analogy with the previous ones, the subscript "+n" of the vector means that we are considering a vector with coordinates from N_{+n}.

Theorem 3.2. *Let a chain be regular and $c = 0$. Denote $\hat{a}_n^\rho = \max_{i \neq n} a_i^{\rho,*}$. Then the following statements hold*

a) *There exist $a_{+n}^{0,*} = \lim_{\rho \to 0} a_{+n}^{\rho,*}$; numbers $g > 0$, $\mu > 0$, and an integer number $m_0 \geq 0$ such that $\lim_{\rho \to 0} \dfrac{a_n^{\rho,*} - \hat{a}_n^\rho}{a(\rho)} = \dfrac{1}{\mu}$, where $a(\rho) > m_0$ is the root of the equation $[a(\rho)]^{m_0} e^{-a(\rho)} = g\rho$; for $0 \leq x < \infty$ there exists $U^0(x) = \lim_{\rho \to 0} U^\rho(x)$ such that $U_n^0(x) = -P_n$ and the function $U_{+n}^0(x)$ for $0 \leq x < \infty$ is a unique continuously differentiable solution of the equation*

$$\frac{dU_{+n}^0(x)}{dx} = \max\left(\Lambda_{+n} U_{+n}^0(x) - \Lambda^{(n)} P_n, \mathbf{0}_{+n}\right), \quad U_{+n}^0(0) = -P_{+n}, \tag{3.6}$$

where $\Lambda^{(n)}$ is the column-vector with elements $\lambda_{i,n}$, $1 \leq i \leq n-1$. The numbers $a_i^{0,*}$, $i \neq n$, and $U_{+n}^0(x)$ are constructed in accordance with the algorithm from Proposition 2.1.

b) *In the problem of minimization of functional (2.3) $V^* = P_n$ and there exists $W(x) = \lim_{\rho \to 0} \left[V^\rho(x) - \left(a_n^\rho + \dfrac{1}{\rho} \right) P_n I \right]$, which is an invariant function that for $i \neq n$ is equal to the minimal expected costs until the hitting time $\mathcal{T}^{(n)}$ of state n minus $P_n E_i\{\mathcal{T}^{(n)}\}$. Herewith*

$$W_n(x) = -xP_n, \qquad W_{+n}(x) = -xP_n I_{+n} - \int_x^\infty \left[U_{+n}^0(v) + P_n I_{+n}\right] dv, \tag{3.7}$$

where $U_{+n}^0(v) + P_n I_{+n}$ exponentially converges to $\mathbf{0}_{+n}$ as $v \to \infty$.

c) *For any $i \in N_{+n}$ the $a_i^{0,*}$-threshold control is optimal in the problem of minimization of the right hand side of (3.2) (where $c = 0$, $V^* = P_n$ and $G(x)) = W(x)$). There is no optimal control for $i = n$. For any $\varepsilon > 0$, there exists such $y(\varepsilon)$ that for $y > y(\varepsilon)$ the y-threshold control is ε-optimal for the state $i = n$. In the problem of minimization of functional (3.1) for any control, the value of the functional is infinity.*

3.2 Uniqueness

When studying functional (2.3), it makes no sense to talk about the uniqueness of the optimal control, since an arbitrary control on any fixed time interval with subsequent use of the optimal control gives the same value of the functional. But it makes sense to talk about the uniqueness of the optimal control for solving problem (3.1) with $G(x) = W(x)$, which, as mentioned, also gives a solution to problem (2.3).

After constructing $W(x)$ you can define numbers $\bar{a}_i^{0,*}$ and sets $I_0^{0,(l)}$, $N_0^{0,(l-1)}$, $1 \leq l \leq r$, just as it was done for $\rho > 0$.

Theorem 3.3. *The statements of Theorem 2.2 and Proposition 2.2 are also true in the optimization problem (3.2) with $G(x) = W(x)$ for $i \in N$ in the case $c > 0$ and for $i \in N_{+n}$ in the case $c = 0$.*

Remark 5. In [14] it was noted that, as a rule, optimal control is unique. If we fix all parameters except ρ, then non-uniqueness is possible only for a finite set of values of ρ at which some of b_i (see (2.5)) vanish, and even then, provided that the corresponding $\lambda_{i,j}$ equals zero. It follows directly from (2.5) that for $\rho = 0$ it may turn out that $N_0^{0,\{1\}} \neq \emptyset$ and therefore optimal control is not unique, while for sufficiently small ρ it is always true that $N_0^{\rho,\{1\}} = \emptyset$ and optimal control is unique.

4 Proofs

4.1 Proof of Theorem 3.1

In this section, we consider a regular Markov chain, for which with positive probability it is possible to go from any state to any other one, possibly after some jumps.

4.1.1. Let us first show how (3.4) yields 3.2) and statement b) of Theorem 3.1. Let us write down the obvious identity:

$$\begin{aligned} V_i^\rho(x) &= \inf_{u\in\mathcal{U}(x)} E_{x,i}\left\{\int_0^{\mathcal{T}} e^{-\rho t}\left(cx^u(t)dt + P_{m(t)}du(t)\right) + e^{-\rho\mathcal{T}}V^\rho_{m(\mathcal{T})}(x^u(\mathcal{T}))\right\} \\ &= E_{x,i}\left\{\int_0^{\mathcal{T}} e^{-\rho t}\left(cx^{a^{\rho,*}}(t)dt + P_{m(t)}du^{\rho,*}(t)\right) + e^{-\rho\mathcal{T}}V^\rho_{m(\mathcal{T})}(x^{a^{\rho,*}}(\mathcal{T}))\right\}. \end{aligned} \tag{4.1}$$

Using the equality $\int_0^{\mathcal{T}} e^{-\rho t}dt = \dfrac{1-e^{-\rho\mathcal{T}}}{\rho}$, relation (4.1) can be rewritten as:

$$\begin{aligned} W_i^\rho(x) &= \inf_{u\in\mathcal{U}(x)} E_{x,i}\left\{\int_0^{\mathcal{T}} e^{-\rho t}\left((cx^u(t) - V^*)dt + P_{m(t)}du(t)\right) + e^{-\rho\mathcal{T}}W^\rho_{m(\mathcal{T})}(x^u(\mathcal{T}))\right\} \\ &= E_{x,i}\left\{\int_0^{\mathcal{T}} e^{-\rho t}\left((cx^{a^{\rho,*}}(t) - V^*)dt + P_{m(t)}du^{\rho,*}(t)\right) + e^{-\rho\mathcal{T}}W^\rho_{m(\mathcal{T})}(x^{a^{\rho,*}}(\mathcal{T}))\right\}. \end{aligned} \tag{4.2}$$

We know the structure of the functions $W_i^\rho(x)$, and therefore we can take a limit as $\rho \to 0$. So we get (3.2) and assertion b) of Theorem 3.1.

4.1.2. Let us move on to the proof of the existence of the limit vector function $W(x)$.

First, we show that using the algorithm from Proposition 2.1, one can construct a vector $a^{0,*}$ such that $a^{0,*} = \lim\limits_{\rho\to 0} a^{\rho,*}$, and a vector-function $U^0(x)$ defined for $0 \le x \le a^{0,m} = \max\limits_{i\in N} a_i^{0,*}$, such that $U^0(x) = \lim\limits_{\rho\to 0} U^\rho(x)$ for $0 \le x \le a^{0,m}$, $U^0(x)$ is the only continuously differentiable solution of equation (3.5).

It is easy to see that for this it suffices to prove the existence of a finite limit $\lim\limits_{\rho\to 0} a^{\rho,*}$, because then relation (3.5) and assertion d) of Theorems 3.1 follow from the continuous dependence on the parameter of the solution of the system of linear differential equations. To prove the existence of a finite limit $\lim\limits_{\rho\to 0} a^{\rho,*}$, in turn, it suffices to prove that for any $1 \le l < r^0$, there exists $i^{(l)} \in N_-^{(l)}$ such that $\lim\limits_{x\to\infty} f_{i^{(l)}}^{\rho,(l)}(x) > 0$ for all $\rho \ge 0$ because, according to the algorithm, $f_i^{\rho,(l)}(a^{\rho,(l)}) < 0$ for any $i \in N_-^{(l)}$, and the positiveness of the corresponding limit guarantees the existence of $a^{\rho,(l+1)}$.

In [14] the proof of the existence of $a^{\rho,*}$ was based on the study of the optimality equation and made essential use of the property $\rho > 0$. Here we give an independent proof of the existence of $a^{\rho,*}$, which is valid for both $\rho > 0$ and $\rho = 0$. Therefore, later in this section we will omit index ρ.

For $f_-(x)$ formula (2.14) can be written as

$$f_-(x) = cI_- + A_\mp F_+(x) - A_-P_- \tag{4.3}$$

where, according to (2.13), for $x \geq a^{(l)}$

$$F_+(a^{(l)} + x) = (-A_+)^{-1} [cI_+ - \Lambda_\pm P_-] + (-A_+)^{-1} e^{A_+(x-a^{(l)})} b_+(a^{(l)}). \tag{4.4}$$

In Remark 2, it was said that since A_+ corresponds to a Markov chain with state set $N_+^{(l)}$ and the killing rate in state i is equal to $\rho_i = \rho + \sum_{j \in N_-^{(l)}} \lambda_{,j}$, the existence of the inverse matrix $(A_+)^{-1}$ follows. For $\rho = 0$, for some $i \in N_+^{(l)}$ the corresponding sum may turn out to be zero. However, the following lemma will be proved in Appendix **A1**.

Lemma 4.1. *If $\rho > 0$ or $\rho = 0$ and a chain $m(t)$ is regular, then*
a) *there exists $(A_+)^{-1}$, and all elements of this matrix are nonpositive,*
b) *the eigenvalue of the matrix A_+ with the maximal real part is negative, and therefore $e^{A_+ x} \to 0$ for $x \to \infty$, and the convergence is exponential.*

From this lemma and from (4.4) we get

$$F_+ =: \lim_{x \to \infty} F_+(x) = (-A_+)^{-1} [cI_+ - A_\pm P_-], \tag{4.5}$$

and from this and from (4.3) it follows that

$$f_- =: \lim_{x \to \infty} f_-(x) = c[I_- + A_\mp (-A_+)^{-1} I_+] - BP_-, \tag{4.6}$$

where

$$B = A_- + A_\mp (-A_+)^{-1} A_\pm, \tag{4.7}$$

In Appendix **A2** an interpretation of the elements of the matrices $(A_+)^{-1}$ and B will be given and the following lemma will be proved.

Lemma 4.2. *If $\rho > 0$, or $\rho = 0$ and the chain $m(t)$ is regular, then*
a) *all off-diagonal elements of the matrix B are non-negative, and the sum of the off-diagonal elements over each row is positive,*
b)

$$BI_- + \rho \left(I_- + A_\mp (-A_+)^{-1} I_+ \right) = \mathbf{0}_-. \tag{4.8}$$

Define $i^{(l)}$ from the condition $P_{i^{(l)}} = \max_{j \in N_-^{(l)}} P_j$. Multiplying (4.8) by $P_{i^{(l)}}$ and adding with (4.6) we get

$$f_- = (c + P_{i^{(l)}} \rho)[I_- + A_\mp (-A_+)^{-1} I_+] + B(P_{i^{(l)}} I_- - P_-). \tag{4.9}$$

All elements of the matrix $(-A_+)^{-1}$ are nonnegative, consequently all elements in the first square brackets are positive. All elements of the matrices $A_\mp$, and $A_\pm$ are also non-negative. Therefore, all coordinates of the vector in square brackets on the right side of (4.9) are non-negative. As for the last term on the right side of (4.9), its component corresponding to $i^{(l)}$ is obtained by multiplying the row corresponding to $i^{(l)}$ by the vector $P_{i^{(l)}} I_- - P_-$, whose coordinate corresponding to $i^{(l)}$ is equal to zero, and all other coordinates are strictly positive. Therefore, it follows from assertion a) of Lemma 4.2 that $f_{i^{(l)}} > 0$. This proves the existence of $a^{(l+1)}$, and hence the existence of the limit $a^{0,*}$ and the limit function $U^0(x)$ on the interval $[0, a^{0,m}]$, while $U^0(x)$ is the only continuously differentiable solution of equation (3.5). Thus, we have proved also both statements c) and d) of Theorem 3.1. To complete the proof of Theorem 3.1, it remains to consider the interval $[a^{\rho,m}, \infty)$.

4.1.3. Note now that on the interval $[a^{\rho,m}, \infty)$ the situation becomes more complicated for $\rho = 0$. On this interval, for $\rho > 0$, the function $V(x)$ is given by formula (2.11), which contains the matrix $(\Lambda-\rho E)^{-1} e^{(\Lambda-\rho E)(x-a^m)}$. Therefore, to carry out the passage to the limit as $\rho \to 0$, we need to consider the structure of the matrix $(\Lambda-\rho E)^{-1} e^{(\Lambda-\rho E)x}$.

Let a Markov chain $m(t)$ be regular, and let $\mu_i,\ i=1,\dots,N$ be the eigenvalues of the matrix Λ. Then the eigenvalue with the maximal real part (we will assume that this is μ_1) is single and equal to zero. It corresponds to the right column eigenvector I, consisting of ones, and the left row eigenvector $\bar{\Lambda}$, which defines the unique invariant distribution of the chain. All other $\mu_i, i \neq 1$, have a negative real part. For the case $N=3$ in [14] it is shown that among them there can be both complex conjugate and coinciding.

Let us first consider the situation in which all roots are different and real, and hence $\mu_i < 0$ for $i \neq 1$. In such a case the following representation is valid:

$$\begin{aligned} e^{(\Lambda-\rho E)x}(\Lambda-\rho E)^{-1} &= X \operatorname{diag}\left(\frac{1}{\mu_i-\rho} e^{(\mu_i-\rho)x}\right) X^{-1} \\ &= -\frac{1}{\rho} e^{-\rho x} X_1 Y_1 + \sum_{i=2}^{N} \frac{1}{\mu_i-\rho} e^{(\mu_i-\rho)x} X_i Y_i, \end{aligned}$$

where $\operatorname{diag}(f_i)$ is a diagonal matrix with diagonal elements f_i; X_i is the i-th column vector of the matrix X, which is the right eigenvector of the matrix Λ corresponding to the eigenvalue μ_i (with $X_1 = I$); Y_i is the i-th row vector of the matrix $Y = X^{-1}$, which is the left eigenvector of the matrix Λ corresponding to the eigenvalue μ_i (where $Y_1 = \bar{\Lambda}$).

In the general case, instead of the diagonal matrix, there are the corresponding Jordan cells, in which there are decreasing exponentials (with sines and cosines in the case of complex eigenvalues), multiplied by polynomials in the case of multiple eigenvalues.

In the general case, the following representation takes place

$$e^{(\Lambda-\rho E)x}(\Lambda-\rho E)^{-1} = -\frac{e^{-\rho x}}{\rho} I\bar{\Lambda} + B^{\rho}(x), \quad B^{\rho}(x)I = \mathbf{0}. \tag{4.10}$$

where the matrix $B^{\rho}(x)$ has a limit as $\rho \to 0$, while the elements of the limit matrix $B^0(x)$ are combinations of decreasing exponents, possibly multiplied by sines, cosines and polynomials, and the last equality follows from the fact that $Y = X^{-1}$.

Hence, using (2.11), (2.10), and (4.10) we obtain that for $x \geq a^{\rho,m}$

$$\begin{aligned} V^{\rho}(x) &= c\frac{\left(e^{-\rho(x-a^{\rho,m})}+\rho x-1\right)}{\rho^2} I - \left[\frac{e^{-\rho(x-a^{\rho,m})}}{\rho} I\bar{\Lambda} - B^{\rho}\left(x-a^{\rho,m}\right)\right] U^{\rho}\left(a^{\rho,m}\right) \\ &= c\frac{\left(e^{-\rho(x-a^{\rho,m})}+\rho x-1\right)}{\rho^2} I - \frac{e^{-\rho(x-a^{\rho,m})}}{\rho} I\bar{\Lambda}U^{\rho}(a^{\rho,m}) + B^{\rho}\left(x-a^{\rho,m}\right) U^{\rho}\left(a^{\rho,m}\right) \\ &= \frac{V^*}{\rho} I + \frac{c}{2}\left(x-a^{0,m}\right)^2 I - \left(x-a^{0,m}\right) I\bar{\Lambda}U^0\left(a^{0,m}\right) + B^0\left(x-a^{0,m}\right) U^0\left(a^{0,m}\right) + o(\rho) I, \end{aligned} \tag{4.11}$$

where $V^* = ca^{0,m} - \bar{\Lambda}U^0\left(a^{0,m}\right)$. Thus, for $x \geq a^{\rho,m}$

$$W(x) = \frac{c}{2}\left(x-a^{0,m}\right)^2 I - \left(x-a^{0,m}\right) I\bar{\Lambda}U^0\left(a^{0,m}\right) + B^0\left(x-a^{0,m}\right) U^0\left(a^{0,m}\right). \tag{4.12}$$

If $0 \leq x \leq a^{0,m}$, then

$$W(x) = -\int_x^{a^{0,m}} U^0(y)dy + W\left(a^{0,m}\right),$$

where the vector-function $U^0(x)$ was constructed in **Section 4.1.2**. □

4.2 Proof of Theorem 3.2

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First we will prove statement a) of the theorem.

For $c = 0$, the proof that there exists $a^{0,(l)} = \lim_{\rho\to 0} a^{\rho,(l)}$ for $1 \le l < r$, and for $0 \le x \le a^{0,(r-1)} = \hat{a}_n^\rho$ there exists a limit $U^0(x) = \lim_{\rho\to 0} U^\rho(x)$ is not different from the corresponding proof for $c > 0$.

In order to prove that $\lim_{\rho\to 0} a_n^\rho = \infty$ we first prove by induction that $b_n^\rho(a^{\rho,(l)}) < 0$ for sufficiently small ρ for any $1 \le l < r$. For $l = 1$, $a^{\rho,(1)} = 0$ holds, and, according to (2.5), $b_n^\rho(0) = \rho P_n + \sum\limits_{j\neq n} \lambda_{n,j}(P_n - P_j)$. Since $P_n < P_j$ for any $j \neq n$, then for sufficiently small ρ this expression is negative. Thus, for $l = 1$ the induction hypothesis is satisfied. If $r = 2$, then everything is proved. Let $r > 2$ and we have proven that $b_n^\rho(a^{\rho,(l)}) < 0$ for some $1 \le l < r-1$ for sufficiently small ρ.

Let us rewrite (4.6) for $c = 0$ and (4.8) in the form

$$f_- = -BP_-, \tag{4.13}$$

$$\mathbf{0}_- = BI_- + \rho\left(I_- + A_\mp\left(-A_+\right)^{-1} I_+\right). \tag{4.14}$$

Multiplying (4.14) by P_n and adding with (4.13) we get:

$$f_- = \rho[I_- + A_\mp\left(-A_+\right)^{-1} I_+]P_n + B(P_nI_- - P_-). \tag{4.15}$$

We will show that the last coordinate of the vector $B(P_nI_- - P_-)$, which corresponds to the state of the chain with the number n, is negative. In fact, this last coordinate is equal to the product of the last row of the matrix B by a vector whose last element is zero, and all the others are negative. But, according to Lemma 4.2, for $\rho \ge 0$ all coordinates other than the last one of the last row of the matrix B are non-negative, and at least one is positive. This proves the negativity of the last coordinate of the vector $B(P_nI_- - P_-)$. From here and from (4.14) it follows that for sufficiently small ρ, the last coordinate of the vector f_- is also negative. Thus, in the interval $a^{0,(l)} \le x < \infty$, the function $f_n(x)$ increases from the value $b_n^\rho(a^{\rho,(l)}) < 0$ to a negative value set by formula (4.14), remaining negative. According to the algorithm, on the interval $a^{\rho,(l)} \le x \le a^{\rho,(l+1)}$ the equality $b_n^\rho(x) = f_n(x)$ holds, and therefore $b_n^\rho(a^{\rho,(l+1)}) < 0$, which completes the proof of the induction assumption.

Thus, for $x > \hat{a}_n^\rho$ and a sufficiently small ρ, the set $N_-^{(r-1)}$ consists of one element n, and due to Lemma 4.1 for such ρ the matrix $(A_{+n}^\rho)^{-1}$ exists, is continuous at $\rho = 0$ and all its elements are nonpositive.

We define the matrix $A(\rho)$, column vectors $G(\rho)$ and $H(\rho)$, and row vector $\Lambda_{(n)}$ as follows:

$$\begin{gathered} A(\rho) = (A_{+n}^\rho)^{-1}, \quad G(\rho) = A(\rho)\Lambda^{(n)}, \quad \Lambda_{(n)} = (\lambda_{n,1}, \ldots, \lambda_{n,n-1}), \\ H(\rho) = U_{+n}(\hat{a}_n^\rho) - P_nG(\rho) = A(\rho)\left[A_{+n}^\rho U(\hat{a}_n^\rho) - P_n\Lambda^{(n)}\right] = A(\rho)b_{+n}^\rho(\hat{a}_n^\rho) \le \mathbf{0}. \end{gathered} \tag{4.16}$$

It is evident, that $\Lambda_{(n)}I_{+n} = \lambda_n$, $G(\rho) = -I_{+n} - \rho A(\rho)I_{+n}$ (the last equality is obtained from the elementary equality $\Lambda^{(n)} = -\left[A_{+n}^\rho + \rho E_{+n}\right]I_{+n}$ by multiplying from the left by $A(\rho)$).

Using these notations and relations, we can rewrite (4.4) and (4.3) for $c = 0$ in the following form: for $x \ge 0$

$$\begin{aligned} F_{+n}^\rho(\hat{a}_n^\rho + x) &= P_nG(\rho) + e^{A_{+n}^\rho x}H(\rho) \\ &= -P_nI_{+n} + \rho P_nA(\rho)I_{+n} + A(\rho)e^{A_{+n}^\rho x}b_{+n}^\rho(\hat{a}_n^\rho) \end{aligned} \tag{4.17}$$

$$\begin{aligned} f_n^\rho(\hat{a}_n^\rho + x) &= P_n(\lambda_n + \rho) + \Lambda_{(n)}F_{+n}^\rho(\hat{a}_n^\rho + x) \\ &= \rho P_n\left(1 - \Lambda_{(n)}A(\rho)I_{+n}\right) + \Lambda_nA(\rho)e^{A_{+n}^\rho x}b_{+n}^\rho(\hat{a}_n^\rho). \end{aligned} \tag{4.18}$$

Note also that according to Remark 3, all coordinates of the vector $F^{\rho}_{+n}(\hat{a}^{\rho}_n + x)$ are strictly increasing.

The function $f^{\rho}_n(\hat{a}^{\rho}_n + x)$ is strictly increasing and according to Proposition 2.2 $f^{\rho}_n(\hat{a}^{\rho}_n) = b^{\rho}_n(\hat{a}^{\rho}_n) < 0$ and, according to (4.18) and due to Lemma 4.1 $f^{\rho}_{-} = \lim_{x\to\infty} f^{\rho}_n(x) > 0$. The threshold $a^{\rho,*}_n$ is determined from the condition $f^{\rho}_n(a^{\rho,*}_n) = 0$, which, according to (4.18), can be written as

$$\rho P_n \left(1 - \Lambda_n A(\rho) I_{+n}\right) = -\Lambda_n e^{A^{\rho}_{+n}(a^{\rho,*}_n - \hat{a}^{\rho}_n)} H(\rho). \tag{4.19}$$

It follows from here that $a^{\rho,*}_n \to \infty$ as $\rho \to 0$, therefore we got that for $x \geq a^{\rho,(r-1)}$ there exists a finite limit $F^0_{+n}(x) = \lim\limits_{\rho\to 0} F^{\rho}_{+n}(x)$. According to Proposition 2.2 for $a^{\rho,(r-1)} \leq x \leq a^{\rho}_n$ we have $F^{\rho}_{+n}(x) = U^{\rho}_{+n}(x),\ U^{\rho}_n(x) = -P_n$.

Thus, we proved that for $0 \leq x < \infty$ there exists a limit $U^0_{+n}(\hat{a}^0_n + x) = \lim\limits_{\rho\to 0} U^{\rho}_{+n}(\hat{a}^{\rho}_n + x)$, where

$$U^0_{+n}(x) = P_n G(0) + e^{\Lambda_{+n}(x)}(U^0_{+n}(\hat{a}^0_n) - P_n G(0)) = P_n I_+ - \rho A(\rho) I_+ + A(\rho) e^{\Lambda_{+n}(x)} b^{\rho}_{+n}(\hat{a}^{\rho}_n),$$

while $U^0_n(x) = -P_n$.

To complete the proof of statement a) of Theorem 3.2 it remains to study the limiting behavior of a^{ρ}_n as $\rho \to 0$. In Appendix **A3**, the following lemma will be proved, which completes the proof of statement a) of Theorem 3.2.

Lemma 4.3. a) *There exist numbers $g > 0$, $\mu > 0$, and integer number $m_0 \geq 0$ such that $\lim\limits_{\rho\to 0} \dfrac{a^{\rho,*}_n - \hat{a}^{\rho}_n}{a(\rho)} = \dfrac{1}{\mu}$, where $a(\rho)$ is the root of the equation $[a(\rho)]^{m_0} e^{-a(\rho)} = g\rho$.*

b) *There exists a finite limit $\lim\limits_{\rho\to 0} \dfrac{1}{\rho} e^{A^{\rho}_{+} a(\rho)}$.*

Let us proceed to the proof of statement b) of Theorem 3.2. For $c = 0$ and $0 \leq x < a^{\rho,m}$, expression (2.11) can be rewritten as

$$V^{\rho}(x) = -\int_x^{a^{\rho,*}_n} (U^{\rho}(y) + P_n I) dy + V^{\rho}(a^{\rho,*}_n) + (a^{\rho,*}_n - x)) P_n I \text{ for } 0 \leq x < a^{\rho,*}_n.$$

It follows from this expression that to prove the existence of $W(x)$, it suffices to prove the existence and finiteness of two limits

$$\lim_{\rho\to 0} \int_x^{a^{\rho,*}_n} (U^{\rho}(y) + P_n I) dy = \int_x^{\infty} (U^0(y) + P_n I) dy < \infty, \tag{4.20}$$

$$\lim_{\rho\to 0} \left[V^{\rho}(a^{\rho}_n) - \frac{1}{\rho} P_n I \right] < \infty\,. \tag{4.21}$$

For $i = n$, the integrands in both parts of (4.20) are zero, since $U^{\rho}_n(x) = -P_n$ for $x < a^{\rho}_n,\ \rho \geq 0$. For $i \neq n$, convergence and finiteness in (4.20) follows from the proven statement a) of Theorems 3.2, from the second equality in (4.17), and from the exponential decay of integrands which in turn follows from Lemma 4.1.

Let us move on to the proof of relation (4.21). From the second equality in (4.11) it follows that for $c = 0$

$$V^{\rho}(a^{\rho}_n) - \frac{P_n}{\rho} I = -I\bar{\Lambda} \frac{U^{\rho}(a^{\rho}_n) + P_n I}{\rho} + B^{\rho}(0) U^{\rho}(a^{\rho}_n).$$

Since $V_n^\rho(a_n^\rho) = P_n$ it follows from here that to prove (4.20), it suffices to prove the existence and finiteness of the limit

$$\lim_{\rho\to 0} \frac{U_{+n}^\rho(a_n^\rho) + P_n I_{+n}}{\rho} < \infty. \tag{4.22}$$

This follows from the second equality in (4.17), and from statement b) of Lemma 4.3.

To complete the proof of statement b) of Theorem 3.2, it remains to verify that $W(x)$ is an invariant function. The proofs of this fact and that the $a_i^{0,*}$-threshold control in state $i \neq n$ is optimal in the problem of minimization of the right hand side of (3.2) is similar to the proof of the corresponding facts for the case $\rho > 0$, which is given in Section 4.1.1. One only needs to substitute $V_n + \rho a_n^{\rho,*}$ instead of V^* in (4.2) and take into account that $\rho a_n^{\rho,*} \to 0$ as $\rho \to 0$.

Let us show that the y-threshold control is ε-optimal for the state $i = n$ in the problem (3.2). Indeed, let us first consider the case in which $\mathcal{T}$ is the moment of the first jump of the process $m(t)$. Then the application of y-threshold control in the state (x, n) with $y > x$ is reduced to a one-time purchase of $y - x$ and a subsequent purchase with intensity P_n up to the moment τ. Therefore, the difference between the value of the functional in (3.2), corresponding to the y-threshold control, and the optimal value (equal to $-xP_n$), is

$$\Delta_y = E_{x,n}\left[yP_n + G_{m(\tau)}(y)\right] = \sum_{j=1}^{n-1} \frac{\lambda_{n,j}}{\lambda_n}\left[yP_n + G_j(y)\right] = -\sum_{j=1}^{n-1} \frac{\lambda_{n,j}}{\lambda_n} \int_y^\infty [sP_n + U_j^0(s)ds, \tag{4.23}$$

where the last equality follows from (3.7). Now ε-optimality for sufficiently large y follows from the exponential convergence to zero of the integrands in (4.23), proved in statement b) of Theorem 3.2.

For an arbitrary τ with a finite mathematical expectation, it is necessary to split the interval from zero to τ into a random number of intervals between successive hits of the chain in state n. On each such interval, except for the last one, the increment of the functional value will be equal to Δ_y, and on the last one it will be only less. □

4.3 Proof of Theorem 3.3

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According to Remark 4, the value of V^* in (3.1) is uniquely determined. At the same time, if the coordinates of the vector-function $G(x)$ satisfy (3.1), then for any constant C the vector $G(x) + CI$ also satisfies relation (3.1). Therefore, without loss of generality, we will assume that $G(0)$ is the same as the value obtained from (3.4).

Let us write relation (3.1) for the moment τ of the first jump of the process $m(t)$

$$G_i(x) = \inf_{z\in\mathcal{Z}(x)} E_{x,i}\left\{\int_0^\tau cx^z(t)dt - V^*\tau + P_i z(\tau) + G_{m(\tau)}(x^z(\tau))\right\}, \tag{4.24}$$

where $\mathcal{Z}(x)$ is the set of admissible controls between jumps, i.e., the set of nondecreasing left continuous deterministic functions $z =: z(t)$ such that $z(0) = 0$ and $x^z(t) =: x - t + z(t) \geq 0$ for any $t > 0$.

Substituting here the expression $x^z(\tau) + \tau - x$ instead of $z(\tau)$ and taking into account that the moment τ of the first jump of the process $m(t)$ with the initial value i has exponential distribution with expectation equal to $\dfrac{1}{\lambda_i}$, we get

$$\begin{aligned} G_i(x) = \frac{P_i - V^*}{\lambda_i} - xP_i + \inf_{y\in\mathcal{A}(x)} \Bigg[&\int_0^\infty \lambda_i e^{-\lambda_i t}\left(\int_0^t cy(s)ds\right) dt + P_i \int_0^\infty \lambda_i e^{-\lambda_i t} y(t)dt \\ &+ \int_0^\infty e^{-\lambda_i t}\left(\sum_{j\neq i} \lambda_{i,j} G_j(y(t))\right) dt \Bigg], \end{aligned} \tag{4.25}$$

where $\mathcal{A}(x)$ is the set of all admissible trajectories between jumps, i.e. the set of deterministic functions $y =: y(t)$ such that $y(0) = x$, $y(t) \geq 0$ for any $t > 0$ and $z(t) =: y(t) - x + t$ is a non-decreasing left continuous function. Changing the order of all integration in the first term under the inf sign, we obtain

$$G_i(x) = \frac{P_i - V^*}{\lambda_i} - xP_i + \inf_{y \in \mathcal{A}(x)} \int_0^\infty e^{-\lambda_i t} \left(y(t)(c + \lambda_i P_i) + \sum_{j \neq i} \lambda_{i,j} G_j(y(t)) \right) dt\,. \tag{4.26}$$

Let $H_i(y) = y(c+\lambda_i P_i) + \sum_{j \neq i} \lambda_{i,j} W_j(y)$. The functions $W_j(y), i \in N$, are convex as limits of convex functions, and from (4.12) it follows that as $y \to \infty$ they tend to $+\infty$ (quadratically for $c > 0$ and linearly for $c = 0$). For each $i \in N$ the function $H_i(y)$ is convex, finite at zero, and tends to plus infinity at infinity, and hence reaches a minimum. It follows that if the function reaches a minimum at one point a_i^*, then the optimal control until the moment of the first jump of the Markov process is unique and is a threshold control with a threshold a_i^*, because any other admissible trajectory gives a greater value of the functional. If the minimum is reached on the interval $[a_i^*, \bar{a}_i]$, then the subset of quasi-threshold strategies for which on the interval $[a_i^*, \bar{a}_i]$ you can use any admissible control that does not go beyond this interval, and on the interval $[0, a_i^*)$ you need to make a one-time purchase in order to jump to any point in the interval $[a_i^*, \bar{a}_i]$.

It remains to prove that $a_i^* \neq \bar{a}_i$ implies that $a_i^* = 0$, $i \in N_0^{0,-}$, and investigate the properties of the solution to equation (3.5). This is done in exactly the same way as done in [14] and we will not dwell on it. □

5 Conclusion

In the paper there is considered the inventory problem, in which a manufacturer who needs to consume an intermediate product (goods) with a constant intensity for production buys this product at a price that depends on the value of a Markov process with continuous time, a finite number of states and known transition intensities. The case of discounted integral costs, consisting of purchase and storage costs, was considered in [14]. In this paper, we study the case with the long-run average cost functional. A passage to the limit is carried out with the discounting parameter tending to zero. As a result, an analogue of the canonical triple, known in the theory of controlled Markov chains with discrete time and a finite number of states, arises. It is shown that, as in the case of discounting, there is an optimal threshold strategy and an algorithm for constructing optimal thresholds is given.

Appendix

A1. Proof of Lemma 4.1. The matrix A_+ is the intensity matrix for a Markov chain with the killing (getting into a fictitious absorbing state) at the time of the first exit of the chain $\{m(t)\}_{0 \leq t < \infty}$ from the set $N_+^{(l)}$, and the exit moment is associated both with the entry of the original chain into a fictitious state (for $\rho > 0$) and with the entry of the original chain into states from $N_-^{(l)}$.

Let us show that the matrix $(-A_+)^{-1}$ exists and all its elements are non-negative. To do this, we represent the matrix $-A_+$ as $-A_+ = D\left(E_+ - \bar{A}_+\right)$, where D is a diagonal matrix with entries $d_{i,i} = \rho + \lambda_i = -a_{i,i}$, $i \in N_+^{(l)}$, and the matrix $\bar{A}_+$ has zeros on the diagonal, and for the remaining elements

$$\bar{a}_{i,j} = \frac{a_{i,j}}{(\rho + \lambda_i)} = \frac{\lambda_{i,j}}{\rho + \sum_{k \in N} \lambda_{i,k}}, \; i, j \in N_+^{(l)}, \, i \neq j.$$

The matrix $\bar{A}_+$ is the transition matrix for an embedded Markov chain (with discrete time) with respect to the chain with continuous time corresponding to A_+.

If $\rho > 0$, then for each row of the matrix $\bar{A}_+$ the sum of the elements is less than one, so there exists $(E_+ - \bar{A}_+)^{-1}$. If $\rho = 0$, then for some $i \in N_+^{(l)}$ it may turn out that the sum of the elements is equal to one. However, if the chain $m(t)$ is regular, then for each row the sum of the elements of the matrix $(\bar{A}_+)^n$ is less than one, because for a regular chain the probability of exiting the set $N_+^{(l)}$ after the nth jump is positive for any initial state $i \in N_+^{(l)}$. Therefore, in both cases, we have the representation

$$(-A_+)^{-1} = \left(E_+ - \bar{A}_+\right)^{-1} D^{-1} = \left(\sum_{k=0}^{\infty} \left(\bar{A}_+\right)^k\right) D^{-1}, \tag{A.1}$$

where the series converges. This representation implies the existence of the matrix $(-A_+)^{-1}$ and the non-negativity of all its elements, i.e., statement a) of Lemma 4.1.

If the intensity matrix of a killable Markov chain is such that the corresponding transition matrix of the imbedded chain has the property that for some power of this transition matrix the sum of the elements for all rows is less than one, then the real part of any eigenvalue of the initial intensity matrix is negative. □

A2. Proof of Lemma 4.2. $\Lambda I = 0$ implies $AI = -\rho I$. This ratio can be written as:

$$A_+ I_+ + A_\pm I_- = -\rho I_+ \tag{A.2}$$

$$A_- I_- + A_\mp I_+ = -\rho I_- \tag{A.3}$$

From (A.3) and (4.7) it follows that

$$\begin{aligned} BI_- &= A_- I_- + A_\mp \left(A_+\right)^{-1} A_\pm I_- = -\rho I_- - A_\mp I_+ + A_\mp \left(A_+\right)^{-1} A_\pm I_- \\ &= -\rho I_- + A_\mp \left(-A_+\right)^{-1} A_+ I_+ + A_\mp \left(A_+\right)^{-1} A_\pm I_- \\ &= -\rho I_- + A_\mp \left(-A_+\right)^{-1} \left(A_+ I_+ + A_\pm I_-\right). \end{aligned} \tag{A.4}$$

The matrix B is the transition matrix of the state-set $N_-^{(l)}$ Markov chain, which is obtained from the original chain (with killing for $\rho > 0$) by discarding time intervals when the original Markov chain is in states from $N_+^{(l)}$. The intensities of transitions in this chain increased compared to the intensities of the original chain, due to the exclusion of time intervals when the original circuit was in states from $N_-^{(l)}$. The first term in (A.2) corresponds to direct transitions inside $N_-^{(l)}$, and the second term corresponds to transitions after entering, staying, and leaving the set $N_+^{(l)}$, while the first factor of the second term corresponds to the transition from $N_-^{(l)}$ to $N_+^{(l)}$, the second factor corresponds to staying in $N_+^{(l)}$, and the third factor corresponds to returning from $N_+^{(l)}$ to $N_-^{(l)}$.

Both terms have off-diagonal elements that are non-negative. If for some $i \in N_-^{(l)}$ all elements of the string were equal to zero, then this would mean that from this state to go to other states from $N_-^{(l)}$ is impossible either directly or after visiting $N_+^{(l)}$, and this contradicts the regularity of the chain. □

A3. Limit properties of a_n^ρ and $e^{A(\rho)(a_n^\rho - \hat{a}_n^\rho)}$. It follows from Lemma 4.1 and (4.18) that

$$f_n^\rho(\hat{a}_n^\rho + x) = \rho P_n \left(1 - \Lambda_{(n)} A(\rho)\right) I_{+n} - q_\rho (\mu x)^{m_0} Q\left(\frac{1}{x}\right) e^{-(\mu+\rho)x} + e^{-(\mu+\rho+\bar{\mu})x} g(x), \tag{A.5}$$

where $-\mu$ is an eigenvalue of the matrix Λ_+ with the maximal real part (it is known that $\mu > 0$); m_0 is the maximal size among the Jordan blocks corresponding to $-\mu$; $Q(y)$ is a polynomial of degree m_0, where $Q(0) = 1$, $q_\rho > 0$ for $\rho \geq 0$; $\mu + \bar{\mu} < \mu_1$, where $-\mu_1$ is the real part of the second eigenvalue in absolute value; $g(x)$ is a bounded function. Recall that the right-hand side in (A.5) is a function that increases from some negative value to a positive value, since all elements of the matrix $A(\rho)$ are non-positive, and vanishes at the point $x = (a_n^\rho - \hat{a}_n^\rho)$.

Let $a(\rho) > m_0$ be the solution to the equation

$$[a(\rho)]^{m_0} e^{-a(\rho)} = \frac{\rho}{q_0} P_n \left(1 - \Lambda_{(n)} A(0)\right) I_{+n}. \tag{A.6}$$

From (A.5), (4.20) and the fact that $f_n^\rho(a_n^\rho) = 0$, statement a) of Lemma 4.3 follows. Statement b) follows from the Jordan representation of the matrix $\frac{1}{\rho} e^{A_+^\rho a(\rho)}$ in which in the limit all elements vanish, with the exception of the elements corresponding to $[a(\rho)]^{m_0} e^{-a(\rho)}$.

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