EURASIAN MATHEMATICAL JOURNAL
volume 16 , number 3, 2025

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Events

EURASIAN NATIONAL UNIVERSITY

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

EURASIAN MATHEMATICAL JOURNAL

VOLUME 16, NUMBER 3 2025

ISSN 2077-9879
9 772077 987003

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2025, Volume 16, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

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From time to time the EMJ publishes survey papers.

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The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Republic of Kazakhstan

The Moscow Editorial Office
The Patrice Lumumba Peoples' Friendship University of Russia
(RUDN University)
Room 473
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MIKHAIL L'VOVICH GOLDMAN

Doctor of physical and mathematical sciences, Professor Mikhail L'vovich Goldman passed away on July 5, 2025, at the age of 80 years.

Mikhail L'vovich was an internationally known expert in scienceand education. His fundamental scientific articles and text books in variousfields of the theory of functions of several variablesand functional analysis, the theory of approximation of functions, embedding theorems and harmonic analysis are a significant contribution to the development of mathematics.

Mikhail L'vovich was born on Aprill 13, 1945 in Moscow. In 1963, he graduated from School No. 128 in Moscow with a gold medal and entered the Physics Faculty of the Lomonosov Moscow State University. He graduated in 1969 and became a postgraduate student in the Mathematics Department. In 1972, he defended his PhD thesis "On integral representations and Fourier series of differentiable functions of several variables" under the supervision of Professor Ilyin Vladimir Aleksandrovich, and in 1988, his doctoral thesis "Study of spaces of differentiable functions of several variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor of Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an Assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he was a Professor of the Department of Theory of Functions and Differential Equations, then of the S.M. Nikol'skii Mathematical Institute at the Patrice Lumumba Peoples' Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman were: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators.Among the most important scientific achievements of M.L. Goldman, we note his research related to the optimal embedding of spaces with generalized smoothness, exact conditions for the convergence of spectral decompositions, descriptions of the integral and differential properties of generalized potentials of the Bessel and Riesz types, exact estimates for operators on cones, descriptions of optimal spaces for cones of functions with monotonicity properties.

M.L. Goldman has published more than 150 scientific articles in central mathematical journals. He is a laureate of the Moscow government competition, a laureate of the RUDN University Prize in Science and Innovation, and a laureate of the RUDN University Prize for supervision of postgraduate students. Under the supervision of Mikhail L'vovich 11 PhD theses were defended. His pupilss are actively involved in professional work at leading universities and research institutes in Russia, Kazakhstan, Ethiopia, Rwanda, Colombia, and Mongolia.

Mikhail L'vovich has repeatedly been a guest lecturer and guest professor at universities in Russia, Germany, Sweden, Great Britain, etc., and an invited speaker at many international conferences. Mikhail L'vovich was not only an excellent mathematician and teacher (he always spoke about mathematics and its teaching with great passion), but also a man of the highest culture and erudition, with a deep knowledge of history, literature and art, a very bright, kind and responsive person. This is how he will remain in the hearts of his family, friends, colleagues and students.

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Mikhail L'vovich Goldman.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 3 (2025), 09 – 19

ON SOME GEOMETRIC ASPECTS
OF EVOLUTION VARIATIONAL PROBLEMS

V.M. Filippov, V.M. Savchin

Communicated by V.I. Burenkov

Key words: Christoffel symbols, evolution equations, geodesics, dynamical systems.

AMS Mathematics Subject Classification: 34A55, 35A15, 58C40.

Abstract. The main objective of the work is to identify the relationship between evolution equations with potential operators and geometries of related configuration spaces of the given systems. Using the Hamilton principle, a wide class of such equations is derived. Their structural analysis is carried out, containing operator analogues of the Christoffel symbols of both the 1st and 2nd kind. It is shown that the study of the obtained evolution equations can be associated, in general, with an extended configuration space, the metric of which is determined by the kinetic energy of the given system.

DOI: https://doi.org/10.32523/2077-9879-2025-16-3-09-19

1 Introduction

One of the main properties of the metric tensor is that it completely defines the geometry of the space to which it belongs. The relationship of expressions for the metric tensor and the kinetic energy allows determining the components of the metric tensor of the configuration space of the system by the type of kinetic energy for the system and constructing its geometric model. The subject of the present paper lies between analytical mechanics, geometry and variational calculus. Tensor methods have long been applied in the dynamics of finite-dimensional systems [11]. They were initially aimed at using the ideas of Riemannian geometry in dynamics. In turn, the problems of mechanics contributed to the development of geometry. Significant results have been obtained over more than a hundred years (see, for example, [1, 4, 3, 7, 11, 12] and references therein). In particular, it was shown that the curvature of a manifold — an invariant distinguishing Riemannian metrics $a_{ij}\left(u^1,\dots,u^n\right)$, $i,j=\overline{1,n}$, — significantly affects the form of geodesics on it, i.e. the motion in the corresponding dynamical system [2].

Geodesics are lines $u^i=u^i\left(t\right)$, $t\in[t_0,t_1]$, $i=\overline{1,n}$, which are solutions to the equations

$$\frac{d^2u^j}{dt^2}+\Gamma^j_{ki}\frac{du^k}{dt}\frac{du^i}{dt}=0,\ j=\overline{1,n},$$

where Γ^j_{ki} are the Christoffel symbols of the second kind.

Here and below, summation is implied by repeating indices of factors located at different levels.

If the metric $a_{ij}\left(u^1,\dots,u^n\right)$ is non-degenerate (i.e. $\det\left(a_{ij}\right)\neq 0$), then

$$\Gamma^k_{ij}=\frac{1}{2}a^{kl}\left(\frac{\partial a_{lj}}{\partial u^i}+\frac{\partial a_{il}}{\partial u^j}-\frac{\partial a_{ij}}{\partial u^l}\right), \tag{1.1}$$

where $\left(a^{kl}\right)$ is the inverse matrix of the matrix $\left(a_{lk}\right)$.

The Christoffel symbols of the first kind are found through the components of the metric tensor by the formulas

$$\Gamma_{k,ij} = \frac{1}{2}\left(\frac{\partial a_{kj}}{\partial u^i} + \frac{\partial a_{ik}}{\partial u^j} - \frac{\partial a_{ji}}{\partial u^k}\right). \tag{1.2}$$

As noted in work [6], in problems of mechanics it is natural to choose as a Riemannian metric the metric that is determined by the kinetic energy of the system.

2 Statement of the problem. Geodesic equations

Let $U = C^2([t_0, t_1], U_1)$, $V = C([t_0, t_1], V_1)$, where U_1, V_1 are normed linear spaces over the field of real numbers $\mathbb{R}$, $U_1 \subseteq V_1$.

Let the state of an infinite-dimensional dynamical system be determined by a function $u \in U$, satisfying the conditions $u|_{t=t_0} = u_0$, $u|_{t=t_1} = u_1$, where u_0, u_1 are given elements from U_1. A curve u in U_1 is a mapping $u : [t_0, t_1] \to U_1$.

We will follow the notation and terminology of [4, 5].

Let be given a symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle : V_1 \times V_1 \to \mathbb{R}$ and the kinetic energy of the system

$$T[t, u, u_t] = \frac{1}{2}\langle u_t, A_u u_t\rangle + \langle u_t, B(t, u)\rangle + \langle u, C(t, u)\rangle,$$

where A_u is a linear Gâteaux differential operator, in general, depending nonlinearly on t and u; $u_t = \frac{du}{dt} = \lim\limits_{\Delta t \to 0} \frac{u(t+\Delta t) - u(t)}{\Delta t} \in U_1$. Operators B, C are differentiable with respect to t, and u in the sense of Gâteaux.

$A'_u(h; g) = \left(\frac{d}{d\varepsilon} A_{u+\varepsilon g} h\right)\Big|_{\varepsilon=0}$; $F[u] = \int\limits_{t_0}^{t_1} T[t, u, u_t]\, dt$, $u \in D(F) = \{u \in U : u|_{t=t_0} = u_0, u|_{t=t_1} = u_1\}$; the Gâteaux differential $\delta F[u, h] = \frac{d}{d\varepsilon} F[u + \varepsilon h]\big|_{\varepsilon=0}$. The construction of adjoint operators in the work is based on the Lagrange identity [8].

Definition 1. A function $u \in D(F)$ is called stationary for a functional F if $\delta F[u, h] = 0$ $\forall h \in D(F'_u)$.

Theorem 2.1. *The stationary function of the functional $F[u]$ is a solution to the operator equation*

$$\begin{aligned} N(u) \equiv \frac{1}{2}(A_u + A_u^*) u_{tt} + \frac{1}{2}\left[A'_u(u_t; u_t) + A_u^{*\prime}(u_t; u_t) - A_u^{\prime *}(u_t; u_t)\right] - \\ - \left(B_u^{\prime *} - B_u^{\prime}\right) u_t + \frac{1}{2}\left(\frac{\partial A_u}{\partial t} + \frac{\partial A_u^*}{\partial t}\right) u_t + \frac{\partial B}{\partial t} - C - C_u^{\prime *} u = 0, \end{aligned} \tag{2.1}$$

where $(\dots)^*$ is the operator adjoint to the operator $(\dots)$ with respect to the given bilinear form, $u_{tt} = \frac{d^2 u}{dt^2}$, $A_u^{\prime *}(u_t; u_t) = (A'_u(u_t; \cdot))^* u_t$.

Proof. For further use, we note that if the Gâteaux derivative N'_u of N exists, then [9]

$$N(u + \varepsilon h) = N(u) + \varepsilon N'_u h + r(u, \varepsilon h),\ u \in D(N), \tag{2.2}$$

where

$$\lim_{\varepsilon \to 0} \frac{r(u, \varepsilon h)}{\varepsilon} = 0.$$

Let us denote

$$F_1[u] = \frac{1}{2}\int\limits_{t_0}^{t_1} \langle u_t, A_u u_t\rangle \, dt,$$

$$F_2[u] = \int\limits_{t_0}^{t_1} \left[\langle u_t, B\left(t,u\right)\rangle + \langle u, C\left(t,u\right)\rangle\right] dt, \;\; u \in D\left(F\right) = D\left(F_1\right) = D\left(F_2\right).$$

Using equality (2.2), we obtain

$$\begin{aligned} F_1[u+\varepsilon h] &= \frac{1}{2}\int\limits_{t_0}^{t_1} \langle u_t + \varepsilon h_t, A_{u+\varepsilon h}\left(u_t+\varepsilon h_t\right)\rangle \, dt \\ &= \frac{1}{2}\int\limits_{t_0}^{t_1} \langle u_t + \varepsilon h_t, A_{u+\varepsilon h} u_t + A_{u+\varepsilon h}\varepsilon h_t\rangle \, dt \\ &= \frac{1}{2}\int\limits_{t_0}^{t_1} \langle u_t + \varepsilon h_t, A_u u_t + A'_u\left(u_t;\varepsilon h\right) + A_u \varepsilon h_t + A'_u\left(\varepsilon h_t;\varepsilon h\right) + r(u,\varepsilon h)\rangle \, dt. \end{aligned}$$

From here we find

$$\begin{aligned} \delta F_1[u,h] &= \frac{1}{2}\int\limits_{t_0}^{t_1} \left[\langle h_t, A_u u_t\rangle + \langle u_t, A'_u\left(u_t;h\right) + A_u h_t\rangle\right] dt \\ &= \frac{1}{2}\int\limits_{t_0}^{t_1} \left[D_t \langle h, A_u u_t\rangle - \langle h, D_t\left(A_u u_t\right)\rangle + \right. \\ &\left. + \left\langle A_u^{\prime *}\left(u_t;\cdot\right) u_t, h\right\rangle + \langle A_u^* u_t, h_t\rangle\right] dt \;\; \forall u \in D\left(F\right), \forall h \in D\left(F'_u\right), \end{aligned} \tag{2.3}$$

where D_t is a total derivative with respect to t.

Since

$$\begin{aligned} \langle A_u^* u_t, h_t\rangle &= D_t \langle A_u^* u_t, h\rangle - \langle D_t\left(A_u^* u_t\right), h\rangle \\ &= D_t \langle A_u^* u_t, h\rangle - \left\langle \frac{\partial A_u^*}{\partial t} u_t + A_u^{*'}\left(u_t;u_t\right) + A_u^* u_{tt}, h\right\rangle, \end{aligned}$$

then from (2.3) we get

$$\begin{aligned} \delta F_1[u,h] = \frac{1}{2}\langle\left(A_u + A_u^*\right) u_t, h\rangle \Big|_{t=t_0}^{t=t_1} + \frac{1}{2}\int\limits_{t_0}^{t_1} \Big[\Big\langle A_u^{\prime *}\left(u_t;\cdot\right) u_t - A_u^{*'}\left(u_t;u_t\right) - \\ - A'_u\left(u_t;u_t\right) - \left(A_u + A_u^*\right) u_{tt} - \left(\frac{\partial A_u}{\partial t} + \frac{\partial A_u^*}{\partial t}\right) u_t, h\Big\rangle\Big] \, dt. \end{aligned} \tag{2.4}$$

Taking into account that

$$h\big|_{t=t_0} = h\big|_{t=t_1} = 0,$$

from (2.4) we find

$$\delta F_1[u,h] = -\frac{1}{2}\int\limits_{t_0}^{t_1}\Big[\Big\langle (A_u + A_u^*)\, u_{tt} + A_u'(u_t;u_t) + A_u^{*'}(u_t;u_t) - A_u'^*(u_t;u_t) + \\ + \left(\frac{\partial A_u}{\partial t} + \frac{\partial A_u^*}{\partial t}\right) u_t, h\Big\rangle\Big]\, dt \;\forall u \in D(F), \forall h \in D(F_u').$$

Using equality (2.2), in a similar way we get

$$F_2[u+\varepsilon h] = \int\limits_{t_0}^{t_1} [\langle u_t + \varepsilon h_t, B(t, u+\varepsilon h)\rangle + \langle u+\varepsilon h, C(t, u+\varepsilon h)\rangle]\, dt,$$

$$\delta F_2[u,h] = \int\limits_{t_0}^{t_1} \left[\langle h_t, B(t,u)\rangle + \langle u_t, B'h\rangle + \langle h, C(t,u)\rangle + \left\langle u, C_u' h\right\rangle\right] dt.$$

From here we obtain

$$\delta F_2[u,h] = \int\limits_{t_0}^{t_1} \Big[D_t\langle h, B(t,u)\rangle - \langle h, D_t B(t,u)\rangle + \left\langle h, B_u'^* u_t\right\rangle + \\ + \langle h, C(t,u)\rangle + \left\langle h, C_u'^*\right\rangle\Big]\, dt = \\ = \langle h, B(t,u)\rangle\Big|_{t_0}^{t_1} + \int\limits_{t_0}^{t_1}\left\langle h, \left(B_u'^* - B_u'\right) u_t - \frac{\partial B}{\partial t} + C(t,u) + C_u'^* u\right\rangle dt. \tag{2.5}$$

Since $h\big|_{t=t_0} = h\big|_{t=t_1} = 0$, from (2.5) we find

$$\delta F_2[u,h] = \int\limits_{t_0}^{t_1}\left\langle h, \left(B_u'^* - B_u'\right) u_t - \frac{\partial B}{\partial t} + C(t,u) + C_u'^* u\right\rangle dt.$$

From the condition

$$\delta F[u,h] \equiv \delta F_1[u,h] + \delta F_2[u,h] = 0, u \in D(F), \forall h \in D(F_u')$$

we obtain operator equation (2.1). □

Corollary 2.1. *If $A_u^* = A_u$, then equation (2.1) takes the form*

$$A_u u_{tt} + \frac{1}{2}\left[A_u'(u_t;u_t) + A_u^{*'}(u_t;u_t) - A_u'^*(u_t;u_t)\right] - \\ - \left(B_u'^* - B_u'\right) u_t + \frac{\partial A_u}{\partial t} u_t + \frac{\partial B}{\partial t} - C - C_u'^* = 0. \tag{2.6}$$

Corollary 2.2. *If $A_u^* = A_u$, $B_u'^* = B_u'$, A_u and B are independent of t, $C = 0$ and there is an inverse operator A_u^{-1}, then equation (2.1) takes the form*

$$u_{tt} + \frac{1}{2}A_u^{-1}\left[A_u'(u_t;u_t) + A_u^{*'}(u_t;u_t) - A_u'^*(u_t;u_t)\right] = 0. \tag{2.7}$$

Consider a finite-dimensional system with coordinates $(u^1, \ldots, u^n)$, $u^i(t_0) = u_0^i$, $u^i(t_1) = u_1^i$, $t \in [t_0, t_1]$, $i = \overline{1, n}$, and the kinetic energy $T = \frac{1}{2}\dot{u}^i a_{ij}(u)\dot{u}^j$, where $(a_{ij})_{i,j=1}^n$ is a symmetric matrix, $\det(a_{ij})_{i,j=1}^n \neq 0$, $\dot{u}^i = \frac{du^i}{dt}$.

Theorem 2.2. *If $T = \frac{1}{2}\dot{u}^i a_{ij}(u)\dot{u}^j$, then equation (2.7) coincides with the geodesic equation*

$$\frac{d^2u^j}{dt^2} + \Gamma^j_{ik}\dot{u}^i\dot{u}^k = 0, \ j = \overline{1, n}, \tag{2.8}$$

where

$$\Gamma^j_{ik} = \frac{1}{2}a^{jl}\left(\frac{\partial a_{lk}}{\partial u^i} + \frac{\partial a_{il}}{\partial u^k} - \frac{\partial a_{ik}}{\partial u^l}\right)$$

are the Christoffel symbols.

Proof. In the case under consideration

$$\langle u_t, A_u u_t\rangle = \dot{u}^i a_{ij}(u)\dot{u}^j, F[u] = \frac{1}{2}\int\limits_{t_0}^{t_1} \dot{u}^i a_{ij}(u)\dot{u}^j dt.$$

We have

$$F[u + \varepsilon h] = \frac{1}{2}\int\limits_{t_0}^{t_1} \left(\dot{u}^i + \varepsilon\dot{h}^i\right)(a_{ij}(u + \varepsilon h))\left(\dot{u}^j + \varepsilon\dot{h}^j\right) dt.$$

From here we find

$$\delta F[u, h] = \frac{d}{d\varepsilon}F[u + \varepsilon h]\Big|_{\varepsilon=0} = \frac{1}{2}\int\limits_{t_0}^{t_1} \left[\dot{h}^i\dot{u}^j a_{ij}(u) + \dot{u}^i\dot{h}^j a_{ij}(u) + \dot{u}^i\dot{u}^j\frac{\partial a_{ij}(u)}{\partial u^k}h^k\right] dt.$$

Integrating by parts, we obtain

$$\delta F[u, h] = \frac{1}{2}\left[h^i\dot{u}^j a_{ij} + h^j\dot{u}^i a_{ij}\right]\Bigg|_{t=t_0}^{t=t_1} +$$
$$+ \frac{1}{2}\int\limits_{t_0}^{t_1}\left[\dot{u}^i\dot{u}^j\frac{\partial a_{ij}}{\partial u^k}h^k - h^i\left(\ddot{u}^j a_{ij} + \dot{u}^j\frac{\partial a_{ij}}{\partial u^k}\dot{u}^k\right) - h^j\left(\ddot{u}^i a_{ij} + \dot{u}^i\frac{\partial a_{ij}}{\partial u^k}\dot{u}^k\right)\right] dt.$$

Since $h^i(t_0) = h^i(t_1) = 0$, $i = \overline{1, n}$, then changing the summation indices in the terms under the integral sign, we find

$$\delta F[u, h] = \frac{1}{2}\int\limits_{t_0}^{t_1}\left[-h^k\ddot{u}^j(a_{kj} + a_{jk}) + h^k\dot{u}^i\dot{u}^j\left(\frac{\partial a_{ij}}{\partial u^k} - \frac{\partial a_{kj}}{\partial u^i} - \frac{\partial a_{ik}}{\partial u^j}\right)\right] dt.$$

Taking into account the symmetry of the matrix $(a_{ij})_{i,j=1}^n$, we arrive at the equality

$$\delta F[u, h] = -\int\limits_{t_0}^{t_1} h^k\left[a_{kj}\ddot{u}^j + \frac{1}{2}\left(\frac{\partial a_{kj}}{\partial u^i} + \frac{\partial a_{ik}}{\partial u^j} - \frac{\partial a_{ij}}{\partial u^k}\right)\dot{u}^i\dot{u}^j\right] dt.$$

From the condition $\delta F[u, h] = 0$, $u \in D(F)$, $\forall h \in D(F'_u)$ we conclude that u is a solution to the system of equations

$$a_{kj}\ddot{u}^j + \Gamma_{k,ij}\dot{u}^i\dot{u}^j = 0, \tag{2.9}$$

where $\Gamma_{k,ij}$ are the Christoffel symbols of the first kind (1.2).

Since $\det (a_{ij})_{i,j=1}^{n} \neq 0$, system of equations (2.9) can be solved with respect to $\ddot{u}^j (j = \overline{1,n})$. As a result, we arrive at system of equations (2.8).

Thus, equations of geodesics (2.8) are obtained. □

In the absence of forces, the motion of a system with kinetic energy $\frac{1}{2}\langle u_t, A_u u_t\rangle$ can be interpreted as motion in U by inertia with the metric

$$ds^2 = \langle u_t, A_u u_t\rangle \, dt^2.$$

Borrowing terminology from mechanics, for such a motion the trajectories are called geodesic lines with respect to indicated metric. Thus, the problem of inertial motion is reduced to finding geodesic lines. Operator equation (2.6) expresses a far-reaching generalization of this fact.

Corollary 2.3. [10] *Equation (2.7) is an operator analogue of geodesic equations (2.8), while the operator*

$$K_{1u}[\cdot] = \frac{1}{2}\left[A'_u(\cdot;\cdot) + A^{*'}_u(\cdot;\cdot) - A'^{*}_u(\cdot;\cdot)\right] \tag{2.10}$$

defines an analogue of the Christoffel symbols of the first kind $\Gamma_{k,ij}$, and

$$K_{2u}[\cdot] = A_u^{-1} K_{1u}[\cdot] \tag{2.11}$$

is an analogue of the Christoffel symbols of the second kind Γ_{ij}^{k}.

The operator $\frac{D}{dt}$, defined by the formula

$$\frac{Du_t}{dt} = u_{tt} + A_u^{-1} K_{1u}[u_t],$$

is an analogue of the covariant derivative of u_t with respect to t.

The above analogues are of particular interest in terms of their relationship with Riemannian geometry, as well as the geometry defined by the pseudo-Riemannian metric.

Using now operators (2.10), (2.11), we get the following.

Corollary 2.4. *If $A_u^* = A_u$ and there exists the inverse operator A_u^{-1}, then evolution equation (2.1) can be represented in the form*

$$N_1(u) \equiv u_{tt} + K_{2u}[u_t] + A_u^{-1}\left[\frac{\partial A_u}{\partial t}u_t - \left(B'^{*} - B'_u\right)u_t + \frac{\partial B}{\partial t} - C - C'^{*}_u u\right] = 0, \qquad u \in D(N) = D(F). \tag{2.12}$$

It is an interesting problem to interpret this operator evolution equation in terms of rheonomic geometry with the metric

$$ds^2 = \frac{1}{2}\langle u_t, A_u u_t\rangle \, dt^2 + \langle u_t, B(t,u)\rangle \, dt^2 + \langle u, C(t,u)\rangle \, dt^2,$$

associated with the given kinetic energy $T[t, u, u_t]$.

3 Evolution equation and relative integral invariant

Let us establish the connection between evolution equation (2.1) and an relative integral invariant of the first order.

Let

$$u = u(\lambda; t), \ \ \lambda \in \Lambda = [0, 1] \tag{3.1}$$

be an arbitrary one-parameter set of elements from U continuously differentiable with respect to λ. It can be considered as a curve γ in U. We assume that $u(0; t) = u(1; t)$, i.e. γ is a closed curve.

Let us introduce the notation

$$\delta u = \frac{\partial u(\lambda; t)}{\partial \lambda} d\lambda.$$

Let us consider the functional

$$F[u(\lambda; t)] = \int_{\tau_0}^{\tau_1} T[t, u(\lambda; t), u_t(\lambda; t)]\, dt,$$

where $[\tau_0, \tau_1]$ is an arbitrary segment from $[t_0, t_1]$.

We have

$$\begin{aligned} \delta F &= \frac{\partial F[u(\lambda; t)]}{\partial \lambda} d\lambda = \int_{\tau_0}^{\tau_1} \frac{\partial T}{\partial \lambda} d\lambda dt = \int_{\tau_0}^{\tau_1} \delta T dt = \\ &= \frac{1}{2} \int_{\tau_0}^{\tau_1} [\langle \delta u_t, A_u u_t \rangle + \langle u_t, A'_u(u_t; \delta u) + A_u \delta u_t \rangle + \langle \delta u_t, B(t, u) \rangle + \\ &\qquad + \langle u_t, B'_u \delta u \rangle + \langle \delta u, C(t, u) \rangle + \langle u, C'_u \delta u \rangle]\, dt. \end{aligned} \tag{3.2}$$

Since $\delta u_t = \frac{d}{dt} \delta u$, from (3.2) we get

$$\begin{aligned} \delta F = \int_{\tau_0}^{\tau_1} \Big\{ \frac{1}{2} \Big[& D_t \langle \delta u, A_u u_t \rangle - \langle \delta u, D_t(A_u u_t) \rangle + \Big\langle A_u'^{*}(u_t; \cdot)\, u_t, \delta u \Big\rangle + \\ & + \langle A_u^* u_t, \delta u_t \rangle \Big] + D_t \langle \delta u, B(t, u) \rangle - \langle \delta u, D_t B(t, u) \rangle + \\ & + \Big\langle B_u'^{*} u_t, \delta u \Big\rangle + \langle \delta u, C(t, u) \rangle + \Big\langle C_u'^{*} u, \delta u \Big\rangle \Big\}\, dt. \end{aligned} \tag{3.3}$$

Bearing in mind that

$$\begin{aligned} D_t(A_u u_t) &= \frac{\partial A_u}{\partial t} u_t + A'_u(u_t; u_t) + A_u u_{tt}, \\ \langle A_u^* u_t, \delta u_t \rangle &= D_t \langle A_u^* u_t, \delta u \rangle - \langle D_t(A_u^* u_t), \delta u \rangle \\ &= D_t \langle A_u^* u_t, \delta u \rangle - \left\langle \frac{\partial A_u^*}{\partial t} + A_u^{*'}(u_t; u_t) + A_u^* u_{tt}, \delta u \right\rangle, \\ D_t B(t, u) &= \frac{\partial B}{\partial t} + B'_u u_t, \end{aligned}$$

from (3.3) we obtain

$$\begin{aligned} \delta F = &\left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B(t,u), \delta u \right\rangle \Bigg|_{t=\tau_0}^{t=\tau_1} - \\ &- \int\limits_{\tau_0}^{\tau_1} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_{tt} + \frac{1}{2}\left[A_u'\left(u_t; u_t\right) + A_u^{*'}\left(u_t; u_t\right) - A_u'^*\left(u_t; \cdot\right) u_t\right] + \right. \\ &\left. + \left(B_u' - B_u'^*\right) u_t + \frac{1}{2}\left(\frac{\partial A_u}{\partial t} + \frac{\partial A_u^*}{\partial t}\right) u_t + \frac{\partial B}{\partial t} - C - C_u'^* u, \delta u \right\rangle = \\ &= \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B(t,u), \delta u \right\rangle \Bigg|_{t=\tau_0}^{t=\tau_1} - \int\limits_{\tau_0}^{\tau_1} \langle N(u), \delta u \rangle \, dt. \end{aligned} \tag{3.4}$$

Along the real trajectories, the solutions to evolution equation (2.1), we have

$$\delta F = \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B(t,u), \delta u \right\rangle \Bigg|_{t=\tau_0}^{t=\tau_1}.$$

Integrating this equality termwise with respect to λ from $\lambda = 0$ to $\lambda = 1$, we obtain

$$\begin{aligned} 0 = F\left[u(1;t)\right] - F\left[u(0;t)\right] &= \int\limits_0^1 \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B(t,u), \delta u \right\rangle \Bigg|_{t=\tau_0}^{t=\tau_1} = \\ = \int\limits_0^1 \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle \Bigg|_{t=\tau_1} &- \int\limits_0^1 \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle \Bigg|_{t=\tau_0} = \\ = \oint\limits_{\gamma_1} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle &- \oint\limits_{\gamma_0} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle, \end{aligned}$$

i.e.

$$\oint\limits_{\gamma_1} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle = \oint\limits_{\gamma_0} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle,$$

where γ_0, γ_1 are arbitrary closed curves, embracing the tube of trajectories.

Thus we proved the following.

Theorem 3.1. *Equation (2.1) has the relative integral invariant*

$$I = \oint \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle.$$

4 Example

Let us denote $U = C^2\left(\left[t_0, t_1\right], U_1\right)$, $V = C\left(\left[t_0, t_1\right], V_1\right)$. Let Ω be a bounded domain in R^3 with piecewise smooth boundary $\partial\Omega$, $U_1 = C^4\left(\overline{\Omega}\right)$, $V_1 = C\left(\overline{\Omega}\right)$, $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2}$ the Laplace operator, $x = (x^1, x^2, x^3)$. Let $A_u = \Delta^2 + \alpha u + \beta u^2$, $\Delta^2 = \Delta\Delta$, where $\alpha, \beta \in C^1\left[t_0, t_1\right]$. We will assume that the domain of definition $D(A_u)$ of the operator A_u consists of all those functions $u \in U$ that satisfy the conditions

$$\begin{aligned} &u|_{t=t_0} = u_0, u|_{t=t_1} = u_1,\\ &u|_\Gamma = \psi(t,x),\\ &\left.\frac{\partial u}{\partial n}\right|_\Gamma = \varphi(t,x), \end{aligned}$$

where $\Gamma = [t_0, t_1] \times \partial\Omega$, $u_i \in C^4\left(\overline{\Omega}\right)$ $(i = 0, 1)$, $\psi, \varphi \in C(\Gamma)$.

Let us define the bilinear form

$$\langle v, g\rangle = \int\limits_\Omega v(t,x)g(t,x)dx.$$

Let us define

$$T = \frac{1}{2}\langle u_t, A_u u_t\rangle,$$

which we will interpret as the kinetic energy of some system.

We will find the form of equation (2.1) for this case.

For this purpose we obtain

$$\begin{aligned} &A_u v = \Delta^2 v + \alpha u v + \beta u^2 v,\\ &A_{u+\varepsilon h} v = \Delta^2 v + \alpha v\,(u+\varepsilon h) + \beta v\,(u+\varepsilon h)^2,\\ &A'_u(v;h) = \frac{d}{d\varepsilon} A_{u+\varepsilon h} v\Big|_{\varepsilon=0} = \alpha v h + 2\beta v u h = (\alpha v + 2\beta v u)\,h. \end{aligned}$$

Let us find A^*_u.

We have

$$\int\limits_{t_0}^{t_1}\int\limits_\Omega h\cdot A_u g dxdt = \int\limits_{t_0}^{t_1}\int\limits_\Omega h\left(\Delta^2 g + \alpha u g + \beta u^2 g\right)dxdt =$$

$$= \int\limits_{t_0}^{t_1}\int\limits_\Omega g\left(\Delta^2 h + \alpha u h + \beta u^2 h\right)dxdt = \int\limits_{t_0}^{t_1}\int\limits_\Omega g\cdot A_u h dxdt \;\forall u \in D(A_u), \forall g, h \in D(A'_u).$$

Thus,

$$A^*_u = A_u \;\forall u \in D(A_u).$$

Next, we get

$$A'^*_u(v;\cdot)h = (\alpha v + 2\beta u v)h, \frac{\partial A_u}{\partial t} = \alpha_t u + \beta_t u^2.$$

According to formula (2.10), we find

$$\begin{aligned} K_{1u}[u_t] &= \frac{1}{2}\left[\alpha u_t + 2\beta u u_t\right]u_t + \frac{1}{2}\left[\alpha u_t + 2\beta u u_t\right]u_t - \frac{1}{2}\left[\alpha u_t + 2\beta u u_t\right]u_t =\\ &= \frac{1}{2}\left[\alpha + 2\beta u\right]u_t^2. \end{aligned}$$

Thus, in the case under consideration, equation (2.1) takes the form

$$\left(\Delta^2 + \alpha u + \beta u^2\right)u_{tt} + \frac{1}{2}(\alpha + 2\beta u)\,u_t^2 + \alpha_t u + \beta_t u^2 = 0.$$

It has the following relative integral invariant

$$\oint\int\limits_\Omega \left(\Delta^2 + \alpha u + \beta u^2\right)u_t\,dx\,\delta u.$$

5 Conclusion

In the work there is identified the relationship between evolution equations with potential operators and geometries of related configuration spaces of the given systems. Using the Hamilton principle, a wide class of such equations is derived. Their structural analysis is carried out, containing operator analogues of the Christoffel symbols of both the 1st and 2nd kind. It is shown that the study of the obtained evolution system can be associated, in general, with an extended configuration space, the metric of which is determined by the kinetic energy of the given system. It is shown that the obtained evolution operator equation has a relative integral invariant.

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Vladimir Mikhailovich Filippov, Vladimir Mikhailovich Savchin
RUDN University
6 Miklukho-Maklay St,
117198 Moscow, Russian Federation
E-mails: filippov-vm@rudn.ru, savchin-vm@rudn.ru

Received: 09.12.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 3 (2025), 20 – 41

OSCILLATORY AND SPECTRAL ANALYSIS OF HIGHER-ORDER DIFFERENTIAL OPERATORS

A. Kalybay, R. Oinarov

Communicated by V.I. Burenkov

Key words: higher-order differential equation, differential operator, oscillation, non-oscillation, spectrum, variational method, weighted inequality.

AMS Mathematics Subject Classification: 34C10, 34K08, 26D10.

Abstract. In the paper there are investigated the oscillatory properties of a $2n$th order differential equation and the spectral properties of a $2n$th order differential operator. These properties are established using the variational method, which relies on verifying a specific nth order differential inequality. Here, the coefficients of both the equation and the operator are the weights in this inequality. Furthermore, the characterization of the inequality occurs when the weights satisfy conditions, ensuring the existence of a certain combination of boundary values at infinity and at zero for the function involved in this inequality.

DOI: https://doi.org/10.32523/2077-9879-2025-16-3-20-41

1 Introduction

Let $I = (0, \infty)$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\lambda > 0$, and $n > 1$ be an integer. Let u be a positive function continuous on the interval I. Suppose that v is a positive function infinitely differentiable on I.

Let $W^n_{p,v} \equiv W^n_{p,v}(I)$ represent the space of functions $f : I \to \mathbb{R}$ possessing weak derivatives up to the nth order on the interval I, satisfying $\|f^{(n)}\|_{p,v} < \infty$, where $\|f\|_{p,v} = \left(\int\limits_0^\infty v(t)|f(t)|^p dt\right)^{\frac{1}{p}}$ denotes the norm of the weighted space $L_{p,v}(I)$. Under certain conditions on the function v, we observe that $C_0^\infty(I) \subset W^n_{p,v}(I)$, where $C_0^\infty(I)$ denotes the set of all functions infinitely differentiable and compactly supported on I. Let $\mathring{W}^n_{p,v} \equiv \mathring{W}^n_{p,v}(I)$ denote the closure of the set $C_0^\infty(I)$ with respect to the norm $\|f^{(n)}\|_{p,v}$.

In the paper there are discussed oscillatory properties of the following $2n$th order differential equation

$$(-1)^n (v(t) y^{(n)}(t))^{(n)} - \lambda u(t) y(t) = 0, \quad t \in I, \tag{1.1}$$

and spectral properties of the self-adjoint differential operator L generated by the differential expression

$$ly(t) = (-1)^n \frac{1}{u(t)} (v(t) y^{(n)}(t))^{(n)}, \tag{1.2}$$

in the space $L_{2,u}(I)$ equipped with the inner product $(f, g)_{2,u} = \int\limits_0^\infty f(t) g(t) u(t) dt$.

In the qualitative analysis of differential equations, there exist effective techniques for determining the oscillatory behavior of second-order equations of the form:

$$(v(t)y'(t))' - u(t)y(t) = 0,\ t \in I.$$

However, extending these methods to higher-order equations poses challenges. Recent studies have explored approaches suited for higher-order equations, often by selecting one of the coefficients to be a power function (see, e.g., [2], [3], [20], and [21]). In this paper, we employ the variational method. This method relies on establishing a connection between the oscillatory properties of equation (1.1) and characterizations of the following inequality:

$$\left(\int_0^\infty u(t)|f(t)|^q dt\right)^{\frac{1}{q}} \le C \left(\int_0^\infty v(t)|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}, \quad f \in \mathring{W}^n_{p,v}(I). \tag{1.3}$$

This approach allows us to relax the requirement that the weights in the equation must be power functions exclusively. Furthermore, we derive explicit conditions for oscillation and spectral properties in terms of the coefficients u and v of equation (1.1) and the operator L. Inequality (1.3) is a generalization of the famous Hardy inequality, which has a long-standing history (see, e.g., [9]). Its different extensions and applications have evolved into an independent area known as the "theory of Hardy-type inequalities" with numerous papers being published annually (see, e.g., the most recent works [12], [17], and [22]).

The investigation of inequality (1.3) hinges on the behavior of the function v at the endpoints of the interval I. According to [10] and [16], if $v^{1-p'} \notin L_1(1,\infty)$, then there exists $f \in W^n_{p,v}$ such that the limits $\lim\limits_{t\to\infty} f^{(i)}(t)$ do not exist for all $i = 0, 1, ..., n-1$; if $v^{1-p'} \in L_1(0,1)$, then for any $f \in W^n_{p,v}$ the limits $\lim\limits_{t\to 0^+} f^{(i)}(t) \equiv f^{(i)}(0)$ exist for all $i = 0, 1, ..., n-1$. The oscillation of equation (1.1) under the conditions $v^{1-p'} \notin L_1(1,\infty)$ and $v^{1-p'} \in L_1(0,1)$ was investigated in [14] using the variational method, as will be done here. This case can be termed the "standard case", since for the nth order inequality (1.3), there exist precisely n boundary conditions at the endpoints of the interval I, namely no conditions at infinity and n finite limits at zero. The spectral properties of the operator L in this "standard case" were examined in paper [18].

From [10] and [16] it also follows that if $v^{1-p'} \in L_1(1,\infty)$ and $t^{p'}v^{1-p'} \notin L_1(1,\infty)$, then for any $f \in W^n_{p,v}$ there exists exactly one limit $\lim\limits_{t\to\infty} f^{(n-1)}(t) \equiv f^{(n-1)}(\infty)$. Therefore, together with the above condition for v at zero $v^{1-p'} \in L_1(0,1)$, they entail $n+1$ conditions at the endpoints:

$$f^{(i)}(0) = 0,\ i = 0, 1, ..., n-1, \quad \text{and} \quad f^{(n-1)}(\infty) = 0.$$

This "overdetermined" case was studied in work [7].

In our study, we explore equation (1.1) and the operator L under the conditions:

$$t^{p'(n-1)}v^{1-p'} \in L_1(1,\infty),\ t^{p'(n-2)}v^{1-p'} \notin L_1(0,1),\ \text{and}\ t^{p'(n-1)}v^{1-p'} \in L_1(0,1), \tag{1.4}$$

which, according to [10] and [16], guaranty the existence of another $n+1$ values $\lim\limits_{t\to 0^+} f(t) \equiv f(0)$ and $\lim\limits_{t\to\infty} f^{(i)}(t) \equiv f^{(i)}(\infty)$, $i = 0, 1, ..., n-1$, at the endpoints of the interval I, so that

$$\mathring{W}^n_{p,v}(I) = \{f \in W^n_{p,v}(I) : f(0) = 0 \ \text{ and } \ f^{(i)}(\infty) = 0,\ i = 0, 1, ..., n-1\}. \tag{1.5}$$

Note that the same problems as here, but specifically for $n = 2$, where the differential equation and operator are of fourth-order, were considered in the paper [15]. Consequently, this paper expands its scope to include the problems for any $n \ge 2$.

The paper is organized as follows. In Section 2, we present all the main results regarding the oscillatory properties of equation (1.1) and the spectral properties of the operator L. Additionally, Section 2 encompasses the characterizations of inequality (1.3). Section 3 offers a proof concerning inequality (1.3). In Section 4, we compile the proofs of the main results concerning equation (1.1) and the operator L. In Section 5, we improve some results obtained earlier.

Let us present notations used in the paper. Assume that $\overline{v}(t) = \frac{v(t)}{t^{p(n-1)}}$, $t \in I$. Since

$$t^{p'(n-1)}v^{1-p'} = t^{p'(n-1)}\overline{v}^{1-p'}t^{p(n-1)(1-p')} = \overline{v}^{1-p'}t^{(n-1)(p'+p-pp')} = \overline{v}^{1-p'},$$

from (1.4) we have that $\overline{v}^{1-p'} \in L_1(I)$. Therefore, for any $\tau \in I$ there exists k_τ such that

$$\int_0^\tau \overline{v}^{1-p'}(t)dt = k_\tau \int_\tau^\infty \overline{v}^{1-p'}(t)dt, \tag{1.6}$$

in addition, k_τ increases in τ, $\lim\limits_{\tau\to 0^+} k_\tau = 0$, and $\lim\limits_{\tau\to\infty} k_\tau = \infty$.

The symbol $A \ll B$ means $A \le CB$ with some constant C. Additionally, we define χ_M as the characteristic function of a set M.

2 Oscillatory properties of equation (1.1) and spectral properties of the operator L

Equation (1.1) is termed oscillatory at zero if, for any $T > 0$, there exists a (non-trivial) solution of this equation possessing more than one zero with multiplicity n to the left of T ([4, p. 69]). Otherwise, equation (1.1) is termed non-oscillatory at zero.

Equation (1.1) is termed strongly oscillatory or non-oscillatory at zero if it is oscillatory or non-oscillatory at zero for all values $\lambda > 0$, respectively.

The oscillatory properties of differential equation (1.1) can be established using the variational method, relying on the following well-known statement.

Lemma A. *Equation (1.1) is non-oscillatory at zero if and only if there exists $T > 0$ such that*

$$\int_0^T \left(v(t)|f^{(n)}(t)|^2 - \lambda u(t)|f(t)|^2\right) dt \ge 0, \quad f \in \overset{\circ}{W}{}^n_{2,v}(0,T).$$

It is obvious that Lemma A can be reformulated as follows.

Lemma 2.1. *(i) Equation (1.1) is non-oscillatory at zero if and only if there exists $T > 0$ and $C_T > 0$, depending only on T, such that the inequality*

$$\int_0^T \lambda u(t)|f(t)|^2 dt \le \lambda C_T \int_0^T v(t)|f^{(n)}(t)|^2 dt, \quad f \in \overset{\circ}{W}{}^n_{2,v}(0,T), \tag{2.1}$$

holds with the least constant λC_T such that $0 < \lambda C_T \le 1$;
(ii) Equation (1.1) is oscillatory at zero if and only if for any $T > 0$ the least constant in (2.1) is such that $\lambda C_T > 1$.

Inequality (2.1) is a particular case of inequality (1.3), the characterizations of which are provided in the following theorem.

Theorem 2.1. *Let* $1 < p \leq q < \infty$ *and* (1.4) *hold. For* $\tau \in I$ *suppose that*

$$B_1(\tau) = \sup_{z>\tau} \left(\int_\tau^z u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty (s-z)^{p'(n-1)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B_2(\tau) = \sup_{z>\tau} \left(\int_\tau^z (z-t)^{q(n-1)} u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B_3(\tau) = \frac{1}{\tau} \left(\int_0^\tau t^q u(t)dt \right)^{\frac{1}{q}} \left(\int_\tau^\infty (s-\tau)^{p'(n-1)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B_4(\tau) = \frac{1}{\tau} \left(\int_0^\tau t^q (\tau-t)^{q(n-2)} u(t)dt \right)^{\frac{1}{q}} \left(\int_\tau^\infty (s-\tau)^{p'} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$F_1(\tau) = \sup_{0<z<\tau} \frac{1}{\tau^{n-1}} \left(\int_0^z t^q (\tau-t)^{q(n-2)} u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\tau (\tau-s)^{p'} s^{p'(n-2)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$F_2(\tau) = \sup_{0<z<\tau} \frac{1}{\tau^{n-1}} \left(\int_z^\tau (\tau-t)^{q(n-1)} u(t)dt \right)^{\frac{1}{q}} \left(\int_0^z s^{p'(n-1)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B(\tau) = \max\{B_1(\tau), B_2(\tau), B_3(\tau), B_4(\tau)\}, \quad F(\tau) = \max\{F_1(\tau), F_2(\tau)\},$$

$$BF = \inf_{\tau \in I} \max\{B(\tau), F(\tau)\},$$

$$\varepsilon_l(n) = \frac{4^{-\frac{1}{p}}}{(n-1)!}, \quad \varepsilon_r(n) = \frac{1}{(n-1)!} \left((n-1)2^{n-2} + (n+8)p^{\frac{1}{q}} (p')^{\frac{1}{p'}} \right).$$

Then for the least constant C *in* (1.3) *the estimates*

$$\varepsilon_l(n) BF \leq C \leq \varepsilon_r(n) BF, \tag{2.2}$$

$$\frac{1}{(n-1)!} \sup_{\tau \in I} (1 + k_\tau^{p-1})^{-\frac{1}{p}} F(\tau) \leq C \leq \varepsilon_r(n) F(\tau_0) \tag{2.3}$$

hold, where

$$\tau_0 = \inf\{\tau > 0 : B(\tau) \leq F(\tau)\}. \tag{2.4}$$

By following the same steps as in the proof of Lemma 4.3 in [7], using Lemma 2.1, we can deduce the following statement.

Lemma 2.2. *Let* C_T *be the least constant in* (2.1).
(i) Equation (1.1) *is strongly non-oscillatory at zero if and only if* $\lim_{T \to 0^+} C_T = 0$.
(ii) Equation (1.1) *is strongly oscillatory at zero if and only if* $C_T = \infty$ *for any* $T > 0$.

Based on Lemma 2.2 and Theorem 2.1, we establish the criteria for strong oscillation and non-oscillation of equation (1.1) as follows:

Theorem 2.2. *Let $t^{2(n-1)}v^{-1} \in L_1(I)$ and $t^{2(n-2)}v^{-1} \notin L_1(0,1)$.*
(i) Equation (1.1) is strongly non-oscillatory at zero if and only if

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_0^z t^2 u(t)dt \int_z^\tau s^{2(n-2)}v^{-1}(s)ds = 0, \tag{2.5}$$

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_z^\tau u(t)dt \int_0^z s^{2(n-1)}v^{-1}(s)ds = 0. \tag{2.6}$$

(ii) Equation (1.1) is strongly oscillatory at zero if and only if

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_0^z t^2 u(t)dt \int_z^\tau s^{2(n-2)}v^{-1}(s)ds = \infty \tag{2.7}$$

or

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_z^\tau u(t)dt \int_0^z s^{2(n-1)}v^{-1}(s)ds = \infty. \tag{2.8}$$

Let the minimal differential operator $L_{\min}$ be generated by differential expression (1.2), i.e., $L_{\min}y = ly$ is an operator with the domain $D(L_{\min}) = C_0^\infty(I)$. It is known that all self-adjoint extensions of the minimal differential operator $L_{\min}$ have the same spectrum ([4]).

Now, we present conditions under which any self-adjoint extension L of the operator $L_{\min}$ has a spectrum which is discrete and bounded below. The significance of studying these spectral properties is fully elucidated in [5].

The relationship between the non-oscillation of equation (1.1) and the above spectral properties of the operator L is expounded in the following statement ([4]).

Lemma B. *The operator L is bounded below and has a discrete spectrum if and only if equation (1.1) is strongly non-oscillatory.*

On the basis of Lemma B and Theorem 2.2, we obtain the following statement.

Theorem 2.3. *Let the assumptions of Theorem 2.2 hold. Then the operator L has a spectrum discrete and bounded below if and only if both (2.5) and (2.6) hold.*

If the operator $L_{\min}$ is nonnegative, it possesses the Friedrichs extension L_F. According to Theorem 2.3, the operator L_F exhibits a discrete spectrum if and only if both conditions (2.5) and (2.6) are satisfied.

For $p = q = 2$ inequality (1.3) can be rewritten as $(f, f)_2 C^{-2} \leq (L_F f, f)_{2,u}$. Then from Theorem 2.3 we have the following theorem, where the introduced above values BF, $\varepsilon_l(n)$, and $\varepsilon_r(n)$ are taken for $p = q = 2$.

Theorem 2.4. *Let the assumptions of Theorem 2.2 hold. Then the operator L_F is positive definite if and only if $BF < \infty$. Moreover, $\varepsilon_l(n)BF \leq \lambda_1^{-\frac{1}{2}} \leq \varepsilon_r(n)BF$ holds for the smallest eigenvalue λ_1 of the operator L_F.*

By Rellih's lemma ([11, p. 183]), the operator L_F^{-1} possesses a spectrum that is discrete and bounded below in $L_{2,u}$ if and only if the space equipped with the norm $(L_F f, f)_{2,u}^{\frac{1}{2}}$ is compactly embedded into the space $L_{2,u}$. Consequently, we derive another statement from Theorem 2.3.

Theorem 2.5. *Under the assumptions of Theorem 2.2, the embedding* $\overset{\circ}{W}{}^n_{2,v}(I) \hookrightarrow L_{2,u}$ *is compact, and the operator* L_F^{-1} *is uniformly continuous on* $L_{2,u}$ *if and only if both conditions (2.5) and (2.6) are satisfied.*

Let the operator L_F^{-1} be completely continuous on $L_{2,u}$. Suppose that $\{\lambda_k\}_{k=1}^{\infty}$ are the eigenvalues and $\{\varphi_k\}_{k=1}^{\infty}$ is the corresponding complete orthonormal system of eigenfunctions of the operator L_F^{-1}. Assume that

$$D(t) = \int_t^{\infty} \left(\int_0^t (s-x)^{n-2} dx \right)^2 v^{-1}(s) ds + \int_0^t \left(\int_0^s (s-x)^{n-2} dx \right)^2 v^{-1}(s) ds.$$

Theorem 2.6. *Let the assumptions of Theorem 2.2 hold. Let (2.5) and (2.6) hold.*
(i)

$$\frac{1}{((n-2)!)^2} D(t) \le \sum_{k=1}^{\infty} \frac{|\varphi_k(t)|^2}{\lambda_k} \le \frac{2}{((n-2)!)^2} D(t). \tag{2.9}$$

(ii) The operator L_F^{-1} *is nuclear if and only if* $\int_0^{\infty} u(t)D(t)dt < \infty$, *and for the nuclear norm* $\|L_F^{-1}\|_{\sigma_1}$ *of the operator* L_F^{-1} *the relation*

$$\frac{1}{((n-2)!)^2} \int_0^{\infty} u(t)D(t)dt \le \|L_F^{-1}\|_{\sigma_1} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \le \frac{2}{((n-2)!)^2} \int_0^{\infty} u(t)D(t)dt \tag{2.10}$$

holds.

3 Proof of Theorem 2.1

Let $-\infty \le a < b \le \infty$. To prove Theorem 2.1 we use characterizations of the standard weighted Hardy inequality provided in the following statement (see, e.g., [9]).

Theorem A. *Let* $1 < p \le q < \infty$.
(i) The inequality

$$\left(\int_a^b u(t) \left| \int_a^t f(s) ds \right|^q dt \right)^{\frac{1}{q}} \le C \left(\int_a^b v(t) |f(t)|^p dt \right)^{\frac{1}{p}} \tag{3.1}$$

holds if and only if

$$A^+ = \sup_{a<z<b} \left(\int_z^b u(t) dt \right)^{\frac{1}{q}} \left(\int_a^z v^{1-p'}(s) ds \right)^{\frac{1}{p'}} < \infty,$$

moreover,

$$A^+ \le C \le p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A^+,$$

where C *is the least constant in (3.1).*
(ii) The inequality

$$\left(\int_a^b u(t) \left| \int_t^b f(s) ds \right|^q dt \right)^{\frac{1}{q}} \le C \left(\int_c^b v(t) |f(t)|^p dt \right)^{\frac{1}{p}} \tag{3.2}$$

holds if and only if

$$A^- = \sup_{a<z<b} \left(\int_a^z u(t)dt \right)^{\frac{1}{q}} \left(\int_z^b v^{1-p'}(s)ds \right)^{\frac{1}{p'}} < \infty,$$

moreover,

$$A^- \leq C \leq p^{\frac{1}{q}} (p')^{\frac{1}{p}} A^-,$$

where C *is the least constant in* (3.2).

We also need the statement, which follows from the results of the works [19] and [6]. Let

$$B_1 = \sup_{a<z<b} \left(\int_a^z (z-t)^{q(n-1)} u(t)dt \right)^{\frac{1}{q}} \left(\int_z^b v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B_2 = \sup_{a<z<b} \left(\int_a^z u(t)dt \right)^{\frac{1}{q}} \left(\int_z^b (s-z)^{p'(n-1)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}}.$$

Theorem B. *Let* $1 < p \leq q < \infty$. *The inequality*

$$\left(\int_a^b u(t) \left| \int_t^b (s-t)^{n-1} f(s)ds \right|^q dt \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(t) \left| f(t) \right|^p dt \right)^{\frac{1}{p}} \tag{3.3}$$

holds if and only if $\max\{B_1, B_2\} < \infty$. *Moreover,*

$$\max\{B_1, B_2\} \leq C \leq 8p^{\frac{1}{q}} (p)^{\frac{1}{p'}} \max\{B_1, B_2\},$$

where C *is the least constant in* (3.3).

To establish Theorem 2.1, we adopt the approach outlined in the proof of Theorem 2.2 in [13].

Proof of Theorem 2.1. *Sufficiency.* By the conditions, we have (1.5). Let $\tau \in I$. We assume that $f(t) = \int_0^t f'(x)dx$ for $0 < t < \tau$, $f(t) = -\int_t^\infty f'(x)dx$ for $t > \tau$ and $f'(x) = \frac{(-1)^{n-1}}{(n-2)!} \int_x^\infty (s-x)^{n-2} f^{(n)}(s)ds$ for $x \in I$. Then for $f \in \mathring{W}^n_{p,v}(I)$ we have

$$f(t) = \frac{(-1)^n}{(n-1)!} \int_t^\infty (s-t)^{n-1} f^{(n)}(s)ds \tag{3.4}$$

for $t > \tau$. Moreover, we have

$$f(t) = \frac{(-1)^{n-1}}{(n-2)!} \int_0^t \int_x^\infty (s-x)^{n-2} f^{(n)}(s)dsdx = \frac{(-1)^{n-1}}{(n-2)!} \left[\int_0^t \int_x^t (s-t)^{n-2} f^{(n)}(s)dsdx \right.$$

$$\left. + \int_0^t \int_t^\tau (s-x)^{n-2} f^{(n)}(s)dsdx + \int_0^t \int_\tau^\infty (s-x)^{n-2} f^{(n)}(s)dsdx \right]$$

$$= \frac{(-1)^{n-1}}{(n-2)!}\left[\int_0^t f^{(n)}(s)\int_0^s (s-x)^{n-2}dxds + \int_t^\tau f^{(n)}(s)\int_0^t (s-x)^{n-2}dxds \right.$$
$$\left. + \int_\tau^\infty f^{(n)}(s)\int_0^t (s-x)^{n-2}dxds\right] \quad (3.5)$$

$$= \frac{(-1)^{n-1}}{(n-2)!}\left[\int_0^t f^{(n)}(s)\frac{s^{n-1}}{n-1}ds + \int_t^\tau f^{(n)}(s)s^{n-1}\frac{\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)}{n-1}ds \right.$$
$$\left. + \int_\tau^\infty f^{(n)}(s)s^{n-1}\frac{\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)}{n-1}ds\right].$$

Assuming $g(s) = f^{(n)}(s)s^{n-1}$, the last equality gives that

$$f(t) = \frac{(-1)^n}{(n-1)!}\left[-\int_\tau^\infty g(s)\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)ds \right.$$
$$\left. -\int_t^\tau g(s)\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)ds - \int_0^t g(s)ds\right]. \quad (3.6)$$

Since $\int_0^\infty f'(x)dx = 0$, we get

$$f(t) = c_1\int_0^\infty\int_x^\infty (s-x)^{n-2}f^{(n)}(s)dsdx = c_2\int_0^\infty f^{(n)}(s)s^{n-1}ds = 0,$$

which gives that $\int_0^\infty g(s)ds = 0$. Therefore, for $f \in \overset{\circ}{W}{}^n_{p,v}(I)$ from (3.6) we get

$$f(t) = \frac{(-1)^n}{(n-1)!}\left[-\int_\tau^\infty g(s)\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)ds \right.$$
$$\left. -\int_t^\tau g(s)\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)ds - \int_0^t g(s)ds + \left(1-\left(1-\frac{t}{\tau}\right)^{n-1}\right)\int_0^\infty g(s)ds\right]$$
$$= \frac{(-1)^n}{(n-1)!}\left[\int_\tau^\infty g(s)\left(\left(1-\frac{t}{s}\right)^{n-1} - \left(1-\frac{t}{\tau}\right)^{n-1}\right)ds \right.$$
$$\left. -\int_t^\tau g(s)\left(\left(1-\frac{t}{\tau}\right)^{n-1} - \left(1-\frac{t}{s}\right)^{n-1}\right)ds - \left(1-\frac{t}{\tau}\right)^{n-1}\int_0^t g(s)ds\right] \quad (3.7)$$

for $0<t<\tau$. Then, for $f\in \mathring{W}^n_{p,v}(I)$ from (3.4) and (3.7) we obtain

$$\frac{(n-1)!}{(-1)^n}f(t)=\chi_{(0,\tau)}(t)\left[\int_\tau^\infty g(s)\left(\left(1-\frac{t}{s}\right)^{n-1}-\left(1-\frac{t}{\tau}\right)^{n-1}\right)ds\right.$$
$$\left.-\int_t^\tau g(s)\left(\left(1-\frac{t}{\tau}\right)^{n-1}-\left(1-\frac{t}{s}\right)^{n-1}\right)ds-\left(1-\frac{t}{\tau}\right)^{n-1}\int_0^t g(s)ds\right]$$
$$+\chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)^{n-1}\frac{g(s)}{s^{n-1}}ds. \quad (3.8)$$

Since

$$\int_0^\infty v(s)|f^{(n)}(s)|^p ds=\int_0^\infty \frac{v(s)}{s^{p(n-1)}}|f^{(n)}(s)s^{n-1}|^p ds=\int_0^\infty \overline{v}(s)|g(s)|^p,$$

the condition $f\in \mathring{W}^n_{p,v}(I)$ is equivalent to the condition $g\in \widetilde{L}_{p,\overline{v}}(I)$, where $\widetilde{L}_{p,\overline{v}}(I)=\Big\{g\in L_{p,\overline{v}}(I): \int_0^\infty g(s)ds=0\Big\}$. Taking into account that for $s>\tau$

$$\left(1-\frac{t}{s}\right)^{n-1}-\left(1-\frac{t}{\tau}\right)^{n-1}\le (n-1)\frac{(s-t)^{n-2}}{s^{n-2}}\left(\frac{t}{\tau}-\frac{t}{s}\right)$$
$$\le (n-1)2^{n-3}\frac{\left[(s-\tau)^{n-2}+(\tau-t)^{n-2}\right]t(s-\tau)}{s^{n-1}\tau}$$
$$=(n-1)2^{n-3}\left[\frac{(s-\tau)^{n-1}t}{s^{n-1}\tau}+\frac{(s-\tau)(\tau-t)^{n-2}t}{s^{n-1}\tau}\right]$$

and for $\tau>s$

$$\left(1-\frac{t}{\tau}\right)^{n-1}-\left(1-\frac{t}{s}\right)^{n-1}\le (n-1)\frac{(\tau-t)^{n-2}(\tau-s)t}{\tau^{n-1}s},$$

by (3.8) inequality (1.3) can be written in the form

$$\frac{1}{(n-1)!}\left[\left(\int_0^\tau u(t)\left|(n-1)2^{n-3}\frac{t}{\tau}\int_\tau^\infty (s-\tau)^{n-1}\frac{g(s)}{s^{n-1}}ds\right.\right.\right.$$
$$+(n-1)2^{n-3}\frac{t}{\tau}(\tau-t)^{n-2}\int_\tau^\infty (s-\tau)\frac{g(s)}{s^{n-1}}ds$$
$$\left.-\frac{(\tau-t)^{n-1}}{\tau^{n-1}}\int_0^t g(s)ds-(n-1)\frac{t}{\tau^{n-1}}(\tau-t)^{n-2}\int_t^\tau (\tau-s)\frac{g(s)}{s}ds\right|^q dt$$
$$\left.\left.+\int_\tau^\infty u(t)\left|\int_t^\infty (s-t)^{n-1}\frac{g(s)}{s^{n-1}}ds\right|^q dt\right)^{\frac{1}{q}}\right]\le C\left(\int_0^\infty \overline{v}(s)|g(s)|^p ds\right)^{\frac{1}{p}}. \quad (3.9)$$

In the left-hand side of (3.9) applying the Minkowski inequality for sums, then the Hölder inequality, Theorem A, and Theorem B, we get

$$\left(\int_0^\infty u(t)|f(t)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{(n-1)!}\left(p^{\frac{1}{q}}(p')^{\frac{1}{p'}}F_1(\tau)+(n-1)p^{\frac{1}{q}}(p')^{\frac{1}{p'}}F_2(\tau)\right)\left(\int_0^\tau v(s)|f^{(n)}(s)|^p ds\right)^{\frac{1}{p}}$$
$$+\frac{1}{(n-1)!}\left((n-1)2^{n-3}B_3(\tau)+(n-1)2^{n-3}B_4(\tau)\right.$$
$$\left.+8\,p^{\frac{1}{q}}(p')^{\frac{1}{p'}}\max\{B_1(\tau),B_2(\tau)\}\right)\left(\int_\tau^\infty v(s)|f^{(n)}(s)|^p ds\right)^{\frac{1}{p}}$$
$$\leq \frac{1}{(n-1)!}\left(n\,p^{\frac{1}{q}}(p')^{\frac{1}{p'}}F(\tau)+\left((n-1)2^{n-2}+8p^{\frac{1}{q}}(p')^{\frac{1}{p'}}\right)B(\tau)\right)\left(\int_0^\infty v(s)|f^{(n)}(s)|^p ds\right)^{\frac{1}{p}}$$
$$\leq \varepsilon_r(n)\max\{B(\tau),F(\tau)\}\left(\int_0^\infty v(s)|f^{(n)}(s)|^p ds\right)^{\frac{1}{p}}. \quad (3.10)$$

Since the left-hand side of (3.10) is independent of $\tau\in I$, (3.10) implies the right estimate in (2.2).

Now, let us prove the right estimate in (2.3). Since

$$\lim_{\tau\to\infty}F_1(\tau)=\lim_{\tau\to\infty}\sup_{0<z<\tau}\left(\int_0^z t^q\left(1-\frac{t}{\tau}\right)^{q(n-2)}u(t)dt\right)^{\frac{1}{q}}$$
$$\times\left(\int_z^\tau\left(1-\frac{s}{\tau}\right)^{p'}s^{p'(n-2)}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}=\sup_{z>0}\left(\int_0^z t^q u(t)dt\right)^{\frac{1}{q}}\left(\int_z^\infty s^{p'(n-2)}v^{1-p'}(s)ds\right)^{\frac{1}{p'}},$$

we have that

$$B_4(\tau)=\left(\int_0^\tau t^q\left(1-\frac{t}{\tau}\right)^{q(n-2)}u(t)dt\right)^{\frac{1}{q}}\tau^{n-3}\left(\int_\tau^\infty(s-\tau)^{p'}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$
$$<\left(\int_0^\tau t^q u(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty s^{p'(n-2)}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}\leq\lim_{\tau\to\infty}F_1(\tau). \quad (3.11)$$

For $0 < N < \tau$ we obtain

$$B_3(\tau) < \left(\int_0^N \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} + \left(\int_N^\tau \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$
$$\leq \left(\int_0^N \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} + \left(\int_N^\tau u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}.$$

Since

$$\lim_{\tau\to\infty} \left(\int_0^N \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} = 0,$$

then

$$B_3(\tau) \ll \left(\int_N^\tau u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$

for a sufficiently large $\tau > N$. If $\lim\limits_{\tau\to\infty} F_2(\tau) = \infty$, where

$$\lim_{\tau\to\infty} F_2(\tau) = \lim_{\tau\to\infty} \sup_{0<z<\tau} \left(\int_z^\tau \left(1-\frac{t}{\tau}\right)^{q(n-1)} u(t)dt\right)^{\frac{1}{q}} \left(\int_0^z s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} = \sup_{z>0} \left(\int_z^\infty u(t)dt\right)^{\frac{1}{q}} \left(\int_0^z s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}},$$

then $\int_z^\infty u(t)dt = \infty$ for any $z > 0$. Therefore, $B_3(\tau) < \lim\limits_{\tau\to\infty} F_2(\tau) = \infty$ for a sufficiently large $\tau > N$. If $\lim\limits_{\tau\to\infty} F_2(\tau) < \infty$, then $\int_z^\infty u(t)dt < \infty$, which implies that $\lim\limits_{\tau\to\infty} B_3(\tau) = 0$, and we find that $B_3(\tau) < F(\tau)$ for a sufficiently large τ. It is also obvious that $B_i(\tau) < F(\tau)$, $i = 1, 2$. Combining these estimates with the obtained estimates $B_3(\tau) < F(\tau)$ and (3.11), we have that $B(\tau) \leq F(\tau)$ in some neighborhood of infinity. Therefore, in relation (2.4) there exists $\tau_0 > 0$ such that $B(\tau_0) \leq F(\tau_0)$. Consequently,

$$BF = \inf_{\tau\in I} \max\{B(\tau), F(\tau)\} \leq F(\tau_0)$$

and the right estimate in (2.3) holds.

Necessity. Since $\overline{v}^{1-p'} \in L_1(I)$, then (1.6) holds. For $\tau \in I$ we consider two sets $\mathcal{L}_1 = \{g \in L_{p,\overline{v}}(0,\tau) : g \leq 0\}$ and $\mathcal{L}_2 = \{g \in L_{p,\overline{v}}(\tau,\infty) : g \geq 0\}$. Repeating the proof of the necessary part of Theorem 1 in [8], for each $g_1 \in \mathcal{L}_1$ we construct a function $g_2 \in \mathcal{L}_2$ and for each $g_2 \in \mathcal{L}_2$ we construct a function $g_1 \in \mathcal{L}_1$ such that $g(t) = g_1(t)$ for $0 < t \le \tau$ and $g(t) = g_2(t)$ for $t > \tau$ belongs to the set $\widetilde{L}_{p,\overline{v}}(I)$. For the constructed function g we have (see [8, (26)])

$$\int_0^\infty \overline{v}(t)|g(t)|^p dt = (1 + k_\tau^{p-1}) \int_0^\tau \overline{v}(t)|g_1(t)|^p dt = (1 + k_\tau^{1-p}) \int_\tau^\infty \overline{v}(t)|g_2(t)|^p dt < \infty. \tag{3.12}$$

Taking into account (3.8) for the function $g \in \widetilde{L}_{p,\overline{v}}(I)$, we have

$$\frac{1}{(n-1)!}\left[\left(\int_0^\tau u(t)\left|\int_\tau^\infty g_2(s)\left(\left(1-\frac{t}{s}\right)^{n-1} - \left(1-\frac{t}{\tau}\right)^{n-1}\right)ds\right.\right.\right.$$
$$\left. + \int_t^\tau |g_1(s)|\left(\left(1-\frac{t}{\tau}\right)^{n-1} - \left(1-\frac{t}{s}\right)^{n-1}\right)ds + \left(1-\frac{t}{\tau}\right)^{n-1}\int_0^t |g_1(s)|ds\right|^q dt$$
$$\left.\left. + \int_\tau^\infty u(t)\left|\int_t^\infty (s-t)^{n-1}\frac{g_2(s)}{s^{n-1}}ds\right|^q dt\right)^{\frac{1}{q}}\right] \le C\left(\int_0^\infty \overline{v}(s)|g(s)|^p ds\right)^{\frac{1}{p}}. \tag{3.13}$$

In the left-hand side of (3.13), all terms are nonnegative. Using the estimate for $s > \tau$

$$\left(1-\frac{t}{s}\right)^{n-1} - \left(1-\frac{t}{\tau}\right)^{n-1} \ge \left(1-\frac{t}{s}\right)^{n-1} - \left(1-\frac{t}{s}\right)^{n-2}\left(1-\frac{t}{\tau}\right)$$
$$= \frac{(s-t)^{n-2}(s-\tau)t}{s^{n-1}\tau} \ge \max\left[\frac{(s-\tau)^{n-1}t}{s^{n-1}\tau}, \frac{(s-\tau)(\tau-t)^{n-2}t}{s^{n-1}\tau}\right],$$

assuming that the function $g \in \widetilde{L}_{p,\overline{v}}(I)$ is constructed by the function $g_2 \in \mathcal{L}_2$, from (3.12) and (3.13), we have

$$\frac{1}{(n-1)!}\left(\int_0^\tau u(t)\left(\frac{t}{\tau}\int_\tau^\infty (s-\tau)^{n-1}\frac{g_2(s)}{s^{n-1}}ds\right)^q dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{(n-1)!}\left(\int_0^\tau t^q u(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{\tau}\int_\tau^\infty (s-\tau)^{n-1}\frac{g_2(s)}{s^{n-1}}ds\right)$$
$$\le C(1+k_\tau^{1-p})^{\frac{1}{p}}\left(\int_\tau^\infty \overline{v}(t)|g_2(t)|^p dt\right)^{\frac{1}{p}},$$

$$\frac{1}{(n-1)!}\left(\int_0^\tau u(t)\left(\frac{t}{\tau}(\tau-t)^{n-2}\int_\tau^\infty (s-\tau)\frac{g_2(s)}{s^{n-1}}ds\right)^q dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{(n-1)!}\left(\int_0^\tau t^q(\tau-t)^{q(n-2)}u(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{\tau}\int_\tau^\infty (s-\tau)\frac{g_2(s)}{s^{n-1}}ds\right)$$
$$\le C(1+k_\tau^{1-p})^{\frac{1}{p}}\left(\int_\tau^\infty \overline{v}(t)|g_2(t)|^p dt\right)^{\frac{1}{p}},$$

$$\frac{1}{(n-1)!}\left(\int_{\tau}^{\infty} u(t)\left(\int_{t}^{\infty}(s-t)^{n-1}\frac{g_2(s)}{s^{n-1}}ds\right)^{q}dt\right)^{\frac{1}{q}} \leq C(1+k_\tau^{1-p})^{\frac{1}{p}}\left(\int_{\tau}^{\infty}\overline{v}(t)|g_2(t)|^p dt\right)^{\frac{1}{p}}.$$

Due to the arbitrariness of $g_2 \in \mathcal{L}_2$, applying the reverse Hölder inequality to the first two inequalities and Theorem B to the last inequality, we obtain

$$\frac{1}{(n-1)!}\max\{B_1(\tau), B_2(\tau), B_3(\tau)B_4(\tau)\} = \frac{1}{(n-1)!}B(\tau) \leq C(1+k_\tau^{1-p})^{\frac{1}{p}}. \tag{3.14}$$

Similarly, using the estimate for $\tau > s$

$$\left(1-\frac{t}{\tau}\right)^{n-1} - \left(1-\frac{t}{s}\right)^{n-1} \geq \frac{(\tau-t)^{n-2}(\tau-s)t}{\tau^{n-1}s},$$

for the function $g \in \widetilde{L}_{p,\overline{v}}(I)$ constructed by the function $g_1 \in \mathcal{L}_1$, from (3.12) and (3.13) we have

$$\frac{1}{(n-1)!}\left(\int_0^{\tau} u(t)\left(\frac{(\tau-t)^{n-1}}{\tau^{n-1}}\int_0^{t}|g_1(s)|ds\right)^{q}dt\right)^{\frac{1}{q}} \leq C(1+k_\tau^{p-1})^{\frac{1}{p}}\left(\int_0^{\tau}\overline{v}(t)|g_1(t)|^p dt\right)^{\frac{1}{p}},$$

$$\frac{1}{(n-1)!}\left(\int_0^{\tau} u(t)\left(\frac{t}{\tau^{n-1}}(\tau-t)^{n-2}\int_t^{\tau}(\tau-s)\frac{|g_1(s)|}{s}ds\right)^{q}dt\right)^{\frac{1}{q}}$$

$$\leq C(1+k_\tau^{p-1})^{\frac{1}{p}}\left(\int_0^{\tau}\overline{v}(t)|g_1(t)|^p dt\right)^{\frac{1}{p}}.$$

The latter, due to the arbitrariness of $g_1 \in \mathcal{L}_1$, by Theorem A, gives that

$$\frac{1}{(n-1)!}F(\tau) \leq C(1+k_\tau^{p-1})^{\frac{1}{p}}. \tag{3.15}$$

From (3.14) and (3.15) we find that

$$\frac{1}{(n-1)!}BF \leq C\inf_{\tau\in I}\left[\max\{(1+k_\tau^{p-1})(1+k_\tau^{1-p})\}\right]^{\frac{1}{p}} \leq 4^{\frac{1}{p}}C,$$

which yields the left estimate in (2.2). From (3.15) we get the left estimate in (2.3). □

4 Proofs of Theorems 2.2 and 2.6

Theorems 2.3, 2.4, and 2.5 directly follow as corollaries from the combination of results presented in Section 2 and Theorem 2.1 proved above. Here we present the proofs of Theorems 2.2 and 2.6.

For clarity, let us write the squared values $F_1(\tau)$ and $F_2(\tau)$ for $p = q = 2$ in the form:

$$F_1^2(\tau) = \sup_{0<z<\tau}\int_0^z t^2\left(1-\frac{t}{\tau}\right)^{2(n-2)}u(t)dt\int_z^{\tau}\left(1-\frac{s}{\tau}\right)^2 s^{2(n-2)}v^{-1}(s)ds,$$

$$F_2^2(\tau) = \sup_{0<z<\tau}\int_z^{\tau}\left(1-\frac{t}{\tau}\right)^{2(n-1)}u(t)dt\int_0^z s^{2(n-1)}v^{-1}(s)ds,$$

$$F^2(\tau) = \max\{F_1^2(\tau), F_2^2(\tau)\}.$$

Proof of Theorem 2.2. (i) Suppose that equation (1.1) is strongly non-oscillatory at zero. Then by Lemma 2.2, we have that $\lim\limits_{T\to 0^+} C_T = 0$. From the left estimate in (2.3) we have

$$\frac{1}{(n-1)!}\sup_{0<\tau<T}(1+k_\tau)^{-1}F^2(\tau)\le C_T,$$

which gives that

$$\lim_{T\to 0^+}\sup_{0<\tau<T}(1+k_\tau)^{-1}F^2(\tau)=0.$$

Hence,

$$\lim_{\tau\to 0^+}(1+k_\tau)^{-1}F^2(\tau)=\lim_{\tau\to 0^+}F^2(\tau)=0,$$

i.e., $\lim\limits_{\tau\to 0^+}F_1^2(\tau)=\lim\limits_{\tau\to 0^+}F_2^2(\tau)=0$. Thus,

$$0=\lim_{\tau\to 0^+}F_1^2(\tau)\ge\lim_{\tau\to 0^+}\sup_{0<z<\frac{\tau}{2}}\int\limits_0^z t^2\left(1-\frac{t}{\tau}\right)^{2(n-2)}u(t)dt\int\limits_z^{\frac{\tau}{2}}\left(1-\frac{s}{\tau}\right)^2 s^{2(n-2)}v^{-1}(s)ds$$

$$\ge 4^{-(n-1)}\lim_{\tau\to 0^+}\sup_{0<z<\frac{\tau}{2}}\int\limits_0^z t^2u(t)dt\int\limits_z^{\frac{\tau}{2}}s^{2(n-2)}v^{-1}(s)ds,$$

i.e., (2.5) holds. Similarly, we prove that (2.6) also holds.

Inversely, let (2.5) and (2.6) hold. Since $1-\frac{t}{\tau}\le 1$ for $0<t<\tau$, we obtain

$$0=\lim_{\tau\to 0^+}\sup_{0<z<\tau}\int\limits_0^z t^2u(t)dt\int\limits_z^{\tau}s^{2(n-2)}v^{-1}(s)ds$$

$$\ge\lim_{\tau\to 0^+}\sup_{0<z<\tau}\int\limits_0^z t^2\left(1-\frac{t}{\tau}\right)^{2(n-2)}u(t)dt\int\limits_z^{\tau}\left(1-\frac{s}{\tau}\right)^2 s^{2(n-2)}v^{-1}(s)ds=\lim_{\tau\to 0^+}F_1^2(\tau).$$

Similarly, we find that $\lim\limits_{\tau\to 0^+}F_2^2(\tau)=0$, i.e., $\lim\limits_{\tau\to 0^+}F^2(\tau)=0$. From the right estimate in (2.3) we have

$$C_T\le\varepsilon_r(n)F^2(\tau_0),\ 0<\tau_0<T. \tag{4.1}$$

Therefore, we get

$$0=\varepsilon_r(n)\lim_{T\to 0^+}F^2(\tau_0)=\varepsilon_r(n)\lim_{\tau\to 0^+}F^2(\tau)\ge\lim_{T\to 0^+}C_T.$$

Thus, $\lim\limits_{T\to 0^+}C_T=0$ and, by Lemma 2.2, equation (1.1) is strongly non-oscillatory at zero.

(ii) Let equation (1.1) be strongly oscillatory at zero, then by Lemma 2.2, we have $C_T=\infty$ for any $T>0$. Consequently, from (4.1), we deduce $\lim\limits_{T\to 0^+}F(\tau_0)=\lim\limits_{\tau\to 0^+}F(\tau)=\infty$. This indicates that at least one of conditions (2.7) or (2.8) holds.

Inversely, let (2.7) hold. Then

$$\infty = \lim_{\frac{\tau}{2}\to 0^+} \sup_{0<z<\frac{\tau}{2}} \int_0^z t^2 u(t)dt \int_z^{\frac{\tau}{2}} s^{2(n-2)} v^{-1}(s)ds$$

$$= \lim_{\frac{\tau}{2}\to 0^+} \sup_{0<z<\frac{\tau}{2}} \int_0^z t^2 u(t) 4^{-(n-2)} dt \int_z^{\frac{\tau}{2}} 4^{-1} s^{2(n-2)} v^{-1}(s)ds$$

$$\leq \lim_{\frac{\tau}{2}\to 0^+} \sup_{0<z<\frac{\tau}{2}} \int_0^z t^2 \left(1-\frac{t}{\tau}\right)^{2(n-2)} u(t)dt \int_z^{\frac{\tau}{2}} \left(1-\frac{s}{\tau}\right)^2 s^{2(n-2)} v^{-1}(s)ds$$

$$= \lim_{\frac{\tau}{2}\to 0^+} F_1^2\left(\frac{\tau}{2}\right) = \lim_{\tau\to 0^+} F_1^2(\tau).$$

Thus, $\lim\limits_{\tau\to 0^+} F_1^2(\tau) = \infty$. Since $\frac{1}{(n-1)!} \sup\limits_{0<\tau<T} (1+k_\tau)^{-1} F_1^2(\tau) \leq C_T$ and

$$\frac{1}{(n-1)!} \lim_{T\to 0^+} \sup_{0<\tau<T} (1+k_\tau)^{-1} F_1^2(\tau) \geq \frac{1}{(n-1)!} \lim_{\tau\to 0^+} (1+k_\tau)^{-1} F_1^2(\tau) = \lim_{\tau\to 0^+} F_1^2(\tau),$$

from $\lim\limits_{\tau\to 0^+} F_1^2(\tau) = \infty$ we get that $C_T = \infty$ for any $T > 0$. Therefore, by Lemma 2.2, we conclude that equation (1.1) is strongly oscillatory at zero. Arguing similarly, we prove that if (2.8) holds, then equation (1.1) is strongly oscillatory at zero. □

To prove Theorem 2.6 we need the following lemma.

Lemma 4.1. *Let the assumptions of Theorem 2.2 hold. Then for $t \in I$*

$$\frac{1}{(n-1)!} \sup_{\tau\in I} D(t,\tau) \leq \sup_{f\in \mathring{W}_{2,v}^n} \frac{|f(t)|}{\|f^{(n)}\|_{2,v}} \leq \frac{\sqrt{2}}{(n-1)!} \inf_{\tau\in I} D(t,\tau), \tag{4.2}$$

where

$$D(t,\tau) = \Bigg\{ \chi_{(0,\tau)}(t)(n-1)^2 \int_\tau^\infty \left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds$$

$$+ \chi_{(\tau,\infty)}(t) \int_t^\infty (s-t)^{2(n-1)} v^{-1}(s)ds + \chi_{(0,\tau)}(t)(n-1)^2 \int_t^\tau \left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds$$

$$+ \chi_{(0,\tau)}(t)(n-1)^2 \int_0^t \left(\int_0^s (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds \Bigg\}^{\frac{1}{2}}.$$

Proof of Lemma 4.1. From (3.4) and (3.5) for the function $f \in \overset{\circ}{W}{}^n_{2,v}$ we have

$$f(t) = \frac{(-1)^{n-1}}{(n-1)!}\left\{(n-1)\chi_{(0,\tau)}\left[\int_0^t f^{(n)}(s)\int_0^s (s-x)^{n-2}dxds + \int_t^\tau f^{(n)}(s)\int_0^t (s-x)^{n-2}dxds + \int_\tau^\infty f^{(n)}(s)\int_0^t (s-x)^{n-2}dxds\right] - \chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)^{n-1}f^{(n)}(s)ds\right\}. \quad (4.3)$$

Applying the Hölder inequality, we obtain

$$|f(t)| \le \frac{1}{(n-1)!}\left\{\left[(n-1)\chi_{(0,\tau)}(t)\left(\int_\tau^\infty\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac{1}{2}} + \chi_{(\tau,\infty)}(t)\left(\int_t^\infty (s-t)^{2(n-1)}v^{-1}(s)ds\right)^{\frac{1}{2}}\right] \times \left(\int_\tau^\infty v(s)|f^{(n)}(s)|^2 ds\right)^{\frac{1}{2}}\right.$$

$$\left. + (n-1)\chi_{(0,\tau)}(t)\left[\left(\int_t^\tau\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac{1}{2}} + \left(\int_0^t\left(\int_0^s (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac{1}{2}}\right] \times \left(\int_0^\tau v(s)|f^{(n)}(s)|^2 ds\right)^{\frac{1}{2}}\right\}$$

$$\le \frac{1}{(n-1)!}\left\{\left[(n-1)\chi_{(0,\tau)}(t)\left(\int_\tau^\infty\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac{1}{2}} + \chi_{(\tau,\infty)}(t)\left(\int_t^\infty (s-t)^{2(n-1)}v^{-1}(s)ds\right)^{\frac{1}{2}}\right]^2\right.$$

$$+\chi_{(0,\tau)}(t)(n-1)^2\left[\left(\int_t^\tau\left(\int_0^t(s-x)^{n-2}dx\right)^2v^{-1}(s)ds\right)^{\frac{1}{2}}\right.$$

$$\left.\left.+\left(\int_0^t\left(\int_0^s(s-x)^{n-2}dx\right)^2v^{-1}(s)ds\right)^{\frac{1}{2}}\right]^2\right\}^{\frac{1}{2}}\|f^{(n)}\|_{2,v}$$

$$\leq\frac{1}{(n-1)!}\left\{\chi_{(0,\tau)}(t)(n-1)^2\int_\tau^\infty\left(\int_0^t(s-x)^{n-2}dx\right)^2v^{-1}(s)ds\right.$$

$$+\chi_{(\tau,\infty)}(t)\int_t^\infty(s-t)^{2(n-1)}v^{-1}(s)ds+2\chi_{(0,\tau)}(t)(n-1)^2\int_t^\tau\left(\int_0^t(s-x)^{n-2}dx\right)^2v^{-1}(s)ds$$

$$\left.+2\chi_{(0,\tau)}(t)(n-1)^2\int_0^t\left(\int_0^s(s-x)^{n-2}dx\right)^2v^{-1}(s)ds\right\}^{\frac{1}{2}}\|f^{(n)}\|_{2,v}$$

$$\leq\frac{\sqrt{2}}{(n-1)!}D(t,\tau)\|f^{(n)}\|_{2,v}.$$

Therefore, $|f(t)|\leq\frac{\sqrt{2}}{(n-1)!}\inf_{\tau\in I}D(t,\tau)\|f^{(n)}\|_{2,v}$ and the right estimate in (4.2) holds.

Let us prove the left estimate in (4.2). We fix $t\in I$ in (4.3) and select a function $f^{(n)}$ depending on t as follows:

$$f_t^{(n)}(s)=\begin{cases}\chi_{(0,t)}(s)(n-1)\int\limits_0^s(s-x)^{n-2}dx\,v^{-1}(s) & \text{if } 0<t<\tau,\\ \chi_{(t,\tau)}(s)(n-1)\int\limits_0^t(s-x)^{n-2}dx\,v^{-1}(s) & \text{if } 0<t<\tau,\\ \chi_{(\tau,\infty)}(s)(n-1)\int\limits_0^t(s-x)^{n-2}dx\,v^{-1}(s) & \text{if } 0<t<\tau,\\ -\chi_{(t,\infty)}(s)(s-t)^{n-1}v^{-1}(s) & \text{if } t>\tau.\end{cases}$$

Placing this function in (4.3), we get

$$f_t(t)=\frac{(-1)^{n-1}}{(n-1)!}\left\{\chi_{(0,\tau)}(t)(n-1)^2\int_\tau^\infty\left(\int_0^t(s-x)^{n-2}dx\right)^2v^{-1}(s)ds\right.$$

$$+(n-1)^2\int_t^\tau\left(\int_0^t(s-x)^{n-2}dx\right)^2v^{-1}(s)ds+(n-1)^2\int_0^t\left(\int_0^s(s-x)^{n-2}dx\right)^2v^{-1}(s)ds$$

$$\left.+\chi_{(\tau,\infty)}(t)\int_t^\infty(s-t)^{2(n-1)}v^{-1}(s)ds\right\}=\frac{(-1)^{n-1}}{(n-1)!}D^2(t,\tau).\quad(4.4)$$

Let us calculate $\|f_t^{(n)}\|_{2,v}$:

$$\left(\int_0^\infty v(s)|f_t^{(n)}(s)|^2 ds\right)^{\frac{1}{2}} = \left(\int_0^\tau v(s)|f_t^{(n)}(s)|^2 ds + \int_\tau^\infty v(s)|f_t^{(n)}(s)|^2 ds\right)^{\frac{1}{2}}$$

$$= \left\{\chi_{(0,\tau)}(t)(n-1)^2 \int_\tau^\infty \left(\int_0^t (s-x)^{n-2} dx\right)^2 v^{-1}(s) ds\right.$$

$$+ \chi_{(0,\tau)}(t)(n-1)^2 \int_t^\tau \left(\int_0^t (s-x)^{n-2} dx\right)^2 v^{-1}(s) ds$$

$$+ \chi_{(0,\tau)}(t)(n-1)^2 \int_0^t \left(\int_0^s (s-x)^{n-2} dx\right)^2 v^{-1}(s) ds$$

$$\left. + \chi_{(\tau,\infty)}(t) \int_t^\infty (s-t)^{2(n-1)} v^{-1}(s) ds\right\}^{\frac{1}{2}} = D(t,\tau). \quad (4.5)$$

From (4.4) and (4.5) we get

$$\sup_{f\in \mathring{W}^n_{2,v}} \frac{|f(t)|}{\|f^{(n)}\|_{2,v}} \geq \frac{|f_t(t)|}{\|f_t^{(n)}\|_{2,v}} = \frac{1}{(n-1)!} D(t,\tau)$$

for any $\tau \in I$. This relation proves the validity of the left estimate in (4.2). □

Together with Lemma 4.1 we need the following statement from work [1].

Lemma C. *Let $H = H(I)$ be a Hilbert function space, and $C[0,\infty)\bigcap H$ be dense in it. For any point $t \in I$, we define the operator $E_t f = f(t)$ on $C[0,\infty)\bigcap H$, which acts to the space of complex numbers. We assume that E_t is a closed operator. Then, the norm of this operator is equal to the value $\left(\sum_{k=1}^\infty |\varphi_k(t)|^2\right)^{\frac{1}{2}}$ (finite or infinite), where $\{\varphi_k(\cdot)\}_{k=1}^\infty$ is any complete orthonormal system of continuous functions in H.*

Proof of Theorem 2.6. By the condition, the operator L_F^{-1} is completely continuous on $L_{2,u}$. We assume that the space $\mathring{W}^n_{2,v}(I)$ with the norm $\|f^{(n)}\|_{2,v}$ is the space $H(I)$ of Lemma C. Since the system of functions $\{\lambda_k^{-\frac{1}{2}}\varphi_k\}_{k=1}^\infty$ is a complete orthonormal system in the space $\mathring{W}^n_{2,v}(I)$, then by Lemma C we have

$$\|E_t\|^2 = \left(\sup_{f\in \mathring{W}^n_{2,v}(I)} \frac{|f(t)|}{\|f^{(n)}\|_{2,v}}\right)^2 = \sum_{k=1}^\infty \frac{|\varphi_k(t)|^2}{\lambda_k},$$

where $E_t f = f(t)$. The latter and (4.2) give

$$\frac{1}{((n-1)!)^2} \sup_{\tau\in I} D^2(t,\tau) \leq \sum_{k=1}^\infty \frac{|\varphi_k(t)|^2}{\lambda_k} \leq \frac{2}{((n-1)!)^2} \inf_{\tau\in I} D^2(t,\tau). \quad (4.6)$$

Since

$$\inf_{\tau\in I} D^2(t,\tau) \leq \lim_{\tau\to\infty} D^2(t,\tau) = (n-1)^2 D(t) \leq \sup_{\tau\in I} D^2(t,\tau),$$

from (4.6) we have (2.9). Multiplying both sides of (2.9) by u and integrating them from zero to infinity, we get (2.10). □

5 Remarks

As pointed out in Introduction, from [10] and [16] it follows that if

$$v^{-1} \in L_1(0,1), \quad v^{-1} \in L_1(1,\infty), \quad \text{and} \quad t^2 v^{-1} \notin L_1(1,\infty), \tag{5.1}$$

then for any $f \in W^n_{2,v}$ there exist the limits $\lim\limits_{t\to 0^+} f^{(i)}(t) \equiv f^{(i)}(0)$ for all $i = 0, 1, ..., n-1$, and $\lim\limits_{t\to\infty} f^{(n-1)}(t) \equiv f^{(n-1)}(\infty)$. In paper [7] there are investigated oscillatory properties of equation (1.1) and spectral properties of the operator L under conditions (5.1), which give that

$$\overset{\circ}{W}{}^n_{2,v}(I) = \{f \in W^n_{2,v}(I) : f^{(i)}(0) = 0, i = 0, 1, ..., n-1, f^{(n-1)}(\infty) = 0\}.$$

Item (i) of Theorem 4.2 in [7] can be equivalently rewritten in the form.

Theorem 5.1. *Let assumption (5.1) hold. Then the operator L has a spectrum discrete and bounded below if and only if*

$$\lim_{z\to\infty} \int\limits_z^\infty t^{2(n-2)} u(t)dt \int\limits_0^z s^2 v^{-1}(s)ds = 0, \tag{5.2}$$

$$\lim_{z\to\infty} \int\limits_0^z t^{2(n-1)} u(t)dt \int\limits_z^\infty v^{-1}(s)ds = 0. \tag{5.3}$$

Theorem 4.6 in [7] can be also rewritten in the following simpler form.

Theorem 5.2. *Let assumption (5.1) hold. Let (5.2) and (5.3) hold.*
(i)

$$\frac{1}{((n-2)!)^2}\mathcal{D}(t) \le \sum_{k=1}^\infty \frac{|\varphi_k(t)|^2}{\lambda_k} \le \frac{2}{((n-2)!)^2}\mathcal{D}(t), \tag{5.4}$$

where

$$\mathcal{D}(t) = \int\limits_0^t \left(\int\limits_0^s (t-x)^{n-2}dx\right)^2 v^{-1}(s)ds + \frac{1}{(n-1)^2} t^{2(n-1)} \int\limits_t^\infty v^{-1}(s)ds.$$

(ii) The operator L_F^{-1} is nuclear if and only if $\int\limits_0^\infty u(t)\mathcal{D}(t)dt < \infty$ and for the nuclear norm $\|L_F^{-1}\|_{\sigma_1}$ of the operator L_F^{-1} the relation

$$\frac{1}{((n-2)!)^2}\int\limits_0^\infty u(t)\mathcal{D}(t)dt \le \|L_F^{-1}\|_{\sigma_1} = \sum_{k=1}^\infty \frac{1}{\lambda_k} \le \frac{2}{((n-2)!)^2}\int\limits_0^\infty u(t)\mathcal{D}(t)dt \tag{5.5}$$

holds.

This statement follows from the relation

$$\frac{1}{(n-1)!}\sup_{\tau\in I}\mathcal{D}(t,\tau) \le \sup_{f\in \overset{\circ}{W}{}^n_{2,v}} \frac{|f(t)|}{\|f^{(n)}\|_{2,v}} \le \frac{\sqrt{2}}{(n-1)!}\inf_{\tau\in I}\mathcal{D}(t,\tau), \tag{5.6}$$

where

$$\mathcal{D}(t,\tau) = \Bigg\{ \chi_{(0,\tau)}(t) \int_0^t (t-s)^{2(n-1)} v^{-1}(s)ds + \chi_{(\tau,\infty)}(t)(n-1)^2 \int_0^\tau \left(\int_s^\tau (t-x)^{n-2}dx \right)^2 v^{-1}(s)ds + \chi_{(\tau,\infty)}(t)(n-1)^2 \int_\tau^t \left(\int_\tau^s (t-x)^{n-2}dx \right)^2 v^{-1}(s)ds + \chi_{(\tau,\infty)}(t)(t-\tau)^{2(n-1)} \int_t^\infty v^{-1}(s)ds \Bigg\}^{\frac{1}{2}},$$

found in [7, Lemma 4.5]. Arguing similarly as in the proof of Lemma 4.1 and taking into account that

$$\inf_{\tau\in I} \mathcal{D}^2(t,\tau) \le \lim_{\tau\to 0^+} \mathcal{D}^2(t,\tau) = (n-1)^2\mathcal{D}(t) \le \sup_{\tau\in I} \mathcal{D}^2(t,\tau),$$

from (5.6) we get (5.4). Multiplying both sides of (5.4) by u and integrating them from zero to infinity, we obtain (5.5).

Acknowledgments

This work was supported by the Ministry of Science and Higher Education of the Republic of Kazakhstan, grant no. AP19677836.

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Aigerim Kalybay
KIMEP University
4 Abay Ave
Almaty, Republic of Kazakhstan
E-mail: kalybay@kimep.kz

Ryskul Oinarov
L.N. Gumilyov Eurasian National University
13 Kazhymukan St
010008 Astana, Republic of Kazakhstan
and
Institute of Mathematics and Mathematical Modeling
125 Pushkin St
050010 Almaty, Republic of Kazakhstan
E-mail: o_ryskul@mail.ru

Received: 06.06.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 3 (2025), 42 – 56

ALGEBRAS OF BINARY FORMULAS FOR WEAKLY CIRCULARLY MINIMAL THEORIES WITH EQUIVALENCE RELATIONS

B.Sh. Kulpeshov

Communicated by J.A. Tussupov

Key words: algebra of binary formulas, $\aleph_0$-categorical theory, weak circular minimality, circularly ordered structure, convexity rank.

AMS Mathematics Subject Classification: 03C64, 03C07, 03C40.

Abstract. Algebras of binary isolating formulas are described for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure, having only equivalence relations.

DOI: https://doi.org/10.32523/2077-9879-2025-16-3-42-56

1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of a one-type at the binary level with respect to the superposition of binary definable sets. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in papers [27, 28]. In recent years, algebras of binary formulas have been studied intensively and have been continued in works [1], [3], [7]–[14], [26], [29].

Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

Let $M = \langle M, \leq\rangle$ be a linearly ordered set. If we connect two endpoints of M (possibly, $-\infty$ and $+\infty$), then we obtain a circular order. More formally, the *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x\forall y\forall z(K(x, y, z) \to K(y, z, x))$;

(co2) $\forall x\forall y\forall z(K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;

(co3) $\forall x\forall y\forall z(K(x, y, z) \to \forall t[K(x, y, t) \vee K(t, y, z)])$;

(co4) $\forall x\forall y\forall z(K(x, y, z) \vee K(y, x, z))$.

The following observation relates linear and circular orders.

Fact 1.1. [4] (i) *If $\langle M, \leq\rangle$ is a linear ordering and K is the ternary relation derived from $\leq$ by the rule*

$$K(x, y, z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x),$$

then K is a circular order relation on M.

(ii) *If $\langle N, K\rangle$ is a circular ordering and $a \in N$, then the relation $\leq_a$ defined on $M := N \setminus \{a\}$ by the rule $y \leq_a z :\Leftrightarrow K(a, y, z)$ is a linear order.*

Thus, any linearly ordered structure is circularly ordered, since the relation of circular order is $\emptyset$-definable in an arbitrary linearly ordered structure. However, the opposite is not true. The following example shows that there are circularly ordered structures not being linearly ordered (in the sense that a linear ordering relation is not $\emptyset$-definable in an arbitrary circularly ordered structure).

Example 1. [5, 6] Let $\mathbb{Q}_2^* := \langle \mathbb{Q}_2, K, L\rangle$ be a circularly ordered structure, where $L = \{\sigma_0^2, \sigma_1^2\}$, for which the following conditions hold:

(i) its domain $\mathbb{Q}_2$ is a countable dense subset of the unit circle, no two points making the central angle π;

(ii) for distinct $a, b \in \mathbb{Q}_2$

$$(a, b) \in \sigma_0 \Leftrightarrow 0 < \arg(a/b) < \pi,$$

$$(a, b) \in \sigma_1 \Leftrightarrow \pi < \arg(a/b) < 2\pi,$$

where $\arg(a/b)$ means the value of the central angle between a and b clockwise.

Indeed, one can check that the linear order relation is not $\emptyset$-definable in this structure.

The notion of *weak circular minimality* was studied initially in [15]. Let $A \subseteq M$, where M is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a, c, b)$, $c \in A$ holds, or for any $c \in M$ with $K(b, c, a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $M = \langle M, K, \ldots\rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. The study of weakly circularly minimal structures was continued in papers [16]–[22].

Let M be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(M)$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on M is an arbitrary G-invariant equivalence relation on M. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on M.

Notation 1. (1) $K_0(x, y, z) := K(x, y, z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n\rangle$ having the length 3 (in ascending order) satisfy K; similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure M. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend naturally that notation, using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

Further, we need the notion of the definable completion of a circularly ordered structure, introduced in [15]. Its linear analogue was introduced in [25]. A *cut* $C(x)$ in a circularly ordered structure M is the maximal consistent set of formulas of the form $K(a, x, b)$, where $a, b \in M$. A cut is said to be *algebraic* if there exists $c \in M$ that realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for all $b \in M$ the formula $K(a, x, b) \in C(x)$, or for all $b \in M$ the formula $K(b, x, a) \in C(x)$, then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*. A *definable cut* in M is a cut $C(x)$ with the following property: there exist $a, b \in M$ such that $K(a, x, b) \in C(x)$ and the set $\{c \in M \mid K(a, c, b) \text{ and } K(a, x, c) \in C(x)\}$ is definable. The *definable completion* $\overline{M}$ of a structure M consists of M together with all definable cuts in M that are irrational (essentially $\overline{M}$ consists of endpoints of definable subsets of the structure M).

Notation 2. [15] Let $F(x, y)$ be an L-formula such that $F(M, b)$ is convex infinite co-infinite for each $b \in M$. Let $F^\ell(y)$ be the formula saying y is a left endpoint of $F(M, y)$:

$$\exists z_1 \exists z_2 [K_0(z_1, y, z_2) \wedge \forall t_1 (K(z_1, t_1, y) \wedge t_1 \neq y \rightarrow \neg F(t_1, y)) \wedge$$

$$\forall t_2 (K(y, t_2, z_2) \wedge t_2 \neq y \rightarrow F(t_2, y))].$$

We say that $F(x, y)$ is *convex-to-right* if

$$M \models \forall y \forall x[F(x, y) \to F^l(y) \land \forall z(K(y, z, x) \to F(z, y))].$$

If $F_1(x, y), F_2(x, y)$ are arbitrary convex-to-right formulas we say F_2 is *bigger than* F_1 if there is $a \in M$ with $F_1(M, a) \subset F_2(M, a)$. If M is 1-transitive and this holds for some a, it holds for all a. This gives a total ordering on the (finite) set of all convex-to-right formulas $F(x, y)$ (viewed up to equivalence modulo $Th(M)$).

Consider $F(M, a)$ for arbitrary $a \in M$. In general, $F(M, a)$ has no the right endpoint in M. For example, if $dcl(a) = \{a\}$ holds for some $a \in M$, then for any convex-to-right formula $F(x, y)$ and any $a \in M$ the formula $F(M, a)$ has no the right endpoint in M. We write $f(y) := \text{rend } F(M, y)$, assuming that $f(y)$ is the right endpoint of the set $F(M, y)$ that lies, in general, in the definable completion $\overline{M}$ of M. Then, f is a function mapping M in $\overline{M}$.

Let $F(x, y)$ be a convex-to-right formula. We say that $F(x, y)$ is *equivalence-generating* if for any $a, b \in M$ such that $M \models F(b, a)$ the following holds:

$$M \models \forall x(K(b, x, a) \land x \neq a \to [F(x, a) \leftrightarrow F(x, b)]).$$

Lemma 1.1. [22] *Let M be an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, $F(x, y)$ be a convex-to-right formula that is equivalence-generating. Then $E(x, y) := F(x, y) \lor F(y, x)$ is an equivalence relation partitioning M into infinite convex classes.*

Let M, N be circularly ordered structures. The 2-*reduct* of M is a circularly ordered structure with the same universe of M and consisting of predicates for each $\emptyset$-definable relation on M of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure M is *isomorphic* to N *up to binarity* or *binarily isomorphic to* N if the 2-reduct of M is isomorphic to the 2-reduct of N.

The following definition can be used in a circular ordered structure as well.

Definition 1. [23], [24] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set A* ($RC(A)$) is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if A is finite and non-empty.

3) $RC(A) \geq 1$ if A is infinite.

4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- for every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- for every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

The following theorem characterizes up to binarity $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures of convexity rank greater than 1 having both a trivial definable closure and the condition that any convex-to-right formula is equivalence-generating:

Theorem 1.1. [16] *Let M be an $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1 with $dcl(a) = \{a\}$ for some $a \in M$ such that any convex-to-right formula is equivalence-generating.*

Then, M is isomorphic up to binarity to $M_{s,m} := \langle M, K^3, E_1^2, E_2^2, \dots, E_s^2, E_{s+1}^2 \rangle$, where M is a circularly ordered structure, M is densely ordered, $s, m \geq 1$; E_{s+1} is an equivalence relation, partitioning M into m infinite convex classes without endpoints; E_i for every $1 \leq i \leq s$ is an equivalence relation, partitioning each E_{i+1}-class into infinitely many infinite convex E_i-subclasses without endpoints so that the induced ordering on E_i-subclasses is dense without endpoints.

In [9] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. In [11] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [12]–[13] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [14] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure.

Here, we describe algebras of binary isolating formulas for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure.

2 Results

Definition 2. [28] Let $p \in S_1(\emptyset)$ be non-algebraic. The algebra $\mathcal{P}_{\nu(p)}$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_{\nu(p)}$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_{\nu(p)}$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_{\nu(p)}$. We also say that an m-deterministic algebra $\mathcal{P}_{\nu(p)}$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic.

We say that the algebra $\mathcal{P}_{\nu(p)}$ is *$\exists$-maximally absorbing* if there exist $u_1, u_2 \in \rho_{\nu(p)}$ such that $u_1 \cdot u_2$ consists of all the labels of $\mathcal{P}_{\nu(p)}$.

Example 2. Consider the structure $M_{1,1} := \langle M, K^3, E_1^2 \rangle$ from Theorem 1.1. We assert that $Th(M_{1,1})$ has four binary isolating formulas:

$$\theta_0(x, y) := x = y,$$

$$\theta_1(x, y) := E_1(x, y) \land x \neq y \land \forall t[K(x, t, y) \to E_1(x, t)],$$

$$\theta_2(x, y) := \neg E_1(x, y),$$

$$\theta_3(x, y) := E_1(x, y) \land x \neq y \land \forall t[K(y, t, x) \to E_1(x, t)].$$

Clearly,

$$K_0(\theta_0(a, M), \theta_1(a, M), \theta_2(a, M), \theta_3(a, M))$$

holds for every $a \in M$.

Define the labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x, y), \text{ where } 0 \leq k \leq 3.$$

It easy to check that for the algebra $\mathfrak{P}_{M_{1,1}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{0,1,3\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2,3\}$	$\{2\}$
3	$\{3\}$	$\{0,1,3\}$	$\{2\}$	$\{3\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,1}}$ is commutative and strictly 4-deterministic.

Example 3. Consider now the structure $M_{1,2} := \langle M, K^3, E_1^2, E_2^2\rangle$ from Theorem 1.1. We assert that $Th(M_{1,2})$ has six binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$
$$\theta_2(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(x,t,y) \to E_2(x,t)],$$
$$\theta_3(x,y) := \neg E_2(x,y),$$
$$\theta_4(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(y,t,x) \to E_2(x,t)],$$
$$\theta_5(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

Clearly,

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M))$$

holds for every $a \in M$.

Define the labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \leq k \leq 5.$$

It easy to check that for the algebra $\mathfrak{P}_{M_{1,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{0,1,5\}$
2	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{0,1,2,4,5\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,1,2,4,5\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\{0,1,2,4,5\}$	$\{3\}$	$\{4\}$	$\{4\}$
5	$\{5\}$	$\{0,1,5\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,2}}$ is commutative and strictly 5-deterministic.

Proposition 2.1. *The algebra $\mathfrak{P}_{M_{1,m}}$ of binary isolating formulas has $m+4$ labels, is commutative and strictly 5-deterministic for every natural number $m \geq 2$.*

Proof. The universe M of the structure $M_{1,m}$ is partitioned by the equivalence relation E_2 into m infinite convex classes. Take an arbitrary element $a \in M$. It belongs to one of these convex classes. In this convex class five binary isolating formulas appear:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$
$$\theta_2(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(x,t,y) \to E_2(x,t)],$$
$$\theta_{m+2}(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(y,t,x) \to E_2(x,t)],$$

$$\theta_{m+3}(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

There remain $m-1$ convex classes, where there are no elements lying in the algebraic closure of the element a, defining additionally $m-1$ binary isolating formulas. These formulas are defined as follows:

$$\theta_i(x,y) := \neg E_2(x,y) \wedge \forall t[K(x,t,y) \wedge \neg E_1(x,t) \wedge \neg E_2(t,y) \to \vee_{s=2}^{i-1}\theta_s(x,t)], 3 \leq i \leq m+1.$$

Thus, there are $5+(m-1) = m+4$ binary isolating formulas, and we have defined the formulas so that for any $a \in M$ the following holds:

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \ldots, \theta_m(a,M), \theta_{m+1}(a,M), \theta_{m+2}(a,M), \theta_{m+3}(a,M)).$$

Prove now the commutativity. First, it is obvious, $0 \cdot k = k \cdot 0 = \{k\}$ for every $0 \leq k \leq m+3$. Suppose further that both $k_1 \neq 0$ and $k_2 \neq 0$.

Case 1. $k_1 + k_2 = m+4$.

If $k_1 = 1$, then $k_2 = m+3$. In this case each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains, as a conjunctive member, the formula $E_1(x,y)$, i.e. the formula $E_1(x,y)$ is compatible with

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)].$$

We have: for any t, satisfying the formula $\theta_{k_1}(x,t)$, it follows that $t \in E_1(x,M)$ and t is in this class to the right of the element x. Considering an arbitrary element y, satisfying the formula $\theta_{k_2}(t,y)$, we obtain that $y \in E_1(t,M)$ and y is in this class to the left of the element t, i.e. the formula

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with every formula from the list of formulas with labels $\{0,1,m+3\}$. Consequently, $k_1 \cdot k_2 = \{0,1,m+3\}$. We can show similarly that $k_2 \cdot k_1 = \{0,1,m+3\}$.

If $k_1 = 2$, then $k_2 = m+2$. In this case each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains as a conjunctive member the formula $E_2(x,y) \wedge \neg E_1(x,y)$, i.e. the formula $E_2(x,y) \wedge \neg E_1(x,y)$ is compatible with

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)].$$

We have: for any t, satisfying the formula $\theta_{k_1}(x,t)$, it follows that $t \in E_2(x,M)$, $t \notin E_1(x,M)$, and t is in this class to the right of the element x. Considering an arbitrary element y, satisfying the formula $\theta_{k_2}(t,y)$, we obtain that $y \in E_2(t,M)$, $y \notin E_1(t,M)$, and y is in this class to the left of the element t, i.e. the formula

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with every formula from the list of formulas with labels $\{0,1,2,m+2,m+3\}$. Consequently, $k_1 \cdot k_2 = \{0,1,2,m+2,m+3\}$. We can show similarly that

$$k_2 \cdot k_1 = \{0,1,2,m+2,m+3\}.$$

Let now $2 < k_1 < m+2$. Then, we also have that $2 < k_2 < m+2$. Consequently, each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains as a conjunctive member the formula $\neg E_2(x,y)$. We have: t lies in the (k_1-1)-th E_2-class from $E_2(x,M)$ (i.e. the E_2-class, containing x is the first E_2-class; the next clockwise E_2-class is the second, etc.); y lies in the (k_2-1)-th E_2-class from $E_2(t,M)$. Then, we obtain that y lies in the (k_1+k_2-2)-th E_2-class from $E_2(x,M)$. But $k_1+k_2-2 = m+2$, i.e. y falls into $E_2(x,M)$. Therefore, we get that

$$k_1 \cdot k_2 = k_2 \cdot k_1 = \{0,1,2,m+2,m+3\}.$$

Case 2. $k_1 + k_2 < m+4$.

Let us first assume that $k_1 = 1$. If $k_2 = 1$, then

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with the formula $E_1(x,y)$. We have: t lies in the same E_1-class with x and in this class to the right of it; y lies in the same E_1-class with t and also to the right of it in this class. Consequently, y lies in the same E_1-class with x and in this class to the right of it, i.e. $1 \cdot 1 = \{1\}$.

Suppose now that $k_1 = 2$. If $k_2 = 2$ then

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with the formula $E_2(x,y)$. We have: t lies in the same E_2-class with x and in this class to the right of it; y lies in the same E_2-class with t and also in this class to the right of it. Consequently, y lies in the same E_2-class with x and in this class to the right of it, i.e. $2 \cdot 2 = \{2\}$.

Let now $k_2 > 2$. Clearly, $k_2 < m+2$ (since $k_1 + k_2 < m+4$). We have: t lies in the same E_2-class with x and in this class to the right of it; y lies in the (k_2-1)-th E_2-class from $E_2(t,M)$. Consequently, y lies in the (k_2-1)-th E_2-class from $E_2(x,M)$, i.e. $2 \cdot k_2 = \{k_2\}$. We can show similarly that $k_2 \cdot 2 = \{k_2\}$.

Suppose now that $k_1 > 2$ and $k_2 > 2$. Clearly, $k_1 < m+2$ and $k_2 < m+2$. Then each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains as a conjunctive member the formula $\neg E_2(x,y)$. We have: t lies in the (k_1-1)-th E_2-class from $E_2(x,M)$; y lies in the (k_2-1)-th E_2-class from $E_2(t,M)$. Then, we obtain that y lies in the (k_1+k_2-2)-th E_2-class from $E_2(x,M)$, i.e. $k_1 \cdot k_2 = \{k_1+k_2-2\}$. We can show similarly that $k_2 \cdot k_1 = \{k_1+k_2-2\}$.

Case 3. $k_1 + k_2 > m+4$.

In this case $k_1 > 1$ and $k_2 > 1$ (since otherwise we would obtain that $k_1 + k_2 \le m+4$).

Suppose first that $k_1 = 2$. Then, we unambiguously obtain that $k_2 = m+3$. We have: t lies in $E_2(x,M)$ and t is in this class to the right of the element x; y lies in $E_1(t,M)$ and t is in this class to the left of the element t, whence we obtain that $k_1 \cdot k_2 = \{k_1\}$. We can show similarly that $k_2 \cdot k_1 = \{k_1\}$.

Let now $k_1 > 2$. We have: t lies in the (k_1-1)-th E_2-class from $E_2(x,M)$. In this case $k_2 \ge m+2$, i.e. k_2 can take only the following values: $m+2$ and $m+3$. Then, we obtain: y lies in $E_2(t,M) \setminus E_1(t,M)$ or $E_1(t,M)$ and t is in the corresponding class to the left of the element t, whence we obtain that $k_1 \cdot k_2 = \{k_1\}$. We can show similarly that $k_2 \cdot k_1 = \{k_1\}$.

Suppose now that $k_1 = m+2$. We have: t lies in $E_2(x,M)$ and t is in this class to left of the element x. In this case $k_2 > 2$. If $k_2 \ge m+2$, then again we get that $k_1 \cdot k_2 = \{k_1\}$. We can show similarly that $k_2 \cdot k_1 = \{k_1\}$.

Further, suppose that $2 < k_1 < m+2$ and $2 < k_2 < m+2$. We have: t lies in the (k_1-1)-th E_2-class from $E_2(x,M)$; y lies in the (k_2-1)-th E_2-class from $E_2(t,M)$, but at the same time y jumps over $E_2(x,M)$ that is consistent with five binary isolating formulas. Therefore, y lies in the $(k_1+k_2+2)[\text{mod } m+4]$-th E_2-class from $E_2(x,M)$. Consequently, the formula

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

uniquely determines the formula $\theta_{(k_1+k_2+2)[\text{mod } m+4]}(x,y)$. We can show similarly that

$$k_2 \cdot k_1 = (k_1+k_2+2)[\text{mod } m+4].$$

□

Example 4. Consider now the structure $M_{2,1} := \langle M, K^3, E_1^2, E_2^2 \rangle$ from Theorem 1.1. We assert that $Th(M_{2,1})$ has six binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$
$$\theta_2(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(x,t,y) \to E_2(x,t)],$$
$$\theta_3(x,y) := \neg E_2(x,y),$$
$$\theta_4(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(y,t,x) \to E_2(x,t)],$$
$$\theta_5(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

Clearly, $K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M))$ holds for every $a \in M$. Define the labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \leq k \leq 5.$$

It easy to check that for the algebra $\mathfrak{P}_{M_{2,1}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{0,1,5\}$
2	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{0,1,2,4,5\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,1,2,3,4,5\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\{0,1,2,4,5\}$	$\{3\}$	$\{4\}$	$\{4\}$
5	$\{5\}$	$\{0,1,5\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{2,1}}$ is commutative and strictly 6-deterministic.

Proposition 2.2. *The algebra $\mathfrak{P}_{M_{s,1}}$ of binary isolating formulas has $2s+2$ labels, is commutative and strictly $(2s+2)$-deterministic for every natural number $s \geq 1$.*

Proof. The universe M of the structure $M_{s,1}$ is partitioned by the equivalence relation E_s into infinitely many infinite convex classes, so that the induced ordering on E_s-classes is dense without endpoints; in addition, for any $2 \leq i \leq s$, each E_i-class is partitioned into infinitely many convex E_{i-1}-subclasses, so that the induced order on E_{i-1}-subclasses is dense without endpoints.

We have the following binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$
$$\theta_i(x,y) := E_i(x,y) \wedge \neg E_{i-1}(x,y) \wedge \forall t[K(y,t,x) \to E_i(x,t)], 2 \leq i \leq s,$$
$$\theta_{s+1}(x,y) := \neg E_s(x,y),$$
$$\theta_j(x,y) := E_{2s+2-j}(x,y) \wedge \neg E_{2s+1-j}(x,y) \wedge \forall t[K(y,t,x) \to E_{2s+2-j}(x,t)], s+2 \leq j \leq 2s,$$
$$\theta_{2s+1}(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

Thus, there exist $2s+2$ binary isolating formulas, and we have defined the formulas so that

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \ldots, \theta_{2s}(a,M), \theta_{2s+1}(a,M))$$

holds for any $a \in M$.

Prove now the commutativity. First, it is obvious that $0 \cdot k = k \cdot 0 = \{k\}$ for any $0 \leq k \leq 2s+1$. Suppose further that $k_1 \neq 0$ and $k_2 \neq 0$.

Case 1. $k_1 + k_2 = 2s+2$.

If $k_1 = l$ for some $1 \leq l \leq s$, then $k_2 = 2s + 2 - l$. Then, each of the formulas $\theta_{k_1}(x, y)$ and $\theta_{k_2}(x, y)$ contains, as a conjunctive member, the formula $E_l(x, y)$, i.e. the formula $E_l(x, y)$ is compatible with

$$\exists t[\theta_{k_1}(x, t) \wedge \theta_{k_2}(t, y)].$$

We have: for any t, satisfying the formula $\theta_{k_1}(x, t)$, it follows that $t \in E_l(x, M) \setminus E_{l-1}(x, M)$ (if $l = 1$, then $t \in E_1(x, M)$) and t is in this class to the right of the element x. Considering an arbitrary element y, satisfying the formula $\theta_{k_2}(t, y)$, we obtain that $y \in E_l(t, M) \setminus E_{l-1}(t, M)$ (if $l = 1$, then $y \in E_1(t, M)$) and y is in this class to the left of the element t, i.e. the formula

$$\exists t[\theta_{k_1}(x, t) \wedge \theta_{k_2}(t, y)]$$

is compatible with every formula from the list of formulas with labels $\{0, 1, \ldots, l, 2s+2-l, \ldots, 2s+1\}$. Consequently, $k_1 \cdot k_2 = \{0, 1, \ldots, l, 2s + 2 - l, \ldots, 2s + 1\}$. We can show similarly that

$$k_2 \cdot k_1 = \{0, 1, \ldots, l, 2s + 2 - l, \ldots, 2s + 1\}.$$

Let now $k_1 = s + 1$. Then, we also have that $k_2 = s + 1$ and each of the formulas $\theta_{k_1}(x, y)$ and $\theta_{k_2}(x, y)$ contains as a conjunctive member the formula $\neg E_s(x, y)$.

We have: for any t satisfying the formula $\theta_{k_1}(x, t)$, $\neg E_s(x, t)$ holds. Considering an arbitrary element y, satisfying the formula $\theta_{k_2}(t, y)$, we obtain that $\neg E_s(t, y)$. Thus, both $\neg E_s(x, y)$ and $E_s(x, y)$ are possible. Consequently, $k_1 \cdot k_2 = \{0, 1, 2, \ldots, 2s, 2s + 1\}$. We can show similarly that

$$k_2 \cdot k_1 = \{0, 1, 2, \ldots, 2s, 2s + 1\}.$$

If $k_1 = l$ for some $s+2 \leq l \leq 2s+1$, then $k_2 = 2s+2-l$, i.e. $1 \leq k_2 \leq l$. We can show similarly that

$$k_1 \cdot k_2 = \{0, 1, \ldots, l, 2s + 2 - l, \ldots, 2s + 1\}$$

and

$$k_2 \cdot k_1 = \{0, 1, \ldots, l, 2s + 2 - l, \ldots, 2s + 1\}.$$

Thus, in the case $k_1 = k_2 = s + 1$ we obtain that the product of labels k_1 and k_2 contains all the labels of the algebra, whence we conclude that the algebra $\mathfrak{P}_{M_{s,1}}$ is strictly $(2s + 2)$-deterministic.

Case 2. $k_1 + k_2 < 2s + 2$.

Suppose first that $1 \leq k_1, k_2 \leq s$. If $k_1 = k_2$, then since each of the formulas $\theta_{k_1}(x, y)$ and $\theta_{k_2}(x, y)$ contains as a conjunctive member the formula $E_l(x, y)$ for some $1 \leq l \leq s$, we obtain that $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l\}$. If $k_1 < k_2$, then since $\theta_{k_1}(x, y)$ contains as a conjunctive member the formula $E_{l_1}(x, y)$, and $\theta_{k_2}(x, y)$ contains as a conjunctive member the formula $E_{l_2}(x, y)$ for some $1 \leq l_1 < l_2 \leq s$, we obtain that $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$. Similar reasoning is for the case $k_1 > k_2$.

Suppose now that $1 \leq k_1 \leq s$ and $k_2 > s$. If $k_2 = s + 1$, then for any t satisfying the formula $\theta_{k_1}(x, t)$, it follows that $t \in E_l(x, M)$ for some $1 \leq l \leq s$; while for any y, satisfying the formula $\theta_{k_2}(t, y)$, $\neg E_s(t, y)$ holds. Whence we conclude that $k_1 \cdot k_2 = k_2 \cdot k_1 = \{s + 1\}$. If $k_2 \neq s + 1$, then $s + 2 \leq k_2 < 2s + 1$ and for any y satisfying the formula $\theta_{k_2}(t, y)$, it follows that $y \in E_{l_2}(t, M)$ for some $1 \leq l_2 \leq s$ (here $l_2 = 2s + 2 - k_2$).

If $l > l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l\}$. If $l < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$. The case $l = l_2$ is impossible, since otherwise we obtain $l + l_2 = 2s + 2$.

The case in which $k_1 > s$ is considered similarly (in this case $1 \leq k_2 < s$).

Case 3. $k_1 + k_2 > 2s + 2$.

In this case $k_1 > 1$ and $k_2 > 1$ (indeed, if we suppose that $k_1 = 1$, then k_2 must be greater than $2s + 1$ that is impossible). If $2 \leq k_1 \leq s$, then $k_2 > s + 2$, i.e. $s + 3 \leq k_2 \leq 2s + 1$.

We have: $t \in E_{l_1}(x, M)$ for some $2 \leq l_1 \leq s$, $y \in E_{l_2}(t, M)$ for some $1 \leq l_2 \leq s - 1$.

If $l_1 > l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

The case $l_1 = l_2$ is also impossible, since otherwise we obtain $l_1 + l_2 = 2s + 2$.

Let now $k_1 > s$. In this case $s + 2 \leq k_2 \leq 2s + 1$. If $k_1 = s + 1$, then we obtain $\neg E_s(x,t)$. Consequently, $k_1 \cdot k_2 = k_2 \cdot k_1 = \{s + 1\}$.

If $k_1 \geq s + 2$, then $s + 1 \leq k_2 \leq 2s + 1$. If $k_2 = s + 1$, then we obtain $k_1 \cdot k_2 = k_2 \cdot k_1 = \{s + 1\}$. If $k_2 \geq s + 2$, then we have: $t \in E_{l_1}(x, M)$ for some $1 \leq l_1 \leq s$, $y \in E_{l_2}(t, M)$ for some $1 \leq l_2 \leq s$. If $l_1 \geq l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$. $\square$

Corollary 2.1. *The algebra $\mathfrak{P}_{M_{s,1}}$ of binary isolating formulas is $\exists$-maximally absorbing for every natural number $s \geq 1$.*

Example 5. Consider now the structure $M_{2,2} := \langle M, K^3, E_1^2, E_2^2, E_3^2 \rangle$ from Theorem 1.1. Here $E_3(x,y)$ is an equivalence relation partitioning the universe of the structure into two infinite convex classes. We assert that $Th(M_{2,2})$ has eight binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$

$$\theta_2(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(x,t,y) \to E_2(x,t)],$$

$$\theta_3(x,y) := E_3(x,y) \wedge \neg E_2(x,y) \wedge \forall t[K(x,t,y) \to E_3(x,t)],$$

$$\theta_4(x,y) := \neg E_3(x,y),$$

$$\theta_5(x,y) := E_3(x,y) \wedge \neg E_2(x,y) \wedge \forall t[K(y,t,x) \to E_3(x,t)],$$

$$\theta_6(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(y,t,x) \to E_2(x,t)],$$

$$\theta_7(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

Clearly,

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M), \theta_6(a,M), \theta_7(a,M))$$

holds for every $a \in M$.

Define the labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \leq k \leq 7.$$

It easy to check that for the algebra $\mathfrak{P}_{M_{2,2}}$ the following equalities hold:
$0 \cdot k = k \cdot 0 = \{k\}$ for every $0 \leq k \leq 7$,
$1 \cdot k = k \cdot 1 = \{k\}$ for every $1 \leq k \leq 6$, and $1 \cdot 7 = \{0, 1, 7\}$,
$2 \cdot k = k \cdot 2 = \{k\}$ for every $2 \leq k \leq 5$, $2 \cdot 6 = \{0, 1, 2, 6, 7\}$, and $2 \cdot 7 = \{2\}$,
$3 \cdot k = k \cdot 3 = \{k\}$ for every $3 \leq k \leq 4$, $3 \cdot 5 = \{0, 1, 2, 3, 5, 6, 7\}$, and
$3 \cdot 6 = 6 \cdot 3 = \{3\}$, $3 \cdot 7 = 7 \cdot 3 = \{3\}$,
$4 \cdot k = k \cdot 4 = \{4\}$ for every $1 \leq k \leq 3$, $4 \cdot 4 = \{0, 1, 2, 3, 5, 6, 7\}$, and
$4 \cdot 5 = 5 \cdot 4 = \{4\}$, $4 \cdot 6 = 6 \cdot 4 = \{4\}$, $4 \cdot 7 = 7 \cdot 4 = \{4\}$,
$5 \cdot k = k \cdot 5 = \{5\}$ for every $5 \leq k \leq 7$, and $5 \cdot 3 = \{0, 1, 2, 3, 5, 6, 7\}$,
$6 \cdot 6 = \{6\}$, $6 \cdot 7 = 7 \cdot 6 = \{6\}$, and $6 \cdot 2 = \{0, 1, 2, 6, 7\}$,
$7 \cdot 7 = \{7\}$, and $7 \cdot 1 = \{0, 1, 7\}$.

According to these equalities, the algebra $\mathfrak{P}_{M_{2,2}}$ is commutative and strictly 7-deterministic.

Theorem 2.1. *The algebra $\mathfrak{P}_{M_{s,m}}$ of binary isolating formulas has $2s+m+2$ labels, is commutative and strictly $(2s+3)$-deterministic for any natural numbers $s, m \geq 1$.*

Proof. The universe M of the structure $M_{s,m}$ is partitioned by the equivalence relation E_{s+1} into m infinite convex classes. Take an arbitrary element $a \in M$. It falls into one of these convex classes. In this convex class, $2s+3$ binary isolating formulas arise:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$

$$\theta_i(x,y) := E_i(x,y) \wedge \neg E_{i-1}(x,y) \wedge \forall t[K(x,t,y) \to E_i(x,t)],\ 2 \leq i \leq s+1,$$

$$\theta_j(x,y) := E_{2s+m+2-j}(x,y) \wedge \neg E_{2s+m+1-j}(x,y) \wedge \forall t[K(x,t,y) \to E_{2s+m+2-j}(x,t)],$$

$$\text{where } s+m+1 \leq j \leq 2s+m,$$

$$\theta_{2s+m+1}(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

There remain $m-1$ convex classes, where there are no elements lying in the algebraic closure of the element a, defining additionally $m-1$ binary isolating formulas. These formulas are defined as follows:

$$\theta_l(x,y) := \neg E_{s+1}(x,y) \wedge \forall t[K(x,t,y) \wedge \neg E_s(x,t) \wedge \neg E_{s+1}(t,y) \to \vee_{k=s+1}^{l-1}\theta_k(x,t)],$$

$$\text{where } s+2 \leq l \leq s+m.$$

Thus, we get $2s+3+(m-1) = 2s+m+2$ binary isolating formulas, and we have defined the formulas, so that

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \ldots, \theta_{2s+m}(a,M), \theta_{2s+m+1}(a,M)).$$

holds for any $a \in M$.

Prove now the commutativity. First, it is obvious that $0 \cdot k = k \cdot 0 = \{k\}$ for any $0 \leq k \leq 2s+m+1$. Suppose further that $k_1 \neq 0$ and $k_2 \neq 0$.

Case 1. $k_1 + k_2 = 2s+m+2$.

If $k_1 = 1$, then clearly $k_2 = 2s+m+1$ and each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains, as a conjunctive member, the formula $E_1(x,y)$, i.e. the formula $E_1(x,y)$ is compatible with

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)].$$

We have: for any t, satisfying the formula $\theta_{k_1}(x,t)$, it follows that $t \in E_1(x,M)$ and t is to the right of the element x. Considering an arbitrary element y satisfying the formula $\theta_{k_2}(t,y)$, we obtain that $y \in E_1(t,M)$ and y is to the left of the element t, i.e. we obtain that the formula

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with every formula of the list of formulas with labels $\{0,1,2s+m+1\}$. Consequently, $k_1 \cdot k_2 = \{0,1,2s+m+1\}$. We can show similarly that $k_2 \cdot k_1 = \{0,1,2s+m+1\}$.

If $k_1 = l$ for some $2 \leq l \leq s+1$, we have $k_2 = 2s+m+2-l$. Then, each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains as a conjunctive member the formula

$$E_l(x,y) \wedge \neg E_{l-1}(x,y).$$

We have the following: $t \in E_l(x,M) \setminus E_{l-1}(x,M)$ and t is in this class to the right of the element x; $y \in E_l(t,M) \setminus E_{l-1}(t,M)$ and y is in this class to the left of the element t. Whence we obtain that

$$k_1 \cdot k_2 = k_2 \cdot k_1 = \{0,1,\ldots,l,2s+m+2-l,\ldots,2s+m+1\}.$$

We can show similarly that

$$k_2 \cdot k_1 = \{0, 1, \ldots, l, 2s + m + 2 - l, \ldots, 2s + m + 1\}.$$

Suppose now that $s + 2 \le k_1 \le s + m$. Then, $k_2 = 2s + m + 2 - k_1$, i.e. we also have $s + 2 \le k_2 \le s + m$ and each of the formulas $\theta_{k_1}(x, y)$ and $\theta_{k_2}(x, y)$ contains as a conjunctive member the formula $\neg E_{s+1}(x, y)$.

We have the following: t lies in the $(k_1 - s)$-th E_{s+1}-class from $E_{s+1}(x, M)$; y lies in the $(k_2 - s)$-th E_{s+1}-class from $E_{s+1}(t, M)$. Then, we obtain that y lies in the $(k_1 + k_2 - 2s - 1)$-th E_{s+1}-class from $E_{s+1}(x, M)$. But $k_1 + k_2 - 2s - 1 = m + 1$, i.e. y falls into $E_{s+1}(x, M)$, whence

$$k_1 \cdot k_2 = \{0, 1, \ldots, s + 1, s + m + 1, \ldots, 2s + m + 1\}.$$

We can show similarly that

$$k_2 \cdot k_1 = \{0, 1, \ldots, s + 1, s + m + 1, \ldots, 2s + m + 1\}.$$

Let now $s + m + 1 \le k_1 \le 2s + m + 1$. Then, obviously $1 \le k_2 \le s + 1$. If $k_1 = l$ for some $s + m + 1 \le l \le 2s + m + 1$, we can show similarly that

$$k_1 \cdot k_2 = k_2 \cdot k_1 = \{0, 1, \ldots, 2s + m + 2 - l, s + m + 1, \ldots, l\}.$$

Case 2. $k_1 + k_2 < 2s + m + 2$.

First, suppose that $1 \le k_1 \le s + 1$. If $1 \le k_2 \le s + 1$, then we have: $t \in E_{l_1}(x, M)$ for some $1 \le l_1 \le s + 1$ and t is in this class to the right of the element x; $y \in E_{l_2}(t, M)$ for some $1 \le l_2 \le s + 1$ and y is in this class to the right of the element t. Then, we obtain that if $l_1 \ge l_2$, $y \in E_{l_1}(x, M)$ and consequently $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $y \in E_{l_2}(x, M)$, and consequently $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

If $s + 2 \le k_2 \le s + m$, then we have: $t \in E_l(x, M)$ for some $1 \le l \le s + 1$, and $y \in \neg E_{s+1}(t, M)$, whence we obtain $\neg E_{s+1}(x, y)$, i.e. $k_1 \cdot k_2 = k_2 \cdot k_1 = \{k_2\}$.

Suppose now that $k_2 > s + m$. We have the following: $t \in E_{l_1}(x, M)$ for some $1 \le l_1 \le s + 1$ and t is in this class to the right of the element x; $y \in E_{l_2}(t, M)$ for some $1 \le l_2 \le s + 1$ and y is in this class to the left of the element t. And the case $l_1 = l_2$ is impossible, since $k_1 + k_2 < 2s + m + 2$. If $l_1 > l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

Other cases are considered similarly.

Case 3. $k_1 + k_2 > 2s + m + 2$.

In this case $k_1 > 1$ and $k_2 > 1$ (since otherwise we would obtain that $k_1 + k_2 \le 2s + m + 2$).

If $2 \le k_2 \le s+1$ then $k_2 > s+m+1$. We have the following: $t \in E_{l_1}(x, M)$ for some $2 \le l_1 \le s+1$ and t is in this class to the right of the element x; $y \in E_{l_2}(t, M)$ for some $2 \le l_2 \le s$ and y is in this class to the left of the element t. And the case $l_1 = l_2$ is impossible, since $k_1 + k_2 > 2s + m + 2$. If $l_1 > l_2$ then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$ then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

Suppose now that $s+2 \le k_1 \le s+m$. Then, $k_2 > s+m$. We have the following: $t \in \neg E_{s+1}(x, M)$ and $y \in E_l(t, M)$ for some $2 \le l \le s + 1$, whence we obtain $k_1 \cdot k_2 = k_2 \cdot k_1 = \{k_1\}$.

Let now $s + m + 1 \le k_1 \le 2s + m + 1$. If $2 \le k_2 \le s + 1$, we have that $t \in E_{l_1}(x, M)$ for some $2 \le l_1 \le s + 1$ and t is in the this class to the left of the element x; $y \in E_{l_2}(t, M)$ for some $2 \le l_2 \le s$ and y is in this class to the right of the element t. And the case $l_1 = l_2$ is impossible, since $k_1 + k_2 > 2s + m + 2$. If $l_1 > l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

If $s + 2 \le k_2 \le s + m$, then we have: $t \in E_l(x, M)$ for some $1 \le l \le s + 1$, and $y \in \neg E_{s+1}(t, M)$, whence we obtain $\neg E_{s+1}(x, y)$, i.e. $k_1 \cdot k_2 = k_2 \cdot k_1 = \{k_2\}$.

Suppose now that $k_2 > s + m$. We have the following: $t \in E_{l_1}(x, M)$ for some $1 \le l_1 \le s + 1$ and t is in this class to the left of the element x; $y \in E_{l_2}(t, M)$ for some $1 \le l_2 \le s + 1$ and y is in this class to the left of the element t. If $l_1 \ge l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$. □

Corollary 2.2. *The algebra $\mathfrak{P}_{M_{s,m}}$ is $\exists$-maximally absorbing if and only if $m = 1$.*

Acknowledgments

This work was supported by Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan, grant no. AP23484665.

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Beibut Shaiykovich Kulpeshov
Institute of Mathematics and Mathematical Modeling
28 Shevchenko St,
050010, Almaty, Republic of Kazakhstan
and
Kazakh-British Technical University
59 Tole bi St,
050000, Almaty, Republic of Kazakhstan
E-mails: b.kulpeshov@kbtu.kz, kulpesh@mail.ru

Received: 12.05.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 3 (2025), 57 – 67

RECONSTRUCTION OF THE WEIGHTED DIFFERENTIAL OPERATOR WITH POINT δ-INTERACTION

M.Dzh. Manafov, A. Kablan

Communicated by B.E. Kangyzhin

Key words: differential operator, uniqueness results, point δ-interactions.

AMS Mathematics Subject Classification: 34A55, 34B24, 47E05.

Abstract. In this article, we provide various uniqueness results for inverse problems of weighted differential operator with point $\delta-$interaction. In terms of the method of spectral mappings, we also offer step by step strategies for finding their potential and boundary conditions basing either on the Weyl function, on spectral data, or on two spectra.

DOI: https://doi.org/10.32523/2077-9879-2025-16-3-57-67

1 Introduction

In this work, we study inverse problems for the boundary value problems generated by the differential equation

$$ly := -y'' + q(x)\,y = \lambda y, \qquad x \in (0,a) \cup (a,T) \tag{1.1}$$

with the Robin boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(T) + Hy(T) = 0, \tag{1.2}$$

and the transmission conditions at the point $x = a$

$$I(y) := \begin{cases} y(a+0) = y(a-0) \equiv y(a) \\ y'(a+0) - y'(a-0) = -\alpha\lambda y(a), \end{cases} \tag{1.3}$$

where $q(x)$ is a real function belonging to the space $L_2[0,T]$, λ is a spectral parameter and h, H, and α are real numbers with $\alpha > 0$. Denote the boundary value problems, defined above, by $L(q(x), h, H)$.

It is important to note that, we can interpret problem (1.1) and (1.3) as analyzing the equation

$$-y'' + q(x)\,y = \lambda\rho(x)\,y, \quad x \in (0,T), \tag{1.4}$$

when $\rho(x) = 1 + \alpha\delta(x)$ where $\delta(x)$ is the Dirac Delta-function (see [1]).

One type of problems, the direct problem, consists of examining the spectral properties of an operator. But some problems in mathematical physics require the investigation of inverse problems of spectral analysis for various differential operators, which require the recovery of operators from some of their given spectral data. Such problems are often considered in mathematics and various branches of natural science and technical science. Direct and inverse problems for the classical Sturm-Liouville operators have been comprehensively investigated in [6, 10, 15] and references therein. Some classes of direct and inverse problems for discontinuous boundary value problems in various statements have been considered in [2, 7, 8, 12, 13, 16, 17, 18]. Notice that, spectral characteristics for weighted Sturm-Liouville operator with point $\delta-$interactions have been investigated in [9, 14]. Here, we provide procedures for finding the potential of a problem and its boundary conditions basing either on the Weyl function, on spectral data, or on two spectra in terms of the method of spectral mappings.

2 Constructing the Hilbert space relevant to the problem and some of its spectral properties

We will start this section by defining the Hilbert space $\mathbb{H} := L_2[0,T] \oplus \mathbb{C}$ of the two component vectors, equipped with the inner product

$$\langle f, g\rangle_{\mathbb{H}} := \int_0^T f_1(x)\overline{g}_1(x)dx + \frac{1}{\alpha} f_2\overline{g}_2$$

for

$$f = \begin{pmatrix} f_1(x) \\ f_2 \end{pmatrix}, g = \begin{pmatrix} g_1(x) \\ g_2 \end{pmatrix},$$

where $f_1(x)$, $g_1(x) \in L_2(0,T)$ and $f_2, g_2 \in \mathbb{C}$. In the space $\mathbb{H}$, we define the operator L

$$L : \mathbb{H} \to \mathbb{H}$$

with the domain

$$D(L) = \{f \in \mathbb{H} | f_1, f_1' \in AC((0,a) \cup (a,T)), lf_1 \in L_2[(0,T) \setminus \{a\}], f_2 = \alpha f_1(a), U(f_1) = V(f_1) = 0\}$$

and the operator rule

$$L(f) = \begin{pmatrix} lf_1 \\ f_1'(a-0) - f'(a+0) \end{pmatrix}.$$

Here, $AC(\cdot)$ stands for the set of all functions that are absolutely continuous on a related interval.

Theorem 2.1. *The operator L is symmetric.*

Proof. We obtain the equality $\langle Lf, g\rangle_{\mathbb{H}} = \langle f, Lg\rangle_{\mathbb{H}}$ for $f, g \in D(L)$ immediately from the conditions at the point $x = a$ and the fact that f and $\overline{g}$ satisfy the same boundary conditions (1.2). So, L is symmetric.

□

Corollary 2.1. *The function W defined by $W\{f, g; x\} = f(x)g'(x) - f'(x)g(x)$ is continuous on $(0,T)$.*

Lemma 2.1. *If $y(x,\lambda)$ and $z(x,\mu)$ are solutions to the equations $ly = \lambda y$ and $lz = \mu z$, respectively, then*

$$\frac{d}{dx} W\{y, z; x\} = (\lambda - \mu) yz.$$

Let $C(x,\lambda)$, $S(x,\lambda)$, $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be solutions to equation (1.1) under the following initial conditions:

$$C(0,\lambda) = 1, \ C'(0,\lambda) = 0, \ S(0,\lambda) = 0, \ S'(0,\lambda) = 1,$$

$$\varphi(0,\lambda) = 1, \ \varphi'(0,\lambda) = h, \ \psi(T,\lambda) = 1, \ \psi'(T,\lambda) = -H$$

and under transmission conditions (1.3). Then,

$$U(\varphi) = V(\psi) = 0.$$

Let us denote

$$\Delta(\lambda) = W\{\varphi, \psi; x\}. \tag{2.1}$$

Due to Corollary 1 and the Ostrogradskii-Liouville theorem (see [4, p. 83]) $W\{\varphi, \psi; x\}$ does not depend on x. Here, the function $\Delta(\lambda)$ is called the *characteristic function* of L. It is easly seen that

$$\Delta(\lambda) = -V(\varphi) = U(\psi), \tag{2.2}$$

and $\Delta(\lambda)$ is an entire function of λ, so it has at most countable set of zeros $\{\lambda_n\}_{n\geq 0}$.

Lemma 2.2. *The zeros $\{\lambda_n\}_{n\geq 0}$ of the characteristic function are the eigenvalues of the boundary value problem L. Also the functions $\varphi(x, \lambda_n)$ and $\psi(x, \lambda_n)$ are the eigenfunctions, and there exists a sequence $\{\beta_n\}$ such that*

$$\psi(x, \lambda_n) = \beta_n.\varphi(x, \lambda_n), \;\; \beta_n \neq 0.$$

Denote

$$\alpha_n := \int_0^T \varphi^2(x, \lambda_n)dx + \alpha\varphi^2(a, \lambda_n). \tag{2.3}$$

The set $\Omega = \{\lambda_n, \alpha_n\}_{n\geq 0}$ is called *the spectral data* associated with problem (1.1)–(1.3).

Lemma 2.3. *The following relation holds*

$$\dot{\Delta}(\lambda_n) = \alpha_n\beta_n,$$

where $\dot{\Delta}(\lambda) = d\Delta(\lambda)/d\lambda$.

We omit the proofs of Lemma 2.2 and Lemma 2.3 since they are similiar to those for the classical Sturm-Liouville operators (see [11]).

Corollary 2.2. *The eigenvalues $\{\lambda_n\}$ and the eigenfunctions $\varphi(x, \lambda_n)$, $\psi(x, \lambda_n)$ are real. Also all zeros of $\Delta(\lambda)$ are simple, i.e. $\dot{\Delta}(\lambda_n) \neq 0$.*

Now, consider the solution $\varphi(x, \lambda)$. Let $C_0(x, \lambda)$ and $S_0(x, \lambda)$ be smooth solutions to equation (1.1) on the interval $[0, T]$ under the initial condition

$$C_0(x, \lambda) = S'(0, \lambda) = 1, S_0(x, \lambda) = C_0'(0, \lambda) = 0. \tag{2.4}$$

Then,

$$C(x, \lambda) = C_0(x, \lambda), \;\; S(x, \lambda) = S_0(x, \lambda), \;\; 0 < x < a \tag{2.5}$$

$$\begin{aligned} C(x, \lambda) &= A_1C_0(x, \lambda) + B_1S_0(x, \lambda), \\ S(x, \lambda) &= A_2C_0(x, \lambda) + B_2S_0(x, \lambda), \;\; a < x < T, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} A_1 &= 1 + \alpha\lambda C_0(a, \lambda)S_0(a, \lambda), \quad B_1 = -\alpha\lambda C_0^2(a, \lambda), \\ A_2 &= \alpha\lambda S_0^2(a, \lambda), \quad B_2 = 1 - \alpha\lambda C_0(a, \lambda)S_0(a, \lambda). \end{aligned} \tag{2.7}$$

Let $\lambda = \rho^2$, $\rho = \sigma + i\tau$. It is easy to show that, the function $C_0(x, \lambda)$ satisfies the following integral equation:

$$C_0(x, \lambda) = \cos\rho x + \frac{1}{\rho}\int_0^x \sin\rho(x-t)q(t)C_0(t, \lambda)dt. \tag{2.8}$$

Using the method of successive approximations to solve problem (2.8), we obtain

$$
\begin{aligned}
C_0(x,\lambda) = \cos\rho x + \frac{\sin\rho x}{2\rho}\int_0^x q(t)dt + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)dt \\
+ O\left(\frac{1}{\rho^2}\exp(|\tau|)x\right).
\end{aligned} \tag{2.9}
$$

Analogously,

$$
\begin{aligned}
S_0(x,\lambda) = \frac{\sin\rho x}{\rho} - \frac{\cos\rho x}{2\rho^2}\int_0^x q(t)dt + \frac{1}{2\rho^2}\int_0^x q(t)\cos\rho(x-2t)dt \\
+ O\left(\frac{1}{\rho^3}\exp(|\tau|)x\right).
\end{aligned} \tag{2.10}
$$

By virtue of (2.7) and (2.9)-(2.10),

$$
A_1 = \frac{\alpha}{2}\rho\sin 2\rho a - 1 - \frac{\alpha}{2}\cos 2\rho a\int_0^a q(t)dt + O\left(\frac{1}{\rho}\right),
$$

$$
B_1 = -\frac{\alpha}{2}\rho^2\left(1+\cos 2\rho a\right) - \frac{\alpha}{2}\rho\sin\rho a\int_0^a q(t)dt + O\left(1\right),
$$

$$
A_2 = \frac{\alpha}{2}\left(1-\cos 2\rho a\right) + O\left(\frac{1}{\rho}\right), B_2 = -\frac{\alpha}{2}\rho\sin 2\rho a + O\left(1\right)
$$

Since $\varphi(x,\lambda) = C(x,\lambda) + hS(x,\lambda)$, by using (2.5)-(2.10), we find

$$
\varphi(x,\lambda) = \cos\rho x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin\rho x}{\rho} + O\left(\frac{1}{\rho}\exp\left(|\tau|x\right)\right), 0<x<a, \tag{2.11}
$$

$$
\varphi'(x,\lambda) = -\rho\sin\rho x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\cos\rho x + O\left(\exp\left(|\tau|x\right)\right), 0<x<a, \tag{2.12}
$$

$$
\begin{aligned}
\varphi(x,\lambda) = \frac{\alpha}{2}\rho\left(\sin\rho\left(2a-x\right) - \sin\rho x\right) + f_1(x)\cos\rho x + f_2(x)\cos\rho\left(2a-x\right) \\
+ O\left(\exp\left(|\tau|x\right)\right), \ a<x<T,
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
\varphi'(x,\lambda) = -\frac{\alpha}{2}\rho^2\left(\cos\rho x + \cos\rho\left(2a-x\right)\right) - \rho f_1(x)\sin\rho x + \rho f_2(x)\sin\rho\left(2a-x\right) \\
+ O\left(\rho\exp\left(|\tau|x\right)\right), \ a<x<T,
\end{aligned} \tag{2.14}
$$

where

$$
f_1(x) = 1 + \frac{\alpha}{2}h + \frac{\alpha}{4}\int_0^x q(t)dt, \quad f_2(x) = \frac{\alpha}{4}\left(-2h + \int_a^x q(t)dt - \int_0^a q(t)dt\right).
$$

It follows from (2.2),(2.13), and (2.14) that

$$
\begin{aligned}
\Delta(\lambda) = \frac{\alpha}{2}\rho^2\left(\cos\rho T + \cos\rho\left(2a-T\right)\right) + \omega_1\rho\sin\rho T + \omega_2\rho\sin\rho\left(2a-T\right) \\
+ O\left(\rho\exp\left(|\tau|T\right)\right),
\end{aligned} \tag{2.15}
$$

where

$$
\omega_1 = -\left(1 + \frac{\alpha}{2}h + \frac{\alpha}{2}H + \frac{\alpha}{4}\int_0^T q(t)dt\right)
$$

$$
\omega_2 = \frac{\alpha}{2}\left(h - H - \frac{1}{2}\int_a^T q(t)dt + \frac{1}{2}\int_0^a q(t)dt\right).
$$

Let $\lambda_n^0 = (\rho_n^0)^2$ and $\lambda_n = (\rho_n)^2$ be the zeros of the functions $\Delta_0(\lambda) = \frac{\alpha}{2}\rho^2(\cos\rho T + \cos\rho\,(2a-T))$ and $\Delta(\lambda)$, respectively.

The following properties of the characteristic function $\Delta(\lambda)$ and the eigenvalues $\lambda_n = \rho_n^2$ of the boundary value problem of L can be discovered using (2.15) and the well-known methods (see, for example [3]).

(i) Denote $G_\delta =\{\rho : |\rho - \rho_n^0| \geq \delta, n \geq 0\}$. There is a constant $C_\delta > 0$ such that

$$|\Delta_0(\lambda)| \geq C_\delta |\lambda| \exp\left(|\tau|T|\right),\ \ \rho \in G_\delta.$$

(ii) For sufficiently large value of n, the following inequality is valid

$$|\Delta(\lambda) - \Delta_0(\lambda)| \leq \frac{1}{2}C_\delta \exp\left(|\tau|T\right),\ \ \rho \in \Gamma_n = \{\rho : |\rho| = |\rho_n^0| + \frac{1}{2}\inf_{n\neq m} |\rho_n^0 - \rho_m^0|\}.$$

Thus, for sufficiently large natural number n and $\rho \in \Gamma_n$,

$$|\Delta_0(\lambda)| \geq C_\delta |\lambda| \exp\left(|\tau|T\right) > \frac{1}{2}C_\delta |\lambda| \exp\left(|\tau|T\right) > |\Delta(\lambda) - \Delta_0(\lambda)|.$$

Then by Rouche's theorem, the number of zeros of $\Delta_0(\lambda)$, counting multiplicities, inside circuit G_n coincides with the number of zeros of $\Delta(\lambda)$. Analogously, applying Rouche's theorem, we say that for sufficiently large values of n, the function $\Delta(\lambda)$ has exactly one zero ρ_n inside each circle $G_\delta = \{\rho : |\rho - \rho_n^0| \leq \delta\}$. Since δ is arbitrary sufficiently small number, we have

$$\rho_n = \rho_n^0 + \epsilon_n,\ \ \epsilon_n = o\,(1)\,,\ \ n \to \infty \tag{2.16}$$

Since the function $\Delta_0(\lambda)$ is type of *sinus* (see [5, p. 119]), there exist the number $d_\delta > 0$ such that, for all n, $|\frac{d}{d\lambda}\Delta_0(\lambda)|_{\lambda=\lambda_n} \geq d_\delta > 0$. Since ρ_n are zeros of $\Delta(\lambda)$, from (2.15) we get

$$\epsilon_n = -\frac{2}{\alpha\rho_n^0}\left[\omega_1 \sin\rho_n^0 T + \omega_2 \sin\rho_n^0\,(2a-T)\right]\left[\frac{d}{d\lambda}\Delta_0(\lambda)|_{\lambda=\lambda_n^0}\right]. \tag{2.17}$$

Substituting (2.17) into (2.16) we get

$$\rho_n = \rho_n^0 + \frac{d_n}{\rho_n^0} + \frac{\gamma_n}{\rho_n^0}, \tag{2.18}$$

where

$$d_n = -\frac{2}{\alpha}\left[\omega_1 \sin\rho_n^0 T + \omega_2 \sin\rho_n^0\,(2a-T)\right]\left[\frac{d}{d\lambda}\Delta_0(\lambda)|_{\lambda=\lambda_n^0}\right]$$

and $\gamma_n = o\,(1)$.

Finally, using (2.11),(2.12) and (2.18) into (2.3) we obtain

$$\alpha_n = \alpha_n^0 + o\,(1)\,,\ \ n \to \infty, \tag{2.19}$$

where

$$\alpha_n^0 = \int_0^T \varphi^2(x, \lambda_n^0)dx + \alpha\varphi^2(a, \lambda_n^0).$$

3 Algorithm of solving the inverse problem

In this section, we first give the spectral characteristics of the boundary value problem L and then demonstrate relationships between their spectral characteristics. Moreover, we provide the formula to solve the inverse problem of the reconstruction of the problem L basing on the Weyl function, on the spectral data, and on two spectra.

We define the Weyl function by

$$M(\lambda) = \frac{\psi(0,\lambda)}{\Delta(\lambda)}. \tag{3.1}$$

Here the function $\psi(0,\lambda)$ is the characteristic function of the boundary value problem which consists of equation (1.1) along with the boundary conditions $y(0) = V(y) = 0$ and transmission conditions (1.3). Let $\{\mu_n\}_{n\geq 0}$ be the zeros of the entire function $\psi(0,\lambda)$. It is clear that, $\psi(0,\lambda)$ and $\Delta(\lambda)$ have no common zeros. Thus, the Weyl function $M(\lambda)$ is meromorphic which has poles at the points $\{\lambda_n\}_{n\geq 0}$ and zeros at the points $\{\mu_n\}_{n\geq 0}$.

The following lemma gives the relationships between the spectral characteristic of L : the spectral data Ω, the Weyl function $M(\lambda)$ and the two spectra $\{\lambda_n, \mu_n\}_{n\geq 0}$.

Lemma 3.1. *Let $M(\lambda)$, Ω and $\Delta(\lambda)$ be defined as above. Then the following representation holds:*

$$M(\lambda) = \sum_{h=0}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}. \tag{3.2}$$

Moreover, $\Delta(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta(\lambda) = T(\lambda_0 - \lambda)\prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{\lambda_n^0}. \tag{3.3}$$

Since the arguments for proving this lemma are similar to those in [2], we skip the proof. Now, we will consider the following inverse problems of recovering L :

- *Inverse problem* 1*:* constructing $q(x)$, h, and H when the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$ is given.
- *Inverse problem* 2*:* constructing $q(x)$, h and H when the Weyl function $M(\lambda)$ is given.
- *Inverse problem* 3*:* constructing $q(x)$, h and H when the two spectra $\Omega = \{\lambda_n, \mu_n\}_{n\geq 0}$ are given.

Let us note that, according to (3.1), (3.2) and (3.3), the inverse problems of recovering L basing on the spectral data and on the two spectra are specifications of the inverse problem of recovering L from the Weyl function. Consequently, the inverse problems 1–3 are equivalent.

The inverse problems studied here can be seen as generalizations of the inverse problems for the classical Sturm-Liouville operators, see [6, Chapter 1]. In the next section, using results stated above, we provide a constructive procedure for solving these inverse problems.

4 Finding solutions to inverse problems

In this section, with the help of Cauchy's integral formula and the Residue theorem, we will solve the inverse problems of recovering the Sturm-Liouville problem $L(q(x), h, H)$ using the spectrum mappings approach. We first reduce an inverse problem to the so-called main equation which is a linear equation in a corresponding Banach space of sequences. Finally, we provide the algorithms for solving the inverse problems by using the solution of the main equation.

For this purpose we introduce a new problem with new notations: together with L we consider a boundary value problem $\tilde{L}$ of the same form but with different coefficients $\tilde{q}(x)$, $\tilde{h}$, $\tilde{H}$. Throughout next sections, if a certain symbol e denotes an object related to L, then the symbol $\tilde{e}$ with tilde denotes the analogous object related to $\tilde{L}$. Now we introduce the following notations for convenience of further discussions.

$$\lambda_{n0} = \lambda_n, \quad \lambda_{n1} = \tilde{\lambda}_n, \quad \alpha_{n0} = \alpha_n, \quad \alpha_{n1} = \tilde{\alpha}_n,$$

$$\varphi_{ni}(x) = \varphi(x, \lambda_{ni}), \quad \tilde{\varphi}_{ni}(x) = \tilde{\varphi}(x, \lambda_{ni}),$$

$$Q_{kj}(x, \lambda) = \frac{< \varphi(x, \lambda), \varphi_{kj}(x) >}{\alpha_{kj}(\lambda - \lambda_{kj})} = \frac{1}{\alpha_{kj}} \int_0^x \varphi(t, \lambda)\varphi_{kj}(t)dt,$$

$$Q_{ni,kj}(x) = Q_{kj}(x, \lambda_{ni}),$$

for $i, j = 0, 1$ and $n, k \geq 0$. Here $\tilde{\varphi}(x, \lambda)$ is the solution of (1.4) with the potential $\tilde{q}$ under the initial conditions $\tilde{\varphi}(0, \lambda) = 1$, $\tilde{\varphi}'(0, \lambda) = \tilde{h}$. Similarly, we can define $\tilde{Q}_{kj}(x, \lambda)$ by replacing φ by $\tilde{\varphi}$ in the above definition.

Using the Schwartz lemma, see [5, p. 130] and (2.11)-(2.14), (2.17) we obtain the following asymptotic estimates:

$$|\varphi_{ni}(x)| \leq C(|\rho_n^0| + 1), \quad |\varphi_{n0}(x) - \varphi_{n1}(x)| \leq C(|\rho_n^0| + 1)^{\frac{1}{2}}, \tag{4.1}$$

$$|Q_{ni,kj}(x)| \leq \frac{C(|\rho_n^0| + 1)}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)},$$

$$|Q_{ni,k0}(x) - Q_{ni,k1}(x)| \leq \frac{C(|\rho_n| + 1)}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)^{\frac{3}{2}}}, \tag{4.2}$$

$$|Q_{n0,kj}(x) - Q_{n1,j1}(x)| \leq \frac{C(|\rho_n| + 1)^{\frac{1}{2}}}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)}$$

where $n, k \geq 0, 1$ and $C > 0$ is independent of n, k, i, j. Similar estimates are also valid for $\tilde{\varphi}_{ni}(x)$, $\tilde{Q}_{ni,kj}(x)$.

Lemma 4.1. *Let $\varphi_{ni}(x)$ and $Q_{ni,kj}(x)$ be defined as above. Then the following representations hold for $i, j = 0, 1$ and $n, k \geq 0$:*

$$\tilde{\varphi}_{ni}(x) = \varphi_{ni}(x) + \sum_{l=1}^{\infty} (\tilde{Q}_{ni,l0}(x)\varphi_{k0}(x) - \tilde{Q}_{ni,l1}(x)\varphi_{k1}(x)), \tag{4.3}$$

$$Q_{ni,kj}(x) - \tilde{Q}_{ni,kj}(x) + \sum_{l=0}^{\infty} (\tilde{Q}_{ni,l0}(x)Q_{l0,kj}(x) - \tilde{Q}_{ni,l1}(x)Q_{l1,kj}(x)) = 0, \tag{4.4}$$

Both series converge absolutely and uniformly with respect to $x \in [0, T] \setminus \{a\}$.

The proof of this lemma is similar to that of the lemma given in [13] and, hence, is omitted.

From all arguments mentioned above, it is seen that, for each fixed $x \in (0,T) \setminus \{a\}$, relation (4.3) can be thought as a system of linear equations with respect to $\varphi_{ni}(x)$ for $n \geq 0$ and $i = 0, 1$. But the series in (4.3) converges only with brackets. So, it is not convenient to use (4.3) as a main equation of the inverse problem. Below, we will transfer (4.3) to a linear equation in a corresponding Banach space of sequences.

Let V be a set of all indices $u = (n,i)$, $n \geq 0$, $i = 0, 1$. For each fixed $x \in (0,T) \setminus \{a\}$, we define the vector

$$\phi(x) = [\phi_u(x)] = \begin{bmatrix} \phi_{n0}(x) \\ \phi_{n1}(x) \end{bmatrix}_{n\geq 0}$$

by the formulas

$$\begin{bmatrix} \phi_{n0}(x) \\ \phi_{n1}(x) \end{bmatrix} = \begin{bmatrix} \frac{\rho_n^0+1}{\rho_n} & -\frac{\rho_n^0+1}{\rho_n} \\ 0 & \frac{1}{\rho_n^0} \end{bmatrix} \begin{bmatrix} \varphi_{n0}(x) \\ \varphi_{n1}(x) \end{bmatrix} = \begin{bmatrix} \frac{(\rho_n^0+1)(\varphi_{n0}(x)-\varphi_{n1}(x))}{\rho_n} \\ \frac{\varphi_{n1}(x)}{\rho_n} \end{bmatrix} \tag{4.5}$$

Further, we define the block matrix

$$H(x) = [H_{u,v}(x)]_{u,v\in V} = \begin{bmatrix} H_{n0,k0}(x) & H_{n0,k1}(x) \\ H_{n1,k0}(x) & H_{n1,k1}(x) \end{bmatrix}_{n,k\geq 0},$$

where $u = (n,i)$, $v = (k,j)$ and

$$\begin{bmatrix} H_{n0,k0}(x) & H_{n0,k1}(x) \\ H_{n1,k0}(x) & H_{n1,k1}(x) \end{bmatrix} = \begin{bmatrix} \frac{\rho_n^0+1}{\rho_n} & -\frac{\rho_n^0+1}{\rho_n} \\ 0 & \frac{1}{\rho_n^0} \end{bmatrix} \begin{bmatrix} Q_{n0,k0}(x) & Q_{n0,k1}(x) \\ Q_{n1,k0}(x) & Q_{n1,k1}(x) \end{bmatrix} \begin{bmatrix} \frac{\rho_k}{\rho_k^0+1} & \rho_k \\ 0 & -\rho_k \end{bmatrix}.$$

Analogously, we define $\tilde{\varphi}(x)$, $\tilde{H}(x)$ by replacing in the previous definitions $\varphi_{ni}(x)$, $Q_{ni,kj}(x)$ by $\tilde{\varphi}_{ni}(x)$, $\tilde{Q}_{ni,kj}(x)$, respectively. It follows from (2.11) - (2.14), (2.17), (2.18), (4.1), (4.2) and the Schwarz lemma that

$$|\phi_{nj}(x)|, |\tilde{\phi}_{nj}(x)| \leq C, \tag{4.6}$$

and

$$|H_{ni,kj}(x)|, |\tilde{H}_{ni,kj}(x)| \leq \frac{C}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)}, \tag{4.7}$$

where $C > 0$ is independent of n, k, i, j.

Let us consider the Banach space B of bounded sequences $\alpha = [\alpha_u]_{u\in V}$ with the norm $\|\alpha\|_B = sup_{u\in V}|\alpha_u|$ and consider the operator $E + \tilde{H}(x)$ acting from B to B. Here E is the identity operator. It follows from (4.6), (4.7) that for each fixed x, this operator is a linear bounded operator, and

$$\|H(x)\|, \|\tilde{H}(x)\| \leq C \sup_n \sum_{k=0}^{\infty} \frac{1}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)} < \infty.$$

Now we are ready to give the main result of this section.

Theorem 4.1. *For each fixed $x \in (0,T) \setminus \{a\}$, the vector $\varphi(x) \in B$ satisfies the equation*

$$\tilde{\varphi}(x) = (E + \tilde{H}(x))\varphi(x), \tag{4.8}$$

in the Banach space B. Moreover, the operator $E + \tilde{H}(x)$ has a bounded inverse operator, hence, equation (4.8) is uniquely solvable.

Proof. Using the notation $\tilde{\phi}(x)$, rewrite (4.3) as

$$\tilde{\phi}_{ni}(x) = \phi_{ni} + \sum_{n,j} \tilde{H}_{ni,kj}(x)\phi_{kj}(x), (n,i) \in V, (k,j) \in V,$$

which is equivalent to (4.3). Interchanging places for L and $\tilde{L}$, we obtain analogously the equalities

$$\phi(x) = (E - H(x))\tilde{\varphi}(x), (E - H(x))(E + \tilde{H}(x)) = E.$$

Hence, the operator $(E + \tilde{H}(x))^{-1}$ exist, and it is a linear bounded operator. □

Equation (4.8) is named a *basic equation* of the inverse problem. Solving (4.8) we find the vector $\phi(x)$, and hence, the functions $\varphi_{ni}(x)$. Thus, we get the following algorithms to find the solution of an inverse problem.

Algorithm 1. *When the spectral data* $\{\lambda_n, \alpha_n\}_{n\geq 0}$ *is given, to construct* $q(x)$, h *and* H, *we follow the steps:*

(i) first contruct $\tilde{L}$ *and then calculate* $\tilde{\phi}(x)$ *and* $\tilde{H}(x)$,

(ii) by solving equation (4.8) *find* $\phi(x)$ *and calculate* $\varphi_{n0}(x)$ *by using* (4.5),

(iii) choose some n *(for example,* $n = 0$*) and construct* $q(x)$, h *and* H *by using the following formulas:*

$$q(x) = \frac{\varphi''_{n0}(x)}{\varphi_{n0}(x)} + \lambda_n, \ h = \varphi'_{n0}(0), \ H = -\frac{\varphi_{n0}(T)}{\varphi_{n0}(T)}.$$

Algorithm 2. *When the function* $M(\lambda)$ *is given, to construct* $q(x)$, h *and* H, *we follow the steps:*

(i) construct the spectral data Ω *by using* (3.2),

(ii) construct $q(x)$, h *and* H *by using Algorithm 1.*

Algorithm 3. *When two spectra* $\{\lambda_n, \mu_n\}_{n\geq 0}$ *are given, to construct* $q(x)$, h *and* H, *we follow the steps:*

(i) calculate $M(\lambda)$ *by using* (3.1),

(ii) construct $q(x)$, h *and* H *by using Algorithm* 2.

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Manaf Dzh. Manafov
Faculty of Arts and Sciences, Department of Mathematics
Adıyaman University
02040 Adıyaman, Turkey
E-mail: mmanafov@adiyaman.edu.tr,

Institute of Mathematics and Mechanics
Ministry of Science and Education
AZ1141 Baku, Azerbaijan

and

Scientific and Innovation Center
Western Caspian University
AZ1001 Baku, Azerbaijan

Abdullah Kablan
Faculty of Arts and Sciences, Department of Mathematics
Gaziantep University
27310 Gaziantep, Turkey
E-mail: kablan@gantep.edu.tr

Received: 05.01.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 3 (2025), 68 – 89

EXACT SOLUTION TO A STEFAN-TYPE PROBLEM FOR A GENERALIZED HEAT EQUATION WITH THE THOMSON EFFECT

T.A. Nauryz, S.N. Kharin, A.C. Briozzo, J. Bollati

Communicated by Ya.T. Sultanaev

Key words: Stefan problem, generalized heat equation, Thomson effect, similarity solution, nonlinear integral equations, nonlinear thermal coefficient, fixed point theorem.

AMS Mathematics Subject Classification: 80A22, 80A05, 35C11.

Abstract. We study a one-dimensional Stefan type problem which models the behavior of electromagnetic fields and heat transfer in closed electrical contacts that arises, when an instantaneous explosion of the micro-asperity occurs. This model involves vaporization, liquid and solid zones, in which the temperature satisfies a generalized heat equation with the Thomson effect. Accounting for the nonlinear thermal coefficient, the model also incorporates temperature-dependent electrical conductivity. By employing a similarity transformation, the Stefan-type problem is reduced to a system of coupled nonlinear integral equations. The existence of a solution is established using the fixed point theory in Banach spaces.

DOI: https://doi.org/10.32523/2077-9879-2025-16-3-68-89

1 Introduction

Stefan problems are fundamental in understanding the phase transition phenomena, particularly in situations involving heat transfer and solidification processes. They were first introduced by J. Stefan in his seminal work in [24]. These problems concern the determination of the moving boundary between phases during the process of solidification or melting.

The classical Stefan problem arises in scenarios, in which a material undergoes a phase change, such as freezing or melting, subject to certain boundary conditions and physical constraints. One of the key aspects of Stefan problems is the existence of a sharp interface, known as the Stefan interface, which separates the regions of different phases.

Significant theoretical contributions to Stefan problems have been done in [21], [1]. Further, the study of free and moving boundary problems, including Stefan problems, has garnered considerable attention. In [6] J. Crank provides a comprehensive treatment of such problems, offering valuable insights into their mathematical formulation and solution techniques.

Stefan problems, which traditionally deal with phase-change phenomena under classical heat conduction assumptions, have seen extensions to encompass more complex physical scenarios. These extensions, often referred to as non-classical Stefan problems, involve variations in thermal coefficients, boundary conditions, or latent heat dependencies, among other factors. The investigation of non-classical Stefan problems has significant implications in various fields, including materials science, engineering, and mathematical physics.

One avenue of research in non-classical Stefan problems involves the consideration of thermal coefficients that vary with temperature or position. In [4], [2] were explored Stefan problems for diffusion-convection equations with temperature-dependent thermal coefficients, providing insights into the behavior of phase-change processes under such conditions. Similarly, in [18], [17] A. Kumar et al. investigated Stefan problems with variable thermal coefficients, highlighting the impact of these variations on the phase-change dynamics. Furthermore, exact and approximate solutions to the Stefan problem in ellipsoidal coordinates were obtained in [8]

Another aspect of non-classical Stefan problems involves incorporating convective boundary conditions or heat flux conditions on fixed faces. In paper [4] there is examined the existence of exact solutions for one-phase Stefan problems with nonlinear thermal coefficients, incorporating Tirskii's method to handle such complexities. Additionally, paper [5] is devoted to the one-phase Stefan problem for a non-classical heat equation with a heat flux condition on the fixed face, contributing to the understanding of phase-change phenomena under non-standard boundary conditions.

Non-linear Stefan problems offer a valuable mathematical framework to model and analyze complex phenomena, providing insights, for example, into heat transfer processes during phase transitions within electrical contacts [3], [12]-[20].

Thermal phenomena in electrical apparatus, such as welding, arcing, and bridging, contribute to their failure and are highly complex. These phenomena depend on various factors including current, voltage, contact force, contact material properties, and arc duration [23], [7]. Experimental investigations usually focus on cumulative probability representations of resulting values as direct experimental observation of these processes is often challenging or even impossible due to their extremely short duration.

Hence, mathematical modelling plays a crucial role in understanding the dynamics of such processes, improving the endurance and reliability of contact systems, and predicting and preventing failures in electrical apparatus.

Efforts have been made in [22], [9]-[11] to address these aspects and the study of electrical contacts involves intricate thermal dynamics influenced by non-linearities in material properties and heat generation mechanisms.

This paper aims to further develop the existing models to models, also incorporating the Thomson effect.

The Thomson effect refers to the phenomenon, in which a temperature difference is created across an electrical conductor when an electric current flows through it. This effect occurs due to the interaction between the current-carrying electrons and the lattice structure of the conductor.

In the context of a closure of electrical contact after the instantaneous explosion of a micro-asperity, it is important to take into account that micro-asperities are tiny protrusions or irregularities on the surface of a material. An explosion or sudden release of energy can cause these micro-asperities to rupture or deform.

After such an explosion, the closure of electrical contact can manifest itself in several ways. The intense energy release can lead to the melting or vaporization of micro-asperities, altering the surface characteristics of the contact. This can potentially disrupt the normal flow of electric current and create temperature variations due to the Thomson effect.

The Thomson effect in this scenario could result in localized heating or cooling at the contact points, depending on the direction of the current flow. This temperature difference might affect the electrical conductivity and overall performance of the closure of electrical contact.

In the initial phase of a closed electrical contact, when a micro-asperity undergoes sudden ignition, the contact region comprises both a metallic vaporization zone and a liquid domain, see Figure 1. Modelling the metallic vapour zone, denoted as Z_0 with a height range of $0 < z < s(t)$, is a complex undertaking. We propose that the temperature within this region decreases linearly from the ionization temperature of the metallic vapour, denoted as T_{ion}, which occurs after the explosion at the fixed face $z = 0$, to the boiling temperature T_b at the free boundary that separates the vapour and liquid phases. The temperature field within the vapour zone Z_0 exhibits a gradual and linear

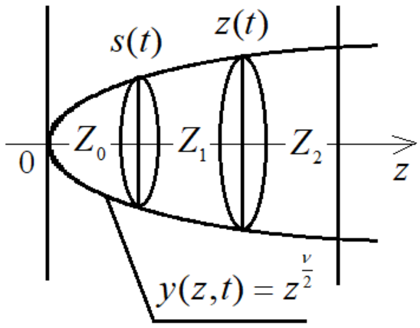

Figure 1: Contact zones: $Z_0 : (0 < z < s(t))$-vaporization zone, $Z_1 : (s(t) < z < r(t))$-liquid zone, $Z_2 : (r(t) < z)$-solid zone.

decrease

$$T_V(z,t) = \frac{z}{s(t)}(T_b - T_{ion}) + T_{ion}, \quad 0 \le z \le s(t), \tag{1.1}$$

where the following boundary conditions hold

$$T_V(0,t) = T_{ion}, \tag{1.2}$$

$$T_V(s(t),t) = T_b. \tag{1.3}$$

Temperature distribution and electrical potential field of the zones Z_1 and Z_2 are defined by the following relations:

$$c(T_1)\gamma(T_1)\frac{\partial T_1}{\partial t} = \frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\lambda(T_1)z^\nu \frac{\partial T_1}{\partial z}\right] + \sigma_{T_1}\frac{\partial T_1}{\partial z}\frac{\partial \varphi_1}{\partial z} + \frac{1}{\rho(T_1)}\left(\frac{\partial \varphi_1}{\partial z}\right)^2, \tag{1.4}$$

$$\frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\frac{1}{\rho(T_1)}z^\nu \frac{\partial \varphi_1}{\partial z}\right] = 0, \quad s(t) < z < r(t), \quad t > 0, \quad 0 < \nu < 1, \tag{1.5}$$

$$c(T_2)\gamma(T_2)\frac{\partial T_2}{\partial t} = \frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\lambda(T_2)z^\nu \frac{\partial T_2}{\partial z}\right] + \sigma_{T_2}\frac{\partial T_2}{\partial z}\frac{\partial \varphi_2}{\partial z} + \frac{1}{\rho(T_2)}\left(\frac{\partial \varphi_2}{\partial z}\right)^2, \tag{1.6}$$

$$\frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\frac{1}{\rho(T_2)}z^\nu \frac{\partial \varphi_2}{\partial z}\right] = 0, \quad r(t) < z, \quad t > 0, \quad 0 < \nu < 1, \tag{1.7}$$

$$T_1(s(t),t) = T_b, \quad t > 0, \tag{1.8}$$

$$-\lambda\left(T_1(s(t),t)\right)\left.\frac{\partial T_1}{\partial z}\right|_{z=s(t)} = \frac{Q_0 e^{-s_0^2}}{2a\sqrt{\pi t}}, \quad t > 0, \tag{1.9}$$

$$\varphi_1(s(t),t) = 0, \quad t > 0, \tag{1.10}$$

$$T_1(r(t),t) = T_2(r(t),t) = T_m > 0, \quad t > 0, \tag{1.11}$$

$$\varphi_1(r(t),t) = \varphi_2(r(t),t), \quad t > 0, \tag{1.12}$$

$$-\lambda\left(T_1(r(t),t)\right)\left.\frac{\partial T_1}{\partial z}\right|_{z=r(t)} + \lambda\left(T_2(r(t),t)\right)\left.\frac{\partial T_2}{\partial z}\right|_{z=r(t)} = l_m\gamma_m\frac{dr}{dt}, \quad t > 0, \tag{1.13}$$

$$\frac{1}{\rho(T_1(r(t),t))}\left.\frac{\partial \varphi_1}{\partial z}\right|_{z=r(t)} = \frac{1}{\rho(T_2(r(t),t))}\left.\frac{\partial \varphi_2}{\partial z}\right|_{z=r(t)}, \quad t > 0, \tag{1.14}$$

$$T_2(+\infty,t) = 0, \quad t > 0, \tag{1.15}$$

$$\varphi_2(+\infty,t)=\frac{U_c}{2},\quad t>0, \tag{1.16}$$

$$T_2(z,0)=\varphi_2(z,0)=0,\quad z>0,\qquad s(0)=r(0)=0, \tag{1.17}$$

where T_1, T_2 and φ_1, φ_2 are temperatures and electrical potential fields for liquid and solid zones, $c(T_i)$,$\gamma(T_i)$ and $\lambda(T_i)$ are specific heat, density and thermal conductivity which depend on the temperature, σ_{T_i} is the Thomson coefficient, $\rho(T_i)$ is the electrical resistivity, $Q_0>0$ is the power of the heat flux, T_m is the melting temperature, U_c is the contact voltage, $s(t)$ and $r(t)$ are locations of the boiling and melting interfaces.

This paper is structured as follows. In Section 2, we use the similarity transformation to obtain an equivalent system of coupled integral equations for problem (1.4)-(1.17). In Section 3, we define proper spaces in order to apply the fixed point Banach theorem to prove the existence of a solution to the system of coupled integral equations.

The contribution of the problem addressed in our paper has significant implications for electrical engineering. By developing a mathematical model that captures the behavior of electromagnetic fields and heat transfer in closed electrical contacts, particularly during instantaneous micro-asperity explosions, we offer valuable insights into the complex dynamics of these systems.

Our model accounts for the non-linear nature of thermal coefficients and temperature-dependent electrical conductivity, factors that are crucial in accurately representing real-world scenarios. By considering vaporization, liquid, and solid zones within the contact, we provide a comprehensive framework for analyzing the thermal and electromagnetic effects associated with such phenomena.

Furthermore, our approach, which utilizes similarity transformations to reduce the Stefan-type problem to a system of nonlinear integral equations, offers practical methodologies for analysing and predicting the closure of electrical contacts under extreme conditions. The rigorous establishment of the validity of this approach through discussions and proofs supported by the fixed point theory in Banach spaces enhances the reliability and applicability of our proposed solutions.

2 Integral formulation

In this section, taking into account that problem (1.4)-(1.17) can be thought as a Stefan-type problem, we look for similarity type solutions that depend on the similarity variable

$$\eta=\frac{z}{2a\sqrt{t}},$$

with $a=\sqrt{\frac{\lambda_0}{\rho_0 c_0}}$ where λ_0, ρ_0 and c_0 are reference thermal coefficients.

We propose the following transformation

$$f_i(\eta)=\frac{T_i(z,t)-T_m}{T_m},\qquad \phi_i(\eta)=\varphi_i(z,t),\quad ,\quad i=1,2. \tag{2.1}$$

According to this transformation, the location of the boiling and melting fronts are given by

$$s(t)=2as_0\sqrt{t},\qquad r(t)=2ar_0\sqrt{t}, \tag{2.2}$$

where s_0 and r_0 must be determined as a part of the solution.

Therefore, problem (1.4)-(1.17) can be rewritten in the following form:

$$[L(f_i)\eta^\nu f_i']'+2a\eta^{\nu+1}N(f_i)f_i'+\frac{\sigma_{f_i}}{c_0\gamma_0 a}\eta^\nu f_i'\phi_i'+\frac{\eta^\nu}{c_0\gamma_0 T_m aK(f_i)}(\phi_i')^2=0, \tag{2.3}$$

$$\left[\frac{1}{K(f_i)}\eta^\nu\phi_i'\right]'=0, \tag{2.4}$$

$$i = 1: \quad s_0 < \eta < r_0, \qquad i = 2: \quad \eta > r_0,$$

$$f_1(s_0) = B, \tag{2.5}$$

$$L(f_1(s_0))f_1'(s_0) = -Qe^{-s_0^2}, \tag{2.6}$$

$$\phi_1(s_0) = 0, \tag{2.7}$$

$$f_1(r_0) = f_2(r_0) = 0, \tag{2.8}$$

$$\phi_1(r_0) = \phi_2(r_0), \tag{2.9}$$

$$-L(f_1(r_0))f_1'(r_0) = -L(f_2(r_0))f_2'(r_0) + Mr_0, \tag{2.10}$$

$$\frac{1}{K(f_1(r_0))}\phi_1'(r_0) = \frac{1}{K(f_2(r_0))}\phi_2'(r_0), \tag{2.11}$$

$$f_2(+\infty) = -1, \tag{2.12}$$

$$\phi_2(+\infty) = \frac{U_c}{2}, \tag{2.13}$$

where

$$B = \frac{T_b - T_m}{T_m}, \qquad Q = \frac{Q_0}{\lambda_0 T_m \sqrt{\pi}} > 0, \quad M = \frac{2l_m \gamma_m a^2}{\lambda_0 T_m} > 0 \tag{2.14}$$

and for $i = 1, 2$:

$$N(f_i) = \frac{c(f_i T_m + T_m)\gamma(f_i T_m + T_m)}{c_0 \gamma_0}, \tag{2.15}$$

$$L(f_i) = \frac{\lambda(f_i T_m + T_m)}{\lambda_0}, \tag{2.16}$$

$$K(f_i) = \rho(f_i T_m + T_m), \tag{2.17}$$

$$\sigma_{f_i} = \sigma_{T_i}, \tag{2.18}$$

From (2.4), (2.7), (2.9), (2.11) and (2.13), we obtain the solution for electrical potential field for liquid and solid zones explicitly depending on f_1, f_2, s_0 and r_0 as

$$\phi_1(\eta, s_0, r_0, f_1, f_2) = \frac{U_c F_1(\eta, s_0, f_1)}{2H(r_0, s_0, f_1, f_2)}, \quad s_0 \le \eta \le r_0, \tag{2.19}$$

$$\phi_2(\eta, s_0, r_0, f_1, f_2) = \frac{U_c\left(F_1(r_0, s_0, f_1) + F_2(\eta, r_0, f_2)\right)}{2H(r_0, s_0, f_1, f_2)}, \quad \eta \ge r_0, \tag{2.20}$$

where

$$F_1(\eta, s_0, f_1) = \int_{s_0}^{\eta} \frac{K(f_1(v))}{v^\nu} dv, \quad s_0 \le \eta \le r_0, \tag{2.21}$$

$$F_2(\eta, r_0, f_2) = \int_{r_0}^{\eta} \frac{K(f_2(v))}{v^\nu} dv, \quad \eta \ge r_0, \tag{2.22}$$

and

$$H(r_0, s_0, f_1, f_2) = F_1(r_0, s_0, f_1) + F_2(+\infty, r_0, f_2). \tag{2.23}$$

In addition, from (2.3), (2.6) and (2.8), we get

$$f_1(\eta) = s_0^\nu Q \exp(-s_0^2)\left[\Phi_1(r_0, s_0, f_1, f_2) - \Phi_1(\eta, s_0, f_1, f_2)\right]$$

$$+\frac{D_1^*}{H^2(r_0,s_0,f_1,f_2)}\left[G_1(r_0,s_0,f_1,f_2)-G_1(\eta,s_0,f_1,f_2)\right],\quad s_0\le\eta\le r_0, \tag{2.24}$$

and from (2.3), (2.8) and (2.12) we get

$$f_2(\eta)=\left[\frac{D_2^*}{H^2(r_0,s_0,f_1,f_2)}G_2(+\infty,r_0,f_1,f_2)-1\right]\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}$$

$$-\frac{D_2^*}{H^2(r_0,s_0,f_1,f_2)}G_2(\eta,r_0,f_1,f_2),\quad \eta\ge r_0. \tag{2.25}$$

Moreover, from conditions (2.5) and (2.10) we obtain the following equalities:

$$s_0^\nu Q\exp(-s_0^2)\Phi_1(r_0,s_0,f_1,f_2)+\frac{D_1^*}{H^2(r_0,s_0,f_1,f_2)}G_1(r_0,s_0,f_1,f_2)=B, \tag{2.26}$$

and

$$E_1(r_0,s_0,f_1,f_2)\left[Q\exp(-s_0^2)s_0^\nu+\frac{D_1^*}{H^2(r_0,s_0,f_1,f_2)}H_1(r_0,s_0,f_1,f_2)\right]$$

$$-\frac{1}{\Phi_2(+\infty,r_0,f_1,f_2)}\left[1-\frac{D_2^*}{H^2(r_0,s_0,f_1,f_2)}G_2(+\infty,r_0,f_1,f_2)\right]=Mr_0^{\nu+1}, \tag{2.27}$$

where

$$\Phi_1(\eta,s_0,f_1,f_2)=\int_{s_0}^{\eta}\frac{E_1(v,s_0,f_1,f_2)}{L(f_1(v))v^\nu}dv,\quad s_0\le\eta\le r_0, \tag{2.28}$$

$$\Phi_2(\eta,r_0,f_1,f_2)=\int_{r_0}^{\eta}\frac{E_2(v,r_0,f_1,f_2)}{L(f_2(v))v^\nu}dv,\quad \eta\ge r_0, \tag{2.29}$$

$$G_1(\eta,s_0,f_1,f_2)=\int_{s_0}^{\eta}\frac{E_1(v,s_0,f_1,f_2)}{L(f_1(v))v^\nu}H_1(v,r_0,f_1,f_2)dv,\quad s_0\le\eta\le r_0, \tag{2.30}$$

$$G_2(\eta,r_0,f_1,f_2)=\int_{r_0}^{\eta}\frac{E_2(v,s_0,f_1,f_2)}{L(f_2(v))v^\nu}H_2(v,r_0,f_1,f_2)dv\quad \eta\ge r_0 \tag{2.31}$$

$$H_1(\eta,s_0,f_1,f_2)=\int_{s_0}^{\eta}\frac{K(f_1(v))}{v^\nu E_1(v,s_0,f_1,f_2)}dv,\quad s_0\le\eta\le r_0, \tag{2.32}$$

$$H_2(\eta,r_0,f_1,f_2)=\int_{r_0}^{\eta}\frac{K(f_2(v))}{v^\nu E_2(v,r_0,f_1,f_2)}dv\quad \eta\ge r_0 \tag{2.33}$$

$$E_1(\eta,s_0,f_1,f_2)=\exp\left(-\int_{s_0}^{\eta}\left[2av\frac{N(f_1(v))}{L(f_1(v))}+\frac{D_1}{H(r_0,s_0,f_1,f_2)}\frac{K(f_1(v))}{L(f_1(v))v^\nu}\right]dv\right),\quad s_0\le\eta\le r_0, \tag{2.34}$$

$$E_2(\eta,r_0,f_1,f_2)=\exp\left(-\int_{r_0}^{\eta}\left[2av\frac{N(f_2(v))}{L(f_2(v))}+\frac{D_2}{H(r_0,s_0,f_1,f_2)}\frac{K(f_2(v))}{L(f_2(v))v^\nu}\right]dv\right),\quad \eta\ge r_0, \tag{2.35}$$

and the coefficients D_i and D_i^* for $i = 1, 2$ are given by

$$D_i = \frac{\sigma_{f_i} U_c}{2c_0\gamma_0 a}, \qquad D_i^* = \frac{U_c D_i}{2}. \tag{2.36}$$

In conclusion, to find a similarity solution to problem (1.4)-(1.17) is equivalent to obtain f_1, f_2, s_0 and r_0 such that (2.24), (2.25), (2.26) and (2.27) hold. Notice that the electric potential fields ϕ_1 and ϕ_2 are explicitly given by (2.19) and (2.20) as functions of f_1, f_2, s_0 and r_0.

In the next section, to address the existence and uniqueness of solutions, we employ a rigorous analytical approach. We leverage similarity transformations to reduce the problem to a set of ordinary differential equations, facilitating a more tractable analysis. Additionally, we draw upon the fixed point theory in Banach spaces to establish the validity of our proposed solutions.

3 Existence of solution

In order to prove the existence and uniqueness of solution f_1, f_2 to equations (2.24) and (2.25), we fix positive constants $0 < s_0 < r_0$ and consider the Banach space

$$\mathcal{C} = C[s_0, r_0] \times C_b[r_0, +\infty) \tag{3.1}$$

endowed with the norm

$$||\vec{f}|| = \|(f_1, f_2)\| = \max\left\{||f_1||_{C[s_0,r_0]}, ||f_2||_{C_b[r_0,+\infty)}\right\},$$

where $C[s_0, r_0]$ denotes the space of all continuous functions defined on the interval $[s_0, r_0]$ and $C_b[r_0, +\infty)$ represents the space of all continuous and bounded functions on the interval $[r_0, +\infty)$. We define the closed subset $\mathcal{M}$ of $C_b[r_0, +\infty)$ by

$$\mathcal{M} = \{f_2 \in C_b[r_0, +\infty) : f_2(r_0) = 0, f_2(+\infty) = -1\}.$$

We consider the operator Ψ on $\mathcal{K} = C[s_0, r_0] \times \mathcal{M}$ given by

$$\Psi(\vec{f}) = (V_1(\vec{f}), V_2(\vec{f})), \tag{3.2}$$

where $V_1(\vec{f})$, $V_2(\vec{f})$ are defined by

$$\begin{aligned} V_1(\vec{f})(\eta) &= s_0^{\nu} Q \exp(-s_0^2) \left[\Phi_1(r_0, s_0, f_1, f_2) - \Phi_1(\eta, s_0, f_1, f_2)\right] \\ &+ \tfrac{D_1^*}{H^2(r_0,s_0,f_1,f_2)} \Big[G_1(r_0, s_0, f_1, f_2) - G_1(\eta, s_0, f_1, f_2)\Big], \quad s_0 \le \eta \le r_0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} V_2(\vec{f})(\eta) &= \left[\tfrac{D_2^*}{H^2(r_0,s_0,f_1,f_2)} G_2(+\infty, r_0, f_1, f_2) - 1\right] \tfrac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)} \\ &- \tfrac{D_2^*}{H^2(r_0,s_0,f_1,f_2)} G_2(\eta, r_0, f_1, f_2), \quad \eta \ge r_0. \end{aligned} \tag{3.4}$$

Notice that solving the system of equations (2.24) and (2.25) is equivalent to obtaining a fixed point to the operator Ψ.

Taking into account that $\mathcal{K}$ is a closed subset of $\mathcal{C}$ we will prove that $\Psi(\mathcal{K}) \subset \mathcal{K}$ and Ψ is a contraction mapping in order to apply the fixed point Banach theorem.

For this purpose we will assume that there exists positive coefficients μ, L_{im}, L_{iM}, N_{im} and N_{iM}, $\widetilde{L}_i$, $\widetilde{N}_i$ and $\widetilde{K}_i$ for $i = 1, 2$ such that

(A1) for each $f_1 \in C[s_0, r_0] : s_0 \leq v \leq r_0$

$$L_{1m}\eta^{\mu} \leq L(f_1)(\eta) \leq L_{1M}\eta^{\mu}, \tag{3.5}$$

$$N_{1m}\eta^{-\mu} \leq N(f_1)(\eta) \leq N_{1M}\eta^{-\mu}, \tag{3.6}$$

$$K_{1m}\eta^{-\mu} \leq K(f_1)(\eta) \leq K_{1M}\eta^{-\mu}, \tag{3.7}$$

(A2) for each $f_2 \in \mathcal{M},\ \eta \geq r_0$:

$$L_{2m}\eta^{\mu} \leq L(f_2)(\eta) \leq L_{2M}\eta^{\mu}, \tag{3.8}$$

$$N_{2m}\eta^{-\mu} \leq N(f_2)(\eta) \leq N_{2M}\eta^{-\mu}, \tag{3.9}$$

$$K_{2m}\eta^{-\mu} \leq K(f_2)(\eta) \leq K_{2M}\eta^{-\mu}, \tag{3.10}$$

(A3) for each $f_1, g_1 \in C[s_0, r_0], s_0 \leq \eta \leq r_0$:

$$|L(f_1(\eta)) - L(g_1(\eta))| \leq \widetilde{L}_1||f_1 - g_1||, \tag{3.11}$$

$$|N(f_1(\eta)) - N(g_1(\eta))| \leq \widetilde{N}_1||f_1 - g_1||, \tag{3.12}$$

$$|K(f_1(\eta)) - K(g_1(\eta))| \leq \widetilde{K}_1\eta^{-\mu}||f_1 - g_1||, \tag{3.13}$$

(A4) for each $f_2, g_2 \in \mathcal{M}, \eta \geq r_0$:

$$|L(f_2(\eta)) - L(g_2(\eta))| \leq \widetilde{L}_2||f_2 - g_2||, \tag{3.14}$$

$$|N(f_2(\eta)) - N(g_2(\eta))| \leq \widetilde{N}_2||f_2 - g_2||, \tag{3.15}$$

$$|K(f_2(\eta)) - K(g_2(\eta))| \leq \widetilde{K}_2\eta^{-\mu}||f_2 - g_2||, \tag{3.16}$$

(A5) $\mu > 2$.

From now on, hypothesis (A1)-(A5) will be assumed to hold throughout the paper.

We will present preliminary results that will be useful to prove the existence and uniqueness of a fixed point of the operator Ψ.

Lemma 3.1. *For every $\vec{f} = (f_1, f_2)$, $\vec{g} = (g_1, g_2) \in \mathcal{K}$, the following inequalities hold:*

$$H(r_0, s_0, f_1, f_2) \geq H_{inf}(r_0, s_0), \tag{3.17}$$

$$H(r_0, s_0, f_1, f_2) \leq H_{sup}(r_0, s_0), \tag{3.18}$$

$$|H(r_0, s_0, f_1, f_2) - H(r_0, s_0, g_1, g_2)| \leq \widetilde{H}(r_0, s_0)||\vec{f} - \vec{g}||, \tag{3.19}$$

where

$$H_{inf}(r_0, s_0) := \tfrac{K_{1m}}{\mu+\nu-1}\left(\tfrac{1}{s_0^{\mu+\nu-1}} - \tfrac{1}{r_0^{\mu+\nu-1}}\right), \tag{3.20}$$

$$H_{sup}(r_0, s_0) := \tfrac{1}{\mu+\nu-1}\left(\tfrac{K_{1M}}{s_0^{\mu+\nu-1}} + \tfrac{K_{2M}}{r_0^{\mu+\nu-1}}\right), \tag{3.21}$$

$$\widetilde{H}(r_0, s_0) := \tfrac{1}{\mu+\nu-1}\left(\tfrac{\widetilde{K}_1}{s_0^{\mu+\nu-1}} + \tfrac{\widetilde{K}_2}{r_0^{\mu+\nu-1}}\right). \tag{3.22}$$

Proof. Taking into account the definition of H given by (2.23) and assumptions (3.7) and (3.10), we have

$$H(r_0,s_0,f_1,f_2) \geq \frac{K_{1m}}{\mu+\nu-1}\left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{r_0^{\mu+\nu-1}}\right) + \frac{K_{2m}}{\mu+\nu-1}\frac{1}{r_0^{\mu+\nu-1}}$$

$$\geq \frac{K_{1m}}{\mu+\nu-1}\left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{r_0^{\mu+\nu-1}}\right),$$

and then we get (3.17). Inequality (3.18) follows analogously. In addition, taking into account assumptions (3.13) and (3.16), we get

$$|H(r_0,s_0,f_1,f_2) - H(r_0,s_0,g_1,g_2)|$$

$$\leq \left(\widetilde{K}_1 \int_{s_0}^{r_0} \frac{1}{v^{\mu+\nu}}dv + \widetilde{K}_2 \int_{r_0}^{+\infty} \frac{1}{v^{\mu+\nu}}dv\right) ||\vec{f} - \vec{g}||$$

$$\leq \frac{1}{\mu+\nu-1}\left(\widetilde{K}_1\left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{r_0^{\mu+\nu-1}}\right) + \frac{\widetilde{K}_2}{r_0^{\mu+\nu-1}}\right) ||\vec{f} - \vec{g}||$$

$$\leq \frac{1}{\mu+\nu-1}\left(\frac{\widetilde{K}_1}{s_0^{\mu+\nu-1}} + \frac{\widetilde{K}_2}{r_0^{\mu+\nu-1}}\right) ||\vec{f} - \vec{g}||,$$

and, as a corollary, inequality (3.19) holds. □

Lemma 3.2. *For every $\vec{f} = (f_1, f_2)$, $\vec{g} = (g_1, g_2) \in \mathcal{K}$, the following inequalities hold:*

1)

$$E_1(\eta,s_0,f_1,f_2) \geq E_{1inf}(r_0,s_0), \tag{3.23}$$

$$E_1(\eta,s_0,f_1,f_2) \leq 1, \tag{3.24}$$

$$|E_1(\eta,s_0,f_1,f_2) - E_1(\eta,s_0,g_1,g_2)| \leq \widetilde{E}_1(r_0,s_0)||\vec{f} - \vec{g}||, \tag{3.25}$$

where

$$E_{1inf}(r_0,s_0) := \exp\left(-\left[a\frac{N_{1M}}{L_{1m}(\mu-1)}\frac{1}{s_0^{2\mu-2}} + \frac{D_1K_{1M}}{H_{inf}(r_0,s_0)L_{1m}(2\mu+\nu-1)}\frac{1}{s_0^{2\mu+\nu-1}}\right]\right), \tag{3.26}$$

$$\begin{aligned}\widetilde{E}_1(r_0,s_0) &:= 2a\left[\frac{\widetilde{N}_1}{L_{1m}(\mu-2)}\frac{1}{s_0^{\mu-2}} + \frac{N_{1M}\widetilde{L}_1}{L_{1m}^2(3\mu-2)}\frac{1}{s_0^{3\mu-2}}\right] \\ &+D_1\Bigg(\frac{\widetilde{K}_1}{H_{inf}(r_0,s_0)L_{1m}(2\mu+\nu-1)}\frac{1}{s_0^{2\mu+\nu-1}} \\ &+\frac{K_{1M}}{H_{inf}(r_0,s_0)L_{1m}}\left(\frac{\widetilde{H}(r_0,s_0)}{H_{inf}(r_0,s_0)(2\mu+\nu-1)}\frac{1}{s_0^{2\mu+\nu-1}} + \frac{\widetilde{L}_1}{L_{1m}(3\mu+\nu-1)}\frac{1}{s_0^{3\mu+\nu-1}}\right)\Bigg);\end{aligned} \tag{3.27}$$

2)

$$|\Phi_1(\eta,s_0,f_1,f_2) - \Phi_1(\eta,s_0,g_1,g_2)| \leq \widetilde{\Phi}_1(r_0,s_0)||\vec{f} - \vec{g}||, \tag{3.28}$$

where

$$\widetilde{\Phi}_1(r_0,s_0) := \frac{\widetilde{E}_1(r_0,s_0)}{L_{1m}(\mu+\nu-1)}\frac{1}{s_0^{\nu+\mu-1}} + \frac{\widetilde{L}_1}{L_{1m}^2(2\mu+\nu-1)}\frac{1}{s_0^{\nu+2\mu-1}}; \tag{3.29}$$

3)

$$H_1(\eta,s_0,f_1,f_2) \geq H_{1inf}(\eta,s_0), \tag{3.30}$$

$$H_1(\eta,s_0,f_1,f_2) \leq H_{1sup}(r_0,s_0), \tag{3.31}$$

$$|H_1(\eta,s_0,f_1,f_2) - H_1(\eta,s_0,g_1,g_2)| \leq \widetilde{H}_1(r_0,s_0)||\vec{f} - \vec{g}||, \tag{3.32}$$

where

$$H_{1inf}(\eta, s_0) := \frac{K_{1m}}{(\mu+\nu-1)}\left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}}\right), \tag{3.33}$$

$$H_{1sup}(r_0, s_0) := \frac{K_{1M}}{E_{1inf}(r_0,s_0)}\frac{1}{(\mu+\nu-1)}\frac{1}{s_0^{\mu+\nu-1}}, \tag{3.34}$$

$$\widetilde{H}_1(r_0, s_0) := \left(\widetilde{K}_1 + \frac{K_{1M}\widetilde{E}_1(r_0,s_0)}{E_{1inf}(r_0,s_0)}\right)\frac{1}{E_{1inf}(r_0,s_0)(\mu+\nu-1)}\frac{1}{s_0^{\mu+\nu-1}}; \tag{3.35}$$

4)

$$G_1(\eta, s_0, f_1, f_2) \geq G_{1inf}(\eta, r_0, s_0) \tag{3.36}$$

$$G_1(\eta, s_0, f_1, f_2) \leq G_{1sup}(r_0, s_0) \tag{3.37}$$

$$|G_1(\eta, s_0, f_1, f_2) - G_1(\eta, s_0, g_1, g_2)| \leq \widetilde{G}_1(r_0, s_0)||\vec{f} - \vec{g}|| \tag{3.38}$$

where

$$G_{1inf}(\eta, r_0, s_0) := \frac{K_{1m}E_{1inf}(r_0,s_0)}{2L_{1M}(\mu+\nu-1)^2}\left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}}\right)^2, \tag{3.39}$$

$$G_{1sup}(r_0, s_0) := \frac{H_{1sup}(r_0,s_0)}{L_{1m}}\frac{1}{(\mu+\nu-1)}\frac{1}{s_0^{\mu+\nu-1}}, \tag{3.40}$$

$$\widetilde{G}_1(r_0, s_0) := H_{1sup}(r_0, s_0)\widetilde{\Phi}_1(r_0, s_0) + \frac{\widetilde{H}_1(r_0,s_0)}{L_{1m}}\frac{1}{(\mu+\nu-1)}\frac{1}{s_0^{\mu+\nu-1}}. \tag{3.41}$$

Proof. From the definition of E_1 given by (2.34), assumptions (3.5)-(3.7) and inequality (3.17) we obtain that

$$\begin{aligned}
&\int_{s_0}^{\eta} 2av\frac{N(f_1(v))}{L(f_1(v)} + \frac{D_1}{H(r_0,s_0,f_1,f_2)}\frac{K(f_1(v))}{L(f_1(v))v^{\nu}}dv \\
&\leq \int_{s_0}^{\eta} 2a\frac{N_{1M}}{L_{1m}}\frac{1}{v^{2\mu-1}} + \frac{D_1K_{1M}}{H_{inf}(r_0,s_0)L_{1m}}\frac{1}{v^{2\mu+\nu}}dv \\
&\leq 2a\frac{N_{1M}}{L_{1m}(2\mu-2)}\frac{1}{s_0^{2\mu-2}} + \frac{D_1K_{1M}}{H_{inf}(r_0,s_0)L_{1m}(2\mu+\nu-1)}\frac{1}{s_0^{2\mu+\nu-1}}.
\end{aligned}$$

As a corollary it follows that $E_1(\eta, s_0, f_1, f_2) \geq E_{1inf}(r_0, s_0)$ with $E_{1inf}(r_0, s_0)$ given by (3.26). In addition, as E_1 is a negative exponential function, it follows that $E_1(\eta, s_0, f_1, f_2) \leq 1$.

Let us analyse the difference of E_1. From the inequality $|\exp(-x) - \exp(-y)| \leq |x - y|$ we have

$$\begin{aligned}
&|E_1(\eta, s_0, f_1, f_2) - E_1(\eta, s_0, g_1, g_2)| \leq \int_{s_0}^{\eta} 2av\left|\frac{N(f_1(v))}{L(f_1(v))} - \frac{N(g_1(v))}{L(g_1(v))}\right|dv \\
&+D_1\int_{s_0}^{\eta}\left|\frac{K(f_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^{\nu}} - \frac{K(g_1(v))}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^{\nu}}\right|dv.
\end{aligned} \tag{3.42}$$

On one hand,

$$\begin{aligned}
&\int_{s_0}^{\eta} 2av\left|\frac{N(f_1(v))}{L(f_1(v))} - \frac{N(g_1(v))}{L(g_1(v))}\right|dv \\
&\leq 2a\int_{s_0}^{\eta}\left|v\frac{N(f_1(v))L(g_1(v)) - N(g_1(v))L(g_1(v)) + N(f_1(v))L(g_1(v)) - N(g_1(v))L(f_1(v))}{L(f_1(v))L(g_1(v))}\right|dv \\
&\leq 2a\left[\int_{s_0}^{\eta}\frac{v\widetilde{N}_1||\vec{f}-\vec{g}||}{L_{1m}v^{\mu}}dv + \int_{s_0}^{\eta}\frac{N_{1M}\widetilde{L}_1v^{1-\mu}||\vec{f}-\vec{g}||}{L_{1m}^2v^{2\mu}}dv\right] \\
&\leq 2a\left[\frac{\widetilde{N}_1}{L_{1m}(\mu-2)}\frac{1}{s_0^{\mu-2}} + \frac{N_{1M}\widetilde{L}_1}{L_{1m}^2(3\mu-2)}\frac{1}{s_0^{3\mu-2}}\right]||\vec{f} - \vec{g}||.
\end{aligned} \tag{3.43}$$

On the other hand,

$$\begin{aligned}
&D_1\int_{s_0}^{\eta}\left|\tfrac{K(f_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^{\nu}}-\tfrac{K(g_1(v))}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^{\nu}}\right|dv\\
&\leq D_1\Bigg(\int_{s_0}^{\eta}\left|\tfrac{K(f_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^{\nu}}-\tfrac{K(g_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^{\nu}}\right|dv\\
&+\int_{s_0}^{\eta}\left|\tfrac{K(g_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^{\nu}}-\tfrac{K(g_1(v))}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^{\nu}}\right|dv\Bigg)\\
&\leq D_1\Bigg(\int_{s_0}^{\eta}\tfrac{|K(f_1(v))-K(g_1(v))|}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^{\nu}}dv\\
&+\int_{s_0}^{\eta}|K(g_1(v))|\left|\tfrac{1}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^{\nu}}-\tfrac{1}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^{\nu}}\right|dv\Bigg).
\end{aligned}\tag{3.44}$$

From assumptions (3.5), (3.7), (3.13) and inequalities (3.17), (3.19) we get that

$$\int_{s_0}^{\eta}\tfrac{|K(f_1(v))-K(g_1(v))|}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^{\nu}}dv\leq\tfrac{\widetilde{K}_1}{H_{inf}(r_0,s_0)L_{1m}(2\mu+\nu-1)}\tfrac{1}{s_0^{2\mu+\nu-1}}||\vec{f}-\vec{g}|| \tag{3.45}$$

and

$$\begin{aligned}
&\int_{s_0}^{\eta}|K(g_1(v))|\left|\tfrac{1}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^{\nu}}-\tfrac{1}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^{\nu}}\right|dv\\
&\leq K_{1M}\Bigg(\int_{s_0}^{\eta}\tfrac{|H(r_0,s_0,g_1,g_2)-H(r_0,s_0,f_1,f_2)|}{H(r_0,s_0,f_1,f_2)L(f_1(v))H(r_0,s_0,g_1,g_2)}\tfrac{dv}{v^{\mu+\nu}}\\
&+\int_{s_0}^{\eta}\tfrac{|L(g_1(v))-L(f_1(v))|}{L(f_1(v))H(r_0,s_0,g_1,g_2)L(g_1(v))}\tfrac{dv}{v^{\mu+\nu}}\Bigg)\\
&\leq\tfrac{K_{1M}}{H_{inf}(r_0,s_0)L_{1m}}\left(\tfrac{\widetilde{H}(r_0,s_0)}{H_{inf}(r_0,s_0)(2\mu+\nu-1)}\tfrac{1}{s_0^{2\mu+\nu-1}}+\tfrac{\widetilde{L}_1}{L_{1m}(3\mu+\nu-1)}\tfrac{1}{s_0^{3\mu+\nu-1}}\right)||\vec{f}-\vec{g}||.
\end{aligned}\tag{3.46}$$

Then inequalities (3.42)-(3.46) imply that

$$|E_1(\eta,s_0,f_1,f_2)-E_1(\eta,s_0,g_1,g_2)|\leq\widetilde{E}_1(r_0,s_0)||\vec{f}-\vec{g}||$$

with $\widetilde{E}_1$ given by (3.27).

From the definition of Φ_1 given by (2.19) we have that

$$\begin{aligned}
&|\Phi_1(\eta,s_0,f_1,f_2)-\Phi_1(\eta,s_0,g_1,g_2)|\leq\int_{s_0}^{\eta}\left|\tfrac{E_1(v,s_0,f_1,f_2)}{L(f_1(v))}-\tfrac{E_1(v,s_0,g_1,g_2)}{L(g_1(v))}\right|\tfrac{dv}{v^{\nu}}\\
&\leq\int_{s_0}^{\eta}\tfrac{|E_1(v,s_0,f_1,f_2)-E_1(v,s_0,g_1,g_2)|}{L(f_1(v))}\tfrac{dv}{v^{\nu}}+\int_{s_0}^{\eta}\tfrac{E_1(v,s_0,g_1,g_2)|L(f_1(v))-L(g_1(v))|}{L(f_1(v))L(g_1(v))}\tfrac{dv}{v^{\nu}}\\
&\leq\left(\tfrac{\widetilde{E}_1(r_0,s_0)}{L_{1m}}\int_{s_0}^{\eta}\tfrac{1}{v^{\mu+\nu}}dv+\tfrac{\widetilde{L}_1}{L_{1m}^2}\int_{s_0}^{\eta}\tfrac{1}{v^{2\mu+\nu}}dv\right)||\vec{f}-\vec{g}||\leq\widetilde{\Phi}_1(r_0,s_0)||\vec{f}-\vec{g}||,
\end{aligned}$$

where $\widetilde{\Phi}_1(r_0, s_0)$ is given by (3.29).

Taking into account the definition of H_1 given by (2.32), we easily obtain that

$$H_1(\eta, s_0, f_1, f_2) \geq K_{1m} \int_{s_0}^{\eta} \frac{1}{v^{\mu+\nu}} dv \geq \frac{K_{1m}}{(\mu+\nu-1)} \left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}} \right),$$

$$|H_1(\eta, s_0, f_1, f_2)| \leq \frac{K_{1M}}{E_{1inf}(r_0,s_0)} \int_{s_0}^{\eta} \frac{1}{v^{\mu+\nu}} dv \leq \frac{K_{1M}}{E_{1inf}(r_0,s_0)} \frac{1}{(\mu+\nu-1)} \frac{1}{s_0^{\mu+\nu-1}},$$

then (3.30) and (3.31) hold. In addition,

$$|H_1(\eta, s_0, f_1, f_2) - H_1(\eta, s_0, g_1, g_2)| \leq \int_{s_0}^{\eta} \left| \frac{K(f_1(v))}{E_1(v,s_0,f_1,f_2)} - \frac{K(g_1(v))}{E_1(v,s_0,g_1,g_2)} \right| \frac{dv}{v^\nu}$$

$$\leq \int_{s_0}^{\eta} \frac{|K(f_1(v))-K(g_1(v))|}{E_1(v,s_0,f_1,f_2)} \frac{dv}{v^\nu} + \int_{s_0}^{\eta} \frac{K(g_1(v))|E_1(v,s_0,f_1,f_2)-E_1(v,s_0,g_1,g_2)|}{E_1(v,s_0,f_1,f_2)E_1(v,s_0,g_1,g_2)} \frac{dv}{v^\nu}$$

$$\leq \frac{1}{E_{1inf}(r_0,s_0)} \left(\widetilde{K}_1 + \frac{K_{1M}\widetilde{E}_1(r_0,s_0)}{E_{1inf}(r_0,s_0)} \right) \int_{s_0}^{\eta} \frac{1}{v^{\mu+\nu}} dv \quad ||\vec{f} - \vec{g}|| \leq \widetilde{H}_1(r_0, s_0)||\vec{f} - \vec{g}||,$$

where $\widetilde{H}_1$ is given by (3.35).

From the definition of G_1 it follows that

$$\begin{aligned} G_1(\eta, s_0, f_1, f_2)| &\geq \frac{E_{1inf}(r_0,s_0)}{L_{1M}} \int_{s_0}^{\eta} \frac{H_{1inf}(v,s_0)}{v^{\mu+\nu}} dv \\ &\geq \frac{K_{1m}E_{1inf}(r_0,s_0)}{L_{1M}(\mu+\nu-1)} \int_{s_0}^{\eta} \frac{1}{v^{\mu+\nu}} \left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{v^{\mu+\nu-1}} \right) dv \geq G_{1inf}(\eta, r_0, s_0), \end{aligned}$$

where G_{1inf} is given by (3.39) and

$$|G_1(\eta, s_0, f_1, f_2)| \leq \int_{s_0}^{\eta} \left| \frac{E_1(v,s_0,f_1,f_2)H_1(v,s_0,f_1,f_2)}{L(f_1(v))} \right| \frac{dv}{v^\nu}$$

$$\leq \frac{H_{1sup}(r_0,s_0)}{L_{1m}} \int_{s_0}^{\eta} \frac{dv}{v^{\mu+\nu}} \leq G_{1sup}(r_0, s_0),$$

where G_{1sup} is given by (3.40). Moreover,

$$|G_1(\eta, s_0, f_1, f_2) - G_1(\eta, s_0, g_1, g_2)|$$

$$\leq \int_{s_0}^{\eta} |H_1(v, s_0, f_1, f_2)| \left| \frac{E_1(v,s_0,g_1,g_2)}{L(g_1(v))} - \frac{E_1(v,s_0,f_1,f_2)}{L(f_1(v))} \right| \frac{dv}{v^\nu}$$

$$+ \int_{s_0}^{\eta} \frac{E_1(v,s_0,g_1,g_2)|H_1(v,s_0,f_1,f_2)-H_1(v,s_0,g_1,g_2)|}{L(g_1(v))} \frac{dv}{v^\nu}$$

$$\leq \widetilde{G}_1(r_0, s_0)||\vec{f} - \vec{g}||,$$

with $\widetilde{G}_1$ defined by (3.41).

□

Lemma 3.3. *For every $\vec{f} = (f_1, f_2)$, $\vec{g} = (g_1, g_2) \in \mathcal{K}$, the following inequalities hold:*

1)

$$E_2(\eta, r_0, f_1, f_2) \geq E_{2inf}(r_0, s_0), \tag{3.47}$$

$$E_2(\eta, r_0, f_1, f_2) \leq 1, \tag{3.48}$$

$$|E_2(\eta, r_0, f_1, f_2) - E_2(\eta, r_0, g_1, g_2)| \leq \widetilde{E}_2(r_0, s_0)||\vec{f} - \vec{g}||, \tag{3.49}$$

where

$$E_{2inf}(r_0,s_0) := \exp\left(-\left[a\frac{N_{2M}}{L_{2m}(\mu-1)}\frac{1}{r_0^{2\mu-2}} + \frac{D_2 K_{2M}}{H_{inf}(r_0,s_0)L_{2m}(2\mu+\nu-1)}\frac{1}{r_0^{2\mu+\nu-1}}\right]\right), \tag{3.50}$$

$$\widetilde{E}_2(r_0,s_0) := 2a\left[\frac{\widetilde{N}_2}{L_{2m}(\mu-2)}\frac{1}{r_0^{\mu-2}} + \frac{N_{2M}\widetilde{L}_2}{L_{2m}^2(3\mu-2)}\frac{1}{r_0^{3\mu-2}}\right]$$

$$+D_2\left(\frac{\widetilde{K}_2}{H_{inf}(r_0,s_0)L_{2m}(2\mu+\nu-1)}\frac{1}{r_0^{2\mu+\nu-1}}\right. \tag{3.51}$$

$$\left.+\frac{K_{2M}}{H_{inf}(r_0,s_0)L_{2m}}\left(\frac{\widetilde{H}(r_0)}{H_{inf}(r_0,s_0)(2\mu+\nu-1)}\frac{1}{r_0^{2\mu+\nu-1}} + \frac{\widetilde{L}_2}{L_{2m}(3\mu+\nu-1)}\frac{1}{r_0^{3\mu+\nu-1}}\right)\right);$$

2)

$$\Phi_2(\eta,r_0,f_1,f_2) \geq \Phi_{2inf}(\eta,r_0,s_0), \tag{3.52}$$

$$\Phi_2(\eta,r_0,f_1,f_2) \leq \Phi_{2sup}(r_0), \tag{3.53}$$

$$|\Phi_2(\eta,r_0,f_1,f_2) - \Phi_2(\eta,r_0,g_1,g_2)| \leq \widetilde{\Phi}_2(r_0,s_0)||\vec{f}-\vec{g}||, \tag{3.54}$$

where

$$\Phi_{2inf}(\eta,r_0,s_0) := \frac{E_{2inf}(r_0,s_0)}{L_{2M}}\frac{1}{(\mu+\nu-1)}\left(\frac{1}{r_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}}\right), \tag{3.55}$$

$$\Phi_{2sup}(r_0) := \frac{1}{L_{2m}}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}, \tag{3.56}$$

$$\widetilde{\Phi}_2(r_0,s_0) := \frac{\widetilde{E}_2(r_0,s_0)}{L_{2m}}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}} + \frac{\widetilde{L}_2}{L_{2m}^2}\frac{1}{(2\mu+\nu-1)}\frac{1}{r_0^{2\mu+\nu-1}}; \tag{3.57}$$

3)

$$H_2(\eta,r_0,f_1,f_2) \leq H_{2inf}(\eta,r_0,s_0), \tag{3.58}$$

$$H_2(\eta,r_0,f_1,f_2) \leq H_{2sup}(r_0,s_0), \tag{3.59}$$

$$|H_2(\eta,r_0,f_1,f_2) - H_2(\eta,r_0,g_1,g_2)| \leq \widetilde{H}_2(r_0,s_0)||\vec{f}-\vec{g}||, \tag{3.60}$$

where

$$H_{2inf}(\eta,r_0) := \frac{K_{2m}}{(\mu+\nu-1)}\left(\frac{1}{r_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}}\right), \tag{3.61}$$

$$H_{2sup}(r_0,s_0) := \frac{K_{2M}}{E_{2inf}(r_0,s_0)}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}, \tag{3.62}$$

$$\widetilde{H}_2(r_0,s_0) := \left(\widetilde{K}_2 + \frac{K_{2M}\widetilde{E}_2(r_0,s_0)}{E_{2inf}(r_0,s_0)}\right)\frac{1}{E_{2inf}(r_0,s_0)(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}; \tag{3.63}$$

4)

$$G_2(\eta,r_0,f_1,f_2) \leq G_{2inf}(\eta,r_0,s_0), \tag{3.64}$$

$$G_2(\eta,r_0,f_1,f_2) \leq G_{2sup}(r_0,s_0), \tag{3.65}$$

$$|G_2(\eta,r_0,f_1,f_2) - G_2(\eta,r_0,g_1,g_2)| \leq \widetilde{G}_2(r_0,s_0)||\vec{f}-\vec{g}||, \tag{3.66}$$

where

$$G_{2inf}(\eta,r_0,s_0) := \frac{K_{2m}E_{2inf}(r_0,s_0)}{2L_{2M}(\mu+\nu-1)^2}\left(\frac{1}{r_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}}\right)^2, \tag{3.67}$$

$$G_{2sup}(r_0,s_0) := \frac{H_{2sup}(r_0,s_0)}{L_{2m}}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}, \tag{3.68}$$

$$\widetilde{G}_2(r_0,s_0) := H_{2sup}(r_0,s_0)\widetilde{\Phi}_2(r_0,s_0) + \frac{\widetilde{H}_2(r_0,s_0)}{L_{2m}}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}. \tag{3.69}$$

Proof. The proof follows analogously to the previous lemma. □

Lemma 3.4. *For every* $\vec{f}=(f_1,f_2), \vec{g}=(g_1,g_2)\in\mathcal{K}$ *it follows that*

$$||V_1(\vec{f})-V_1(\vec{g})||_{C[s_o,r_0]}\leq \varepsilon_1(r_0,s_0)||\vec{f}-\vec{g}||,$$

where

$$\begin{aligned}\varepsilon_1(r_0,s_0)&=2s_0^{\nu}Q\exp(-s_0^2)\widetilde{\Phi}_1(r_0,s_0)\\&+2D_1^*\left(\tfrac{G_{1sup}(r_0,s_0)2H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)}+\tfrac{\widetilde{G}_1(r_0,s_0)}{H^2_{inf}(r_0,s_0)}\right).\end{aligned}\tag{3.70}$$

Proof. Taking into account that

$$\begin{aligned}&\left|\tfrac{G_1(\eta,s_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}-\tfrac{G_1(\eta,s_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\right|\leq\tfrac{|G_1(\eta,s_0,f_1,f_2)||H^2(r_0,s_0,f_1,f_2)-H^2(r_0,s_0,g_1,g_2)|}{H^2(r_0,s_0,f_1,f_2)H^2(g_1,g_2)}\\&+\tfrac{|G_1(\eta,s_0,f_1,f_2)-G_1(\eta,s_0,g_1,g_2)|}{H^2(r_0,s_0,g_1,g_2)}\leq\left[\tfrac{G_{1sup}(r_0,s_0)2H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)}+\tfrac{\widetilde{G}_1(r_0,s_0)}{H^2_{inf}(r_0,s_0}\right]||\vec{f}-\vec{g}||,\end{aligned}\tag{3.71}$$

for each $\eta\in[s_0,r_0]$ it follows that

$$\begin{aligned}&|V_1(\vec{f})(\eta)-V_1(\vec{g})(\eta)|\leq\\&\leq s_0^{\nu}Q\exp(-s_0^2)\big[|\Phi_1(r_0,s_0,f_1,f_2)-\Phi_1(r_0,s_0,g_1,g_2)|\\&+|\Phi_1(\eta,s_0,f_1,f_2)-\Phi_1(\eta,s_0,g_1,g_2)|\big]\\&+\left|\tfrac{D_1^*G_1(r_0,s_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}-\tfrac{D_1^*G_1(r_0,s_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\right|+\left|\tfrac{D_1^*G_1(\eta,s_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}-\tfrac{D_1^*G_1(\eta,s_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\right|\\&\leq\Big[2s_0^{\nu}Q\exp(-s_0^2)\widetilde{\Phi}_1(r_0,s_0)\\&+2D_1^*\big(\tfrac{G_{1sup}(r_0,s_0)2H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)}+\tfrac{\widetilde{G}_1(r_0,s_0)}{H^2_{inf}(r_0,s_0)}\big)\Big]||\vec{f}-\vec{g}||=\varepsilon_1(r_0,s_0)||\vec{f}-\vec{g}||.\end{aligned}\tag{3.72}$$

□

Lemma 3.5. *For every* $\vec{f}=(f_1,f_2), \vec{g}=(g_1,g_2)\in\mathcal{K}$ *it follows that*

$$||V_2(\vec{f})-V_2(\vec{g})||_{C_b[r_0,+\infty)}\leq\varepsilon_2(r_0,s_0)||\vec{f}-\vec{g}||,$$

where

$$\varepsilon_2(r_0,s_0)=\varepsilon_{21}(r_0,s_0)+\varepsilon_{22}(r_0,s_0)+\varepsilon_{23}(r_0,s_0)\tag{3.73}$$

with

$$\varepsilon_{21}(r_0,s_0)=\tfrac{2\widetilde{\Phi}_2(r_0,s_0)}{\Phi_{2inf}(+\infty,r_0,s_0)},$$

$$\varepsilon_{22}(r_0,s_0)=\tfrac{\widetilde{G}_2(r_0,s_0)}{H^2_{inf}(r_0,s_0)}+\tfrac{2G_{2sup}(r_0,s_0)H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)},$$

$$\varepsilon_{23}(r_0,s_0)=\tfrac{\Phi_{2sup}(r_0,s_0)}{\Phi_{2inf}(+\infty,r_0,s_0)}\varepsilon_{22}(r_0,s_0)+\tfrac{G_{2sup}(r_0,s_0)}{H^2_{inf}(r_0,s_0)}\varepsilon_{21}(r_0,s_0).$$

Proof. On one hand, we have that

$$\begin{aligned}&\left|\tfrac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\tfrac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right|\leq\tfrac{|\Phi_2(\eta,r_0,f_1,f_2)-\Phi_2(\eta,r_0,g_1,g_2)|}{\Phi_2(+\infty,r_0,f_1,f_2)}\\&+\tfrac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\tfrac{|\Phi_2(+\infty,r_0,f_1,f_2)-\Phi_2(+\infty,r_0,g_1,g_2)|}{\Phi_2(+\infty,r_0,f_1,f_2)}\\&\leq\tfrac{|\Phi_2(\eta,r_0,f_1,f_2)-\Phi_2(\eta,r_0,g_1,g_2)|}{\Phi_2(+\infty,r_0,f_1,f_2)}+\tfrac{|\Phi_2(+\infty,r_0,f_1,f_2)-\Phi_2(+\infty,r_0,g_1,g_2)|}{\Phi_2(+\infty,r_0,f_1,f_2)}\\&\leq\tfrac{2\widetilde{\Phi}_2(r_0,s_0)}{\Phi_{2inf}(+\infty,r_0,s_0)}||\vec{f}-\vec{g}||=\varepsilon_{21}(r_0,s_0)||\vec{f}-\vec{g}||.\end{aligned}\tag{3.74}$$

On the other hand, we obtain that

$$\begin{aligned}
&\left|\frac{G_2(\eta,r_0,f_1,f_2)}{H^2(s_0,r_0,f_1,f_2)}-\frac{G_2(\eta,r_0,g_1,g_2)}{H^2(s_0,r_0,g_1,g_2)}\right| \\
&\leq \frac{|G_2(\eta,r_0,f_1,f_2)-G_2(\eta,r_0,g_1,g_2)|}{H^2(s_0,r_0,f_1,f_2)}+\frac{|G_2(\eta,r_0,g_1,g_2)||H^2(s_0,r_0,g_1,g_2)-H^2(s_0,r_0,f_1,f_2)|}{H^2(s_0,r_0,f_1,f_2)H^2(s_0,r_0,g_1,g_2)} \\
&\leq \left(\frac{\widetilde{G}_2(r_0,s_0)}{H^2_{inf}(r_0,s_0)}+\frac{2G_{2sup}(r_0,s_0)H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)}\right)||\vec{f}-\vec{g}|| \\
&=\varepsilon_{22}(r_0,s_0)||\vec{f}-\vec{g}||.
\end{aligned} \tag{3.75}$$

In addition,

$$\begin{aligned}
&\left|\frac{G_2(+\infty,r_0,f_1,f_2)}{H^2(s_0,r_0,f_1,f_2)}\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\frac{G_2(+\infty,r_0,g_1,g_2)}{H^2(s_0,r_0,g_1,g_2)}\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right| \\
&\leq \frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}\left|\frac{G_2(+\infty,r_0,f_1,f_2)}{H^2(s_0,r_0,f_1,f_2)}-\frac{G_2(+\infty,r_0,g_1,g_2)}{H^2(s_0,r_0,g_1,g_2)}\right| \\
&+\frac{G_2(+\infty,r_0,g_1,g_2)}{H^2(s_0,r_0,g_1,g_2)}\left|\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right| \\
&\leq \left(\frac{\Phi_{2sup}(r_0,s_0)}{\Phi_{2inf}(+\infty,r_0,s_0)}\varepsilon_{22}(r_0,s_0)+\frac{G_{2sup}(r_0,s_0)}{H^2_{inf}(r_0,s_0)}\varepsilon_{21}(r_0,s_0)\right)||\vec{f}-\vec{g}|| \\
&=\varepsilon_{23}(r_0,s_0)||\vec{f}-\vec{g}||.
\end{aligned} \tag{3.76}$$

From the previous inequalities, for each $\eta \geq r_0$, it follows that

$$\begin{aligned}
&|V_2(\vec{f})(\eta)-V_2(\vec{g})(\eta)| \\
&\leq \left|\frac{D_2^*G_2(+\infty,r_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\frac{D_2^*G_2(+\infty,r_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right| \\
&+\left|\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right|+\left|\frac{D_2^*G_2(\eta,r_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}-\frac{D_2^*G_2(\eta,r_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\right| \\
&\leq \varepsilon_2(r_0,s_0)||\vec{f}-\vec{g}||.
\end{aligned} \tag{3.77}$$

□

Theorem 3.1. *For every $\vec{f}=(f_1,f_2)$, $\vec{g}=(g_1,g_2)\in\mathcal{K}$ it follows that*

$$||\Psi(\vec{f})-\Psi(\vec{g})|| \leq \varepsilon(r_0,s_0)||\vec{f}-\vec{g}||$$

with

$$\varepsilon(r_0,s_0)=\max\{\varepsilon_1(r_0,s_0),\varepsilon_2(r_0,s_0)\}, \tag{3.78}$$

where $\varepsilon_1(r_0,s_0)$ and $\varepsilon_2(r_0,s_0)$ are given by (3.70) and (3.73), respectively.

Proof. From the previous lemmas we have that

$$\begin{aligned}
||\Psi(\vec{f})-\Psi(\vec{g})|| &= \max\left\{||V_1(\vec{f})-V_1(\vec{g})||_{C[s_0,r_0]},||V_2(\vec{f})-V_2(\vec{g})||_{C_b[r_0,+\infty)}\right\} \\
&= \max\left\{\varepsilon_1(r_0,s_0)||\vec{f}-\vec{g}||,\varepsilon_2(r_0,s_0)||\vec{f}-\vec{g}||\right\}=\varepsilon(r_0,s_0)||\vec{f}-\vec{g}||.
\end{aligned}$$

□

Now we will look for conditions that guarantee that Ψ is a contraction mapping.

For each $s_0 > 0$ fixed, we define the functions

$$\varepsilon_{1,s_0}(r_0) = \varepsilon_1(r_0, s_0) \text{ and } \varepsilon_{2,s_0}(r_0) = \varepsilon_2(r_0, s_0), \text{ for all } r_0 > s_0,$$

where ε_1, ε_2 are given by (3.70) and (3.73), respectively. The following results hold.

Lemma 3.6. a) *The function ε_{1,s_0} is a decreasing function that satisfies $\varepsilon_{1,s_0}(s_0) = +\infty$ and $\varepsilon_{1,s_0}(+\infty) = j_1(s_0)$, where*

$$\begin{aligned} j_1(s_0) &= 2s_0^{\nu} Q \exp(-s_0^2)\widetilde{\Phi}_1(+\infty, s_0) \\ &+2D_1^*\left(\tfrac{G_{1sup}(+\infty,s_0)2H_{sup}(+\infty,s_0)\widetilde{H}(+\infty,s_0)}{H_{inf}^4(+\infty,s_0)} + \tfrac{\widetilde{G}_1(+\infty,s_0)}{H_{inf}^2+\infty,s_0)}\right). \end{aligned} \tag{3.79}$$

b) *If*

$$\tfrac{2D_1^*\widetilde{K}_1}{L_{1m}K_{1m}^2}\left(\tfrac{2K_{1M}}{K_{1m}^2} + 1\right) < 1, \tag{3.80}$$

then there exists a unique $s_1 > 0$ such that $j_1(s_0) < 1$ for all $s_0 > s_1$.

Moreover, for each $s_0 > s_1$ there exists $r_1 = r_1(s_0) > s_0$ such that $\varepsilon_{1,s_0}(r_1) = 1$ and $\varepsilon_{1,s_0}(r_0) < 1$ for all $r_0 > r_1$.

Proof. a) According to the definition of ε_1 given by (3.70), the proof follows straightforwardly from Lemmas 3.1 and 3.2.

b) From the definition of j_1 given by (3.79), we have that it is a decreasing function that satisfies $j_1(0) = +\infty$ and $j_1(+\infty) = \tfrac{2D_1^*\widetilde{K}_1}{L_{1m}K_{1m}^2}\left(\tfrac{2K_{1M}}{K_{1m}^2} + 1\right)$. Then, assuming (3.80), it follows that there exists a unique $s_1 > s_0$ such that $j_1(s_1) = 1$ and $j_1(s_0) < 1$ for all $s_0 > s_1$. Moreover, from item a), for each $s_0 > s_1$ there exists $r_1 = r_1(s_0) > s_0$ such that $\varepsilon_{1,s_0}(r_1) = 1$ and $\varepsilon_{1,s_0}(r_0) < 1$ for all $r_0 > r_1$.

□

Lemma 3.7. a) *The function ε_{2,s_0} is a decreasing function that satisfies the equalities $\varepsilon_{2,s_0}(s_0) = +\infty$ and $\varepsilon_{2,s_0}(+\infty) = 0$.*

b) *For each $s_0 > 0$ there exists $r_2 = r_2(s_0) > s_0$ such that $\varepsilon_{2,s_0}(r_2) = 1$ and $\varepsilon_{2,s_0}(r_0) < 1$ for all $r_0 > r_2$.*

Proof. a) It follows from Lemmas 3.1 and 3.3, by taking into account that ε_2 is defined by (3.73).

b) It clearly follows from item a).

□

Theorem 3.2. *If inequality (3.80) holds, then for each $(r_0, s_0) \in \Sigma$ with*

$$\Sigma = \{(r_0, s_0) : \ s_0 > s_1, \ r_0 > \overline{r}_0(s_0)\} \tag{3.81}$$

we have that $\varepsilon(r_0, s_0) < 1$, where ε is given by (3.78) and

$$\overline{r}_0(s_0) = \max\{r_1(s_0), r_2(s_0)\} \tag{3.82}$$

with s_1, r_1 and r_2 defined in Lemmas 3.6 and 3.7, respectively.

Proof. The proof follows immediately by Lemmas 3.6 and 3.7. □

Corollary 3.1. *Under assumption (3.80), for each $(r_0, s_0) \in \Sigma$, the operator Ψ defined by (3.2) is a contraction mapping.*

Theorem 3.3. *Under assumption (3.80), for each $(r_0, s_0) \in \Sigma$, there exists a unique fixed point $(f_1^*, f_2^*) \in \mathcal{K}$ of the operator Ψ.*

Proof. First, notice that $\mathcal{K}$ is a closed subset of the Banach space $\mathcal{C}$ given by (3.1). In addition, it is easy to see that $\Psi(\vec{f}) \in \mathcal{K}$ given that $V_1(\vec{f}) \in C[s_0, r_0]$, $V_2(\vec{f}) \in C_b[r_0, +\infty)$, $V_2(\vec{f})(r_0) = 0$ and $V_2(\vec{f})(+\infty) = 0$. Finally, according to Corollary 3.1, under assumption (3.80), for each $(r_0, s_0) \in \Sigma$ it follows that Ψ is a contraction mapping. As a corollary, applying the fixed point Banach theorem, we get that there exists a unique fixed point $(f_1^*, f_2^*) \in \mathcal{K}$ of the operator Ψ for each $(r_0, s_0) \in \Sigma$. □

Corollary 3.2. *If (3.80) holds, for each $(r_0, s_0) \in \Sigma$, then there exists a unique solution (f_1^*, f_2^*) to the system of equations (2.24)-(2.25).*

It remains to prove the existence of solution $(r_0, s_0) \in \Sigma$ to the system of equations given by (2.26) and (2.27), where $f_1 = f_1^*$ and $f_2 = f_2^*$ are the unique solutions to equations (2.24)-(2.25). For that purpose we will need some preliminary results.

Let us notice that equation (2.26) can be rewritten as

$$X(r_0, s_0) = Y(r_0, s_0), \tag{3.83}$$

where

$$X(r_0, s_0) = Z(r_0, s_0) - B, \quad Z(r_0, s_0) = \frac{D_1^* G_1(r_0, s_0, f_1^*, f_2^*)}{H^2(r_0, s_0, f_1^*, f_2^*)}, \tag{3.84}$$

and

$$Y(r_0, s_0) = -Q s_0^{\nu} \exp(-s_0^2) \Phi_1(r_0, s_0, f_1^*, f_2^*). \tag{3.85}$$

Lemma 3.8. *The following properties hold:*

a) *$Y(r_0, s_0) < 0$ for each $(r_0, s_0) \in \Sigma$,*

b) *$Z(r_0, s_0) > Z_{inf}(r_0, s_0)$ for each $(r_0, s_0) \in \Sigma$, where*

$$Z_{inf}(r_0, s_0) = \frac{D_1^* E_{1inf}(r_0, s_0) K_{1m}}{2L_{1M}} \left(\frac{r_0^{\mu+\nu-1} - s_0^{\mu+\nu-1}}{K_{1M} r_0^{\mu+\nu-1} + K_{2M} s_0^{\mu+\nu-1}} \right)^2,$$

c) *for a fixed $s_0 > s_1$, if we assume that*

$$X(\overline{r}_0(s_0), s_0) < Y(\overline{r}_0(s_0), s_0), \tag{3.86}$$

then $Z_{inf}(\cdot, s_0)$ is an increasing function that satisfies the conditions

$$Z_{inf}(\overline{r_0}(s_0), s_0) < B, \qquad Z_{inf}(+\infty, s_0) = j_2(s_0), \tag{3.87}$$

where

$$j_2(s_0) = \frac{D_1^* E_{1inf}(+\infty, s_0) K_{1m}}{L_{1M} K_{1M}^2} \tag{3.88}$$

is an increasing function that satisfies the equality

$$j_2(+\infty) = \frac{D_1^* K_{1m}}{2L_{1M} K_{1M}^2},$$

d) *if we assume that*

$$\frac{D_1^* K_{1m}}{2L_{1M} K_{1M}^2} > B, \tag{3.89}$$

then there exists a unique $s_2 = \min\{s_0 \geq s_1 : j_2(s_0) \geq B\}$. Moreover, for each $s_0 > s_2$, we have that $j_2(s_0) > B$,

e) *if* (3.86) *and* (3.89) *hold for each* $s_0 > s_2$, *then there exists a unique* $r_B(s_0) > s_0$ *such that* $Z_{inf}(r_0, s_0) > B$ *for all* $r_0 > r_B(s_0)$.

Proof.

a) It is clear from the definition of the function Y given by (3.85).

b) It follows from the inequalities obtained in Lemmas 3.1 and 3.2.

c) From the definition of Z_{inf} it it easy to see that $Z_{inf}(\cdot, s_0)$ is an increasing function for each fixed $s_0 > s_1$. In addition, assumption (3.86) and item a) lead to the inequalities

$$Z_{inf}(\overline{r}_0(s_0), s_0) - B < Z(\overline{r}_0(s_0), s_0) - B < Y(\overline{r}_0(s_0), s_0) < 0.$$

Hence, it follows that $Z_{inf}(\overline{r}_0(s_0), s_0) < B$. Finally, taking a limit gives that $Z_{inf}(+\infty, s_0) = j_2(s_0)$ for each $s_0 > s_1$.

d) First, notice that hypothesis (3.89) can be rewritten as $j_2(+\infty) > B$. From the fact that j_2 is an increasing function, we can conclude that there exists a unique $s_2 = \min\{s_0 \geq s_1 : j_2(s_0) \geq B\}$. Notice that $s_2 = s_1$ in the case $j_2(s_1) > B$. As a corollary, for each $s_0 > s_2$, we get that $j_2(s_0) > B$.

e) For each fixed $s_0 > s_2$, we have that $Z_{inf}(\overline{r}_0(s_0), s_0) < B$ from item c) and $Z_{inf}(+\infty, s_0) > B$ from item d). Then, there exists a unique $r_B = r_B(s_0) > \overline{r}_0(s_0)$ such that $Z_{inf}(r_B(s_0), s_0) = B$ and $Z_{inf}(r_0, s_0) > B$ for all $r_0 > r_B(s_0)$. □

Lemma 3.9. *For each* $s_0 > s_2$, *if we assume that inequalities* (3.86) *and* (3.89) *hold, then there exists at least one solution* $r_0^* = r_0^*(s_0, f_1^*, f_2^*) \in (\overline{r}_0(s_0), r_B(s_0))$ *to equation* (2.26).

Proof. For each $s_0 > s_2$, taking into account assumption (3.86) and the fact that from item e) of Lemma 3.8 the following inequality holds

$$X(r_B(s_0), s_0) \geq Z_{inf}(r_B(s_0), s_0) - B = 0 > Y(r_B(s_0), s_0),$$

we obtain that there exists at least one solution $r_0^* \in (\overline{r}_0(s_0), r_B(s_0))$ to equation (2.26). □

Now we will analyze equation (2.27). If we replace r_0 by $r_0^*(s_0)$ and (f_1, f_2) by (f_1^*, f_2^*), the resulting equation is equivalent to the equation

$$W(r_0^*(s_0), s_0) = M, \tag{3.90}$$

where

$$\begin{aligned} W(r_0^*(s_0), s_0) = {} & \tfrac{E_1(r_0^*(s_0), s_0, f_1^*, f_2^*)}{r_0^{\nu+1}} \Big[Q \exp(-s_0^2) s_0^{\nu} \\ & + \tfrac{D_1^*}{H^2(r_0^*(s_0), s_0, f_1^*, f_2^*)} H_1(r_0^*(s_0), s_0, f_1^*, f_2^*) \Big] \\ & - \tfrac{1}{r_0^*(s_0)^{\nu+1} \Phi_2(+\infty, r_0^*(s_0), f_1^*, f_2^*)} \left[1 - \tfrac{D_2^*}{H^2(r_0^*(s_0), s_0, f_1^*, f_2^*)} G_2(+\infty, r_0^*(s_0), f_1^*, f_2^*) \right]. \end{aligned} \tag{3.91}$$

Lemma 3.10. *If any of the following two systems of inequalities hold*

$$\begin{cases} W_{inf}(s_2) > M \\ W_{sup}(+\infty) < M \end{cases} \quad \text{or} \quad \begin{cases} W_{sup}(s_2) < M \\ W_{inf}(+\infty) > M, \end{cases} \tag{3.92}$$

then there exists at least one solution $\widehat{s}_0 > s_2$ *to equation* (3.90)*, where*

$$W_{inf}(s_0) = \tfrac{E_{1inf}(r_0^*(s_0),s_0)}{r_B^{\nu+1}(s_0)}\left[Q\exp(-s_0^2)s_0^\nu + \tfrac{D_1^*}{H_{sup}^2(r_0^*(s_0),s_0)}H_{1inf}(r_0^*(s_0),s_0)\right]$$
$$-\tfrac{1}{\overline{r_0}^{\nu+1}(s_0)\Phi_{2inf}(+\infty,r_0^*(s_0),s_0)} + \tfrac{1}{r_B^{\nu+1}(s_0)\Phi_{2sup}(r_0^*(s_0))}\cdot\tfrac{D_2^*}{H_{sup}^2(r_0^*(s_0),s_0)}G_{2inf}(+\infty,r_0^*(s_0),s_0), \tag{3.93}$$

$$W_{sup}(s_0) = \tfrac{1}{\overline{r_0}^{\nu+1}(s_0)}\Bigg[Q\exp(-s_0^2)s_0^\nu + \tfrac{D_1^*}{H_{inf}^2(r_0^*(s_0),s_0)}H_{1sup}(r_0^*(s_0),s_0)$$
$$+\tfrac{1}{\Phi_{2inf}(+\infty,r_0^*(s_0),s_0)}\tfrac{D_2^*}{H_{inf}^2(r_0^*(s_0),s_0)}G_{2sup}(r_0^*(s_0),s_0).\Bigg]. \tag{3.94}$$

The above analysis allows to establish the following existence theorem.

Theorem 3.4. *If hypotheses* $(A1)-(A5)$ *and inequalities* (3.80), (3.86), (3.89) *and* (3.92) *hold, then there exists at least one solution* $(\widehat{s}_0, r_0^*(\widehat{s}_0), f_1^*, f_2^*)$ *to the system of equations* (2.24)-(2.27)*, where* (f_1^*, f_2^*) *is the unique fixed point of the operator* Ψ *corresponding to* $(\widehat{s}_0, r_0^*(\widehat{s}_0)) \in \Sigma$.

Corollary 3.3. *If hypotheses* $(A1)-(A5)$ *and inequalities* (3.80), (3.86), (3.89) *and* (3.92) *hold, then there exists at least one solution to problem* (1.4)-(1.17)*, where*

$$\begin{cases} T_1(z,t) = T_m f_1^*\left(\frac{z}{2a\sqrt{t}}\right) + T_m, & s(t) \le z \le r(t),\ t>0, \\ T_2(z,t) = T_m f_2^*\left(\frac{z}{2a\sqrt{t}}\right) + T_m, & z \ge r(t),\ t>0, \\ \varphi_1(z,t) = \dfrac{U_c}{2}\cdot\dfrac{F_1\left(\frac{z}{2a\sqrt{t}},\widehat{s}_0,f_1^*\right)}{H(r_0^*(\widehat{s}_0),\widehat{s}_0,f_1^*,f_2^*)}, & s(t) \le z \le r(t),\ t>0, \\ \varphi_2(z,t) = \dfrac{U_c}{2}\cdot\dfrac{F_1(r_0^*(\widehat{s}_0),\widehat{s}_0,f_1^*) + F_2\left(\frac{z}{2a\sqrt{t}},r_0^*(\widehat{s}_0),f_2^*\right)}{H(r_0^*(\widehat{s}_0),\widehat{s}_0,f_1^*,f_2^*)}, & z \ge r(t),\ t>0, \end{cases}$$

with $s(t) = 2a\widehat{s}_0\sqrt{t}$ *and* $r(t) = 2ar_0^*(\widehat{s}_0)\sqrt{t}$.

Conclusion

We have considered a two-phase Stefan type problem governed by the generalized heat equation with the Thomson effect and nonlinear thermal coefficients, that models the dynamics of electromagnetic fields and heat transfer within closed electrical contacts, particularly focusing on the instantaneous explosion of micro-asperities.

By employing similarity transformations, we have effectively reduced the problem to a set of coupled ordinary differential equations, thereby facilitating tractable analysis and solution.

The validity and utility of our approach have been rigorously demonstrated through discussions and proofs grounded on the fixed point theory within the framework of Banach spaces. This theoretical underpinning not only enhances our confidence in the proposed solutions, but also provides a solid foundation for future research endeavors in related domains.

Furthermore, the insights gained from this study hold significant implications for various practical applications involving electrical contacts, such as in the design and optimization of electronic devices, electrical connectors, and power transmission systems. By elucidating the intricate interplay between electromagnetic fields and heat transfer phenomena, our work contributes to advancing the understanding and engineering of such systems in both industrial and academic contexts.

Acknowledgments

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. AP19675480) and projects O06-24CI1901 and O06-24CI1903 of the Austral University, Rosario, Argentina.

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Targyn Nauryz
International School of Economics
Kazakh-British Technical University
59 Tole Bi St,
050005 Almaty, Republic of Kazakhstan,
Department of Mathematical Physics and Modelling,
Institute of Mathematics and Mathematical Modelling
125 Pushkin St,
and
050010 Almaty, Republic of Kazakhstan,
School of Digital Technologies,
Narxoz University
55 Zhandosov St,
050035 Almaty, Republic of Kazakhstan
E-mail: targyn.nauryz@gmail.com

Stanislav Nikolaevich Kharin
Department of Mathematical Physics and Modelling,
Institute of Mathematics and Mathematical Modelling
125 Pushkin St,
050010 Almaty, Republic of Kazakhstan
E-mail: staskharin@yahoo.com

Adriana Briozzo, Julieta Bollati,
Department of Matematics
Universidad Austral
CONICET
1950 Paraguay St,
S2000FZF, Rosario, Argentina
E-mails: ABriozzo@austral.edu.ar, JBollati@austral.edu.ar

Received: 28.08.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 3 (2025), 90 – 101

ASYMPTOTICS OF SOLUTIONS OF THE STURM-LIOUVILLE EQUATION IN VECTOR-FUNCTION SPACE

Ya.T. Sultanaev, N.F. Valeev, A. Yeskermessuly

Communicated by I.N. Parasidis

Key words: asymptotic methods, rapidly oscillating coefficients, vector-function, L-diagonal system, resonance case.

AMS Mathematics Subject Classification: 34L05, 34E05.

Abstract. In this paper, we study the asymptotic behaviour of fundamental systems of solutions to the Sturm-Liouville equation with rapidly oscillating potentials in a two-dimensional vector-function space. We consider different cases in which the coefficients do not satisfy the regularity conditions. Additionally, we investigate the asymptotic behaviour of solutions in resonance cases.

DOI: https://doi.org/10.32523/2077-9879-2025-16-3-90-101

1 Introduction

Numerous studies have focused on the asymptotic properties of solutions to singular Sturm-Liouville equations and differential equations of arbitrary orders, as discussed in papers [1, 2, 12] and papers cited there. These studies predominantly assumed that the equation's coefficients exhibit regular growth to infinity. In contrast, works [3, 4, 5, 7, 8, 10, 11] explored the asymptotic properties of solutions to ordinary differential equations with coefficients from broader classes, particularly those that do not meet the Titchmarsh-Levitan conditions.

In [11], a method was proposed to study the asymptotic behaviour of solutions of the Sturm-Liouville equation

$$y'' + (1 + q(x))y = 0, \quad x_0 < x < \infty \tag{1.1}$$

for the case in which $q(x)$ is a rapidly oscillating function belonging to the class σ as defined in [11]. This method enables the construction of asymptotic formulas for solutions whether $q(x)$ influences the leading term of the asymptotic expansion or not. However, this method does not address the classes in which $q(x)$ oscillates but does not belong to the class described in [11]. An example of such a function is $\sin(x)/x^\alpha$, where $\alpha > 0$.

In [9], this approach was modified to construct the asymptotics of perturbations of the form $p(x)/x^\alpha$, where $\alpha > 0$ and $p(x)$ is a quasi-periodic function.

Note that for $\alpha > 1$ the condition $\int |p(x)/x^\alpha| dx < \infty$ is satisfied, hence, due to Theorem 1 in [1] (p. 133), all solutions of equation (1.1) are bounded. Therefore, further study of this case is not of interest.

In this paper, we extend the methods of [8, 9] to construct asymptotic formulas for the solution of the Sturm-Liouville equation in the two-dimensional vector-function space:

$$\vec{y}'' + \left(A_0 + \frac{p(x)}{x^\alpha} A_1\right)\vec{y} = 0, \quad x_0 < x < \infty,$$

where

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A_j = \left(a_{lk}^j\right), \; A_j = const, \; j = \overline{0,1}, \; A_0^* = A_0 > 0,$$

and $p(x)$ is a quasi-periodic function.

2 Construction of asymptotic formulas

We consider the Sturm-Liouville equation in the two-dimensional vector-function space:

$$\vec{\varphi}'' + \left(A_0 + \frac{p(x)}{x^\alpha} A_1\right) \vec{\varphi} = 0, \quad A_0^* = A_0 > 0, \quad \alpha > 0, \tag{2.1}$$

$$p(x) = \sum_{k=1}^{m} s_k e^{ip_k x}, \quad s_k \in \mathbb{C}, \, p_k \in \mathbb{R} \setminus \{0\}. \tag{2.2}$$

The substitution

$$\vec{\varphi} = T\vec{y}, \tag{2.3}$$

transforms equation (2.1) to the equation

$$\vec{y}'' + \begin{pmatrix} \mu_1^2 & 0 \\ 0 & \mu_2^2 \end{pmatrix} \vec{y} + \frac{p(x)}{x^\alpha} B\vec{y} = 0, \quad x_0 \le x < \infty, \quad \alpha > 0, \tag{2.4}$$

where

$$T^{-1}A_0T = \begin{pmatrix} \mu_1^2 & 0 \\ 0 & \mu_2^2 \end{pmatrix}, \quad B = T^{-1}A_1T = \left(b_{jk}\right), \; j,k = \overline{1,2}.$$

We present the main result of this paper.

Theorem 2.1. *Let $\alpha > 1/3$, and let a function $p(x)$ have form (2.2). Moreover, suppose that the following conditions hold.*

1. *For any set of numbers $\{c_1, ..., c_m\}$, where $c_j \in 0 \cup \mathbb{N}$, $\sum\limits_{j=1}^{m} c_j \neq 0$, the following condition is satisfied:*

$$\sum_{k=1}^{m} c_k p_k \neq 0. \tag{2.5}$$

2. *For any p_k, $k = 1, \dots, m$, it is true that*

$$p_k \notin \{\pm 2\mu_1, \, \pm 2\mu_2, \, \pm\mu_1 \pm \mu_2\}. \tag{2.6}$$

Then, for the fundamental system of solutions of equation (2.4), as $x \to +\infty$, the following asymptotic relation holds:

$$\vec{y} \sim \begin{pmatrix} c_{11}e^{i\mu_1 x} & c_{12}e^{-i\mu_1 x} \\ c_{21}e^{i\mu_2 x} & c_{22}e^{-i\mu_2 x} \end{pmatrix} (I + o(1))\vec{y}_0, \; c_{jk} = const, \; j,k = \overline{1,2}, \; \vec{y}_0 = const.$$

Proof. We reduce equation (2.4) to an equivalent first-order system of equations.

Let us introduce the following vector-function:

$$\vec{z}(x,\mu) = col(z_1, z_2, z_3, z_4): \; z_1 = y_1, \; z_2 = y_2, \; z_3 = y_1', \; z_4 = y_2'.$$

Then, equation (2.4) transforms into the following form:

$$\vec{z}' = (L + \frac{p(x)}{x^\alpha} A)\vec{z}, \tag{2.7}$$

where

$$L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\mu_1^2 & 0 & 0 & 0 \\ 0 & -\mu_2^2 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -b_{11} & -b_{12} & 0 & 0 \\ -b_{21} & -b_{22} & 0 & 0 \end{pmatrix}.$$

The substitution

$$\vec{z}(x) = T_1\vec{u}, \quad T_1 = \begin{pmatrix} -\frac{i}{\mu_1} & \frac{i}{\mu_1} & 0 & 0 \\ 0 & 0 & -\frac{i}{\mu_2} & \frac{i}{\mu_2} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{2.8}$$

transforms system (2.7) into the system

$$\vec{u}' = i\Lambda_0\vec{u} + \frac{1}{x^\alpha}\tilde{B}(x)\vec{u}, \tag{2.9}$$

$$\Lambda_0 = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & -\mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & -\mu_2 \end{pmatrix}, \quad \tilde{B}(x) = \frac{ip(x)}{2}\begin{pmatrix} \frac{b_{11}}{\mu_1} & \frac{-b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2} & \frac{-b_{12}}{\mu_2} \\ \frac{b_{11}}{\mu_1} & \frac{-b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2} & \frac{-b_{12}}{\mu_2} \\ \frac{b_{21}}{\mu_1} & \frac{-b_{21}}{\mu_1} & \frac{b_{22}}{\mu_2} & \frac{-b_{22}}{\mu_2} \\ \frac{b_{21}}{\mu_1} & \frac{-b_{21}}{\mu_1} & \frac{b_{22}}{\mu_2} & \frac{-b_{22}}{\mu_2} \end{pmatrix}.$$

We apply the following substitution:

$$\vec{u} = C(x)\vec{v}, \quad C(x) = C_0(x) + \frac{1}{x^\alpha}C_1(x), \tag{2.10}$$

which leads us to the following system:

$$C'(x)\vec{v} + C(x)\vec{v}' = i\Lambda_0 C(x)\vec{v} + \frac{1}{x^\alpha}\tilde{B}(x)C(x)\vec{v}. \tag{2.11}$$

We seek the matrices $C_0(x)$ and $C_1(x)$ from the following system of matrix equations:

$$\begin{cases} C_0'(x) = i\Lambda_0 C_0(x), \\ C_1'(x) = i\Lambda_0 C_1(x) + \tilde{B}(x)C_0(x). \end{cases} \tag{2.12}$$

From (2.12), we obtain

$$C_0(x) = e^{i\Lambda_0 x} = \begin{pmatrix} e^{i\mu_1 x} & 0 & 0 & 0 \\ 0 & e^{-i\mu_1 x} & 0 & 0 \\ 0 & 0 & e^{i\mu_2 x} & 0 \\ 0 & 0 & 0 & e^{-i\mu_2 x} \end{pmatrix}.$$

Also, from (2.12), we have

$$C_1(x) = C_0(x) - C_0(x)D(x), \quad D'(x) = D_1(x) = -C_0^{-1}(x)\tilde{B}(x)C_0(x), \tag{2.13}$$

$$D_1(x) = \frac{ip(x)}{2}\begin{pmatrix} \frac{b_{11}}{\mu_1} & -\frac{b_{11}}{\mu_1}e^{-2i\mu_1 x} & \frac{b_{12}}{\mu_2}e^{i(-\mu_1+\mu_2)x} & -\frac{b_{12}}{\mu_2}e^{i(-\mu_1-\mu_2)x} \\ \frac{b_{11}}{\mu_1}e^{2i\mu_1 x} & -\frac{b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2}e^{i(\mu_1+\mu_2)x} & -\frac{b_{12}}{\mu_2}e^{i(\mu_1-\mu_2)x} \\ \frac{b_{21}}{\mu_1}e^{i(\mu_1-\mu_2)x} & -\frac{b_{21}}{\mu_1}e^{i(-\mu_1-\mu_2)x} & \frac{b_{22}}{\mu_2} & -\frac{b_{22}}{\mu_2}e^{-2i\mu_2 x} \\ \frac{b_{21}}{\mu_1}e^{i(\mu_1+\mu_2)x} & -\frac{b_{21}}{\mu_1}e^{i(-\mu_1+\mu_2)x} & \frac{b_{22}}{\mu_2}e^{2i\mu_2} & -\frac{b_{22}}{\mu_2} \end{pmatrix}.$$

We define the matrix $D(x)$ as the antiderivative of the matrix function $D_1(x)$

$$D(x) = \begin{pmatrix} p_{11}(x,0) & -p_{11}(x,-2\mu_1) & p_{12}(x,-\mu_1+\mu_2) & -p_{12}(x,-\mu_1-\mu_2) \\ p_{11}(x,2\mu_1) & -p_{11}(x,0) & p_{12}(x,\mu_1+\mu_2) & -p_{12}(x,\mu_1-\mu_2) \\ p_{21}(x,\mu_1-\mu_2) & -p_{21}(x,-\mu_1-\mu_2) & p_{22}(x,0) & -p_{22}(x,-2\mu_2) \\ p_{21}(x,\mu_1+\mu_2) & -p_{21}(x,-\mu_1+\mu_2) & p_{22}(x,2\mu_2) & -p_{22}(x,0) \end{pmatrix},$$

where

$$p_{jk}(x,\sigma_{ml}) = \int \frac{ib_{jk}p(x)}{2\mu_k} e^{i\sigma_{ml}x} dx,\ j,k=\overline{1,2},\ m,l=\overline{1,4},$$

$$\sigma_{ml} \in \{0, \pm 2\mu_1, \pm 2\mu_2, \pm\mu_1 \pm \mu_2\}.$$

Thus, the solution $C_1(x)$ of system (2.12) has the form

$$C_1(x) = C_0(x) \cdot (I - D(x)). \tag{2.14}$$

It is easy to prove that due to conditions (2.5) and (2.6) of Theorem 2.1, all elements of the matrix $D(x)$ are bounded. Hence, the matrices $C_0(x)$ and $C_1(x)$ are bounded. Taking into account the last expressions, for the matrix $C(x)$ we obtain

$$C(x) = C_0(x)\left(I + \frac{1}{x^\alpha}(I - D(x))\right). \tag{2.15}$$

Since $C_0(x)$ is a diagonal matrix, $D(x)$ is bounded, and $x^{-\alpha} \to 0$ as $x \to \infty$, the matrix $C(x)$ admits a bounded inverse.

Considering (2.12) and (2.15), we can rewrite system (2.11) in the following form:

$$(\vec{v})' = \frac{1}{x^{2\alpha}}\left(C_0(x) + \frac{1}{x^\alpha}C_1(x)\right)^{-1} \tilde{B}(x)C_1(x)\vec{v} + \frac{\alpha}{x^{\alpha+1}}\left(C_0(x) + \frac{1}{x^\alpha}C_1(x)\right)^{-1} C_1(x)\vec{v}. \tag{2.16}$$

From the boundedness of the matrices $C_0(x)$ and $C_1(x)$, it follows that the matrices $C^{-1}\tilde{B}C_1$, $C^{-1}C_1$ are also bounded.

Let us consider the case $\alpha > 1/2$. We rewrite system (2.16) as follows:

$$(\vec{v})' = \tilde{C}(x)\vec{v}, \tag{2.17}$$

where

$$\tilde{C}(x) = \frac{1}{x^{2\alpha}}C^{-1}(x)\tilde{B}(x)C_1(x) + \frac{\alpha}{x^{\alpha+1}}C^{-1}(x)C_1(x).$$

If $\alpha > 1/2$, then the boundedness of $C^{-1}\tilde{B}C_1$ and $C^{-1}C_1$ obviously implies that all elements of the matrix $\tilde{C}(x)$ are summable, i.e., $\|\tilde{C}(x)\| \in L^1(x_0,\infty)$. Therefore, using successive approximations for system (2.17), we obtain

$$\vec{v} = \vec{v}_0 + \int_x^\infty \tilde{C}(\xi)\vec{v}(\xi)d\xi, \quad \vec{v}_0 = const,$$

which implies

$$\vec{v} = (I + o(1))\vec{v}_0.$$

Taking into account substitutions (2.10), (2.8), we obtain the solution to equation (2.7) as follows:

$$\vec{z}(x) = T_1 \cdot \left(C_0(x) + \frac{1}{x^\alpha} C_1(x) \right) \cdot (I + o(1)) \cdot \vec{v}_0, \quad v_0 = const. \tag{2.18}$$

Now, let us consider the case $1/3 < \alpha < 1/2$. In this case, the elements of the matrix $\frac{\alpha}{x^{\alpha+1}} C^{-1} C_1$ are summable. Given that $2\alpha < 1$, the asymptotic relation $x^{\alpha+1} = o(x^{3\alpha})$ as $x \to \infty$ holds. Therefore, we can rewrite system (2.16) by using the Neumann series for the inverse matrix

$$C^{-1}(x) = \left(I + \frac{1}{x^\alpha}(I - D(x)) \right)^{-1} C_0^{-1}(x) = C_0^{-1}(x) + \mathcal{O}(x^{-\alpha}),$$

in the following form:

$$(\vec{v})' = \frac{1}{x^{2\alpha}} C_0^{-1}(x) \tilde{B}(x) C_1(x) \vec{v} + \frac{1}{x^{3\alpha}} F(x, \mu_1, \mu_2) \vec{v}, \tag{2.19}$$

where

$$\frac{1}{x^{3\alpha}} F(x, \mu_1, \mu_2) = \frac{1}{x^{2\alpha}} \mathcal{O}(x^{-\alpha}) + \frac{\alpha}{x^{\alpha+1}} C^{-1}(x) C_1(x),$$

and $\left\| \frac{1}{x^{3\alpha}} F(x, \mu) \right\| \in L^1(x_0, \infty)$. Taking into account (2.13) and (2.14), for the matrix $C_0^{-1}(x) \tilde{B}(x) C_1(x)$ we have

$$C_0^{-1}(x) \tilde{B}(x) C_1(x) = C_0^{-1}(x) \tilde{B}(x) C_0(x) D(x) = D'(x) D(x).$$

Hence, the elements of the matrix $C_0^{-1} \tilde{B} C_1$ are oscillating functions and can be represented as

$$G(x, \mu_1, \mu_2) = \sum G_k e^{i\sigma_k x}, \quad \sigma_k \in \{p_k, \pm 2\mu_1, \pm 2\mu_2, \pm\mu_1 \pm \mu_2\}, \quad G_k = const.$$

Thus, system (2.19) takes the form

$$(\vec{v})' = \frac{1}{x^{2\alpha}} G(x, \mu_1, \mu_2) \vec{v} + \frac{1}{x^{3\alpha}} F(x, \mu_1, \mu_2) \vec{v}. \tag{2.20}$$

The substitution

$$\xi = \frac{x^{1-2\alpha}}{1 - 2\alpha}, \quad x = ((1 - 2\alpha)\xi)^{\frac{1}{1-2\alpha}}, \quad \vec{v}(x) = \vec{w}(\xi), \tag{2.21}$$

$$\beta = \frac{1}{1 - 2\alpha}, \quad \gamma = \frac{\alpha}{1 - 2\alpha},$$

transforms system (2.20) to the system

$$(\vec{w})'_\xi = G\left(a\xi^\beta, \mu_1, \mu_2\right) \vec{w} + \frac{\beta^\gamma}{\xi^\gamma} F\left(a\xi^\beta, \mu_1, \mu_2\right) \vec{w}, \tag{2.22}$$

where $a = \beta^{-\beta}$ is a constant, which does not affect the asymptotic behaviour of the solutions.

The condition $1/3 < \alpha < 1/2$ implies $3 < \beta < \infty$, hence $\gamma = \frac{\alpha}{1-2\alpha} = \beta\alpha > 1$. Therefore, the second term of system (2.22) is summable.

By by integration, from (2.22), we obtain

$$\vec{w}(\xi) = \vec{w}(\xi_0) + \int_\xi^\infty G\left(\tau^\beta, \mu_1, \mu_2\right) \vec{w}(\tau) d\tau + \beta^\gamma \int_\xi^\infty \tau^{-\gamma} F\left(\tau^\beta, \mu_1, \mu_2\right) \vec{w}(\tau) d\tau. \tag{2.23}$$

Integrating by part the second term of expression (2.23), we have

$$\int_{\xi}^{\infty} G\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)d\tau = \hat{G}\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)\Big|_{\xi}^{\infty} - \int_{\xi}^{\infty}\hat{G}\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}'(\tau)d\tau, \tag{2.24}$$

where

$$\left|\hat{G}\left(\xi^{\beta},\mu_1,\mu_2\right)\right| = \left|\int_{\xi}^{\infty} G\left(\tau^{\beta},\mu_1,\mu_2\right)d\tau\right| = \left|\frac{1}{\beta}\int_{\xi}^{\infty} G\left(\tau^{\beta},\mu_1,\mu_2\right)\frac{d\tau^{\beta}}{\tau^{\beta-1}}\right| = O\left(\frac{1}{\tau^{\beta-1}}\right).$$

Hence, $\|\hat{G}\left(\xi,\mu_1,\mu_2\right)\| \in L^1(\xi_0,\infty)$. By using (2.22) for (2.24), we get

$$J := \int_{\xi}^{\infty} G\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)d\tau = \hat{G}\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)\Big|_{\xi}^{\infty}$$
$$-\int_{\xi}^{\infty}\hat{G}\left(\tau^{\beta},\mu_1,\mu_2\right)\left(G\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)+\frac{\beta^{\gamma}}{\tau^{\gamma}}F\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)\right)d\tau.$$

Taking into account the last expressions and (2.23), we obtain the following estimate:

$$\|\vec{w}-\vec{w}_0\|_{C(\xi_0,\infty)} \le K\|\vec{w}\|_{C(\xi_0,\infty)}, \quad K = const.$$

Hence, it follows that

$$\vec{w}(\xi) = \vec{w}(\xi_0) + o(1), \tag{2.25}$$

where $\vec{w}(\xi) = \vec{w}_0$. Returning from system (2.25) to system (2.7), taking into account substitutions (2.21), (2.10) and (2.8), we obtain (2.18).

Let us consider the case $\alpha = 1/2$. In this case, system (2.24) takes the form

$$(\vec{v})' = \frac{1}{x}G(x,\mu_1,\mu_2)\vec{v} + \frac{1}{x^{3/2}}F(x,\mu_1,\mu_2)\vec{v}. \tag{2.26}$$

Substituting

$$\xi = \ln x, \quad x = e^{\xi}, \quad \vec{v}(x) = \vec{w}(\xi),$$

converts system (2.26) to the system

$$(\vec{w})'_{\xi} = G(e^{\xi},\mu_1,\mu_2)\vec{w} + e^{-\frac{\xi}{2}}F(e^{\xi},\mu_1,\mu_2)\vec{w}.$$

Similarly to (2.22), by using successive approximations, we obtain

$$\vec{w}(\xi) = \vec{w}(\xi_0) + o(1).$$

Taking into account (2.21), (2.10) and (2.8), we obtain (2.18).

Finally, we obtain the asymptotics of the solutions to system (2.4) in the following form

$$\vec{y} \sim \begin{pmatrix} c_{11}e^{i\mu_1 x} & c_{12}e^{-i\mu_1 x} \\ c_{21}e^{i\mu_2 x} & c_{22}e^{-i\mu_2 x}\end{pmatrix}(I+o(1))\vec{y}_0,$$

where $c_{jk} = const$, $j,k = \overline{1,2}$, $\vec{y}_0 = const$. □

Remark 1. From the proven theorem, it follows that the perturbation $p(x)/x^\alpha$ does not affect the dominant part of the asymptotics of the solution to equation (2.4) provided that conditions (2.5) and (2.6) are satisfied.

Remark 2. The condition $\alpha > 1/3$ arises from the selection of the substitution:

$$C(x) = C_0(x) + \frac{1}{x^\alpha} C_1(x).$$

For the general case, the substitution takes the form:

$$C(x) = \sum_{k=0}^{m} \frac{1}{x^{k\alpha}} C_k(x),$$

as provided in [9] for the scalar case.

Remark 3. If condition (2.5) of Theorem 2.1 is not satisfied, a resonance case occurs, and the asymptotics will significantly differ from those obtained above.

3 Resonance case

In this section, we will show that the conditions of Theorem 2.1 are essential. To do this, let us consider the case in which $p_k \in \{\pm 2\mu_1, \pm 2\mu_2, \pm\mu_1 \pm \mu_2\}$, i.e., condition (2.5) is not satisfied. For the matrix $C(x)$ in (2.15), we obtain

$$C(x) = C_0(x)\left(I - \frac{1}{x^\alpha}\tilde{D}(x) - x^{1-\alpha} D_2(x)\right),$$

where

$$\tilde{D}(x) = \sum \tilde{D}_k e^{i\sigma_k}, \quad D_2(x) = \sum (D_2)_k e^{i\sigma_k}, \quad \tilde{D}_k,\ (D_2)_k = const,$$
$$\sigma_k \in \{\pm 2\mu_1, \pm 2\mu_2, \pm\mu_1 \pm \mu_2, p_k.\}$$

For the case $\alpha < 1$, the matrix $C(x)$ becomes unbounded as $x \to \infty$. This generates the resonance case and the method described in the previous section is no longer applicable. Therefore, we apply a different approach to study the asymptotic behaviour of solutions.

Let $p(x) = \cos(\mu_1 + \mu_2)x$, $\mu_1, \mu_2 \in \mathbb{R} \setminus 0$. Then, system (2.9) takes the following form:

$$\vec{u}' = \frac{i}{2}\Lambda_0 \vec{u} + \frac{i\cos(\mu_1 + \mu_2)x}{2x^\alpha} B\vec{u}, \tag{3.1}$$

where

$$\Lambda_0 = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & -\mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & -\mu_2 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{b_{11}}{\mu_1} & \frac{-b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2} & \frac{-b_{12}}{\mu_2} \\ \frac{b_{11}}{\mu_1} & \frac{-b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2} & \frac{-b_{12}}{\mu_2} \\ \frac{b_{21}}{\mu_1} & \frac{-b_{21}}{\mu_1} & \frac{b_{22}}{\mu_2} & \frac{-b_{22}}{\mu_2} \\ \frac{b_{21}}{\mu_1} & \frac{-b_{21}}{\mu_1} & \frac{b_{22}}{\mu_2} & \frac{-b_{22}}{\mu_2} \end{pmatrix}.$$

In the case $\mu_1 = \mu_2$, system (3.1) becomes equivalent to two second-order scalar linear differential equations. Therefore, we consider the case $\mu_1 \neq \mu_2$.

The substitution

$$\vec{u} = e^{\frac{i}{2}\Lambda_0 x}\vec{v} \tag{3.2}$$

transforms system (3.1) to

$$\vec{v}' = \frac{i\cos(\mu_1 + \mu_2)x}{2x^\alpha} e^{-\frac{i}{2}\Lambda_0 x} B e^{\frac{i}{2}\Lambda_0 x}\vec{v}.$$

After some modifications, we get

$$\vec{v}' = \frac{1}{x^\alpha}\left(B_0 + \sum_{k=1}^{m} e^{i\sigma_k x} B_k\right)\vec{v}, \tag{3.3}$$

where

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & -\frac{ib_{12}}{4\mu_2} \\ 0 & 0 & \frac{ib_{12}}{4\mu_2} & 0 \\ 0 & -\frac{ib_{21}}{4\mu_1} & 0 & 0 \\ \frac{ib_{21}}{4\mu_1} & 0 & 0 & 0 \end{pmatrix}, \quad B_k = const, \quad k = 1, \dots, m,$$

$$\sigma_k \in \{\pm 2\mu_1; \pm 2\mu_2; \pm 2(\mu_1+\mu_2); \pm(\mu_1 \pm \mu_2); \pm(3\mu_1+\mu_2); \pm(\mu_1+3\mu_2)\}.$$

Replacing independent variable x by ξ as in (2.21):

$$\xi = \frac{x^{1-\alpha}}{1-\alpha}, \quad x = ((1-\alpha)\xi)^{\frac{1}{1-\alpha}}, \quad \vec{v}(x) = \vec{w}(\xi), \tag{3.4}$$

$$\beta = \frac{1}{1-\alpha}, \quad a = (1-\alpha)^\beta,$$

we obtain

$$\vec{w}'_\xi = B_0 \vec{w} + \sum_{k=1}^{m} e^{ia\sigma_k \xi^\beta} B_k \vec{w}. \tag{3.5}$$

Denote

$$\phi_k(\xi) = \int_\xi^\infty e^{ia\sigma_k \tau^\beta} d\tau, \quad k = 1, \dots, m.$$

Assume that $1/2 < \alpha < 1$, which implies $\beta > 2$. Then

$$\phi_k(\xi) = \int_\xi^\infty e^{ia\sigma_k \tau^\beta} d\tau = \frac{1}{\beta}\int_\xi^\infty \frac{e^{ia\sigma_k \tau^\beta}}{\tau^{\beta-1}} d\tau^\beta \in L^1(\xi_0, \infty). \tag{3.6}$$

Applying the substitution

$$\vec{w} = e^{-\phi_1(\xi)B_1}\vec{w}_1, \tag{3.7}$$

we get

$$\vec{w}_1'(\xi) = e^{\phi_1(\xi)B_1} B_0 e^{-\phi_1(\xi)C_1}\vec{w}_1 + e^{\phi_1(\xi)B_1} \cdot \left(\sum_{k=2}^{m} e^{ia\sigma_k \xi^\beta} B_k\right) \cdot e^{-\phi_1(\xi)B_1}\vec{w}_1. \tag{3.8}$$

Using the properties of the matrix exponent, from (3.6) we obtain

$$e^{\phi_1(\xi)B_1} = I + F_1(\xi), \quad \|F_1(\xi)\| \in L^1(\xi_0, \infty),$$

and

$$e^{-\phi_1(\xi)B_1} = I + F_2(\xi), \quad \|F_2(\xi)\| \in L^1(\xi_0, \infty).$$

Therefore,

$$e^{\phi_1(\xi)B_1} B_0 e^{-\phi_1(\xi)B_1} = B_0 + F_3(\xi), \quad \|F_3(\xi)\| \in L^1(\xi_0, \infty),$$

where

$$F_3(\xi) = B_0 F_2(\xi) + F_1(\xi) B_0 + F_1(\xi) B_0 F_2(\xi).$$

For the remaining terms of system (3.8), we have

$$e^{\phi_1(\xi)B_1}\cdot\left(e^{ia\sigma_k\xi^\beta}B_k\right)\cdot e^{-\phi_1(\xi)B_1}=(I+F_1(\xi))e^{ia\sigma_k\xi^\beta}B_k(I+F_2(\xi))=e^{ia\sigma_k\xi^\beta}B_k+G_k(\xi),$$

$$G_k(\xi)=e^{ia\sigma_k\xi^\beta}\left(B_kF_2(\xi)+F_1(\xi)B_k+F_1(\xi)B_kF_2\right),\quad k=2,\dots,m.$$

The matrices $e^{ia\sigma_k\xi^\beta}B_k$ are bounded, and the matrices $F_1(\xi)$, $F_2(\xi)$ are summable. Hence, the matrices $G_k(\xi)$ are summable. On the base of these relations, system (3.8) takes the form

$$\vec{w}_1'(\xi)=B_0\vec{w}_1+\sum_{k=2}^{m}e^{ia\sigma_k\xi^\beta}B_k\vec{w}_1+G(\xi)\vec{w}_1, \tag{3.9}$$

where

$$G(\xi)=\sum_{k=2}^{m}G_k(\xi),\quad \|G(\xi)\|\in L^1(\xi_0,\infty).$$

Using the substitutions

$$\vec{w}_{k-1}=e^{ia\sigma_k\xi^\beta B_k}\vec{w}_k,\quad k=2,\dots,m, \tag{3.10}$$

one by one and conducting similar calculations, we finally obtain

$$\vec{w}_m'(\xi)=B_0\vec{w}_m+P(\xi)\vec{w}_m,\quad \|P(\xi)\|\in L^1(\xi_0,\infty). \tag{3.11}$$

Applying Levinson's Theorem to (3.11) (see [6], p. 292), we obtain the solution

$$\vec{w}_m(\xi)=e^{\xi B_0}\cdot(I+M\cdot o(1)),\quad M=const.$$

Using substitutions (3.9), (3.7), (3.4), (3.2) and (2.8), we obtain the solution for system (2.7) in the following form:

$$\vec{z}=T_1\cdot e^{\frac{i}{2}\Lambda_0\xi^\beta}\cdot\prod_{k=1}^{m}e^{-\phi_k(\xi)B_k}e^{\xi B_0}(I+M\cdot o(1))\vec{w}_m(\xi_0).$$

The dominant part of asymptotics of solutions is

$$\vec{z}\sim e^{\frac{i}{2}\Lambda_0 x}\cdot\exp\left\{\frac{x^{1-\alpha}}{1-\alpha}B_0\right\}(I+M\cdot o(1))z_0,\quad z_0=const.$$

Let us consider the case $\alpha=1$. Then, system (3.3) takes the form

$$\vec{v}'=\frac{1}{x}\left(B_0+\sum_{k=1}^{m}e^{i\sigma_k x}B_k\right)\vec{v}.$$

By using the substitution

$$\xi=\ln x,\quad x=e^\xi,\quad \vec{v}(x)=\vec{w}(\xi) \tag{3.12}$$

we obtain the system

$$\vec{w}_\xi'=\left(B_0+\sum_{k=1}^{m}e^{i\sigma_k e^\xi}B_k\right)\vec{w}.$$

Denote

$$\psi_k(\xi)=\int_\xi^\infty e^{i\sigma_k e^\tau}d\tau,\quad k=1,\dots,m.$$

As $\psi_k(\xi) \in L^1(\xi_0, \infty)$, $j = 1, \dots, m$, using the substitutions

$$\vec{w} = e^{-\psi_1(\xi)B_1}\vec{w}_1, \quad \vec{w}_{k-1} = e^{-\psi_k(\xi)B_k}\vec{w}_k, \quad k = 2, \dots, m, \tag{3.13}$$

and conducting similar calculations as in the previous case, we obtain

$$\vec{w}_m'(\xi) = B_0\vec{w}_m + G(\xi)\vec{w}_m, \quad \|G(\xi)\| \in L^1(\xi_0, \infty).$$

Applying Levinson's Theorem (see [6], p. 292) and using substitutions (3.13), (3.12), (3.2) and (2.8), we obtain the following expression for the solution to system (2.7):

$$\vec{z} = T_1 \cdot e^{\frac{i}{2}\Lambda_0 e^{\xi}} \cdot \prod_{k=1}^{m} e^{-\psi_k(\xi)B_k} \cdot e^{\xi B_0} \cdot (I + M \cdot o(1)) \cdot \vec{w}_m(\xi_0).$$

The dominant part of the asymptotics of the solution is given by

$$\vec{z} \sim e^{\frac{i}{2}\Lambda_0 x} \cdot e^{\ln x B_0} \cdot (I + M \cdot o(1))\vec{z}_0, \quad \vec{z}_0 = const.$$

Remark 4. In both cases $\alpha < 1$ and $\alpha = 1$, the asymptotics of solutions to system (2.7), as described by equation (2.4), will depend on the elements of the constant matrix B.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. AP23486342).

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Yaudat Talgatovich Sultanayev
Chair of Mathematics and Statistics
M. Akmulla Bashkir State Pedagogical University
3a October Revolution St
450000 Ufa, Russian Federation
and
Faculty of Mechanics and Mathematics, Chair of Mathematical Analysis
Center for Applied and Fundamental Mathematics, M.V. Lomonosov Moscow State University
1 Leninskie gory
119991 Moscow, Russian Federation
E-mail: sultanaevyt@gmail.com

Nurmukhamet Fuatovich Valeev
Chair of Mathematics and Statistics
M. Akmulla Bashkir State Pedagogical University
3a October Revolution St
450000 Ufa, Russian Federation
E-mail: valeevnf@yandex.ru

Alibek Yeskermessuly
Faculty of Natural Sciences and Informatization
Altynsarin Arkalyk Pedagogical Institute
17 Auelbekov St
110300 Arkalyk, Republic of Kazakhstan
E-mail: alibek.yeskermessuly@gmail.com

Received: 14.06.2024

Events

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 3 (2025), 102 – 104

ONLINE WORKSHOP ON DIFFERENTIAL EQUATIONS AND FUNCTION SPACES, DEDICATED TO THE 80-TH ANNIVERSARY OF D.SC., PROFESSOR MIKHAIL L'VOVICH GOLDMAN

The workshop on differential equations and function spaces was held online from April 15 to 17, 2025 in the Vladikavkaz Scientific Center (VSC) of the Russian Academy of Siences (RAS).

This workshop is the second in 2025 in a series of workshops in the framework of the Operator Theory and Differential Equations (OTDE)-Workshops project: "Workshops on the operator theory, differential equations and their applications".

Workshop Organizers

- Department of General Control Problems, Faculty of Mechanics and Mathematics, M.V. Lomonosov Moscow State University (MSU);
- S.M. Nikolskii Mathematical Institute, Patrice Lumumba Peoples' Friendship University of Russia (RUDN University);
- Astana International University;
- North Caucasus Center for Mathematical Research of the VSC RAS;
- Southern Mathematical Institute of the VSC RAS.

The event was held with the support of the Russian Ministry of Education and Science, agreement No. 075-02-2025-1633.

Organizing Committee

- D.Sc., Professor **Anatoly Geogievich Kusraev** (VSC RAS);
- D.Sc., Professor **Georgii Georgievich Magaril-Il'yaev** (M.V. Lomonosov Moscow State University);
- D.Sc., Professor **Andrey Borisovich Muravnik** (S.M. Nikolskii Mathematical Institute, RUDN University).

The event was held with the support of the Russian Ministry of Education and Science, agreement No. 075-02-2025-1633.

Working group:

- Dr. **Elza Gizarovna Bakhtigareeva** (the S.M. Nikolskii Mathematical Institute, RUDN University);

- Dr. **Gulden Zhumabekkyzy Karshigina** (Astana International University);
- **Victoria Amurkhanovna Tamaeva** (North Caucasus Center for Mathematical Research of the VSC RAS), the secretary of the Organizing Committee;
- Dr. **Batradz Botazovich Tasoev** (Southern Mathematical Institute of the VSC RAS).

WORKSHOP PROGRAMME
April 15, 2025 / Tuesday

Moderator: D.Sc., Professor Georgii Georgievich Magaril-Ilyaev

15:00 – 15:15 Opening of the Workshop

Professor Georgii Georgievich Magaril-Ilyaev delivered a welcoming speech on behalf of the Organizing Committee.

Professor Tikhomirov Vladimir Mikhailovich spoke about the activities of Professor Mikhail L'vovich Goldman

15:20 – 15:55
D.Sc., Professor, Corresponding Member of the RAS
Besov Oleg Vladimirovich
"Estimates of the entropy numbers of the Sobolev embedding operator on a Hölder domain"

15:55 – 16:30 D.Sc., Professor, Foreign Member of the NAS of Kazakhstan
Burenkov Victor Ivanovich
"On joint scientific work with M.L. Goldman"

16:30 – 17:05 D.Sc., Professor Nazarov Alexander Ilyich
"Hardy-type inequalities with general cylindrical-spherical weights"

17:05 – 17:40 D.Sc., Professor, Corresponding Member of the RAS
Stepanov Vladimir Dmitrievich
"The space of fractional Riemann–Liouville potentials on the half-axis"

17:40 Discussions

April 16, 2025 / Wednesday

Moderator: D.Sc., Professor Kusraev Anatoly Georgievich

15:00 – 15:35 D.Sc., Professor, Academician of the NAS of Kazakhstan
Oinarov Ryskul
"Boundedness of a class of integral operators in Lebesgue spaces"

15:35 – 16:10 Dr. Tikhonov Sergey Yuryevich
"Absolute convergence of Fourier series"

16:10 – 16:45 D.Sc., Professor Muratov Mustafa Abdureshitovich
"Lower and upper Marcinkiewicz classes of measurable functions on the half-axis"

16:45 – 17:20 D.Sc., Professor Nursultanov Erlan Dautbekovich
"Network spaces and Hölder inequality"

17:20 Discussions.

April 17, 2025 / Thursday

Moderator: Dr. Pliev Marat Amurkhanovich

15:00 – 15:35 D.Sc., Associate Professor Avsyankin Oleg Gennadievich
"Some issues in the theory of integral operators in Morrey spaces"

15:35 – 16:10 D.Sc., Professor Bokayev Nurzhan Adilkhanovich
"On the embedding of the space of generalized fractional maximal functions and the space of the generalized Riesz potentials in rearrangement-invariant spaces"

16:10 – 16:45 Dr. Gogatishvili Amiran
"Weighted inequalities containing Hardy operators"

16:45 – 17:20 D.Sc., Professor, Corresponding Member of the NAS of Azerbaijan
Guliyev Vagif Sabirovich
"Lorentz boundedness criteria for commutators of the maximal operator on spaces of homogeneous type"

17:20 Discussions

More detailed information on the delivered talks can be found in
https://smath.ru/upload/iblock/001/Program_Workshop_G-ML80_Rus_1.pdf (in Russian).

V.I. Burenkov, A.G. Kusraev, G.G. Magaril-Ilyaev

EURASIAN MATHEMATICAL JOURNAL

2025 – Том 16, № 3 – Астана: ЕНУ. – 105 с.

Подписано в печать 30.09.2025 г. Тираж – 40 экз.

Адрес редакции: 010008, Астана, ул. Кажымукан, 13,
Евразийский национальный университет имени Л.Н. Гумилева,
корпус № 3, каб. 306а.
Тел.: +7-7172-709500, добавочный 33312.

Дизайн: К. Булан

Отпечатано в типографии ЕНУ имени Л.Н. Гумилева

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Зарегистрировано
Министерством информации и общественного развития Республики Казахстан.
Свидетельство о постановке на переучет печатного издания
№ KZ30VPY00032663 от 19.02.2021 г.
(Дата и номер первичной постановки на учет: № 10330 – Ж от 25.09.2009 г.)