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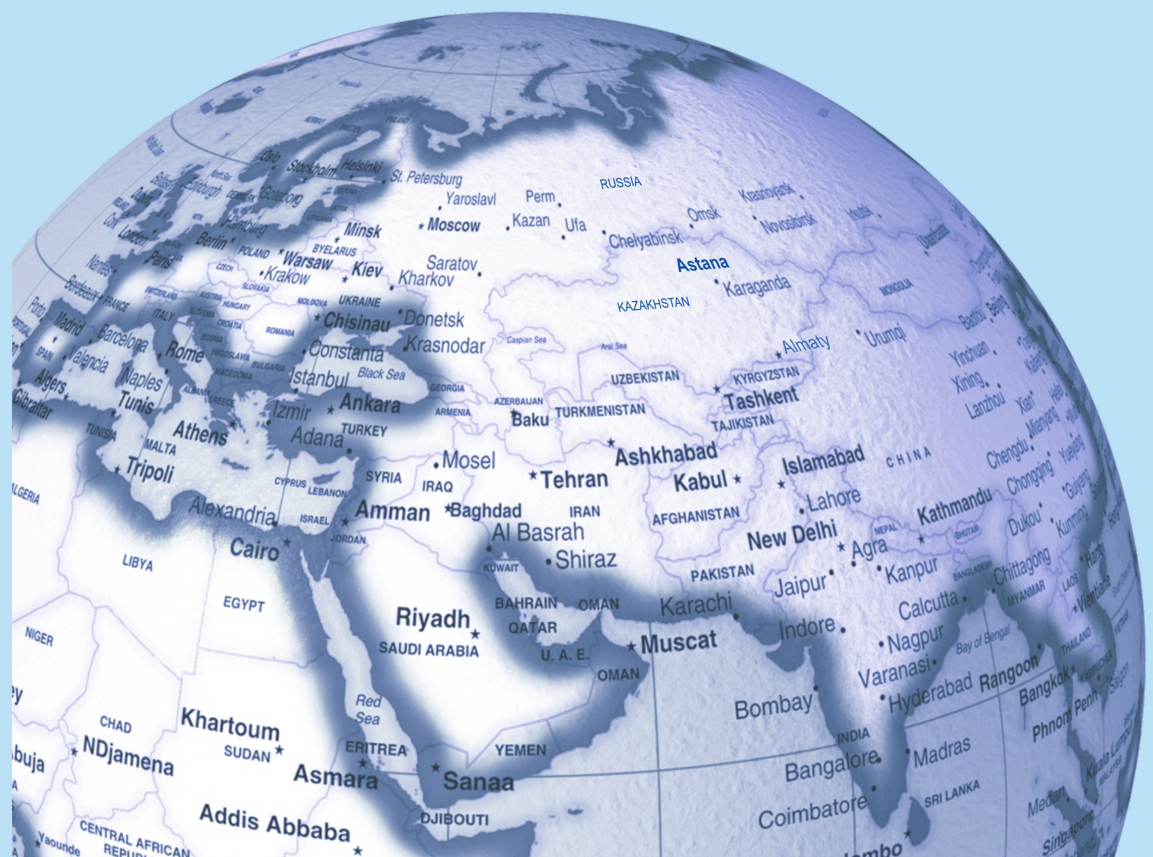

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BOUNDARY VALUE PROBLEM FOR HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS OF MIXED TYPE

A.T. Assanova, Z.S. Kobeyeva, R.A. Medetbekova

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Key words: hyperbolic integro-differential equations, nonlocal conditions, solvability criteria, parametrization method.

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Abstract. The boundary value problem for a system of hyperbolic integro-differential equations of mixed type with degenerate kernels is considered on a rectangular domain. This problem is reduced to a family of boundary value problems for a system of integro-differential equations of mixed type and integral relations. The system of integro-differential equations of mixed type is transferred to a system of Fredholm integro-differential equations. For solving the family of boundary value problems for integro-differential equations Dzhumabaev's parametrization method is applied. A new concept of a general solution to a system of integro-differential equations with parameter is developed. The domain is divided into N subdomains by a temporary variable, the values of a solution at the interior lines of the subdomains are considered as additional functional parameters, and a system of integro-differential equations is reduced to a family of special Cauchy problems on the subdomains for Fredholm integro-differential equation with functional parameters. Using the solutions to these problems, a new general solutions to a system of Fredholm integro-differential equations with parameter is introduced and its properties are established. Based on a general solution, boundary conditions, and the continuity conditions of a solution at the interior lines of the partition, a system of linear functional equations with respect to parameters is composed. Its coefficients and right-hand sides are found by solving the family of special Cauchy problems for Fredholm integro-differential equations on the subdomains. It is shown that the solvability of the family of boundary value problems for Fredholm integro-differential equations is equivalent to the solvability of the composed system. Methods for solving boundary value problems are proposed, which are based on the construction and solving of these systems. Conditions for the existence and uniqueness of a solution to the boundary value problem for a system of hyperbolic integro-differential equations of mixed type with degenerate kernels are obtained.

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1 Introduction and statement of problem

Boundary value problems for systems of hyperbolic integro-differential equations of mixed type arise in various scientific and engineering fields when a phenomena exhibits both hyperbolic and integral characteristics.

Hyperbolic equations often model wave propagation, and the presence of integro-differential terms can account for the effects of heterogeneous media. Applications include seismology, acoustics, and electromagnetic wave propagation in complex environments [13, 31, 32].

Fluid flow problems involving memory effects, such as viscoelastic or non-Newtonian fluids, can be described using hyperbolic integro-differential equations. This is relevant in modeling flows with memory-dependent constitutive relationships [12]. Modeling the dynamic behavior of structures with distributed parameters, viscoelastic materials, or memory effects in the constitutive relations can lead to hyperbolic integro-differential equations. This is important in understanding the vibrations and responses of complex structures [14, 21, 24].

Hyperbolic integro-differential equations with mixed types can appear in the modeling of systems with time delays, which is common in control theory. These equations can be used to study the stability and control of systems with delays [22, 23].

The spread of infectious diseases, predator-prey interactions, or other ecological systems may be modeled using hyperbolic integro-differential equations. The integral terms can represent memory effects or non-local interactions within populations [25, 33, 34].

Modeling heat conduction in materials with complex structures, like composites or materials with memory effects, can lead to hyperbolic integro-differential equations. This is crucial in designing materials with specific thermal properties [5, 26].

In financial mathematics, models with memory effects, stochastic processes, or non-local interactions can be described using hyperbolic integro-differential equations. This is particularly relevant in option pricing and risk management [29, 30].

Non-local interactions in image processing, such as image denoising or inpainting, can be modeled using hyperbolic integro-differential equations. These equations allow for the consideration of information from distant pixels. Modeling biological systems involving neural dynamics, drug delivery, or reaction-diffusion processes can lead to hyperbolic integro-differential equations. These equations can help simulate and understand complex interactions in biological systems [10, 27].

Hyperbolic integro-differential equations are used to model various geophysical phenomena, including heat conduction in the Earth's crust, seismic wave propagation, and groundwater flow in heterogeneous media [11].

The solutions to these problems provide insights into the behavior of complex systems and aid in the design and optimization of processes in a wide range of scientific and engineering applications. Solving these equations often requires a combination of analytical and numerical techniques tailored to the specific characteristics of the problem at hand.

Therefore, the study of new methods for solving boundary value problems for hyperbolic integro-differential equations is driven by the need to address the complexities of real-world problems, improve computational efficiency, enhance accuracy, and adapt to diverse applications across various disciplines. It reflects the dynamic nature of scientific inquiry and the ongoing quest to develop more robust tools for understanding and manipulating complex systems.

This issue can be resolved by developing constructive methods. In present paper we propose an effective method for solving the boundary value problem for the second order system of hyperbolic integro-differential equations of mixed type. This method is based on the method of introducing new unknown functions [3, 7], the parametrization method [15] and a new concept of a general solution [17].

On the rectangular domain $\Omega = [0,T] \times [0,\omega]$, we consider the boundary value problem for the following second order system of hyperbolic integro-differential equations of mixed type:

$$\frac{\partial^2 u}{\partial x \partial t} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x) +$$

$$+\Phi_1(t,x)\int\limits_0^T \Psi_1(s,x)\frac{\partial u(s,x)}{\partial x}ds + \Xi_1(t,x)\int\limits_0^t \Theta_1(s,x)\frac{\partial u(s,x)}{\partial x}ds +$$

$$+\Phi_2(t,x)\int\limits_0^T \Psi_2(s,x)u(s,x)ds + \Xi_2(t,x)\int\limits_0^t \Theta_2(s,x)u(s,x)ds, \quad (t,x)\in\Omega, \tag{1.1}$$

$$P_1(x)\frac{\partial u(0,x)}{\partial x} + P_2(x)u(0,x) + S_1(x)\frac{\partial u(T,x)}{\partial x} + S_2(x)u(T,x) = \varphi(x), \quad x\in[0,\omega], \tag{1.2}$$

$$u(t,0) = \psi(t), \qquad t\in[0,T]. \tag{1.3}$$

Here $u = col(u_1, u_2, ..., u_n)$ is the unknown vector-function, the $n\times n$ matrices $A(t,x)$, $B(t,x)$, $C(t,x)$ and n-vector $f(t,x)$ are continuous on Ω; the $n\times n$ matrices $\Phi_i(t,x)$, $\Psi_1(t,x)$, $\Xi_i(t,x)$, $\Theta_i(t,x)$, $i = 1,2$, are continuous on Ω; the $n\times n$ matrices $P_j(x)$, $S_j(x)$, $j = 1,2$, and n-vector $\varphi(x)$ are continuous on $[0,\omega]$; the n-vector $\psi(t)$ is continuously differentiable on $[0,T]$.

A vector-function $u(t,x)\in C(\Omega,\mathbb{R}^n)$, which has partial derivatives $\frac{\partial u(t,x)}{\partial x}\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial u(t,x)}{\partial t}\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial^2 u(t,x)}{\partial x\partial t}\in C(\Omega,\mathbb{R}^n)$, is called a *solution* to problem (1.1)–(1.3) if it satisfies system (1.1) for all $(t,x)\in\Omega$, the nonlocal condition (1.2) for all $x\in[0,\omega]$ and the condition on the characteristics (1.3) for all $t\in[0,T]$.

2 Reduction to a family of problems for first order integro-differential equations

Previously, the relationship between nonlocal problems for hyperbolic equations and families of problems for ordinary differential equations was shown in [3, 4, 28]. With the help of new unknown functions, the problem under consideration was reduced to a family of problems for differential equations and integral relations. To solve a family of problems for differential equations, Dzhumabaev parametrization method was used [15] and criteria for the unique solvability of the problem under investigation were obtained in terms of coefficients and boundary data. This has made it possible to establish necessary and sufficient conditions for the well-posed solvability of nonlocal problems for hyperbolic equations in terms of the original data [3, 4]. These results were extended to nonlocal problems for loaded hyperbolic equations [19]. An application of this approach to problems for hyperbolic integro-differential equations leads to a new class of problems for integro-differential equations of mixed type. This, in turn, requires the development of new approaches and methods for solving them.

In this Section by method of introduction of new functions we transfer problem (1.1)–(1.3) to a family of problems for integro-differential equations of mixed type.

We introduce new functions $v(t,x) = \frac{\partial u(t,x)}{\partial x}$ and $w(t,x) = \frac{\partial u(t,x)}{\partial t}$ for all $(t,x)\in\Omega$ [4]. Problem (1.1)–(1.3) transfers to a family of boundary value problems for the following integro-differential equations of mixed type and integral relations

$$\frac{\partial v}{\partial t} = A(t,x)v(t,x) + F(t,x,u,w) +$$

$$+\Phi_1(t,x)\int\limits_0^T \Psi_1(s,x)v(s,x)ds + \Xi_1(t,x)\int\limits_0^t \Theta_1(s,x)v(s,x)ds, \qquad (t,x)\in\Omega, \tag{2.1}$$

$$P_1(x)v(0,x) + S_1(x)v(T,x) = \phi(x,u), \qquad x\in[0,\omega], \tag{2.2}$$

$$u(t,x) = \psi(t) + \int\limits_0^x v(t,\xi)d\xi, \qquad w(t,x) = \dot\psi(t) + \int\limits_0^x \frac{\partial v(t,\xi)}{\partial t}d\xi, \tag{2.3}$$

where

$$F(t,x,u,w) = f(t,x) + B(t,x)w(t,x) + C(t,x)u +$$

$$+\Phi_2(t,x)\int_0^T \Psi_2(s,x)u(s,x)ds + \Xi_2(t,x)\int_0^t \Theta_2(s,x)u(s,x)ds,$$

$$\phi(x,u) = \varphi(x) - P_2(x)u(0,x) - S_2(x)u(T,x).$$

A triple of functions $\{v(t,x), u(t,x), w(t,x)\}$, where $v(t,x) \in C(\Omega,\mathbb{R}^n)$, $u(t,x) \in C(\Omega,\mathbb{R}^n)$, $w(t,x) \in C(\Omega,\mathbb{R}^n)$ is called a solution to problem (2.1)–(2.3) if it satisfies integro-differential equations of mixed type with parameters (2.1), condition (2.2) and integral relations (2.3).

Let $u^*(t,x)$ be a classical solution to problem (1.1)–(1.3).

We construct a triple of functions $\{v^*(t,x),\, u^*(t,x),\, w^*(t,x)\}$, where $v^*(t,x) = \frac{\partial u^*(t,x)}{\partial x}$, $w^*(t,x) = \frac{\partial u^*(t,x)}{\partial t}$.

Then

$$u^*(t,x) = u^*(t,0) + \int_0^x \frac{\partial u^*(t,\xi)}{\partial \xi}d\xi = \psi(t) + \int_0^x v^*(t,\xi)d\xi$$

and taking into account that $u^*(t,x)$ is a solution to problem (1.1)–(1.3), we have

$$\frac{\partial^2 u^*(t,x)}{\partial x\partial t} = \frac{\partial^2 u^*(t,x)}{\partial t\partial x},$$

$$w^*(t,x) = \frac{\partial u^*(t,x)}{\partial t} = \frac{\partial u^*(t,0)}{\partial t} + \int_0^x \frac{\partial^2 u^*(t,\xi)}{\partial \xi\partial t}d\xi =$$

$$= \frac{\partial u^*(t,0)}{\partial t} + \int_0^x \frac{\partial^2 u^*(t,\xi)}{\partial t\partial \xi}d\xi = \dot{\psi}(t) + \int_0^x \frac{\partial v^*(t,\xi)}{\partial t}d\xi,$$

$$\frac{\partial v^*}{\partial t} = \frac{\partial^2 u^*}{\partial t\partial x} = A(t,x)\frac{\partial u^*}{\partial x} + B(t,x)\frac{\partial u^*}{\partial t} + C(t,x)u^* + f(t,x) +$$

$$+\Phi_1(t,x)\int_0^T \Psi_1(s,x)\frac{\partial u^*(s,x)}{\partial x}ds + \Xi_1(t,x)\int_0^t \Theta_1(s,x)\frac{\partial u^*(s,x)}{\partial x}ds +$$

$$+\Phi_2(t,x)\int_0^T \Psi_2(s,x)u^*(s,x)ds + \Xi_2(t,x)\int_0^t \Theta_2(s,x)u^*(s,x)ds =$$

$$= A(t,x)v^* + F(t,x,w^*(t,x),u^*(t,x)) +$$

$$+\Phi_1(t,x)\int_0^T \Psi_1(s,x)v^*(s,x)ds + \Xi_1(t,x)\int_0^t \Theta_1(s,x)v^*(s,x)ds,$$

$$P_1(x)v^*(0,x) + S_1(x)v^*(T,x) = P_1(x)\frac{\partial u^*(0,x)}{\partial x} + S_1(x)\frac{\partial u^*(T,x)}{\partial x} =$$

$$= \varphi(x) - P_2(x)u^*(0,x) - S_2(x)u^*(T,x) = \phi(x,u^*),$$

i.e. the triple of functions $\{v^*(t,x),\, u^*(t,x),\, w^*(t,x)\}$ obtained in this way is a solution to problem (2.1)–(2.3).

Conversely, if a triple of functions $\{v^{**}(t,x), u^{**}(t,x), w^{**}(t,x)\}$ is a solution to problem (2.1)–(2.3), then from functional relations (2.3) we obtain that the function $u^{**}(t,x)$ satisfies the condition $u^{**}(t,0) = \psi(t)$ and has continuous partial derivatives of first order

$$\frac{\partial u^{**}(t,x)}{\partial x} = v^{**}(t,x), \quad \frac{\partial u^{**}(t,x)}{\partial t} = \dot{\psi}(t) + \int_0^x \frac{\partial v^{**}(t,\xi)}{\partial t} d\xi = w^{**}(t,x),$$

and continuous partial derivatives of second order

$$\frac{\partial^2 u^{**}(t,x)}{\partial t \partial x} = \frac{\partial v^{**}(t,x)}{\partial t}, \qquad \frac{\partial^2 u^{**}(t,x)}{\partial x \partial t} = \frac{\partial v^{**}(t,x)}{\partial t}.$$

Substituting them into (2.1), (2.2), we obtain that the function $u^{**}(t,x)$ satisfies system of hyperbolic integro-differential equations of mixed type (1.1), boundary condition (1.2), respectively for all $(t,x) \in \Omega$, $x \in [0,\omega]$. Since it also satisfies initial condition (1.3), then $u^{**}(t,x)$ is a classical solution to problem (1.1)–(1.3).

Thus, the original problem for the second order system of hyperbolic integro-differential equations of mixed type (1.1)–(1.3) is reduced to an equivalent family of boundary value problems for integro-differential equations of mixed type and integral relations (2.1)–(2.3).

Here, the vector-function $v(t,x)$ is a solution to the family of boundary value problems for integro-differential equations of mixed type with parameters (2.1), (2.2), where the functional parameters $u(t,x)$ and $w(t,x)$ are related to $v(t,x)$ and $\frac{\partial v(t,x)}{\partial t}$ by integral relations (2.3).

Now, let us introduce the notations

$$z_{(1)}(t,x) = v(t,x), \qquad z_{(2)}(t,x) = \int_0^t \Theta_1(s,x) v(s,x) ds, \qquad (t,x) \in \Omega.$$

Then we move on to a family of two-point boundary value problems for Fredholm integro-differential equations with unknown parameters:

$$\frac{\partial z}{\partial t} = \widetilde{A}(t,x) z(t,x) + \widetilde{\Phi}_1(t,x) \int_0^T \widetilde{\Psi}_1(s,x) z(s,x) ds + \widetilde{F}(t,x,\widetilde{u},\widetilde{w}), \quad (t,x) \in \Omega, \tag{2.4}$$

$$\widetilde{P}_1(x) z(0,x) + \widetilde{S}_1(x) z(T,x) = \widetilde{\phi}(x,\widetilde{u}), \qquad x \in [0,\omega], \tag{2.5}$$

$$\widetilde{u}(t,x) = \widetilde{\psi}(t) + \int_0^x z(t,\xi) d\xi, \qquad \widetilde{w}(t,x) = \dot{\widetilde{\psi}}(t) + \int_0^x \frac{\partial z(t,\xi)}{\partial t} d\xi, \tag{2.6}$$

where $z(t,x) = \begin{pmatrix} z_{(1)}(t,x) \\ z_{(2)}(t,x) \end{pmatrix}$ is the unknown vector-function,

$$\widetilde{A}(t,x) = \begin{pmatrix} A(t,x) & \Xi_1(t,x) \\ \Theta_1(t,x) & O_n \end{pmatrix}, \ \widetilde{\Phi}_1(t,x) = \begin{pmatrix} \Phi_1(t,x) & O_n \\ O_n & O_n \end{pmatrix},$$

$$\widetilde{\Psi}_1(s,x) = \begin{pmatrix} \Psi_1(s,x) & O_n \\ O_n & O_n \end{pmatrix}, \ \widetilde{u}(t,x) = \begin{pmatrix} u(t,x) \\ O_n \end{pmatrix}, \ \widetilde{w}(t,x) = \begin{pmatrix} w(t,x) \\ O_n \end{pmatrix},$$

$$\widetilde{F}(t,x,\widetilde{u},\widetilde{w}) = \begin{pmatrix} F(t,x,u,w) \\ O_n \end{pmatrix}, \widetilde{P}_1(x) = \begin{pmatrix} P_1(x) & O_n \\ O_n & I_n \end{pmatrix},$$

$$\widetilde{S}_1(x) = \begin{pmatrix} S_1(x) & O_n \\ O_n & O_n \end{pmatrix}, \ \widetilde{\phi}(x,\widetilde{u}) = \begin{pmatrix} \phi(x,u) \\ O_n \end{pmatrix}, \ \widetilde{\psi}(t) = \begin{pmatrix} \psi(t) \\ O_n \end{pmatrix},$$

O_n and I_n are the zero and identity matrices of dimension $n \times n$.

A triple of functions $\{z(t,x), \widetilde{u}(t,x), \widetilde{w}(t,x)\}$, where $z(t,x) \in C(\Omega, \mathbb{R}^{2n})$, $\widetilde{u}(t,x) \in C(\Omega, \mathbb{R}^{2n})$, $\widetilde{w}(t,x) \in C(\Omega, \mathbb{R}^{2n})$ is called a solution to problem (2.4)–(2.6) if it satisfies the family of Fredholm integro-differential equations with parameters (2.4), two-point condition (2.5) and integral relations (2.6).

For fixed $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$ problem (2.4), (2.5) is the family of two-point boundary value problems for first order Fredholm integro-differential equations [8]. The unknown functions $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$ are determined from integral relations (2.6).

It is well known that linear ordinary differential equations and Volterra integro-differential equations are solvable for any right-hand side and have classical general solutions. Note that there are linear Fredholm integro-differential equations that do not have classical general solutions [16]. An important problem arises: is it possible to construct general solutions that would exist for all differential and integro-differential equations and use them to solve boundary value problems? A new approach to defining a general solution was proposed in [17]. Based on Dzhumabaev's parametrization method [15], a new general solution is proposed, which, unlike the classical general solution, exists for all linear Fredholm integro-differential equations. Using a new general solution, criteria for the solvability of linear boundary value problems for Fredholm integro-differential equations were established and numerical and approximate methods for finding their solutions were constructed [18]. Further, these results were extended to problems with parameter for Fredholm integro-differential equations [2, 6, 9], problems for a system of differential equations with piecewise-constant argument of generalized type [1], problems for nonlinear Fredholm integro-differential equations [20].

3 Scheme of the parametrization method and Δ_N general solution

Consider the following family of problems for Fredholm integro-differential equations:

$$\frac{\partial z}{\partial t} = \widetilde{A}(t,x)z(t,x) + \widetilde{\Phi}_1(t,x)\int_0^T \widetilde{\Psi}_1(s,x)z(s,x)ds + F(t,x), \tag{3.1}$$

$$\widetilde{P}_1(x)z(0,x) + \widetilde{S}_1(x)z(T,x) = g(x), \qquad x \in [0,\omega], \tag{3.2}$$

where $z(t,x) = col(z_1(t,x),...,z_{2n}(t,x))$ is the unknown vector-function, the $2n$ vector-function $F(t,x)$ is continuous on Ω, the $2n$ vector-function $g(x)$ is continuous on $[0,\omega]$.

A vector-function $z(t,x) = col(z_1(t,x),...,z_{2n}(t,x)) \in C(\Omega,\mathbb{R}^n)$, which has a continuous partial derivative with respect to t is called a solution to the family of problems (3.1), (3.2), if it satisfies Fredholm integro-differential equations (3.1) for all $(t,x) \in \Omega$ and two-point conditions (3.2) for all $x \in [0,\omega]$.

The domain Ω is divided into subdomains and this partition is denoted by Δ_N:

$$\Omega = \bigcup_{r=1}^{N} \Omega_r, \quad \Omega_r = [t_{r-1}, t_r) \times [0,\omega], \quad r = \overline{1,N}, \quad 0 = t_0 < t_1 < ... < t_N = T.$$

Let $C(\Omega, \Delta_N, \mathbb{R}^{2nN})$ be the space of all vector-functions $z([t],x) = col(z_1(t,x), z_2(t,x), ..., z_N(t,x))$, where the notation $[t]$ means partition by t, the functions

$z_r : \Omega_r \to \mathbb{R}^{2n}$ are continuous and have finite left-sided limits $\lim\limits_{t\to t_r-0} z_r(t,x)$ uniformly with respect to $x \in [0,\omega]$ for all $r = \overline{1,N}$, with the norm

$$||v([\cdot],x)||_2 = \max_{r=\overline{1,N}} \sup_{t\to t_r-0} ||v_r(t,x)||.$$

We denote by $z_r(t,x)$ the restriction of the solution $z(t,x)$ to the subdomain Ω_r, i.e. $z_r(t,x) = z(t,x)$ for $(t,x) \in \Omega_r$, $r = \overline{1,N}$.

Then the vector-functions $z([t],x) = col(z_1(t,x),...,z_N(t,x)) \in C(\Omega,\Delta_N,\mathbb{R}^{2nN})$ with elements $z_r(t,x)$, $r = \overline{1,N}$, satisfy the following Fredholm integro-differential equations

$$\frac{\partial z_r}{\partial t} = \widetilde{A}(t,x)z_r(t,x) + \widetilde{\Phi}_1(t,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)z_j(s,x)ds + F(t,x), \tag{3.3}$$

$$(t,x) \in \Omega_r, \qquad r = \overline{1,N}.$$

Let us introduce functional parameters $\lambda_r(x) = z_r(t_{r-1},x)$, $r = \overline{1,N}$, $x \in [0,\omega]$. By replacing $\widetilde{z}_r(t,x) = z_r(t,x) - \lambda_r(x)$ on each r-th domain Ω_r, we obtain the following system Fredholm integro-differential equations with parameters

$$\frac{\partial \widetilde{z}_r}{\partial t} = \widetilde{A}(t,x)\widetilde{z}_r(t,x) + \widetilde{\Phi}_1(t,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)\widetilde{z}_j(s,x)ds + F(t,x)+$$

$$+\widetilde{A}(t,x)\lambda_r(x) + \widetilde{\Phi}_1(t,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\widetilde{\Psi}_1(s,x)ds\lambda_j(x), \quad (t,x)\in\Omega_r,\ r=\overline{1,N}. \tag{3.4}$$

and initial conditions

$$\widetilde{z}_r(t_{r-1},x) = 0, \qquad x \in [0,\omega], \qquad r = \overline{1,N}. \tag{3.5}$$

For fixed $\lambda_r(x) \in C([0,\omega],\mathbb{R}^{2n})$, a special Cauchy problem for the system of Fredholm integro-differential equations (3.4), (3.5) is obtained. The family of problems (3.4), (3.5) has a unique solution is a system functions $\widetilde{z}([t],x,\lambda) = col(\widetilde{z}_1(t,x,\lambda_1),\widetilde{z}_2(t,x,\lambda_2),...,\widetilde{z}_N(t,x,\lambda_N))$ with elements $\widetilde{z}_r(t,x,\lambda_r)$ belongs to $C(\Omega,\Delta_N,\mathbb{R}^{2nN})$.

A vector-function $\widetilde{z}([t],x,\lambda)$ is called a solution special Cauchy problem with parameters (3.4), (3.5).

Let us now introduce a new general solution to the family of integro-differential equations (3.1).

Definition 1. Let $\widetilde{z}([t],x,\lambda) = col(\widetilde{z}_1(t,x,\lambda_1),\widetilde{z}_2(t,x,\lambda_2),\ldots,\widetilde{z}_N(t,x,\lambda_N))$ be a solution to a special Cauchy problem (3.4), (3.5) for the parameter $\lambda(x) = (\lambda_1(x),\lambda_2(x),...,\lambda_N(x)) \in C([0,\omega],\mathbb{R}^{2nN})$. Then the function $z(\Delta_N,t,x,\lambda)$, given by the equalities

$z(\Delta_N,t,x,\lambda) = \lambda_r(x) + \widetilde{z}_r(t,x,\lambda_r)$, for $(t,x)\in\Omega_r$, $r=\overline{1,N}$,

and

$z(\Delta_N,T,x,\lambda) = \lambda_N(x) + \lim\limits_{t\to T-0}\widetilde{z}_N(t,x,\lambda_N)$,

is called a Δ_N general solution to family of Fredholm integro-differential equations (3.1).

From Definition 3.1 it is clear that a Δ_N general solution depends on N arbitrary functions $\lambda_r(x) \in C([0,\omega],\mathbb{R}^{2n})$, $x\in[0,\omega]$, $r=\overline{1,N}$, and satisfies family of integro-differential equations (3.1) for all $(t,x) \in (0,T)\backslash\{t_p, p=\overline{1,N-1}\}\times[0,\omega]$.

Using a fundamental matrix $U_r(t,x)$ of the family of differential equations

$$\frac{\partial z_r}{\partial t} = \widetilde{A}(t,x)z_r(t,x), \qquad (t,x) \in \Omega_r, \quad r = \overline{1,N},$$

we write the solution to the family of special Cauchy problems with parameters (3.4), (3.5) in the following form

$$\widetilde{z}_r(t,x) = U_r(t,x) \int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)\widetilde{\Phi}_1(\tau,x) \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} \widetilde{\Psi}_1(s,x)\widetilde{z}_j(s,x)dsd\tau +$$

$$+U_r(t,x) \int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)F(\tau,x)d\tau + U_r(t,x) \int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)\widetilde{A}(\tau,x)d\tau\lambda_r(x)+$$

$$+ U_r(t,x) \int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)\widetilde{\Phi}_1(\tau,x) \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} \widetilde{\Psi}_1(s,x)ds\lambda_j(x), \quad (t,x) \in \Omega_r, \ r = \overline{1,N}. \tag{3.6}$$

Consider the following family of Cauchy problems on subdomains

$$\frac{\partial z_r}{\partial t} = \widetilde{A}(t,x)z_r(t,x) + P(t,x), \quad z(t_{r-1},x) = 0, \quad (t,x) \in \Omega_r, \quad r = \overline{1,N}, \tag{3.7}$$

where $P(t,x)$ is a square matrix or a vector of dimension $2n$, continuous on Ω.

Let us denote by $a_r(P,t,x)$ the unique solution to family of Cauchy problems (3.7) on each r-th domain. It has the following form

$$a_r(P,t,x) = U_r(t,x) \int\limits_{t_{r-1}}^{t} U_r^{-1}(\tau,x)P(\tau,x)d\tau, \quad (t,x) \in \Omega_r, \quad r = \overline{1,N}.$$

We introduce the notation $\mu(x) = \sum\limits_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} \widetilde{\Psi}_1(s,x)\widetilde{z}_j(s,x)ds$. Then we can rewrite (3.7) in the form

$$\widetilde{z}_r(t,x) = a_r(\widetilde{\Phi}_1,t,x)\mu(x) + a_r(F,t,x) + a_r(\widetilde{A},t,x)\lambda_r(x)+$$

$$+ a_r(\widetilde{\Phi}_1,t,x) \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} \widetilde{\Psi}_1(s,x)ds\lambda_j(x), \quad (t,x) \in \Omega_r, \ r = \overline{1,N}. \tag{3.8}$$

First multiplying both sides by $\widetilde{\Psi}_1(t,x)$, integrating over t from t_{r-1} to t_r, summing over $r = \overline{1,N}$, we obtain from (3.8) the following system of equations:

$$[I - G(N,x)]\mu(x) = \sum_{r=1}^{N} \int\limits_{t_{r-1}}^{t_r} \widetilde{\Psi}_1(t,x)a_r(F,t,x)dt + \sum_{r=1}^{N} \int\limits_{t_{r-1}}^{t_r} \widetilde{\Psi}_1(t,x)a_r(\widetilde{A},t,x)dt\lambda_r(x)+$$

$$+ \sum_{r=1}^{N} \int\limits_{t_{r-1}}^{t_r} \widetilde{\Psi}_1(t,x)a_r(\widetilde{\Phi}_1,t,x)dt \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} \widetilde{\Psi}_1(s,x)ds\lambda_j(x), \quad (t,x) \in \Omega_r, \ r = \overline{1,N}, \tag{3.9}$$

where I is a unit matrix on dimension $2n$, $G(N,x) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \widetilde{\Psi}_1(t,x) a_r(\widetilde{\Phi}_1,t,x)dt$.

Assuming the invertibility of the $2n \times 2n$ matrix $I - G(N,x)$, from (3.9) for all $x \in [0,\omega]$ we uniquely define $\mu(x)$ in terms of $\lambda_r(x)$, $r = \overline{1,N}$, and $F(t,x)$. Then, substituting the found expression instead of $\mu(x)$ in (3.8), we obtain a representation of $\widetilde{z}_r(t,x)$ via $\lambda_r(x)$, $(t,x) \in \Omega_r$, $r = \overline{1,N}$.

Corollary 3.1. *Let $z^*(t,x)$ be a solution to system of equations (3.1) and $z(\Delta_N,t,x,\lambda)$ be a Δ_N general solution to family integro-differential equations (3.1).*

Then there exists a unique $\lambda^*(x) = col(\lambda_1^*(x), \lambda_2^*(x), \ldots, \lambda_N^*(x)) \in C([0,\omega], \mathbb{R}^{2nN})$ such that the equality $z(\Delta_N,t,x,\lambda^*) = z^*(t,x)$ holds for all $(t,x) \in \Omega$.

If $z(t,x)$ is a solution to system (3.1), and $z([t],x) = col(z_1(t,x), z_2(t,x), ..., z_N(t,x))$ is the vector-function composed of its restrictions to the subdomains Ω_r, $r = \overline{1,N}$, then the following equalities

$$\lim_{t \to t_p - 0} z_p(t,x) = z_{p+1}(t_p,x), \qquad x \in [0,\omega], \qquad p = \overline{1,N-1}, \tag{3.10}$$

hold. These equalities are the continuity conditions for the solution to system (3.1) at the interior lines of the partition Δ_N.

Theorem 3.1. *Let a vector-function $z([t],x) = col(z_1(t,x), z_2(t,x), ..., z_N(t,x))$ belong to $C(\Omega, \Delta_N, \mathbb{R}^{2nN})$. Assume that the functions $z_r(t,x)$, $r = \overline{1,N}$, satisfy system (3.1) and continuity conditions (3.10). Then the function $z^*(t,x)$, given by the equalities*

$z^*(t,x) = z_r(t,x)$ *for* $t \in (t,x) \in \Omega_r, \quad r = \overline{1,N}$,

and

$z^*(T,x) = \lim_{t \to T-0} z_N(t,x), \quad x \in [0,\omega]$,

is continuously differentiable on Ω and satisfies system (3.1).

Now, we consider family of problems for systems of $2n$ Fredholm integro-differential equations (3.1), (3.2). Using notations above, we obtain the following family of problems

$$\frac{\partial \widetilde{z}_r}{\partial t} = \widetilde{A}(t,x)\widetilde{z}_r(t,x) + \widetilde{\Phi}_1(t,x) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \widetilde{\Psi}_1(s,x)\widetilde{z}_j(s,x)ds + F(t,x) +$$

$$+ \widetilde{A}(t,x)\lambda_r(x) + \widetilde{\Phi}_1(t,x) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \widetilde{\Psi}_1(s,x)ds\lambda_j(x), \quad (t,x) \in \Omega_r, \ r = \overline{1,N}. \tag{3.11}$$

$$\widetilde{z}_r(t_{r-1},x) = 0, \qquad x \in [0,\omega], \qquad r = \overline{1,N}. \tag{3.12}$$

$$\widetilde{P}_1(x)\lambda_1(x) + \widetilde{S}_1(x) \lim_{t \to T-0} \widetilde{z}_N(t,x) + \widetilde{S}_1(x)\lambda_N(x) = g(x), \qquad x \in [0,\omega], \tag{3.13}$$

$$\lim_{t \to t_p - 0} \widetilde{z}_p(t,x) + \lambda_p(x) = \lambda_{p+1}(x), \qquad x \in [0,\omega], \qquad p = \overline{1,N-1}. \tag{3.14}$$

A solution to problem (3.11)–(3.14) is the pair $\{\widetilde{z}([t],x), \lambda(x)\}$, where the vector-functions $\widetilde{z}([t],x) = col(\widetilde{z}_1(t,x), \widetilde{z}_2(t,x), ..., \widetilde{z}_N(t,x)) \in C(\Omega, \Delta_N, \mathbb{R}^{2nN})$, $\lambda(x) = col(\lambda_1(x), \lambda_2(x), ..., \lambda_N(x)) \in C([0,\omega], \mathbb{R}^{2nN})$ with the elements $\widetilde{z}_r(t,x)$, $\lambda_r(x)$, $r = \overline{1,N}$, satisfy system (3.11), initial conditions (3.12), boundary conditions (3.13), continuity conditions (3.14).

Using the results of this section, we obtain a representation of $\widetilde{z}_r(t,x)$ in terms of $\lambda_r(x)$, $(t,x) \in \Omega_r$, $r = \overline{1,N}$. From the resulting representation, determining the values of the left-hand limits

$\lim\limits_{t\to T-0}\widetilde{z}_N(t,x)$, $\lim\limits_{t\to t_p-0}\widetilde{z}_p(t,x)$, $p=\overline{1,N-1}$, and substituting into conditions (3.13), (3.14), we obtain a system of functional equations of the form

$$Q(\Delta_N,x)\lambda(x)=-E(\Delta_N,x,g,F), \qquad \lambda(x)\in C([0,\omega],\mathbb{R}^{2nN}), \tag{3.15}$$

where $Q(\Delta_N,x)$ is a $2nN\times 2nN$ matrix, composed of the functions $\lambda_r(x)\in C([0,\omega],\mathbb{R}^{2n})$, $r=\overline{1,N}$, and $E(\Delta_N,x,g,F)$ contains the right-hand sides F and g.

Theorem 3.2. *Let the $2n\times 2n$ matrix $I-G(N,x)$ and the $2nN\times 2nN$ matrix $Q(\Delta_N,x)$ be invertible for all $x\in[0,\omega]$. Then family of problems (3.11) − (3.14) has a unique solution $\{\widetilde{z}^*([t],x),\lambda^*(x)\}$.*

From the equivalence of problems (3.1), (3.2) and (3.11)–(3.14) follows

Theorem 3.3. *Let the $2n\times 2n$ matrix $I-G(N,x)$ and the $2nN\times 2nN$ matrix $Q(\Delta_N,x)$ be invertible for all $x\in[0,\omega]$. Then family of boundary value problems for system of Fredholm integro-differential equations (3.1), (3.2) has a unique solution $z^*(t,x)$.*

The proofs of these theorems are similar to the proofs of the corresponding theorems in [8].

4 Algorithm and main results

Based on the results of Section 3, we offer the following algorithm for finding a solution to the family of two-point boundary value problems for system of Fredholm integro-differential equations with functional parameters (2.4)–(2.6).

Algorithm.

Step 1. i) Assume that $\widetilde{u}^{(0)}(t,x)=\widetilde{\psi}(t)$, $\widetilde{w}^{(0)}(t,x)=\dot{\widetilde{\psi}}(t)$ in the left-hand side of (2.4), (2.5). Solving the family of two-point boundary value problems for system of Fredholm integro-differential equations, we find of a function $z^{(1)}(t,x)$ for all $(t,x)\in\Omega$. ii) From integral relations (2.6) we determine $\widetilde{u}^{(1)}(t,x)$ and $\widetilde{w}^{(1)}(t,x)$ for $z(t,x)=z^{(1)}(t,x)$ and $\frac{\partial z(t,x)}{\partial t}=\frac{\partial z^{(1)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

And so on.

Step k. i) Assume that $\widetilde{u}(t,x)=\widetilde{u}^{(k-1)}(t,x)$, $\widetilde{w}(t,x)=\widetilde{w}^{(k-1)}(t,x)$ in the left-hand side of (2.4), (2.5). Solving the family of two-point boundary value problems for system of Fredholm integro-differential equations, we find the function $z^{(k)}(t,x)$ for all $(t,x)\in\Omega$. ii) From integral relations (2.6) we determine the functions $\widetilde{u}^{(k)}(t,x)$ and $\widetilde{w}^{(k)}(t,x)$ for $z(t,x)=z^{(k)}(t,x)$ and $\frac{\partial z(t,x)}{\partial t}=\frac{\partial z^{(k)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

$k=1,2,...,$

The algorithm for finding a solution to the family of two-point boundary value problems for system of Fredholm integro-differential equations with functional parameters (2.4)–(2.6) consists of two stages: 1) the family of two-point boundary value problems for system of Fredholm integro-differential equations (2.4), (2.5) is solved and the unknown function $z(t,x)$ is found for fixed $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$; 2) $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$ are determined from integral relations (2.6) by using $z(t,x)$ and $\frac{\partial z(t,x)}{\partial t}$.

We show that the conditions for unique solvability of the family of two-point boundary value problems for system of Fredholm integro-differential equations (3.1), (3.2) are the convergence conditions of the proposed algorithm.

For fixed $\widetilde{u}(t,x)$ and $\widetilde{w}(t,x)$ the family of two-point boundary value problems for system of Fredholm integro-differential equations with functional parameters (2.4)–(2.6) is the family of boundary value problems for system of Fredholm integro-differential equations (3.1), (3.2) with

$F(t,x) = \widetilde{F}(t,x,\widetilde{u},\widetilde{w})$, $g(x) = \widetilde{\phi}(x,\widetilde{u})$. From Theorem 3.1 it follows that the family of boundary value problems for system of Fredholm integro-differential equations (3.1), (3.2) has a unique solution $z^*(t,x)$. Moreover, similarly to Theorem 2.1 in [16], the conditions of Theorem 3.1 ensure that the estimate

$$\max_{t\in[0,T]} ||z^*(t,x)|| \leq \mathcal{N}(x)\max\Big(||g(x)||, \max_{t\in[0,T]}||F(t,x)||\Big), \tag{4.1}$$

holds, where

$$\mathcal{N}(x) =$$

$$e^{\alpha(x)\theta}\Big\{\widetilde{\Phi}_1^*(x)\Big[||[I-G(N,x)]^{-1}||\widetilde{\Psi}_1^*(x)\Big(e^{\alpha(x)\theta}-1+e^{\alpha(x)\theta}\widetilde{\Phi}_1^*(x)\widetilde{\Psi}_1^*(x)\Big)+\widetilde{\Psi}_1^*(x)\Big]+1\Big\}$$

$$\times||[Q(\Delta_N,x)]^{-1}||(1+||\widetilde{S}_1(x)||)\max\Big\{1,\theta e^{\alpha(x)\theta}\Big[1+e^{\alpha(x)\theta}\widetilde{\Phi}_1^*(x)||[I-G(N,x)]^{-1}||\widetilde{\Psi}_1^*(x)\Big]\Big\}$$

$$+e^{\alpha(x)\theta}\Big[\widetilde{\Phi}_1^*(x)||[I-G(N,x)]^{-1}||\widetilde{\Psi}_1^*(x)e^{\alpha(x)\theta}+1\Big],$$

$$\alpha(x) = \max_{t\in[0,T]}||\widetilde{A}(t,x)||, \qquad \theta = \max_{r=\overline{1,N}}(t_r - t_{r-1}),$$

$$\widetilde{\Phi}_1^*(x) = \max_{r=\overline{1,N}}\int_{t_{r-1}}^{t_r}||\widetilde{\Phi}_1(t,x)||dt, \qquad \widetilde{\Psi}_1^*(x) = \int_0^T||\widetilde{\Psi}_1(t,x)||dt.$$

Suppose $\widetilde{u}^{(k-1)}(t,x)$ and $\widetilde{w}^{(k-1)}(t,x)$ are known. According to the Step k of the algorithm, we have

$$\max_{t\in[0,T]}||z^{(k)}(t,x)|| \leq \mathcal{N}(x)\max\Big(||\widetilde{\phi}(x,\widetilde{u}^{(k-1)})||, \max_{t\in[0,T]}||\widetilde{F}(t,x,\widetilde{u}^{(k-1)},\widetilde{w}^{(k-1)})||\Big), \tag{4.2}$$

$$\max_{t\in[0,T]}\Big|\Big|\frac{\partial z^{(k)}(t,x)}{\partial t}\Big|\Big| \leq \Big\{\max\Big(\alpha(x)+\max_{t\in[0,T]}||\widetilde{\Phi}_1(t,x)||\widetilde{\Psi}_1^*(x)\Big)\mathcal{N}(x)+1\Big\}$$
$$\times\max\Big(||\widetilde{\phi}(x,\widetilde{u}^{(k-1)})||, \max_{t\in[0,T]}||\widetilde{F}(t,x,\widetilde{u}^{(k-1)},\widetilde{w}^{(k-1)})||\Big), \tag{4.3}$$

$k = 1, 2, \ldots$.

Once $z^{(k)}(t,x)$ is found the successive approximations for $\widetilde{u}(t,x)$ and $\widetilde{u}(t,x)$ are found from relations (2.6):

$$\widetilde{u}^{(k)}(t,x) = \widetilde{\psi}(t) + \int_0^x z^{(k)}(t,\xi)d\xi, \qquad \widetilde{w}^{(k)}(t,x) = \dot{\widetilde{\psi}}(t) + \int_0^x \frac{\partial z^{(k)}(t,\xi)}{\partial t}d\xi, \tag{4.4}$$

We construct the differences $\Delta z^{(k)}(t,x) = z^{(k)}(t,x) - z^{(k-1)}(t,x)$, $\Delta\widetilde{u}^{(k)}(t,x) = \widetilde{u}^{(k)}(t,x) - \widetilde{u}^{(k-1)}(t,x)$, $\Delta\widetilde{w}^{(k)}(t,x) = \widetilde{w}^{(k)}(t,x) - \widetilde{w}^{(k-1)}(t,x)$, and by using the unique solvability of family problems (3.1), (3.2), and estimates (4.2), (4.3), we establish estimates

$$\max\Big\{\max_{t\in[0,T]}||\Delta z^{(k+1)}(t,x)||, \max_{t\in[0,T]}\Big|\Big|\frac{\partial\Delta z^{(k+1)}(t,x)}{\partial t}\Big|\Big|\Big\}$$

$$\leq \max\Big\{\mathcal{N}(x), \max\Big(\alpha(x)+\max_{t\in[0,T]}||\widetilde{\Phi}_1(t,x)||\widetilde{\Psi}_1^*(x)\Big)\mathcal{N}(x)+1\Big\}\mathcal{N}_1(x)$$

$$\times\max\Big\{\max_{t\in[0,T]}||\Delta\widetilde{w}^{(k)}(t,x)||, \max_{t\in[0,T]}||\Delta\widetilde{u}^{(k)}(t,x)||\Big\}, \tag{4.5}$$

$$\max\Big\{\max_{t\in[0,T]}||\Delta w^{(k)}(t,x)||, \max_{t\in[0,T]}||\Delta u^{(k)}(t,x)||\Big\}$$

$$\le \int\limits_0^x \max\Big\{\max_{t\in[0,T]}||\Delta z^{(k)}(t,\xi)||, \max_{t\in[0,T]}\Big|\Big|\frac{\partial \Delta z^{(k)}(t,\xi)}{\partial t}\Big|\Big|\Big\}d\xi, \tag{4.6}$$

where

$$\mathcal{N}_1(x) = \max\Big\{||P_2(x)|| + ||S_2(x)||, \max_{t\in[0,T]}||B(t,x)|| + \max_{t\in[0,T]}||C(t,x)||$$

$$+\max_{t\in[0,T]}||\Phi_2(t,x)||T\max_{t\in[0,T]}||\Psi_2(t,x)|| + \max_{t\in[0,T]}||\Xi_2(t,x)||T\max_{t\in[0,T]}||\Theta_2(t,x)||\Big\}.$$

This implies the main inequality

$$\max\Big\{\max_{t\in[0,T]}||\Delta z^{(k+1)}(t,x)||, \max_{t\in[0,T]}\Big|\Big|\frac{\partial \Delta z^{(k+1)}(t,x)}{\partial t}\Big|\Big|\Big\}$$

$$\le \max\Big\{\mathcal{N}(x), \max\Big(\alpha(x) + \max_{t\in[0,T]}||\widetilde{\Phi}_1(t,x)||\widetilde{\Psi}_1^*(x)\Big)\mathcal{N}(x)+1\Big\}\mathcal{N}_1(x)$$

$$\times \int\limits_0^x \max\Big\{\max_{t\in[0,T]}||\Delta z^{(k)}(t,\xi)||, \max_{t\in[0,T]}\Big|\Big|\frac{\partial \Delta z^{(k)}(t,\xi)}{\partial t}\Big|\Big|\Big\}d\xi. \tag{4.7}$$

From (4.7) it follows that the sequences $\{z^{(k)}(t,x)\}$ and $\{\frac{\partial z^{(k)}(t,x)}{\partial t}\}$ are convergent in the space $C(\Omega,\mathbb{R}^{2n})$ as $k\to\infty$. Then the uniform convergence on Ω of the sequences $\{\widetilde{u}^{(k)}(t,x)\}$ and $\{\widetilde{w}^{(k)}(t,x)\}$ follows from estimate (4.6).

In this case, the limit functions $z^*(t,x)$, $\frac{\partial z^*(t,x)}{\partial t}$, $\widetilde{u}^*(t,x)$ and $\widetilde{w}^*(t,x)$ are continuous on Ω, and the triple $\{z^*(t,x), \widetilde{u}^*(t,x), \widetilde{w}^*(t,x)\}$ is a solution to problem (2.4)-(2.6).

The uniqueness of a solution to problem (2.4)-(2.6) is proved assuming the contrary.

Now, using the constructed solution to the family of problems (2.4)–(2.6), the triple of functions $\{z^*(t,x), \widetilde{u}^*(t,x), \widetilde{w}^*(t,x)\}$, we verify the validity of the following equalities:

$$z^*(t,x) = col(z^*_{(1)}(t,x), z^*_{(2)}(t,x)),$$

$$\widetilde{u}^*(t,x) = \widetilde{\psi}(t) + \int\limits_0^x z^*(t,\xi)d\xi, \qquad \widetilde{w}^*(t,x) = \dot{\widetilde{\psi}}(t) + \int\limits_0^x \frac{\partial z^*(t,\xi)}{\partial t}d\xi,$$

$$u^*(t,x) = \psi(t) + \int\limits_0^x z^*_{(1)}(t,\xi)d\xi \qquad \text{for all} \qquad (t,x)\in\Omega.$$

The function $u^*(t,x)$ is the desired solution to problem (1.1)–(1.3).

Theorem 4.1. *Let the $2n\times 2n$ matrix $I - G(N,x)$ and the $2nN\times 2nN$ matrix $Q(\Delta_N,x)$ be invertible for all $x\in[0,\omega]$. Then boundary value problem for system of hyperbolic integro-differential equations of mixed type (3.1) − (3.3) has the unique solution $u^*(t,x)$.*

The proof of this theorem follows from the above algorithm and is similar to the proof of Theorem 3.2 in [4].

Conclusion. In the paper, we propose an effective method of solving the boundary value problem for a second order system of hyperbolic integro-differential equations of mixed type with degenerate kernels. This method is based on the method of introducing new functions, Dzhumabaev's parametrization method and a new concept of a general solution to a family Fredholm integro-differential equations. New general solution enables us to establish qualitative properties of the boundary value problems for second order systems of hyperbolic integro-differential equations and to develop algorithms for solving them. The algorithms are based on constructing and solving systems of linear functional equations with respect to the new general solution and integral equations. Further, we will study the boundary value problem for second order systems of hyperbolic integro-differential equations of mixed type in general case. The obtained results can be used to solve the boundary value problems for impulsive hyperbolic integro-differential equations of mixed type.

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NOTES ON THE GENERALIZED GAUSS REDUCTION ALGORITHM

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Abstract. The hypothetical possibility of building a quantum computer in the near future has forced a revision of the foundations of modern cryptography. The fact is that many difficult algorithmic problems, such as the discrete logarithm, factoring a (large) natural number into prime factors, etc., on the complexity of which many cryptographic protocols are based these days, have turned out to be relatively easy to solve using quantum algorithms.

Intensive research is currently underway to find problems that are difficult even for a quantum computer and have potential applications for cryptographic protocols. Our article contains notes related to the so-called generalized Gauss algorithm, which calculates the reduced basis of a two-dimensional lattice [8], [2]. Note that researchers are increasingly putting forward difficult algorithmic problems from lattice theory as candidates for the foundation of post-quantum cryptography. The majority of algorithmic problems related to lattice reduction become NP-hard as the lattice dimension increases [3], [1]. Fundamental problems such as the Shortest Vector Problem (SVP), the Closest Vector Problem (CVP), and Bounded Distance Decoding (BDD) are conjectured to remain hard even for quantum algorithms [4], [6]. Although the generalized Gauss reduction algorithm applies to two-dimensional lattices, where exact analysis is feasible (dimensions 3 and 4 are studied in [7], [5]), understanding such low-dimensional reductions provides important insights into the structure and complexity of lattice-based cryptographic constructions.

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1 Preliminaries

All necessary information on the basics of lattice theory can be found in [8]. For those who are familiar with the group theory, a *lattice* is a finitely generated subgroup of the additive group of the Euclidean space $\mathbb{R}^n$. In this note we will limit ourselves to considering the 2-generated lattice $L \in \mathbb{R}^n$. Any pair of vectors generating L is called a *basis* of the lattice.

The Euclidean space metric $\mathbb{R}^n$, obtained by the standard dot product, induces a metric on L. Let us clarify the notation associated with this metric: for vectors $a, b \in L$, let us denote by (a, b) their dot product, by $\|a\|$ the length of vector a, and by $[a]$ the square of this length, that is, $[a] = (a, a) = \|a\|^2$.

Definition 1. Vectors $a, b \in L$ will be called an *ordered basis* and denoted by $\langle a, b\rangle$ if the following conditions are satisfied:
(1) $\|a\| \leq \|b\|$;
(2) $\|a - b\| \leq \|a + b\|$.

Note that for any lattice basis it is easy to obtain an ordered basis: if the vectors $a, b \in L$ form a basis, then first we arrange them in increasing length, and if we already have $\|a\| \leq \|b\|$, and $\|a-b\| \leq \|a+b\|$ is not satisfied, then we change b to $-b$. Therefore, in what follows only ordered bases of the lattice L are considered.

Definition 2. (1) If $\|a\| \leq \|a-b\| < \|b\|$, then the ordered basis $\langle a, b\rangle$ is called *well-ordered*.
(2) An ordered basis $\langle a, b\rangle$ is called *reduced* if $\|b\| \leq \|a-b\|$.

In Sections 2 and 3 we present results that are valid for any normed lattices, that is, for lattices with a norm which their norm is obtained by restricting a certain norm on the space $\mathbb{R}^n$.

Definition 3. A function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of all non-negative real numbers, is called a *norm* if it satisfies the following conditions for any vectors $x, y \in \mathbb{R}^n$ and for any real number $\alpha \in \mathbb{R}$:
(1) $\|x\| = 0$ if and only if x is the zero vector;
(2) $\|x+y\| \leq \|x\| + \|y\|$ (*the triangle inequality*);
(3) $\|\alpha x\| = |\alpha| \cdot \|x\|$.

We will call a norm *strict* if the equality in condition (2) is satisfied only when at least one of the vectors x, y is the zero vector or the vectors x, y are collinear and co-directional.

The following corollary of the triangle inequality is often useful.

Corollary 1.1. *For any $x, y \in \mathbb{R}^n$ we have $\big|\|x\| - \|y\|\big| \leq \|x-y\|$.*

Definition 4. (1) $\lambda_1 = \min\{\|a\| : 0 \neq a \in L\}$
(2) $\lambda_2 = \min\{\|b\| : \langle a, b\rangle \text{ is an ordered basis for some } a \in L\}$.

The numbers λ_1, λ_2 are always defined, since the lattice L is a discrete group: any ball of finite radius centered at the zero vector contains only a finite number of lattice elements [8].

The following theorem, the proof of which can be found in [8, Theorem 16] (see also [2, Theorem 4]), explains why a reduced basis is sometimes called a *minimal basis*.

Theorem 1.1. *An ordered basis $\langle a, b\rangle$ is reduced if and only if $\|a\| = \lambda_1$ and $\|b\| = \lambda_2$.*

The following useful lemma was also proven in [8, Lemma 17].

Lemma 1.1. *Consider three vectors on a line: x, $x+y$ and $x+\alpha y$, where $\alpha \in (1, \infty)$. For any norm $\|\cdot\|$ from the inequality $\|x\| \leq \|x+y\|$ it follows that $\|x+y\| \leq \|x+\alpha y\|$, and from the inequality $\|x\| < \|x+y\|$ it follows that $\|x+y\| < \|x+\alpha y\|$.*

Note that using Lemma 1.1 one can prove that if a basis $\langle a, b\rangle$ is well-ordered, then $\|a\| \leq \|a-b\| < \|b\| < \|a+b\|$ (see [2]).

2 About the function $l(\tau) = \|b - \tau a\|$

In this section, we study the properties of the function $l(\tau) = \|b - \tau a\|$, $\tau \in \mathbb{R}$, where a, b are vectors of some real space with the norm $\|\cdot\|$. If a is the zero vector, then $l(\tau) \equiv \|b\|$ is a constant function, and if b is the zero vector, then $l(\tau) = \|a\| \cdot |\tau|$ is the absolute value function multiplied by the constant $\|a\|$. A similar function will be obtained if the vectors a, b are linearly dependent: for example, if $b = \gamma a$, then $l(\tau) = \|a\| \cdot |\tau - \gamma|$. Therefore, the case is interesting when the vectors a, b are linearly independent.

Theorem 2.1. *Let a, b be linearly independent vectors of some real space with the norm $\|\cdot\|$. Then the function $l(\tau) = \|b - \tau a\|, \tau \in \mathbb{R}$, has the following properties:*
(1) l is continuous on the entire real line;
(2) l is not bounded from above: $\lim_{\tau\to-\infty} l(\tau) = +\infty$ and $\lim_{\tau\to+\infty} l(\tau) = +\infty$;
(3) there exists $\mu_0 \stackrel{def}{=} \min\{l(\tau) : \tau \in \mathbb{R}\} > 0$ and there exists a closed interval of minimality $[\tau_0, \tau_1] \stackrel{def}{=} \{\tau \in \mathbb{R} : l(\tau) = \mu_0\}$;
(4) on the interval $(-\infty, \tau_0)$ the function l strictly decreases, and on the interval $(\tau_1, +\infty)$ it strictly increases.

Proof. (1) Let us prove the continuity of the function l at an arbitrary point $\tau_0 \in \mathbb{R}$. By Corollary 1.1 we have

$$|l(\tau + \tau_0) - l(\tau_0)| = |\|b - (\tau + \tau_0)a\| - \|b - \tau_0 a\|| \le \|\tau a\| = \|a\| \cdot |\tau|.$$

Therefore, $|l(\tau + \tau_0) - l(\tau_0)| < \varepsilon$ holds for $|\tau| < \delta = \frac{\varepsilon}{\|a\|}$.

(2) Using Corollary 1.1 again and property (3) of the norm, we obtain

$$l(\tau) = \|b - \tau a\| \ge \|a\| \cdot |\tau| - \|b\|,$$

which obviously implies $\lim_{\tau\to-\infty} l(\tau) = +\infty$ and $\lim_{\tau\to+\infty} l(\tau) = +\infty$.

(3) Let us choose numbers $\alpha_0 < 0 < \beta_0 \in \mathbb{R}$ so that $l(\tau) > l(0) = \|b\|$ for any real number τ lying outside the interval $[\alpha_0, \beta_0]$: this is possible according to (2). According to Weierstrass's theorem, the function l reaches its minimum at a point τ_0 of the interval $[\alpha_0, \beta_0]$, which we denote by $\mu_0 = l(\tau_0)$. Obviously, this μ_0 will be the minimum of the function over the entire $\mathbb{R}$.

Let us call $\tau \in \mathbb{R}$ a *point of monotonicity* (of the function l), if $l(\tau) > \mu_0$. Let $\gamma < \tau_0$ be a point of monotonicity. Then note that each $\delta < \gamma$ is a point of monotonicity, since $l(\delta) > l(\gamma) > \mu_0$ holds (apply Lemma 1.1 for the vectors $x = b - \tau_0 a$ and $y = (\tau_0 - \gamma)a$). So, the interval $(-\infty, \gamma]$ consists entirely of monotonicity points. In addition, due to the continuity of the function l, some neighborhood of the point γ will consist entirely of monotonicity points. This means that each monotonicity point $\gamma < \tau_0$ is contained in a certain interval of the form $(-\infty, \alpha)$, consisting entirely of monotonicity points. Since the union of intervals of this type again gives an open interval of the same type, we conclude that the monotonicity points located to the left of τ_0 form an interval of this type, which we will denote without loss of generality by $(-\infty, \tau_0)$. Similar reasoning shows that monotonicity points located to the right of τ_0 form an interval $(\tau_1, +\infty)$ for some $\tau_1 \ge \tau_0$.

(4) In the last paragraph of the proof of point (3), in fact, it was proven that $l(\delta) > l(\gamma)$ holds for $\delta < \gamma < \tau_0$, that is, that the function l strictly decreases on the interval $(-\infty, \tau_0)$. Similarly, using Lemma 1.1 we prove the second statement of this point, namely, that the function l is strictly increasing on the interval $(\tau_1, +\infty)$. □

Example. The norm defined for $\mathbb{R}^2$ as follows is not strict: for $(\alpha, \beta) \in \mathbb{R}^2$ we set

$$\|(\alpha, \beta)\| \stackrel{def}{=} \max\{|\alpha|, |\beta|\}.$$

With $a = (0, 1), b = (1, 0)$ for the function $l(\tau) = \|b - \tau a\|$ we have $\mu_0 \stackrel{def}{=} \min\{l(\tau) : \tau \in \mathbb{R}\} = 1$, and the interval of minimality is $[-1, 1]$. □

Note that it may well be $\tau_0 = \tau_1$, that is, the interval $[\tau_0, \tau_1]$ can consist of only one point. This situation occurs if the norm $\|\cdot\|$ on the subspace generated by the vectors a, b is strict. Indeed, if

$\tau_0 \neq \tau_1$ and the norm $\|\cdot\|$ is strict, then the vectors $b - \tau_0 a$, $b - \tau_1 a$ are not collinear, therefore the sum of their lengths is strictly greater than the length of their sum:

$$2\mu_0 = \|b - \tau_0 a\| + \|b - \tau_1 a\| > \|2b - (\tau_0 + \tau_1)a\| = 2\left\|b - \frac{\tau_0 + \tau_1}{2}a\right\|.$$

We obtain $l(\frac{\tau_0+\tau_1}{2}) = \|\frac{\tau_0+\tau_1}{2}a + b\| < \mu_0$, which contradicts the minimality of the value μ_0.

In particular, we have $\tau_0 = \tau_1$, when the norm $\|\cdot\|$ is generated by the dot product in $\mathbb{R}^n$. In addition, in this case the value of τ_0 is explicitly calculated. Indeed, we have

$$l(\tau)^2 = [b - \tau a] = (b - \tau a, b - \tau a) = [a]\tau^2 - 2(a, b)\tau + [b],$$

and this quadratic function reaches a minimum at point $\tau_0 = \frac{(a,b)}{(a,a)} = \frac{(a,b)}{[a]}$.

In the next section we use an oracle that solves the following problem.

Problem. For a given ordered basis $\langle a, b\rangle$, find an integer $\mu = \mu(a, b)$ such that $\|b - \mu a\| = \min\{\|b - na\| : n \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers.

By Theorem 2.1 for the function $l(\tau) = \|b - \tau a\|$ it follows that the problem is correct, that is, it always has a solution. In general, if the interval $[\tau_0, \tau_1]$ contains an integer, then any integer from it will be a solution, if not, then $\mu = \lfloor\tau_0\rfloor$ or $\mu = \lceil\tau_1\rceil$, where $\lfloor x\rfloor$ ($\lceil x\rceil$) is the largest (smallest) integer from the interval $(-\infty, x]$ ($[x, +\infty)$. Thus, this problem can be solved effectively if we can efficiently calculate an approximate value of some number from $[\tau_0, \tau_1]$. This is the case when, for example, the norm $\|\cdot\|$ is defined by the scalar product in $\mathbb{R}^n$, in this case $\tau_0 = \tau_1 = \frac{(a,b)}{(a,a)} = \frac{(a,b)}{[a]}$.

As noted in [8], if we know a not very large interval of real numbers containing $[\tau_0, \tau_1]$, then the above problem can be effectively solved using the binary search algorithm. It is also proved there that $\mu(a, b) \in [1, 2\|b\|/\|a\|)$ provided $\|b\| > \|b - a\|$.

3 On the generalized Gauss reduction algorithm

In this section we will give some notes about the generalized Gaussian reduction algorithm, which allows to find a minimal lattice basis from an initial ordered basis. This algorithm is described in sufficient detail in [8] and [2].

First, we will describe the introductory part of the algorithm, during which we obtain from a given ordered basis, in the worst case, some well-ordered basis, and in the best case, a solution to our problem, i.e. we find some reduced basis.

Let us assume that an ordered basis $\langle a, b\rangle$ is given. Recall that by the definition of an ordered basis we have $\|a\| \le \|b\|$ and $\|a - b\| \le \|a + b\|$. Let us consider possible cases:

(1)$\|b\| \le \|a - b\|$.

In this case, the basis $\langle a, b\rangle$ is reduced and our problem is solved.

(2) $\|a - b\| < \|a\|$.

If $\|a\| = \|b\|$, then $\langle a - b, a\rangle$ is a reduced basis and our problem is solved again:

$$\|a - b\| < \|a\| = \| - b\| = \|(a - b) - a\|$$

$$= 2\|a\| - \|b\| \le \|2a - b\| = \|(a - b) + a\|.$$

If $\|a\| < \|b\|$, then $\langle b - a, b\rangle$ is a well-ordered basis:

$$\|b - a\| = \|a - b\| < \|a\| = \| - a\| = \|(b - a) - b\|$$

$$< \|b\| < 2\|b\| - \|a\| \le \|2b - a\| = \|(b - a) + b\|.$$

(3) $\|a\| \le \|a-b\| < \|b\|$.
In this case, the basis $\langle a, b\rangle$ is well-ordered. □

We would like to evaluate the complexity of the generalized Gaussian algorithm, so we must consider worst-case scenarios in all stages of the algorithm. We assume that having received an ordered basis at the input, after the introductory part of the algorithm described above, we obtain a well-ordered basis at the output. The time spent on the introductory part will be short, since the main operations in it are to compare the lengths of some specific vectors.

We move on to describe the next, main stage of the algorithm, which consists of cyclically repeating the same procedure. Let us assume that before the start of this stage we have a well-ordered basis $\langle a, b\rangle$. A cyclically repeated procedure updates this basis as follows. First, using the oracle described in section 2, we find $\mu = \mu(a, b)$ and consider the basis consisting of the vectors a and $b - \mu a$. We correct the second vector of this basis, multiplying it by $\varepsilon \in \{-1, +1\}$ so that the sum of vectors a and $\varepsilon(b-\mu a)$ has a norm no less than the norm of their difference. Further,

(1) if $\|a\| \le \|b-\mu a\|$, then $\langle a, \varepsilon(b-\mu a)\rangle$ is a reduced basis and the algorithm terminates,

(2) if $\|b-\mu a\| < \|a\|$, then according to the analysis from the introductory part of the algorithm, the ordered basis $\langle \varepsilon(b-\mu a), a\rangle$ will be either reduced or well-ordered, since case (2) from the introductory part of the algorithm for the basis $\langle \varepsilon(b-\mu a), a\rangle$ is impossible.

So, the procedure, having obtained a well-ordered basis $\langle a, b\rangle$ at the input, produces a new well-ordered basis $\langle \varepsilon(b-\mu a), a\rangle$ at the output (in an unsuccessful scenario). Since each time the procedure is executed, the length of one of the vectors of the well-ordered basis decreases, after a certain finite number of steps the procedure, due to the discreteness of the lattice, will produce the reduced basis and the algorithm completes its work.

Finally, let us move on to estimating the number of repetitions of the procedure of the main stage of the algorithm. Let k be the number of repetitions and $\langle a, b\rangle = \langle a_k,\ a_{k+1}\rangle$ be a well-ordered basis at the beginning of the stage. Let us represent the results of cyclic procedures as a sequence of ordered bases

$$\langle a_k,\ a_{k+1}\rangle,\ \langle a_{k-1},\ a_k\rangle, \ldots, \left\langle a_1,\ a_2^0\right\rangle,$$

where $\langle a_1,\ a_2^0\rangle$ is a reduced basis. Then the following lemma, proven in [8], is true.

Lemma 3.1. *For $i \ge 3$, the inequality $2\|a_i\| < \|a_{i+1}\|$ holds.*

The notation a_2^0 is introduced due to the following circumstances. There are two possibilities for completing the algorithm by obtaining the reduced basis $\langle a_1,\ a_2^0\rangle$ from the well-ordered basis $\langle a_2,\ a_3\rangle$. It may well be $a_1 = \varepsilon(a_3 - \mu a_2)$, $a_2^0 = a_2$, if case (2) occurred during the last update of the basis by the main stage procedure. But there could also be case (1), then $a_1 = a_2$, $a_2^0 = \varepsilon(a_3 - \mu a_2)$.

Note that in any case we have $\|a_2^0\| = \lambda_2 < \|a_3\|$. Therefore, we get

$$\frac{\|b\|}{\lambda_2} = \frac{\|a_{k+1}\|}{\lambda_2} > \frac{2^{k-2}\|a_3\|}{\lambda_2} > 2^{k-2},$$

which implies the estimate $k < 2 + \log_2\left(\frac{\|b\|}{\lambda_2}\right)$. □

Finally, the last remark concerns the minimality intervals of the functions $l(\tau) = \|b - \tau a\|$, $\tau \in \mathbb{R}$, for well-ordered bases $\langle a, b\rangle$. It is clear that long minimality intervals can significantly reduce the running time of the Gaussian reduction algorithm. Without going into complex computational analysis, we will limit ourselves to just one simple example confirming this fact.

Lemma 3.2. *If the minimality interval of the function $l(\tau) = \|b - \tau a\|$, $\tau \in \mathbb{R}$, for the basis $\langle a, b\rangle$ contains an integer n_0, then $\|b - n_0 a\| = \lambda_1$ or $\|a\| = \lambda_1$.*

Proof. So, assume that $\|b - n_0 a\| = \mu_0 \stackrel{def}{=} \min\{l(\tau) : \tau \in \mathbb{R}\}$. On the other hand, for some $\alpha, \beta \in \mathbb{Z}$ we have $\|\alpha a + \beta b\| = \lambda_1$. If $\beta = 0$, then obviously $|\alpha| = 1$ and $\|a\| = \lambda_1$. Therefore, let us assume that $\beta \neq 0$. Then, $\lambda_1 = |\beta| \cdot \|\frac{\alpha}{\beta} a + b\| = |\beta| \cdot l(-\frac{\alpha}{\beta}) \geq |\beta| \cdot \mu_0 = |\beta| \cdot \|b - n_0 a\|$, which implies $|\beta| = 1$ and $\|b - n_0 a\| = \lambda_1$. □

Thus, if during the execution of the procedure of the main stage of the algorithm, a well-ordered basis $\langle a, b \rangle$ is given as input, satisfying the condition of Lemma 3.2, then at the output we obtain an ordered basis $\langle c, d \rangle$ with $\|c\| = \lambda_1$, and, if $\langle c, d \rangle$ is not a reduced basis, then at the next step the result of the procedure falling into case (1) will be a reduced basis. Therefore, the number k of repetitions of the procedure will not exceed 2.

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NIKOL'SKII-BESOV SPACES WITH A DOMINANT MIXED DERIVATIVE AND WITH A MIXED METRIC: INTERPOLATION PROPERTIES, EMBEDDING THEOREMS, TRACE AND EXTENSION THEOREMS

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Abstract. In this work, we define Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric. The interpolation properties of these spaces with respect to the anisotropic interpolation method are studied, sharp embedding theorems of different metrics are proved, and sharp trace and extension theorems are proved.

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1 Introduction

The theory of embeddings of spaces of differentiable functions originated in the work of S.L. Sobolev [26]. This theory studies important connections between differential (smoothness) properties of functions in various metrics. Further development of this theory is assosicated with new classes of function spaces introduced by S.M. Nikol'skii, O.V. Besov, P.I. Lizorkin, H. Triebel, and others. This development was driven both by intrinsic questions of the theory and by applications to the theory of boundary value problems of mathematical physics, approximation theory, and other areas of analysis (see, for example, monographs [13, 17, 22, 28]).

In the 1960s, the study of spaces with a dominant mixed derivative was initiated in the works of S.M. Nikol'skii [23], A.D. Dzhabrailov [20], and T.I. Amanov [3]. Further research of these spaces in connection with the theory of embeddings, interpolation, and approximation theory, is associated with the works of A.P. Uninskii, O.V. Besov, V.N. Temlyakov, E.D. Nursultanov, D.B. Bazarkhanov, A.S. Romanyuk, G.A. Akishev, K.A. Bekmaganbetov, Ye. Toleugazy, and others (see, for example, [29, 30, 15, 16, 27, 24, 5, 6, 25, 1, 2, 12]).

In Section 2, we define Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric, and study some elementary embedding properties. In Section 3, we study interpolation properties of these spaces with respect to the anisotropic interpolation method. In Section 4, we prove sharp embedding theorems of different metrics for the introduced spaces and anisotropic Lorentz spaces. In Section 5, we prove trace and extension theorems for the spaces under consideration.

2 Main definitions

By generalizing the construction in [22, Chapter 8], we define the Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric $S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$.

Let $\mathbf{1} \leq \mathbf{p} = (p_1, \dots, p_n) \leq \infty$. The Lebesgue space with a mixed metric $L_{\mathbf{p}}(\mathbb{R}^n)$ is the set of measurable functions for which the following norm is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{R}^n)} = \left(\int_{-\infty}^{\infty} \left(\dots \left(\int_{-\infty}^{\infty} |f(x_1, \dots, x_n)|^{p_1} \, dx_1 \right)^{p_2/p_1} \dots \right)^{p_n/p_{n-1}} dx_n \right)^{1/p_n}.$$

Here, for $p = \infty$ the expression $\left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p}$ is understood as $\operatorname{esssup}_{t \in \mathbb{R}} |f(t)|$.

A generalized function f is called regular in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$ if for some $\rho_0 > 0$

$$I_{\rho_0} f = F \in L_{\mathbf{p}}(\mathbb{R}^n),$$

where

$$I_{\rho_0} f = \mathfrak{F}^{-1} \left(\left(1 + |\xi|^2\right)^{-\rho_0/2} \mathfrak{F}(f) \right),$$

and $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ are the direct and inverse Fourier transforms, respectively.

Let f be a regular function in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$. A regular expansion of a function f in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$ over the Vallee-Poussin sums is the following representation

$$f = \sum_{\mathbf{s} \in \mathbb{Z}_+^n} Q_{\mathbf{s}}(f),$$

where

$$Q_{\mathbf{s}}(f) = \frac{1}{\pi^n} I_{-\rho} \left(\prod_{i=1}^{n} \left(V_{2^{s_i}}(x_i) - V_{\left[2^{s_i - 1}\right]}(x_i) \right) * I_\rho f \right),$$

where $\rho > 0$ is sufficiently large so that $I_\rho f \in L_{\mathbf{p}}(\mathbb{R}^n)$, $V_M(t) = \dfrac{1}{M} \displaystyle\int_M^{2M} \frac{\sin \lambda t}{t} d\lambda$ is an analogue of the Vallee-Poussin kernel for the parameter $M > 0$ and $V_0(t) \equiv 0$.

Let further $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $\mathbf{1} \leq \mathbf{q} = (q_1, \dots, q_n) \leq \infty$. The Nikol'skii-Besov space $S_{\mathbf{p}}^{\alpha \mathbf{q}} B(\mathbb{R}^n)$ with a dominant mixed derivative and with a mixed metric is the set of regular in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$ functions f for which the following norm is finite

$$\|f\|_{S_{\mathbf{p}}^{\alpha \mathbf{q}} B(\mathbb{R}^n)} = \left\| \left\{ 2^{(\alpha, \mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\mathbf{s} \in \mathbb{Z}_+^{\mathbf{n}}} \right\|_{l_{\mathbf{q}}},$$

where $(\alpha, \mathbf{s}) = \displaystyle\sum_{j=1}^{n} \alpha_j s_j$ is the inner product and $\| \cdot \|_{l_{\mathbf{q}}}$ is the norm of the discrete Lebesgue space $l_{\mathbf{q}}$ with a mixed metric.

Remark 1. In the case in which $\alpha = (\alpha_1, \dots, \alpha_n) > \mathbf{0}$, these spaces with the parameter $\mathbf{q} = (\infty, \dots, \infty)$ were introduced and studied in the works [29, 30]. The case of $\mathbf{p} = (p, \dots, p)$ and $\mathbf{q} = (q, \dots, q)$ was considered in the works [23, 20, 3]. Periodic analogues of these spaces were studied in the series of work by K.A. Bekmaganbetov, K.Ye. Kervenev and Ye. Toleugazy [7, 8, 9].

The following lemma shows some elementary embeddings of Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric.

Lemma 2.1. a) *Let* $\mathbf{1} \leq \mathbf{q_0} = (q_1^0, \dots, q_n^0) \leq \mathbf{q_1} = (q_1^1, \dots, q_n^1) \leq \infty$, *then*

$$S_{\mathbf{p}}^{\alpha \mathbf{q_0}} B(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha \mathbf{q_1}} B(\mathbb{R}^n).$$

b) *Let* $\alpha_{\mathbf{0}} = (\alpha_1^0, \dots, \alpha_n^0) < \alpha_{\mathbf{1}} = (\alpha_1^1, \dots, \alpha_n^1)$ *and* $\mathbf{1} \leq \mathbf{q_0} = (q_1^0, \dots, q_n^0), \mathbf{q_1} = (q_1^1, \dots, q_n^1) \leq \infty$, *then*

$$S_{\mathbf{p}}^{\alpha_{\mathbf{1}} \mathbf{q_1}} B(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha_{\mathbf{0}} \mathbf{q_0}} B(\mathbb{R}^n).$$

Proof. The proof of statement a) follows from Jensen's inequality.

Let us prove statement b). According to paragraph a), for $\alpha_{\mathbf{0}} < \alpha_{\mathbf{1}}$ it suffices to prove the embedding

$$S_{\mathbf{p}}^{\alpha_{\mathbf{1}}\infty}B(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha_{\mathbf{0}}\mathbf{1}}B(\mathbb{R}^n). \tag{2.1}$$

We have

$$\|f\|_{S_{\mathbf{p}}^{\alpha_{\mathbf{0}}\mathbf{1}}B(\mathbb{R}^n)} = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} 2^{(\alpha_0,\mathbf{s})}\|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)}$$

$$\leq \sup_{\mathbf{s}\in\mathbb{Z}_+^n} 2^{(\alpha_1,\mathbf{s})}\|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \sum_{\mathbf{s}\in\mathbb{Z}_+^n} 2^{(\alpha_0-\alpha_1,\mathbf{s})} = C_1\|f\|_{S_{\mathbf{p}}^{\alpha_{\mathbf{1}}\infty}B(\mathbb{R}^n)}.$$

This inequality shows that embedding (2.1) holds. □

3 Interpolation

Let us give the definition of the anisotropic interpolation method. Let $E = \{0,1\}^n$, $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}$ be a family of Banach spaces that are subspaces of some linear Hausdorff space. This family $\mathbf{A}$ is called a compatible family of Banach spaces (see [10, 21, 24]). For $\mathbf{t} \in \mathbb{R}_+^n$, we define the functional

$$K(\mathbf{t}, a; \mathbf{A}) = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^\varepsilon \|a_\varepsilon\|_{A_\varepsilon},$$

where a is an element of the space $\sum_{\varepsilon\in E} A_\varepsilon$ and $\mathbf{t}^\varepsilon = t_1^{\varepsilon_1}\cdot\ldots\cdot t_n^{\varepsilon_n}$.

Let $\mathbf{0} < \theta = (\theta_1,\ldots,\theta_n) < \mathbf{1}$ and $\mathbf{0} < \mathbf{r} = (r_1,\ldots,r_n) \leq \infty$. Let $\mathbf{A}_{\theta\mathbf{r}} = (A_\varepsilon; \varepsilon\in E)_{\theta\mathbf{r}}$ denote the linear subspace of the space $\sum_{\varepsilon\in E} A_\varepsilon$ such that

$$\|a\|_{\mathbf{A}_{\theta\mathbf{r}}} =$$

$$= \left(\int_0^\infty \left(\ldots\left(\int_0^\infty \left(t_1^{-\theta_1}\ldots t_n^{-\theta_n} K(\mathbf{t}, a; \mathbf{A})\right)^{r_1} \frac{dt_1}{t_1}\right)^{r_2/r_1} \ldots\right)^{r_n/r_{n-1}} \frac{dt_n}{t_n}\right)^{1/r_n} < \infty.$$

Lemma 3.1 ([4, 24]). *Let $\mathbf{0} < \theta < \mathbf{1}$, $\mathbf{0} < \mathbf{r} \leq \infty$, and let $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}, \mathbf{B} = \{B_\varepsilon\}_{\varepsilon\in E}$ be two compatible families of Banach spaces. If there are two vectors $\mathbf{M_0} = (M_1^0,\ldots,M_n^0), \mathbf{M_1} = (M_1^1,\ldots,M_n^1)$ with positive components such that for a linear operator T holds $T : \mathbf{A}_\varepsilon \to \mathbf{B}_\varepsilon$ with the operator norm bounded by $C_\varepsilon \prod_{i=1}^n M_i^{\varepsilon_i}$ for any $\varepsilon \in E$, where $C_\epsilon > 0$ is independent of $M_i^{\varepsilon_i}$, $i = 1,\ldots,n$, then*

$$T : \mathbf{A}_{\theta\mathbf{r}} \to \mathbf{B}_{\theta\mathbf{r}},$$

with the norm

$$\|T\|_{\mathbf{A}_{\theta\mathbf{r}}\to\mathbf{B}_{\theta\mathbf{r}}} \leq \max_{\varepsilon\in E} C_\varepsilon \prod_{i=1}^n \left(M_i^0\right)^{1-\theta_i} \left(M_i^1\right)^{\theta_i}.$$

Let multi-indices $\mathbf{p} = (p_1,\ldots,p_n)$, $\mathbf{r} = (r_1,\ldots,r_n)$ be such that if $1 \leq p_i < \infty$, then $1 \leq r_i \leq \infty$, and if $p_i = \infty$, then $r_i = \infty$ $(i = 1,\ldots,n)$.

The anisotropic Lorentz space $L_{\mathbf{pr}}(\mathbb{R}^n)$ is the set of all functions $f(\mathbf{x}) = f(x_1,\ldots,x_n)$ such that

$$\|f\|_{L_{\mathbf{pr}}(\mathbb{R}^n)} =$$

$$= \left(\int_0^\infty \left(\dots \left(\int_0^\infty \left(t_1^{1/p_1} \dots t_n^{1/p_n} f^{*_1,\dots,*_n}(t_1,\dots,t_n) \right)^{r_1} \frac{dt_1}{t_1} \right)^{r_2/r_1} \dots \right)^{r_n/r_{n-1}} \frac{dt_n}{t_n} \right)^{1/r_n} < \infty,$$

where $f^*(\mathbf{t}) = f^{*_1,\dots,*_n}(t_1,\dots,t_n)$ is the repeated non-increasing rearrangement of the function f (see [18]).

Let us denote $\mathbf{b}_\varepsilon = (b_1^{\varepsilon_1},\dots,b_n^{\varepsilon_n})$ for multi-indices $\mathbf{b}_0 = (b_1^0,\dots,b_n^0), \mathbf{b}_1 = (b_1^1,\dots,b_n^1)$, and $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in E$.

Lemma 3.2 ([24]). *Let* $\mathbf{1} \le \mathbf{p_0} = (p_1^0,\dots,p_n^0) \ne \mathbf{p_1} = (p_1^1,\dots,p_n^1) \le \infty$. *Then for* $\mathbf{0} < \theta = (\theta_1,\dots,\theta_n) < \mathbf{1}$ *and* $\mathbf{1} \le \mathbf{r} = (r_1,\dots,r_n) \le \infty$ *holds*

$$\left(L_{\mathbf{p}_\varepsilon}(\mathbb{R}^n); \varepsilon \in E\right)_{\theta\mathbf{r}} = L_{\mathbf{pr}}(\mathbb{R}^n),$$

where $\mathbf{1}/\mathbf{p} = (\mathbf{1}-\theta)/\mathbf{p_0} + \theta/\mathbf{p_1}$.

Let $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$ and $\mathbf{1} \le \mathbf{q} = (\mathbf{q_1},\dots,\mathbf{q_n}) \le \infty$. We will denote by $l_{\mathbf{q}}^\alpha(A)$ the set of multi-sequences $\{a_\mathbf{k}\}_{\mathbf{k}\in\mathbb{Z}^n}$ with values in a Banach space A for which the following norm is finite:

$$\|a\|_{l_{\mathbf{q}}^\alpha(A)} = \left(\sum_{\mathbf{k}\in\mathbb{Z}^n} \left(2^{(\alpha,\mathbf{k})} \|a_\mathbf{k}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1}/\mathbf{q}}.$$

Remark 2. The norm of the space $S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$ can be written as

$$\|f\|_{S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)} = \left\| \{Q_\mathbf{s}(f)\}_{\mathbf{s}\in\mathbb{Z}_+^\mathbf{n}} \right\|_{l_{\mathbf{q}}^\alpha(L_\mathbf{p}(\mathbb{R}^n))}.$$

We will need this form of the norm when describing interpolation properties of the spaces $S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$.

Lemma 3.3 ([7]). *Let* $\alpha_0 = (\alpha_1^0,\dots,\alpha_n^0) \ne \alpha_1 = (\alpha_1^1,\dots,\alpha_n^1)$, $\mathbf{1} \le \mathbf{q_0} = (q_1^0,\dots,q_n^0), \mathbf{q_1} = (q_1^1,\dots,q_n^1) \le \infty$. *Then for* $\mathbf{0} < \theta = (\theta_1,\dots,\theta_n) < \mathbf{1}$, $\mathbf{1} \le \mathbf{q} = (q_1,\dots,q_n) \le \infty$

$$\left(l_{\mathbf{q}_\varepsilon}^{\alpha_\varepsilon}(A); \varepsilon \in E \right)_{\theta\mathbf{q}} = l_{\mathbf{q}}^\alpha(A),$$

where $\alpha = (\mathbf{1}-\theta)\alpha_\mathbf{0} + \theta\alpha_\mathbf{1}$.

Definition 1. Let A and B be Banach spaces. An operator $R \in L(A,B)$ is called a *retraction* if there exists an operator $S \in L(B,A)$ such that

$$RS = E \quad \text{(the identity operator in } L(B,B)).$$

In this case, the operator S is called a *coretraction* (corresponding to R).

Lemma 3.4. *Let* $-\infty < \alpha = (\alpha_1,\dots,\alpha_n) < \infty$, $\mathbf{1} \le \mathbf{p} = (p_1,\dots,p_n) < \infty$, *and* $\mathbf{1} \le \mathbf{q} = (q_1,\dots,q_n) \le \infty$. *Then the space* $S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$ *is a retraction of the space* $l_{\mathbf{q}}^\alpha(L_\mathbf{p}(\mathbb{R}^n))$.

Proof. First step. For a function $f \in S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$ we define the operator S by

$$Sf = \{Q_\mathbf{s}(f)\}_{\mathbf{s}\in\mathbb{Z}_+^n}.$$

Therefore, according to the definition, we have

$$\|Sf\|_{l_{\mathbf{q}}^\alpha(L_\mathbf{p}(\mathbb{R}^n))} = \|\{Q_\mathbf{s}(f)\}\|_{l_{\mathbf{q}}^\alpha(L_\mathbf{p}(\mathbb{R}^n))} = \|f\|_{S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)},$$

which means that the S-property is satisfied.

Second step. For a sequence $G = \{g_{\mathbf{s}}\}_{\mathbf{s}\in\mathbb{Z}_+^n}$, we define the operator R by

$$RG = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} U_{\mathbf{s}} * g_{\mathbf{s}},$$

where

$$U_{\mathbf{s}}(\mathbf{x}) = \frac{1}{\pi^n}\prod_{i=1}^n \left(V_{2^{s_i+1}}(x_i) - V_{\left[2^{s_i-2}\right]}(x_i)\right).$$

Since $V_M \in L_1(\mathbb{R})$, we obtain

$$\|U_{\mathbf{s}} * g\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \le C_2 \|g\|_{L_{\mathbf{p}}(\mathbb{R}^n)},$$

where C_2 is an absolute constant, and then

$$\|RG\|_{S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)} = \|\{Q_{\mathbf{s}}(U_{\mathbf{s}} * g_{\mathbf{s}})\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{R}^n))} = \|\{Q_{\mathbf{s}}(g_{\mathbf{s}})\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{R}^n))} \le$$

$$\le C_2 \|\{g_{\mathbf{s}}\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{R}^n))} = C_2 \|G\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{R}^n))}.$$

The last inequality means that the R-property holds.

Third step. Let us show that $RS = E$. Indeed,

$$RSf = R(\{Q_{\mathbf{s}}(f)\}) = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} U_{\mathbf{s}} * Q_{\mathbf{s}}(f) = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} Q_{\mathbf{s}}(f) = f.$$

□

Theorem 3.1. *Let $\mathbf{1} \le \mathbf{p} = (p_1,\dots,p_n) < \infty$, $\alpha_{\mathbf{0}} = (\alpha_1^0,\dots,\alpha_n^0) \ne \alpha_{\mathbf{1}} = (\alpha_1^1,\dots,\alpha_n^1)$, $\mathbf{1} \le \mathbf{q_0} = (q_1^0,\dots,q_n^0), \mathbf{q_1} = (q_1^1,\dots,q_n^1) \le \infty$, $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in E$. Then for $\mathbf{0} < \theta = (\theta_1,\dots,\theta_n) < \mathbf{1}$ and $\mathbf{1} \le \mathbf{q} = (q_1,\dots,q_n) \le \infty$*

$$\left(S_{\mathbf{p}}^{\alpha_\varepsilon\mathbf{q}_\varepsilon}B(\mathbb{R}^n); \varepsilon \in E\right)_{\theta\mathbf{q}} = S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n),$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

Proof. The proof of the theorem follows by Lemmas 3.3 and 3.4. □

4 Embedding theorems

In this section, the sharp embedding theorems for Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric and for anisotropic Lorentz spaces are proved.

Lemma 4.1 (Inequality of different metrics, [22]). *Let $Q_{\mathbf{s}}(\mathbf{x})$ be an entire function of exponential type of order $\mathbf{s} = (s_1,\dots,s_n)$ by $\mathbf{x} = (x_1,\dots,x_n)$. Then for $\mathbf{1} \le \mathbf{p}_0 = (p_1^0,\dots,p_n^0) < \mathbf{p}_1 = (p_1^1,\dots,p_n^1) < \infty$ holds*

$$\|Q_{\mathbf{s}}\|_{L_{\mathbf{p}_1}(\mathbb{R}^n)} \le C_3 \prod_{i=1}^n s_i^{1/p_i^0 - 1/p_i^1} \|Q_{\mathbf{s}}\|_{L_{\mathbf{p}_0}(\mathbb{R}^n)},$$

where C_3 is a positive constant independent of $\mathbf{s}$.

Theorem 4.1. *Let $-\infty < \alpha_0 = (\alpha_1^0,\dots,\alpha_n^0) \le \alpha_1 = (\alpha_1^1,\dots,\alpha_n^1) < \infty$, $\mathbf{1} \le \tau = (\tau_1,\dots,\tau_n) \le \infty$, and $\mathbf{1} \le \mathbf{p}_0 = (p_1^0,\dots,p_n^0), \mathbf{p}_1 = (p_1^1,\dots,p_n^1) < \infty$. Then*

$$S_{\mathbf{p_1}}^{\alpha_1\tau}B(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p_0}}^{\alpha_0\tau}B(\mathbb{R}^n)$$

for $\alpha_0 - 1/\mathbf{p}_0 = \alpha_1 - 1/\mathbf{p}_1$.

Proof. Let $f \in S^{\alpha_1 \tau}_{\mathbf{p}_1} B(\mathbb{R}^n)$. Then, according to the inequality of different metrics (Lemma 4.1), we obtain

$$\|f\|_{S^{\alpha_0 \tau}_{\mathbf{p}_0} B(\mathbb{R}^n)} = \left\| \left\{ 2^{(\alpha_0, \mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}_0}(\mathbb{R}^n)} \right\} \right\|_{l_\tau}$$
$$\leq C_3 \left\| \left\{ 2^{(\alpha_0 + 1/\mathbf{p}_1 - 1/\mathbf{p}_0, \mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}_1}(\mathbb{R}^n)} \right\} \right\|_{l_\tau}$$
$$= C_3 \left\| \left\{ 2^{(\alpha_1, \mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}_1}(\mathbb{R}^n)} \right\} \right\|_{l_\tau} = C_3 \|f\|_{S^{\alpha_1 \tau}_{\mathbf{p}_1} B(\mathbb{R}^n)}.$$

□

Theorem 4.2. *Let* $\mathbf{1} \leq \mathbf{p} = (p_1, \ldots, p_n) < \mathbf{q} = (q_1, \ldots, q_n) < \infty$ *and* $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. *Then*

$$S^{\alpha \tau}_{\mathbf{p}} B(\mathbb{R}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{R}^n)$$

for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$.

Proof. According to Minkowski's inequality and the inequality of different metrics (Lemma 4.1), we obtain

$$\|f\|_{L_{\mathbf{q}}(\mathbb{R}^n)} = \left\| \sum_{\mathbf{s}=\mathbf{0}}^{\infty} Q_{\mathbf{s}}(f) \right\|_{L_{\mathbf{q}}(\mathbb{R}^n)}$$

$$\leq \sum_{\mathbf{s}=\mathbf{0}}^{\infty} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{q}}(\mathbb{R}^n)} \leq C_3 \sum_{\mathbf{s}=\mathbf{0}}^{\infty} 2^{(1/\mathbf{p} - 1/\mathbf{q}, \mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} = C_3 \|f\|_{S^{\alpha \mathbf{1}}_{\mathbf{p}} B(\mathbb{R}^n)},$$

where $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$.

Therefore, for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$ we get

$$S^{\alpha \mathbf{1}}_{\mathbf{p}} B(\mathbb{R}^n) \hookrightarrow L_{\mathbf{q}}(\mathbb{R}^n).$$

Let us fix $\mathbf{p} = (p_1, \ldots, p_n)$ and let us choose $\alpha_i = (\alpha^i_1, \ldots, \alpha^i_n)$ and $\mathbf{q}_i = (q^i_1, \ldots, q^i_n)$ such that $\alpha^i_j = 1/p_j - 1/q^i_j$, where $i = 0, 1$ and $j = 1, \ldots, n$. Then for every $\varepsilon \in E$ we have

$$S^{\alpha_\varepsilon \mathbf{1}}_{\mathbf{p}} B(\mathbb{R}^n) \hookrightarrow L_{\mathbf{q}_\varepsilon}(\mathbb{R}^n).$$

According to Lemma 3.2 and Theorem 3.1 we obtain

$$\left(S^{\alpha_\varepsilon \mathbf{1}}_{\mathbf{p}} B(\mathbb{R}^n); \varepsilon \in E \right)_{\theta \tau} \hookrightarrow \left(L_{\mathbf{q}_\varepsilon}(\mathbb{R}^n); \varepsilon \in E \right)_{\theta \tau}$$

or

$$S^{\alpha \tau}_{\mathbf{p}} B(\mathbb{R}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{R}^n),$$

where $\alpha = (1-\theta)\alpha_0 + \theta \alpha_1$, $1/\mathbf{q} = (1-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1$.

Let us check the relationship between the parameters α, $\mathbf{p}$ and $\mathbf{q}$

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1 = (1-\theta)\left(1/\mathbf{p} - 1/\mathbf{q}_0\right) + \theta\left(1/\mathbf{p} - 1/\mathbf{q}_1\right) =$$

$$= ((1-\theta)/\mathbf{p} + \theta/\mathbf{p}) - ((1-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1) = 1/\mathbf{p} - 1/\mathbf{q}.$$

□

Theorem 4.3. *Let* $\mathbf{1} < \mathbf{q} = (q_1, \ldots, q_n) < \mathbf{p} = (p_1, \ldots, p_n) < \infty$ *and* $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. *Then*

$$L_{\mathbf{q}\tau}(\mathbb{R}^n) \hookrightarrow S^{\alpha \tau}_{\mathbf{p}} B(\mathbb{R}^n),$$

where $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$.

Proof. According to the inequality of different metrics (Lemma 4.1) since $V_M \in L_1(\mathbb{R})$, we obtain

$$\|f\|_{S^{\alpha\infty}_{\mathbf{p}}B(\mathbb{R}^n)} = \sup_{\mathbf{s}\geq\mathbf{0}} 2^{(\alpha,\mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \leq C_3 \sup_{\mathbf{s}\geq\mathbf{0}} 2^{(\alpha+1/\mathbf{q}-1/\mathbf{p},\mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{q}}(\mathbb{R}^n)}$$

$$= C_3 \sup_{\mathbf{s}\geq\mathbf{0}} \left\| \frac{1}{\pi^n} \prod_{i=1}^{n} \left(V_{2^{s_i}}(\cdot) - V_{\left[2^{s_i-1}\right]}(\cdot) \right) * f \right\|_{L_{\mathbf{q}}(\mathbb{R}^n)} \leq C_4 \|f\|_{L_{\mathbf{q}}(\mathbb{R}^n)},$$

for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$, where $C_4 > 0$ is independent of f.

Thus, for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$ we have

$$L_{\mathbf{q}}(\mathbb{R}^n) \hookrightarrow S^{\alpha\infty}_{\mathbf{p}}B(\mathbb{R}^n).$$

Let us fix $\mathbf{p} = (p_1, \ldots, p_n)$ and let us choose parameters $\alpha_i = (\alpha^i_1, \ldots, \alpha^i_n)$ and $\mathbf{q}_i = (q^i_1, \ldots, q^i_n)$ such that $\alpha^i_j = 1/p_j - 1/q^i_j$, where $i = 0, 1$ and $j = 1, \ldots, n$. Then for every $\varepsilon \in E$ we obtain

$$L_{\mathbf{q}_\varepsilon}(\mathbb{R}^n) \hookrightarrow S^{\alpha_\varepsilon\infty}_{\mathbf{p}}B(\mathbb{R}^n).$$

According to Lemma 3.2 and Theorem 3.1 we obtain

$$\left(L_{\mathbf{q}_\varepsilon}(\mathbb{R}^n); \varepsilon \in E\right)_{\theta\tau} \hookrightarrow \left(S^{\alpha_\varepsilon\infty}_{\mathbf{p}}B(\mathbb{R}^n); \varepsilon \in E\right)_{\theta\tau}$$

or

$$L_{\mathbf{q}\tau}(\mathbb{R}^n) \hookrightarrow S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n),$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $1/\mathbf{q} = (1-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1$.

Let us check the relationship between the parameters α, $\mathbf{p}$ and $\mathbf{q}$

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1 = (1-\theta)\left(1/\mathbf{p} - 1/\mathbf{q}_0\right) + \theta\left(1/\mathbf{p} - 1/\mathbf{q}_1\right) =$$

$$= \left((1-\theta)/\mathbf{p} + \theta/\mathbf{p}\right) - \left((1-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1\right) = 1/\mathbf{p} - 1/\mathbf{q}.$$

□

Remark 3. It is possible to show that the conditions of Theorems 4.1 – 4.3 are sharp. The proof of these facts can be carried out by analogy with the corresponding proofs in the papers [8, 11].

5 The theorems about trace and extension

In this section, trace and extension theorems for functions belonging to Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric are proved.

Let $1 \leq m < n$. For $\mathbf{a} = (a_1, \ldots, a_m, a_{m+1}, \ldots, a_n)$, we denote $\bar{\mathbf{a}} = (a_1, \ldots, a_m)$ and $\tilde{\mathbf{a}} = (a_{m+1}, \ldots, a_n)$.

Lemma 5.1 (Inequality of different dimensions, [22]). *Let* $\mathbf{1} \leq \mathbf{p} = (p_1, \ldots, p_m, p_{m+1}, \ldots, p_n) < \infty$ *and let* $Q_{\mathbf{s}}(\mathbf{x})$ *be an entire function of exponential type of order* $\mathbf{s} = (s_1, \ldots, s_m, s_{m+1}, \ldots, s_n)$ *by* $\mathbf{x} = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$*. Then for an arbitrary fixed point* $\tilde{\mathbf{x}} \in \mathbb{R}^{n-m}$ *holds the inequality*

$$\|Q_{\mathbf{s}}(\cdot, \tilde{\mathbf{x}})\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \leq C_5 \prod_{i=m+1}^{n} s_i^{1/p_i} \|Q_{\mathbf{s}}\|_{L_{\mathbf{p}}(\mathbb{R}^n)},$$

where C_5 *is a positive constant independent of* $\mathbf{s}$ *and* $\tilde{\mathbf{x}}$*.*

Theorem 5.1. *Let* $\mathbf{1} \le \mathbf{p} = (p_1, \dots, p_m, p_{m+1}, \dots, p_n) < \infty$, $\alpha = (\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_n)$, *and* $\mathbf{1} \le \tau = (\tau_1, \dots, \tau_m, \tau_{m+1}, \dots, \tau_n) \le \infty$ *with* $\alpha_i = 1/p_i$ *and* $\tau_i = 1$ *for* $i = m+1, \dots, n$. *Then the trace operator* $T : f \mapsto f(\cdot, \tilde{\mathbf{0}})$ *is well-defined and satisfies*

$$T : S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n) \to S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m).$$

Proof. Fix any $f \in S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n)$. We will show that

$$f \in L_\infty(\mathbb{R}^{n-m}; S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m)) \tag{5.1}$$

and

$$\|f(\cdot, \cdot + \tilde{\mathbf{h}}) - f(\cdot, \cdot)\|_{L_\infty(\mathbb{R}^{n-m}; S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m))} \to 0 \tag{5.2}$$

as $\tilde{\mathbf{h}} \to \tilde{\mathbf{0}}$. By properties (5.1) and (5.2) it follows that f coincides almost everywhere with a unique bounded uniformly continuous function $g : \mathbb{R}^{n-m} \to S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m)$. The trace operator is then well-defined by $Tf := g(\tilde{\mathbf{0}})$.

We now show (5.1). According to the inequality of different dimensions (Lemma 5.1) and Minkowski's inequality, for almost everywhere $\tilde{\mathbf{x}} \in \mathbb{R}^{n-m}$ holds

$$\begin{aligned}
\|f(\cdot, \tilde{\mathbf{x}})\|_{S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m)} &= \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} Q_{(\bar{\mathbf{s}}, \tilde{\mathbf{s}})}(f)(\cdot, \tilde{\mathbf{x}}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}} \\
&\le \left\| \left\{ \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| Q_{(\bar{\mathbf{s}}, \tilde{\mathbf{s}})}(f)(\cdot, \tilde{\mathbf{x}}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}} \\
&\le \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| Q_{(\bar{\mathbf{s}}, \tilde{\mathbf{s}})}(f)(\cdot, \tilde{\mathbf{x}}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}} \\
&= \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| Q_{\mathbf{s}}(f)(\cdot, \tilde{\mathbf{x}}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)} \right\} \right\|_{l_{\bar{\tau}}} \\
&\le C_5 \sum_{\tilde{\mathbf{s}} \in \mathbb{Z}^{n-m}_+} 2^{(\mathbf{1}/\tilde{\mathbf{p}}, \tilde{\mathbf{s}})} \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| Q_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\} \right\|_{l_{\bar{\tau}}} \\
&= C_5 \left\| \left\{ 2^{(\alpha, \mathbf{s})} \left\| Q_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\} \right\|_{l_\tau} = C_5 \|f\|_{S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n)}.
\end{aligned} \tag{5.3}$$

We now show (5.2).

Since $f \in S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n)$, for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$I^2_{N(\varepsilon)} = \left\| \left\{ 2^{(\alpha, \mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\{\mathbf{s} : (\mathbf{s}, \mathbf{1}) > N(\varepsilon)\}} \right\|_{l_\tau} < \frac{\varepsilon}{3C_5}. \tag{5.4}$$

Applying inequality (5.3) and the Minkowski inequality, according to estimate (5.4) we obtain

$$\|f(\cdot, \cdot + \tilde{\mathbf{h}}) - f(\cdot, \cdot)\|_{L_\infty(\mathbb{R}^{n-m}; S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}} B(\mathbb{R}^m))} \le C_5 \|f(\cdot, \cdot + \tilde{\mathbf{h}}) - f(\cdot, \cdot))\|_{S^{\alpha\tau}_{\mathbf{p}} B(\mathbb{R}^n)}$$

$$\le C_5 \left(\left\| \left\{ 2^{(\alpha, \mathbf{s})} \left\| Q_{\mathbf{s}}(f(\cdot, \cdot + \tilde{\mathbf{h}})) - Q_{\mathbf{s}}(f(\cdot, \cdot)) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\{\mathbf{s} : (\mathbf{s}, \mathbf{1}) \le N(\varepsilon)\}} \right\|_{l_\tau} \right.$$

$$+ \left\| \left\{ 2^{(\alpha,\mathbf{s})} \left\| Q_{\mathbf{s}}(f(\cdot,\cdot+\tilde{\mathbf{h}})) - Q_{\mathbf{s}}(f(\cdot,\cdot)) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\{\mathbf{s}:(\mathbf{s},\mathbf{1})>N(\varepsilon)\}} \right\|_{l_\tau} \Bigg)$$

$$\leq C_5 \left(\left\| \left\{ 2^{(\alpha,\mathbf{s})} \left\| Q_{\mathbf{s}}(f(\cdot,\cdot+\tilde{\mathbf{h}})) - Q_{\mathbf{s}}(f(\cdot,\cdot)) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\{\mathbf{s}:(\mathbf{s},\mathbf{1})\leq N(\varepsilon)\}} \right\|_{l_\tau} \right.$$

$$\left. + 2 \left\| \left\{ 2^{(\alpha,\mathbf{s})} \left\| Q_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\{\mathbf{s}:(\mathbf{s},\mathbf{1})>N(\varepsilon)\}} \right\|_{l_\tau} \right)$$

$$= C_5 \left(I^1_{N(\varepsilon)} + 2I^2_{N(\varepsilon)} \right) < C_5 I^1_{N(\varepsilon)} + \frac{2\varepsilon}{3}. \tag{5.5}$$

In order to evaluate $I^1_{N(\varepsilon)}$, we will use the following inequality (see [3])

$$\left\| Q_{\mathbf{s}}(f(\cdot,\cdot+\tilde{\mathbf{h}})) - Q_{\mathbf{s}}(f(\cdot,\cdot)) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \leq C_6 2^{(\tilde{\mathbf{s}},\tilde{\mathbf{1}})} \max_{i=m+1,\ldots,n} |h_i| \left\| Q_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)},$$

where $C_6 > 0$ is independent of f.

Hence, we get

$$I_1(N(\varepsilon)) \leq C_6 2^{N(\varepsilon)} \max_{i=m+1,\ldots,n} |h_i| \left\| \left\{ 2^{(\alpha,\mathbf{s})} \left\| Q_{\mathbf{s}}(f) \right\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\{\mathbf{s}:(\mathbf{s},\mathbf{1})\leq N(\varepsilon)\}} \right\|_{l_\tau}$$

$$\leq C_6 2^{N(\varepsilon)} |\tilde{\mathbf{h}}| \|f\|_{S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n)}.$$

We now choose $|\tilde{\mathbf{h}}| < \dfrac{\varepsilon}{3C_5C_6 2^{N(\varepsilon)} \|f\|_{S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n)}}$, then

$$I^1_{N(\varepsilon)} < \frac{\varepsilon}{3C_5}. \tag{5.6}$$

Plugging estimate (5.6) into (5.5), we obtain

$$\|f(\cdot,\cdot+\tilde{\mathbf{h}}) - f(\cdot,\cdot)\|_{L_\infty(\mathbb{R}^{n-m}; S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}}B(\mathbb{R}^m))} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (5.2) is proved. □

Remark 4. Trace theorems for Nikol'skii-Besov spaces with a dominant mixed derivative were previously obtained in [23, 20, 3, 30] under the condition $\alpha_i > 1/p_i$ for $i = m+1,\ldots,n$. Compared to the above mentioned works, in Theorem 5.1 we allow a weaker condition $\alpha_i = 1/p_i$ with $\tau_i = 1$ (this effect was previously seen, for instance, in [14, 15] and [11]).

Theorem 5.2. *Let $\alpha = (\alpha_1,\ldots,\alpha_m,\alpha_{m+1},\ldots,\alpha_n)$, $\mathbf{1} \leq \tau = (\tau_1,\ldots,\tau_m,\tau_{m+1},\ldots,\tau_n) \leq \infty$ with $\alpha_i = 1/p_i, \tau_i = 1$ for $i = m+1,\ldots,n$, and $\mathbf{1} \leq \mathbf{p} = (p_1,\ldots,p_m,p_{m+1},\ldots,p_n) < \infty$. Then for any function $\varphi(\bar{\mathbf{x}}) \in S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}}B(\mathbb{R}^m)$ there exists a function $f(\bar{\mathbf{x}},\tilde{\mathbf{x}})$ having the following properties:*

$$f \in S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n);$$

$$\|f\|_{S^{\alpha\tau}_{\mathbf{p}}B(\mathbb{R}^n)} \leq C_7 \|\varphi\|_{S^{\bar{\alpha}\bar{\tau}}_{\bar{\mathbf{p}}}B(\mathbb{R}^m)},$$

where $C_7 > 0$ is independent of φ;

$$f(\bar{\mathbf{x}},\tilde{\mathbf{0}}) = \varphi(\bar{\mathbf{x}}),\ \bar{\mathbf{x}} \in \mathbb{R}^m.$$

Proof. Let $\varphi \in S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}}B(\mathbb{R}^m)$. This function can be represented as a series

$$\varphi(\bar{\mathbf{x}}) = \sum_{\bar{\mathbf{s}}=0}^{\infty} Q_{\bar{\mathbf{s}}}(\varphi)(\bar{\mathbf{x}})$$

and

$$\|\varphi\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}}B(\mathbb{R}^m)} = \left\|\left\{2^{(\bar{\alpha},\bar{\mathbf{s}})}\left\|Q_{\bar{\mathbf{s}}}\left(\varphi\right)\right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)}\right\}\right\|_{l_{\bar{\tau}}}.$$

Fix any functions $f_i(x_i) \in C_0^\infty(\mathbb{R})$ with $f_i(0) = 1$, $i = m+1, \dots, n$. We introduce a new function $f(\mathbf{x})$ by

$$f(\bar{\mathbf{x}}, \tilde{\mathbf{x}}) = \varphi(\bar{\mathbf{x}}) \cdot \prod_{i=m+1}^{n} f_i(x_i).$$

Clearly, $Q_{\mathbf{s}}(f) = Q_{\bar{\mathbf{s}}}(\varphi) \prod_{i=m+1}^{n} Q_{s_i}(f_i)$. Therefore,

$$\begin{aligned}\|f\|_{S_{\mathbf{p}}^{\alpha\tau}B(\mathbb{R}^n)} &= \left\|\left\{2^{(\alpha,\mathbf{s})}\left\|Q_{\mathbf{s}}\left(f\right)\right\|_{L_{\mathbf{p}}(\mathbb{R}^n)}\right\}\right\|_{l_\tau} \\ &= \left\|\left\{2^{(\bar{\alpha},\bar{\mathbf{s}})}\left\|Q_{\bar{\mathbf{s}}}\left(\varphi\right)\right\|_{L_{\bar{\mathbf{p}}}(\mathbb{R}^m)}\right\}\right\|_{l_{\bar{\tau}}} \prod_{i=m+1}^{n} \left\|\left\{2^{s_i/p_i}\left\|Q_{s_i}(f_i)\right\|_{L_{p_i}(\mathbb{R})}\right\}\right\|_{l_1} \\ &= C_7\left\|\varphi\right\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}}B(\mathbb{R}^m)}.\end{aligned}$$

Here $C_7 < \infty$ since the norm $\left\|\left\{2^{s_i/p_i}\|Q_{s_i}(\cdot)\|_{L_{p_i}(\mathbb{R})}\right\}\right\|_{l_1}$ is equivalent to the Besov norm $\|\cdot\|_{B_{p_i}^{1/p_i,1}(\mathbb{R})}$ (see [22]), and $f_i \in C_0^\infty(\mathbb{R}) \subset B_{p_i}^{1/p_i,1}(\mathbb{R})$.

Further, we have

$$\lim_{\tilde{\mathbf{h}}\to\tilde{\mathbf{0}}}\left\|f(\cdot,\tilde{\mathbf{h}}) - \varphi(\cdot)\right\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}}B(\mathbb{R}^m)} = \lim_{\tilde{\mathbf{h}}\to\tilde{\mathbf{0}}}\left\|\varphi(\cdot)\left(\prod_{i=m+1}^{n} f_i(h_i) - 1\right)\right\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}}B(\mathbb{R}^m)}$$

$$= \|\varphi\|_{S_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\tau}}B(\mathbb{R}^m)} \cdot \lim_{\tilde{\mathbf{h}}\to\tilde{\mathbf{0}}}\left|\prod_{i=m+1}^{n} f_i(h_i) - 1\right| = 0.$$

These arguments show that φ is the trace of the function f. □

Remark 5. The extension operator constructed in the proof of Theorem 5.2 is linear. It should be noted that in the work of V.I. Burenkov and M.L. Gol'dman [19] it was shown that in the limiting case for Nikol'skii-Besov spaces it is possible to construct only a nonlinear extension operator, but this effect is not observed for Nikol'skii-Besov spaces with a dominant mixed derivative.

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LOCAL AND 2-LOCAL $\frac{1}{2}$-DERIVATIONS OF SOLVABLE LEIBNIZ ALGEBRAS

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Abstract. We show that any local $\frac{1}{2}$-derivation on solvable Leibniz algebras with model or abelian nilradicals, whose dimensions of complementary spaces are maximal, is a $\frac{1}{2}$-derivation. We show that solvable Leibniz algebras with abelian nilradicals, which have 1-dimensional complementary spaces are $\frac{1}{2}$-derivations. Moreover, a similar problem concerning 2-local $\frac{1}{2}$-derivations of such algebras is investigated.

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1 Introduction

In recent years non-associative analogues of classical constructions have become of interest in connection with their applications in many branches of mathematics and physics. The notions of local and 2-local derivations have also become popular for some non-associative algebras such as Lie and Leibniz algebras.

The notions of local derivations were introduced in 1990 by R.V. Kadison [17] and D.R. Larson, A.R. Sourour [18]. Later in 1997, P. Šemrl introduced the notions of 2-local derivations and 2-local automorphisms on algebras [16]. The main problems concerning these notions are to find conditions under which all local (2-local) derivations become (global) derivations and to present examples of algebras with local (2-local) derivations that are not derivations.

Investigation of local derivations on Lie algebras was initiated in papers [7] and [14]. Sh.A. Ayupov and K.K. Kudaybergenov have proved that every local derivation on a semi-simple Lie algebra is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations which are not derivations. In [8] local derivations and automorphisms of complex finite-dimensional simple Leibniz algebras are investigated. They proved that all local derivations on finite-dimensional complex simple Leibniz algebras are automatically derivations and it is shown that filiform Leibniz algebras admit local derivations which are not derivations.

Several papers have been devoted to similar notions and corresponding problems for 2-local derivations and automorphisms of finite-dimensional Lie and Leibniz algebras [5, 8, 9, 14]. Namely, in [9] it is proved that every 2-local derivation on a semi-simple Lie algebra is a derivation and that each finite-dimensional nilpotent Lie algebra, with dimension larger than two admits a 2-local derivation which is not a derivation. Concerning 2-local automorphisms, Z. Chen and D. Wang in [14] proved that if $\mathcal{L}$ is a simple Lie algebra of type A_l, D_l or $E_k, (k = 6, 7, 8)$ over an algebraically closed field of characteristic zero, then every 2-local automorphism of $\mathcal{L}$ is an automorphism. Finally, in [5]

Sh.A. Ayupov and K.K. Kudaybergenov generalized this result of [14] and proved that every 2-local automorphism of a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is an automorphism. Moreover, they also showed that every nilpotent Lie algebra of finite dimension greater than two admits a 2-local automorphism which is not an automorphism.

In [3] local derivations of solvable Lie algebras are investigated and it is shown that in the class of solvable Lie algebras there exist algebras which admit local derivations which are not derivations and also algebras for which every local derivation is a derivation. Moreover, it is proved that every local derivation on a finite-dimensional solvable Lie algebra with model nilradical and maximal dimension of complementary space is a derivation. Sh.A. Ayupov, A.Kh. Khudoyberdiyev and B.B. Yusupov proved similar results concerning local derivations on solvable Leibniz algebras in their recent paper [4]. The results of paper [10] show that p-filiform Leibniz algebras as a rule admit local derivations which are not derivations. Similar results concerning local derivations on direct sum null-filiform Leibniz algebras were obtained in [2].

In [13], [21] Sh.A. Ayupov and B.B. Yusupov investigated 2-local derivations on infinite-dimensional Lie algebras over a field of characteristic zero. They proved that all 2-local derivations on a Witt algebra as well as on a positive Witt algebra are (global) derivations, and gave an example of an infinite-dimensional Lie algebra with a 2-local derivation which is not a derivation. In [11] they have proved that every 2-local derivation on a generalized Witt algebra $W_n(\mathbb{F})$ over a vector space $\mathbb{F}^n$ is a derivation. In [15] Y. Chen, K. Zhao and Y. Zhao studied local derivations on generalized Witt algebras. They proved that every local derivation a Witt algebra is a derivation and that every local derivation on a centerless generalized Virasoro algebra of higher rank is a derivation. In [12] Sh.A. Ayupov, K.K. Kudaybergenov and B.B. Yusupov studied local and 2-local derivations of locally finite split simple Lie algebras. They proved that every local and 2-local derivation on a locally finite split simple Lie algebra is a derivation.

In the present paper we study local and 2-local $\frac{1}{2}$-derivations of solvable Leibniz algebras. We show that any local $\frac{1}{2}$-derivation on a solvable Leibniz algebra with model or abelian nilradicals, whose dimension of the complementary space is maximal, is a $\frac{1}{2}$-derivation. Moreover, similar problems concerning 2-local $\frac{1}{2}$-derivations of such algebras are investigated.

2 Preliminaries

In this section we give some necessary definitions and preliminary results.

Definition 1. A vector space with a bilinear bracket $(\mathcal{L}, [\cdot,\cdot])$ is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big],$$

holds, or equivalently, $[[x,y],z] = [[x,z],y] + [x,[y,z]]$.

Here, we adopt the right Leibniz identity; since the bracket is not skew-symmetric, there exists the version corresponding to the left Leibniz identity,

$$\big[[x,y],z\big] = \big[x,[y,z]\big] - \big[y,[x,z]\big]\,.$$

Let $\mathcal{L}$ be a Leibniz algebra. For a Leibniz algebra $\mathcal{L}$ consider the following lower central and derived sequences:

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}^1], \quad k \geq 1,$$
$$\mathcal{L}^{[1]} = \mathcal{L}, \quad \mathcal{L}^{[s+1]} = [\mathcal{L}^{[s]}, \mathcal{L}^{[s]}], \quad s \geq 1.$$

Definition 2. A Leibniz algebra $\mathcal{L}$ is called nilpotent (respectively, solvable), if there exists $k \in \mathbb{N}$ ($s \in \mathbb{N}$) such that $\mathcal{L}^k = 0$ (respectively, $\mathcal{L}^{[s]} = 0$).The minimal number k (respectively, s) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra $\mathcal{L}$.

Note that any Leibniz algebra $\mathcal{L}$ contains a unique maximal solvable (respectively nilpotent) ideal, called the radical (respectively nilradical) of the algebra.

A $\frac{1}{2}$-derivation on a Leibniz algebra $\mathcal{L}$ is a linear map $D : \mathcal{L} \to \mathcal{L}$ which satisfies the Leibniz rule:

$$D([x,y]) = \frac{1}{2}\left([D(x),y] + [x,D(y)]\right) \quad \text{for any} \quad x,y \in \mathcal{L}. \tag{2.1}$$

The set of all $\frac{1}{2}$-derivations of a Leibniz algebra $\mathcal{L}$ is a Lie algebra with respect to the usual matrix commutator and it is denoted by $\frac{1}{2}Der(\mathcal{L})$.

For a finite-dimensional nilpotent Leibniz algebra N and for the matrix of the linear operator ad_x denote by $C(x)$ the descending sequence of its Jordan blocks' dimensions. Consider the lexicographical order on the set $C(N) = \{C(x) \mid x \in N\}$.

Definition 3. The sequence

$$\left(\max_{x \in N \setminus N^2} C(x)\right)$$

is said to be the characteristic sequence of a nilpotent Leibniz algebra N.

Definition 4. A linear operator Δ is called a local $\frac{1}{2}$-derivation, if for any $x \in \mathcal{L}$, there exists a $\frac{1}{2}$-derivation $D_x : \mathcal{L} \to \mathcal{L}$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local $\frac{1}{2}$-derivations on $\mathcal{L}$ we denote by $Loc\frac{1}{2}Der(\mathcal{L})$.

Definition 5. A map $\nabla : \mathcal{L} \to \mathcal{L}$ (not necessary linear) is called a 2-local $\frac{1}{2}$-derivation, if for any $x, y \in \mathcal{L}$, there exists a $\frac{1}{2}$-derivation $D_{x,y} \in \frac{1}{2}Der(\mathcal{L})$ such that

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

2.1 Solvable Leibniz algebras with abelian nilradical

Let $\mathbf{a}_n$ be an n-dimensional abelian algebra and let R be a solvable Leibniz algebra with the nilradical $\mathbf{a}_n$. Take a basis $\{f_1, f_2, \ldots, f_n, x_1, x_2, \ldots x_k\}$ of R, such that $\{f_1, f_2, \ldots, f_n\}$ is a basis of $\mathbf{a}_n$. In [1] such solvable algebras in the case of $k = n$ are classified and it is proved that any $2n$-dimensional solvable Leibniz algebra with the nilradical $\mathbf{a}_n$ is isomorphic to the direct sum of two dimensional algebras, i.e., isomorphic to the algebra

$$\mathcal{L}_t : [f_j, x_j] = f_j, \quad [x_j, f_j] = \alpha_j f_j, \quad 1 \le j \le n,$$

where $\alpha_j \in \{-1, 0\}$ and t is the number of zero parameters α_j.

Moreover, in the following theorem a classification of $(n+1)$-dimensional solvable Leibniz algebras with n-dimensional abelian nilradical is given.

Theorem 2.1. *[1] Let R be an $(n+1)$-dimensional solvable Leibniz algebra with n-dimensional abelian nilradical. If R has a basis $\{f_1, f_2, \ldots, f_n, x\}$ such that the operator $ad_x|_{\mathbf{a}_n}$ has Jordan block form, then it is isomorphic to one of the following two non-isomorphic algebras:*

$$R_1 : \begin{cases} [f_i, x] = f_i + f_{i+1}, & 1 \le i \le n-1, \\ [f_n, x] = f_n, \end{cases} \qquad R_2 : \begin{cases} [f_i, x] = f_i + f_{i+1}, & 1 \le i \le n-1, \\ [f_n, x] = f_n, \\ [x, f_i] = -f_i - f_{i+1}, & 1 \le i \le n-1, \\ [x, f_n] = -f_n. \end{cases}$$

2.2 Solvable Leibniz algebras with model nilradical

Let N be a nilpotent Leibniz algebra with the characteristic sequence $(m_1, \dots, m_s)$, and with the table of multiplication

$$N_{m_1,\dots,m_s} : [e_i^t, e_1^1] = e_{i+1}^t, \ 1 \le t \le s, \ 1 \le i \le m_t - 1.$$

The algebra $N_{m_1,\dots,m_s}$ is usually said to be a model Leibniz algebra. For solvable Leibniz algebras with nilradical $N_{m_1,\dots,m_s}$ and the complement dimension space equal to s, we will use the notation $R(N_{m_1,\dots,m_s}, s)$.

Theorem 2.2. *[20] A solvable Leibniz algebra $R(N_{m_1,\dots,m_s}, s)$ with nilradical $N_{m_1,\dots,m_s}$, such that $DimR(N_{m_1,\dots,m_s}, s) - DimN_{m_1,\dots,m_s} = s$, is isomorphic to the algebra:*

$$R(N_{m_1,\dots,m_s}, s) : \begin{cases} [e_i^t, e_1^1] = e_{i+1}^t, & 1 \le t \le s, \ 1 \le i \le m_t - 1, \\ [e_i^1, x_1] = ie_i^1, & 1 \le i \le m_1, \\ [e_i^t, x_1] = (i-1)e_i^t, & 2 \le t \le s, \ 2 \le i \le m_t, \\ [e_i^t, x_t] = e_i^t, & 2 \le t \le s, \ 1 \le i \le m_t, \\ [x_1, e_1^1] = -e_1^1, & \end{cases}$$

where $\{x_1, \dots x_s\}$ is a basis of the complementary vector space.

3 $\frac{1}{2}$-derivation of solvable Leibniz algebras

In the following propositions, we present a general form of the $\frac{1}{2}$-derivation of the algebras $R(N_{m_1,\dots,m_s}, s)$, $\mathcal{L}_t$, R_1 and R_2.

Proposition 3.1. *Any $\frac{1}{2}$-derivation D of the algebra $\frac{1}{2}Der(R(N_{m_1,\dots,m_s}, s))$ has the following form:*

$$\begin{aligned} &D(e_i^1) = \alpha_1 e_i^1, && 1 \le i \le m_1, \\ &D(e_i^t) = \tfrac{1}{2^{i-1}}((2^{i-1} - 1)\alpha_1 + \alpha_t)e_i^t, && 2 \le t \le s, \quad 1 \le i \le m_t, \\ &D(x_i) = \alpha_i x_i, \quad 1 \le i \le s. \end{aligned}$$

Proof. Let $\{e_1^1, e_1^2, \dots, e_1^s, x_1, \dots, x_s\}$ be a basis elements of the algebra $R(N_{m_1,\dots,m_s}, s)$.

Let d be a $\frac{1}{2}$-derivation of the algebra $R(N_{m_1,\dots,m_s}, s)$.

We put

$$D(e_1^p) = \sum_{t=1}^{s}\sum_{i=1}^{m_t} \alpha_{t,i}^p e_i^t + \sum_{i=1}^{s} \beta_{1,i}^p x_i, \quad D(x_p) = \sum_{t=1}^{s}\sum_{i=1}^{m_t} \gamma_{t,i}^p e_i^t + \sum_{i=1}^{s} \beta_{2,i}^p x_i, \quad 1 \le p \le s.$$

The following restrictions follow from the equality

$$D([e_1^1, x_1]) = \frac{1}{2}([D(e_1^1), x_1] + [e_1^1, D(x_1)]) :$$

$\alpha_{1,i}^1 = 0, \quad 3 \le i \le m_1, \quad \gamma_{1,1}^1 = 0, \qquad \beta_{2,1}^1 = \alpha_{1,1}^1,$
$\alpha_{t,i}^1 = 0, \quad 2 \le t \le s, \quad i = 1, 2, \quad 4 \le i \le m_t.$
$\beta_{1,i}^1 = 0, \quad 1 \le i \le s.$

Consider the equality

$$D([e_1^p, x_1]) = \frac{1}{2}([D(e_p^1), x_1] + [e_p^1, D(x_1)]), \quad \text{for} \quad 2 \le p \le s.$$

Then we get

$$\begin{cases} \alpha_{1,i}^p = 0, & 1 \leq i \leq m_1, \\ \alpha_{t,i}^p = 0, & 2 \leq t \leq s, \qquad 2 \leq i \leq m_t, \\ \beta_{2,p}^1 = 0. \end{cases}$$

Similarly, from the equality

$$0 = D([x_p, x_1]) = \frac{1}{2}([D(x_p), x_1] + [x_p, D(x_1)]),$$

with $1 \leq p \leq s$ we have

$$\begin{cases} \gamma_{t,i}^p = 0, & 1 \leq t \leq s, \quad 2 \leq i \leq m_t, \quad 1 \leq p \leq s, \\ \gamma_{1,1}^p = 0, & 2 \leq p \leq s. \end{cases}$$

The equality

$$D([e_1^1, x_p]) = \frac{1}{2}([D(e_1^1), x_p] + [e_1^1, D(x_p)]),$$

for $2 \leq p \leq s$ which imply

$$\alpha_{p,3}^1 = \beta_{2,1}^p, \quad 2 \leq p \leq s.$$

Consequently,

$$D(e_1^1) = \alpha_{1,1}^1 e_1^1 + \alpha_{1,2}^1 e_2^1, \quad D(x_p) = \sum_{t=2}^{s} \gamma_{t,1}^p e_1^t + \sum_{i=2}^{s} \beta_{2,i}^p x_i, \quad 2 \leq p \leq s.$$

From the equality

$$0 = D([x_p, x_j]) = \frac{1}{2}([D(x_p), x_j] + [x_p, D(x_j)]),$$

for $2 \leq p, j \leq s$ we obtain the following restrictions:

$$\gamma_{j,1}^p = 0, \quad 1 \leq p, j \leq s.$$

From the relations

$$D([x_1, e_1^1]) = \frac{1}{2}([D(x_1), e_1^1] + [x_1, D(e_1^1)]), \quad D([e_1^p, x_j]) = \frac{1}{2}([D(e_1^p), x_j] + [e_1^p, D(x_j)]),$$

for $2 \leq p, j \leq s$, we have

$$\begin{cases} \alpha_{1,2}^1 = 0, & \gamma_{t,1}^1 = 0, & 2 \leq t \leq s, \\ \alpha_{t,1}^p = 0, & 2 \leq p, t \leq s, & p \neq t, \\ \beta_{1,j}^p = 0, & \beta_{2,p}^p = \alpha_{p,1}^p, & 2 \leq p \leq s, \quad 1 \leq j \leq s, \\ \beta_{2,p}^j = 0, & 2 \leq j, p \leq s & j \neq p. \end{cases}$$

Consequently,

$$\begin{cases} D(e_1^p) = \alpha_{p,1}^p e_1^p, & 1 \leq p \leq s, \\ D(x_p) = \alpha_{p,1}^p x_p, & 1 \leq p \leq s. \end{cases}$$

From the chain of equalities

$$D(e_i^p) = D([e_{i-1}^p, e_1^1]) = \frac{1}{2}[D(e_{i-1}^p), e_1^1] + \frac{1}{2}[e_{i-1}^p, D(e_1^1)], \quad 1 \le p \le s, \quad 2 \le i \le m_p,$$

and the restrictions obtained above, it is easy to establish that

$$\begin{aligned} &D(e_i^1) = \alpha_{1,1}^1 e_i^1, \quad 1 \le i \le m_1, \\ &D(e_i^p) = \tfrac{1}{2^{i-1}}((2^{i-1}-1)\alpha_{1,1}^1 + \alpha_{p,1}^p)e_i^p, \quad 2 \le p \le s, \quad 1 \le i \le m_p. \end{aligned}$$

□

Proposition 3.2. *Any $\frac{1}{2}$-derivation D of the algebra $\mathcal{L}_t$ has the following form:*

$$D(f_j) = a_j f_j, \quad D(x_j) = \alpha_j b_j f_j, \quad 1 \le j \le n.$$

Proof. The proof is similar to the proof of Proposition 3.1. □

Proposition 3.3. *Any $\frac{1}{2}$-derivation D of the algebras R_1 and R_2 have the following forms, respectively:*

$$Der(R_1) : \begin{cases} D(f_i) &= \alpha_1 f_i, \quad 1 \le i \le n, \\ D(x) &= \alpha_1 x. \end{cases}$$

$$Der(R_2) : \begin{cases} D(f_i) &= \alpha_1 f_i, \quad 1 \le i \le n, \\ D(x) &= \sum\limits_{j=1}^{n} \beta_j f_j + \alpha_1 x. \end{cases}$$

Proof. The proof is similar to the proof of Proposition 3.1. □

4 Local $\frac{1}{2}$-derivation of solvable Leibniz algebras

4.1 Local $\frac{1}{2}$-derivation of solvable Leibniz algebra $R(N_{m_1,\dots,m_s}, s)$

Now we shall give the main result concerning local $\frac{1}{2}$-derivations of the solvable Leibniz algebra $R(N_{m_1,\dots,m_s}, s)$.

Theorem 4.1. *Any local $\frac{1}{2}$-derivation on the solvable Leibniz algebra $R(N_{m_1,\dots,m_s}, s)$ is a $\frac{1}{2}$-derivation.*

Proof. Let Δ be a local $\frac{1}{2}$-derivation on $R(N_{m_1,\dots,m_s}, s)$, then we have

$$\Delta(x_i) = \sum_{j=1}^{s} a_{i,j} x_j + \sum_{p=1}^{s} \sum_{j=1}^{m_p} b_{i,j}^p e_j^p, \quad \Delta(e_i^t) = \sum_{j=1}^{s} c_{i,j}^t x_j + \sum_{p=1}^{s} \sum_{j=1}^{m_p} d_{i,j}^{t,p} e_j^p.$$

Let D be a $\frac{1}{2}$-derivation on $R(N_{m_1,\dots,m_s}, s)$, then by Proposition 3.1, we obtain

$$\begin{aligned} &D(e_i^1) = \alpha_{1,e_i^1} e_i^1, && 1 \le i \le m_1, \\ &D(e_i^t) = \tfrac{1}{2^{i-1}}((2^{i-1}-1)\alpha_{1,e_i^t} + \alpha_{t,e_i^t})e_i^t, && 2 \le t \le s, \quad 1 \le i \le m_t, \\ &D(x_i) = \alpha_{i,x_i} x_i, \quad 1 \le i \le s. \end{aligned}$$

Considering the equalities

$$\Delta(x_j) = D_{x_j}(x_j), \quad 1 \le j \le s,$$

$$\Delta(e_i^t) = D_{e_i^t}(e_i^t), \quad 1 \le t \le s, \ 1 \le i \le m_t,$$

we have

$$\begin{cases} \sum\limits_{j=1}^{s} c_{i,j}^{1} x_j + \sum\limits_{p=1}^{s}\sum\limits_{j=1}^{m_p} d_{i,j}^{1,p} e_j^p = \alpha_{1,e_i^1} e_i^1, & 1 \le i \le m_1 \\ \sum\limits_{j=1}^{s} c_{i,j}^{t} x_j + \sum\limits_{p=1}^{s}\sum\limits_{j=1}^{m_p} d_{i,j}^{t,p} e_j^p = \frac{1}{2^{i-1}}((2^{i-1}-1)\alpha_{1,e_i^t} + \alpha_{t,e_i^t}) e_i^t, & 2 \le t \le s,\ 1 \le i \le m_t, \\ \sum\limits_{j=1}^{s} a_{i,j} x_j + \sum\limits_{p=1}^{s}\sum\limits_{j=1}^{m_p} b_{i,j}^{p} e_j^p = \alpha_{i,x_i} x_i, & 1 \le i \le n. \end{cases}$$

From the previous restrictions, we get that

$$\begin{aligned} \Delta(e_i^1) &= d_{i,i}^{1,1} e_i^1, \quad 1 \le i \le m_1, \\ \Delta(e_i^t) &= d_{i,i}^{t,t} e_i^t, \quad 2 \le t \le s, \quad 1 \le i \le m_t, \\ \Delta(x_i) &= a_{i,i} x_i \quad 1 \le i \le s. \end{aligned}$$

Considering $\Delta(e_1^1 + e_i^1)$ for $2 \le i \le m_1$, we have

$$\Delta(e_1^1 + e_i^1) = d_{1,1}^{1,1} e_1^1 + d_{i,i}^{1,1} e_i^1.$$

On the other hand,

$$\begin{aligned} \Delta(e_1^1 + e_i^1) = D_{e_1^1+e_i^1}(e_1^1 + e_i^1) = D_{e_1^1+e_i^1}(e_1^1) + D_{e_1^1+e_i^1}(e_i^1) = \\ = \alpha_{1,e_1^1+e_i^1} e_1^1 + \alpha_{i,e_1^1+e_i^1} e_i^1 \end{aligned}$$

Comparing the coefficients at the basis elements e_1^1 and e_i^1, we get the equalities $\alpha_{1,e_1^1+e_i^1} = d_{1,1}^{1,1}$, $\alpha_{1,e_1^1+e_i^1} = d_{i,i}^{1,1}$, which imply

$$d_{i,i}^{1,1} = d_{1,1}^{1,1}, \quad 2 \le i \le m_1.$$

Now for $2 \le t \le s$, $1 \le i \le m_t$, we consider

$$\Delta(e_i^t + e_1^1 + x_1 + x_t) = d_{i,i}^{t,t} e_i^t + d_{1,1}^{1,1} e_1^1 + a_{1,1} x_1 + a_{t,t} x_t.$$

On the other hand,

$$\begin{aligned} \Delta(e_i^t + e_1^1 + x_1 + x_t) = D_{e_i^t+e_1^1+x_1+x_t}(e_i^t + e_1^1 + x_1 + x_t) = \\ = \frac{1}{2^{i-1}}((2^{i-1}-1)\alpha_{1,e_i^t+e_1^1+x_1+x_t} + \alpha_{i,e_i^t+e_1^1+x_1+x_t}) e_i^t + \\ + \alpha_{1,e_i^t+e_1^1+x_1+x_t} e_1^1 + \alpha_{1,e_i^t+e_1^1+x_1+x_t} x_1 + \alpha_{t,e_i^t+e_1^1+x_1+x_t} x_t \end{aligned}$$

Comparing the coefficients at the basis elements e_i^t, e_1^1, x_1 and x_t, we get the equalities

$$\alpha_{1,e_i^t+e_1^1+x_1+x_t} = d_{1,1}^{1,1} = a_{1,1}, \quad \frac{1}{2^{i-1}}((2^{i-1}-1)\alpha_{1,e_i^t+e_1^1+x_1+x_t} + \alpha_{i,e_i^t+e_1^1+x_1+x_t}) = d_{i,i}^{t,t},$$

$$\alpha_{t,e_i^t+e_1^1+x_1+x_t} = a_{t,t},$$

which imply

$$d_{i,i}^{t,t} = \frac{1}{2^{i-1}}((2^{i-1}-1)d_{1,1}^{1,1} + a_{t,t}), \quad a_{1,1} = d_{1,1}^{1,1},\ 2 \le t \le s, \quad 1 \le i \le m_t.$$

Thus, we obtain that the local $\frac{1}{2}$-derivation Δ has the following form:

$$\begin{array}{ll}\Delta(e_i^1) = d_{1,1}^{1,1}e_i^1, & 1 \le i \le m_1,\\ \Delta(e_i^t) = \frac{1}{2^{i-1}}((2^{i-1}-1)d_{1,1}^{1,1} + a_{t,t})e_i^t, & 2 \le t \le s, \quad 1 \le i \le m_t,\\ \Delta(x_1) = d_{1,1}^{1,1}x_1, & \\ \Delta(x_i) = a_{t,t}x_i, & i \le t \le s.\end{array}$$

Proposition 3.1 implies that Δ is a $\frac{1}{2}$-derivation. Hence, every local $\frac{1}{2}$-derivation on $R(N_{m_1,\dots,m_s}, s)$ is a $\frac{1}{2}$-derivation. □

4.2 Local $\frac{1}{2}$-derivation of solvable Leibniz algebras with abelian nilradical

Now we shall give the main result concerning local $\frac{1}{2}$-derivations on solvable Leibniz algebras with abelian nilradicals.

Theorem 4.2. *Any local $\frac{1}{2}$-derivation on the algebra $\mathcal{L}_t$ is a $\frac{1}{2}$-derivation.*

Proof. For any local $\frac{1}{2}$-derivation Δ on the algebra $\mathcal{L}_t$, we put the $\frac{1}{2}$-derivation D, such that:

$$D(f_j) = a_j f_j, \quad D(x_j) = \alpha_j b_j f_j, \quad 1 \le j \le n,$$

Then, we get

$$\Delta(f_j) = D_{f_j}(f_j) = a_j f_j, \quad \Delta(x_j) = D_{x_j}(x_j) = \alpha_j b_j f_j.$$

Hence, Δ is a $\frac{1}{2}$-derivation. □

In the following theorem, we show that $(n+1)$-dimensional solvable Leibniz algebras with n-dimensional abelian nilradical have a local derivation which is not a derivation.

Theorem 4.3. *Consider the $(n+1)$-dimensional solvable Leibniz algebras R_1 and R_2 (see Theorem 2.1). Any local $\frac{1}{2}$-derivation on the algebras R_1 and R_2 is a $\frac{1}{2}$-derivation.*

Proof. We prove the theorem for the algebra R_1, and for the algebra R_2 the proof is similar.

Let Δ be a local $\frac{1}{2}$-derivation on R_1, then we have

$$\begin{aligned}\Delta(f_i) &= \sum_{j=1}^{n} a_{i,j} f_j + c_i x, \quad 1 \le i \le n,\\ \Delta(x) &= \sum_{j=1}^{n} b_j f_j + dx.\end{aligned} \tag{4.1}$$

Let D be a $\frac{1}{2}$-derivation on R_1, then by Proposition 3.3, we obtain

$$\begin{cases} D(f_i) & = \alpha_{1,f_i} f_i, \quad 1 \le i \le n,\\ D(x) & = \alpha_{1,x} x.\end{cases}$$

Considering the equalities

$$\Delta(x) = D_x(x), \quad \Delta(f_i) = D_{f_i}(f_i), \quad 1 \le i \le n,$$

we have

$$\begin{cases} \sum\limits_{j=1}^{n} a_{i,j} f_j + c_i x = \alpha_{1,f_i} f_i, \quad 1 \le i \le n\\ \sum\limits_{j=1}^{n} b_j f_j + dx = \alpha_{1,x} x.\end{cases}$$

From the previous restrictions, we get that

$$\Delta(f_i) = a_{i,i} f_i, \quad 1 \leq i \leq n,$$
$$\Delta(x) = dx.$$

For $2 \leq i \leq n$, we have

$$\Delta(f_1 + f_i) = a_{1,1} f_1 + a_{i,i} f_i.$$

On the other hand,

$$\Delta(f_1 + f_i) = D_{f_1+f_i}(f_1 + f_i) = D_{f_1+f_i}(f_1) + D_{f_1+f_i}(f_i) = \\ = \alpha_{1,f_1+f_i} f_1 + \alpha_{1,f_1+f_i} f_i.$$

Comparing the coefficients at the basis elements f_1 and f_i, we get the equalities $\alpha_{1,f_1+f_i} = a_{1,1}$, $\alpha_{1,f_1+f_i} = a_{i,i}$, which imply

$$a_{i,i} = a_{1,1}, \quad 2 \leq i \leq n.$$

Similarly, the equalities

$$\begin{aligned}\Delta(f_1 + x) &= a_{1,1} f + dx \\ &= D_{f_1+x}(f_1 + x) = D_{f_1+x}(f_1) + D_{f_1+x}(x) \\ &= \alpha_{1,f_1+x} f_1 + \alpha_{1,f_1+x} x,\end{aligned}$$

imply

$$d = a_{1,1}.$$

Thus, we obtain that the local $\frac{1}{2}$-derivation Δ has the following form:

$$\Delta(f_i) = a_{1,1} f_i, \quad 1 \leq i \leq n,$$
$$\Delta(x) = a_{1,1} x$$

Proposition 3.3 implies that Δ is a $\frac{1}{2}$-derivation. Hence, every local $\frac{1}{2}$-derivation on R_1 is a $\frac{1}{2}$-derivation.

□

5 2-local $\frac{1}{2}$-derivation of solvable Leibniz algebras

5.1 2-local $\frac{1}{2}$-derivation of solvable Leibniz algebra $R(N_{m_1,\ldots,m_s}, s)$

Now we shall give the main result concerning of the 2-local $\frac{1}{2}$-derivations of the solvable Leibniz algebra $R(N_{m_1,\ldots,m_s}, s)$.

Consider an element $q = \sum\limits_{t=1}^{s} x_t$ of $R(N_{m_1,\ldots,m_s}, s)$.

Theorem 5.1. *Any 2-local $\frac{1}{2}$-derivation of the solvable Leibniz algebra $R(N_{m_1,\ldots,m_s}, s)$ is a $\frac{1}{2}$-derivation.*

Proof. Let ∇ be a 2-local $\frac{1}{2}$-derivation on $R(N_{m_1,\ldots,m_s}, s)$ such that $\nabla(q) = 0$. Then for any element

$$p = \sum_{t=1}^{s} \sum_{i=1}^{m_t} \xi_i^t e_i^t + \sum_{t=1}^{s} \zeta_t x_t \in R(N_{m_1,\ldots,m_s}, s),$$

there exists a $\frac{1}{2}$-derivation $D_{q,p}(p)$, such that

$$\nabla(q) = D_{q,p}(q), \quad \nabla(p) = D_{q,p}(p).$$

Hence,

$$0 = \nabla(q) = D_{q,p}(q) = \sum_{t=1}^{s} \alpha_t x_t,$$

which implies, $\alpha_t = 0,\ 1 \le t \le s$.

Consequently, from the description of the $\frac{1}{2}$-derivation $R(N_{m_1,\dots,m_s}, s)$, we conclude that $D_{q,p} = 0$. Thus, we obtain that if $\nabla(q) = 0$, then ∇ is a zero.

Let now ∇ be an arbitrary 2-local $\frac{1}{2}$-derivation of $R(N_{m_1,\dots,m_s}, s)$. Take a $\frac{1}{2}$-derivation $D_{q,p}$, such that

$$\nabla(q) = D_{q,p}(q) \ \text{ and } \ \nabla(p) = D_{q,p}(p).$$

Set $\nabla_1 = \nabla - D_{q,p}$. Then ∇_1 is a 2-local $\frac{1}{2}$-derivation, such that $\nabla_1(q) = 0$. Hence $\nabla_1(p) = 0$ for all $\xi \in R(N_{m_1,\dots,m_s}, s)$, which implies $\nabla = D_{q,p}$. Therefore, ∇ is a $\frac{1}{2}$-derivation.

□

5.2 2-local $\frac{1}{2}$-derivation of solvable Leibniz algebras with alebian nilradical

Now we shall give the result concerning of 2-local $\frac{1}{2}$-derivations of solvable Leibniz algebras with abelian nilradical.

Proposition 5.1. *Any 2-local $\frac{1}{2}$-derivation of the algebra R_1 is a derivation.*

Proof. Let ∇ be a 2-local $\frac{1}{2}$-derivation on R_1, such that $\nabla(f_1) = 0$. Then for any element $\xi = \sum\limits_{i=1}^{n} \xi_i f_i + \xi_{n+1} x \in R_1$, there exists a $\frac{1}{2}$-derivation $D_{f_1,\xi}(\xi)$, such that

$$\nabla(f_1) = D_{f_1,\xi}(f_1), \quad \nabla(\xi) = D_{f_1,\xi}(\xi).$$

Hence,

$$0 = \nabla(f_1) = D_{f_1,\xi}(f_1) = \alpha_1 f_1,$$

which implies, $\alpha_1 = 0$.

Consequently, from the description of the $\frac{1}{2}$-derivation of R_1, we conclude that $D_{f_1,\xi} = 0$. Thus, we obtain that if $\nabla(f_1) = 0$, then ∇ is a zero.

Let now ∇ be an arbitrary 2-local $\frac{1}{2}$-derivation of R_1. Take a $\frac{1}{2}$-derivation $D_{f_1,\xi}$, such that

$$\nabla(f_1) = D_{f_1,\xi}(f_1) \ \text{ and } \ \nabla(\xi) = D_{f_1,\xi}(\xi).$$

Set $\nabla_1 = \nabla - D_{f_1,\xi}$. Then ∇_1 is a 2-local $\frac{1}{2}$-derivation, such that $\nabla_1(f_1) = 0$. Hence, $\nabla_1(\xi) = 0$ for all $\xi \in R_1$, which implies that $\nabla = D_{f_1,\xi}$. Therefore, ∇ is a $\frac{1}{2}$-derivation. □

Theorem 5.2. *The solvable Leibniz algebra R_2 admits a 2-local $\frac{1}{2}$-derivation which is not a $\frac{1}{2}$-derivation.*

Proof. Let us define a homogeneous non-additive function f on $\mathbb{C}^2$ as follows

$$f(z_1, z_2) = \begin{cases} \frac{z_1^2}{z_2}, & \text{if } z_2 \neq 0, \\ 0, & \text{if } z_2 = 0, \end{cases}$$

where $(z_1, z_2) \in \mathbb{C}^2$.

Define the operator ∇ on R_2, such that

$$\nabla(\xi) = f(\xi_1, \xi_{n+1}) f_1, \tag{5.1}$$

for any element $\xi = \sum\limits_{i=1}^{n} \xi_i f_i + \sum\limits_{i=1}^{n} \xi_{n+i} x_i$,

The operator ∇ is not a $\frac{1}{2}$-derivation, since it is not linear.

Let us show that ∇ is a 2-local $\frac{1}{2}$-derivation. For this purpose, define a $\frac{1}{2}$-derivation D on R_2 by

$$D(\xi) = (a\xi_1 + b\xi_2) f_n.$$

For each pair of elements ξ and η, we choose a and b, such that $\nabla(\xi) = D(\xi)$ and $\nabla(\eta) = D(\eta)$. Let us rewrite the above equalities as system of linear equations with respect to the unknowns a, b as follows

$$\begin{cases} \xi_1 a + \xi_2 b = f(\xi_1, \xi_2), \\ \eta_1 a + \eta_2 b = f(\eta_1, \eta_2). \end{cases} \tag{5.2}$$

Case 1. $\xi_1 \eta_2 - \xi_2 \eta_1 = 0$. In this case, since the right-hand side of system (5.2) is homogeneous, it has infinitely many solutions.

Case 2. $\xi_1 \eta_2 - \xi_2 \eta_1 \neq 0$. In this case, system (5.2) has a unique solution.

□

Theorem 5.3. *The algebra $\mathcal{L}_t$ admits a 2-local $\frac{1}{2}$-derivation which is not a $\frac{1}{2}$-derivation.*

Proof. The proof is similar to the proof of Theorem 5.2. □

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FACTORIZATION METHOD FOR SOLVING SYSTEMS OF SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Key words: systems of ordinary differential equations, nonlocal boundary value problems, multi-point boundary problems, integral boundary conditions, exact solution, correct problems, factorization.

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Abstract. We consider in a Banach space the following two abstract systems of first-order and second-order linear ordinary differential equations with general boundary conditions, respectively,

$$X'(t) - A_0(t)X(t) = F(t), \quad \Phi(X) = \sum_{j=1}^{n} M_j \Psi_j(X),$$

and

$$X''(t) - S(t)X'(t) - Q(t)X(t) = F(t),$$
$$\Phi(X) = \sum_{i=j}^{n} M_j \Psi_j(X), \quad \Phi(X') = C\Phi(X) + \sum_{j=1}^{r} N_j \Theta_j(X),$$

where $X(t) = \mathrm{col}\,(x_1(t), \dots, x_m(t))$ denotes a vector of unknown functions, $F(t)$ is a given vector and $A_0(t), S(t), Q(t)$ are given matrices, $\Phi, \Psi_1, \dots, \Psi_n$, $\Theta_1, \dots, \Theta_r$ are vectors of linear bounded functionals, and $M_1, \dots, M_n, C$, $N_1, \dots, N_r$ are constant matrices. We first provide solvability conditions and a solution formula for the first-order system. Then we construct in closed form the solution of a special system of $2m$ first-order linear ordinary differential equations with constant coefficients when the solution of the associated system of m first-order linear ordinary differential equations is known. Finally, we construct in closed form the solution of the second-order system in the case in which it can be factorized into first-order systems.

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1 Introduction

Boundary value problems (BVPs) for ordinary differential equations (ODEs) appear in a wide range of sciences. Many of these are nonlocal problems with integral and multipoint boundary conditions, such as in the modeling of power networks, telecommunication lines, electric railway systems, kinetic reaction problems in chemistry, elasticity, and elsewhere [19, 18, 15].

Perhaps the first problem with nonlocal integral boundary conditions for a system of linear first order ODEs was Hilb's problem

$$LY = PY' + QY = f, \quad \int_0^1 K(\xi)Y(\xi)d\xi + \gamma Y(0) - \Gamma Y(1) = 0,$$

which was investigated in 1911 [12]. The multipoint boundary value problems for a system of Transferable Differential-Algebraic Equations was investigated in [15]. In [4] an approach is given to solving the overdetermined problem for a system of the first and second order ODEs. The unique exact solution to the BVP

$$Y'(t) - M(t)Y(t) = F(t), \quad \Phi(Y) = \vec{c},$$

was obtained in [10]. The solvability condition and exact solution to the BVP

$$Y'(t) - AY(t) = F(t), \quad \sum_{i=1}^{m} A_i Y(t_i) + \sum_{j=0}^{s} B_j \int_{z_j}^{z_{j+1}} C_j(t)Y(t)dt = \vec{0},$$

where A, A_i, B_j are constant matrices, are given in [5]. Necessary and sufficient conditions are established in [7] for the existence of a unique holomorphic solution of the BVP

$$X'(t) = T(t)X(t) + F(t), \quad \sum_{i=1}^{m} A_i X(t_i) + \sum_{j=0}^{m} \int_{t_{i-1}}^{t_i} \Phi_i(t)X(t)dt = h$$

with holomorphic coefficients and general linear boundary conditions. The existence of positive solutions of nonlocal BVPs for ordinary second order differential systems is given in [8]. The existence of solutions of nonlocal BVPs for ordinary differential systems of higher order was investigated in [9]. A numerical method for solving systems of linear nonautonomous ODEs with nonseparated multipoint and integral conditions was considered in [1]. Numerical solutions of systems of loaded ordinary differential equations are given in [2]. Ordinary differential equations and systems of various types were studied by the parametrization method in [13], [3] (see also [16]). The factorization (decomposition) method is a powerful tool for finding solutions to systems of ODEs. The factorization method proposed here for systems of ODEs is essentially different from other factorization methods in the relevant literature, where usually approximate solutions to ordinary differential systems are found by using the Adomian decomposition method and its many modifications [17], [6]. Note that finding of the fundamental and particular solutions for the following system of linear second order ODEs

$$B_2 X(t) = X''(t) - S(t)X'(t) - Q(t)X(t) = F,$$

with nonlocal boundary conditions, is usually a difficult problem. Our goal is to find special cases that allow factorization like $B_2 X(t) = B^2 X(t)$, where an operator B corresponds to a system of linear first order ODEs with a simpler nonlocal boundary condition. The technique proposed in this article is simple to use and can be easily incorporated to any Computer Algebra System (CAS).

2 Preliminaries

Let $\mathcal{X}$ be a Banach space such as the space of continuous functions $C[0,1]$ or the space of Lebesgue integrable functions $L_p(0,1)$. Let $\mathcal{X}_m$ be the space of column vectors $X(t) = \mathrm{col}(x_1(t), \ldots, x_m(t))$, $x_i(t) \in \mathcal{X}, i = 1, \ldots, m$, i.e. $\mathcal{X}_m = C_m = C_m[0,1]$ or $\mathcal{X}_m = L_{p,m} = L_{p,m}(0,1)$ with the norm

$$||X(t)||_{\mathcal{X}_m} = \sum_{i=1}^{m} ||x_i(t)||_{\mathcal{X}}.$$

In addition, let $\mathcal{X}^k, k > 0$, be the space $C^k[0,1]$ with the norm

$$||x(t)||_{\mathcal{X}^k} = \sum_{\ell=0}^{k} ||x^{(\ell)}(t)||_C,$$

or the Sobolev space $\hat{W}_p^k(0,1)$ with the norm

$$||x(t)||_{\mathcal{X}^k} = \sum_{\ell=0}^{k} ||x^{(\ell)}(t)||_{L_p},$$

(in the case of the Sobolev spaces $x^{(l)}$ are weak derivatives), and $\mathcal{X}_m^k$ be the space $C_m^k[0,1]$ or $\hat{W}_{p,m}^k(0,1)$ with the norm

$$||X(t)||_{\mathcal{X}_m^k} = \sum_{\ell=0}^{k} ||X^{(\ell)}(t)||_{\mathcal{X}_m}.$$

Let $\mathcal{X}^*$ be the adjoint space of $\mathcal{X}$, i.e. the set of all linear and bounded functionals Φ on $\mathcal{X}$. We denote by $\Phi(x)$ the value of $\Phi \in \mathcal{X}^*$ on $x \in \mathcal{X}$. Let $\Psi_j \in \mathcal{X}^*, j = 1,\ldots,n$, and the vector $\Psi = \mathrm{col}(\Psi_1,\ldots,\Psi_n) \in [\mathcal{X}_m]^*$. For $X \in \mathcal{X}_m$ we write

$$\Phi(X) = \begin{pmatrix} \Phi(x_1) \\ \vdots \\ \Phi(x_m) \end{pmatrix}, \quad \Psi_j(X) = \begin{pmatrix} \Psi_j(x_1) \\ \vdots \\ \Psi_j(x_m) \end{pmatrix}, \quad \Psi(X) = \begin{pmatrix} \Psi_1(X) \\ \vdots \\ \Psi_n(X) \end{pmatrix}.$$

Remark 1. Let $m = 2$, $k = 1$, $X(t) = \mathrm{col}(x_1(t), x_2(t)) \in \mathcal{X}_2$ and the functional vector $\Theta\,(X(t)) = \mathrm{col}(\Theta(x_1), \Theta(x_2))$. Then $\Theta \in [\mathcal{X}_2]^*$ if there exists a constant $c_1 > 0$, such that

$$\begin{aligned} |\Theta(X)| = \sqrt{[\Theta(x_1)]^2 + [\Theta(x_2)]^2} &\leq |\Theta(x_1)| + |\Theta(x_2)| \\ &\leq c_1||x_1||_{\mathcal{X}} + c_1||x_2||_{\mathcal{X}} = c_1||X(t)||_{\mathcal{X}_2}. \end{aligned}$$

Similarly $\Theta \in [\mathcal{X}_2^1]^*$ if there exists a constant $c_2 > 0$, such that

$$\begin{aligned} |\Theta(X)| = \sqrt{[\Theta(x_1)]^2 + [\Theta(x_2)]^2} &\leq |\Theta(x_1)| + |\Theta(x_2)| \\ &\leq c_2(||x_1||_{\mathcal{X}} + ||x_2||_{\mathcal{X}} + ||x_1'||_{\mathcal{X}} + ||x_2'||_{\mathcal{X}}) \\ &= c_2(||X(t)||_{\mathcal{X}_2} + ||X'(t)||_{\mathcal{X}_2}) = c_2||X(t)||_{\mathcal{X}_2^1}. \end{aligned}$$

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces as above. Let the operator $A : \mathcal{X} \to \mathcal{Y}$ and let $D(A)$ and $R(A)$ denote its domain and the range, respectively. The operator A is said to be *injective or uniquelly solvable* if for all $x_1, x_2 \in D(A)$ such that $Ax_1 = Ax_2$, it follows that $x_1 = x_2$. Recall that a linear operator A is injective if and only if $\ker A = \{0\}$. The operator A is called *surjective* or *everywhere solvable* if $R(A) = \mathcal{Y}$. The operator A is called *bijective* if it is both injective and surjective. Finally, the operator A is said to be *correct* if A is bijective and its inverse A^{-1} is bounded on $\mathcal{Y}$. Recall that *the problem $Au = f$ is said to be well-posed* if the operator A is correct.

We denote by 0_m and I_m the $m \times m$ zero and identity matrix, respectively, $0_{m,n}$ the $m \times n$ zero matrix, and $\vec{0}$ the zero column vector.

Definition 1. Two $n \times m$ matrices $P = P(t) = (P_1(t),\ldots,P_m(t))$ and $G = G(t) = (G_1(t),\ldots,G_m(t))$, where $P_i(t) = \mathrm{col}(p_{1i}(t),\ldots,p_{ni}(t))$ and $G_i(t) = \mathrm{col}(g_{1i}(t),\ldots,g_{ni}(t))$, $i = 1,\ldots,m$, respectively, are said to be linearly independent if the vectors $P_1(t),\ldots,P_m(t), G_1(t),\ldots,G_m(t)$ are linearly independent, that is, if $\vec{c}_1, \vec{c}_2$ are two m-dimensional constant column vectors and $P(t)\vec{c}_1 + G(t)\vec{c}_2 = \vec{0}$, then $\vec{c}_1 = \vec{c}_2 = \vec{0}$.

3 General systems of m first-order ODEs

Let $A : \mathcal{X}_m \to \mathcal{X}_m$ be the differential operator defined by

$$AX(t) = X'(t) - A_0(t)X(t), \quad X(t) \in D(A) = \mathcal{X}_m^1, \tag{3.1}$$

where $A_0(t)$ is an $m \times m$ matrix with entries from $\mathcal{X}$. Let the $m \times m$ matrix $Z = Z(t) = (Z_1(t), \ldots, Z_m(t)) = (z_{ij}(t)), i, j = 1, \ldots, m$, be a fundamental matrix of the homogeneous system

$$AX(t) = \vec{0}, \tag{3.2}$$

such that

$$\Phi(Z) = \left(\begin{array}{c} \Phi(Z_1), \ldots, \Phi(Z_m) \end{array} \right) = \begin{pmatrix} \Phi(z_{11}) & \ldots & \Phi(z_{1m}) \\ \vdots & \ddots & \vdots \\ \Phi(z_{m1}) & \ldots & \Phi(z_{mm}) \end{pmatrix} = I_m,$$

where $\Phi \in \mathcal{X}^*$.

Lemma 3.1. *Let the operator A be defined as in (3.1), Z be a fundamental matrix of the homogeneous system (3.2), and $F = F(t) = \mathrm{col}(f_1(t), \ldots, f_m(t)) \in \mathcal{X}_m$. Then:*

(i) the operator $\widehat{A} : \mathcal{X}_m \to \mathcal{X}_m$, corresponding to the problem

$$\hat{A}X(t) = AX(t) = F(t), \ D(\hat{A}) = \{X(t) \in D(A) = \mathcal{X}_m^1 : \Phi(X) = \vec{0}\}, \tag{3.3}$$

is correct and the unique solution $X(t)$ of equation (3.3) is given by

$$\begin{aligned} X(t) &= \widehat{A}^{-1}F(t) \\ &= -Z(t)\Phi\left(Z(t)\int_0^t Z^{-1}(s)F(s)ds\right) + Z(t)\int_0^t Z^{-1}(s)F(s)ds, \end{aligned} \tag{3.4}$$

(ii) if in (i), $\Phi(X) = X(0)$ then

$$X(t) = \widehat{A}^{-1}F(t) = Z(t)\int_0^t Z^{-1}(s)F(s)ds. \tag{3.5}$$

Proof. (i) It is well known that every solution of the system $AX(t) = F(t)$, is given by

$$X(t) = Z(t)\vec{c} + Z(t)\int_0^t Z^{-1}(s)F(s)ds, \tag{3.6}$$

where $\vec{c}$ is an arbitrary m-dimensional constant column vector. Acting by functional vector Φ on both sides of (3.6) and taking into account the boundary condition in (3.3) and that $\Phi(Z) = I_m$, we obtain

$$\begin{aligned} \Phi(X) &= \vec{c} + \Phi\left(Z(t)\int_0^t Z^{-1}(s)F(s)ds\right) = \vec{0}, \\ \vec{c} &= -\Phi\left(Z(t)\int_0^t Z^{-1}(s)F(s)ds\right). \end{aligned}$$

Substituting $\vec{c}$ into (3.6), we get (3.4).

(ii) Equation (3.5) is derived directly from (3.4) using $\Phi(X) = X(0) = \vec{0}$. $\square$

Theorem 3.1. *Let the operators A and $\widehat{A}$, the vector F and the matrix Z be defined as in Lemma 3.1. In addition, let the $m \times (mn)$ constant matrix $M = (M_1, \ldots, M_n)$, where $M_j, j = 1, \ldots, n$, are $m \times m$ constant matrices, the functionals $\Phi, \Psi_j \in \mathcal{X}^*, j = 1, \ldots, n$, and the functional vector $\Psi = \mathrm{col}(\Psi_1, \ldots, \Psi_n)$ are given. Then:*

(i) the operator $B : \mathcal{X}_m \to \mathcal{X}_m$, corresponding to the problem

$$\begin{aligned} BX(t) &= AX(t) = F(t), \\ D(B) &= \{X(t) \in D(A) = \mathcal{X}_m^1 : \Phi(X) = \sum_{j=1}^{n} M_j \Psi_j(X)\} \end{aligned} \tag{3.7}$$

is injective if and only if

$$\det W = \det[I_{mn} - \Psi(Z)M] \neq 0, \tag{3.8}$$

(ii) if the operator B is injective, then it is also correct and the unique solution to problem (3.7) is given by

$$X(t) = B^{-1}F(t) = \widehat{A}^{-1}F(t) + ZMW^{-1}\Psi(\widehat{A}^{-1}F), \tag{3.9}$$

where $\widehat{A}^{-1}F(t)$ is the solution of system (3.3) given in (3.4).

Proof. (i) Let $\det W \neq 0$ and $X(t) \in \ker B$. Then from problem (3.7) we get

$$AX(t) = \vec{0}, \quad \Phi(X) = M\Psi(X), \tag{3.10}$$

which, since $Z \in kerA$ and $\Phi(Z) = I_m$, can be written as

$$A\,(X(t) - ZM\Psi(X)) = \vec{0}, \quad \Phi(X(t) - ZM\Psi(X)) = \vec{0}. \tag{3.11}$$

From the second equation of (3.11) by taking into account (3.3) we get $X(t) - ZM\Psi(X) \in D(\widehat{A})$ and then from the first equation of (3.11), since $\ker \widehat{A} = \{0\}$ and A is the extension of $\widehat{A}$, it follows that

$$X(t) = ZM\Psi(X). \tag{3.12}$$

Acting by the functional vector Ψ on both sides we get

$$[I_{mn} - \Psi(Z)M]\Psi(X) = W\Psi(X) = \vec{0},$$

and since $\det W \neq 0$, it is implied that $\Psi(X) = \vec{0}$. Substitution into (3.10) yields $\widehat{A}X(t) = \vec{0}$. This means that $X(t) = \vec{0}$ and therefore the operator B is injective.

Conversely, let $\det W = 0$. Then there exists a nonzero vector $\vec{c} = \mathrm{col}(c_1, \ldots, c_{mn})$, such that $W\vec{c} = \vec{0}$. Consider the element

$$X_0(t) = Z(t)M\vec{c}, \tag{3.13}$$

and note that $X_0(t) \neq \vec{0}$, since otherwise $W\vec{c} = [I_{mn} - \Psi(Z)M]\vec{c} = \vec{c} - \Psi(Z)M\vec{c} = \vec{c} = \vec{0}$. Then

$$\begin{aligned} BX_0(t) &= AX_0(t) = \vec{0}, \\ \Phi(X_0) - M\Psi(X_0) &= M\vec{c} - M\Psi(Z)M\vec{c} = M[I_{mn} - \Psi(Z)M]\vec{c} = MW\vec{c} = \vec{0}, \end{aligned}$$

and, hence, $X_0(t) \in \ker B$. Therefore, B is not injective. Thus, we proved that if B is injective, then $\det W \neq 0$.

(ii) Let $\det W \neq 0$, then the operator B is injective. Problem (3.7) can be written as

$$A\,(X(t) - Z(t)M\Psi(X)) = F(t), \quad \Phi\,(X(t) - Z(t)M\Psi(X)) = \vec{0}. \tag{3.14}$$

Then, since (3.3) we get $X(t) - Z(t)M\Psi(X) \in D(\widehat{A})$, and from (3.14) it follows that

$$X(t) = Z(t)M\Psi(X) + \widehat{A}^{-1}F. \tag{3.15}$$

Acting by the functional vector Ψ on both sides of the above equation we get

$$[I_{mn} - \Psi(Z)M]\Psi(X) = \Psi(\widehat{A}^{-1}F),$$
$$\Psi(X) = [I_{mn} - \Psi(Z)M]^{-1}\Psi(\widehat{A}^{-1}F) = W^{-1}\Psi(\widehat{A}^{-1}F).$$

Substituting into (3.12), we get solution (3.9). Since the functionals $\Psi_1, \ldots, \Psi_n$ and the operator $\widehat{A}^{-1}$ in (3.9) are bounded, then the operator B^{-1} is also bounded. Note that solution (3.9) is obtained for any arbitrary vector $F(t) \in \mathcal{X}_m$. This means that $R(B) = \mathcal{X}_m$, i.e. the operator B is everywhere solvable. So, the operator B is correct. □

Lemma 3.2. *Let the operators $A, \widehat{A}$ and the $m \times m$ fundamental matrix Z be defined as in Lemma 3.1. Then:*

(i) the set $Z \cup \widehat{A}^{-1}Z$, with $\widehat{A}^{-1}Z = (\widehat{A}^{-1}Z_1(t), \ldots, \widehat{A}^{-1}Z_m(t))$, is linearly independent,

(ii) the set $Z \cup tZ, 0 < t < 1$, is also linearly independent.

Proof. (i) The vectors $Z_1(t), \ldots, Z_m(t)$ are linearly independent since they are the columns of the fundamental matrix. Furthermore, the vectors $\widehat{A}^{-1}Z_1(t), \ldots, \widehat{A}^{-1}Z_m(t)$ are also linearly independent since $\ker \widehat{A} = \{0\}$. Let $Z\vec{c}_1 + \widehat{A}^{-1}Z\vec{c}_2 = \vec{0}$, where $\vec{c}_1, \vec{c}_2$ are two m-dimensional constant column vectors. Then, since $\ker A \cap D(\widehat{A}) = \{0\}$ [11], we have $Z\vec{c}_1 = \vec{0}$ and $\widehat{A}^{-1}Z\vec{c}_2 = \vec{0}$, and hence $\vec{c}_1 = \vec{c}_2 = \vec{0}$ since which since $Z_1, \ldots, Z_m$ and $\widehat{A}^{-1}Z_1(t), \ldots, \widehat{A}^{-1}Z_m(t)$ are linealy independent. Thus, the set $Z \cup \widehat{A}^{-1}Z$ is linearly independent.

(ii) Let $\Phi(X) = X(0)$. Then from (3.5) and (i) it follows that $\widehat{A}^{-1}Z(t) = tZ(t)$ and so the set $Z \cup tZ, 0 < t < 1$, is linearly independent. □

4 Special type systems of $2m$ first-order ODEs with constant coefficients

In this section, we consider the solvability and the construction of the exact solution of a special type of systems of $2m$ first-order ODEs with constant coefficients.

Lemma 4.1. *Let the operator A, where A_0 is an $m \times m$ nonsingular constant matrix, and the associated fundamental matrix $Z = Z(t)$ be defined as in Lemma 3.1. Let the operator $\mathbf{A} : \mathcal{X}_{2m} \to \mathcal{X}_{2m}$ be defined by*

$$\mathbf{A}U(t) = U'(t) - D_0U(t) = \vec{0}, \quad D(\mathbf{A}) = \mathcal{X}^1_{2m}, \tag{4.1}$$

where the vector $U = U(t) = \operatorname{col}(u_1(t), \ldots, u_{2m}(t)) \in \mathcal{X}^1_{2m}$ and the $2m \times 2m$ constant matrix D_0 has the special form

$$D_0 = \begin{pmatrix} 2A_0 & -A_0^2 \\ I_m & 0_m \end{pmatrix}.$$

Then the $2m \times 2m$ matrix

$$\mathbf{Z}(t) = \begin{pmatrix} Z(t) & tZ(t) \\ \int_0^t Z(s)ds + A_0^{-1} & \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix} \tag{4.2}$$

is a fundamental matrix of the homogeneous system (4.1).

Proof. The two $2m \times m$ matrices

$$\begin{pmatrix} Z \\ \int_0^t Z(s)ds + A_0^{-1} \end{pmatrix}, \quad \begin{pmatrix} tZ \\ \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix} \tag{4.3}$$

satisfy equation (4.1). Indeed, since $Z' = A_0 Z$ and $Z(0) = I_m$, we have

$$\begin{aligned}
&\begin{pmatrix} Z \\ \int_0^t Z(s)ds + A_0^{-1} \end{pmatrix}' - \begin{pmatrix} 2A_0 & -A_0^2 \\ I_m & 0_m \end{pmatrix} \begin{pmatrix} Z \\ \int_0^t Z(s)ds + A_0^{-1} \end{pmatrix} \\
&= \begin{pmatrix} -A_0 Z + A_0 \int_0^t A_0 Z(s)ds + A_0 \\ Z - Z \end{pmatrix} \\
&= \begin{pmatrix} -A_0 Z + A_0 \int_0^t Z'(s)ds + A_0 \\ 0_m \end{pmatrix} = \begin{pmatrix} 0_m \\ 0_m \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&\begin{pmatrix} tZ \\ \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix}' - \begin{pmatrix} 2A_0 & -A_0^2 \\ I_m & 0_m \end{pmatrix} \begin{pmatrix} tZ \\ \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix} \\
&= \begin{pmatrix} Z + tZ' \\ tZ \end{pmatrix} - \begin{pmatrix} 2A_0 tZ - A_0 \int_0^t sA_0 Z(s)ds + I_m \\ tZ \end{pmatrix} \\
&= \begin{pmatrix} Z + tZ' - 2A_0 tZ + A_0 \int_0^t sZ'(s)ds - I_m \\ 0_m \end{pmatrix} \\
&= \begin{pmatrix} Z + t(Z' - A_0 Z) - A_0 tZ + A_0 [sZ(s)]_0^t - \int_0^t A_0 Z(s)ds - I_m \\ 0_m \end{pmatrix} \\
&= \begin{pmatrix} Z - \int_0^t Z'(s)ds - I_m \\ 0_m \end{pmatrix} \\
&= \begin{pmatrix} Z(t) - [Z(t) - Z(0)] - I_m \\ 0_m \end{pmatrix} = \begin{pmatrix} 0_m \\ 0_m \end{pmatrix}.
\end{aligned}$$

Furthermore, as shown below, the two matrices in (4.3) are linearly independent. Let

$$\begin{pmatrix} Z \\ \int_0^t Z(s)ds + A_0^{-1} \end{pmatrix} \vec{c}_1 + \begin{pmatrix} tZ \\ \int_0^t sZ(s)ds - [A_0^{-1}]^2 \end{pmatrix} \vec{c}_2 = \vec{0},$$

where $\vec{c}_1, \vec{c}_2$ are m-dimensional constant column vectors. Then we have $Z\vec{c}_1 + tZ\vec{c}_2 = \vec{0}$ and since Z, tZ, are linearly independent by Lemma 3.2, it follows that $\vec{c}_1 = \vec{c}_2 = \vec{0}$. Thus, the two matrices in (4.3) are linearly independent and $\mathbf{Z}$ is a fundamental matrix for the system (4.1). □

Theorem 4.1. *Let the operator* $\mathbf{A}$ *and the* $2m \times 2m$ *fundamental matrix* $\mathbf{Z}$ *be defined as in Lemma 4.1. Let the vector* $\mathbf{F} = \mathbf{F}(t) = \mathrm{col}\,(f_1(t), \ldots, f_{2m}(t)) \in \mathcal{X}_{2m}$*, the vector of functionals* $\Psi = (\Psi_1, \ldots, \Psi_n)$*,* $\Psi_j \in \mathcal{X}^*, j = 1, \ldots, n$*, and* $\mathbf{M}$ *be a* $2m \times 2mn$ *constant matrix. Then:*

(i) the operator $\mathbf{B} : \mathcal{X}_{2m} \to \mathcal{X}_{2m}$ *defined by the problem*

$$\mathbf{B}U(t) = \mathbf{A}U(t) = \mathbf{F}, \quad D(\mathbf{B}) = \{U(t) \in D(\mathbf{A}) : U(0) = \mathbf{M}\Psi(U)\} \tag{4.4}$$

is injective if and only if

$$det\mathbf{W} = det[\mathbf{Z}(0) - \mathbf{M}\Psi(\mathbf{Z})] \neq 0, \tag{4.5}$$

where

$$\mathbf{Z}(0) = \begin{pmatrix} I_m & 0_m \\ A_0^{-1} & -[A_0^{-1}]^2 \end{pmatrix}, \tag{4.6}$$

(ii) the unique solution of problem (4.4) for every $\mathbf{F} \in \mathcal{X}_{2m}$ is given by

$$U(t) = \mathbf{Z}\mathbf{W}^{-1}\mathbf{M}\Psi\left(\mathbf{Z}(t)\int_0^t \mathbf{Z}^{-1}(s)\mathbf{F}(s)ds\right) + \mathbf{Z}(t)\int_0^t \mathbf{Z}^{-1}(s)\mathbf{F}(s)ds. \tag{4.7}$$

Proof. The proof follows the same procedure as for the proof of Theorem 3.1. □

5 Factorization of systems of second-order ODEs

In this section, we present the main results regarding the factorization method for solving nonlocal systems of second-order linear differential equations.

Lemma 5.1. *Let the operators $A, \widehat{A}$, where the elements of $A_0(t)$ belong to $\mathcal{X}^1$ and the functional $\Phi \in [\mathcal{X}^1]^*$, the vectors X, F and the fundamental matrix Z be defined as in Lemma 3.1. Then:*

(i) for the operator $A^2 : \mathcal{X}_m \to \mathcal{X}_m$ defined as

$$A^2X(t) = X''(t) - 2A_0(t)X'(t) + [A_0^2(t) - A_0'(t)]X(t), \quad D(A^2) = \mathcal{X}_m^2, \tag{5.1}$$

(ii) the operator $\widehat{A}^2$ defined by

$$\widehat{A}^2X = A^2X = F, \quad D(\widehat{A}^2) = \{X(t) \in D(A^2) : \Phi(X) = \vec{0},\ \Phi(AX) = \vec{0}\} \tag{5.2}$$

is correct and the unique solution of system (5.2) is given by

$$\begin{aligned} X(t) &= \widehat{A}^{-2}F(t) = \widehat{A}^{-1}Y(t) \\ &= -Z(t)\Phi\left(Z(t)\int_0^t Z^{-1}(s)Y(s)ds\right) + Z(t)\int_0^t Z^{-1}(s)Y(s)ds, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} Y(t) &= \widehat{A}^{-1}F(t) \\ &= -Z(t)\Phi\left(Z(t)\int_0^t Z^{-1}(s)F(s)ds\right) + Z(t)\int_0^t Z^{-1}(s)F(s)ds, \end{aligned} \tag{5.4}$$

(iii) in the case that $\Phi(X) = X(0)$, $Z, tZ \in ker A^2$ and $(Z,\, tZ), 0 < t < 1$, is a fundamental matrix of the homogeneous system

$$A^2X(t) = \vec{0}, \tag{5.5}$$

and

$$\widehat{A}^{-2}F(t) = Z(t)\int_0^t (t-s)Z^{-1}(s)F(s)ds. \tag{5.6}$$

Proof. (i) Let $Y(t) = AX(t) = X'(t) - A_0(t)X(t)$. Then

$$\begin{aligned} A^2X(t) &= AY(t) = Y'(t) - A_0(t)Y(t) \\ &= [X'(t) - A_0(t)X(t)]' - A_0(t)[X'(t) - A_0(t)X(t)] \\ &= X''(t) - A_0'(t)X(t) - A_0(t)X'(t) - A_0(t)X'(t) + A_0^2(t)X(t) \\ &= X''(t) - 2A_0(t)X'(t) + [A_0^2(t) - A_0'(t)]X(t). \end{aligned} \tag{5.7}$$

It easily follows that if $D(A) = \mathcal{X}_m^1$, then $D(A^2) = \mathcal{X}_m^2$.

(ii) By using (5.7) system (5.2) can be factorized into the following two systems of first order differential equations

$$\widehat{A}Y(t) = AY(t) = Y'(t) - A_0(t)Y(t) = F(t), \quad \Phi(Y) = \vec{0},$$

$$\widehat{A}X(t) = AX(t) = X'(t) - A_0(t)X(t) = Y(t), \quad \Phi(X) = \vec{0},$$

which, by Lemma 3.1, are well-posed and their solutions are given by $Y(t) = \widehat{A}^{-1}F(t)$ and $X(t) = \widehat{A}^{-1}Y(t)$, respectively, from where (5.3) and (5.4) are derived. The operator $\widehat{A^2}$ is correct because it is a superposition of two correct operators [14].

(iii) Let $A^2X = \vec{0}$. Setting $Y = AX$ we get $AY = \vec{0}$. Then $Y = Z\vec{c}_1$ or $AX = Z\vec{c}_1$, which gives $X = Z\vec{c}_2 + \widehat{A}^{-1}Y = Z\vec{c}_2 + \widehat{A}^{-1}Z\vec{c}_1$, where $\vec{c}_1, \vec{c}_2$ are m-dimensional constant column vectors. From here, taking into account (5.3) and $\Phi(X) = X(0)$, for $F = Z\vec{c}_1$ we obtain $\widehat{A}^{-1}Z = tZ$ and $X(t) = Z(t)\vec{c}_2 + tZ(t)\vec{c}_1 \in ker\, A^2$. By Lemma 3.2, the system $Z \cup tZ$ is linearly independent. Hence $Z, tZ \in ker\, A^2$ and the system (Z, tZ) constitutes a fundamental solution to (5.5). From (5.3), (5.4), because of $\Phi(0) = X(0) = \vec{0}$, by Fubini's theorem, equality (5.6) easily follows. □

Theorem 5.1. *Let the operator $\mathcal{A} : \mathcal{X}_m \to \mathcal{X}_m$ be defined by*

$$\mathcal{A}X(t) = X''(t) - S(t)X'(t) - Q(t)X(t), \quad D(\mathcal{A}) = \mathcal{X}_m^2, \tag{5.8}$$

where $Q(t)$ and $S(t)$ are $m \times m$ matrices with entries from $\mathcal{X}$ and $\mathcal{X}^1$, respectively, and the operator $B_2 : \mathcal{X}_m \to \mathcal{X}_m$ be defined as

$$\begin{aligned} B_2X(t) &= \mathcal{A}X(t) = F(t), \\ D(B_2) &= \{X(t) \in \mathcal{X}_m^2 : \Phi(X) = \sum_{i=1}^{n} M_i\Psi_i(X), \\ &\qquad \Phi(X') = \Phi(TX) + \sum_{j=1}^{r} N_j\Theta_j(X)\}, \end{aligned} \tag{5.9}$$

where $F \in \mathcal{X}_m$, $T(t)$ is an $m \times m$ matrix with entries from $\mathcal{X}$, $M_j, j = 1, \ldots, n$, and $N_j, j = 1, \ldots, r$, are $m \times m$ constant matrices, $\Phi \in [\mathcal{X}^1]^*$, $\Psi_j \in \mathcal{X}^*, j = 1, \ldots, n$, and $\Theta_j \in \mathcal{X}^*, j = 1, \ldots, r$. Then:

(i) if

$$Q(t) = \frac{1}{2}S'(t) - \frac{1}{4}S^2(t), \tag{5.10}$$

the operator $\mathcal{A}$ can be factorized as follows

$$\mathcal{A}X(t) = A^2X(t), \quad X(t) \in D(\mathcal{A}), \tag{5.11}$$

where

$$AX(t) = X'(t) - \frac{1}{2}S(t)X(t), \quad D(A) = \mathcal{X}_m^1, \tag{5.12}$$

(ii) if, in addition to (i), we have $T(t) = \frac{1}{2}S(t)$, the operator B_2 is injective if and only if

$$\det W_2 = \det \begin{pmatrix} I_{mn} - \Psi(Z)M & -\Psi(\widehat{A}^{-1}Z)N \\ -\Theta(Z)M & I_{mk} - \Theta(\widehat{A}^{-1}Z)N \end{pmatrix} \neq 0, \tag{5.13}$$

where W_2 is an $m(n+r) \times m(n+r)$ matrix, Z is a fundamental matrix of the system $AX = \vec{0}$,

$$\widehat{A}X(t) = AX(t) = F(t), \quad D(\widehat{A}) = \{X(t) \in D(A) : \Phi(X) = \vec{0}\}, \tag{5.14}$$

and $\Psi = \mathrm{col}(\Psi_1, \ldots, \Psi_n)$ and $\Theta = \mathrm{col}(\Theta_1, \ldots, \Theta_r)$, *(iii) under (ii), the operator B_2 is correct and the unique solution of system (5.9) is given by*

$$X(t) = B_2^{-1}F(t) = \widehat{A}^{-2}F(t) + \left(Z(t)M, \widehat{A}^{-1}Z(t)N\right) W_2^{-1} \begin{pmatrix} \Psi(\widehat{A}^{-2}F) \\ \Theta(\widehat{A}^{-2}F) \end{pmatrix}, \tag{5.15}$$

where $\widehat{A}^{-2}F(t)$, $\widehat{A}^{-1}F(t)$ are given by (5.3), (5.4), respectively.

Proof. (i) Denote $Y(t) = X'(t) - \frac{1}{2}S(t)X(t)$. Then since (5.10) and (5.12), we get

$$\begin{aligned} \mathcal{A}X(t) &= X''(t) - S(t)X'(t) - Q(t)X(t) \\ &= X''(t) - S(t)X'(t) - \left[\frac{1}{2}S'(t) - \frac{1}{4}S^2(t)\right] X(t) \\ &= X''(t) - \frac{1}{2}\left(S(t)X(t)\right)' - \frac{1}{2}S\left(X' - \frac{1}{2}SX\right) \\ &= \left(X' - \frac{1}{2}SX\right)' - \frac{1}{2}S\left(X' - \frac{1}{2}SX\right) = Y' - \frac{1}{2}SY = AY = A^2X. \end{aligned}$$

From $D(A) = \mathcal{X}_m$ it easily follows that $D(A^2) = \mathcal{X}_m^2$. Thus, we proved that $B_2X(t) = \mathcal{A}X(t) = A^2X(t)$.

(ii) If $T(t) = \frac{1}{2}S(t)$, then $\Phi(X') - \Phi(TX) = \Phi\left(X' - \frac{1}{2}SX\right) = \Phi(AX)$ and problem (5.9) is reduced to

$$B_2X(t) = A^2X(t) = F(t), \quad \Phi(X) = M\Psi(X), \quad \Phi(AX) = N\Theta(X). \tag{5.16}$$

Let $\det W_2 \neq 0$ and $X(t) \in \ker B_2$. Then from problem (5.16) we get

$$B_2X(t) = A^2X(t) = \vec{0}, \quad \Phi(X) = M\Psi(X), \quad \Phi(AX) = N\Theta(X), \tag{5.17}$$

which, since $\Phi(Z) = I_m$ and $AZ = 0_m$, can be represented as

$$A\left(AX(t) - ZN\Theta(X)\right) = \vec{0}, \tag{5.18}$$

$$\Phi(X(t) - ZM\Psi(X)) = \vec{0}, \tag{5.19}$$

$$\Phi(AX(t) - ZN\Theta(X)) = \vec{0}. \tag{5.20}$$

Further taking into account (3.3), we get $X(t) - ZM\Psi(X),\ AX(t) - ZN\Theta(X) \in D(\widehat{A})$ and from (5.18), because of A is an extension of $\widehat{A}$ and $\ker \widehat{A} = \{0\}$, it follows that

$$\begin{gathered} AX(t) = ZN\Theta(X), \\ A(X(t) - ZM\Psi(X)) = ZN\Theta(X), \\ \widehat{A}(X(t) - ZM\Psi(X)) = ZN\Theta(X), \\ X(t) = ZM\Psi(X) + \widehat{A}^{-1}ZN\Theta(X). \end{gathered}$$

Then acting by functional vectors Ψ, Θ on both sides of the above equation we get

$$[I_{mn} - \Psi(Z)M]\Psi(X) - \Psi(\widehat{A}^{-1}Z)N\Theta(X) = \vec{0}, \tag{5.21}$$

$$-\Theta(Z)M\Psi(X) + [I_{mk} - \Theta(\widehat{A}^{-1}Z)N]\Theta(X) = \vec{0}. \tag{5.22}$$

From the last system, since $\det W_2 \neq 0$, it follows that $\Psi(X) = \vec{0}$, $\Theta(X) = \vec{0}$. Substituting these values into (5.17), we obtain that $\widehat{A}^2X(t) = \vec{0}$, and so, because $\widehat{A}^2$ is correct, we have $X(t) = \vec{0}$. Then $\ker B_2 = \{\vec{0}\}$ and B_2 is injective.

Conversely, let $\det W_2 = 0$. Then there exists a nonzero constant vector $\vec{c} = \mathrm{col}(\mathbf{c}_1, \mathbf{c}_2)$, where $\mathbf{c}_1 = \mathrm{col}(c_{11}, \ldots, c_{1,mn})$, $\mathbf{c}_2 = \mathrm{col}(c_{21}, \ldots, c_{2,mk})$, such that

$$W_2\vec{c} = \begin{pmatrix} I_{mn} - \Psi(Z)M & -\Psi(\widehat{A}^{-1}Z)N \\ -\Theta(Z)M & I_{mk} - \Theta(\widehat{A}^{-1}Z)N \end{pmatrix} \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}. \tag{5.23}$$

Consider the vector

$$X_0(t) = Z(t)M\mathbf{c}_1 + \widehat{A}^{-1}Z(t)N\mathbf{c}_2. \tag{5.24}$$

Note that $X_0(t) = \vec{0}$, if and only if $M\mathbf{c}_1 = \vec{0}$, $N\mathbf{c}_2 = \vec{0}$, since $Z(t)$ is the fundamental matrix and the set $Z(t) \cup \widehat{A}^{-1}Z(t)$, by Lemma 3.2, is linearly independent. But if $M\mathbf{c}_1 = \vec{0}$, $N\mathbf{c}_2 = \vec{0}$, then from (5.23) follows that $\mathbf{c}_1 = \vec{0}$, $\mathbf{c}_2 = \vec{0}$. Thus, we obtain $\vec{c} = \vec{0}$. But by hypothesis $\vec{c} \neq \vec{0}$. So $X_0(t) \neq \vec{0}$. Further using (5.24) and taking into account (5.23), we find

$$\begin{gathered} B_2X_0(t) = A^2X_0(t) = \vec{0}, \\ \Phi(X_0) - M\Psi(X_0) = M[I_{mn} - \Psi(Z)M]\mathbf{c}_1 - M\Psi(\widehat{A}^{-1}Z)N\mathbf{c}_2 = \vec{0}, \\ AX_0(t) = Z(t)N\mathbf{c}_2, \\ \Phi(AX_0) - N\Theta(X_0) = -N\Theta(Z)M\mathbf{c}_1 + N[I_{mk} - \Theta(\widehat{A}^{-1}Z)N]\mathbf{c}_2 = \vec{0}. \end{gathered}$$

From here it follows that $X_0(t) \in \ker B_2$ and B_2 is not injective. Thus, by way of contradiction we proved that if B_2 is injective, then $\det W \neq 0$.

(iii) Let $\det W_2 \neq 0$, then the operator B_2 is injective. From (5.16) we obtain

$$A\left(AX(t) - Z(t)N\Theta(X)\right) = F(t), \quad \Phi\left(AX(t) - Z(t)N\Theta(X)\right) = \vec{0}. \tag{5.25}$$

Then since $\widehat{A}$ is a restriction of A and (3.3), we get $AX(t) - Z(t)N\Theta(X) \in D(\widehat{A})$. From (5.25) and first boundary condition (5.16) it follows that

$$AX(t) = Z(t)N\Theta(X) + \widehat{A}^{-1}F(t), \quad \Phi(X(t) - Z(t)M\Psi(X)) = \vec{0}. \tag{5.26}$$

By means (3.3) we get $X(t) - Z(t)M\Psi(X) \in D(\widehat{A})$. Then from (5.26), taking into account that $\widehat{A}$ is a restriction of A, we get

$$\begin{gathered} \widehat{A}[X(t) - Z(t)M\Psi(X)] - Z(t)N\Theta(X) = \widehat{A}^{-1}F(t), \\ X(t) - Z(t)M\Psi(X) - \widehat{A}^{-1}Z(t)N\Theta(X) = \widehat{A}^{-2}F(t). \end{gathered} \tag{5.27}$$

Acting by functional vectors Ψ, Θ on both sides of the above equation, obtain

$$[I_{mn} - \Psi(Z)M]\Psi(X) - \Psi(\widehat{A}^{-1}Z)N\Theta(X) = \Psi(\widehat{A}^{-2}F), \tag{5.28}$$

$$-\Theta(Z)M\Psi(X) + [I_{mk} - \Theta(\widehat{A}^{-1}Z)N]\Theta(X) = \Theta(\widehat{A}^{-2}F), \tag{5.29}$$

or

$$W_2\begin{pmatrix} \Psi(X) \\ \Theta(X) \end{pmatrix} = \begin{pmatrix} \Psi(\widehat{A}^{-2}F) \\ \Theta(\widehat{A}^{-2}F) \end{pmatrix}.$$

The last equation yelds

$$\begin{pmatrix} \Psi(X) \\ \Theta(X) \end{pmatrix} = W_2^{-1}\begin{pmatrix} \Psi(\widehat{A}^{-2}F) \\ \Theta(\widehat{A}^{-2}F) \end{pmatrix}.$$

Substituting this value into (5.27), we get solution (5.15). Since the functionals $\Psi_1, \ldots, \Psi_n, \Theta_1, \ldots, \Theta_k$ and the operators $\widehat{A}^{-1}, \widehat{A}^{-2}$ in (5.15) are bounded, then the operator B_2^{-1} is also bounded. Note that formula (5.15) was proved for any arbitrary vector $F(t) \in \mathcal{X}_m$. This means that $R(B_2) = \mathcal{X}_m$, i.e. the operator B_2 is everywhere solvable. Before we proved that B_2 is injective and B_2^{-1} is bounded. Hence, B_2 is correct. □

Corollary 5.1. *In Theorem 5.1 let $\Phi(X) = X(t_0)$, $t_0 \in [0,1]$, $r = n$, $\Psi_j, \Theta_j \in [\mathcal{X}^1]^*$, $j = 1, \ldots, n$, $Q(t)$ satisfies (5.10) and T be a constant matrix. Then*

$$\begin{aligned} B_2 X(t) &= \mathcal{A}X(t) = F(t), \\ D(B_2) &= \{X(t) \in \mathcal{X}_m^2 : X(t_0) = \sum_{j=1}^n M_j \Psi_j(X), \\ &\qquad X'(t_0) = TX(t_0) + \sum_{j=1}^n N_j \Theta_j(X)\}. \end{aligned} \tag{5.30}$$

(i) If

$$T = \frac{1}{2} S(t_0), \quad N_j = M_j, \quad \Theta_j(X) = \Psi_j(AX),\ i = 1, \ldots, n, \tag{5.31}$$

then there exists an operator $B : \mathcal{X}_m \to \mathcal{X}_m$ defined by

$$BX(t) = AX(t), \quad D(B) = \{X(t) \in D(A) : X(t_0) = \sum_{j=1}^n M_j \Psi_j(X)\}, \tag{5.32}$$

such that B_2 can be factorized into $B_2 = B^2$,

(ii) in addition, problem (5.30) is uniquely solvable if and only if

$$\det W_3 = \det[I_{mn} - \Psi(Z)M] \neq 0, \tag{5.33}$$

and its unique solution for all $F \in \mathcal{X}_m$ is given by

$$X(t) = B_2^{-1} F(t) = \widehat{A}^{-1} Y(t) + Z M W_3^{-1} \Psi(\widehat{A}^{-1} Y), \tag{5.34}$$

where

$$Y(t) = \widehat{A}^{-1} F(t) + Z M W_3^{-1} \Psi(\widehat{A}^{-1} F), \tag{5.35}$$

$$\widehat{A}^{-1} F(t) = -Z(t) Z(t_0) \int_0^{t_0} Z^{-1}(s) F(s) ds + Z(t) \int_0^t Z^{-1}(s) F(s) ds, \tag{5.36}$$

$$\widehat{A}^{-1} Y(t) = -Z(t) Z(t_0) \int_0^{t_0} Z^{-1}(s) Y(s) ds + Z(t) \int_0^t Z^{-1}(s) Y(s) ds, \tag{5.37}$$

$Z = Z(t)$ is a fundamental matrix of $AX(t) = \vec{0}$, satisfying $Z(t_0) = I_m$, and

$$\widehat{A} X(t) = AX(t), \quad D(\widehat{A}) = \{X(t) \in D(A) : X(t_0) = \vec{0}\}. \tag{5.38}$$

Proof. (i) Consider the operator B defined by (5.32), namely

$$BX(t) = AX(t) = X'(t) - \frac{1}{2} S(t) X(t), \quad X(t) \in D(B).$$

Then for $X(t) \in D(B^2) \cap D(B_2)$, since (5.10), the following formula is valid

$$B^2 X(t) = A^2 X(t) = X''(t) - S(t) X'(t) - \left[\frac{1}{2} S'(t) - \frac{1}{4} S^2(t)\right] X(t) = B_2 X(t).$$

It remains to prove that $D(B^2) = D(B_2)$ for T, N and $\Theta(X)$, satisfying (5.31). Indeed, because of the equality $BX = AX$, $X \in D(B)$, we obtain

$$\begin{aligned} D(B^2) &= \{X(t) \in D(B) : BX(t) \in D(B)\} \\ &= \{X(t) \in D(A^2) : X(t_0) = M\Psi(X),\ (BX)(t_0) = M\Psi(BX)\} \\ &= \{X(t) \in D(A^2) : X(t_0) = M\Psi(X),\ (AX)(t_0) = M\Psi(AX)\}, \end{aligned} \tag{5.39}$$

where

$$(AX)(t_0) = \Phi(AX) = \Phi\left(X'(t) - \frac{1}{2}S(t)X(t)\right) = X'(t_0) - \frac{1}{2}S(t_0)X(t_0) = X'(t_0) - TX(t_0).$$

Then from (5.39) we get

$$\begin{aligned} D(B^2) &= \{X(t) \in D(A^2) : X(t_0) = M\Psi(X),\ X'(t_0) = TX(t_0) + M\Psi(AX)\} \\ &= D(B_2). \end{aligned}$$

(ii) By Theorem 5.1, the operator B_2 is injective if and only if (5.13) is fulfilled, where $k = n$, $N = M$, $\Theta(Z) = \Psi(AZ)$ and $\Theta(\widehat{A}^{-1}Z) = \Psi(A\widehat{A}^{-1}Z)$, or if and only if

$$\det W_2 = \det \begin{pmatrix} I_{mn} - \Psi(Z)M & -\Psi(\widehat{A}^{-1}Z)M \\ -\Psi(AZ)M & I_{mn} - \Psi(A\widehat{A}^{-1}Z)M \end{pmatrix} \neq 0,$$

or

$$\det \begin{pmatrix} I_{mn} - \Psi(Z)M & -\Psi(\widehat{A}^{-1}Z)M \\ 0_{mn} & I_{mn} - \Psi(Z)M \end{pmatrix} = [\det (I_{mn} - \Psi(Z)M)]^2 = [\det W_3]^2 \neq 0.$$

Thus, $B_2 = B^2$ is injective if and only if $\det W_3 \neq 0$. The problem $B^2X(t) = F(t)$ by substituting $BX(t) = Y(t)$ is reduced to two systems $BY(t) = F(t)$ and $BX(t) = Y(t)$. By Theorem 3.1, a unique solution to the first system is given by (5.35), where $\widehat{A}^{-1}F(t)$ is given by (3.4) or (5.36). Substituting the value $Y(t)$ from (5.35) into the system $BX(t) = Y(t)$ and again using Theorem 3.1, we obtain (5.34). □

6 Examples

Example 1 In the function space $C^1[0,1]$, the following system of four first-order differential equations with four homogeneous initial conditions

$$\begin{aligned} y_1'(t) &+ 2\pi y_2(t) + \pi^2 y_3(t) = \cos \pi t, \\ y_2'(t) &- 2\pi y_1(t) - \pi^2 y_4(t) = \sin \pi t, \\ y_3'(t) &- y_1(t) = 2\sin \pi t, \\ y_4'(t) &- y_2(t) = -\cos \pi t, \\ y_1(0) &= y_2(0) = y_3(0) = y_4(0) = 0, \end{aligned} \tag{6.1}$$

has the unique solution

$$\begin{aligned} y_1(t) &= \frac{1}{4}\left[t(\pi+4)\cos \pi t + \left(\pi t^2(3\pi - 2) - 1\right)\sin \pi t\right], \\ y_2(t) &= \frac{1}{4}\left[\pi t^2(2 - 3\pi)\cos \pi t + t(\pi+4)\sin \pi t\right], \\ y_3(t) &= \frac{1}{4}\left[t^2(2-3\pi)\cos \pi t + 7t\sin \pi t\right], \\ y_4(t) &= \frac{1}{4\pi}\left[\left(3 - \pi t^2(3\pi - 2)\right)\sin \pi t - 7\pi t\cos \pi t\right]. \end{aligned} \tag{6.2}$$

Proof. Let $Y = Y(t) = \operatorname{col}(y_1(t), y_2(t), y_3(t), y_4(t))$ and write (6.1) in the matrix form

$$Y'(t) - D_0Y(t) = \mathbf{F}, \tag{6.3}$$

where

$$D_0 = \begin{pmatrix} 0 & -2\pi & \pi^2 & 0 \\ 2\pi & 0 & 0 & \pi^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \cos\pi t \\ \sin\pi t \\ 2\sin\pi t \\ -\cos\pi t \end{pmatrix}.$$

Note that D_0 can be written as

$$D_0 = \begin{pmatrix} 2A_0 & -A_0^2 \\ I_2 & 0_2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}, \quad A_0^2 = \begin{pmatrix} -\pi^2 & 0 \\ 0 & -\pi^2 \end{pmatrix}.$$

Let $\mathcal{X} = C[0,1]$, $\mathcal{X}^1 = C^1[0,1]$, $m = 2$, $X = \operatorname{col}(x_1(t), x_2(t))$. Consider the homogeneous system

$$X'(t) - A_0 X(t) = \vec{0}.$$

It can be easily shown that the fundamental matrix of this system is

$$Z = \begin{pmatrix} \cos\pi t & -\sin\pi t \\ \sin\pi t & \cos\pi t \end{pmatrix}.$$

Then from (4.2) it follows that the fundamental matrix of the homogeneous system

$$Y'(t) - D_0 Y(t) = \vec{0}$$

is

$$\mathbf{Z}(t) = \begin{pmatrix} \cos\pi t & -\sin\pi t & t\cos\pi t & -t\sin\pi t \\ \sin\pi t & \cos\pi t & t\sin\pi t & t\cos\pi t \\ \frac{1}{\pi}\sin\pi t & \frac{1}{\pi}\cos\pi t & \frac{1}{\pi^2}\cos\pi t + \frac{t}{\pi}\sin\pi t & -\frac{1}{\pi^2}\sin\pi t + \frac{t}{\pi}\cos\pi t \\ -\frac{1}{\pi}\cos\pi t & \frac{1}{\pi}\sin\pi t & -\frac{t}{\pi}\cos\pi t + \frac{1}{\pi^2}\sin\pi t & \frac{1}{\pi^2}\cos\pi t + \frac{t}{\pi}\sin\pi t \end{pmatrix}. \tag{6.4}$$

Since $\mathbf{M} \equiv \mathbf{0}$ it follows from (4.5) and (6.4) that $\det\mathbf{W} = \det\mathbf{Z}(0) = 1/\pi^4 \neq 0$ and hence by Theorem 4.1 problem (6.3) is uniquely solvable and its solution is given by (4.7), i.e.

$$Y(t) = \mathbf{Z}\int_0^t \mathbf{Z}^{-1}(s)\mathbf{F}(s)ds,$$

where

$$\mathbf{Z}^{-1}(t) = \begin{pmatrix} \cos\pi t - \pi t\sin\pi t & \pi t\cos\pi t + \sin\pi t & -\pi^2 t\cos\pi t & -\pi^2 t\sin\pi t \\ -\pi t\cos\pi t - \sin\pi t & \cos\pi t - \pi t\sin\pi t & \pi^2 t\sin\pi t & -\pi^2 t\cos\pi t \\ \pi\sin\pi t & -\pi\cos\pi t & \pi^2\cos\pi t & \pi^2\sin\pi t \\ \pi\cos\pi t & \pi\sin\pi t & -\pi^2\sin\pi t & \pi^2\cos\pi t \end{pmatrix}.$$

After performing the calculations, we get solution (6.2). □

Example 2 Let $X(t) = \operatorname{col}(x(t), y(t))$, $F(t) = \operatorname{col}(f_1(t), f_2(t))$. Find the unique solution of the problem $B_2 X(t) = F(t)$ on $C[0,1]$ defined by

$$\begin{gathered} x''(t) - 2x'(t) - 4y'(t) + 9x(t) + 8y(t) = f_1(t), \\ y''(t) - 8x'(t) - 6y'(t) + 16x(t) + 17y(t) = f_2(t), \\ x(0) = 3x(1), \quad y(0) = -2y(1), \\ x'(0) = x(0) + 2y(0) + 3x'(1) - 3x(1) - 6y(1), \\ y'(0) = 4x(0) + 3y(0) - 2y'(1) + 8x(1) + 6y(1). \end{gathered} \tag{6.5}$$

Proof. First we rewrite problem (6.5) in the matrix form

$$B_2\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix} x''(t)\\ y''(t)\end{pmatrix} - \begin{pmatrix} 2 & 4\\ 8 & 6\end{pmatrix}\begin{pmatrix} x'(t)\\ y'(t)\end{pmatrix} + \begin{pmatrix} 9 & 8\\ 16 & 17\end{pmatrix}\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix} f_1(t)\\ f_2(t)\end{pmatrix}, \tag{6.6}$$

$$\begin{pmatrix} x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix} 3 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix} x(1)\\ y(1)\end{pmatrix},$$

$$\begin{pmatrix} x'(0)\\ y'(0)\end{pmatrix} = \begin{pmatrix} 1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix} x(0)\\ y(0)\end{pmatrix} + \begin{pmatrix} 3 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix} x'(1)-x(1)-2y(1)\\ y'(1)-4x(1)-3y(1)\end{pmatrix}.$$

If we compare problem (6.6) with (5.30), it is natural to take $\mathcal{X} = C = C[0,1]$, $\mathcal{X}^1 = C^1[0,1] = C_1$, $\mathcal{X}^2 = C^2[0,1]$, $\mathcal{X}_2^1 = C_2^1[0,1] = C_2^1$, $m = 2$, $n = 1$, $t_0 = 0$,

$S(t) = \begin{pmatrix} 2 & 4\\ 8 & 6\end{pmatrix}$, $Q(t) = -\begin{pmatrix} 9 & 8\\ 16 & 17\end{pmatrix}$, $T = \begin{pmatrix} 1 & 2\\ 4 & 3\end{pmatrix}$,

$M = N = \begin{pmatrix} 3 & 0\\ 0 & -2\end{pmatrix}$, $X(t_0) = \begin{pmatrix} x(0)\\ y(0)\end{pmatrix}$, $X'(t_0) = \begin{pmatrix} x'(0)\\ y'(0)\end{pmatrix}$,

$\Psi(X) = X(1) = \begin{pmatrix} x(1)\\ y(1)\end{pmatrix}$, $\Theta(X) = \begin{pmatrix} x'(1)-x(1)-2y(1)\\ y'(1)-4x(1)-3y(1)\end{pmatrix} = \begin{pmatrix} x'(1)\\ y'(1)\end{pmatrix} - \begin{pmatrix} 1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix} x(1)\\ y(1)\end{pmatrix}$.

By Remark 1, it follows that $\Psi \in [C_2[0,1]]^*$ and $\Theta \in [C_2^1[0,1]]^*$, since Ψ_i, Θ_i, $i = 1,2$ are linear and $|\Psi(X)| \le ||X(t)||_{C_2}$ and

$$|\Theta(X)| \le 5(||x'(t)||_C + ||y'(t)||_C + ||x(t)||_C + ||y(t)||_C) = 5(||X'(t)||_{C_2} + ||X(t)||_{C_2}) = 5||X(t)||_{C_2^1}.$$

It is easy to verify that Q and S satisfy (5.10), then by Theorem 5.1, there exists the operator A defined by (5.12), namely

$$AX(t) = X'(t) - \frac{1}{2}S(t)X(t) = \begin{pmatrix} x'(t)\\ y'(t)\end{pmatrix} - \begin{pmatrix} 1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix} x(t)\\ y(t)\end{pmatrix}.$$

Note that $\Theta(X) = \Psi(AX) = (AX)(1)$, $M = N$, $T = \frac{1}{2}S(0)$, i.e. conditions (5.31) are fulfilled. Then, by Corollary 5.1, problem (6.6) is uniquelly solved if and only if (5.33) holds, namely $\det W_3 = \det[I_2 - \Psi(Z)M] \neq 0$. It is easy to verify that the fundamental matrix $Z = Z(t)$, $Z(0) = I_2$ for the system $AX(t) = \vec{0}$ has the form

$$Z = \frac{1}{3}\begin{pmatrix} e^{5t}+2e^{-t} & e^{5t}-e^{-t}\\ 2e^{5t}-2e^{-t} & 2e^{5t}+e^{-t}\end{pmatrix}, \quad Z^{-1} = \frac{1}{3}\begin{pmatrix} e^{-5t}+2e^{t} & e^{-5t}-e^{t}\\ 2e^{-5t}-2e^{t} & 2e^{-5t}+e^{t}\end{pmatrix},$$

$$\Phi(Z) = I_2, \ \det W_3 \neq 0,$$

$$W_3^{-1} = \frac{1}{e^6 - 18e^5 + 3e - 4}\begin{pmatrix} 4e^6+3e+2) & 2(1-e^6)\\ 6(e^6-1) & -3(e^6-e+2)\end{pmatrix}.$$

By Corollary 5.1, problem (6.5) has the unique solution which is given by (5.34), where $\widehat{A}^{-1}F(t) = Z(t)\int_0^t Z^{-1}(s)F(s)ds$, $\quad \Psi(\widehat{A}^{-1}F) = (\widehat{A}^{-1}F)(1)$,

$$Y(t) = \widehat{A}^{-1}F(t) + ZMW_3^{-1}(\widehat{A}^{-1}F)(1), \quad \widehat{A}^{-1}Y(t) = Z(t)\int_0^t Z^{-1}(s)Y(s)ds,$$

$$\Psi(\widehat{A}^{-1}Y) = (\widehat{A}^{-1}Y)(1).$$

Substituting these values into (5.34), we obtain the unique solution to (6.5)

$$X(t) = \widehat{A}^{-1}Y(t) + ZMW_3^{-1}(\widehat{A}^{-1}Y)(1).$$

□

Example 3 The following system of two second-order differential equations with nonlocal boundary conditions

$$\begin{aligned} y''(t) &+ 2\pi x'(t)-\pi^2 y(t)=\sin\pi t,\\ x''(t) &- 2\pi y'(t)-\pi^2 x(t)=\cos\pi t,\\ y(0) &= -2y(1)+2x(1),\\ x(0) &= x(1),\\ y'(0) &= -\pi x(0)-2y'(1)-2\pi x(1)+2x'(1)-2\pi y(1),\\ x'(0) &= \pi y(0)+x'(1)-\pi y(1) \end{aligned} \tag{6.7}$$

has the unique solution

$$\begin{aligned} y(t) &= \frac{t-2}{2\pi}\cos\pi t-\frac{1}{2\pi^2}\sin\pi t,\\ x(t) &= \frac{t-2}{2\pi}\sin\pi t. \end{aligned} \tag{6.8}$$

Proof. First we write problem (6.7) in the matrix form

$$\begin{aligned} \begin{pmatrix} y''(t)\\ x''(t)\end{pmatrix} &- \begin{pmatrix} 0 & -2\pi\\ 2\pi & 0\end{pmatrix}\begin{pmatrix} y'(t)\\ x'(t)\end{pmatrix}-\begin{pmatrix} \pi^2 & 0\\ 0 & \pi^2\end{pmatrix}\begin{pmatrix} y(t)\\ x(t)\end{pmatrix}=\begin{pmatrix} \sin\pi t\\ \cos\pi t\end{pmatrix},\\ \begin{pmatrix} y(0)\\ x(0)\end{pmatrix} &= \begin{pmatrix} -2 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix} y(1)\\ x(1)\end{pmatrix},\\ \begin{pmatrix} y'(0)\\ x'(0)\end{pmatrix} &= \begin{pmatrix} 0 & -\pi\\ \pi & 0\end{pmatrix}\begin{pmatrix} y(0)\\ x(0)\end{pmatrix}+\begin{pmatrix} -2 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix} y'(1)+\pi x(1)\\ x'(1)-\pi y(1)\end{pmatrix}. \end{aligned} \tag{6.9}$$

If we compare problem (6.9) with (5.30), it is natural to take $\mathcal{X}=C[0,1]$, $\mathcal{X}^1=C^1[0,1]$, $\mathcal{X}^2=C^2[0,1]$, $m=2$, $n=1$, $t_0=0$,
$S(t)=\begin{pmatrix} 0 & -2\pi\\ 2\pi & 0\end{pmatrix}$, $Q(t)=\begin{pmatrix} \pi^2 & 0\\ 0 & \pi^2\end{pmatrix}$, $T=\begin{pmatrix} 0 & -\pi\\ \pi & 0\end{pmatrix}$,
$M=N=\begin{pmatrix} -2 & 2\\ 0 & 1\end{pmatrix}$, $X(t)=\begin{pmatrix} y(t)\\ x(t)\end{pmatrix}$, $X(t_0)=\begin{pmatrix} y(0)\\ x(0)\end{pmatrix}$, $X'(t_0)=\begin{pmatrix} y'(0)\\ x'(0)\end{pmatrix}$,
$\Psi(X)=\begin{pmatrix} y(1)\\ x(1)\end{pmatrix}$, $\Theta(X)=\begin{pmatrix} y'(1)+\pi x(1)\\ x'(1)-\pi y(1)\end{pmatrix}$, $F(t)=\begin{pmatrix} \sin\pi t\\ \cos\pi t\end{pmatrix}$.
By Remark 1, it follows that $\Psi\in[C_2^1[0,1]]^*$ and $\Theta\in[C_2^1[0,1]]^*$, since Ψ_i,Θ_i, $i=1,2$ are linear and

$$|\Psi(X)|\le ||X(t)||_{C_2},\quad |\Theta(X)|\le\pi||X(t)||_{C_2^1}.$$

It is easy to verify that Q and S satisfy (5.10), then, by Theorem 5.1, there exists the operator A defined by (5.12), namely

$$AX(t)=X'(t)-\frac{1}{2}S(t)X(t)=\begin{pmatrix} y'(t)\\ x'(t)\end{pmatrix}-\begin{pmatrix} 0 & -\pi\\ \pi & 0\end{pmatrix}\begin{pmatrix} y(t)\\ x(t)\end{pmatrix}.$$

Let $Z=Z(t)$, $Z(0)=I_2$ be a fundamental matrix to the system $AX(t)=\vec{0}$. Note that $\Theta(X)=\Psi(AX)=(AX)(1)$, $M=N$, $T=\frac{1}{2}S(0)$, i.e. conditions (5.31) are fulfilled. Then, by Corollary 5.1, problem (6.7) is uniquelly solved if and only if (5.33) holds, namely $\det W_3=\det[I_2-\Psi(Z)M]\neq 0$. It is easy to verify that

$$Z=\begin{pmatrix} \cos\pi t & -\sin\pi t\\ \sin\pi t & \cos\pi t\end{pmatrix},\ Z^{-1}=\begin{pmatrix} \cos\pi t & \sin\pi t\\ -\sin\pi t & \cos\pi t\end{pmatrix},\ \Psi(Z)=\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix},$$

$$\Phi(Z) = Z(0) = I_2, \quad \det W_3 = \det \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix} \neq 0, \quad W_3^{-1} = -\begin{pmatrix} 1 & 1 \\ 0 & 1/2 \end{pmatrix}.$$

By Corollary 5.1, problem (6.7) has solution given by (5.34), where $\widehat{A}^{-1}F(t) = Z(t)\int_0^t Z^{-1}(s)F(s)ds = \begin{pmatrix} 0 \\ \frac{1}{\pi}\sin\pi t \end{pmatrix}$, $\Psi(\widehat{A}^{-1}F) = (\widehat{A}^{-1}F)(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
$Y(t) = \widehat{A}^{-1}F(t) + ZMW_3^{-1}\Psi(\widehat{A}^{-1}F) = \begin{pmatrix} 0 \\ \frac{1}{\pi}\sin\pi t \end{pmatrix}$,
$\widehat{A}^{-1}Y(t) = Z(t)\int_0^t Z^{-1}(s)Y(s)ds = \begin{pmatrix} \frac{t}{2\pi}\cos\pi t - \frac{1}{2\pi^2}\sin\pi t \\ \frac{t}{2\pi}\sin\pi t \end{pmatrix}$,
$\Psi(\widehat{A}^{-1}Y) = (\widehat{A}^{-1}Y)(1) = \begin{pmatrix} -\frac{1}{2\pi} \\ 0 \end{pmatrix}$.
Substituting these values into (5.34) we obtain the unique solution to (6.7)

$$X(t) = \widehat{A}^{-1}Y(t) + ZMW_3^{-1}\Psi(\widehat{A}^{-1}Y) = \begin{pmatrix} \frac{t-2}{2\pi}\cos\pi t - \frac{1}{2\pi^2}\sin\pi t \\ \frac{t-2}{2\pi}\sin\pi t \end{pmatrix},$$

which gives (6.8). □

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AN INVERSE PROBLEM FOR 1D FRACTIONAL INTEGRO-DIFFERENTIAL WAVE EQUATION WITH FRACTIONAL TIME DERIVATIVE

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Abstract. This paper is devoted to obtaining a unique solution to an inverse problem for a one-dimensional time-fractional integro-differential equation. First, we consider the direct problem, and the unique existence of the weak solution is established, after that, the smoothness conditions for the solution are obtained. Secondly, we study the inverse problem of determining the unknown coefficient and kernel, and the well-posedness of this inverse problem is proved. The local existence and global uniqueness results are based on the Fourier method, fractional calculus, properties of the Mittag-Leffler function, and Banach fixed point theorem in a suitable Sobolev space.

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1 Introduction and setting up the problem

For studying objects or processes in the surrounding world, methods of mathematical modeling are extensively used. An efficient way to study processes by mathematical methods is by modeling these processes in the form of fractional differential equations.

Fractional differential equations have excited, in recent years, a considerable interest both in mathematics and in applications. They were used in the modeling of many physical and chemical processes and engineering (see, e.g., [2]-[4]). Other studies [7]-[6] demonstrate several interesting features of the fractional diffusion-wave equations, which represent a peculiar union of properties typical for second-order parabolic and wave differential equations. Fractional evolution inclusions are an important form of differential inclusions within nonlinear mathematical analysis. They are generalizations of the much more widely developed fractional evolution equations (such as time-fractional diffusion equations) seen through the lens of multivariate analysis. Compared with fractional evolution equations, research on the theory of fractional differential inclusions is however only in its initial stage of development. This is important because differential models with the fractional derivative provide an excellent instrument for the description of memory and hereditary properties, and have recently been proven valuable tools in the modeling of many physical phenomena (see, [20] and the references therein).

According to the fractional order α, the diffusion process can be specified as sub-diffusion ($\alpha \in (0,1)$) and super-diffusion ($\alpha \in (1,2)$), respectively. There is abundant literature on the studies of fractional equations on various aspects, such as physical backgrounds, weak solutions, maximum principle and numerical methods (see, [19] and the references therein).

Practical needs often lead to problems in determining the coefficients, kernel, or the right-hand side of a differential equation from certain known information about its solution. Such problems have received the name inverse problems of mathematical physics. Inverse problems arise in various domains of human activity, such as seismology, prospecting for mineral deposits, biology, medical visualization, computer-aided tomography, the remote sounding of the Earth, spectral analysis, nondestructive control, etc., (see [3], [10]-[17]). In this paper, we discuss an inverse problem of determining a coefficient and kernel depending on the time in a fractional-differential equation by the measurement data of time trace at a fixed point x_i.

Let $Q_0^T := (0,1) \times (0,T)$ for a given time $T > 0$. We consider the following fractional integro-differential equation with a fractional derivative in time t:

$$\partial_t^\alpha u(x,t) + \mathcal{L}u(x,t) = q(t)u_t(x,t) + k * u(x,\cdot) + f(x,t), \quad (x,t) \in Q_0^T, \tag{1.1}$$

where $1 < \alpha < 2$ and $\partial_t^\alpha u(x,t)$ is the left Gerasimov-Caputo fractional derivative with respect to t and is defined in [9] as

$$\partial_t^\alpha v(t) = \mathcal{K} * v'',$$

here the kernel function $\mathcal{K}$ is given by

$$\mathcal{K}(t) = \begin{cases} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

$\Gamma(\cdot)$ is the Gamma function and $\mathcal{L}$ is the differential operator defined by

$$\mathcal{L}u \equiv -(\varrho(x)u')' + c(x)u,$$

where the coefficients belong to the set:

$$\Lambda := \left\{(\varrho, c) \in C^1[0,1] \times C[0,1] : \, c(x) > 0, \; \varrho(x) \geq \varrho_0 > 0\right\},$$

and $*$ denotes the Laplace convolution

$$f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Note that if $\alpha = 1$ and $\alpha = 2$, then equation (1.1) represents a parabolic and a hyperbolic integro-differential equations, respectively. Since we are interested mainly in the fractional cases, we restrict the order α to $1 < \alpha < 2$.

We supplement the above fractional wave equation with the following initial conditions:

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 < x < 1, \tag{1.2}$$

and the zero boundary condition:

$$u(0,t) = u(1,t) = 0, \quad 0 < t < T. \tag{1.3}$$

For convenience of the reader, we present here the necessary definitions from functional analysis and fractional calculus theory.

For integers m, we denote $H^m(0,1) = W^{m,2}(0,1)$ and $W^{k,1}(0,T)$ the usual Sobolev spaces defined for spatial and time variables respectively (see [1]), and $H_0^m(0,1)$ is the closure of $C_0^\infty(0,1)$ in the norm of space $H^m(0,1)$. For a given Banach space V on $(0,1)$, we use the notation $C^m([0,T];V)$ to denote the following space:

$$C^m([0,T];V) := \left\{u : [0,T] \to V : \, \|\partial_t^j u(t)\|_V \, \text{is continuous in} \, t \, \text{on} \, [0,T] \, \text{for all} \, 0 \leq j \leq m\right\}.$$

We endow $C^m([0,T];V)$ with the following norm, making it a Banach space:

$$\|u\|_{C^m([0,T];V)} = \sum_{j=0}^{m} \left(\max_{0\le t\le T} \|\partial_t^j u(t)\|_V \right).$$

In addition, we define the Banach space X_0^T by

$$X_0^T := C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})) \cap C^1([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))$$

with the norm

$$\|u\|_{X_0^T} := \|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} + \|u\|_{C^1([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}.$$

The notation X_0^T indicates that zero represents the initial state of the time variable, i.e., $t = 0$. Furthermore, we set

$$Y_0^T = X_0^T \times C^1[0,T] \times C[0,T]$$

endowed with the norm

$$\|(u,q,k)\|_{Y_0^T} := \|u\|_{X_0^T} + \|q\|_{C^1[0,T]} + \|k\|_{C[0,T]}.$$

We denote the domain of $\mathcal{L}$ by $\mathcal{D}(\mathcal{L}) = H^2(0,1)\cap H_0^1(0,1)$. It is well known that, if the coefficients ϱ and c of the operator $\mathcal{L}$ are in the set Λ, then the operator $\mathcal{L}$ has only real and simple eigenvalues λ_n, and with suitable numbering, we have $0 < \lambda_1 \le \lambda_2 \le \cdots$, $\lim_{k\to\infty} \lambda_k = \infty$. By e_k, we denote the eigenfunction corresponding to λ_k, which satisfies $\|e_k\|^2_{L^2(0,1)} = (e_k, e_k) = 1$, where $(\cdot,\cdot)$ denotes the inner product in the Hilbert space $L^2(0,1)$ and λ_k, e_k satisfy $\mathcal{L}e_k = \lambda_k e_k$, $e_k(0) = e_k(1) = 0$, $\{e_k\} \subset H^2(0,1) \cap H_0^1(0,1)$ is an orthonormal basis of $L^2(0,1)$.

Now we define the fractional power operator $\mathcal{L}^\gamma$ for $\gamma \in \mathbb{R}$ (e.g. [13]) and the Hilbert space $\mathcal{D}(\mathcal{L}^\gamma)$ by

$$\mathcal{D}(\mathcal{L}^\gamma) := \Big\{ u \in L^2(0,1) : \sum_{k=1}^{\infty} \lambda_k^{2\gamma} |(u,e_k)|^2 < \infty \Big\}, \quad \mathcal{L}^\gamma u = \sum_{k=1}^{\infty} \lambda_k^{\gamma} (u,e_k) e_k$$

with the inner product $(u,v)_{\mathcal{D}(\mathcal{L}^\gamma)} = (\mathcal{L}^\gamma u, \mathcal{L}^\gamma v)_{L^2(0,1)}$ and, respectively, the norm

$$\|u\|_{\mathcal{D}(\mathcal{L}^\gamma)} = \|\mathcal{L}^\gamma u\| = \left(\sum_{k=1}^{\infty} \lambda_k^{2\gamma} |(u,e_k)|^2 \right)^{1/2}.$$

Moreover, we shall use the Mittag-Leffler function (see [9]):

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad z \in \mathbb{C}$$

with $\mathrm{Re}(\rho) > 0$ and $\mu \in \mathbb{C}$. It is known that $E_{\rho,\mu}(z)$ is an entire function in $z \in \mathbb{C}$.

Lemma 1.1. *Let $0 < \rho < 2$, $\mu \in \mathbb{R}$ be arbitrary and θ satisfy $\frac{\pi\rho}{2} < \theta < \min\{\pi, \pi\rho\}$. Then there exists a constant $c = c(\rho,\mu,\theta) > 0$ such that*

$$|E_{\rho,\mu}(z)| \le \frac{c}{1+|z|}, \qquad \theta \le |arg(z)| \le \pi,$$

and the asymptotic behavior of $E_{\rho,\mu}(z)$ at infinity is as follows: for any $N \in \mathbb{N}$

$$E_{\rho,\mu}(z) = -\sum_{n=1}^{N} \frac{z^{-n}}{\Gamma(\mu - \rho n)} + O(z^{-N-1}).$$

For the proof, we refer, for example, to [5].

Remark 1. In the paper, the Mittag-Leffler function is used only for real negative z, in which case the constant c depends only on ρ and μ.

Proposition 1.1. *(see [9]) For $\lambda > 0$, $\alpha > 0, \beta \in \mathbb{C}$ and positive integer $m \in \mathbb{N}$, we have*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0,$$

$$\frac{d}{dt}\left(t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right) = t^{\beta-2} E_{\alpha,\beta-1}(-\lambda t^\alpha), \quad t > 0$$

and

$$\partial_t^\alpha (E_{\alpha,1}(-\lambda t^\alpha)) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t \ge 0.$$

Also, we shall use the following simple equality

$$\max_{y \ge 0} \frac{y^\theta}{1+y} = \theta^\theta (1-\theta)^{1-\theta} < 1 \quad \text{for} \quad 0 < \theta < 1. \tag{1.4}$$

If $q(t)$, $k(t)$, $f(x,t)$, $\varphi(x)$ and $\psi(x)$ are known, then problem (1.1)-(1.3) is called a direct problem. The inverse problem in this paper is to reconstruct $q(t)$ and $k(t)$ according to the additional data

$$u(x_i, t) = h_i(t), \quad t \in [0, T], \tag{1.5}$$

where $x_i \in (0,1)$, $i = 1,2$ are fixed points, $h_i(t)$, $i = 1,2$ are given functions.

We investigate the following inverse problem.

Inverse problem. Find $u \in X_0^T$, $q \in C^1[0,T]$ and $k \in C[0,T]$ to satisfy (1.1)-(1.3) and additional measurements (1.5), where $\mathcal{D}(\mathcal{L}^\gamma)$ is a Hilbert space with some positive constant γ satisfying inequality (1.6).

We now give a similar definition of a weak solution to (1.1)-(1.3), which is introduced in [15].

Definition 1. We call u a weak solution to (1.1)-(1.3) if (1.1) holds in $L^2(0,1)$ and $u(\cdot,t) \in H_0^1(0,1)$ for almost all $t \in (0,T)$, $u, \partial_t u \in C([0,T]; \mathcal{D}(\mathcal{L}^{-\gamma}))$ and

$$\lim_{t\to 0} \|u(\cdot,t) - \varphi\|_{\mathcal{D}(\mathcal{L}^{-\gamma})} = \lim_{t\to 0} \|u_t(\cdot,t) - \psi\|_{\mathcal{D}(\mathcal{L}^{-\gamma})} = 0$$

with some $\gamma > 0$.

Throughout this paper, we assume that $\frac{3}{2} < \gamma_0$ and

$$\frac{5}{4} < \gamma \le \gamma_0. \tag{1.6}$$

We make the following assumptions:

(C1) $\partial_t^\alpha h_i \in C^1[0,T]$ $(i = 1,2)$, $\varphi \in \mathcal{D}(\mathcal{L}^{\gamma_0 + \frac{1}{\alpha}})$, $\psi \in \mathcal{D}(\mathcal{L}^{\gamma_0})$, $f \in C^1([0,T]; \mathcal{D}(\mathcal{L}^\gamma))$;
(C2) $h_i'(0) q(0) = \partial_t^\alpha h_i(0) + \mathcal{L}\varphi(x_i) - \tilde{f}_i(0)$, where $\tilde{f}_i(t) = f(x_i, t)$, $(i = 1,2)$;
(C3) $\varphi(x_i) = h_i(0)$, $\psi(x_i) = h_i'(0)$, $(i = 1,2)$;
(C4) $p(t) = h_1'(t) h_2(0) - h_2'(t) h_1(0) \ne 0$ and $p \in C^1[0,T]$ satisfies the following inequality:

$$|p(t)| \ge \frac{1}{p_0},$$

where p_0 is a positive constant.

Remark 2. In (C1), $\partial_t^\alpha h \in C^1[0,T]$ implies $h_i \in W^{2,1}(0,T) \hookrightarrow H^1(0,T)$ (see [17]).

Remark 3. (C2)-(C3) are the consistency conditions for our problem (1.1)-(1.3), (1.5), which guarantees that inverse problem (1.1)-(1.3), (1.5) is equivalent to (2.36) and (2.38) (see Lemma 2.6).

Remark 4. In order to guarantee that $\partial_t^\alpha h \in C^1[0,T]$, we could give the usual regularity condition $h_i \in C^3[0,T]$, such that $h_i''(0) = 0$ (see [17]).

The main result of this paper is the following local existence and uniqueness result for an inverse problem.

Theorem 1.1. *Let the assumptions* (C1)-(C4) *hold. Then, the inverse problem has a unique solution* $(u,q,k) \in Y_0^T$ *for sufficiently small* $T > 0$.

The outline of the paper is as follows. Section 2 presents preliminary results, including the existence and uniqueness of the direct problem (1.1)-(1.3), along with an equivalent problem. In Section 3, we establish the local existence and global uniqueness of the solution to the inverse problem (1.1)-(1.3), (1.5) using the Fourier method and the Banach fixed-point theorem. In Section 4, we provide examples of the inverse problem (1.1)-(1.3), (1.5).

2 Preliminary results

This section presents some preliminary results, including the well-posedness for a fractional differential equation, an equivalent lemma for our inverse problem, and a technical result, which will be used to prove our main results.

Let

$$\begin{cases} Z_1(t)\eta(x) = \sum\limits_{n=1}^{\infty} (\eta, e_n) E_{\alpha,1}(-\lambda_n t^\alpha) e_n(x), \\ Z_2(t)\eta(x) = \sum\limits_{n=1}^{\infty} (\eta, e_n) t E_{\alpha,2}(-\lambda_n t^\alpha) e_n(x), & (x,t) \in Q_0^T, \\ Z_3(t)\eta(x) = -\sum\limits_{n=1}^{\infty} \lambda_n (\eta, e_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) e_n(x). \end{cases}$$

for $\eta \in L^2(0,1)$.

We first consider the following initial and boundary value problem:

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{L}u(x,t) = F(x,t), & (x,t) \in Q_0^T, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x), & 0 < x < 1. \end{cases} \tag{2.1}$$

We split (2.1) into the following two initial and boundary value problems:

$$\begin{cases} \partial_t^\alpha v(x,t) + \mathcal{L}v(x,t) = 0, & (x,t) \in Q_0^T, \\ v(0,t) = v(1,t) = 0, & 0 < t < T, \\ v(x,0) = \varphi(x), \ v_t(x,0) = \psi(x), & 0 < x < 1, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \partial_t^\alpha w(x,t) + \mathcal{L}w(x,t) = F(x,t), & (x,t) \in Q_0^T, \\ w(0,t) = w(1,t) = 0, & 0 < t < T, \\ w(x,0) = 0, \ w_t(x,0) = 0, & 0 < x < 1. \end{cases} \tag{2.3}$$

Similarly to Theorem 2.3 in [15], it is easy to obtain the following assertion:

Lemma 2.1. *Let $\varphi \in H^2(0,1) \cap H_0^1(0,1)$ and $\psi \in H_0^1(0,1)$. Let $\gamma > 0$. Then there exists a unique weak solution $v \in C([0,T]; H^2(0,1) \cap H_0^1(0,1)) \cap C^1([0,T]; \mathcal{D}(\mathcal{L}^{-\gamma}))$ to (2.2). Moreover, there exists a constant $c > 0$, depending only on α, γ and λ_1, such that*

$$\|v(\cdot,t)\|_{H^2(0,1)} + \|v_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{-\gamma})} \leq c\left(\|\varphi\|_{H^2(0,1)} + \|\psi\|_{H^1(0,1)}\right), \quad t \in (0,T). \tag{2.4}$$

Furthermore, we have

$$\begin{cases} v(x,t) = Z_1(t)\varphi(x) + Z_2(t)\psi(x), & (x,t) \in Q_0^T, \\ v_t(x,t) = Z_3(t)\varphi(x) + Z_1(t)\psi(x), & (x,t) \in Q_0^T. \end{cases} \tag{2.5}$$

Proof. The uniqueness and existence of a weak solution are verified similarly to Theorem 2.1 in [15], but the statement about the smoothness for the function v given in the lemma differs from the statement about the smoothness given in [15]. Therefore, here we prove only inequality (2.4). Using Lemma 1.1, we have

$$\begin{aligned}\|v(\cdot,t)\|^2_{H^2(0,1)} &\leq 2\sum_{n=1}^{\infty}\lambda_n^2\left|(\varphi,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2 + 2\sum_{n=1}^{\infty}\lambda_n^2\left|(\psi,e_n)tE_{\alpha,2}(-\lambda_n t^\alpha)\right|^2 \\ &\leq 2c^2\|\varphi\|^2_{H^2(0,1)} + 2c^2\sum_{n=1}^{\infty}\lambda_n|(\psi,e_n)|^2\left(\frac{(\lambda_n t^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n t^\alpha}\right)^2\lambda_n^{1-\frac{2}{\alpha}},\end{aligned}$$

where $c > 0$, depending only on α, is given in Lemma 1.1. Since $\lambda_n^{1-\frac{2}{\alpha}} \leq \lambda_1^{1-\frac{2}{\alpha}}, n = 1,2,...,$ by (1.4) we have

$$\|v(\cdot,t)\|^2_{H^2(0,1)} \leq 2c^2\max\{1,\lambda_1^{1-\frac{2}{\alpha}}\}\left(\|\varphi\|^2_{H^2(0,1)} + \|\psi\|^2_{H^1(0,1)}\right). \tag{2.6}$$

Further, by the second formula of (2.5), we have

$$\begin{aligned}\|v_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{-\gamma})} &\leq 2\sum_{n=1}^{\infty}\lambda_n^{-2\gamma}\left|\lambda_n t^{\alpha-1}(\varphi,e_n)E_{\alpha,\alpha}(-\lambda_n t^\alpha)\right|^2 \\ &\quad + 2\sum_{n=1}^{\infty}\lambda_n^{-2\gamma}\left|(\psi,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2 \\ &\leq 2c^2\sum_{n=1}^{\infty}\lambda_n^2|(\varphi,e_n)|^2\left(\frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^\alpha}\right)^2\lambda_n^{-2(\gamma+1-\frac{1}{\alpha})} + 2c^2\sum_{n=1}^{\infty}\lambda_n|(\psi,e_n)|^2\lambda_n^{-2(\gamma+\frac{1}{2})}.\end{aligned} \tag{2.7}$$

Now, using Lemma 1.1 and (1.4), we have

$$\|v_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{-\gamma})} \leq 2c^2\max\{\lambda_1^{-2(\gamma+1-\frac{1}{\alpha})},\lambda_1^{-2(\gamma+\frac{1}{2})}\}\left(\|\varphi\|^2_{H^2(0,1)} + \|\psi\|^2_{H^1(0,1)}\right). \tag{2.8}$$

□

We introduce the following auxiliary lemmas to obtain the main results.

Lemma 2.2. *Let $F \in C([0,T]; \mathcal{D}(\mathcal{L}^{1/\alpha}))$. Then there exists a unique weak solution $w \in C([0,T]; H^2(0,1) \cap H_0^1(0,1))$ to (2.3) with $\partial_t^\alpha w \in C([0,T]; L^2(0,1))$. Moreover, for any $\gamma > 0$, we have $w_t \in C([0,T]; \mathcal{D}(\mathcal{L}^{-\gamma}))$,*

$$\lim_{t\to 0}\|w(\cdot,t)\|_{H^2(0,1)} = \lim_{t\to 0}\|w_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{-\gamma})} = 0. \tag{2.9}$$

Furthermore, there exists a constant $c>0$, depending only on α,γ and λ_1, such that

$$\|w(\cdot,t)\|_{H^2(0,1)}+\|w_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{-\gamma})}\le c\big(t+t^{\alpha-1}\big)\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))} \tag{2.10}$$

and we have

$$\begin{cases} w(x,t)=-\int_0^t \mathcal{L}^{-1}Z_3(t-s)F(x,s)ds, & (x,t)\in Q_0^T,\\ w_t(x,t)=\int_0^t \mathcal{L}^{-1}Z_4(t-s)F(x,s)ds, & (x,t)\in Q_0^T, \end{cases} \tag{2.11}$$

where

$$Z_4(t)\eta(x)=\sum_{n=1}^{\infty}\lambda_n(\eta,e_n)t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)e_n(x)$$

and the function w belongs to the space $C([0,T];H^2(0,1)\cap H_0^1(0,1))\cap C^1([0,T];\mathcal{D}(\mathcal{L}^{-\gamma}))$.

Proof. By Theorem 2.2 in [15], for $F\in C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))$, the unique solution $w\in C([0,T];H^2(0,1)\cap H_0^1(0,1))$ to (2.3) can be expressed by (2.11). As above, the uniqueness and existence of the weak solution are verified similarly to Theorem 2.1 in [15]. Therefore, here we omitted it and we prove only equality (2.9) and inequality (2.10).

We first have

$$\|w(\cdot,t)\|^2_{L^2(0,1)}=\sum_{n=1}^{\infty}\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)ds\right|^2$$
$$\le c^2\sum_{n=1}^{\infty}\left|\int_0^t \lambda_n^{\frac{1}{\alpha}}|(F(\cdot,s),e_n)|\frac{(\lambda_n(t-s)^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n(t-s)^\alpha}\lambda_n^{-1}ds\right|^2,$$

or, by virtue of the generalized Minkowski inequality, we have

$$\|w(\cdot,t)\|^2_{L^2(0,1)}\le c^2\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{\frac{2}{\alpha}}|(F(\cdot,s),e_n)|^2\right)^{1/2}\lambda_1^{-1}ds\right|^2$$
$$\le c^2\lambda_1^{-2}\max_{0\le s\le t}\|F(\cdot,s)\|^2_{\mathcal{D}(\mathcal{L}^{1/\alpha})}\left|\int_0^t ds\right|^2\le c^2\lambda_1^{-2}\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))}t^2. \tag{2.12}$$

Furthermore, according to Lemma 2.2, for $F\in C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))$ and by Lemma 1.1, we have

$$\|\omega(\cdot,t)\|^2_{H^2(0,1)}$$
$$\le\|\mathcal{L}\omega(\cdot,t)\|^2_{L^2(0,1)}=\sum_{n=1}^{\infty}\lambda_n^2\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)ds\right|^2$$
$$\le c^2\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{\frac{2}{\alpha}}|(F(\cdot,s),e_n)|^2\right)^{1/2}\frac{(\lambda_n(t-s)^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n(t-s)^\alpha}ds\right|^2$$
$$\le c^2\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))}t^2. \tag{2.13}$$

By (2.3) and (2.13) we can estimate also $\|\partial_t^\alpha\omega(\cdot,t)\|_{C([0,T];L^2(0,1))}$ and we have $\lim_{t\to 0}\|\omega(\cdot,t)\|_{H^2(0,1)}=0$. Next, applying Lemma 1.1, Proposition 1, and the Cauchy-Schwarz inequality, for any $\gamma>0$, we have

$$\|\omega_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{-\gamma})} = \sum_{n=1}^{\infty}\lambda_n^{-2\gamma}\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^\alpha)ds\right|^2$$
$$\le c^2\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{\frac{1}{\alpha}}|(F(\cdot,s),e_n)|^2\lambda_n^{-2\gamma-\frac{2}{\alpha}}\right)^{1/2}(t-s)^{\alpha-2}ds\right|^2$$
$$\le \frac{c^2}{(\alpha-1)^2}\lambda_1^{-2\gamma-\frac{2}{\alpha}}\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))}t^{2\alpha-2}. \quad (2.14)$$

Therefore, $\lim_{t\to 0}\|\omega_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{-\gamma})} = 0$. Inequality (2.10) follows from inequalities (2.13) and (2.14). □

By Lemma 2.1 and 2.2, we get the following assertion:

Lemma 2.3. *Let $\varphi \in H^2(0,1)\cap H_0^1(0,1)$, $\psi \in H_0^1(0,1)$ and $F(x,t) \in C([0,T];D(\mathcal{L}^{1/\alpha}))$. Then there exists a unique weak solution $u \in C([0,T];H^2(0,1)\cap H_0^1(0,1))\cap C^1([0,T];\mathcal{D}(\mathcal{L}^{-\gamma}))$ to (2.1), such that*

$$\|u(\cdot,t)\|_{H^2(0,1)} + \|u_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{-\gamma})}$$
$$\le c\left[\|\varphi\|_{H^2(0,1)} + \|\psi\|_{H^1(0,1)} + (t+t^{\alpha-1})\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{1/\alpha}))}\right] \quad (2.15)$$

for all $t \in [0,T]$, where the constant $c > 0$ depends only on α,γ and λ_1, in particular, does not depend on T. Furthermore, for all $(x,t) \in Q_0^T$ we have

$$\begin{cases} u(x,t) = Z_1(t)\varphi(x) + Z_2(t)\psi(x) - \int_0^t \mathcal{L}^{-1}Z_3(t-s)F(x,s)ds, \\ u_t(x,t) = Z_3(t)\varphi(x) + Z_1(t)\psi(x) + \int_0^t \mathcal{L}^{-1}Z_4(t-s)F(x,s)ds, \end{cases} \quad (2.16)$$

where $Z_j(t)[\cdot](j = 1,2,3,4)$ are defined above.

The next two lemmas are regularity results of the solution u to problem (2.1).

Lemma 2.4. *Let $\varphi \in \mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})$, $\psi \in \mathcal{D}(\mathcal{L}^\gamma)$ and $F \in C([0,T];\mathcal{D}(\mathcal{L}^\gamma)$. Then the unique weak solution $u \in X_0^T$ to (2.1) is such that*

$$\|u(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + \|u_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^\gamma)} \le c\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^\gamma)} + t^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\Big), \quad (2.17)$$

where $c > 0$ depends only on α.

Proof. Obviously, by Lemma 1.1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|u(\cdot,t)\|^2_{D(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} &\le 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}|(\varphi,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)|^2 + 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}t^2|(\psi,e_n)E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \\
&+ 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)ds\right|^2 \\
&\le 3c^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}|(\varphi,e_n)|^2 + 3c^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma}(\psi,e_n)^2\left(\frac{(\lambda_n t^\alpha)^{1/\alpha}}{1+\lambda_n t^\alpha}\right)^2 \\
&+ 3\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(F(\cdot,s),e_n)|^2\right)^{1/2}\lambda_n^{\frac{1}{\alpha}}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)ds\right|^2 \\
&\le 3c^2\|\varphi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + 3c^2\|\psi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma})} \\
&+ 3c^2\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(F(\cdot,s),e_n)|^2\right)^{1/2}\frac{(\lambda_n(t-s)^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n(t-s)^\alpha}(t-s)^{\alpha-2}ds\right|^2 \\
&\le 3c^2\|\varphi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + 3c^2\|\psi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma})} + 3c^2\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\left|\int_0^t (t-s)^{\alpha-2}ds\right|^2. \quad (2.18)
\end{aligned}$$

As a result, we get

$$\|u(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} \le c(\alpha)\left(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma})} + t^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\right), \quad (2.19)$$

where $c(\alpha) = \frac{3c^2}{(\alpha-1)^2}$. Furthermore, by Lemma 2.3, we have

$$\begin{aligned}
u_t(x,t) &= \sum_{n=1}^{\infty}\left\{-\lambda_n t^{\alpha-1}(\varphi,e_n)E_{\alpha,\alpha}(-\lambda_n t^\alpha) + (\psi,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)\right\}e_n(x) \\
&+ \sum_{n=1}^{\infty}\left\{\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^\alpha)ds\right\}e_n(x). \quad (2.20)
\end{aligned}$$

Therefore, by applying (1.4), and Lemma 1.1 again, we have

$$\begin{aligned}
&\|u_t(\cdot,t)\|^2_{\mathcal{D}(\mathcal{L}^{\gamma})} \\
&\le 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma}\lambda_n^2|(\varphi,e_n)|^2|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 + 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(\psi,e_n)|^2|E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \\
&+ 3\sum_{n=1}^{\infty}\lambda_n^{2\gamma}\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^\alpha)ds\right|^2 \\
&\le 3c^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}(\varphi,e_n)^2\left(\frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^\alpha}\right)^2 + 3c^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma}(\psi,e_n)^2 \\
&+ 3c^2\left|\int_0^t\left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(F(\cdot,s),e_n)|^2\right)^{1/2}\frac{(t-s)^{\alpha-2}}{1+\lambda_1(t-s)^\alpha}ds\right|^2 \\
&\le 3c^2\|\varphi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + 3c^2\|\psi\|^2_{\mathcal{D}(\mathcal{L}^{\gamma})} + \frac{3c^2}{(\alpha-1)^2}t^{2(\alpha-1)}\|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}. \quad (2.21)
\end{aligned}$$

Thus,

$$\|u_t(\cdot,t)\|_{\mathcal{D}(\mathcal{L}^\gamma)} \le c_1\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^\gamma)} + t^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\Big) \tag{2.22}$$

for all $t \in [0,T]$, where $c_1 > 0$ depends only on α. Then, we immediately obtain the desired estimate (2.17). □

It is easy to see that

$$\mathcal{L}u(x_i,t) = \mathcal{L}Z_1(t)\varphi(x_i) + \mathcal{L}Z_2(t)\psi(x_i) - \int_0^t Z_3(t-s)F(x_i,s)ds, \quad i=1,2. \tag{2.23}$$

The following lemma is valid.

Lemma 2.5. *Let $\varphi \in \mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})$, $\psi \in \mathcal{D}(\mathcal{L}^{\gamma_0})$, $c_0 = \min_{x\in[0,1]} c(x) > 0$ and $F \in C([0,T];\mathcal{D}(\mathcal{L}^\gamma))$. Then there exists a positive constant $c > 0$, depending only on α, ρ_0, c_0, γ, γ_0, λ_1, such that*

$$\|\mathcal{L}u(x_i,\cdot)\|_{C[0,T]} \le c\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})} + T\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\Big), \quad i=1,2, \tag{2.24}$$

and

$$\|\mathcal{L}u_t(x_i,\cdot)\|_{C[0,T]} \le c\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})} + T^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\Big), \quad i=1,2. \tag{2.25}$$

Proof. An inequality similar to the estimate in (2.24) was derived in [18]. However, the smoothness assumptions differ from those in [18], so we provide a detailed proof of inequalities (2.24) and (2.25).

Recall the following inequality for the fractional power $\mathcal{L}^\beta$ of $\mathcal{L}$ with $\beta \in \mathbb{R}$, $\beta > 0$:

$$\|u\|_{H^{2\beta}(0,1)} \le c_2\|\mathcal{L}^\beta u\|_{L^2(0,1)}$$

where constant $c_2 > 0$ depends only on β and λ_1 (see., [13], p. 208).

Let $\varepsilon_0 = \min\{\varepsilon_{01}, \varepsilon_{02}\}$ with $2\varepsilon_{01} = \gamma_0 + \frac{1}{\alpha} - \frac{3}{2} > 0$ and $2\varepsilon_{02} = \gamma - \frac{1}{4} - \frac{1}{\alpha} > 0$. According to the Sobolev embedding theorem $H^{2\beta}(0,1) \subset C[0,1]$ for $\beta = \frac{1}{4} + \varepsilon_0$, we have

$$\|e_n\|_{C[0,1]} \le c_3\|e_n\|_{H^{2\beta}(0,1)} \le c_3c_2\|\mathcal{L}^\beta e_n\|_{L^2(0,1)} \le c_4\lambda_n^\beta, \tag{2.26}$$

where $c_2, c_3, c_4 > 0$ depend only of β, λ_1.

For convenience, we split $\mathcal{L}u(x_i,t)$ in three parts, namely $\mathcal{L}u(x_i,t) := \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3$, where

$$\mathrm{I}_1 := \mathcal{L}Z_1(t)\varphi(x_i), \quad \mathrm{I}_2 := \mathcal{L}Z_2(t)\psi(x_i), \quad \mathrm{I}_3 := -\int_0^t Z_3(t-s)F(x_i,s)ds,\ i=1,2.$$

Note that

$$\lambda_n \ge c_5 n^2,$$

where $c_5 > 0$ depends only on ρ_0 and c_0 (see [12], p. 190). For I_1, by Lemma 1.1, and , we have

$$\begin{aligned}|\mathrm{I}_1| &\le \sum_{n=1}^\infty \lambda_n|(\varphi,e_n)||E_{\alpha,1}(-\lambda_n t^\alpha)||e_n(x_i)| \le c\sum_{n=1}^\infty \lambda_n^{\gamma_0+\frac{1}{\alpha}}|(\varphi,e_n)|\lambda_n^{-(\gamma_0+\frac{1}{\alpha}-\beta-1)} \\ &\le c\Big(\sum_{n=1}^\infty \lambda_n^{2\gamma_0+\frac{2}{\alpha}}|(\varphi,e_n)|^2\Big)^{1/2}\Big(\sum_{n=1}^\infty \lambda_n^{-2(\gamma_0+\frac{1}{\alpha}-\beta-1)}\Big)^{1/2} \\ &\le cc_5\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})}\Big(\sum_{n=1}^\infty n^{-4(\gamma_0+\frac{1}{\alpha}-\beta-1)}\Big)^{1/2}.\end{aligned} \tag{2.27}$$

By the choice of β, we have $4(\gamma_0 + \frac{1}{\alpha} - \beta - 1) = 1 + 8\varepsilon_{01} - 4\varepsilon_0 > 1$, which implies

$$\sum_{n=1}^{\infty} n^{-4(\gamma_0+\frac{1}{\alpha}-\beta-1)} < c(\alpha, \gamma_0, \gamma).$$

So, we get

$$|\mathrm{I}_1| \leq c(\alpha, \gamma_0, \gamma, \rho_0, c_0)\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})}. \tag{2.28}$$

Further, by Lemma 1.1 and (1.4), we have the following estimate for I_2:

$$\begin{aligned}
|\mathrm{I}_2| &\leq \sum_{n=1}^{\infty} \lambda_n |(\psi, e_n)| t |E_{\alpha,2}(-\lambda_n t^\alpha)||e_n(x_i)| \leq cc_4 \sum_{n=1}^{\infty} |(\psi, e_n)| \frac{\lambda_n t}{1+\lambda_n t^\alpha} \lambda_n^\beta \\
&\leq cc_4 \sum_{n=1}^{\infty} \lambda_n^{\gamma_0} |(\psi, e_n)| \frac{(\lambda_n t^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n t^\alpha} \lambda_n^{-(\gamma_0+\frac{1}{\alpha}-\beta-1)} \\
&\leq cc_4 \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} |(\psi, e_n)|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \lambda_n^{-2(\gamma_0+\frac{1}{\alpha}-\beta-1)} \right)^{1/2} \leq c_6 \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})},
\end{aligned} \tag{2.29}$$

where $c_6 > 0$ depends only on $\alpha, \gamma, \gamma_0, \lambda_1, \rho_0, c_0$. Next, we estimate I_3. The estimate for I_3 is the same as in [18] for $\gamma - \beta - \frac{1}{\alpha} = 2\varepsilon_{02} - \varepsilon_0 > 0$, and we have

$$\begin{aligned}
|\mathrm{I}_3|^2 &= \sum_{n=1}^{\infty} \left| \lambda_n \int_0^t (F(\cdot, s), e_n)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) ds \cdot e_n(x_i) \right|^2 \\
&\leq c^2 c_4^2 \left| \int_0^t \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(F(\cdot, s), e_n)|^2 \right)^{1/2} \frac{(\lambda_n (t-s)^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n (t-s)^\alpha} ds \right|^2 \lambda_n^{-2(\gamma-\beta-\frac{1}{\alpha})} \\
&\leq c^2 c_4^2 \lambda_1^{-2(\gamma-\beta-\frac{1}{\alpha})} \|F\|^2_{C([0,T];\mathcal{D}(\mathcal{L}))} t^2.
\end{aligned} \tag{2.30}$$

So,

$$|\mathrm{I}_3| \leq c(\alpha, \gamma, \gamma_0, \lambda_1, \rho_0, c_0) t \|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}, \qquad \forall t \in [0, T]. \tag{2.31}$$

According to (2.28)-(2.31), we obtain (2.24).

By differentiating (2.20) with respect to the variable t and taking into account Proposition 1, we obtain

$$\begin{aligned}
\frac{d}{dt} \mathcal{L} u(x_i, t) = &- \sum_{n=1}^{\infty} \lambda_n^2 (\varphi, e_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) e_n(x_i) \\
&+ \sum_{n=1}^{\infty} \lambda_n (\psi, e_n) E_{\alpha,1}(-\lambda_n t^\alpha) e_n(x_i) \\
&+ \sum_{n=1}^{\infty} \lambda_n \left(\int_0^t (F(\cdot, s), e_n)(t-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (t-s)^\alpha) ds \right) e_n(x_i) \\
&:= \tilde{\mathrm{I}}_1 + \tilde{\mathrm{I}}_2 + \tilde{\mathrm{I}}_3.
\end{aligned} \tag{2.32}$$

Let $\varepsilon_0 = \min\{\varepsilon_{10}, \varepsilon_{11}\}$ where $2\varepsilon_{10} = \gamma_0 - \frac{3}{2} > 0$ and $2\varepsilon_{11} = \gamma - \frac{5}{4} > 0$.

By the asymptotic property of the eigenvalues $\lambda_n \geq c_5 n^2$, for $\tilde{\mathrm{I}}_1$, using Lemma 1.1 and (1.4), we have

$$|\tilde{\mathrm{I}}_1| \leq \sum_{n=1}^{\infty} \lambda_n^2 |(\varphi, e_n)| t^{\alpha-1} |E_{\alpha,\alpha}(-\lambda_n t^\alpha)||e_n(x_i)|$$
$$\leq cc_4 \sum_{n=1}^{\infty} \lambda_n^{\gamma_0 + \frac{1}{\alpha}} |(\varphi, e_n)| \frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^\alpha} \lambda_n^{-(\gamma_0-\beta-1)}$$
$$\leq cc_4 \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma_0 + \frac{2}{\alpha}} |(\varphi, e_n)|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \lambda_n^{-2(\gamma_0-\beta-1)} \right)^{1/2}$$
$$\leq cc_4 c_5 \|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} \left(\sum_{n=1}^{\infty} n^{-4(\gamma_0-\beta-1)} \right)^{1/2}.$$

By choice of β, we have $4(\gamma_0 - \beta - 1) = 1 + 8\varepsilon_{10} - 4\varepsilon_0 > 1$, which implies

$$\sum_{n=1}^{\infty} n^{-4(\gamma_0-\beta-1)} < c(\gamma, \gamma_0).$$

Thus, we obtain

$$|\tilde{\mathrm{I}}_1| \leq c(\alpha, \gamma_0, \rho_0, c_0) \|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})}. \tag{2.33}$$

Similarly, we have the following estimate for $\tilde{\mathrm{I}}_2$:

$$|\tilde{\mathrm{I}}_2| \leq \sum_{n=1}^{\infty} \lambda_n |(\psi, e_n)||E_{\alpha,1}(-\lambda_n t^\alpha)||e_n(x_i)|$$
$$\leq cc_4 \sum_{n=1}^{\infty} \lambda_n^{\gamma_0} |(\psi, e_n)| \frac{\lambda_n^{-(\gamma_0-\beta-1)}}{1+\lambda_n t^\alpha} \leq cc_4 \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} |(\psi, e_n)|^2 \right)^{1/2}$$
$$\times \left(\sum_{n=1}^{\infty} n^{-4(\gamma_0-\beta-1)} \right)^{1/2} \leq c(\alpha, \gamma, \gamma_0, \rho_0, c_0) \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}. \tag{2.34}$$

Further, we estimate $\tilde{\mathrm{I}}_3$. By Lemma 1.1 and $\gamma - \beta - 1 = 2\varepsilon_{11} - \varepsilon_0 > 0$, we have

$$|\tilde{\mathrm{I}}_3|^2 \leq \sum_{n=1}^{\infty} \left| \lambda_n \int_0^t (F(\cdot, s), e_n)(t-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (t-s)^\alpha) ds \cdot e_n(x_i) \right|^2$$
$$\leq c^2 c_4^2 \left| \int_0^t \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(F(\cdot, s), e_n)|^2 \right)^{1/2} \frac{(t-s)^{\alpha-2}}{1+\lambda_1 (t-s)^\alpha} ds \right|^2 \cdot \lambda_n^{-2(\gamma-\beta-1)}$$
$$\leq c^2 c_4^2 \max_{0 \leq s \leq t} \|F(\cdot, s)\|^2_{\mathcal{D}(\mathcal{L}^\gamma)} \left| \int_0^t s^{\alpha-2} ds \right|^2 \cdot \lambda_1^{-2(\gamma-\beta-1)}.$$

So that

$$|\tilde{\mathrm{I}}_3| \leq c(\alpha, \gamma, \gamma_0, \lambda_1, \rho_0, c_0) \|F\|_{C([0,T], \mathcal{D}(\mathcal{L}^\gamma))} t^{\alpha-1}, \quad \forall t \in [0, T]. \tag{2.35}$$

Finally, by (2.33)-(2.35), we get (2.25), thereby completing the proof of this lemma. □

To study the main problem (1.1)-(1.3), (1.5), we consider the following auxiliary inverse initial and boundary value problem.

Lemma 2.6. *Let* (C1)-(C4) *be held. Then the problem of finding a solution to* (1.1)-(1.3), (1.5) *is equivalent to the problem of determining functions* $u(x,t) \in X_0^T$, $q(t) \in C^1[0,T]$ *and* $k(t) \in C[0,T]$ *satisfying*

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{L}u(x,t) = q(t)u_t(x,t) + (k * u)(t) + f(x,t), & (x,t) \in Q_0^T, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) & 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \end{cases} \tag{2.36}$$

and

$$q(t) = \frac{1}{p(t)}\Big(h_2(0)\mathcal{N}_1[u,l](t) - h_1(0)\mathcal{N}_2[u,l](t)\Big), \qquad 0 \le t \le T, \tag{2.37}$$

$$k(t) = D_t\left[\frac{1}{p(t)}\Big(h_1'(t)\mathcal{N}_2[u,l](t) - h_2'(t)\mathcal{N}_1[u,l](t)\Big)\right], \quad 0 \le t \le T, \tag{2.38}$$

where $D_t := (d/dt)$,

$$\mathcal{N}_i = \partial_t^\alpha h_i(t) + \mathcal{L}u(x_i,t) - (l * h_i')(t) - \tilde{f}_i(t),\ (i = 1,2) \tag{2.39}$$

and

$$l(t) = \int_0^t k(\tau)d\tau. \tag{2.40}$$

Remark 5. By Lemma 2.6, we know that problem (2.36)-(2.38) is an equivalent form of the original inverse problem (1.1)-(1.3), (1.5). Therefore, in the following sections, we will discuss problem (2.36)-(2.38), rather than the original one.

Proof. The solution $(u(x,t), q(t), k(t)) \in Y_0^T$ of our inverse problem (1.1)-(1.3), (1.5) is also a solution to problem (2.36) in Y_0^T, because problem (2.36) is the same as (1.1)-(1.3). Therefore, we should show only (2.37) and (2.38). Let the three $\{u(x,t), q(t), k(t)\}$ functions be a solution to problem (1.1)-(1.3), (1.5). Taking into account the conditions $\partial_t^\alpha h_i(t) \in C[0,T]$ which imply that $h_i \in C^1[0,T]$, and fractional differentiating both sides of (1.5) with respect to t gives

$$\partial_t^\alpha u(x_i,t) = \partial_t^\alpha h_i(t), \quad u_t(x_i,t) = h_i'(t), \quad 0 \le t \le T. \tag{2.41}$$

Set $x = x_i$ in equation (1.1), the procedure yields

$$\partial_t^\alpha u(x_i,t) + \mathcal{L}u(x_i,t) = q(t)u_t(x_i,t) + \int_0^t k(t-\tau)u(x_i,\tau)d\tau + f(x_i,t),\ i = 1,2. \tag{2.42}$$

We note that $l(t) = \int_0^t k(\tau)d\tau$. Then by integration by parts, we get the following equality:

$$\int_0^t k(\tau)h_i(t-\tau)d\tau = h_i(0)l(t) + \int_0^t l(t-\tau)h_i'(\tau)d\tau. \tag{2.43}$$

With the help of (2.41) and (2.43), we can rewrite (2.42) as

$$h_i'(t)q(t) + h_i(0)l(t) = \partial_t^\alpha h_i(t) + \mathcal{L}u(x_i,t) - (l * h_i')(t) - \tilde{f}_i(t), \quad i = 1,2.$$

Due to (C4), we can solve this system to get (2.37) and

$$l(t) = \frac{1}{p(t)}\Big(h_1'(t)\mathcal{N}_2[u,l](t) - h_2'(t)\mathcal{N}_1[u,l](t)\Big). \tag{2.44}$$

Furthermore, by differentiating (2.44) with respect to t, we get (2.38).

Now, assume that (u, q, k) satisfies (2.36)-(2.38). To prove that $\{u, q, k\}$ is a solution to the inverse problem (1.1)-(1.3), (1.5), it suffices to show that $\{u, q, k\}$ satisfies (1.5).

Setting $x = x_i$ in equation (2.36), we have

$$\partial_t^\alpha u(x_i, t) + \mathcal{L}u(x_i, t) = q(t)u_t(x_i, t) + (k * u)(t) + \tilde{f}_i(t). \tag{2.45}$$

On the other hand, from (C2), we easily see that

$$\frac{1}{p(0)}\Big(h_1'(0)\mathcal{N}_2[u, l](0) - h_2'(0)\mathcal{N}_1[u, l](0)\Big) = 0.$$

We get (2.44) by integrating (2.38) over $[0, t]$. From (2.37) and (2.44), we conclude that

$$\begin{aligned} h_i'(t)q(t) &= -h_i(0)l(t) + \partial_t^\alpha h_i(t) + \mathcal{L}u(x_i, t) - (l * h_i')(t) - \tilde{f}_i(t) \\ &= \partial_t^\alpha h_i(t) + \mathcal{L}u(x_i, t) - (k * h_i)(t) - \tilde{f}_i(t) \end{aligned}$$

or

$$\tilde{f}_i(t) = -h_i'(t)q(t) + \partial_t^\alpha h_i(t) + \mathcal{L}u(x_i, t) - (k * h_i)(t). \tag{2.46}$$

Then substituting (2.46) into (2.45), and using (C3), we have that $P_i(t) := u(x_i, t) - h_i(t)$ $(i = 1, 2)$ satisfies

$$\begin{cases} \partial_t^\alpha P_i(t) = q(t)P_i'(t) + (k * P_i)(t), \quad t > 0, \\ P_i(0) = P_i'(0) = 0. \end{cases} \tag{2.47}$$

Then, the fractional initial value problem (2.47) is equivalent to the integral equation (see, [9], p. 199)

$$\begin{aligned} P_i(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(\int_s^t (t-\tau)^{\alpha-1}k(\tau - s)d\tau\right) P_i(s)ds \\ - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q'(s)P_i(s)ds \\ + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}q(s)P_i(s)ds, \quad i = 1, 2. \end{aligned} \tag{2.48}$$

This is a weakly singular homogeneous integral equation, and it has only a trivial solution for $q(t) \in C^1[0, T]$ and $k(t) \in C[0, T]$ (see [8]). Therefore, $u(x_i, t) - h_i(t) = 0$, for $0 \le t \le T$, i.e., condition (1.5) is satisfied. □

At the end of this section, we present a lemma that will be used to estimate q and k.

Lemma 2.7. *Let* (C1) *hold. Then for all* $(u, q, k) \in Y_0^T$ *and* $l \in C^1[0, T]$, *there exists a constant* $c > 0$ *depending only on* $\alpha, \gamma, \gamma_0, \lambda_1, \rho_0, c_0$, *in particular, independent of* T, φ, ψ, *such that*

$$\begin{aligned} \|\mathcal{N}_i[u, l]\|_{C^1[0,T]} \le \|\partial_t^\alpha h_i\|_{C^1[0,T]} + c\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}\Big) + \|\tilde{f}_i\|_{C^1[0,T]} \\ + c(T + T^{\alpha-1})\|q\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))} + c(T^2 + T^\alpha)\|k\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} \\ + c(T + T^{\alpha-1})\|f\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))} + T^{\frac{1}{2}}\|l\|_{C^1[0,T]}\Big], \end{aligned} \tag{2.49}$$

where $\mathcal{N}_i$ $(i = 1, 2)$ *are the same as those in* (2.39) *and* $l(t)$ *as in* (2.40).

Proof. By Lemma 2.5 and condition (C1), we see that

$$\|\mathcal{N}_i[u,l]\|_{C[0,T]} \le \|\partial_t^\alpha h_i\|_{C[0,T]} + \|\mathcal{L}u(x_i,t)\|_{C[0,T]} + \|l * h_i'\|_{C[0,T]}$$
$$+\|f_i\|_{C[0,T]} \le \|\partial_t^\alpha h_i\|_{C[0,T]} + c\Big(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}$$
$$+T^{\frac{\alpha}{2}}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big) + T^{\frac{1}{2}}\|l\|_{C[0,T]}\|h_i'\|_{L^2(0,T)} + \|\tilde{f}_i\|_{C[0,T]}.$$

By the definition of F, the last inequality gives

$$\|\mathcal{N}_i[u,l]\|_{C[0,T]} \le \|\partial_t^\alpha h_i\|_{C[0,T]} + c\Big[\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}$$
$$+ T\Big(\|q\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))} + \lambda_1^{-\frac{1}{\alpha}} T\|k\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}$$
$$+ \|f\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big)\Big] + T^{\frac{1}{2}}\|l\|_{C[0,T]}\|h_i'\|_{L^2(0,T)} + \|\tilde{f}_i\|_{C[0,T]}, \quad (2.50)$$

where we have used that

$$\|v\|^2_{D(\mathcal{L}^{\gamma})} = \sum_{n=1}^{\infty} \lambda_n^{2\gamma+\frac{2}{\alpha}}(v,e_n)^2\lambda_n^{-\frac{2}{\alpha}} \le \lambda_1^{-\frac{2}{\alpha}}\|v\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})}.$$

On the other hand, direct calculations yields

$$D_t\mathcal{N}_i[u,l](t) = (\partial_t^\alpha h_i)' + \mathcal{L}u_t(x_i,t) - (l' * h_i')(t) - \tilde{f}_i'(t). \quad (2.51)$$

Here we have taken into account that $l(0) = 0$. By Lemma 2.5, we have

$$\|D_t\mathcal{N}_i[u,l]\|_{C[0,T]} \le \|(\partial_t^\alpha h_i)'\|_{C[0,T]} + c\Big[\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}$$
$$+T^{\alpha-1}\Big(\|q\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))} + \lambda_1^{-\frac{1}{\alpha}} T\|k\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}$$
$$+ \|f\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))}\Big)\Big] + T^{\frac{1}{2}}\|l'\|_{C[0,T]}\|h_i'\|_{L^2(0,T)} + \|\tilde{f}_i'\|_{C[0,T]}. \quad (2.52)$$

Using (2.50) and (2.52), we obtain the desired estimate given in (2.49). □

3 Well-posedness of the inverse problem

We can now prove the existence of a solution to our inverse problem, i.e. Theorem 1.1, which proceeds by a fixed point argument. First, we define the function set

$$B_{\rho,T} = \Big\{(\bar{u},\bar{q},\bar{k}) \in Y_0^T: \ \bar{u}(x,0) = \varphi(x), \bar{u}_t(x,0) = \psi(x), \ \bar{u}(0,t) = \bar{u}(1,t) = 0,$$
$$\|\bar{u}\|_{X_0^T} + \|\bar{q}\|_{C^1[0,T]} + \|\bar{k}\|_{C[0,T]} \le \rho\Big\}.$$

Here ρ is a large constant that depends on the initial and source data φ, ψ, f, as well on the measurement data h_i. For a given $(\bar{u},\bar{q},\bar{k}) \in B_{\rho,T}$, we consider

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{L}u(x,t) = F(x,t), & (x,t) \in Q_0^T, \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x), & 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \end{cases} \quad (3.1)$$

where

$$F(x,t)=\bar{q}(t)\bar{u}_t(x,t)+(\bar{k}*\bar{u})(t)+f(x,t),$$

and

$$q(t)=\frac{1}{p(t)}\Big(h_2(0)\mathcal{N}_1[u,\bar{l}](t)-h_1(0)\mathcal{N}_2[u,\bar{l}](t)\Big), \tag{3.2}$$

$$k(t)=\frac{d}{dt}\left(\frac{h_1'(t)\mathcal{N}_2[u,\bar{l}](t)-h_2'(t)\mathcal{N}_1[u,\bar{l}](t)}{p(t)}\right) \tag{3.3}$$

to generate (u,q,k), where $\bar{l}(t)=\int_0^t\bar{k}(\tau)d\tau$, $\mathcal{N}_i\,(i=1,2)$ are the same as those in (2.37).

By Hölder's inequality, we have

$$\|(\bar{k}*\bar{u})(t)\|^2_{\mathcal{D}(\mathcal{L}^\gamma)}\le\int_0^t|\bar{k}(t-\tau)|^2d\tau\int_0^t\|u(\cdot,\tau)\|^2_{\mathcal{D}(\mathcal{L}^\gamma)}d\tau\le\lambda_1^{-\frac{2}{\alpha}}t^2\|\bar{k}\|^2_{C[0,t]}\|\bar{u}\|^2_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})} \tag{3.4}$$

which implies

$$\|(\bar{k}*\bar{u})(t)\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\le\lambda_1^{-\frac{1}{\alpha}}\rho^2T.$$

Furthermore

$$\|\bar{q}\bar{u}_t\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}=\max_{0\le t\le T}\left|\sum_{n=1}^\infty\lambda_n^{2\gamma}(\bar{q}(t)\bar{u}_t(\cdot,t),e_n)^2\right|\le\|\bar{q}\|^2_{C[0,T]}\|\bar{u}_t\|^2_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\le\rho^4. \tag{3.5}$$

Using these results, along with $f\in C^1([0,T];\mathcal{D}(\mathcal{L}^\gamma))$, we have

$$\bar{q}(t)\bar{u}_t(x,t)+(\bar{k}*\bar{u})(t)+f(x,t)\in C([0,T];\mathcal{D}(\mathcal{L}^\gamma)).$$

By Lemma 2.4, the unique solution $u\in X_0^T$ to problem (3.1), given by (2.16) satisfies

$$\|u\|_{X_0^T}\le c\left(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})}+\|\psi\|_{\mathcal{D}(\mathcal{L}^\gamma)}+T^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\right), \tag{3.6}$$

where $c>0$ depends only on α. Further, (3.2)-(3.3) define the functions $q(t)$ and $k(t)$ in terms of u. Furthermore, by Lemma 2.7, we have

$$\begin{aligned}\|q\|_{C^1[0,T]}+\|k\|_{C[0,T]}\le{}& c\|1/p\|_{C^1[0,T]}\Big(|h_1(0)|+|h_2(0)|+\|h_1'\|_{C^1[0,T]}+\|h_2'\|_{C^1[0,T]}\Big)\\ &\times\Big(1+(T+T^{\alpha-1})(1+\|\bar{q}\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))})\\ &+(T^2+T^\alpha)\|\bar{k}\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}+T^{\frac{1}{2}}\|\bar{l}\|_{C^1[0,T]}\Big).\end{aligned} \tag{3.7}$$

Note $\bar{l}(t)=\int_0^t\bar{k}(\tau)d\tau$. Then, we get

$$\|\bar{l}\|_{C^1[0,T]}=\left\|\int_0^t\bar{k}(\tau)d\tau\right\|_{C[0,T]}+\|\bar{k}\|_{C[0,T]}\le(1+T)\|\bar{k}\|_{C[0,T]}. \tag{3.8}$$

Substituting (3.8) into (3.7) yields

$$\begin{aligned}\|q\|_{C^1[0,T]}+\|k\|_{C[0,T]}\le c(T)\Big[&1+\|\bar{q}\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))}\\ &+\|\bar{k}\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}})}+\|\bar{k}\|_{C[0,T]}\Big].\end{aligned} \tag{3.9}$$

This implies that $q(t)\in C^1[0,T]$ and $k(t)\in C[0,T]$.

Thus the mapping

$$Z:B_{\rho,T}\to Y_0^T,\qquad(\bar{u},\bar{q},\bar{k})\mapsto(u,q,k) \tag{3.10}$$

given by (3.1)-(3.3) is well defined.

The next lemma shows that Z is a contraction map on $B_{\rho,T}$ for sufficiently small $T>0$. More precisely, we have the following result.

Lemma 3.1. *Let* (C1)-(C4) *be hold. For* $(\bar{u},\bar{q},\bar{k})$, $(\bar{U},\bar{Q},\bar{K}) \in B_{\rho,T}$, *define*

$$(u,q,k) = Z(\bar{u},\bar{q},\bar{k}), \quad (U,Q,K) = Z(\bar{U},\bar{Q},\bar{K}).$$

Then for any sufficiently large ρ *and suitably small* $\tau(\rho) > 0$, *we have*

$$\|(u,q,k)\|_{Y_0^T} \leq \rho$$

and

$$\|(u-U,q-Q,k-K)\|_{Y_0^T} \leq \frac{1}{2}\|(\bar{u}-\bar{U},\bar{q}-\bar{Q},\bar{k}-\bar{K})\|_{Y_0^T} \tag{3.11}$$

for all $T \in (0,\tau(\rho)]$.

In the following proof, we use c_j $(j = 7,...)$ to denote a constant that depends on $\alpha,\gamma,\gamma_0,\lambda_1,\rho_0,c_0$ and the known functions φ, ψ, f and measurement data $h_i, i = 1,2$, but is independent of ρ.

Proof. First, we prove that $Z(B_{\rho,T}) \subset B_{\rho,T}$ for sufficiently large ρ and suitably small T. Without loss of generality, we assume that $\rho \in [1,\infty)$ and $T \in (0,1]$.

By Lemma 2.4 and inequalities (3.4)-(3.6), we have

$$\begin{aligned} \|u\|_{X_0^T} \leq c\lambda_1^{-(\gamma_0-\gamma)}(\|\varphi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0+\frac{1}{\alpha}})} + \|\psi\|_{\mathcal{D}(\mathcal{L}^{\gamma_0})}) \\ + cT^{\alpha-1}\Big[\|\bar{q}(t)\bar{u}_t\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma}))} + \|(\bar{k}*\bar{u})\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma}))} \\ + \|f\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma}))})\Big] \leq c_7\left[1+\rho^2T^{\alpha-1}\right]. \end{aligned} \tag{3.12}$$

Here we have used the assumptions $\rho \in [1,\infty)$ and $T \in (0,1]$ (and we shall use them further on). On the other hand, by (3.2)-(3.3), together with Lemma 2.7 and (3.8), we have

$$\begin{aligned} \|q\|_{C^1[0,T]} + \|k\|_{C[0,T]} &\leq c_8\Big(\|\mathcal{N}_1[u,\bar{l}]\|_{C^1[0,T]} + \|\mathcal{N}_2[u,\bar{l}]\|_{C^1[0,T]}\Big) \\ &\leq c_9\Big[1+T+T^{\alpha-1}+\rho(T+T^{\alpha-1})\|u_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma}))} \\ &\quad + \rho(T^2+T^{\alpha})\|u\|_{C([0,T];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} + \rho T^{\frac{1}{2}}(1+T)\Big] \\ &\leq c_{10}\Big[1+T+T^{\alpha-1}+\rho(T^2+T^{\alpha-1})\|u\|_{X_0^T} + \rho T^{\frac{1}{2}}(1+T)\Big] \\ &\leq c_{11}[1+c_7\rho T^{\alpha-1}(1+\rho^2T^{\alpha-1})+\rho T^{1/2}] \\ &\leq c_{12}[1+\rho^3(T^{\alpha-1}+T^{1/2})]. \end{aligned} \tag{3.13}$$

Adding inequalities (3.12) and (3.13) gives us

$$\begin{aligned} \|(u,q,k)\|_{Y_0^T} \leq c_7\left[1+\rho^2T^{\alpha-1}\right] \\ + c_{12}\left[1+\rho^3(T^{\alpha-1}+T^{1/2})\right] \leq c_{13}\Big[1+\rho^3\left(T^{\alpha-1}+T^{1/2}\right)\Big]. \end{aligned} \tag{3.14}$$

For $\rho > c_{13}$, we choose a sufficiently small $\tau_1(\rho)$ such that, for $\rho > c_{13}$ and $0 < T < \tau_1(\rho)$

$$c_{13}\Big[1+\rho^3\left(T^{\alpha-1}+T^{1/2}\right)\Big] \leq \rho. \tag{3.15}$$

Therefore, for all $T < \min\{1,\tau_1(\rho)\}$, we have

$$\|(u,q,k)\|_{Y_0^T} \leq \rho. \tag{3.16}$$

That is, Z maps $B_{\rho,T}$ into itself for each fixed $\rho > c_{13}$ and $T \in (0, \min\{1, \tau_1(\rho)\}]$.

Next, we check the second condition of contractive mapping Z. Let $(u, q, k) = Z(\bar{u}, \bar{q}, \bar{k})$ and $(U, Q, K) = Z(\bar{U}, \bar{Q}, \bar{K})$. Then we obtain that $(u - U, q - Q, k - K)$ satisfies the equalities

$$u(x,t) - U(x,t) = \int_0^t A^{-1} Y(t-s) \bar{F}(x,s) ds, \quad (x,t) \in Q_0^T, \tag{3.17}$$

and

$$q(t) - Q(t) = \frac{1}{p(t)}\Big(h_2(0)(\mathcal{N}_1[u,\bar{l}](t) - \mathcal{N}_1[U,\bar{L}](t)) - h_1(0)(\mathcal{N}_2[u,\bar{l}](t) - \mathcal{N}_2[U,\bar{L}](t))\Big), \tag{3.18}$$

$$k(t) - K(t) = \frac{d}{dt}\Big(\frac{h_1'(t)(\mathcal{N}_2[u,\bar{l}](t) - \mathcal{N}_2[U,\bar{L}](t))}{p(t)} - \frac{h_2'(t)(\mathcal{N}_1[u,\bar{l}](t) - \mathcal{N}_1[U,\bar{L}](t))}{p(t)}\Big) \tag{3.19}$$

where $\bar{L}(t) = \int_0^t \bar{K}(\tau) d\tau$ and

$$\bar{F} := q(u_t - U_t) + (q - Q)U_t + k * (u - U) + (k - K) * U.$$

Using Lemma 2.4, (3.5) and (3.6), we get

$$\begin{aligned}
\|u - U\|_{X_0^T} &\le c_{14} T^{\alpha-1}\Big[\|(\bar{q} - \bar{Q})\bar{u}_t\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))} + \|(\bar{u}_t - \bar{U}_t)\bar{q}\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))} \\
&\quad + \|(\bar{k} - \bar{K}) * \bar{u})\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))} + \|\bar{k} * (\bar{u} - \bar{U})\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))}\Big] \\
&\le c_{14} T^{\alpha-1}\Big[\|\bar{q} - \bar{Q}\|_{C[0,T]} \|\bar{u}_t\|_{C([0,T];\mathcal{D}(\mathcal{L}^\gamma))} \\
&\quad + \|\bar{u}_t - \bar{U}_t\|_{C([0,T],\mathcal{D}(\mathcal{L}^\gamma))} \|\bar{q}\|_{C[0,T]} + T^2 \lambda_1^{-\frac{1}{\alpha}} \|\bar{k} - \bar{K}\|_{C[0,T]} \|\bar{u}\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} \\
&\quad + T^2 \lambda_1^{-\frac{1}{\alpha}} \|\bar{u} - \bar{U}\|_{C([0,T],\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} \|\bar{k}\|_{C[0,T]}\Big] \\
&\le c_{14} \rho T^{\alpha-1} \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}\}\Big[\|\bar{q} - \bar{Q}\|_{C[0,T]} + \|\bar{u} - \bar{U}\|_{X_0^T} + \|\bar{k} - \bar{K}\|_{C[0,T]}\Big].
\end{aligned} \tag{3.20}$$

Similarly, by (3.18)-(3.19) and Lemma 2.7, we have

$$\begin{aligned}
\|q - Q\|_{C^1[0,T]} + \|k - K\|_{C[0,T]} &\le c_{15} \rho (T^2 + T^{\alpha-1}) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}, T^{\frac{3}{2}}\} \\
&\quad \times \Big[\|\bar{q} - \bar{Q}\|_{C[0,T]} + \|\bar{u} - \bar{U}\|_{X_0^T} + \|\bar{k} - \bar{K}\|_{C[0,T]}\Big].
\end{aligned} \tag{3.21}$$

Therefore, by (3.20) and (3.21), we have

$$\begin{aligned}
\|(u - U, q - Q, k - K)\|_{Y_0^T} &\le c_{16} \rho \Big[T^{\alpha-1} \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}\} \\
&\quad + (T^2 + T^{\alpha-1}) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}, T^{\frac{3}{2}}\}\Big] \|(\bar{u} - \bar{U}, \bar{q} - \bar{Q}, \bar{k} - \bar{K})\|_{Y_0^T}.
\end{aligned} \tag{3.22}$$

Hence, we can choose a sufficiently small τ_2 such that

$$c_{16} \rho \Big[T^{\alpha-1} \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}\} + (T^2 + T^{\alpha-1}) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}, T^{\frac{3}{2}}\}\Big] \le 1/2 \tag{3.23}$$

for all $T \in (0, \tau_2]$ to obtain

$$\|(u - U, q - Q, k - K)\|_{Y_0^T} \le \frac{1}{2} \|(\bar{u} - \bar{U}, \bar{q} - \bar{Q}, \bar{k} - \bar{K})\|_{Y_0^T}. \tag{3.24}$$

Estimates (3.16) and (3.24) show that Z is a contraction map on $B_{\rho,T}$ for all $T \in (0, \tau]$, if we choose $\tau \le \min\{1, \tau_1, \tau_2\}$. □

Let us now prove Theorem 1.1.

Proof. Lemma 3.1 shows that there exists a sufficiently large $\rho > 0$ and a corresponding sufficiently small $\tau(\rho) > 0$, such that, for any $0 < T < \tau(\rho)$, the mapping Z is a contraction on $B_{\rho,T}$. Hence, the Banach fixed point theorem guarantees the existence of a unique solution $(u, q, k) \in B_{\rho,T} \subset Y_0^\tau$ to the system (2.36)-(2.38), for sufficiently small τ. As a consequence, the problem constituted by (1.1)-(1.3) and (1.5) also admits a unique solution $(u, q, k) \in B_{\rho,T} \subset Y_0^\tau$ by Lemma 2.6. □

Now, we present a global uniqueness result in time.

Lemma 3.2. *Under conditions* (C1)-(C4)*, for given measurement data $h_i(t)$ for $i = 1, 2$ in* (1.5)*, if the inverse problem* (1.1)-(1.3)*,* (1.5) *has two solutions $(u_j, q_j, k_j) \in Y_0^T$ $(j = 1, 2)$ for any time, then $(u_1, q_1, k_1) = (u_2, q_2, k_2)$ in $[0, T]$.*

According to Remark 5, we know that (2.36)-(2.38) is equivalent to (1.1)-(1.3), (1.5). In Lemma 3.2, we discuss the global uniqueness of inverse problem (2.36)-(2.38).

Proof. Given any time T, let (u_i, q_i, k_i) $(i = 1, 2)$ be two solutions to inverse problem (2.36)-(2.38) in $[0, T]$ such that $(u_i, q_i, k_i) \in Y_0^T$. This implies

$$\|(u_i, q_i, k_i)\|_{Y_0^T} \le C^*, \quad i = 1, 2, \tag{3.25}$$

where $C^* > 0$ depends only on α, T, initial data φ and ψ, the known function f and measurement data h_i.

Let

$$\tilde{u} = u_1 - u_2, \quad \tilde{q} = q_1 - q_2, \quad \tilde{k} = k_1 - k_2.$$

Then $(\tilde{u}, \tilde{q}, \tilde{k})$ satisfies

$$\begin{cases} \partial_t^\alpha \tilde{u} + \mathcal{L}\tilde{u} = q_1 \tilde{u}_t + \tilde{q} u_{2t} + k_1 * \tilde{u} + \tilde{k} * u_2, & (x,t) \in Q_0^T, \\ \tilde{u}(x,0) = \tilde{u}_t(x,0) = 0, & 0 < x < 1, \\ \tilde{u}(0,t) = \tilde{u}(1,t) = 0, & 0 < t < T, \end{cases} \tag{3.26}$$

and

$$\tilde{q}(t) = \frac{1}{p(t)}\Big(h_2(0)\mathcal{L}\tilde{u}(x_1,t) - h_1(0)\mathcal{L}\tilde{u}(x_2,t) - \tilde{l} * p\Big), \tag{3.27}$$

$$\tilde{k}(t) = \frac{d}{dt}\left[\frac{h_1'(t)\left(\mathcal{L}\tilde{u}(x_2,t) - \tilde{l} * h_2'\right) - h_2'(t)\left(\mathcal{L}\tilde{u}(x_1,t) - \tilde{l} * h_1'\right)}{p(t)}\right], \tag{3.28}$$

where $\tilde{l}(t) = l_1 - l_2$ and $l_i(t) = \int_0^t k_i(s)ds$. We have to show

$$\|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_0^T} = 0. \tag{3.29}$$

Define

$$\sigma = \inf\Big\{t \in (0,T] : \|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_0^t} > 0\Big\}. \tag{3.30}$$

If (3.29) does not hold, then it is clear that σ is well-defined and satisfies $0 \le \sigma \le T$. Moreover, by Theorem 1.1, we have $\sigma > 0$, and $\sigma < T$ follows from the fact that $\|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_0^T} > 0$ and the continuity of the norm with respect to time t.

Let $0 < \epsilon < T - \sigma$. Further, by (2.16), we can write the solution $\tilde{u}$ as

$$\tilde{u}(x,t) = -\int_0^t \mathcal{L}^{-1} Z_3(t-s)\tilde{F}(x,s)ds, \quad (x,t) \in Q_\sigma^{\sigma+\epsilon}, \tag{3.31}$$

where

$$\tilde{F}(x,t) = q_1\tilde{u}_t + \tilde{q}u_{2t} + k_1 * \tilde{u} + \tilde{k} * u_2.$$

Then, similarly to the proofs of Lemma 2.4 and 2.5, we have

$$\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} \leq c\epsilon^{\alpha-1}\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}, \tag{3.32}$$

and

$$\begin{cases} \|\mathcal{L}\tilde{u}(x_i,\cdot)\|_{C[\sigma,\sigma+\epsilon]} \leq c_{17}\epsilon\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}, \\ \|\mathcal{L}\tilde{u}_t(x_i,\cdot)\|_{C[\sigma,\sigma+\epsilon]} \leq c_{18}\epsilon^{\alpha-1}\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}. \end{cases} \tag{3.33}$$

From the definition of σ, we see that

$$\tilde{u} = \tilde{q} = \tilde{k} = 0 \quad \text{in} \quad [0,\sigma]. \tag{3.34}$$

By the definition of $\tilde{F}$, and using (3.4), (3.5) and (3.25), we have

$$\begin{aligned} \|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} &\leq c_{19}\epsilon^{\alpha-1}\Big(\|q_1\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \|\tilde{q}u_{2t}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} \\ &\qquad + \|k_1 * \tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \|\tilde{k} * u_2\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}\Big) \\ &\leq c_{20}C^*\epsilon^{\alpha-1}\Big(\|\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \|\tilde{q}\|_{C[\sigma,\sigma+\epsilon]} \\ &\qquad + \lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))} + \lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}\Big). \end{aligned} \tag{3.35}$$

Due to $\tilde{q}(\sigma) = 0$, then implies

$$\|\tilde{q}\|_{C[\sigma,\sigma+\epsilon]} = \max_{\sigma\leq t\leq\sigma+\epsilon}\Big|\int_\sigma^t \tilde{q}'(s)ds\Big| \leq \epsilon\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]}. \tag{3.36}$$

Substituting (3.36) into (3.35), we have

$$\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} \leq c_{20}C^*\epsilon^{\alpha-1}\max\{1,\epsilon,\lambda_1^{-\frac{1}{\alpha}}\epsilon\}\|(\tilde{u},\tilde{q},\tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}}. \tag{3.37}$$

Note $\|\tilde{q}\|_{C^1[0,\sigma]} = \|\tilde{k}\|_{C[0,\sigma]} = 0$. On the other hand, by (3.27), and using (3.33), we have the following estimate for $\tilde{q}$

$$\begin{aligned} &\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]} \\ &\quad\leq c_{21}(\epsilon+\epsilon^{\alpha-1})\Big(\|h_2(0)/p(t)\|_{C^1[\sigma,\sigma+\epsilon]} + \|h_2(0)/p(t)\|_{C^1[\sigma,\sigma+\epsilon]}\Big)\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} \\ &\quad + \epsilon^{1/2}\|p\|_{C[\sigma,\sigma+\epsilon]}\|\tilde{l}\|_{C[\sigma,\sigma+\epsilon]} \\ &\quad\leq c_{21}C(\|h_1\|_{C^1[0,T]},\|h_2\|_{C^1[0,T]})(\epsilon+\epsilon^{\alpha-1})\Big(\|\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \epsilon\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]} \\ &\qquad + \lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}\Big) + C(\|h_1\|_{C[0,T]},\|h_2\|_{C[0,T]})\epsilon^{3/2}\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}, \end{aligned} \tag{3.38}$$

where we have used that

$$\|\tilde{l}\|_{C[\sigma,\sigma+\epsilon]} = \max_{\sigma\leq t\leq\sigma+\epsilon}\Big|\int_\sigma^t \tilde{k}(s)ds\Big| \leq \epsilon\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}.$$

Similarly to (3.38), by (3.28) we can easily estimate $\tilde{k}$

$$\begin{aligned} \|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]} &\leq C(\|h_1\|_{C^2[0,T]},\|h_2\|_{C^2[0,T]})\Big[c_{21}(\epsilon+\epsilon^{\alpha-1})\big(\|\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} \\ &\qquad + \epsilon\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]} + \lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}\big) + \epsilon^{3/2}\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}\Big]. \end{aligned} \tag{3.39}$$

From (3.37)-(3.39), we obtain

$$\|(\tilde{u},\tilde{q},\tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}} \le C(\|h_i\|_{C^2[0,T]}, C^*)\eta(\epsilon)\|(\tilde{u},\tilde{q},\tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}} \tag{3.40}$$

with

$$\lim_{\epsilon\to+0}\eta(\epsilon) = \lim_{\epsilon\to+0}\left(\epsilon + 2\epsilon^{\alpha-1} + \epsilon^{3/2}\right)\max\{1,\epsilon,\lambda_1^{-\frac{1}{\alpha}}\epsilon\} = 0,$$

and implying

$$\|(\tilde{u},\tilde{q},\tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}} = 0$$

for some sufficiently small positive constant ϵ. This means that $(u_1-u_2, q_1-q_2, k_1-k_2)$ vanishes in $[0,\sigma+\epsilon]$, which contradicts with the definition of σ. Therefore (3.29) is proved. From here, we can conclude that

$$(u_1,q_1,k_1) = (u_2,q_2,k_2) \quad \text{in} \quad [0,T]$$

for any time T. □

4 Examples

In this section, as an illustration, we provide two examples of inverse problem (1.1)-(1.3), (1.5).
Example 1. In this example, we consider inverse problem (1.1)-(1.3), (1.5) with the following input data:

$$\begin{cases} \varphi(x) = \sin 2\pi x, \quad \psi(x) = \left((8-16\sqrt{2})x^2 + (8\sqrt{2}-2)x\right)\sin\pi x, \quad x_1 := \frac{1}{4},\ x_2 := \frac{1}{2}, \\ f(x,t) = -\varphi''(x) - \left[\left(\psi''(x) + (64-74\sqrt{2}+7\pi^2)\psi(x)\right)t + (48-42\sqrt{2}+8\pi^2)t^2\psi(x)\right] \\ \qquad\qquad - \left[(48-42\sqrt{2}+8\pi^2)t + 4\pi^2t^2\right]\varphi(x), \\ h_1(t) := 1+t, \quad h_2(t) := t. \end{cases}$$

It is easy to see that all functions given in Example 1 satisfy conditions (C1)-(C4).

Then, the exact solution of the inverse problem is

$$u(x,t) = \varphi(x) + \psi(x)t, \quad k(t) = 48 - 42\sqrt{2} + 8\pi^2 + 8\pi^2 t,$$

$$q(t) = (64-74\sqrt{2}+7\pi^2)t + (24-21\sqrt{2}+4\pi^2)t^2 - \frac{4\pi^2}{3}t^3.$$

Example 2. In this example, we consider inverse problem (1.1)-(1.3), (1.5) with the following input data:

$$\begin{cases} \varphi(x) = \sin 2\pi x, \quad \psi(x) = \left((8-16\sqrt{2})x^2 + (8\sqrt{2}-2)x\right)\sin\pi x, \quad x_1 := \frac{1}{4},\ x_2 := \frac{1}{2}, \\ f(x,t) = -\varphi''(x) + \varphi(x)\varphi''(x_1) \\ \qquad\qquad -\Big[\psi''(x) + (\varphi(x)-\psi(x))\psi''(x_2) + (\varphi''(x_1)-\psi''(x_1))\varphi(x)\Big]\cos t, \\ h_1(t) := 1+t, \quad h_2(t) := t. \end{cases}$$

Then the exact solution of the inverse problem is

$$u(x,t) = \varphi(x) + \psi(x)t, \quad k(t) = 48 - 42\sqrt{2} + 4\pi^2\sin t,$$
$$q(t) = (-64+74\sqrt{2}-3\pi^2)t + (24-21\sqrt{2})t^2 + 4\pi^2\sin t.$$

Since the inverse problem considered in equations (1.1)-(1.3), (1.5) is nonlinear, hence, an analytical solution cannot be found, however, to obtain the exact solutions provided in Examples 1 and 2, we employed a reverse approach: first, we define the functions φ,ψ,h_1,h_2 that satisfy conditions (C1)-(C4) and the function $u(t,x)$ that satisfies conditions (1.2), (1.3) and (1.5), then we determine the remaining functions q,k,f from the system of equations (2.36)-(2.38).

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Events

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INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF ANALYSIS, DIFFERENTIAL EQUATIONS AND ALGEBRA" (EMJ-2025), DEDICATED TO THE 15TH ANNIVERSARY OF THE EURASIAN MATHEMATICAL JOURNAL

From January 7 to January 11, 2025 at the Non-profit joint-stock company "L.N. Gumilyov Eurasian National University" (ENU) the International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2025) was held. The conference was dedicated to the 15th anniversary of the Eurasian Mathematical Journal (EMJ).

The purposes of the conference were to discuss the current state of development of mathematical scientific directions, expand the number of potential authors of the Eurasian Mathematical Journal and further strengthen the scientific cooperation between the Faculty of Mechanics and Mathematics of the ENU and scientists from other cities of Kazakhstan and abroad.

The partner universities for the organization of the conference were the Peoples' Friendship University of Russia (the RUDN University, Moscow), the M.V. Lomonosov Moscow State University and the University of Padua (Italy).

The conference was attended by more than 100 mathematicians from the cities of Aktobe, Almaty, Arkalyk, Astana, Karaganda, Shymkent, Taraz, Turkestan, Ust-Kamenogorsk, as well as from several foreign countries: from Georgia, Italy, Kyrgyzstan, Pakistan, Russia, Uzbekistan, Czech Republic.

The chairman of the International Programme Committee of the conference was Ye.B. Sydykov, rector of the ENU, co-chairmen were Editors-in-Chief of the EMJ: V.I. Burenkov, professor of the RUDN University, foreign member of the National Academy of Sciences of the Republic of Kazakhstan (NAS RK), M. Otelbaev, academician of the NAS RK, V.A. Sadovnichy, academician of the Russian Academy of Sciences (RAS), rector of the M.V. Lomonosov Moscow State University (MSU).

There were three sections at the conference: "Theory of Functions and Functional Analysis", "Differential Equations and Equations of Mathematical Physics" and "Algebra and Theory of Models". 10 plenary presentations of 40 minutes each and 89 offline and online sectional presentations of 20 minutes each, devoted to contemporary areas of mathematics, were given.

It was decided to recommend selected reports of the participants for publication in the Eurasian Mathematical Journal and the Bulletin of the Karaganda State University (series "Mathematics").

Before the conference, the electronic version of the collection of abstracts of the participants' talks was published.

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Members: A.M. Abylaeva (Kazakhstan), M. Aldai (Kazakhstan), T. Bekzhan (Kazakhstan), K.A. Bekmaganbetov (Kazakhstan), N.A. Bokaev (Kazakhstan), S.K. Burgumbaeva (Kazakhstan), A. Gogatishvili (Czech Republic), B.E. Kanguzhin (Kazakhstan), R.M. Kamatov (Kazakhstan), B.S. Koshkarova (Kazakhstan), P.D. Lamberti (Italy), M.B. Muratbekov (Kazakhstan), I.N. Parasidis (Greece), M.A. Sadybekov (Kazakhstan), A.M. Sarsenbi (Kazakhstan), A.Yu. Seipisheva (Kazakhstan), T.V. Tararykova (Russia), N.T. Tleukhanova (Kazakhstan), B.H. Turmetov (Kazakhstan), J.A. Tusupov (Kazakhstan).

Executive secretary: A.M. Temirkhanova.

Secretariat: R.D. Akhmetkaliyeva, A.N. Beszhanova, Pђ.Pђ. Dzhumabayeva, D.S. Karatay, A.N. Sharipova, D. Matin, Zh.B. Mukanov, B.S. Nurimov, Zh.B. Eskabylova, A. Kankenova, I. Gaidarov, D. Sarsenaly, N. Zhanabergenova, A. Abek, Ye.O. Moldagali.

Conference Schedule:

08.01.2025
09.00 – 10.00 Registration
10.00 – 10.10 Opening of the conference
10.30 – 12.50 Plenary talks
12.50 – 14.00 Lunch
14.00 – 18.10 Session talks

09.01.2025
10.00 – 13.00 Plenary talks
13.00 – 14.00 Lunch
14.00 – 18.50 Session talks
19.00 – Dinner for participants of the conference

10.01.2025
10.00 – 12.30 Plenary talks
12.30 – 13.00 Closing of the conference

At the opening ceremony welcome speeches were given by A.B. Beisenbai, Vice-Rector for Academic Affairs of the ENU, Co-chair of the Organizing Committee of the conference; V.I. Burenkov, professor of the RUDN University, Editor-in-Chief of the EMJ.

Plenary talks were given by

V.I. Burenkov (Russia), E.D. Nursultanov (Kazakhstan), M.A. Sadybekov (Kazakhstan) – on 08.01.2025;

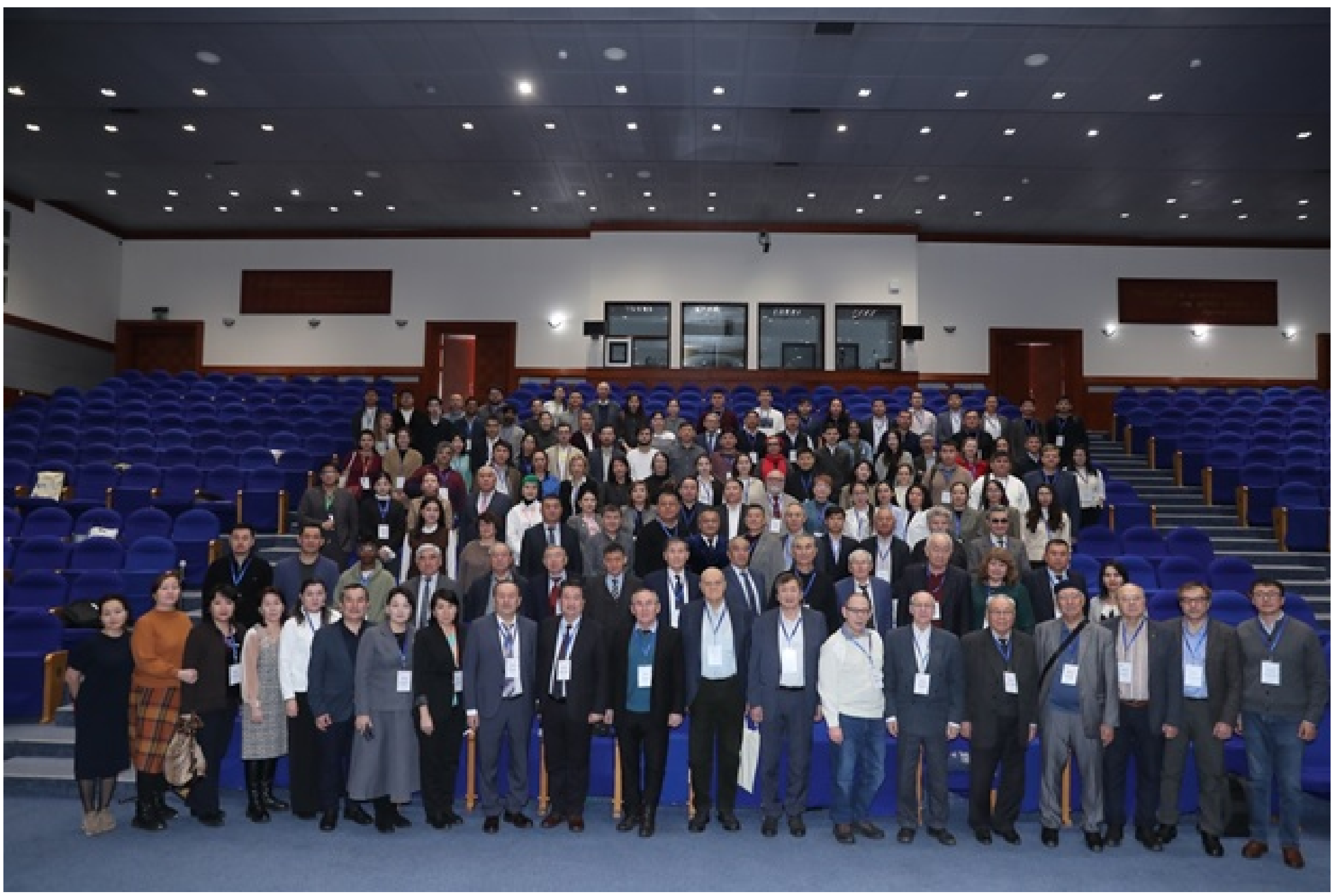
A. Muravnik (Russia), T. Nurlybekuly (Kazakhstan), M.I. Dyachenko (Russia), N. Markhabatov (Kazakhstan) – on 09.01.2025;

D. Suragan (Kazakhstan), D.E. Apushkinskaya (Russia), A. Kashkynbayev (Kazakhstan) – on 10.01.2025.

At the closing ceremony all participants unanimously congratulated the staff of the L.N. Gumilyov Eurasian National University and the Editorial Board of the Eurasian Mathematical Journal with the 15th anniversary of the journal and wished further creative successes.

They expressed hope that the journal will continue to play an important role in the development of mathematical science and education in Kazakhstan in the future.

V.I. Burenkov, K.N. Ospanov, A.M. Temirkhanova



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