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SIMILAR TRANSFORMATION OF ONE CLASS OF WELL-DEFINED RESTRICTIONS

B.N. Biyarov

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Key words: maximal (minimal) operator, well-defined restriction, well-defined extension, similar operators, singular coefficients, Riesz basis with brackets.

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Abstract. In this paper there is considered the description of all well-defined restrictions of a maximal operator in a Hilbert space. A class of well-defined restrictions is found for which a similar transformation has the domain of the fixed well-defined restriction. The resulting theorem is applied to the study of n-order differentiation operator and the Laplace operator.

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1 Introduction

Let a closed linear operator L be given in a Hilbert space H. The linear equation

$$Lu = f \tag{1.1}$$

is said to be *well-definedly solvable* on $R(L)$ if $\|u\| \leq C\|Lu\|$ for all $u \in D(L)$ (where $C > 0$ does not depend on u) and *everywhere solvable* if $R(L) = H$. If (1.1) is simultaneously well-defined and solvable everywhere, then we say that L is a *well-defined operator*. A well-definedly solvable operator L_0 is said to be *minimal* if $R(L_0) \neq H$. A closed operator $\widehat{L}$ is called a *maximal operator* if $R(\widehat{L}) = H$ and $\operatorname{Ker}\widehat{L} \neq \{0\}$. An operator A is called a *restriction* of an operator B and B is said to be an *extension* of A if $D(A) \subset D(B)$ and $Au = Bu$ for all $u \in D(A)$.

Note that if one of the well-defined restriction L of a maximal operator $\widehat{L}$ is known, then the inverses of all well-defined restrictions of $\widehat{L}$ have in the form [9]

$$L_K^{-1} f = L^{-1} f + Kf, \tag{1.2}$$

where K is an arbitrary bounded linear operator in H such that $R(K) \subset \operatorname{Ker}\widehat{L}$.

Let L_0 be a minimal operator, and let M_0 be another minimal operator related to L_0 by the equation $(L_0u, v) = (u, M_0v)$ for all $u \in D(L_0)$ and $v \in D(M_0)$. Then $\widehat{L} = M_0^*$ and $\widehat{M} = L_0^*$ are maximal operators such that $L_0 \subset \widehat{L}$ and $M_0 \subset \widehat{M}$. A well-defined restriction L of a maximal operator $\widehat{L}$ such that L is simultaneously a well-defined extension of the minimal operator L_0 is called a *boundary well-defined extension*. The existence of at least one boundary well-defined extension L was proved by Vishik in [14], that is, $L_0 \subset L \subset \widehat{L}$.

The inverse operators to all possible well-defined restrictions L_K of the maximal operator $\widehat{L}$ have form (1.2), moreover

$$D(L_K) = \left\{u \in D(\widehat{L}) : (I - K\widehat{L})u \in D(L)\right\}$$

is dense in H if and only if $\operatorname{Ker}(I + K^*L^*) = \{0\}$. All possible well-defined extensions M_K of M_0 have inverses of the form

$$M_K^{-1} f = (L_K^*)^{-1} f = (L^*)^{-1} f + K^* f,$$

where K is an arbitrary bounded linear operator in H with $R(K) \subset \operatorname{Ker} \widehat{L}$ such that

$$\operatorname{Ker}(I + K^*L^*) = \{0\}.$$

The main result of this work is the following.

Theorem 1.1. *Let L be a boundary well-defined extension of L_0, that is, $L_0 \subset L \subset \widehat{L}$. If L_K is densely defined in H and*

$$R(K^*) \subset D(L^*) \cap D(L_K^*),$$

where K and L are the operators in representation (1.2), then the operator $\overline{KL}_K$, where the bar denotes the closure of an operator in H, is bounded in H and a well-defined restriction L_K of the maximal operator $\widehat{L}$ is similar to the well-defined extension

$$A_K = L - \overline{KL}_K L \quad on \quad D(A_K) = D(L),$$

of the minimal operator A_0, where $D(A_0) = D(L) \cap Ker\,(\overline{KL}_K L)$ and $A_0 u = Lu$ on $D(A_0)$ (hence, $A_0 \subset L$).

The theory of well-defined restrictions and extensions is intended for the study of unbounded operators in a Hilbert space. A well-defined restriction of a certain maximal operator $\widehat{L}$ is obtained by the domain restriction of the maximal operator. All possible well-defined restrictions L_K are described using one fixed boundary well-defined restriction L in terms of the inverse operator (1.2). Then the direct operator L_K acts as a maximal operator, and its domain is given as a perturbation of the domain of a fixed boundary well-defined restriction L.

The main result of this work is the description of all well-defined restrictions L_K, which are similar to the well-defined extension A_K, of some minimal operator A_0. The domain of A_K coincides with the domain of L, and the action is defined as a perturbation of $\widehat{L}$.

It is clear that the spectra of these similar operators L_K and A_K coincide. Their eigenvectors are different. Further in the work, examples of the application of this abstract theorem to some differential equations are given. We note that a weak perturbation of the boundary condition L_K is equivalent to a singular perturbation of the action of the differential operator $\widehat{L}$.

The study of the properties of singular perturbations of some differential operators and well-defined restrictions is devoted to the works [5], [8].

2 Preliminaries

In this section, we present some results for the well-defined restrictions and extensions [3] which are used in Section 3.

Let A and B be bounded operators in a Hilbert space H. Operators A and B are said to be *similar* if there exist an invertible operator P such that $P^{-1}AP = B$. Similar operators have the same spectrum. If at least one of two operators A and B is invertible, then the operators AB and BA are similar.

Lemma 2.1. *Let L be a densely defined well-defined restriction of a maximal operator $\widehat{L}$ in a Hilbert space H, and K be a bounded linear operator in H. Then the operator KL is bounded on $D(L)$ (hence, $\overline{KL}$ is bounded in H) if and only if*

$$R(K^*) \subset D(L^*).$$

Proof. Let $R(K^*) \subset D(L^*)$. Then, by virtue of the equality $(KL)^* = L^*K^*$, we have that $\overline{KL}$ is bounded in H. Here we have used the boundedness of the operator L^*K^*. Then the operator KL is bounded on $D(L)$. Conversely, let KL be bounded on $D(L)$. Then $\overline{KL}$ is bounded on H, by virtue of the equality $(KL)^* = (\overline{KL})^*$ and, hence, the operator $(KL)^*$ is defined on the whole space H. Moreover, the operator K^* transfers any element f in H to $D(L^*)$. Indeed, for any element g of $D(L)$ we have

$$(Lg, K^*f) = (KLg, f) = (g, (KL)^*f).$$

Therefore, K^*f belongs to the domain $D(L^*)$. □

Lemma 2.2. *Let L_K be a densely defined well-defined restriction of a maximal operator $\widehat{L}$ in a Hilbert space H. Then $D(L^*) = D(L_K^*)$ if and only if $R(K^*) \subset D(L^*) \cap D(L_K^*)$, where L and K are the operators entering representation (1.2).*

Proof. If $D(L^*) = D(L_K^*)$, then by representation (1.2) we easily get

$$R(K^*) \subset D(L^*) \cap D(L_K^*) = D(L^*) = D(L_K^*).$$

Let us prove the converse. If

$$R(K^*) \subset D(L^*) \cap D(L_K^*),$$

then we obtain

$$(L_K^*)^{-1}f = (L^*)^{-1}f + K^*f = (L^*)^{-1}(I + L^*K^*)f, \tag{2.1}$$

$$(L^*)^{-1}f = (L_K^*)^{-1}f - K^*f = (L_K^*)^{-1}(I - L_K^*K^*)f, \tag{2.2}$$

for all f in H. It follows from (2.1) that $D(L_K^*) \subset D(L^*)$, and taking into account (2.2) this implies that $D(L^*) \subset D(L_K^*)$. Thus $D(L^*) = D(L_K^*)$. □

Corollary 2.1. *Let L_K be any densely defined well-defined restriction of a maximal operator $\widehat{L}$ in a Hilbert space H. If $R(K^*) \subset D(L^*)$ and $\overline{KL}$ is a compact operator in H, then*

$$D(L^*) = D(L_K^*).$$

Proof. Compactness of $\overline{KL}$ implies compactness of L^*K^*. Then $R(I + L^*K^*)$ is a closed subspace in H. It follows from the dense definiteness of L_K that $R(I + L^*K^*)$ is a dense set in H. Hence $R(I + L^*K^*) = H$. Then from the equality (2.1) we get $D(L^*) = D(L_K^*)$. □

Lemma 2.3. *If $R(K^*) \subset D(L^*) \cap D(L_K^*)$, then the bounded operators $I + L^*K^*$ and $I - L_K^*K^*$ from (2.1) and (2.2), respectively, have bounded inverses defined on H.*

Proof. By virtue of the density of the domains of the operators L_K^* and L^* it follows that the operators $I + L^*K^*$ and $I - L_K^*K^*$ are invertible. By (2.1) and (2.2) we have $\operatorname{Ker}(I + L^*K^*) = \{0\}$ and $\operatorname{Ker}(I - L_K^*K^*) = \{0\}$, respectively. By representations (2.1) and (2.2) it also follows that

$$R(I + L^*K^*) = H \quad \text{and} \quad R(I - L_K^*K^*) = H,$$

since $D(L^*) = D(L_K^*)$. The inverse operators $(I + L^*K^*)^{-1}$ and $(I - L_K^*K^*)^{-1}$ of the closed operators $I - L_K^*K^*$ and $I + L^*K^*$, respectively, are closed. Then the closed operators $(I + L^*K^*)^{-1}$ and $(I - L_K^*K^*)^{-1}$, defined on the whole of H, are bounded. □

Under the assumptions of Lemma 2.3 the operators KL and KL_K will be (see [2]) restrictions of the bounded operators $\overline{KL}$ and $\overline{KL_K}$, respectively. Thus,

$$(I - L_K^*K^*)^{-1} = I + L^*K^* \quad \text{and} \quad (I - \overline{KL}_K)^{-1} = I + \overline{KL}.$$

In what follows, we need the following theorem.

Theorem 2.1 (Theorem 1.1 [6, p. 307])**.** *A sequence $\{\psi_j\}_{j=1}^{\infty}$ biorthogonal to a basis $\{\phi_j\}_{j=1}^{\infty}$ of a Hilbert space H is also a basis of H.*

3 Proof of Theorem 1.1

In this section we prove our main result Theorem 1.1.

Proof. We transform (1.2) to the form

$$L_K^{-1} = L^{-1} + K = (I + \overline{KL})L^{-1}. \tag{3.1}$$

By Lemma 2.1 and Lemma 2.3 the operators $\overline{KL}$ and $\overline{KL_K}$ are bounded, the operator $I + \overline{KL}$ is invertible and

$$(I + \overline{KL})^{-1} = I - \overline{KL_K}.$$

Then we have

$$\begin{aligned} A_K^{-1} &= (I + \overline{KL})^{-1} L_K^{-1} (I + \overline{KL}) \\ &= (I + \overline{KL})^{-1}(I + \overline{KL})L^{-1}(I + \overline{KL}) = L^{-1}(I + \overline{KL}). \end{aligned}$$

Hence, by Corollary 1 [4, p. 259] we have $D(A_K) = D(L)$ and

$$A_K = (I - \overline{KL_K})L = L - \overline{KL_K}L.$$

Note that $A_K = A_0 = L$ on $D(A_0) = D(L) \cap \operatorname{Ker}(\overline{KL_K}L)$ and A_K is a well-defined extension of the minimal operator A_0. □

Corollary 3.1. *Suppose the hypothesis of Theorem 1.1 is satisfied. Then a well-defined extension L_K^* of a minimal operator M_0 is similar to the well-defined operator*

$$A_K^* = L^*(I - L_K^* K^*)$$

on

$$D(A_K^*) = \big\{ v \in H : (I - L_K^* K^*)v \in D(L^*) \big\}.$$

4 An application of Theorem 1.1 to the differentiation operator of order n

In this section, we give some applications of the main result to differential operators.

As a maximal operator $\widehat{L}$ in $L^2(0,1)$, we consider the operator

$$\widehat{L}y = y^{(n)},$$

with the domain $D(\widehat{L}) = W_2^n(0,1)$, $n \in \mathbb{N}$ ($W_2^n(0,1)$ in the Sobolev space). Then the minimal operator L_0 is the restriction of $\widehat{L}$ on $D(L_0) = \overset{\circ}{W}{}_2^n(0,1)$. As a fixed boundary well-defined extension L of the minimal operator L_0, we take the restriction of $\widehat{L}$ on

$$D(L) = \big\{ y \in W_2^n(0,1) : y^{(\ell)}(0) + y^{(\ell)}(1) = 0, \ \ell = 1, 2, \ldots, n-1 \big\}.$$

We find the inverse to all well-defined restrictions of $L_K \subset \widehat{L}$

$$L_K^{-1} = L^{-1} + K,$$

where

$$Kf=\sum_{\ell=1}^{n}w_\ell(x)\int_0^1 f(t)\overline{\sigma}_\ell(t)\,dt,\quad \sigma_\ell\in L^2(0,1),$$

and $w_\ell\in\operatorname{Ker}\widehat{L},\ \ell=1,2,\dots,n$ are linearly independent functions with the properties

$$w_\ell^{(k-1)}(0)+w_\ell^{(k-1)}(1)=\begin{cases}1, & \ell=k,\\ 0, & \ell\neq k,\end{cases}\qquad \ell,k=1,2,\dots,n.$$

Then the operator L_K is the restriction of $\widehat{L}$ on

$$\begin{aligned}&D(L_K)\\ &=\left\{u\in W_2^n(0,1):\,u^{(k-1)}(0)+u^{(k-1)}(1)=\int_0^1 u^{(n)}(t)\overline{\sigma}_k(t)\,dt,\ k=1,2,\dots,n\right\}.\end{aligned}$$

We will consider restrictions of L_K with dense domains in $L^2(0,1)$, that is,

$$\overline{D(L_K)}=L^2(0,1).$$

If $R(K^*)\subset D(L^*)$, then by Corollary 3.1 the operators $\overline{KL}$ and $\overline{KL_K}$ will be bounded in $L^2(0,1)$. Since $\overline{KL}$ is a compact operator, then by Lemma 2.3 the operator $I+KL$ is invertible and $(I+KL)^{-1}=I-KL_K$. The operator $\overline{KL}$ is bounded if and only if

$$\sigma_\ell\in D(L^*)=\left\{\sigma_\ell\in W_2^n(0,1):\,\sigma_\ell^{(k-1)}(0)+\sigma_\ell^{(k-1)}(1)=0,\quad \ell,k=1,2,\dots,n\right\}.$$

Hence, we have

$$KLy=\sum_{\ell=1}^{n}w_\ell(x)\int_0^1 y^{(n)}(t)\overline{\sigma}_\ell(t)\,dt=(-1)^n\sum_{\ell=1}^{n}w_\ell(x)\int_0^1 y(t)\overline{\sigma}_\ell^{(n)}(t)\,dt.$$

We find the operator KL_K. For this, we invert the operator

$$(I+KL)y=y+(-1)^n\sum_{\ell=1}^{n}w_\ell(x)\int_0^1 y(t)\overline{\sigma}_\ell^{(n)}(t)\,dt=u,$$

where $y\in D(L),\ \ u\in D(L_K)$. Then we can write

$$y=(I-KL_K)u=u-(-1)^n\sum_{\ell=1}^{n}w_\ell(x)\sum_{j=1}^{n}\beta_{\ell j}\int_0^1 u(t)\overline{\sigma}_j^{(n)}(t)\,dt,$$

where $\beta_{\ell j},\ \ell,j=1,2,\dots,n$ are elements of the inverse matrix U^{-1} of the matrix U

$$\begin{aligned}&U\\ &=\begin{pmatrix}1+(-1)^{n-1}\overline{\sigma}_1^{(n-1)}(0) & (-1)^{n-2}\overline{\sigma}_1^{(n-2)}(0) & \cdots & \overline{\sigma}_1^{(2)}(0) & -\overline{\sigma}_1^{(1)}(0) & \overline{\sigma}_1(0)\\ (-1)^{n-1}\overline{\sigma}_2^{(n-1)}(0) & 1+(-1)^{n-2}\overline{\sigma}_2^{(n-2)}(0) & \cdots & \overline{\sigma}_2^{(2)}(0) & -\overline{\sigma}_2^{(1)}(0) & \overline{\sigma}_2(0)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ (-1)^{n-1}\overline{\sigma}_n^{(n-1)}(0) & (-1)^{n-2}\overline{\sigma}_n^{(n-2)}(0) & \cdots & \overline{\sigma}_n^{(2)}(0) & -\overline{\sigma}_n^{(1)}(0) & 1+\overline{\sigma}_n(0)\end{pmatrix}.\end{aligned}$$

Note that the conditions $R(K^*)\subset D(L^*)$ and $\overline{D(L_K)}=L^2(0,1)$ imply that $\det U\neq 0$. Thereby, the operator

$$KL_Ku=(-1)^n\sum_{\ell=1}^{n}w_\ell(x)\sum_{j=1}^{n}\beta_{\ell j}\int_0^1 u(t)\overline{\sigma}_j^{(n)}(t)\,dt,$$

is a bounded operator in $L^2(0,1)$. Then the operator A_K has the form

$$A_K v = Lv - \overline{KL}_K Lv = v^{(n)} - (-1)^n \sum_{\ell=1}^{n} w_\ell(x) \sum_{j=1}^{n} \beta_{\ell j} \int_0^1 v^{(n)}(t)\overline{\sigma}_j^{(n)}(t)\,dt,$$

on

$$D(A_K) = D(L) = \{v \in W_2^n(0,1) : \, v^{(k-1)}(0) + v^{(k-1)}(1) = 0, \quad k = 1,2,\ldots,n\}.$$

The operator A_K can be written as

$$A_K v = v^{(n)} - (-1)^n \sum_{\ell=1}^{n} w_\ell(x) \sum_{j=1}^{n} \beta_{\ell j} F_j(v),$$

where

$$F_j(v) = < F_j, v > = \int_0^1 v^{(n)}(t)\overline{\sigma}_j^{(n)}(t)\,dt, \quad j = 1,2,\ldots,n.$$

It can be seen that $F_j \in W_2^{-n}(0,1)$ in the sense of Lions-Magenes (see [10]).

In this case

$$D(A_0) = \{v \in D(L) : \, F_j(v) = 0, \quad j = 1,2,\ldots,n\},$$

$A_0 \subset L$ and $A_0 \subset A_K$.

We transform the boundary conditions of L_K to the form

$$U \begin{pmatrix} u(0)+u(1) \\ u^{(1)}(0)+u^{(1)}(1) \\ \vdots \\ u^{(n-1)}(0)+u^{(n-1)}(1) \end{pmatrix} = \begin{pmatrix} \int_0^1 u(t)\overline{\sigma}_1^{(n)}(t)\,dt \\ \int_0^1 u(t)\overline{\sigma}_2^{(n)}(t)\,dt \\ \vdots \\ \int_0^1 u(t)\overline{\sigma}_n^{(n)}(t)\,dt \end{pmatrix}.$$

Then we get

$$u^{(\ell-1)}(0) + u^{(\ell-1)}(1) = \sum_{j=1}^{n} \beta_{\ell j} \int_0^1 u(t)\overline{\sigma}_j^{(n)}(t)\,dt, \quad \ell = 1,2,\ldots,n, \tag{4.1}$$

where $u \in D(L_K)$, $\sigma_j^{(n)} \in L^2(0,1)$, $j = 1,2,\ldots,n$. Boundary condition (4.1) is regular in the Shkalikov sense (see [12]). Then, by virtue of [12], the operator L_K has a system of root vectors forming a Riesz basis with brackets in $L^2(0,1)$. Thereby the operator A_K, being similar to the operator L_K, also has a basis with brackets property. The eigenvalues of these operators coincide. If $\{u_k\}_1^\infty$ are eigenfunctions of the operator L_K, then the eigenfunctions v_k of A_K are related to them by the relations

$$u_k = (I + KL)v_k = v_k + (-1)^n \sum_{\ell=1}^{n} w_\ell(x) \int_0^1 v_k(t)\overline{\sigma}_\ell^{(n)}(t)\,dt, \quad k = 1,2,\ldots,n.$$

If, in particular, we take

$$\sigma_\ell^{(n)}(x) = \operatorname{sign}(x - x_\ell), \quad 0 < x_\ell < 1, \quad \ell = 1,2,\ldots,n,$$

then we get

$$F_\ell(v) = -2v^{(n-1)}(x_\ell), \quad \ell = 1,2,\ldots,n.$$

By Corollary 3.1, Theorem 2.1, and [13, p. 928], we can assert that the system of root vectors of the adjoint operator

$$A_K^* v = (-1)^n \frac{d^n}{dx^n}\left[v(x) - (-1)^n \sum_{\ell,j=1}^{n} \overline{\beta}_{\ell j}\sigma_j^{(n)}(x)\int_0^1 v(t)w_\ell(t)\,dt\right],$$

on

$$D(A_K^*) = \left\{v \in L^2(0,1) :\ v(x) - (-1)^n \sum_{\ell,j=1}^{n} \overline{\beta}_{\ell j}\sigma_j^{(n)}(x)\int_0^1 v(t)w_\ell(t)\,dt \in D(L)\right\},$$

forms a Riesz basis with brackets in $L^2(0,1)$.

5 Example in case $n = 2$

If the maximal operator $\widehat{L}$ acts as

$$\widehat{L}y = -y''$$

on the domain $D(\widehat{L}) = W_2^2(0,1)$, then the minimal operator L_0 is a restriction of $\widehat{L}$ on $D(L_0) = \mathring{W}_2^2(0,1)$. As a fixed operator L we take the restriction of $\widehat{L}$ on

$$D(L) = \{y \in W_2^2(0,1) :\ y(0) = y(1) = 0\}.$$

Then

$$\begin{aligned} L_K^{-1}f = L^{-1}f + Kf = -\int_0^x (x-t)f(t)\,dt + x\int_0^1 (1-t)f(t)\,dt \\ + (1-x)\int_0^1 f(t)\overline{\sigma}_1(t)\,dt + x\int_0^1 f(t)\overline{\sigma}_2(t)\,dt, \end{aligned}$$

$$Kf = (1-x)\int_0^1 f(t)\overline{\sigma}_1(t)\,dt + x\int_0^1 f(t)\overline{\sigma}_2(t)\,dt.$$

KL is bounded in $L^2(0,1)$, if $R(K^*) \subset D(L^*) = D(L)$, that is,

$$\sigma_1,\ \sigma_2 \in D(L) = \{\sigma_1, \sigma_2 \in W_2^2(0,1) :\ \sigma_1(0) = \sigma_1(1) = \sigma_2(0) = \sigma_2(1) = 0\},$$

and has the form

$$KLy = -(1-x)\int_0^1 y(t)\overline{\sigma}_1''(t)\,dt - x\int_0^1 y(t)\overline{\sigma}_2''(t)\,dt.$$

The operator KL_K is also bounded in $L^2(0,1)$ and

$$\begin{aligned} KL_K u = -\frac{1-x}{\Delta}\Big[\big(1-\overline{\sigma}_2'(1)\big)\int_0^1 u(t)\overline{\sigma}_1''(t)\,dt + \overline{\sigma}_1'(1)\int_0^1 u(t)\overline{\sigma}_2''(t)\,dt\Big] \\ - \frac{x}{\Delta}\Big[\big(1+\overline{\sigma}_1'(0)\big)\int_0^1 u(t)\overline{\sigma}_2''(t)\,dt - \overline{\sigma}_2'(0)\int_0^1 u(t)\overline{\sigma}_1''(t)\,dt\Big], \end{aligned}$$

where

$$\Delta = \big(1+\overline{\sigma}_1'(0)\big)\big(1-\overline{\sigma}_2'(1)\big) + \overline{\sigma}_2'(0)\,\overline{\sigma}_1'(1).$$

Then the operator A_K has the form

$$A_K v = -v'' - \frac{1}{\Delta}\Big[\big((1-x)(1-\overline{\sigma}_2'(1)) - x\overline{\sigma}_2'(0)\big)\int_0^1 v''(t)\overline{\sigma}_1''(t)\,dt + \big((1-x)\overline{\sigma}_1'(1) + x(1+\overline{\sigma}_1'(0))\big)\int_0^1 v''(t)\overline{\sigma}_2''(t)\,dt\Big],$$

on

$$D(A_K) = D(L) = \big\{v \in W_2^2(0,1) : \; v(0) = v(1) = 0\big\},$$

where $\sigma_1'', \sigma_2'' \in L^2(0,1)$.

We rewrite the operator A_K in the form

$$A_K v = -v'' + a(x)F_1(v) + b(x)F_2(v), \tag{5.1}$$

where

$$a(x) = -\frac{1}{\Delta}\big((1-x)(1-\overline{\sigma}_2'(1)) - x\overline{\sigma}_2'(0)\big), \quad F_1(v) = \int_0^1 v''(t)\overline{\sigma}_1''(t)\,dt,$$

$$b(x) = -\frac{1}{\Delta}\big((1-x)\overline{\sigma}_1'(1) + x(1+\overline{\sigma}_1'(0)\big), \quad F_2(v) = \int_0^1 v''(t)\overline{\sigma}_2''(t)\,dt.$$

Note that $F_1, F_2 \in W_2^{-2}(0,1)$ in the sense of Lions-Magenes (see [10]).

Further, we see that the operator L_K acts as $\widehat{L}$ on the domain

$$D(L_K) = \left\{ u \in W_2^2(0,1) : \right.$$

$$\left. \begin{pmatrix} 1+\overline{\sigma}_1'(0) & 0 & -\overline{\sigma}_1'(1) & 0 \\ \overline{\sigma}_2'(0) & 0 & 1-\overline{\sigma}_2'(1) & 0 \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \\ u(1) \\ u'(1) \end{pmatrix} = \begin{pmatrix} -\int_0^1 u(t)\overline{\sigma}_1''(t)\,dt \\ -\int_0^1 u(t)\overline{\sigma}_2''(t)\,dt \end{pmatrix} \right\},$$

and the determinant of the matrix composed of the first and third columns of the boundary conditions matrix is

$$J_{13} = \big(1+\overline{\sigma}_1'(0)\big)\big(1-\overline{\sigma}_2'(1)\big) + \overline{\sigma}_2'(0)\,\overline{\sigma}_1'(1) = \Delta \neq 0,$$

since $R(K^*) \subset D(L^*)$ and $\overline{D(L_K)} = L^2(0,1)$. Then the left-hand side of this boundary condition is non-degenerate according to Marchenko [11], hence regular according to Birkhoff (see [12]). By virtue of Theorem (see [12, p. 15]), the system of root vectors of the operator L_K forms a Riesz basis with brackets in $L^2(0,1)$. Thus, by virtue of Theorem (1.1) the system of root vectors of A_K also forms a Riesz basis with brackets and the eigenvalues of L_K and A_K coincide, and the eigenfunctions are related to each other as follows

$$u_k = v_k - (1-x)\int_0^1 v_k(t)\overline{\sigma}_1''(t)\,dt - x\int_0^1 v_k(t)\overline{\sigma}_2''(t)\,dt, \;\; k \in \mathbb{N}.$$

If in the particular case we take

$$\begin{aligned} \overline{\sigma}_1''(x) &= \operatorname{sign}(x-x_1) - \operatorname{sign}(x-x_2), \\ \overline{\sigma}_2''(x) &= x\big[\operatorname{sign}(x-x_1) - \operatorname{sign}(x-x_2)\big], \end{aligned} \tag{5.2}$$

where $0 < x_1 < x_2 < 1$, then we get

$$F_1(v) = 2v'(x_2) - 2v'(x_1),$$
$$F_2(v) = 2x_2 v'(x_2) - 2x_1 v'(x_1) - 2v(x_2) + 2v(x_1),$$

in (5.1).

In this case
$$D(A_0) = \{v \in D(L) : \; v(x_1) = v(x_2), \quad v'(x_1) = v'(x_2)\},$$
$A_0 \subset L$ and $A_0 \subset A_K$.

By Corollary 3.1, Theorem 2.1, and [13, p. 928], for $n = 2$, we can assert that the system of root vectors of the operator

$$A_K^* v = (-1)^2 \frac{d^2}{dx^2}\Big[v(x) - c(x)\int_0^1 (1-t)v(t)\,dt - d(x)\int_0^1 tv(t)\,dt\Big],$$

on

$$D(A_K^*) = \Big\{v \in L^2(0,1) : \; v(x) - c(x)\int_0^1 (1-t)v(t)\,dt - d(x)\int_0^1 tv(t)\,dt \in D(L)\Big\},$$

forms a Riesz basis with brackets in $L^2(0,1)$, where

$$c(x) = -\frac{1}{\Delta}\Big[\big(1 - \sigma_2'(1)\big)\sigma_1''(x) + \sigma_1'(1)\sigma_2''(x)\Big],$$
$$d(x) = \frac{1}{\Delta}\Big[\big(1 + \sigma_1'(0)\big)\sigma_2''(x) - \sigma_2'(0)\sigma_1''(x)\Big].$$

Note that
$$\sigma_1'', \sigma_2'' \in L^2(0,1), \quad D(L) = \big\{y \in W_2^2(0,1) : \; y(0) = y(1) = 0\big\}.$$
For clarity, we consider the special case (5.2), then we have

$$c(x) = \frac{\operatorname{sign}(x - x_1) - \operatorname{sign}(x - x_2)}{\Delta}\left[1 + \frac{x_2^3 - x_1^3}{3} - \frac{x_2^2 - x_1^2}{2}x\right],$$
$$d(x) = \frac{\operatorname{sign}(x - x_1) - \operatorname{sign}(x - x_2)}{\Delta}\left[\left(1 + x_2 - x_1 - \frac{x_2^2 - x_1^2}{2}\right)x \right.$$
$$\left. - \frac{x_2^2 - x_1^2}{2} + \frac{x_2^3 - x_1^3}{3}\right].$$

The domain of A_K^* will have the form

$$D(A_K^*) = \Big\{v \in L^2(0,1) \cap W_2^2(0,x_1) \cap W_2^2(x_1,x_2) \cap W_2^2(x_2,1) : \; v(0) = v(1) = 0,$$
$$v(x_1 - 0) - v(x_1 + 0) = -c(x_1 + 0)\int_0^1 (1-t)v(t)\,dt - d(x_1 + 0)\int_0^1 tv(t)\,dt,$$
$$v(x_2 + 0) - v(x_2 - 0) = -c(x_2 - 0)\int_0^1 (1-t)v(t)\,dt - d(x_2 - 0)\int_0^1 tv(t)\,dt,$$
$$v'(x_1 - 0) - v'(x_1 + 0) = -c'(x_1 + 0)\int_0^1 (1-t)v(t)\,dt - d'(x_1 + 0)\int_0^1 tv(t)\,dt,$$
$$v'(x_2 + 0) - v'(x_2 - 0) = -c'(x_2 - 0)\int_0^1 (1-t)v(t)\,dt - d'(x_2 - 0)\int_0^1 tv(t)\,dt\Big\},$$

where

$$
\begin{aligned}
c(x_1+0) &= \frac{2}{\Delta}\Big(1+\frac{x_2^3-x_1^3}{3}-\frac{x_2^2-x_1^2}{2}x_1\Big),\\
d(x_1+0) &= -\frac{2}{\Delta}\Big(\big(1+x_2-x_1-\frac{x_2^2-x_1^2}{2}\big)x_1-\frac{x_2^2-x_1^2}{2}+\frac{x_2^3-x_1^3}{3}\Big),\\
c(x_2-0) &= \frac{2}{\Delta}\Big(1+\frac{x_2^3-x_1^3}{3}-\frac{x_2^2-x_1^2}{2}x_2\Big),\\
d(x_2-0) &= -\frac{2}{\Delta}\Big(\big(1+x_2-x_1-\frac{x_2^2-x_1^2}{2}\big)x_2-\frac{x_2^2-x_1^2}{2}+\frac{x_2^3-x_1^3}{3}\Big),\\
c'(x_1+0) &= -\frac{1}{\Delta}\big(x_2^2-x_1^2\big),\\
d'(x_1+0) &= \frac{2}{\Delta}\Big(1+x_1-x_2+\frac{x_2^2-x_1^2}{2}\Big),\\
c'(x_2-0) &= c'(x_1+0), \quad d'(x_2-0)=d'(x_1+0),\\
\Delta &= 1+x_2-x_1-\frac{x_2^2-x_1^2}{2}+\frac{x_2^3-x_1^3}{3}+\frac{x_2-x_1}{12}\big((x_2-x_1)^3+6x_1x_2\big)\neq 0,
\end{aligned}
$$

since $x_1, x_2 \in (0,1)$. Moreover, the operator A_K^* acts as follows

$$
A_K^* v = -v''(x) + c''(x)\int_0^1 (1-t)v(t)\,dt + d''(x)\int_0^1 t v(t)\,dt,
$$

where

$$
\begin{aligned}
c''(x) = {} & \frac{2}{\Delta}\Big[1+\frac{x_2^3-x_1^3}{3}+\frac{x_2^2-x_1^2}{2}(x_2-x_1)\Big]\big(\delta'(x-x_1)-\delta'(x-x_2)\big)\\
& -\frac{1}{\Delta}\big(x_2^2-x_1^2\big)\big(\delta(x-x_1)-\delta(x-x_2)\big),
\end{aligned}
$$

$$
\begin{aligned}
d''(x) = {} & \frac{2}{\Delta}\Big[\Big(1+x_2-x_1-\frac{x_2^2-x_1^2}{2}\Big)(x_1-x_2)-\frac{x_2^2-x_1^2}{2}+\frac{x_2^3-x_1^3}{3}\Big]\\
& \times\big(\delta'(x-x_1)-\delta'(x-x_2)\big)\\
& -\frac{1}{\Delta}\Big(1+x_2-x_1+\frac{x_2^2-x_1^2}{2}\Big)\big(\delta(x-x_1)-\delta(x-x_2)\big),
\end{aligned}
$$

here δ is the Dirac delta-function.

6 An application to the Laplace operator

In the Hilbert space $L_2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^m$ with infinitely smooth boundary $\partial\Omega$, let us consider the minimal L_0 and maximal $\widehat{L}$ operators generated by the Laplace operator

$$
-\Delta u = -\Big(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\cdots+\frac{\partial^2 u}{\partial x_m^2}\Big). \tag{6.1}
$$

The closure L_0 in the space $L_2(\Omega)$ of the Laplace operator (6.1) with the domain $C_0^\infty(\Omega)$ is called the minimal operator corresponding to the Laplace operator.

The operator $\widehat{L}$, adjoint to the minimal operator L_0 corresponding to the Laplace operator is called the maximal operator (see [7]). Note that

$$D(\widehat{L}) = \{u \in L_2(\Omega) :\ \widehat{L}u = -\Delta u \in L_2(\Omega)\}.$$

Denote by L the operator corresponding to the Dirichlet problem with the domain

$$D(L) = \{u \in W_2^2(\Omega) :\ u|_{\partial\Omega} = 0\}.$$

Then the inverse operators L_K^{-1} to all possible well-defined restrictions of the maximal operator $\widehat{L}$ corresponding to the Laplace operator (6.1) have the following form:

$$u \equiv L_K^{-1} f = L^{-1} f + K f, \tag{6.2}$$

where K is an arbitrary bounded (in $L_2(\Omega)$) linear operator with

$$R(K) \subset \operatorname{Ker} \widehat{L} = \{u \in L_2(\Omega) :\ -\Delta u = 0\}.$$

Then the direct operator L_K is determined from the following problem:

$$\widehat{L}u = f, \quad f \in L_2(\Omega), \tag{6.3}$$

$$D(L_K) = \{u \in D(\widehat{L}) :\ (I - K\widehat{L})u|_{\partial\Omega} = 0\}, \tag{6.4}$$

where I is the unit operator in $L_2(\Omega)$. There are no other linear well-defined restrictions of the operator $\widehat{L}$ (see [1]).

The operators $(L_K^*)^{-1}$, corresponding to the operators L_K^*,

$$v \equiv (L_K^*)^{-1} g = L^{-1} g + K^* g,$$

describe the inverse operators to all possible well-defined extensions of the minimal operator L_0 if and only if K satisfies the condition (see [1])

$$\operatorname{Ker}(I + K^* L) = \{0\}.$$

Note that the last condition is equivalent to the following one: $\overline{D(L_K)} = L_2(\Omega)$. If the additional condition

$$KR(L_0) = \{0\}$$

is imposed on the operator K from (6.2), then the operator L_K corresponding to problem (6.3), (6.4), will turn out to be boundary well-defined. By applying Theorem 1.1 to this particular case we have

Theorem 6.1. *Let the operator K have the form*

$$Kf(x) = \phi(x) \iint\limits_{\Omega} f(\xi)\overline{g(\xi)}d\xi, \quad x,\ \xi \in \Omega \subset \mathbb{R}^m,$$

where ϕ is a harmonic function in $L_2(\Omega)$, $g \in L_2(\Omega)$, and

$$K^* f(x) = g(x) \iint\limits_{\Omega} f(\xi)\overline{\phi(\xi)}d\xi.$$

If K satisfies the assumptions of Theorem 1.1, then $g \in W_2^2(\Omega)$, $g(x)\mid_{\partial\Omega}= 0$,

$$\iint\limits_{\Omega} \phi(\xi)(\Delta\overline{g})(\xi)d\xi \neq 1,$$

and the well-defined operator

$$A_K u(x) = -\Delta u(x) + \frac{\phi(x)}{1+\iint\limits_{\Omega}\phi(\xi)(\Delta\overline{g})(\xi)d\xi}\iint\limits_{\Omega}(\Delta u)(\xi)(\Delta\overline{g})(\xi)d\xi,$$

,

$$D(A_K) = \Big\{u \in W_2^2(\Omega):\ \big(u(x)\mid_{\partial\Omega}= 0\Big\}$$

describes a relatively bounded perturbation of L_K which has the same eigenvalues as L_K.

The system of root vectors of A_K is complete in $L_2(\Omega)$. Morever, if $\{v_k\}$ is a system of eigenfunctions of L_K , then the system of eigenvectors $\{u_k\}$ of A_K has the form

$$u_k(x) = ((I+\overline{KL})v_k)(x) = v_k(x) + \phi(x)\iint\limits_{\Omega} v_k(\xi)(\Delta\overline{g})(\xi)d\xi, \quad k = 1, 2, \ldots$$

We can rewrite

$$A_K u(x) = -\Delta u(x) + \frac{\phi(x)}{1+\iint\limits_{\Omega}\phi(\xi)(\Delta\overline{g})(\xi)d\xi}F(u),$$

where

$$F(u) = \iint\limits_{\Omega}(\Delta u)(\xi)(\Delta\overline{g})(\xi)d\xi.$$

Note that $F \in W_2^{-2}(\Omega)$ in the sense of Lions-Magenes (see [10]).

In this case

$$D(A_0) = \left\{ v \in D(L):\ \iint\limits_{\Omega}(\Delta u)(\xi)(\Delta\overline{g})(\xi)d\xi = 0 \right\},$$

$A_0 \subset L$ and $A_0 \subset A_K$.

Consider a more visual cases when $m = 2$ and $m = 3$, that is, $\Omega \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^3$ respectively. To do this, we define the operator K by using the function g constructed in the following way. Let $x_0 \in \Omega$, be a point lying strictly inside the closed domain $\overline{\Omega}$. As functions $g(x)$ we take the solution to the following Dirichlet problem

$$-(\Delta g)(x) = -\ln|x-x_0|, \quad g|_{\partial\Omega} = 0, \tag{6.5}$$

for $m = 2$ and

$$-(\Delta g)(x) = \ |x-x_0|, \quad g|_{\partial\Omega} = 0, \tag{6.6}$$

for $m = 3$, respectively. Then we get the following:

$$A_K u(x) = -\Delta u(x) + \frac{\phi(x)}{1+\iint\limits_{\Omega}\phi(\xi)(\ln|\xi-x_0|)d\xi}\int\limits_{\partial\Omega}\frac{\partial u}{\partial n}(\xi)(\ln|\xi-x_0|)d\xi$$
$$+\frac{\phi(x)u(x_0)}{1+\iint\limits_{\Omega}\phi(\xi)(\ln|\xi-x_0|)d\xi},$$

$$D(A_K) = \Big\{u \in W_2^2(\Omega):\ u(x)\mid_{\partial\Omega}= 0\Big\}$$

for the case $m = 2$ and

$$A_K u(x) = -\Delta u(x) + \frac{\phi(x)}{1 + \iint\limits_{\Omega} \phi(\xi) \frac{1}{|\xi - x_0|} d\xi} \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(\xi) \frac{1}{|\xi - x_0|} d\xi$$
$$+ \frac{\phi(x) u(x_0)}{1 + \iint\limits_{\Omega} \phi(\xi) \frac{1}{|\xi - x_0|} d\xi},$$
$$D(A_K) = \left\{ u \in W_2^2(\Omega) : \, u(x) \, |_{\partial\Omega} = 0 \right\}$$

for the case $m = 3$. We have obtained a relatively bounded perturbation A_K of L_K which has the same eigenvalues as the operator L_K. The system of root vectors of A_K is complete in the $L_2(\Omega)$. If $\{v_k\}$ is a system of eigenfunctions of L_K, then the system of eigenfunctions $\{u_k\}$ of A_K has the form

$$u_k(x) = ((I + \overline{KL}) v_k)(x) = v_k(x) + \phi(x) \iint\limits_{\Omega} v_k(\xi) \ln |\xi - x_0| d\xi, \quad k = 1, 2, \ldots,$$

in the case $m = 2$ and

$$u_k(x) = ((I + \overline{KL}) v_k)(x) = v_k(x) + \phi(x) \iint\limits_{\Omega} v_k(\xi) \frac{1}{|\xi - x_0} | d\xi, \quad k = 1, 2, \ldots,$$

in the case $m = 3$, respectively.

Thus, we have constructed a singular perturbation A_K of the L_K with a complete system of root vectors. Indeed,

$$L_K^{-1} = L^{-1} + K = (I + KL) L^{-1},$$

where KL is compact operator, $I + KL$ is invertable operator. Selfadjoint operator L^{-1} is positive, compact and belongs to the Neumann-Schatten class. Then by Theorem 8.1 [6, p. 257]the system of root vectors L_K is complete in $L_2(\Omega)$. Hence, by Theorem 1.1 the system of root vectors of A_K is complete in $L_2(\Omega)$.

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SOME NEW APPROACHES IN THE THEORY OF TRIGONOMETRIC SERIES WITH MONOTONE COEFFICIENTS

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Abstract. In this paper, we observe the latest results concerning the trigonometric series whose coefficients are monotone or fractional monotone. We study the asymptotic properties of the sums for both classes of series and also the problems of convergence and integrability for series with fractional monotone coefficients.

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1 Introduction and problem statement

Everywhere below in this paper we denote by

$$f(\boldsymbol{a},x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

and

$$g(\boldsymbol{a},x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

for all x for which the corresponding series converges.

It is known that one of the most important classes of trigonometric series is the class of series with monotone coefficients, i.e. the sequence $\boldsymbol{a} = \{a_n\}_{n=0}^{\infty}$ is such that $a_n \downarrow 0$ as $n \to \infty$. In this case series (1.1) and (1.2) have a lot of very good properties. For instance, the following theorem was proved by G. Hardy and J. Littlewood.

Theorem 1.1 ([14]). *Let $p \in (1,\infty)$. Then $f(\boldsymbol{a},x) \in L_p([0,\pi])$ (or $g(\boldsymbol{a},x) \in L_p([0,\pi])$) if and only if*

$$J_p(\boldsymbol{a}) \equiv \sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty.$$

We mention also the well-known theorems of G. Lorentz (Theorem 1.2) and T. Chaundy and A. Jolliffe (Theorem 1.3).

Theorem 1.2 ([16]). *Let $\alpha \in (0,1)$. Then $f(\boldsymbol{a},x) \in \operatorname{Lip}\alpha$ (or $g(\boldsymbol{a},x) \in \operatorname{Lip}\alpha$) if and only if for some $C > 0$ we have $a_n \leqslant C/n^{1+\alpha}$ for all $n \in \mathbb{N}$.*

Theorem 1.3 ([7]). *Series* (1.2) *uniformly converges if and only if* $na_n \to 0$ *as* $n \to \infty$.

One of the main topics of the present paper is the so-called asymptotic behaviour of the sums of trigonometric series with monotone coefficients in a neighbourhood of zero. The first results in this direction were obtained by R. Salem [22], [23] (see also [6]). His research was continued by S. Izumi [15], S.A. Telyakovskii [28], [29], A.Yu. Popov and A.P. Solodov (see [18]–[21], [24]–[27]), and others. Note that the properties of sine and cosine series differ significantly in this problem. In Section 2 we discuss in detail the asymptotics of the sums of sine series and new approaches to this problem.

As series with monotone coefficients are very interesting because of their properties, many authors introduced the classes of trigonometric series with generalized monotone coefficients. In Section 3 we discuss fractional monotone sequences and the corresponding trigonometric series. M.I. Dyachenko introduced this class in paper [8] and proved some convergence and smoothness properties of cosine and sine series with coefficients belonging to this class. It is necessary to say that many important auxiliary results essential for the study of monotonicity of fractional order were established by A. Andersen [4]. A number of new results in this direction were obtained by M.I. Dyachenko, E.D. Nursultanov, A.P. Solodov, A.B. Mukanov, and E.D. Alferova (see [8]–[13], [17], [2]). Similar questions were also considered in the works [1], [5], [30].

2 New approaches to asymptotic properties

This section is devoted to the study of the asymptotic behavior in the right half-neighbourhood of zero of sums of a sine series with monotone coefficients.

To obtain a two-sided estimate of the sum of a series (1.2), R. Salem [22] defined the following function:

$$v(\boldsymbol{a}, x) = x \sum_{n=1}^{m(x)} n a_n, \quad m(x) = [\pi/x]\,.$$

Under some additional assumptions on the sequence $\boldsymbol{a}$ monotonically tending to zero, he proved the existence of positive constants $C_1(\boldsymbol{a})$, $C_2(\boldsymbol{a})$, and $x_0 > 0$ such that the following estimates hold:

$$C_2(\boldsymbol{a}) v(\boldsymbol{a}, x) \leqslant g(\boldsymbol{a}, x) \leqslant C_1(\boldsymbol{a}) v(\boldsymbol{a}, x), \quad 0 < x \leqslant x_0. \tag{2.1}$$

S.A. Telyakovskii has improved this result by deriving estimate (2.1) with absolute constants C_1 and C_2, freeing the sequence $\boldsymbol{a}$ from additional requirements and showing that the upper bound holds for any monotone sequence $\boldsymbol{a}$, and the lower bound — for any convex sequence $\boldsymbol{a}$ (i.e. $a_n - 2a_{n+1} + a_{n+2} \geqslant 0$, $n \in \mathbb{N}$).

Theorem 2.1 ([28], [29]). *There exists a constant* $C_1 > 0$ *such that for any nonincreasing null sequence* $\boldsymbol{a}$

$$g(\boldsymbol{a}, x) \leqslant C_1 v(\boldsymbol{a}, x), \quad 0 < x \leqslant \pi/11.$$

There exists a constant $C_2 > 0$ *such that for any convex null sequence* $\boldsymbol{a}$

$$g(\boldsymbol{a}, x) \geqslant C_2(\boldsymbol{a}) v(\boldsymbol{a}, x), \quad 0 < x \leqslant \pi/11.$$

A.Yu. Popov calculated the sharp values of the constants in the estimates of Telyakovskii. He proved the following results.

Theorem 2.2 ([18]). *For any nonincreasing null sequence* $\boldsymbol{a}$,

$$g(\boldsymbol{a}, x) \leqslant v(\boldsymbol{a}, x), \quad 0 < x \leqslant \pi. \tag{2.2}$$

Theorem 2.3 ([18]). *For any convex null sequence* $\boldsymbol{a}$*,*

$$g(\boldsymbol{a},x) \geqslant \frac{2}{\pi^2}\, v(\boldsymbol{a},x) - 0.46\, a_{m(x)}, \quad 0 < x \leqslant \frac{\pi}{2}. \tag{2.3}$$

The estimate (2.3), in general, does not hold if there is no second negative term in its right-hand side. The question arises: is it possible to modify the Salem function $v(\boldsymbol{a},x)$ in such way so that the two-sided estimate with constants $C_1 = 1$ and $C_2 = 2\pi^{-2}$ still holds in some right half-neighbourhood of zero? The answer to this question is positive.

In [24] was shown that the estimate (2.2) can be strengthened. As a new majorant, consider the function

$$u(\boldsymbol{a},x) = x \sum_{n=1}^{[(m(x)+1)/2]} n a_n + x \sum_{n=[(m(x)+3)/2]}^{m(x)} \big(m(x)+1-n\big) a_n.$$

The following refinement of Theorem 2.2 is valid.

Theorem 2.4 ([24]). *For any nonincreasing null sequence* $\boldsymbol{a}$*,*

$$g(\boldsymbol{a},x) \leqslant u(\boldsymbol{a},x), \quad 0 < x \leqslant \pi.$$

Under the additional condition of convexity of the sequence $\boldsymbol{a}$, the function $2\pi^{-2}u(\boldsymbol{a},x)$ turns out to be a minorant of the sum of the sine series not only in a certain neighbourhood of zero, but practically over the entire interval $(0,\pi/2]$.

Theorem 2.5 ([24]). *For any convex null sequence* $\boldsymbol{a}$*,*

$$g(\boldsymbol{a},x) \geqslant \frac{2}{\pi^2}\, u(\boldsymbol{a},x), \quad 0 < x \leqslant \frac{9\pi}{20}.$$

In [21], the asymptotic behavior of sums of the particular sine series (1.2) as $x \to 0+$ was studied. Their coefficient sequences not only monotonically tend to zero, but also belong to the following two special classes. First class — let us denote it as $\mathcal{B}\downarrow$ — consists of all sequences $\boldsymbol{a}$ monotonically tending to zero such that the sequence $\{na_n\}_{n=1}^{\infty}$ does not increase, that is $(n+1)a_{n+1} \leqslant na_n$, $n \in \mathbb{N}$. Second class — let us denote it as $\mathcal{B}\uparrow$ — consists of all sequences $\boldsymbol{a}$ monotonically tending to zero such that the sequence $\{na_n\}_{n=1}^{\infty}$ does not decrease, that is $na_n \leqslant (n+1)a_{n+1}$, $n \in \mathbb{N}$.

Theorem 2.6 ([21]). *If* $\boldsymbol{a} \in \mathcal{B}\downarrow$*, then, for any* $x \in (0,\pi/3]$*, the following lower estimate holds:*

$$g(\boldsymbol{a},x) \geqslant \left(\underline{I} - \frac{1}{m(x)}\right) v(\boldsymbol{a},x) - \frac{3}{2}\, a_{m(x)+1} \sin\frac{x}{2},$$

where

$$\underline{I} = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin t}{t}\, dt = 0.451\ldots.$$

Moreover, there exist sequences $\underline{\boldsymbol{a}} \in \mathcal{B}\downarrow$ *and* $\{x_k\}_{k=1}^{\infty}$ *such that*

$$x_k > 0 \quad (\forall k \in \mathbb{N}), \qquad \lim_{k\to\infty} x_k = 0, \qquad g(\underline{\boldsymbol{a}}, x_k) \sim \underline{I}\, v(\underline{\boldsymbol{a}}, x_k), \quad k \to \infty.$$

Theorem 2.7 ([21]). *If* $\boldsymbol{a} \in \mathcal{B}\uparrow$*, then, for any* $x \in (0,\pi)$*, the following upper estimate holds:*

$$g(\boldsymbol{a},x) \leqslant \overline{I}\left(1 + \frac{1}{m(x)}\right) v(\boldsymbol{a},x) + \frac{1}{2}\, a_{m(x)+1} \tan\frac{x}{4},$$

where

$$\overline{I} = \frac{1}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt = 0.589\ldots.$$

Moreover, there exist sequences $\overline{\boldsymbol{a}} \in \mathcal{B}\uparrow$ *and* $\{x_k\}_{k=1}^\infty$ *such that*

$$x_k > 0 \quad (\forall k \in \mathbb{N}), \qquad \lim_{k\to\infty} x_k = 0, \qquad g(\overline{\boldsymbol{a}}, x_k) \sim \overline{I}\, v(\overline{\boldsymbol{a}}, x_k), \quad k \to \infty.$$

In [24], a lower bound for the sums of sine series with convex coefficients was studied. The following result of Popov was refined.

Theorem 2.8 ([18])**.** *For any convex null sequence* $\boldsymbol{a}$*,*

$$g(\boldsymbol{a}, x) \geqslant \frac{2}{\pi^2}\, v(\boldsymbol{a}, x) - \frac{1}{\pi}\, a_{m(x)} - a_{m(x)}\left(\frac{1}{x} - \frac{1}{2}\cot\frac{x}{2}\right), \quad 0 < x \leqslant \frac{\pi}{2}.$$

It has been established that the Salem function with a sharp constant $2\pi^{-2}$ is not, in general, a minorant for the sum of a sine series for the class of all convex sequences $\boldsymbol{a}$.

A sequence $\{\beta_k\}_{k=1}^\infty$ is called *slowly varying* if $\lim_{k\to\infty} \beta_{[\delta k]}/\beta_k = 1$ for any $\delta > 0$.

Theorem 2.9 ([24])**.** *There exists a convex slowly varying null sequence* $\boldsymbol{a}$ *such that*

$$g(\boldsymbol{a}, x_k) < \frac{2}{\pi^2}\, v(\boldsymbol{a}, x_k)$$

for a sequence of points $\{x_k\}_{k=1}^\infty$ *with* $x_k \to +0$.

It is shown that, as an alternative, one can take the modified Salem function

$$v_0(\boldsymbol{a}, x) = x\Biggl(\sum_{n=1}^{m(x)-1} n a_n + \frac{m(x)}{2}\, a_{m(x)}\Biggr).$$

Theorem 2.10 ([24])**.** *Let* $\boldsymbol{a}$ *be a positive convex null sequence. Then for some* $x_0 > 0$

$$g(\boldsymbol{a}, x) > \frac{2}{\pi^2}\, v_0(\boldsymbol{a}, x), \quad 0 < x < x_0.$$

For any $\varepsilon > 0$ *there exists a convex slowly varying null sequence* $\boldsymbol{a}$ *for which there exists a sequence of points* $\{x_k\}_{k=1}^\infty$ *with* $x_k \to +0$ *such that*

$$g(\boldsymbol{a}, x_k) < \frac{2}{\pi^2}\, x_k\Biggl(\sum_{n=1}^{m(x_k)-1} n a_n + \left(\frac{1}{2} + \varepsilon\right) m(x_k) a_{m(x_k)}\Biggr).$$

In other words, the coefficient $1/2$ multiplying the term $m(x)a_{m(x)}$ in the modified Salem majorant is sharp. This shows that in some sense the function $v_0(\boldsymbol{b}, x)$ is optimal for estimating the sum of a sine series with convex coefficients from below.

In [26], the sharp constants were found in the two-sided Telyakovskii estimate for the sum of a sine series with a monotone sequence of coefficients $\boldsymbol{a}$ under the additional condition of convexity.

S.A. Telyakovskii showed that it is convenient to compare the difference between the sum of series (1.2) and the main term of its asymptotic expansion, i.e.

$$g(\boldsymbol{a}, x) - \frac{a_{m(x)}}{2}\cot\frac{x}{2},$$

with the function

$$\sigma(\boldsymbol{a}, x) = \frac{1}{m(x)} \sum_{n=1}^{m(x)-1} n^2 \Delta^1 a_n, \qquad \Delta^1 a_n = a_n - a_{n+1} > 0.$$

Theorem 2.11 ([28], [29]). *There exist positive absolute constants C_1 and C_2 such that*

$$C_1\,\sigma(\boldsymbol{a},x)\leqslant g(\boldsymbol{a},x)-\frac{a_{m(x)}}{2}\cot\frac{x}{2}\leqslant C_2\,\sigma(\boldsymbol{a},x),\quad 0<x\leqslant\frac{\pi}{11},$$

for any convex null sequence $\boldsymbol{a}$.

In the following theorem, the sharp values of the constants C_1 and C_2 are obtained.

Theorem 2.12 ([26]). *The following equalities hold:*

$$\sup_{\boldsymbol{a}}\varlimsup_{x\to+0}\frac{g(\boldsymbol{a},x)-(a_{m(x)}/2)\cot(x/2)}{\sigma(\boldsymbol{a},x)}=\frac{\pi}{2}, \tag{2.4}$$

$$\inf_{\boldsymbol{a}}\varliminf_{x\to+0}\frac{g(\boldsymbol{a},x)-(a_{m(x)}/2)\cot(x/2)}{\sigma(\boldsymbol{a},x)}=\frac{3(\pi-1)}{\pi^2}, \tag{2.5}$$

moreover, the supremum in (2.4) and the infimum in (2.5) are attained for slowly varying sequences.

The following theorems answer the question how large is the deviation between the sum of sine series (1.2) and its asymptotically sharp majorant and minorant for the class of all convex sequences of coefficients.

Theorem 2.13 ([24]). *There exists a convex slowly varying null sequence $\boldsymbol{a}$ such that*

$$0<g(\boldsymbol{a},x_k)-\frac{2}{\pi^2}\,v_0(\boldsymbol{a},x_k)<\frac{1}{2}\sqrt[3]{\pi^2a_1x_ka_{m(x_k)}^2}+\frac{9+\pi^2}{6\pi^2}x_ka_{m(x_k)}$$

for some sequence of points $\{x_k\}_{k=1}^{\infty}$, $x_k\to+0$.

Theorem 2.14 ([25]). *For any $\varepsilon>0$, there exists a convex slowly varying null sequence $\boldsymbol{a}$ such that*

$$0>g(\boldsymbol{a},x_k)-\frac{a_{m(x_k)}}{2}\cot\frac{x_k}{2}-\sin\frac{x_k}{2}\sum_{n=1}^{m(x_k)-1}n(n+1)\Delta^1a_n>-a_1x_k^{3-\varepsilon}$$

for some sequence of points $\{x_k\}_{k=1}^{\infty}$, $x_k\to+0$.

At the end of the section, we present a result that refines the asymptotics of the sum of a sine series (1.1) with a convex slowly varying sequence of coefficients, obtained by S. Aljančić, R. Bojanić and M. Tomić, in the case when the sequence of coefficients satisfies the additional regularity condition.

Theorem 2.15 ([27]). *Let $\boldsymbol{a}$ be a non-negative convex null sequence, and let $\{n\Delta^1a_n\}_{n=1}^{\infty}$ be a convex slowly varying sequence. Then*

$$g(\boldsymbol{a},x)-\frac{a_{m(x)}}{x}\sim(\gamma+\ln\pi)\frac{m(x)\Delta^1a_{m(x)}}{x},\quad x\to+0,$$

where γ — is the Euler constant.

3 The fractional monotonicity

Let us give the corresponding definitions.

Definition 1. *Let $\alpha \in (-\infty, \infty)$. The Cesàro numbers $\{A_n^\alpha\}_{n=0}^\infty$ are defined as the coefficients in the expansion*

$$(1-x)^{-\alpha-1} = \sum_{n=0}^\infty A_n^\alpha x^n \quad \text{for } x \in (0,1).$$

The following properties of the Cesàro numbers are known (see [31]):

(1) $A_n^0 = 1$ for $n = 0, 1, \dots$ and $A_0^\alpha = 1$ for any α.

(2) If $\alpha \neq -1, -2, \dots$, then there are constants $C_1(\alpha) > 0$ and $C_2(\alpha) > 0$ depending only on α such that

$$C_2(\alpha)\, n^\alpha \leqslant |A_n^\alpha| \leqslant C_1(\alpha)\, n^\alpha \quad \text{for all } n > 0.$$

(3) For $\alpha > -1$ and any n, $A_n^\alpha > 0$; for $\alpha > 0$, $A_n^\alpha \uparrow \infty$ as $n \to \infty$; and, for $-1 < \alpha < 0$, $A_n^\alpha \downarrow 0$ as $n \to \infty$.

(4) For all α, β and $n = 0, 1, \dots$

$$\sum_{k=0}^n A_{n-k}^\alpha A_k^\beta = A_n^{\alpha+\beta+1}.$$

In particular, $A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$.

(5) For $\alpha > -1$ and $n = 0, 1, \dots$ we have

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!}.$$

Given a number sequence $\boldsymbol{a} = \{a_n\}_{n=0}^\infty$ and a real number α, we set

$$\Delta^\alpha a_n = \sum_{k=0}^\infty A_k^{-\alpha-1} a_{n+k}$$

for $n = 0, 1, \dots$ if this series converges (this is so, for example, if $\alpha > 0$ and the sequence $\boldsymbol{a}$ is bounded).

Definition 2. *Let $\alpha > 0$, and let $\boldsymbol{a} = \{a_n\}_{n=0}^\infty$ be a sequence of real numbers. We say that $\boldsymbol{a} \in M_\alpha$ if $\lim_{n\to\infty} a_n = 0$ and $\Delta^\alpha a_n \geqslant 0$ for $n = 0, 1, \dots$.*

It follows from Definition 2 that the class M_0 coincides with the class of null sequences of nonnegative numbers, M_1 is the class of monotone nonincreasing null sequences, M_2 is the class of convex null sequences, etc. In addition, in [8, Lemma 1, assertion b)] it was shown that $M_\alpha \subset M_\beta$ for $\alpha > \beta \geqslant 0$.

Definition 3. *Let $\gamma \in (0,1)$. We say that a sequence $\boldsymbol{a} \in P_\gamma$ if $\boldsymbol{a} \in M_0$ and*

$$\sum_{n=1}^\infty n^{-\gamma} a_n < \infty.$$

In [8], M.I. Dyachenko proved the following statements. They were proved for cosine series, but the analogous statements remain valid for sine series.

Theorem 3.1 ([8]). *Let $\alpha \in (0,1)$, a sequence $\boldsymbol{a} \in M_\alpha \cap P_\alpha$. Then a function $f(\boldsymbol{a},x)$ exists for $x \in (0,2\pi)$, such that $f(\boldsymbol{a},x) \in C((0,2\pi))$ and $|f(\boldsymbol{a},x)| \leqslant C(\alpha)x^{-\alpha}$ for $x \in (0,\pi)$, where $C(\alpha)$ depends only on α.*

Theorem 3.2 ([8]). *Let $\alpha \in (0,1)$, a sequence $\boldsymbol{b} \in M_1$ and $\boldsymbol{b} \notin P_\alpha$. Then there exists a sequence $\boldsymbol{a} \in M_\alpha$ such that $a_n \leqslant b_n$ for all n, but series (1.1) diverges at the point $\pi/2$.*

Theorem 3.3 ([8]). *Let $\alpha \in (1,2)$ and $\boldsymbol{a} \in M_\alpha$. Then for any $\gamma \in (0,\pi)$ we have $f(\boldsymbol{a},x+t) - f(\boldsymbol{a},x) = o(t^{\alpha-1})$ as $t \to +0$ uniformly for $x \in [\gamma, 2\pi-\gamma]$.*

Theorem 3.4 ([8]). *Let $\alpha \in (1,2)$ and a function φ be defined on $[0,1]$ and $\varphi(t) \downarrow 0$ as $t \downarrow 0$. Then there exist a sequence $\boldsymbol{a} \in M_\alpha$ and a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \downarrow 0$ as $n \to \infty$ and $|f(\boldsymbol{a},\pi/2+t_n) - f(\boldsymbol{a},\pi/2)| \geqslant Ct_n^{\alpha-1}\varphi(t_n)$ for all n where $C > 0$ does not depend on n.*

In [9], the following statements connected with Theorem 1.1 were obtained.

Theorem 3.5 ([9]). *Let $\alpha \in (0,1)$, $p \in (1/\alpha,\infty)$, a sequence $\boldsymbol{a} \in M_\alpha$ and $J_p(\boldsymbol{a}) < \infty$. Then series (1.1) converges at any $x \in (0,2\pi)$.*

Theorem 3.6 ([9]). *Let $\alpha \in (1/2,1)$, $p \in (1/\alpha,\infty)$, a sequence $\boldsymbol{a} \in M_\alpha$ and $J_p(\boldsymbol{a}) < \infty$. Then the function $f(\boldsymbol{a},x) \in L_p([0,\pi])$.*

Theorem 3.7 ([9]). *Let $\alpha \in (1/2,1)$. Then there exists a sequence $\boldsymbol{a} \in M_\alpha$ such that $J_p(\boldsymbol{a}) < \infty$ for every $p \in (1,1/\alpha)$, but (1.1) is not a Fourier–Lebesgue series.*

It is natural to suppose that the following hypothesis is true.

Hypothesis 3.1. *Let $\alpha \in (1/2,1)$, $p \in (2,1/(1-\alpha))$, a function $f \in L_p([0,\pi])$ and has the Fourier series of type (1.1) or (1.2) with $\boldsymbol{a} \in M_\alpha$. Then $J_p(\boldsymbol{a}) < \infty$.*

This conjecture is still unsolved, but M.I. Dyachenko and E. D. Nursultanov [12] proved, in particular, the following result.

Theorem 3.8 ([12]). *Let $\alpha \in (1/2,1)$ and $p > 1/(1-\alpha)$. Then there exists an even function $f \in L_p([0,\pi])$ such that its Fourier coefficients $\boldsymbol{a} \in M_\alpha$, but $J_p(\boldsymbol{a}) = \infty$.*

As for asymptotic properties of the sums of trigonometric series with fractional monotone coefficients, the results are the following. For cosine series they were established by M.I. Dyachenko [10].

Note that the sums of cosine series are usually estimated using the function

$$q(\boldsymbol{a},x) = \sum_{n=0}^{[\pi/x]}(n+1)(a_n - a_{n+1}).$$

Theorem 3.9 ([10]). *For any $\alpha \in (1,2)$, there exists a sequence $\boldsymbol{a} \in M_\alpha$ and a monotone null sequence $\{t_l\}_{l=1}^\infty$ such that*

$$\lim_{l\to\infty} \frac{q(\boldsymbol{a},t_l)}{f(\boldsymbol{a},t_l)} = 0.$$

Theorem 3.10 ([10]). *Let $\alpha > 2$. Then there exists a constant $C(\alpha) > 0$ such that if a sequence $\boldsymbol{a} \in M_\alpha$, then, for $x \in (0,\pi/6)$, the sum of series (1.1) satisfies the inequality $f(\boldsymbol{a},x) \geqslant C(\alpha)q(\boldsymbol{a},x)$.*

In the same paper [10], an example showing that the condition $\boldsymbol{a} \in M_2$ does not guarantee the validity of the lower bound in terms of $q(\boldsymbol{a},x)$ was given. Of course, the condition $\boldsymbol{a} \in M_2$ is sufficient for the upper bound $f(\boldsymbol{a},x) \leqslant Cq(\boldsymbol{a},x)$, $x \in (0,\pi)$, to hold. So, for cosine series, we need 2-monotonicity for the upper estimate, and $(2+\varepsilon)$-monotonicity for the lower estimate.

For the sine series the situation is quite different. This was shown by M.I. Dyachenko and A.P. Solodov in the paper [13]. They proved the following results.

Theorem 3.11 ([13]). *For any $\alpha \in (0,1)$, there exists a sequence $\boldsymbol{a} \in M_\alpha$ such that series (1.2) diverges almost everywhere.*

Theorem 3.12 ([13]). *Let $\alpha > 1$. Then there exist positive constants $C(\alpha)$ and $x(\alpha)$ such that if a sequence $\boldsymbol{a} \in M_\alpha$, then, for $x \in (0, x(\alpha))$, the sum of series (1.2) satisfies the inequality $g(\boldsymbol{a}, x) \geqslant C(\alpha) v(\boldsymbol{a}, x)$.*

Also, it was shown in [13] that there exists a sequence $\boldsymbol{a} \in M_1$ and a monotone null sequence $\{t_l\}_{l=1}^\infty$ such that

$$\lim_{l \to \infty} \frac{g(\boldsymbol{a}, t_l)}{v(\boldsymbol{a}, t_l)} = 0.$$

In [2], the following analogue of Theorem 1.2 was obtained.

Theorem 3.13 ([2]). *Let an even 2π-periodic function f be in the class $\operatorname{Lip}\beta$ with some $0 < \beta < 1$ and its cosine Fourier coefficients be in the class M_α with some $0 < \alpha < 1$. Then for some $C > 0$ we have $a_n \leqslant C/n^{\alpha+\beta}$ for $n = 1, 2, \dots$.*

This result cannot be improved as it follows from the next statement.

Theorem 3.14 ([2]). *Let $\alpha \in (0,1)$ and $\beta \in (0,1)$. Then there exists an even 2π-periodic function $f \in \operatorname{Lip}\beta$ such that its cosine Fourier coefficients are in the class M_α and also there exists a monotone increasing sequence of natural numbers $\{l_r\}_{r=1}^\infty$ such that the Fourier coefficients $a_{l_r}(f) \geqslant l_r^{-\alpha-\beta}$ for all r.*

Also in [2], the following property of α-monotone sequences was established.

Theorem 3.15 ([2]). *Let $\alpha \in (0,1)$ and $\boldsymbol{a} = \{a_n\}_{n=0}^\infty$ be an α-monotone sequence. Then for any $n \geqslant 1$ holds the inequality $a_k \geqslant a_n\, A_{n-k}^{\alpha-1}$ for all $0 \leqslant k \leqslant n-1$, and this inequality cannot be improved.*

In [11], M.I. Dyachenko proved the following generalization of one part of Theorem 1.3.

Theorem 3.16 ([11]). *Let $\alpha \in (0,1)$, the coefficients of series (1.2) belong to the class M_α and $na_n \to 0$ as $n \to \infty$. Then series (1.2) uniformly converges.*

As for reverse statement, the following is true.

Theorem 3.17 ([11]). *Let $\alpha \in (0,1)$, series (1.2) uniformly converge and its coefficients belong to the class M_α. Then $n^\alpha a_n \to 0$ as $n \to \infty$ and this result cannot be improved.*

Also in [11] the following generalization of Kolmogorov's theorem was obtained.

Theorem 3.18 ([11]). *Let $\alpha > 1$ and a sequence $\boldsymbol{a} \in M_\alpha$. Then the sum of series (1.1) $f(\boldsymbol{a}, x) \in L([0,\pi])$.*

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CONSTRUCTIVE METHOD FOR SOLVING OF ONE CLASS OF CURVILINEAR INTEGRAL EQUATIONS OF THE FIRST KIND

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Abstract. A new method for the construction of a quadrature formula for the normal derivative of the double-layer potential is developed and a method for calculating the approximate solution of the integral equation of the first kind for Dirichlet boundary value problems for the Helmholtz equation in the two-dimensional space is presented in this work.

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1 Introduction and problem statement

It is known that in many cases it is impossible to find an exact solution of Dirichlet boundary value problems for the Helmholtz equation in the two-dimensional space. This generates interest for studying approximate solution of these problems with theoretical justification. One of the methods to solve Dirichlet boundary value problem for the Helmholtz equation in two-dimensional space is to reduce it to an integral equation of the first kind. Note that the main advantage of applying the method of integral equations to exterior boundary value problems is that this method allows reducing the problem for an unbounded domain to the one for a bounded domain of lower dimension.

Let $D \subset R^2$ be a bounded domain with twice continuously differentiable boundary L, and f be a given continuous function on L. Consider the Dirichlet boundary value problems for the Helmholtz equation:

Interior Dirichlet problem. Find a function u, which is twice continuously differentiable on D, continuous on $\bar{D}$, and satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in D and the boundary condition $u = f$ on L, where Δ is the Laplace operator, and k is a wave number with $Im\, k \geq 0$.

Exterior Dirichlet problem. Find a function u, which is twice continuously differentiable on $R^2 \backslash \bar{D}$, continuous on $R^2 \backslash D$, satisfies the Helmholtz equation in $R^2 \backslash \bar{D}$, Sommerfeld radiation condition

$$\left(\frac{x}{|x|}, \operatorname{grad} u(x)\right) - i\, k\, u(x) = o\left(\frac{1}{|x|^{1/2}}\right), \quad x \to \infty,$$

uniformly in all directions $x/|x|$ and the boundary condition $u = f$ on L.

It was shown in [3, p. 87] that the simple-layer potential

$$u(x) = \int_L \Phi(x, y)\, \varphi(y)\, dL_y, \quad x \in R^2 \backslash L,$$

with continuous density φ is a solution of the interior and exterior Dirichlet boundary value problems if φ is a solution of the integral equation of the first kind

$$S\varphi = 2\,f, \tag{1.1}$$

where

$$(S\varphi)\,(x) = \,2\int_L \Phi\,(x,y)\;\varphi\,(y)\;dL_y,\quad x\in L,$$

$\Phi(x,\,y)$ is the fundamental solution of the Helmholtz equation, i.e.

$$\Phi\,(x,\,y) = \begin{cases} \frac{1}{2\pi}\ln\frac{1}{|x-y|} & for\;\; k=0, \\ \frac{i}{4}H_0^{(1)}\,(k\,|x-y|) & for\;\; k\neq 0, \end{cases}$$

where $H_0^{(1)}$ is the zero degree Hankel function of the first kind defined by the formula $H_0^{(1)}\,(z) = J_0\,(z) + i\,N_0\,(z)$,

$$J_0\,(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m}$$

is the Bessel function of zero degree,

$$N_0\,(z) = \frac{2}{\pi}\left(\ln\frac{z}{2}+C\right)\,J_0\,(z) + \sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}}{(m!)^2}\left(\frac{z}{2}\right)^{2m}$$

is the Neumann function of zero degree, and $C = 0.57721...$ is Euler's constant.

Note that the integral equations of the first kind do not fit into the Riesz-Fredholm theory. But, it was proved in [3, p. 89–90] that if $Im\,k>0$, then the operators S and

$$(Tf)\,(x) = 2\,\frac{\partial}{\partial\nu\,(x)}\left(\int_L \frac{\partial\Phi\,(x,\,y)}{\partial\nu\,(y)}\,f(y)dL_y\right),\quad x\in L,$$

are invertible, and

$$T^{-1} = -S\left(I-\tilde{K}\right)^{-1}\left(I+\tilde{K}\right)^{-1},$$

where

$$\left(\tilde{K}\rho\right)(x) = 2\int_L \frac{\partial\Phi\,(x,\,y)}{\partial\nu\,(x)}\,\rho(y)dL_y,\quad x\in L,$$

$\nu\,(x)$ is the outer unit normal at the point $x\in L$, and I is the unit operator in $C\,(L)$, the space of all continuous functions on L with the norm $\|\varphi\|_\infty = \max\limits_{x\in L}|\varphi\,(x)\,|$. Then the inverse operator S^{-1} is defined by

$$S^{-1} = -\left(I-\tilde{K}\right)^{-1}\left(I+\tilde{K}\right)^{-1}T.$$

Consequently, the solution of equation (1.1) has the form

$$\varphi = -2\left(I-\tilde{K}\right)^{-1}\left(I+\tilde{K}\right)^{-1}Tf. \tag{1.2}$$

Note that in spite of invertibility of the operators $I-\tilde{K}$ and $I+\tilde{K}$, the explicit forms of the inverse operators $\left(I-\tilde{K}\right)^{-1}$ and $\left(I+\tilde{K}\right)^{-1}$ are unknown. Besides, Lyapunov's counterexample shows ([6, p. 89–90]) that the derivatives of the double-layer potential with continuous density, in general, do not exist, i.e. the operator S^{-1}, inverse to the compact operator S, is unbounded in $\mathrm{N}(L)$, the space of all continuous functions φ, whose double-layer potential with the density φ has continuous normal derivatives on both sides of the curve L. Note that in [17], quadrature formulas for the simple-layer and double-layer potentials have been constructed using the asymptotic formula for the zero degree Hankel functions of the first kind, which does not allow to find the convergence rate of these quadrature formulas. But, in [11], quadrature formulas for the simple-layer and double-layer potentials have been constructed by using more practical method, and in [12], quadrature formulas for the normal derivative of the simple-layer potential have been constructed and the error estimates have been obtained for the constructed quadrature formulas. Further, in [2, 16], quadrature formulas for the normal derivative of the simple-layer and double-layer logarithmic potentials have been constructed and approximate solutions for integral equations of the exterior Dirichlet boundary value problem and the mixed problem for the Laplace equation have been studied in the two-dimensional space. In [10, 13], a new method for the construction of a cubature formula for the normal derivative of the acoustic double-layer potential has been proposed and justification of the collocation method for the integral equations of exterior Dirichlet and Neumann boundary value problems for the Helmholtz equation has been given in the three-dimensional space. However, it is known that the fundamental solution of the Helmholtz equation in three-dimensional space has the form

$$\Phi_k(x,\, y) = \frac{\exp\left(ik\left|x-y\right|\right)}{4\pi\left|x-y\right|},\ \ x, y \in R^3,\ \ x \neq y,$$

which differs essentially from the fundamental solution of the Helmholtz equation in the two-dimensional space. Also note that in [18, p. 115–116], considering normal derivative of the double-layer potential as a hypersingular integral, i.e. considering integral in the sense of finite value according to Hadamard, quadrature formula for the normal derivative of the double-layer potential has been constructed using subdomain method with an additional condition on the density of f ([18, p. 285–291]). It is known that with this condition the expression for the normal derivative of the double-layer potential can be represented in the form of singular integral ([3, p. 57], [18, p. 100]), i.e. the integral $(Tf)(x)$, $x \in L$, exists in the sense of the Cauchy principal value. Besides, it should be noted that the quadrature formula constructed in [18] is not practical, in other words, its coefficients are singular integrals.

Despite important results in the field of numerical solution of integral equations of the first kind ([4, 5, 7, 8, 20]), due to the above reasons, approximate solving of Dirichlet boundary value problems for the Helmholtz equation in the two-dimensional space has not yet been studied by the method of integral equations of the first kind (1.1). In this work, considering the normal derivative of the double-layer potential as an integral in the sense of the Cauchy principal value, we construct a quadrature formula for the normal derivative of the double-layer potential by a more practical method, and, using formula (1.2), we give a method for calculating an approximate solution to equation (1.1) at some selected points.

2 Approximate solution to equation (1.1)

Assume that the curve L is defined by the parametric equation $x(t) = (x_1(t)\,, x_2(t))\,,\ t \in [a,b]$. Let us divide the interval $[a,b]$ into $n > 2M_0\,(b-a)\,/d$ equal parts: $t_p = a + \frac{(b-a)\,p}{n}$, $p = \overline{0,n}$, where

$$M_0 = \max_{t\in[a,b]} \sqrt{\left(x_1'(t)\right)^2 + \left(x_2'(t)\right)^2} < +\infty$$

(see [19, p. 561]) and d is the standard radius ([21, p. 400]). As control points, we consider $x(\tau_p)$, $p = \overline{1,n}$, where $\tau_p = a + \frac{(b-a)\,(2p-1)}{2n}$. Then the curve L is divided into elementary parts: $L = \bigcup\limits_{p=1}^{n} L_p$, where $L_p = \{\, x(t) :\ \ t_{p-1} \leq t \leq t_p\}$.

It is known ([14]) that
(1) $\forall p \in \{1, 2, ..., n\}$: $r_p(n) \sim R_p(n)$, where

$$r_p\left(n\right) = \min\left\{\,\left|x\left(\tau_p\right) - x\left(t_{p-1}\right)\right|\,,\,\left|x\left(t_p\right) - x\left(\tau_p\right)\right|\,\right\},$$

$$R_p\left(n\right) = \max\left\{\,\left|x\left(\tau_p\right) - x\left(t_{p-1}\right)\right|\,,\,\left|x\left(t_p\right) - x\left(\tau_p\right)\right|\,\right\},$$

and $a\left(n\right) \sim b\left(n\right)$ means $C_1 \le \dfrac{a\left(n\right)}{b\left(n\right)} \le C_2$, with the positive constants C_1 and C_2 independent of n.
(2) $\forall p \in \{1, 2, ..., n\} : R_p\left(n\right) \le d/2$;
(3) $\forall p, j \in \{1, 2, ..., n\} : r_j\left(n\right) \sim r_p\left(n\right)$;
(4) $r\left(n\right) \sim R\left(n\right) \sim \frac{1}{n}$, where $R\left(n\right) = \max\limits_{p=\overline{1,n}} R_p\left(n\right)$, $r\left(n\right) = \min\limits_{p=\overline{1,n}} r_p\left(n\right)$.
The following lemma is true.

Lemma 2.1. [14]. *There exist constants $C_0' > 0$ and $C_1' > 0$, independent of n, such that the inequalities*

$$C_0'\,\left|y - x\left(\tau_p\right)\right| \le \left|x\left(\tau_j\right) - x\left(\tau_p\right)\right| \le C_1'\,\left|y - x\left(\tau_p\right)\right|$$

hold for $\forall p, j \in \{1,\, 2, ...,\, n\}\,, j \ne p$, and $\forall y \in L_j$.

Let

$$\Phi_n(x,\,y) = \frac{i}{4} H_{0,n}^{(1)}\left(k\left|x - y\right|\right), \quad x, y \in L, \quad x \ne y,$$

where

$$H_{0,n}^{(1)}\left(z\right) = J_{0,n}\left(z\right) + i\,N_{0,n}\left(z\right), J_{0,n}\left(z\right) = \sum_{m=0}^{n} \frac{\left(-1\right)^m}{\left(m!\right)^2}\left(\frac{z}{2}\right)^{2m}$$

and

$$N_{0,n}\left(z\right) = \frac{2}{\pi}\left(\ln\frac{z}{2} + C\right)\,J_{0,n}\left(z\right) + \sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}}{\left(m!\right)^2}\left(\frac{z}{2}\right)^{2m}.$$

It is not difficult to show that

$$\frac{\partial\Phi_n\left(x, y\right)}{\partial\nu\left(x\right)} = \frac{i}{4}\left(\frac{\partial J_{0,n}\left(k\left|x - y\right|\right)}{\partial\nu\left(x\right)} + i\,\frac{\partial N_{0,n}\left(k\left|x - y\right|\right)}{\partial\nu\left(x\right)}\right),$$

where

$$\frac{\partial J_{0,n}\left(k\left|x - y\right|\right)}{\partial\nu\left(x\right)} = \left(x - y,\, \nu\left(x\right)\right)\sum_{m=1}^{n}\frac{\left(-1\right)^m k^{2m}\left|x - y\right|^{2m-2}}{2^{2m-1}\left(m - 1\right)!\,m\,!}$$

and

$$\frac{\partial N_{0,n}\left(k\left|x - y\right|\right)}{\partial\nu\left(x\right)} =$$

$$= \frac{2}{\pi}\left(\ln\frac{k\,\left|x - y\right|}{2} + C\right)\frac{\partial J_{0,n}\left(k\left|x - y\right|\right)}{\partial\nu\left(x\right)} + \frac{2\,\left(x - y,\, \nu\left(x\right)\right)}{\pi\,\left|x - y\right|^2} J_{0,n}\left(k\left|x - y\right|\right) +$$

$$+ \left(x - y,\, \nu\left(x\right)\right)\sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1} k^{2m}\left|x - y\right|^{2m-2}}{2^{2m-1}\left(m - 1\right)!\,m\,!}.$$

Consider the matrix $\tilde{K}^n = \left(\tilde{k}_{pj}\right)_{p,j=1}^{n}$ with the elements

$$\tilde{k}_{pj} = \frac{2\,\left|sgn\left(p - j\right)\right|\,\left(b - a\right)}{n}\frac{\partial\Phi_n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2 + \left(x_2'\left(\tau_j\right)\right)^2}.$$

It was proved in [12] that if $\varphi \in C(L)$, then the expression

$$\left(\tilde{K}_n \varphi\right)(x(\tau_p)) = \sum_{\substack{j=1 \\ j \neq p}}^{n} \tilde{k}_{pj}\, \varphi(x(\tau_j))$$

is a quadrature formula for the integral $\left(\tilde{K}\varphi\right)(x)$ at the control points $x(\tau_p)$, $p = \overline{1,\, n}$, with[1]

$$\max_{p=\overline{1,\, n}} \left| \left(\tilde{K}\varphi\right)(x(\tau_p)) - \left(\tilde{K}_n \varphi\right)(x(\tau_p)) \right| \leq M \left(\omega(\varphi,\, 1/n) + \|\varphi\|_\infty \frac{\ln n}{n} \right),$$

where $\omega(\varphi,\, \delta)$ is the modulus of continuity of the function φ, i.e.

$$\omega(\varphi,\, \delta) = \max_{\substack{|x-y| \leq \delta \\ x,\, y \in L}} |\varphi(x) - \varphi(y)|\,, \quad \delta > 0.$$

It is known that if $Im\, k > 0$, then for every right-hand side $g \in C(L)$ the integral equations ([3, p. 81])

$$\varphi \pm \tilde{K}\varphi = g$$

are uniquely solvable in the space $C(L)$. Then, proceeding in the same way as in [9], it is not difficult to prove the following lemmas.

Lemma 2.2. *If $Im k > 0$, then there exists the inverse matrix $\left(I^n + \tilde{K}^n\right)^{-1}$ with*

$$M_1 = \sup_n \left\| \left(I^n + \tilde{K}^n\right)^{-1} \right\| < +\infty$$

and

$$\max_{l=\overline{1,n}} \left| \left(\left(I + \tilde{K}\right)^{-1} g \right)(x(\tau_l)) - \sum_{j=1}^{n} \tilde{k}_{lj}^{+}\, g\,(x(\tau_l)) \right| \leq M \left(\omega\,(g, 1/n) + \|g\|_\infty \frac{\ln n}{n} \right),$$

where I^n is a unit operator in the space C^n, and $\tilde{k}_{lj}^{+}$ is the element of the matrix $\left(I^n + \tilde{K}^n\right)^{-1}$ in the $l-$th row and $j-$th column.

Lemma 2.3. *If $Im k > 0$, then there exists the inverse matrix $\left(I^n - \tilde{K}^n\right)^{-1}$ with*

$$M_2 = \sup_n \left\| \left(I^n - \tilde{K}^n\right)^{-1} \right\| < +\infty$$

and

$$\max_{l=\overline{1,n}} \left| \left(\left(I - \tilde{K}\right)^{-1} g \right)(x(\tau_l)) - \sum_{j=1}^{n} \tilde{k}_{lj}^{-}\, g\,(x(\tau_l)) \right| \leq M \left(\omega\,(g, 1/n) + \|g\|_\infty \frac{\ln n}{n} \right),$$

where $\tilde{k}_{lj}^{-}$ is the element of the matrix $\left(I^n - \tilde{K}^n\right)^{-1}$ in the $l-$th row and $j-$th column.

[1] Hereinafter M denotes different positive constants which can be different in different inequalities.

Now, let us construct a quadrature formula for the normal derivative of the double-layer potential. For this, let us first determine the conditions for the existence of the normal derivative of the double-layer potential and derive the formulas for calculating it.

Lemma 2.4. *Let a function ρ be continuously differentiable on L and*

$$\int_0^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad}\rho,\, t\right)}{t} dt < +\infty.$$

Then the double-layer potential

$$W\left(x\right) = \int_L \frac{\partial\Phi\left(x,\, y\right)}{\partial\nu\left(y\right)}\, \rho\left(y\right) dL_y, \quad x \in L,$$

has the normal derivative in L, with

$$\frac{\partial W\left(x\right)}{\partial\nu\left(x\right)} = \int_L \frac{\partial V\left(x, y\right)}{\partial\nu\left(x\right)} \rho\left(y\right) dL_y - \frac{1}{\pi} \int_L \frac{\left(x-y, \nu\left(y\right)\right)\,\left(x-y, \nu\left(x\right)\right)}{\left|x-y\right|^4} \left(\rho\left(y\right) - \rho\left(x\right)\right) dL_y +$$

$$+ \frac{1}{2\pi} \int_L \frac{\left(\nu\left(y\right),\, \nu\left(x\right)\right)}{\left|x-y\right|^2} \left(\rho\left(y\right) - \rho\left(x\right)\right) dL_y, \;\; x \in L \tag{2.1}$$

and

$$\left| \frac{\partial W(x)}{\partial\nu\left(x\right)} \right| \le M \left(\|\rho\|_\infty + \|\operatorname{grad}\rho\|_\infty + \int_0^d \frac{\omega\left(\operatorname{grad}\rho, t\right)}{t} dt \right), \forall\, x \in L,$$

where

$$V\left(x, y\right) = \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\,\left|x-y\right|}{2} \right) \left(y-x,\, \nu\left(y\right)\right) \sum_{m=1}^{\infty} \frac{\left(-1\right)^m k^{2m} \left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!} -$$

$$- \left(y-x,\, \nu\left(y\right)\right) \sum_{m=1}^{\infty} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{\left(-1\right)^{m+1} k^{2m} \left|x-y\right|^{2m-2}}{2^{2m+1}\left(m-1\right)!\,m!} -$$

$$- \frac{1}{2\pi} \left(y-x,\, \nu\left(y\right)\right) \sum_{m=1}^{\infty} \frac{\left(-1\right)^m k^{2m} \left|x-y\right|^{2m-2}}{2^{2m}\left(m!\right)^2},$$

the first and the second integral terms in (2.1) are weakly singular, and the last integral exists in the sense of the Cauchy principal value.

Proof. It is easy to calculate that

$$\frac{\partial\Phi\left(x, y\right)}{\partial\nu\left(y\right)} = \frac{i}{4} \left(\frac{\partial J_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)} + i\, \frac{\partial N_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)} \right),$$

where

$$\frac{\partial J_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)} = \left(y-x,\, \nu\left(y\right)\right) \sum_{m=1}^{\infty} \frac{\left(-1\right)^m k^{2m} \left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!}$$

and

$$\frac{\partial N_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)} = \frac{2}{\pi} \left(\ln \frac{k\,\left|x-y\right|}{2} + C \right) \frac{\partial J_0\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)} + \frac{2\,\left(y-x,\, \nu\left(y\right)\right)}{\pi\,\left|x-y\right|^2} J_0\left(k\left|x-y\right|\right) +$$

$$+\left(y-x,\nu\left(y\right)\right)\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!}.$$

Then the expression $W\left(x\right)$ can be represented as

$$W\left(x\right)=\int_{L}\left(\frac{\left(x-y,\nu\left(y\right)\right)}{2\pi\left|x-y\right|^{2}}+V\left(x,y\right)\right)\rho\left(y\right)dL_{y},\quad x\in L.$$

It was shown in [15] that if a function ρ is continuously differentiable on L and

$$\int_{0}^{\operatorname{diam}L}\frac{\omega\left(\operatorname{grad}\rho,\,t\right)}{t}dt<+\infty,$$

then the function

$$W_{0}\left(x\right)=\frac{1}{2\pi}\int_{L}\frac{\left(x-y,\nu\left(y\right)\right)}{\left|x-y\right|^{2}}\rho\left(y\right)dL_{y},\quad x\in L,$$

has the normal derivative in L, with

$$\frac{\partial W_{0}\left(x\right)}{\partial\nu\left(x\right)}=-\frac{1}{\pi}\int_{L}\frac{\left(x-y,\nu\left(y\right)\right)\,\left(x-y,\nu\left(x\right)\right)}{\left|x-y\right|^{4}}\left(\rho\left(y\right)-\rho\left(x\right)\right)dL_{y}+$$

$$+\frac{1}{2\pi}\int_{L}\frac{\left(\nu\left(y\right),\nu\left(x\right)\right)}{\left|x-y\right|^{2}}\left(\rho\left(y\right)-\rho\left(x\right)\right)dL_{y},\ \ x\in L \tag{2.2}$$

and

$$\left|\frac{\partial W_{0}\left(x\right)}{\partial\nu\left(x\right)}\right|\leq M\left(\left\|\rho\right\|_{\infty}+\left\|\operatorname{grad}\rho\right\|_{\infty}+\int_{0}^{d}\frac{\omega\left(\operatorname{grad}\rho,t\right)}{t}dt\right),\forall x\in L.$$

The last integral in (2.2) exists in the sense of the Cauchy principal value.

As ([21, p. 403])

$$\left|\left(x-y,\nu\left(x\right)\right)\right|\leq M\left|x-y\right|^{2},\forall x,y\in L, \tag{2.3}$$

taking into account the inequalities

$$\left|J_{0}\left(k\left|x-y\right|\right)\right|=\left|\sum_{m=0}^{\infty}\frac{\left(-1\right)^{m}}{\left(m!\right)^{2}}\left(\frac{k\left|x-y\right|}{2}\right)^{2m}\right|\leq\sum_{m=0}^{\infty}\frac{\left(\left|k\right|\,\operatorname{diam}L\right)^{2m}}{4^{m}\left(m!\right)^{2}},\forall x,y\in L, \tag{2.4}$$

and

$$\left|\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!}\right|\leq$$

$$\leq\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left|k\right|^{2m}\left(\operatorname{diam}L\right)^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!},\forall x,y\in L, \tag{2.5}$$

we obtain

$$\left|V\left(x,y\right)\right|\leq M\,\left|x-y\right|,\quad\forall x,y\in L.$$

Consequently, the function

$$W_{1}\left(x\right)=\int_{L}V\left(x,y\right)\,\rho\left(y\right)dL_{y},\quad x\in L,$$

has the normal derivative in L, with

$$\frac{\partial W_{1}\left(x\right)}{\partial\nu\left(x\right)}=\int_{L}\frac{\partial V\left(x,y\right)}{\partial\nu\left(x\right)}\rho\left(y\right)dL_{y}=$$

$$= \frac{1}{2\pi} \int_L \frac{(y-x,\nu(x))\,(y-x,\nu(y))}{|x-y|^2} \sum_{m=1}^{\infty} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)\,!\,m\,!} \rho(y)\,dL_y -$$

$$- \int_L \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\,|x-y|}{2} \right) (\nu(y), \nu(x)) \sum_{m=1}^{\infty} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)\,!\,m\,!} \rho(y)\,dL_y +$$

$$+ \int_L \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\,|x-y|}{2} \right) (y-x,\nu(y))\,(x-y,\nu(x)) \times$$

$$\times \sum_{m=2}^{\infty} \frac{(-1)^m k^{2m} |x-y|^{2m-4}}{2^{2m-2}(m-2)\,!\,m\,!} \rho(y)\,dL_y +$$

$$+ \int_L (\nu(y), \nu(x)) \sum_{m=1}^{\infty} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1} k^{2m} |x-y|^{2m-2}}{2^{2m+1}(m-1)\,!\,m\,!} \rho(y)\,dL_y -$$

$$- \int_L (x-y,\nu(x))\,(y-x,\nu(y)) \sum_{m=2}^{\infty} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1} k^{2m} |x-y|^{2m-4}}{2^{2m}(m-2)\,!\,m\,!} \rho(y)\,dL_y +$$

$$+ \frac{1}{2\pi} \int_L (\nu(y), \nu(x))\,(y-x,\nu(y)) \sum_{m=1}^{\infty} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m}(m!)^2} \rho(y)\,dL_y -$$

$$- \frac{1}{2\pi} \int_L (x-y,\nu(x))\,(y-x,\nu(y)) \sum_{m=2}^{\infty} \frac{(-1)^m (m-1) k^{2m} |x-y|^{2m-4}}{2^{2m-1}(m!)^2} \rho(y)\,dL_y$$

and

$$\left| \frac{\partial V(x,y)}{\partial \nu(x)} \right| \le M\, |\ln|x-y||\,, \quad \forall\, x,y \in L. \tag{2.6}$$

Hence, we have

$$\left| \frac{\partial W_1(x)}{\partial \nu(x)} \right| \le M\, \|\rho\|_\infty\,, \quad \forall\, x \in L.$$

□

Obviously, there exists a positive integer n_0 such that

$$\sqrt{R(n)} \le \min\{1\,,\, d/2\}\,, \quad \forall n > n_0.$$

Let

$$P_l = \left\{ j \mid 1 \le j \le n\,,\, |x(\tau_l) - x(\tau_j)| \le \sqrt{R(n)} \right\},$$

$$Q_l = \left\{ j \mid 1 \le j \le n\,,\, |x(\tau_l) - x(\tau_j)| > \sqrt{R(n)} \right\}$$

and

$$V_n(x,y) = \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\,|x-y|}{2} \right) (y-x,\nu(y)) \sum_{m=1}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)\,!\,m\,!} -$$

$$- (y-x,\nu(y)) \sum_{m=1}^{n} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1} k^{2m} |x-y|^{2m-2}}{2^{2m+1}(m-1)\,!\,m\,!} -$$

$$- \frac{1}{2\pi} (y-x,\nu(y)) \sum_{m=1}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m}(m!)^2}.$$

It is easy to see that

$$\frac{\partial V_n(x,y)}{\partial \nu(x)} = \frac{1}{2\pi}\frac{(y-x,\nu(x))\,(y-x,\nu(y))}{|x-y|^2}\sum_{m=1}^{n}\frac{(-1)^m k^{2m}|x-y|^{2m-2}}{2^{2m-1}(m-1)!\,m!}-$$

$$-\left(\frac{i}{4}-\frac{C}{2\pi}-\frac{1}{2\pi}\ln\frac{k\,|x-y|}{2}\right)(\nu(y),\nu(x))\sum_{m=1}^{n}\frac{(-1)^m k^{2m}|x-y|^{2m-2}}{2^{2m-1}(m-1)!\,m!}+$$

$$+\left(\frac{i}{4}-\frac{C}{2\pi}-\frac{1}{2\pi}\ln\frac{k\,|x-y|}{2}\right)(y-x,\nu(y))\,(x-y,\nu(x))\sum_{m=2}^{n}\frac{(-1)^m k^{2m}|x-y|^{2m-4}}{2^{2m-2}(m-2)!\,m!}+$$

$$+(\nu(y),\nu(x))\sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}k^{2m}|x-y|^{2m-2}}{2^{2m+1}(m-1)!\,m!}-$$

$$-(x-y,\nu(x))\,(y-x,\nu(y))\sum_{m=2}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}k^{2m}|x-y|^{2m-4}}{2^{2m}(m-2)!\,m!}+$$

$$+\frac{1}{2\pi}(\nu(y),\nu(x))\,(y-x,\nu(y))\sum_{m=1}^{n}\frac{(-1)^m k^{2m}|x-y|^{2m-2}}{2^{2m}(m!)^2}-$$

$$-\frac{1}{2\pi}(x-y,\nu(x))\,(y-x,\nu(y))\sum_{m=2}^{n}\frac{(-1)^m (m-1)\,k^{2m}|x-y|^{2m-4}}{2^{2m-1}(m!)^2}.$$

The following theorem is true.

Theorem 2.1. *Let a function ρ be continuously differentiable on L and*

$$\int_0^{\operatorname{diam} L}\frac{\omega(\operatorname{grad}\rho,\,t)}{t}dt<+\infty.$$

Then the expression

$$(T_n\rho)(x(\tau_p)) = \frac{2\,(b-a)}{n}\sum_{\substack{j=1\\ j\neq p}}^{n}\frac{\partial V_n(x(\tau_p),x(\tau_j))}{\partial\nu(x(\tau_p))}\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,\rho(x(\tau_j))\;-$$

$$-\frac{2\,(b-a)}{\pi\,n}\sum_{\substack{j=1\\ j\neq p}}^{n}\frac{(x(\tau_p)-x(\tau_j),\,\nu(x(\tau_j)))\,(x(\tau_p)-x(\tau_j),\,\nu(x(\tau_p)))}{|x(\tau_p)-x(\tau_j)|^4}\times$$

$$\times\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,(\rho(x(\tau_j))-\rho(x(\tau_p)))+$$

$$+\frac{b-a}{\pi\,n}\sum_{j\in Q_p}\frac{(\nu(x(\tau_j)),\,\nu(x(\tau_p)))}{|x(\tau_j)-x(\tau_p)|^2}\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,(\rho(x(\tau_j))-\rho(x(\tau_p)))$$

is a quadrature formula for $(T\rho)(x)$ at the control points $x(\tau_p)$, $p=\overline{1,n}$, and the following estimate holds:

$$\max_{p=\overline{1,n}}|(T\rho)(x(\tau_p))-(T_n\rho)(x(\tau_p))|\le$$

$$\le M\left[\frac{\|\rho\|_\infty\ln n}{n}+\frac{\|\operatorname{grad}\rho\|_\infty}{\sqrt{n}}+\int_0^{1/\sqrt{n}}\frac{\omega(\operatorname{grad}\rho,\,t)}{t}\,dt\right].$$

Proof. It was proved in [1] that if a function ρ is continuously differentiable on L and

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad}\rho,\, t)}{t} dt < +\infty,$$

then the expression

$$\left(\frac{\partial W_0}{\partial \nu}\right)^n (x(\tau_p)) = -\frac{b-a}{\pi n} \sum_{\substack{j=1\\ j\neq p}}^{n} \frac{(x(\tau_p)-x(\tau_j)\,,\, \nu(x(\tau_j)))\,(x(\tau_p)-x(\tau_j)\,,\,\nu(x(\tau_p)))}{|x(\tau_p)-x(\tau_j)|^4} \times$$

$$\times \sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,(\rho(x(\tau_j))-\rho(x(\tau_p)))+$$

$$+\frac{b-a}{2\pi n} \sum_{j\in Q_p} \frac{(\nu(x(\tau_j))\,,\,\nu(x(\tau_p)))}{|x(\tau_j)-x(\tau_p)|^2} \sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,(\rho(x(\tau_j))-\rho(x(\tau_p)))$$

is a quadrature formula for the integral $\frac{\partial W_0(x)}{\partial \nu(x)}$ at the control points $x(\tau_p)\,,\ p=\overline{1,\,n}$, and the following estimate holds:

$$\max_{p=\overline{1,\,n}} \left| \frac{\partial W_0(x(\tau_p))}{\partial \nu(x(\tau_p))} - \left(\frac{\partial W_0}{\partial \nu}\right)^n (x(\tau_p)) \right| \leq$$

$$\leq M \left[\frac{\|\rho\|_\infty \ln n}{n} + \frac{\|\operatorname{grad}\rho\|_\infty}{\sqrt{n}} + \int_0^{1/\sqrt{n}} \frac{\omega(\operatorname{grad}\rho,\, t)}{t} dt \right].$$

Now, let us show that the expression

$$\left(\frac{\partial W_1}{\partial \nu}\right)^n (x(\tau_l)) = \frac{b-a}{n} \sum_{\substack{j=1\\ j\neq l}}^{n} \frac{\partial V_n(x(\tau_l)\,,x(\tau_j))}{\partial \nu(x(\tau_l))} \sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,\rho(x(\tau_j))$$

is a quadrature formula for the integral $\frac{\partial W_1(x)}{\partial \nu(x)}$ at the control points $x(\tau_l)\,,\ l=\overline{1,\,n}$. It is not difficult to see that

$$\frac{\partial W_1(x(\tau_p))}{\partial \nu(x(\tau_p))} - \left(\frac{\partial W_1}{\partial \nu}\right)^n (x(\tau_p)) = \int_{L_p} \frac{\partial V(x(\tau_p)\,,\, y)}{\partial \nu(x)} \rho(y)\, dL_y +$$

$$+\sum_{\substack{j=1\\ j\neq p}}^{n} \int_{L_j} \left(\frac{\partial V(x(\tau_p)\,,\, y)}{\partial \nu(x(\tau_p))} - \frac{\partial V_n(x(\tau_p)\,,\, x(\tau_j))}{\partial \nu(x(\tau_p))} \right) \rho(y)\, dL_y +$$

$$+\sum_{\substack{j=1\\ j\neq p}}^{n} \int_{L_j} \frac{\partial V_n(x(\tau_p)\,,\, x(\tau_j))}{\partial \nu(x(\tau_p))} (\rho(y)-\rho(x(\tau_j)))\, dL_y +$$

$$+\sum_{\substack{j=1\\ j\neq p}}^{n} \int_{t_{j-1}}^{t_j} \frac{\partial V_n(x(\tau_p)\,,\, x(\tau_j))}{\partial \nu(x(\tau_p))} \times$$

$$\times \left(\sqrt{(x_1'(t))^2+(x_2'(t))^2} - \sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2} \right) \rho(x(\tau_j))\, dt.$$

Denote the terms in the last equality by $\delta_1^n(x(\tau_p))$, $\delta_2^n(x(\tau_p))$, $\delta_3^n(x(\tau_p))$ and $\delta_4^n(x(\tau_p))$, respectively.

Taking into account (2.6) and the formula for calculating a curvilinear integral, we obtain

$$\left|\delta_1^n\left(x\left(\tau_p\right)\right)\right| \le M\ \|\rho\|_\infty \int_0^{R(n)} |\ln\tau|\,d\tau \le M\ \|\rho\|_\infty\, R\left(n\right)\ \left|\ln R\left(n\right)\right|.$$

Let $y \in L_j$ and $j \ne p$. From Lemma 2.1 and inequality (2.3) it is obvious that

$$\left|\left|x\left(\tau_p\right)-y\right|^q - \left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|^q\right| \le M\,q\,R\left(n\right)\ (\mathrm{diam}L)^{q-1},$$

$$\left|\left(\nu\left(y\right), \nu\left(x\left(\tau_p\right)\right)\right) - \left(\nu\left(x\left(\tau_j\right)\right), \nu\left(x\left(\tau_p\right)\right)\right)\right| \le M\,R\left(n\right),$$

$$\left|\left(x\left(\tau_p\right)-y, \nu\left(y\right)\right) - \left(x\left(\tau_p\right)-x\left(\tau_j\right), \nu\left(y\right)\right)\right| = \left|\left(x\left(\tau_j\right)-y, \nu\left(y\right)\right)\right| \le M\ \left(R\left(n\right)\right)^2,$$

$$\left|\left(x\left(\tau_p\right)-y, \nu\left(x\left(\tau_p\right)\right)\right) - \left(x\left(\tau_p\right)-x\left(\tau_j\right), \nu\left(x\left(\tau_p\right)\right)\right)\right| = \left|\left(x\left(\tau_j\right)-y, \nu\left(x\left(\tau_p\right)\right)\right)\right| \le$$
$$\le \left|\left(x\left(\tau_j\right)-y, \nu\left(x\left(\tau_j\right)\right)\right)\right| + \left|\left(x\left(\tau_j\right)-y, \nu\left(x\left(\tau_p\right)\right)-\nu\left(x\left(\tau_j\right)\right)\right)\right| \le M\ \left|y-x\left(\tau_p\right)\right|\ R\left(n\right)$$

and

$$\left|\ln\left(k\left|x\left(\tau_p\right)-y\right|\right) - \ln\left(k\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|\right)\right| = \left|\ln\frac{\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|}{\left|x\left(\tau_p\right)-y\right|}\right| =$$
$$= \left|\ln\left(1+\frac{\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|-\left|x\left(\tau_p\right)-y\right|}{\left|x\left(\tau_p\right)-y\right|}\right)\right| \le \left|\ln\left(1+\frac{\left|x\left(\tau_j\right)-y\right|}{\left|x\left(\tau_p\right)-y\right|}\right)\right| \le M\frac{R\left(n\right)}{\left|x\left(\tau_p\right)-y\right|},$$

where $q \in \mathrm{N}$. Then, taking into account inequalities (2.4) and (2.5), it is not difficult to show that

$$\left|\frac{\partial V\left(x\left(\tau_p\right), y\right)}{\partial\nu\left(x\left(\tau_p\right)\right)} - \frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right| \le M\ \left(R\left(n\right)\ \left|\ln\left|x\left(\tau_p\right)-y\right|\right| + \frac{R\left(n\right)}{\left|x\left(\tau_p\right)-y\right|}\right).$$

Also, by the inequality

$$\left|\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)} - \frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right| \le M\frac{\left|\ln\left|x\left(\tau_p\right)-y\right|\right|}{n\,!}, \tag{2.7}$$

we have

$$\left|\frac{\partial V\left(x\left(\tau_p\right), y\right)}{\partial\nu\left(x\left(\tau_p\right)\right)} - \frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right| \le$$
$$\le M\ \left(R\left(n\right)\ \left|\ln\left|x\left(\tau_p\right)-y\right|\right| + \frac{R\left(n\right)}{\left|x\left(\tau_p\right)-y\right|} + \frac{\left|\ln\left|x\left(\tau_p\right)-y\right|\right|}{n\,!}\right).$$

So, we obtain

$$\left|\delta_2^n\left(x\left(\tau_p\right)\right)\right| \le$$
$$\le M\ \|\rho\|_\infty\left(R\left(n\right)\int_{r(n)}^{\mathrm{diam}\,L}|\ln\tau|\,d\tau + R\left(n\right)\int_{r(n)}^{\mathrm{diam}\,L}\frac{d\tau}{\tau} + \frac{1}{n\,!}\int_{r(n)}^{\mathrm{diam}\,L}|\ln\tau|\,d\tau\right) \le$$
$$\le M\ \|\rho\|_\infty\left(R\left(n\right)\ \left|\ln R\left(n\right)\right| + \frac{1}{n\,!}\right).$$

Let $y \in L_j$ and $j \ne p$. From Lemma 2.1 and inequalities (2.6), (2.7) it is obvious that

$$\left|\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right| \le$$
$$\le \left|\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right| + \left|\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)} - \frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right| \le$$

$$\leq M\left(\left|\ln\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|\right|+\frac{1}{n!}\right), \forall n \in \mathrm{N}. \tag{2.8}$$

Then,

$$\left|\delta_3^n\left(x\left(\tau_p\right)\right)\right| \leq 2\,\omega\left(\rho, R\left(n\right)\right) \sum_{\substack{j=1 \\ j\neq p}}^{n} \int_{L_j}\left|\frac{\partial V_n\left(x\left(\tau_p\right), \, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| dL_y \leq$$

$$\leq 2\,\omega\left(\rho, R\left(n\right)\right) \int_{L}\left|\frac{\partial V_n\left(x\left(\tau_p\right), \, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| dL_y \leq M\omega\left(\rho, R\left(n\right)\right).$$

Besides, taking into account Lemma 2.1, inequality (2.8) and

$$\left|\sqrt{\left(x_1'\left(t\right)\right)^2+\left(x_2'\left(t\right)\right)^2}-\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\right| \leq M\,R\left(n\right), \forall t \in \left[t_{j-1}, t_j\right],$$

we obtain

$$\left|\delta_4^n\left(x\left(\tau_p\right)\right)\right| \leq M\left\|\rho\right\|_\infty R\left(n\right) \sum_{\substack{j=1 \\ j\neq p}}^{n} \int_{t_{j-1}}^{t_j}\left|\frac{\partial V_n\left(x\left(\tau_p\right), \, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| dt \leq$$

$$\leq M\left\|\rho\right\|_\infty R\left(n\right) \sum_{\substack{j=1 \\ j\neq p}}^{n} \int_{L_j}\left|\frac{\partial V_n\left(x\left(\tau_p\right), \, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| dL_y \leq$$

$$\leq M\left\|\rho\right\|_\infty R\left(n\right) \int_{L}\left|\frac{\partial V_n\left(x\left(\tau_p\right), \, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| dL_y \leq M\left\|\rho\right\|_\infty R\left(n\right).$$

Summing up the estimates obtained for the expressions $\delta_1^n\left(x\left(\tau_p\right)\right)$, $\delta_2^n\left(x\left(\tau_p\right)\right)$, $\delta_3^n\left(x\left(\tau_p\right)\right)$ and $\delta_4^n\left(x\left(\tau_p\right)\right)$, and considering the relation $R\left(n\right) \sim \frac{1}{n}$, we obtain

$$\max_{p=\overline{1,\,n}}\left|\frac{\partial W_1\left(x\left(\tau_p\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}-\left(\frac{\partial W_1}{\partial \nu}\right)^n\left(x\left(\tau_p\right)\right)\right| \leq M\left(\omega\left(\rho, \, 1/n\right)+\left\|\rho\right\|_\infty \frac{\ln n}{n}\right).$$

As a result, summing up the quadrature formulas constructed for the integrals $\frac{\partial W_0(x)}{\partial \nu(x)}$ and $\frac{\partial W_1(x)}{\partial \nu(x)}$ at the control points $x\left(\tau_p\right), \; p=\overline{1, \, n}$, we get the validity of Theorem 2.1. $\square$

Now, let us state the main result of this work. Let

$$t_{ll}=\frac{2\left(b-a\right)}{\pi\, n} \sum_{\substack{j=1 \\ j\neq l}}^{n} \frac{\left(x\left(\tau_l\right)-x\left(\tau_j\right), \, \nu\left(x\left(\tau_j\right)\right)\right)\left(x\left(\tau_l\right)-x\left(\tau_j\right), \, \nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_l\right)-x\left(\tau_j\right)\right|^4}\times$$

$$\times\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}-$$

$$-\frac{b-a}{\pi\, n} \sum_{j\in Q_l} \frac{\left(\nu\left(x\left(\tau_j\right)\right), \, \nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_j\right)-x\left(\tau_l\right)\right|^2}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2} \quad \text{for} l=\overline{1, n};$$

$$t_{lj}=\frac{2\,\left(b-a\right)}{n}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\left(\frac{\partial V_n\left(x\left(\tau_l\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_l\right)\right)}-\right.$$

$$\left.-\frac{\left(x\left(\tau_l\right)-x\left(\tau_j\right), \, \nu\left(x\left(\tau_j\right)\right)\right)\left(x\left(\tau_l\right)-x\left(\tau_j\right), \, \nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_l\right)-x\left(\tau_j\right)\right|^4}\right) \quad \text{for } j \in P_l, \, j \neq l;$$

$$t_{l\,j}=\frac{2\,(b-a)}{n}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\,\left(\frac{\partial V_n\left(x\left(\tau_l\right),x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_l\right)\right)}-\right.$$

$$\left.-\frac{\left(x\left(\tau_l\right)-x\left(\tau_j\right),\,\nu\left(x\left(\tau_j\right)\right)\right)\,\left(x\left(\tau_l\right)-x\left(\tau_j\right),\,\nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_l\right)-x\left(\tau_j\right)\right|^4}+\frac{\left(\nu\left(x\left(\tau_j\right)\right),\,\nu\left(x\left(\tau_l\right)\right)\right)}{2\,\left|x\left(\tau_j\right)-x\left(\tau_l\right)\right|^2}\right)\ \text{for } j\in Q_l.$$

From Theorem 2.1 it follows that

$$\left(T_n\rho\right)\left(x\left(\tau_l\right)\right)=\sum_{j=1}^{n}t_{l\,j}\,\rho\left(x\left(\tau_l\right)\right),\,l=\overline{1,n}.$$

Theorem 2.2. *Let $Im k>0$, a function f be continuously differentiable on L and*

$$\int_0^{\operatorname{diam}L}\frac{\omega(\operatorname{grad}f,\,t)}{t}dt<+\infty.$$

Then the expression

$$\varphi_n\left(x\left(\tau_l\right)\right)=-2\sum_{j=1}^{n}\tilde{k}_{l\,j}^{-}\left(\sum_{p=1}^{n}\tilde{k}_{j\,p}^{+}\left(\sum_{m=1}^{n}t_{pm}\,f\left(x\left(\tau_m\right)\right)\right)\right)$$

is an approximate value of the solution $\varphi(x)$ to equation (1.1) at the points $x\left(\tau_l\right),\ l=\overline{1,n}$, with

$$\max_{l=\overline{1,n}}\left|\,\varphi\,\left(x\left(\tau_l\right)\right)-\varphi_n\left(x\left(\tau_l\right)\right)\right|\le$$

$$\le M\,\left[\frac{1}{\sqrt{n}}+\omega\left(\operatorname{grad}f,1/n\right)+\int_0^{1/\sqrt{n}}\frac{\omega\left(\operatorname{grad}f,\,t\right)}{t}\,dt+\frac{1}{n}\int_{1/n}^{\operatorname{diam}L}\frac{\omega\left(\operatorname{grad}f,\,t\right)}{t^2}\,dt\right].$$

Proof. From Lemmas 2.2 and 2.3 we obtain

$$\max_{j=\overline{1,n}}\sum_{l=1}^{n}\left|\tilde{k}_{j\,l}^{+}\right|\le M_1,\ \max_{j=\overline{1,n}}\sum_{l=1}^{n}\left|\tilde{k}_{j\,l}^{-}\right|\le M_2.$$

Besides, taking into account the error estimate for the quadrature formula for $(Tf)\,(x)$,$x\in L$, at the control points $x\left(\tau_l\right),\ l=\overline{1,\,n}$, we have

$$\left|\,\varphi\left(x\left(\tau_l\right)\right)-\varphi_n\left(x\left(\tau_l\right)\right)\right|\le$$

$$\le 2\left|\left(\left(I-\tilde{K}\right)^{-1}\left(I+\tilde{K}\right)^{-1}Tf\right)\,\left(x\left(\tau_l\right)\right)-\sum_{j=1}^{n}\tilde{k}_{lj}^{-}\left(\left(I+\tilde{K}\right)^{-1}Tf\right)\,\left(x\left(\tau_j\right)\right)\right|+$$

$$+2\left|\sum_{j=1}^{n}\tilde{k}_{lj}^{-}\left[\left(\left(I+\tilde{K}\right)^{-1}Tf\right)\,\left(x\left(\tau_j\right)\right)-\sum_{p=1}^{n}\tilde{k}_{j\,p}^{+}\,\left(Tf\right)\left(x\left(\tau_p\right)\right)\right]\right|+$$

$$+2\,\left|\sum_{j=1}^{n}\tilde{k}_{lj}^{-}\left(\sum_{p=1}^{n}\tilde{k}_{j\,p}^{+}\left[\left(Tf\right)\,\left(x\left(\tau_p\right)\right)-\sum_{m=1}^{n}t_{p\,m}\,f\left(x\left(\tau_m\right)\right)\right]\right)\,\right|\le$$

$$\le M\,\left[\left\|\left(I+\tilde{K}\right)^{-1}\right\|\,\left\|Tf\right\|_\infty R\left(n\right)\,\left|\ln R\left(n\right)\right|+\omega\left(\left(I+\tilde{K}\right)^{-1}Tf,R\left(n\right)\right)\right]+$$

$$+M\,M_2\,\left[\,\left\|\,Tf\,\right\|_\infty R\left(n\right)\,\left|\ln R\left(n\right)\right|+\omega\,\left(Tf,R\left(n\right)\right)\,\right]\,+$$

$$+ M\, M_1 M_2 \left[\frac{\|\rho\|_\infty \ln n}{n} + \frac{\|\operatorname{grad}\rho\|_\infty}{\sqrt{n}} + \int_0^{1/\sqrt{n}} \frac{\omega(\operatorname{grad}\rho,\, t)}{t}\, dt \right]. \tag{2.9}$$

From Lemma 2.4 it follows

$$\|Tf\|_\infty \le M \left(\|f\|_\infty + \|\operatorname{grad} f\|_\infty + \int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} f,\, t)}{t}\, dt \right).$$

Further, as the integral $\frac{\partial W_1(x)}{\partial \nu(x)}$, $x \in L$, is weakly singular, it is not difficult to show that

$$\omega\left(\frac{\partial W_1}{\partial \nu}, \delta \right) \le M\ \|\rho\|_\infty\ \delta\ |\ln \delta|\,, \delta > 0.$$

It was shown in [15] that if a function f is continuously differentiable on L and

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} f,\, t)}{t} dt < +\infty,$$

then

$$\omega\left(\frac{\partial W_0}{\partial \nu}, \delta \right) \le$$

$$\le M \left(\delta\ |\ln \delta| + \omega\left(\operatorname{grad} f, \delta\right) + \int_o^\delta \frac{\omega\left(\operatorname{grad} f, t\right)}{t} dt + \delta \int_\delta^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f, t\right)}{t^2} dt \right),$$

where $\delta > 0$. Hence, it follows that

$$\omega\ (Tf, \delta) \le 2 \left(\omega \left(\frac{\partial W_0}{\partial \nu}, \delta \right) + \omega \left(\frac{\partial W_1}{\partial \nu}, \delta \right) \right) \le$$

$$\le M \left(\delta\ |\ln \delta| + \omega\left(\operatorname{grad} f, \delta\right) + \int_o^\delta \frac{\omega\left(\operatorname{grad} f, t\right)}{t} dt + \delta \int_\delta^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f, t\right)}{t^2} dt \right), \delta > 0.$$

It is known ([3, p. 53–54]) that

$$\omega\left(\tilde{K}\rho, \delta \right) \le M\ \|\rho\|_\infty\ \delta\ |\ln \delta|\,, \delta > 0.$$

Then, if a function ρ_* is a solution of the equation $\rho + \tilde{K}\rho = Tf$, we have

$$\omega\left(\left(I + \tilde{K}\right)^{-1} Tf, \delta \right) = \omega\left(\rho_*, \delta\right) = \omega\left(Tf - \tilde{K}\rho_*, \delta \right) \le \omega\left(Tf, \delta\right) + \omega\left(\tilde{K}\rho_*, \delta \right) \le$$

$$\le \omega\left(Tf, \delta\right) + M\ \|\rho_*\|_\infty\ \delta\ |\ln \delta| = \omega\left(Tf, \delta\right) + M \left\| \left(I + \tilde{K}\right)^{-1} Tf \right\|_\infty \delta\ |\ln \delta| \le$$

$$\le \omega\left(Tf, \delta\right) + M \left\| \left(I + \tilde{K}\right)^{-1} \right\| \|Tf\|_\infty\, \delta\ |\ln \delta| \le$$

$$\le M \left(\delta\ |\ln \delta| + \omega\left(\operatorname{grad} f, \delta\right) + \int_o^\delta \frac{\omega\left(\operatorname{grad} f, t\right)}{t} dt + \delta \int_\delta^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f, t\right)}{t^2} dt \right), \delta > 0.$$

So, taking into account the above obtained inequalities in (2.9) and the relation $R(n) \sim \frac{1}{n}$, we get the validity of the theorem. □

Theorem 2.2 has the following corollaries.

Corollary 2.1. *Let $Im\, k > 0$, a function f be continuously differentiable on L and*

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} f,\, t)}{t} dt < +\infty.$$

Then the sequence

$$u_n(x_*) = \frac{b-a}{n} \sum_{l=1}^{n} \Phi_n\left(x_*, x\left(\tau_l\right)\right)\, \varphi_n\left(x\left(\tau_l\right)\right)\, \sqrt{\left(x_1'\left(\tau_l\right)\right)^2 + \left(x_2'\left(\tau_l\right)\right)^2}, \quad x_* \in D,$$

converges to the value $u\left(x_*\right)$ of the solution $u\left(x\right)$ to the interior Dirichlet boundary value problem for the Helmholtz equation at the point x_*, with

$$\left|u_n\left(x_*\right) - u\left(x_*\right)\right| \leq$$

$$\leq M\left[\frac{1}{\sqrt{n}} + \omega\left(\operatorname{grad} f, 1/n\right) + \int_0^{1/\sqrt{n}} \frac{\omega\left(\operatorname{grad} f,\, t\right)}{t}\, dt + \frac{1}{n} \int_{1/n}^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f,\, t\right)}{t^2}\, dt\right],$$

where

$$\varphi_n\left(x\left(\tau_l\right)\right) = -2 \sum_{j=1}^{n} \tilde{k}_{l\,j}^{-} \left(\sum_{p=1}^{n} \tilde{k}_{j\,p}^{+} \left(\sum_{m=1}^{n} t_{pm}\, f\left(x\left(\tau_m\right)\right)\right)\right).$$

Corollary 2.2. *Let $Im\, k > 0$, a function f be continuously differentiable on L and*

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} f,\, t)}{t} dt < +\infty.$$

Then the sequence

$$u_n\left(x^*\right) = \frac{b-a}{n} \sum_{l=1}^{n} \Phi_n\left(x^*, x\left(\tau_l\right)\right)\, \varphi_n\left(x\left(\tau_l\right)\right)\, \sqrt{\left(x_1'\left(\tau_l\right)\right)^2 + \left(x_2'\left(\tau_l\right)\right)^2}, \quad x^* \in R^2 \backslash \bar{D},$$

converges to the value $u\left(x^*\right)$ of the solution $u\left(x\right)$ to the exterior Dirichlet boundary value problem for the Helmholtz equation at the point x^*, with

$$\left|u_n\left(x^*\right) - u\left(x^*\right)\right| \leq$$

$$\leq M\left[\frac{1}{\sqrt{n}} + \omega\left(\operatorname{grad} f, 1/n\right) + \int_0^{1/\sqrt{n}} \frac{\omega\left(\operatorname{grad} f,\, t\right)}{t}\, dt + \frac{1}{n} \int_{1/n}^{\operatorname{diam} L} \frac{\omega\left(\operatorname{grad} f,\, t\right)}{t^2}\, dt\right],$$

where

$$\varphi_n\left(x\left(\tau_l\right)\right) = -2 \sum_{j=1}^{n} \tilde{k}_{l\,j}^{-} \left(\sum_{p=1}^{n} \tilde{k}_{j\,p}^{+} \left(\sum_{m=1}^{n} t_{pm}\, f\left(x\left(\tau_m\right)\right)\right)\right).$$

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MEASURE OF NONCOMPACTNESS APPROACH TO NONLINEAR FRACTIONAL PANTOGRAPH DIFFERENTIAL EQUATIONS

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Abstract. The aim of this manuscript is to explore the existence and uniqueness of solutions for a class of nonlinear Ψ-Caputo fractional pantograph differential equations subject to nonlocal conditions. The proofs rely on key results in topological degree theory for condensing maps, coupled with the method of measures of noncompactness and essential tools in Ψ-fractional calculus. To support the theoretical findings, a nontrivial example is presented as an application.

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1 Introduction

The fractional calculus which permits the integration and differentiation of functions with non-integer orders, is one of the fastest-growing fields of mathematics due to the discovery that fractional operators were utilized in mathematical modeling, see [10, 19, 20, 27, 29]. Fractional differential equations, which can be used to model and describe non-homogeneous physical events, have recently attracted a lot of attention, particularly initial and boundary value problems for nonlinear fractional differential equations. Different researchers have found some interesting solutions to initial and boundary value problems for fractional differential equations involving various fractional derivatives, including their existence and uniqueness, such as Riemann-Liouville [23], Caputo [3], Hilfer [24], Erdelyi-Kober [25] and Hadamard [2]. There is a certain type of kernel dependency included in all those definitions. Therefore, a fractional derivative with respect to another function known as the Ψ-Caputo derivative was introduced in order to study fractional differential equations in a general manner. For specific selections of Ψ, we can obtain some well-known fractional derivatives, such as the Caputo, Caputo-Hadamard, or Caputo-Erdelyi-Kober fractional derivatives, which depend on a kernel. From the viewpoint of applications, this approach also seems appropriate. With the help of a good selection of a "trial" function Ψ, the Ψ-Caputo fractional derivative allows some measure of control over the modeling of the phenomenon under consideration. Almeida et al. [5] investigated the existence and uniqueness results for nonlinear fractional differential equations involving a Ψ-Caputo-type fractional derivative by using fixed point theorems and Picard iteration method. For more details, the reader can also consult [7, 16, 17, 18, 28] and references therein. In particular, the pantograph equation was employed as a useful tool to shed light on some of the modern problems originating from several scientific disciplines, including electrodynamics, probability, quantum mechanics, and number theory. However, a substantial investigation on the characteristics of this type of fractional differential equation, both analytical and numerical, has been done, and intriguing findings have been published in [1, 6, 8, 9, 13, 14, 15].

Inspired by the above recent results, in this paper, we investigate the existence and uniqueness of solutions to the following nonlinear fractional pantograph differential equation with Ψ-Caputo type fractional derivatives of order $\beta \in (1, 2)$:

$$\begin{cases} {}^{C}D_{0^+}^{\beta,\Psi}u(t) = h(t, u(t), u(\varepsilon t)), \qquad t \in J = [0, T], \\ \\ u'(0) = 0, \quad u(0) + \omega(u) = u_0, \end{cases} \tag{1.1}$$

where ${}^{C}D_{0^+}^{\beta,\Psi}$ is the $\Psi-$Caputo fractional derivative of u of order β, $\varepsilon \in (0, 1)$, $T > 0$, $h \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $u_0 \in \mathbb{R}$ and $\omega : C(J) \to \mathbb{R}$ is a nonlocal term satisfying some given conditions, which will be stated in Section 3. For more details we refer the reader to [14, 15].

To the best of the authors' knowledge, topological degree theory for condensing maps has not been applied to nonlinear pantograph differential equations with Ψ-Caputo fractional derivatives.

The structure of this paper is as follows. In Section 2, we give some basic definitions and preliminary results that we will need to prove our main results. In Section 3, we prove the existence of solutions for pantograph equation (1.1). After that, we give a concrete example to illustrate our main results in Section 4 and the last Section 5 contains conclusions on the results obtained in the paper.

2 Basic concepts

This section deals with some preliminaries and notations which are used throughout this paper. For more details we refer the reader to [4].

Definition 1. [5] Let $q > 0$, $g \in L^1(J, \mathbb{R})$ and $\Psi \in C^n(J, \mathbb{R})$ be such that $\Psi'(t) > 0$ for all $t \in J$. The Ψ-Riemann-Liouville fractional integral of order q of a function g is given by

$$I_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(q)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{q-1} g(s)ds, \tag{2.1}$$

where $\Gamma(.)$ is the Euler Gamma function.

Definition 2. [5] Let $q > 0$, $g \in C^{n-1}(J, \mathbb{R})$ and $\Psi \in C^n(J, \mathbb{R})$ such that $\Psi'(t) > 0$ for all $t \in J$. The Ψ-Caputo fractional derivative of order q of a function g is given by

$${}^{C}D_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{n-q-1} g_{\Psi}^{\{n\}}(s)ds, \tag{2.2}$$

where $g_{\Psi}^{\{n\}}(s) = \left(\frac{1}{\Psi'(s)}\frac{d}{ds}\right)^n g(s)$ and $n = [q] + 1$ ($[q]$ denotes the integer part of the real number q).

Remark 1. In particular, if $q \in]0, 1[$, then we have

$${}^{C}D_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} g'(s)ds,$$

and

$${}^{C}D_{0^+}^{q,\Psi}g(t) = I_{0^+}^{1-q,\Psi}\left(\frac{g'(t)}{\Psi'(t)}\right).$$

Proposition 2.1. [5] *Let $q > 0$, if $g \in C^{n-1}(J, \mathbb{R})$, then we have*

1) ${}^C D_{0^+}^{q,\Psi} I_{0^+}^{q,\Psi} g(t) = g(t)$.

2) $I_{0^+}^{q,\Psi} \; {}^C D_{0^+}^{q,\Psi} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g_\Psi^{[k]}(0)}{k!} (\Psi(t) - \Psi(0))^k$.

3) $I_{0^+}^{q,\Psi}$ *is linear and bounded from $\mathcal{C}$ to $\mathcal{C}$.*

Proposition 2.2. [5] *Let $\mu > \nu > 0$ and $t \in J$, then*

1) $I_{0^+}^{\mu,\Psi} (\Psi(t) - \Psi(0))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} (\Psi(t) - \Psi(0))^{\mu+\nu-1}$.

2) $D_{0^+}^{\mu,\Psi} (\Psi(t) - \Psi(0))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)} (\Psi(t) - \Psi(0))^{\nu-\mu-1}$.

3) $D_{0^+}^{\mu,\Psi} (\Psi(t) - \Psi(0))^{k} = 0, \quad \forall k \in \mathbb{N}$.

Definition 3. [11] Let X be a Banach space with the norm $\|.\|$ and $\mathfrak{B}_X$ be the family of all non-empty and bounded subsets of X. The Kuratowski measure of non-compactness is the mapping $\rho : \mathfrak{B}_X \to [0, +\infty[$ defined by: for any $A \in \mathfrak{B}_X$

$\rho(A) = \inf\{ r > 0\colon A$ admits a finite cover by sets of diameter $\leq r\}$.

Proposition 2.3. [11] *The Kuratowski measure of noncompactness ρ satisfies the following assertions: for any $A, A_1, A_2 \in \mathfrak{B}_X$*

1. $\rho(A) = 0$ *if and only if A is relatively compact.*
2. $\rho(kA) = |k|\rho(A), \quad k \in \mathbb{R}$.
3. $\rho(A_1 + A_2) \leq \rho(A_1) + \rho(A_2)$.
4. *If $A_1 \subset A_2$ then $\rho(A_1) \leq \rho(A_2)$.*
5. $\rho(A_1 \cup A_2) = \max\{\rho(A_1), \; \rho(A_2)\}$.
6. $\rho(A) = \rho(\overline{A}) = \rho(\mathrm{conv} A)$ *where $\overline{A}$ and $\mathrm{conv} A$ denote the closure and the convex hull of A, respectively.*

Definition 4. [11] Let $\varphi : \Omega \subset X \to X$ be a continuous bounded map. We say that φ is ρ-Lipschitz if there exists $l \geq 0$ such that

$$\rho(\varphi(A)) \leq l\rho(A), \quad \text{for every} \quad A \subset \Omega.$$

Moreover, if $l < 1$ then we say that φ is a strict ρ-contraction.

Definition 5. [11] We say that a function ω is ρ-condensing if

$$\rho(\omega(A)) < \rho(A),$$

for every bounded subset A of Ω with $\rho(A) > 0$. In other words

$$\rho(\omega(A)) \geq \rho(A) \Rightarrow \rho(A) = 0.$$

Definition 6. [11] We say that a function $g : \Omega \to X$ is Lipschitz if there exists $l > 0$ such that

$$\| g(u) - g(v) \| \leq l \| u - v \|, \quad \text{for all} \quad u, v \in \Omega.$$

Moreover, if $l < 1$ then we say that g is a strict contraction.

Lemma 2.1. [11] *If $L, F : \Omega \to X$ are ρ-Lipschitz mappings with the constants l_1 respectively l_2, then the mapping $F + L : \Omega \to X$ is ρ-Lipschitz with the constant $l_1 + l_2$.*

Lemma 2.2. [11] *If $g : \Omega \to X$ is compact, then g is ρ-Lipschitz with constant $c = 0$.*

Lemma 2.3. [11] *If $g : \Omega \to X$ is Lipschitz with constant l, then g is $\rho-$Lipschitz with the same constant l.*

Theorem 2.1. (See Isaia [22]). *Let $\mathcal{H} : X \to X$ be ρ-condensing and*

$$\mathcal{E}_\gamma = \{x \in X : x = \gamma \mathcal{H} x \text{ for some } 0 \leq \gamma \leq 1\}.$$

If $\mathcal{E}_\gamma$ is a bounded set in X, then there exists $r > 0$ such that $\mathcal{S}_\gamma \subset B_r = \{x \in X : ||x|| \leq r\}, r > 0$, and we have

$$\deg(I - \delta \mathcal{H}, B_r, 0) = 1, \quad \forall \delta \in [0, 1],$$

where $\deg(\cdot, \cdot, \cdot)$ denotes the topological degree in the sense of Leray-Schauder.

As a consequence, the operator $\mathcal{H}$ has at least one fixed point and the set of all fixed points of $\mathcal{H}$ lies in B_r .

3 Main results

We start this section by introducing necessary notations and hypotheses on the functions $\omega \in C(\mathbb{R}, \mathbb{R})$ and $h \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, entering equation (1.1).

• We denote by $\mathcal{C} := C(J, \mathbb{R})$ the space of continuous real-valued functions defined on J provided with the maximum norm

$$\| u \| = \max_{t \in J} |u(t)|.$$

• We denote by B_η the closed ball in $\mathcal{C}$ centred at 0 of radius $\eta > 0$.

(H_1) There exists a constant $L_\omega > 0$ such that

$$|\omega(u) - \omega(v)| \leq L_\omega \|u - v\|, \text{ for each } u, v \in \mathcal{C}.$$

(H_2) There exist two constants $K_\omega, M_\omega > 0$ and $q \in (0, 1)$ such that

$$|\omega(u)| \leq K_\omega \|u\|^q + M_\omega \text{ for each } u \in \mathcal{C}.$$

(H_3) There exist two constants $K_h, M_h > 0$ and $p \in (0, 1)$ such that

$$|h(t, u(t), u(\varepsilon t))| \leq K_h \mid u(t) \mid^p + M_h \text{ for each } u \in \mathcal{C}, \epsilon \in (0, 1).$$

Lemma 3.1. *A function $u \in \mathcal{C}$ is a solution of (1.1) if and only if u satisfies the following fractional integral equation*

$$u(t) = u_0 - \omega(u) + \frac{1}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\beta - 1} h(s, u(s), u(\varepsilon s)) ds, t \in J. \tag{3.1}$$

Proof. Let u be a solution to (1.1), then by applying Ψ-fractional integral $I_{0^+}^{\beta,\Psi}$ to both sides of (1.1) we obtain

$$I_{0^+}^{\beta,\Psi}\ {}^{C}D_{0^+}^{\beta,\Psi} u(t) = I_{0^+}^{\beta,\Psi} h(t, u(t), u(\varepsilon t)),$$

and by employing Proposition 2.1 we get

$$u(t) = c_0 + (\Psi(t) - \Psi(0))c_1 + I_{0^+}^{\beta,\Psi} h(t, u(t), u(\varepsilon t)),$$

where $c_0, c_1 \in \mathbb{R}$, hence,

$$u'(t) = c_1\Psi'(t) + \frac{1}{\Gamma(\beta)} \int_0^t \left(\Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} h(s, u(s), u(\varepsilon s))\right)'_t ds.$$

Since $u(0) + \omega(u) = u_0$ and $u'(0) = 0$, then $c_0 = u_0 - \omega(u)$ and $c_1 = 0$. Hence, (3.1) holds.

Conversely, by simple calculus, it is clear that if u satisfies (3.1), then (1.1) holds. □

To prove that (3.1) has at least one solution $u \in \mathcal{C}$, we consider two operators $\mathcal{B}, \mathcal{A} : \mathcal{C} \to \mathcal{C}$ defined by

$$\mathcal{A}u(t) = u_0 - \omega(u(t)), \quad t \in J, \tag{3.2}$$

and

$$\mathcal{B}u(t) = \frac{1}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} h(s, u(s), u(\varepsilon s))ds, \quad t \in J, \tag{3.3}$$

thus (3.1) can be formulated as follows

$$u(t) = \mathcal{F}u(t) := \mathcal{A}u(t) + \mathcal{B}u(t), \quad t \in J. \tag{3.4}$$

Theorem 3.1. *Assume that hypotheses $(H_1) - (H_3)$ are satisfied, then fractional pantograph differential equation (1.1) has at least one solution $u \in C(J, \mathbb{R})$. Moreover, the set of all solutions to (1.1) is bounded in $C(J, \mathbb{R})$.*

In order to prove the Theorem 3.1, we will need to show some lemmas and preliminary results under the assumption that hypotheses $(H_1) - (H_3)$ are satisfied.

Lemma 3.2. *The operator $\mathcal{A}$ is ρ-Lipschitz with the constant L_ω. Moreover, $\mathcal{A}$ satisfies the following inequality:*

$$\|\mathcal{A}u\| \leq |u_0| + K_\omega \|u\|^q + M_\omega, \text{for every } u \in \mathcal{C}. \tag{3.5}$$

Proof. To prove that the operator $\mathcal{A}$ is Lipschitz with the constant L_ω,we argue as follows.

Let $u, v \in \mathcal{C}$, then we have

$$|\mathcal{A}u(t) - \mathcal{A}v(t)| \leq |\omega(u) - \omega(v)|,$$

by using hypothesis (H_1) we get

$$|\mathcal{A}u(t) - \mathcal{A}v(t)| \leq L_\omega \|u - v\|.$$

Taking supremum over t, we obtain

$$\|\mathcal{A}u - \mathcal{A}v\| \leq L_\omega \|u - v\|,$$

hence, $\mathcal{A}$ is Lipschitz with the constant L_ω. By using Lemma 2.3, it follows that $\mathcal{A}$ is ρ-Lipschitz with the same constant L_ω.

To prove (3.5), let $u \in \mathcal{C}$, then we have

$$|\mathcal{A}u(t)| = |u_0 - \omega(u)| \leq |u_0| + |\omega(u)|,$$

by using assumption (H_2) we get

$$\|\mathcal{A}u\| \leq |u_0| + K_\omega \|u\|^q + M_\omega.$$

□

Lemma 3.3. *The operator $\mathcal{B}$ is continuous and the following inequality holds*

$$\|\mathcal{B}u\| \leq \frac{1}{\Gamma(\beta+1)}(K_\omega \|u\|^p + M_\omega)(\Psi(T) - \Psi(0))^\beta, \quad \forall u \in \mathcal{C}. \tag{3.6}$$

Proof. To prove that the operator $\mathcal{B}$ is continuous, let a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ converge to u in $\mathcal{C}$, it follows that there exists $\delta > 0$ such that $\|u_n\| \leq \delta$ and $\|u\| \leq \delta$. Now let $t \in J$, then we have

$$|\mathcal{B}u_n(t) - \mathcal{B}u(t)| \leq \frac{1}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} \left|h(s, u_n(s), u_n(\varepsilon s)) - h(s, u(s), u(\varepsilon s))\right| ds.$$

Since h is continuous, we have

$$\lim_{n\to\infty} h(s, u_n(s), u_n(\varepsilon s)) = h(s, u(s), u(\varepsilon s)).$$

On the other hand, by using (H_3) we obtain

$$\begin{aligned}\frac{1}{\Gamma(\beta)}(\Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} \left\|h(s, u_n(s), u_n(\varepsilon s)) - h(s, u(s), u(\varepsilon s))\right\| &\leq (K_\omega \delta^p + M_\omega) \\ &\times \frac{1}{\Gamma(\beta)}(\Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1},\end{aligned}$$

since $s \mapsto \frac{1}{\Gamma(\beta)}(\Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1}$ is an integrable function on $[0, t]$, then Lebesgue dominated convergence theorem implies that

$$\lim_{n\mapsto+\infty} \frac{1}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} \left\|h(s, u_n(s), u_n(\varepsilon s)) - h(s, u(s), u(\varepsilon s))\right\| ds = 0.$$

It follows that

$$\lim_{n\mapsto+\infty} \| \mathcal{B}u_n - \mathcal{B}u \| = 0,$$

hence, $\mathcal{B}$ is continuous .

To show (3.6), let $u \in \mathcal{C}$, then we have

$$|\mathcal{B}u(t)| \leq \frac{1}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} \left|h(s, u(s), u(\varepsilon s))\right| ds,$$

from (H_3) we obtain

$$|\mathcal{B}u(t)| \leq \frac{(K_\omega \|u\|^p + M_\omega)}{\Gamma(\beta)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\beta-1} ds,$$

Finally, we obtain

$$\| \mathcal{B}u \| \leq \frac{(K_\omega \|u\|^p + M_\omega)(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)}.$$

□

Lemma 3.4. *$\mathcal{B} : \mathcal{C} \to \mathcal{C}$ is a compact operator.*

Proof. In order to demonstrate the compactness of $\mathcal{B}$ we need to show that $\mathcal{B}B_\eta$ is relatively compact in $\mathcal{C}$ and we use the Arzela-Ascoli Theorem [21]. Let $u \in B_\eta$, then from (3.6) we get

$$\| \mathcal{B}u \| \leq \frac{(K_\omega \eta^p + M_\omega)(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)} := \xi.$$

So, it follows that $\mathcal{B}B_\eta \subset B_\xi$. Hence $\mathcal{B}B_\eta$ is bounded.

To prove that $\mathcal{B}B_\eta$ is is uniformly equicontinuous, let $u \in \mathcal{B}B_\eta$ and $t_1, t_2 \in J$ such that $t_1 < t_2$, then we have

$$|\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| \leq \frac{K_\omega \mid u \mid^p + M_\omega}{\Gamma(\beta)} \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\beta-1} ds,$$

$$|\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| \leq \frac{K_\omega \eta^p + M_\omega}{\Gamma(\beta)} \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\beta-1} ds,$$

$$|\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| \leq \frac{K_\omega \eta^p + M_\omega}{\Gamma(\beta+1)} (\Psi(t_2) - \Psi(t_1))^\beta,$$

$$\sup_{u \in \mathcal{B}B_\gamma} \sup_{|t_1 - t_2| \leq \delta} |\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| \leq \frac{K_\omega \eta^\beta + M_\omega}{\Gamma(\beta+1)} \sup_{|t_1 - t_2| \leq \delta} |\Psi(t_2) - \Psi(t_1)|^\beta.$$

Since Ψ is a continuous function on the closed interval J, then we obtain

$$\lim_{\delta \to 0^+} \sup_{u \in \mathcal{B}B_\gamma} \sup_{|t_1 - t_2| \leq \delta} |\mathcal{B}u(t_1) - \mathcal{B}u(t_2)| = 0.$$

which shows that $\mathcal{B}B_\eta$ is uniformly equicontinuous.

Hence, $\mathcal{B}B_\eta$ is uniformly bounded and is uniformly equicontinuous. Arzelà–Ascoli Theorem [21] permits us to conclude that $\mathcal{B}B_\eta$ is relatively compact, thus $\mathcal{B}$ is compact. □

Corollary 3.1. *$\mathcal{B} : \mathcal{C} \to \mathcal{C}$ is ρ-Lipschitz with zero constant.*

Proof. Since the operator $\mathcal{B}$ is compact and by Lemma 2.2 it follows that $\mathcal{B}$ is ρ-Lipschitz with zero constant. □

Now, we have all tools to establish the proof of Theorem 3.1.
Proof of Theorem 3.1.

Let $\mathcal{A}, \mathcal{B}, \mathcal{F}$: $\mathcal{C} \to \mathcal{C}$ be the operators given by equations (3.2),(3.3) and (3.4) respectively.

The operators $\mathcal{A}, \mathcal{B}, \mathcal{F}$ are continuous and bounded. Moreover, by using Lemma 3.2 we have that $\mathcal{A}$ is ρ-Lipschitz with constant $L_\omega \in [0,1)$ and by using Corollary 3.1 we have that $\mathcal{A}$ is ρ-Lipschitz with zero constant. It follows from Lemma 2.1 that $\mathcal{F}$ is a strict ρ-contraction with constant L_ω.

We consider the following set

$$\mathcal{S}_\gamma = \{u \in \mathcal{C} : u = \gamma \mathcal{F} u \ \text{ for some } \ \gamma \in [0,1]\}.$$

Let us show that $\mathcal{S}_\gamma$ is bounded in $\mathcal{C}$. For this purpose let $u \in \mathcal{S}_\gamma$, then $u = \gamma\mathcal{F}u = \gamma(\mathcal{A}u + \mathcal{B}u)$. It follows that

$$\|u\| = \gamma\|\mathcal{F}u\| \leq \gamma(\|\mathcal{A}u\| + \|\mathcal{B}u\|),$$

by using Lemmas 3.2 and 3.3 we get

$$\|u\| \leq |u_0| + K_\omega\|u\|^q + M_\omega + \frac{(K_h\|u\|^p + M_h)(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)}. \tag{3.7}$$

From inequality (3.7) we deduce that $\mathcal{S}_\gamma$ is bounded in $\mathcal{C}$ with $p < 1$ and $q < 1$.

If this is not the case, we suppose that $\xi := \|u\| \longrightarrow \infty$. Dividing both sides of (3.7) by ξ, and taking $\xi \to \infty$, it follows that

$$1 \leq \lim_{\xi \to \infty} \frac{|u_0| + K_\omega \xi^q + M_\omega + \dfrac{(\mathbb{K}_\omega \xi^p + M_\omega)(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)}}{\xi} = 0,$$

which is a contradiction. By using Theorem 2.1 we conclude that $\mathcal{F}$ has at least one fixed point which is a solution of (1.1) and the set of the fixed points of $\mathcal{F}$ is bounded in $\mathcal{C}$. $\square$

Remark 2. Note that if assumptions (H_2) and (H_3) are formulated for $q = 1$ and $p = 1$, then the conclusions of Theorem 3.1 remain valid provided that

$$K_\omega + \frac{K_h(\Psi(T) - \Psi(0))^\beta}{\Gamma(\beta+1)} < 1.$$

4 An illustrative example

In this section, we give an example to illustrate the usefulness of our main result.
Consider the following problem:

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2}, e^t} u(t) = h(t, u(t), u(\varepsilon t)), \quad t \in J = [0,1], \\ \\ u'(0) = 0, \quad u(0) = \displaystyle\sum_{j=1}^{20} \theta_j |u(t_j)|, \quad \theta_j > 0, \quad 0 < t_j < 1, \quad j = 1, 2, .., 20, \end{cases} \tag{4.1}$$

where $h(t, u(t), u(\varepsilon t)) = \frac{\sin\left(u\left(\frac{t}{\sqrt{2}}\right)\right)}{(9+e^t)\sqrt{2}} \left(\frac{|u(t)|}{1+\left|u\left(\frac{t}{\sqrt{2}}\right)\right|}\right)$.

Here $\varepsilon = \frac{1}{\sqrt{2}}$, $\beta = \frac{3}{2}$, $T = 1$, $\Psi(t) = e^t$, and $\omega(u) = \displaystyle\sum_{j=1}^{20} \theta_j |u(t_j)|$ with $\displaystyle\sum_{j=1}^{20} \theta_j < 1$. Clearly $h \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and (H_1), (H_2) hold with $K_\omega = L_\omega = \displaystyle\sum_{i=j}^{20} \theta_j$, $M_\omega = 0$ and $q = 1$. Indeed, we can write

$$|\omega(u)| \leq \sum_{j=1}^{20} \theta_j \|u\|,$$

thus, $K_\omega = \sum\limits_{j=1}^{20} \theta_j, \quad M_\omega = 0$ and $q = 1$.

Moreover, we have

$$|\omega(u(t)) - \omega(v(t))| = \Big| \sum_{j=1}^{20} \theta_j |u(t_j)| - \sum_{j=1}^{20} \theta_j |v(t_j)| \Big|,$$

hence,

$$|\omega(u) - \omega(v)| \leq \sum_{j=1}^{20} \theta_j \|u - v\|,$$

thus, in (H_2), $L_\omega = \sum\limits_{j=1}^{20} \theta_j$.

To prove (H_3), let $t \in J$ and $u \in \mathbb{R}$, then we have

$$|h(t, u(t), u(\varepsilon t))| = \left| \frac{sin\left(u\left(\frac{t}{\sqrt{2}}\right)\right)}{(9+e^t)\sqrt{2}} \left(\frac{|u(t)|}{1 + \left|u\left(\frac{t}{\sqrt{2}}\right)\right|} \right) \right|,$$

$$|h(t, u(t), u(\varepsilon t))| \leq \frac{1}{10\sqrt{2}} \left(|u| + 1\right).$$

Thus, (H_3) holds with $K_h = M_h = \dfrac{1}{10\sqrt{2}}$ and $p = 1$.

Consequently, Theorem 3.1 implies that problem (4.1) has at least one solution. Moreover, by inequality (3.7) we have

$$\|u\| \leq \frac{(e-1)^{(3/2)}}{10\sqrt{2}\Gamma(8/3) - 1} \approx 0.19,$$

thus, the set of all solutions to (4.1) is bounded.

5 Conclusion

In this paper, we studied the existence of solutions to nonlinear pantograph differential equations involving Caputo type fractional derivative with respect to another function Ψ. The proofs of our proposed model are based on the topological degree theory for condensing maps. We also provided an example to make our results clear.

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TWO-WEIGHT HARDY INEQUALITY ON TOPOLOGICAL MEASURE SPACES

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Abstract We consider a Hardy type integral operator T associated with a family of open subsets $\Omega(t)$ of an open set Ω in a Hausdorff topological space X. In the inequality

$$\left(\int_{\Omega}|Tf(x)|^{q}u(x)d\mu(x)\right)^{1/q}\leq C\left(\int_{\Omega}|f(x)|^{p}v(x)d\nu(x)\right)^{1/p},$$

the measures μ,ν are σ-additive Borel measures; the weights u,v are positive and finite almost everywhere, $1<p<\infty$, $0<q<\infty$, and $C>0$ is independent of f, u, v, μ, ν. We find necessary and sufficient conditions for the boundedness and compactness of the operator T and obtain two-sided estimates for its approximation numbers. All results are proved using domain partitions, thus providing a roadmap for generalizing many one-dimensional results to a Hausdorff topological space.

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1 Introduction

A one-dimensional Hardy inequality

$$\left[\int_{0}^{\infty}u(x)\left(\int_{0}^{x}f(y)\,dy\right)^{q}d\mu(x)\right]^{1/q}\leq C\left(\int_{0}^{\infty}f^{p}(y)\,v(y)\,d\nu(y)\right)^{1/p}$$

has been studied in detail and complete characterizations of its validity for all non-negative functions f have been obtained in terms of pairs of weights u,v and measures μ,ν for all pairs of exponents p,q, see [11], [12], [13], [14], [20] for the history and extensive references. By a characterization we mean obtaining a functional $\Phi(u,v,\mu,\nu)$ such that for all weights and measures the inequality $c_1C\leq\Phi(u,v,\mu,\nu)\leq c_2C$ is true, where C is the best constant in the above inequality and $c_1,\,c_2>0$ can depend on p,q but are not allowed to depend on u,v,μ,ν. Those characterizations are very different for the cases $p\leq q$ and $q<p$.

In the one-dimensional case most researchers have used tools of one-dimensional calculus, such as integration by parts [33]. The lack of such tools has been the main obstacle on the way to multidimensional results. Some general results for $p\leq q$ and for Banach function spaces have been established in [7]. Obtaining full characterizations has been facilitated by the possibility to reduce the multidimensional case to the one-dimensional, by using spherical coordinates [31], [4], the polar decomposition [26], [27] or assuming that the weights are products of functions of one variable [35],

[36]. The result by Sawyer [29] does not allow reduction to dimension one but is limited to a quadrant on the plane R^2.

In a recent paper Sinnamon [32] suggested a very general method that covers totally ordered sets of domains on a measure space. The method relies on a non-increasing rearrangement involving the weights and measures and reduces the multidimensional case to the one-dimensional. Apart from generality, the method allows Sinnamon to improve the constants c_1, c_2. The analysis of ordered cores is of independent interest.

For applications it is desirable to have everything to be expressed in terms of original weights and measures, the most important examples being the Hardy-Steklov type operator [9] and the Hardy inequality on cones of monotone functions [30], [34]. In Sinnamon's method one additional step is required to derive the criteria in terms of original weights and measures from his one-dimensional formulations. K. Mynbaev [21] has obtained results in terms of original weights and measures under the assumptions on the domains that are close to the ones imposed by Sinnamon (see [21, Remark 1] for a more detailed comparison with Sinnamon's paper).

Here we develop a different approach to the norm estimation, compactness conditions and bounds for approximation numbers using domain partitions. The boundedness criteria obtained below can be derived from both [32] and [21]. Nevertheless, we give full proofs of boundedness, compactness and estimates of approximation numbers to show that domain partitions combined with the conditions on the operator T imposed here allow one to extend many of the existing one-dimensional results to the current setup in a Hausdorff space. Possible extensions include results that employ the Oinarov condition [22], [15]. Since Sinnamon's approach covers also discrete Hardy inequalities, it would be interesting to see if the results of [16] can deduced following Sinnamon.

We consider integration over expanding subsets $\Omega(t)$ of an arbitrary open set Ω in a Hausdorff topological space X with σ-additive Borel measures μ, ν. As in [32] and [21], neither $\Omega(t)$ nor their complements $\Omega \backslash \Omega(t)$ need to be connected and there are no requirements on the shape of $\Omega(t)$ when X is a linear space. In the classical case one can notice that the subdomain $\Omega(t) = (0, t)$ of $\Omega = (0, \infty)$ has $\omega(t) = t$ as the boundary in the relative topology and that $\Omega(t) = \{s \in \Omega : \omega(s) < \omega(t)\}$. Our conditions on the family $\{\Omega(t)\}$ are based on this observation.

The existing results on integral Hardy inequalities for R^n or measure metric spaces (in which $\Omega(t)$ are balls, see [4], [2], [26], [27], [28]) follow from ours, as well as from [32] and [21]. Product weights are not included as well as Sawyer's result [29] (his rectangles do not satisfy condition (2.1) below). In papers [26], [27], [28] a metric is required to generate balls and a polar decomposition to use the one-dimensional techniques, while we avoid these requirements. There is a number of situations (homogeneous groups, hyperbolic spaces, Cartan-Hadamard manifolds, and connected Lie groups) when the polar decomposition is available, see also [1] for a study of polarizable metric measure spaces. All such situations are covered by our statements. The authors of the article [28] employ the results from [21].

Unlike [32] and [21], our approach is elementary and does not require any advanced measure theory beyond σ-additivity. Note that binary partitions were used to prove sufficiency for the Hardy operator in the one-dimensional case in [3]. Unlike [3], we avoid their auxiliary functions Φ and Φ_1 and apply discretization both for the upper and lower bounds in terms of the same functional of the weights.

Note that we provide two different proofs of the sufficiency of the compactness conditions: in Sections 3 and 4. Both employ an explicit finite-rank approximation to the Hardy operator.

The study of the approximation numbers (a-numbers) of the Hardy operator in the Lebesgue spaces on the half-line for parameters satisfying $1 < p \leq q < \infty$ started with the papers by D.E. Edmunds, W.D. Evans, D.J. Harris [5], [6]. They found implicit and asymptotic bounds for a-numbers of the operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$. Next D.E. Edmunds, V.D. Stepanov [8] obtained the bounds for singular numbers of the Hardy type operator with a polynomial kernel acting in the spaces $L^2(\mathbb{R}^+)$. Those results were extended by E.N. Lomakina, V.D. Stepanov [19] to the case $1 < p, q < \infty$; besides, two-sided bounds for the Schatten–von-Neumann norm were proved. However, in the case $1 < q < p < \infty$ the upper bound for the a-numbers $a_N(T) \le N^{1/q-1/p}\varepsilon$ was not informative because of its dependence on N. In this paper for $1 < q < p < \infty$ we derive an upper bound that does not depend on N. We do not consider the case $0 < q < 1 < p < \infty$ studied by E.N. Lomakina [17], nor do we attempt to study the Hardy operator acting from the Lebesgue spaces to the Lorentz spaces in the spirit of [18].

2 Hardy operator boundedness

We write $A \asymp B$ to mean that $c_1 A \le B \le c_2 A$ with constants c_1, c_2 that do not depend on weights and measures.

Assumption 1. *Let Ω be an open subset of a Hausdorff topological space X with σ-additive measures μ, ν. The measures are defined on a σ-algebra $\mathfrak{M}$ that contains the Borel-measurable sets. The weights u, v are assumed to be positive and finite almost everywhere.*

Assumption 2. *a) $\{\Omega(t) : t \ge 0\}$ is a one-parametric family of open subsets of Ω which satisfy monotonicity*

$$\text{for } t_1 < t_2,\ \Omega(t_1) \text{ is a proper subset of } \Omega(t_2). \tag{2.1}$$

b) $\Omega(t)$ start at the empty set and eventually cover almost all Ω:

$$\Omega(0) = \bigcap_{t>0} \Omega(t) = \varnothing,\ \nu\left(\Omega \setminus \bigcup_{t>0} \Omega(t)\right) = 0.$$

c) Further, denote by $\omega(t) = \overline{\Omega(t)} \bigcap \overline{(\Omega \setminus \Omega(t))}$ the boundary of $\Omega(t)$ in the relative topology. We require the boundaries to be disjoint and cover almost all Ω:

$$\omega(t_1) \bigcap \omega(t_2) = \varnothing,\ t_1 \ne t_2,\ \nu(\Omega \setminus \bigcup_{t>0} \omega(t)) = 0. \tag{2.2}$$

d) Passing to a different parametrization, if necessary, we can assume that

$$\nu\left(\Omega \setminus \bigcup_{t \le N} \omega(t)\right) > 0 \text{ for any } N < \infty. \tag{2.3}$$

e) Finally, we assume that boundaries are thin in the sense that

$$\nu(\omega(t)) = 0 \text{ for all } t > 0. \tag{2.4}$$

This Assumption has simple implications.

1) (2.2) implies that for ν-almost each $y \in \Omega$ there exists a unique $\tau(y) > 0$ such that $y \in \omega(\tau(y))$, which allows us to define

$$Tf(y) = \int_{\Omega(\tau(y))} f d\nu,\ y \in \Omega, \tag{2.5}$$

for any non-negative $\mathfrak{M}$-measurable f. On the set $\Omega_0 \subset \Omega$ of those y for which $\tau(y)$ is not defined we can put $\tau(\Omega_0) = \varnothing$. (A more general definition of a Hardy-type operator is given in [7]. That definition is more difficult to work with what we call slices.)

2) (2.3) and the fact that $\omega(t) \ne \varnothing,\ t > 0$, lead to the equality $\tau(\Omega) = (0, \infty)$.

3) Because of (2.4) $\int_{\Omega(t)} f d\nu = \int_{\overline{\Omega(t)}} f d\nu$ and up to a set of ν-measure zero

$$\{x \in \Omega : \tau(x) > \tau(y)\} = \Omega \backslash \Omega(\tau(y)). \tag{2.6}$$

For $0 \leq a < b \leq \infty$ we denote $\Omega([a,b]) = \Omega(b) \setminus \Omega(a)$.

Since $\tau(y_1) = \tau(y_2)$ for any $y_1, y_2 \in \omega(t)$, the value $Tf(y)$ is the same for all $y \in \omega(t)$ and we can define $Sf(t) = Tf(y)$ if $y \in \omega(t)$. For a non-negative f, the function Sf is non-decreasing and its jumps are zero due to (2.4). Thus,

$$\text{for each } f \geq 0,\ Sf \text{ is continuous where it is finite, including } t = 0. \tag{2.7}$$

Let $L^p_{vd\nu}(\Omega)$ denote the space with the norm $\|f\|_{L^p_{vd\nu}(\Omega)} = \left(\int_\Omega |f|^p v d\nu\right)^{1/p}$ where v is a weight function and let $\|T\| = \|T\|_{L^p_{vd\nu}(\Omega) \to L^q_{ud\mu}(\Omega)}$ be the norm of a linear operator T acting from $L^p_{vd\nu}(\Omega)$ to $L^q_{ud\mu}(\Omega)$, hence

$$\left[\int_\Omega \left|\int_{\Omega(\tau(x))} f d\nu\right|^q u(x) d\mu(x)\right]^{1/q} \leq \|T\| \left(\int_\Omega |f|^p \, v d\nu\right)^{1/p}.$$

Denote

$$\Psi(t) = \left(\int_{\Omega \backslash \Omega(t)} u d\mu\right)^{1/q} \left(\int_{\Omega(t)} v^{-p'/p} d\nu\right)^{1/p'}.$$

Theorem 2.1. *If $1 < p \leq q < \infty$, then (2.5) is bounded if and only if $A < \infty$, where $A = \sup_{t>0} \Psi(t)$. Moreover, $A \leq \|T\| \leq 4A$.*

Proof. Lower bound. Let an operator $T : L^p_{vd\nu}(\Omega) \to L^q_{ud\mu}(\Omega)$ be bounded, then there exists a constant $C > 0$ such that $\|Tf\|_{L^q_{ud\mu}(\Omega)} \leq C\|f\|_{L^p_{vd\nu}(\Omega)}$.

Put $f_y(z) = v^{-p'/p}(z) \chi_{\Omega(\tau(y))}(z),\ y \in \Omega$. Then

$$Tf_y(x) = \int_{\Omega(\tau(x)) \cap \Omega(\tau(y))} v^{-p'/p} d\nu = \int_{\Omega(\tau(y))} v^{-p'/p} d\nu, \quad \text{for} \quad \tau(x) > \tau(y)$$

and

$$C\|f_y\|_{L^p_{vd\nu}(\Omega)} \geq \|Tf_y\|_{L^q_{ud\mu}(\Omega)}.$$

Therefore, by applying (2.6) and $\tau(\Omega) = (0, \infty)$ we see that

$$\begin{aligned} C &\geq \sup_{y \in \Omega} \frac{\left(\int_\Omega (Tf_y)^q u d\mu\right)^{1/q}}{\left(\int_\Omega f_y^p v d\nu\right)^{1/p}} \geq \sup_{y \in \Omega} \frac{\left(\int_{\{x : \tau(x) > \tau(y)\}} u d\mu\right)^{1/q} \int_{\Omega(\tau(y))} v^{-p'/p} d\nu}{\left(\int_{\Omega(\tau(y))} v^{-p'/p} d\nu\right)^{1/p}} \\ &= \sup_{y \in \Omega} \left(\int_{\Omega \backslash \Omega(\tau(y))} u d\mu\right)^{1/q} \left(\int_{\Omega(\tau(y))} v^{-p'/p} d\nu\right)^{1/p'} = A \end{aligned}$$

and $\|T\|_{L^p_{vd\nu}(\Omega) \to L^q_{ud\mu}(\Omega)} = \inf C \geq A$.

Upper bound. Without loss of generality we suppose that $0 < \sup_{y \in \Omega} Tf(y) < \infty$. Put $t_0 = \infty$, $\Omega(\infty) = \Omega$. By (2.7) $Sf(t) \to 0$, $t \to 0$, so the definition $t_1 = \sup\{t > 0 : 2Sf(t) \leq Sf(t_0)\}$ is correct. By the continuity of Sf, we have $2Sf(t_1) = Sf(t_0)$ and $t_1 < t_0$. By induction, if t_k has been defined, we put $t_{k+1} = \sup\{t > 0 : 2Sf(t) \leq Sf(t_k)\}$. Then

$$2Sf(t_{k+1}) = Sf(t_k),\ t_{k+1} < t_k. \tag{2.8}$$

This can be called a sliding property because it allows us to pass from $Sf(t_k)$ to $Sf(t_{k+j})$. Defining slices

$$s_{k+1} = \Omega(t_k)\backslash\Omega(t_{k+1}),\ k \geq 0,$$

we have

$$\begin{aligned} Sf(t_{k+1}) &= 2Sf(t_{k+1}) - Sf(t_{k+1}) \\ &= Sf(t_k) - Sf(t_{k+1}) = \int_{s_{k+1}} f d\nu,\ k \geq 0. \end{aligned} \tag{2.9}$$

Let $y \in s_{k+1}$ or, equivalently, $t_{k+1} \leq \tau(y) < t_k$. Using (2.8), (2.9) and Hölder's inequality we have

$$\begin{aligned} Tf(y) &= \int_{\Omega(\tau(y))} f d\nu \leq Sf(t_k) = 2Sf(t_{k+1}) = 4Sf(t_{k+2}) \\ &= 4\int_{s_{k+2}} f d\nu \leq 4\left(\int_{s_{k+2}} f^p v d\nu\right)^{1/p}\left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{1/p'}. \end{aligned} \tag{2.10}$$

Denote

$$\alpha_k = \left(\int_{s_{k+1}} u d\mu\right)^{1/q}\left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{1/p'}.$$

We can use (2.10) and the inequality $p \leq q$ to estimate

$$\begin{aligned} \left(\int_\Omega (Tf)^q u d\mu\right)^{1/q} &= \left(\sum_{k\geq 0}\int_{s_{k+1}} (Tf)^q u d\mu\right)^{1/q} \\ &\leq 4\left[\sum_{k\geq 0}\int_{s_{k+1}} u d\mu \left(\int_{s_{k+2}} f^p v d\nu\right)^{q/p}\left(\int_{s_{k+2}} v^{-p'/p} d\nu\right)^{q/p'}\right]^{1/q} \\ &= 4\left[\sum_{k\geq 0}\alpha_k^q\left(\int_{s_{k+2}} f^p v d\nu\right)^{q/p}\right]^{1/q} \leq 4\sup_k \alpha_k\left(\sum_{k\geq 0}\int_{s_{k+2}} f^p v d\nu\right)^{1/p} \\ &\leq 4A\,\|f\|_{L^p_{vd\nu}(\Omega)}. \end{aligned} \tag{2.11}$$

The last transition uses the following inclusions

$$s_{k+1} = \Omega(t_k)\backslash\Omega(t_{k+1}) \subset \Omega\backslash\Omega(t_{k+1}) \quad \text{and} \quad s_{k+2} = \Omega(t_{k+1})\backslash\Omega(t_{k+2}) \subset \Omega(t_{k+1}).$$

If $\sup_{y\in\Omega} Tf(y) = \infty$ we can choose $t < \infty$ such that $\int_{\Omega(t)} f d\nu < \infty$, put $f_t(x) = \chi_{\Omega(t)}(x) f(x)$, and do all calculations leading to (2.11) with f_t in place of f. Since the constant in (2.11) does not depend on t, then we can let $t \to \infty$ thus completing the proof. $\square$

Let u_0, v_0 be non-negative integrable functions such that $u_0 \leq u$, $v_0 \leq v^{1-p'}$. We can assume that $0 < \int_\Omega v_0 d\nu < \infty$ and by analogy with (2.8) define the points $t_0 = \infty > t_1 > ...$, where $t_1 = \sup\{t > 0 : 2Sv_0(t) \leq Sv_0(t_0)\}, ..., t_{k+1} = \sup\{t > 0 : 2Sv_0(t) \leq Sv_0(t_k)\}$ such that $\Omega(\infty) = \Omega$ and

$$\int_{\Omega(t_k)} v_0 d\nu = 2^{-k}\int_\Omega v_0 d\nu,\ k \geq 0. \tag{2.12}$$

This implies the following equality (as before, $s_{k+1} = \Omega(t_k)\backslash\Omega(t_{k+1})$):

$$\int_{s_{k+1}} v_0 d\nu = 2\int_{s_{k+2}} v_0 d\nu.$$

The partition $\{t_k\}$ generates non-negative numbers

$$\begin{aligned} V_k &= \int_{s_{k+1}} v_0 d\nu,\ U_k = \int_{s_{k+1}} u_0 d\mu, \\ x_k &= \int_{\Omega(t_k)} v_0 d\nu = \sum_{j\geq k} V_j,\quad y_k = \int_{\Omega\backslash\Omega(t_{k+1})} u_0 d\mu = \sum_{j\leq k} U_j,\ \ k\geq 0. \end{aligned}$$

Here $\{x_k\}$ is non-increasing and $\{y_k\}$ is non-decreasing. We need the identities

$$\frac{r}{q} - 1 = \frac{r}{p},\quad \frac{r}{p'q} - 1 = \frac{r}{pq'},\quad \frac{r}{p'} - 1 = \frac{r}{q'}. \tag{2.13}$$

The next lemma provides a replacement for the one-dimensional techniques mentioned in the Introduction.

Lemma 2.1. *Let $a \geq 1$.*

a) We have

$$\sum_{j\geq k}\left(\sum_{i\geq j+1} V_i\right)^{a-1} V_{j+1} \geq \frac{1}{a}\left(\sum_{i\geq k+1} V_i\right)^a$$

for any non-negative numbers V_i such that the left side is finite.

b) Moreover,

$$\left(\sum_{i\geq k} V_i\right)^{a-1} V_{k-1} \leq \frac{1}{a}\left[\left(\sum_{i\geq k-1} V_i\right)^a - \left(\sum_{i\geq k} V_i\right)^a\right],\ \ k\geq 1.$$

For this inequality to be true for $k=0$ we formally put $V_{-1}=0$ so that

$$x_{-1} = \sum_{i\geq -1} V_i = \sum_{i\geq 0} V_i = x_0$$

and the inequality holds trivially.

c) For the partition $\{t_k\}$ one has

$$y_{k+1}^{r/q} - y_k^{r/q} \leq \frac{r}{q}\left(\int_{\Omega\backslash\Omega(t_{k+2})} u_0 d\mu\right)^{r/p}\int_{s_{k+2}} u_0 d\mu.$$

Proof. Let $g(x) = x^a$.

a) By the mean value theorem with some $\theta \in (x_{j+2}, x_{j+1})$

$$\begin{aligned} \left(\sum_{i\geq j+1} V_i\right)^{a-1} V_{j+1} &= \frac{1}{a} g'(x_{j+1})(x_{j+1} - x_{j+2}) \\ &\geq \frac{1}{a} g'(\theta)(x_{j+1} - x_{j+2}) = \frac{1}{a}\left(g(x_{j+1}) - g(x_{j+2})\right). \end{aligned}$$

It follows that

$$\begin{aligned}\sum_{j\geq k}\left(\sum_{i\geq j+1}V_i\right)^{a-1}V_{j+1} &\geq \frac{1}{a}\sum_{j\geq k}\left(g(x_{j+1})-g(x_{j+2})\right)\\ &= \frac{1}{a}g(x_{k+1})=\frac{1}{a}\left(\sum_{i\geq k+1}V_i\right)^a.\end{aligned}$$

b) Similarly, with some $\theta\in(x_k,x_{k-1})$

$$\begin{aligned}\left(\sum_{i\geq k}V_i\right)^{a-1}V_{k-1} &= \frac{1}{a}g'(x_k)(x_{k-1}-x_k)\\ &\leq \frac{1}{a}g'(\theta)(x_{k-1}-x_k)=\frac{1}{a}(g(x_{k-1})-g(x_k)).\end{aligned}$$

c) With $a=r/q$ and $\theta\in(y_k,y_{k+1})$ by the first identity (2.13)

$$\begin{aligned}&\left(\int_{\Omega\setminus\Omega(t_{k+2})}u_0d\mu\right)^{r/q}-\left(\int_{\Omega\setminus\Omega(t_{k+1})}u_0d\mu\right)^{r/q}\\ =\;& g(y_{k+1})-g(y_k)=g'(\theta)(y_{k+1}-y_k)\leq g'(y_{k+1})\int_{s_{k+2}}u_0d\mu\\ =\;& \frac{r}{q}\left(\int_{\Omega\setminus\Omega(t_{k+2})}u_0d\mu\right)^{r/p}\int_{s_{k+2}}u_0d\mu.\end{aligned}$$

□

Let $0<q<p$, $1<p<\infty$ and put $1/r=1/q-1/p$,

$$\Phi(y)=\left(\int_{\Omega\setminus\Omega(\tau(y))}ud\mu\right)^{1/p}\left(\int_{\Omega(\tau(y))}v^{-p'/p}d\nu\right)^{1/p'}.$$

For $\Omega=(0,\infty)$ [20], [33] have shown that $c_1\left\|\Phi\right\|_{L^r_{ud\mu}(\Omega)}\leq\|T\|\leq c_2\left\|\Phi\right\|_{L^r_{ud\mu}(\Omega)}$ with constants c_1,c_2 that depend on p,q and do not depend on the weights and measures.

Theorem 2.2. *If* $1<p<\infty$, $0<q<p$ *and* $\frac{1}{q}-\frac{1}{p}=\frac{1}{r}$, *then* (2.5) *is bounded if and only if* $B<\infty$, *where* $B=\left(\int_\Omega\Phi^r ud\mu\right)^{1/r}$. *Moreover,*

$$\frac{q\left(p'/r\right)^{1/p'}2^{1-2r/p'q}}{\left(\left(1+\frac{r}{q}\right)2^{r+r/p'}\right)^{1/p}}B\leq\|T\|\leq 2^{2+1/q}B.$$

Proof. *Upper bound.* As in the proof of Theorem 2.1, it suffices to consider the case $0<\sup\limits_{y\in\Omega}Tf(y)<\infty$. Begin with applying Hölder's inequality with exponents p/q and r/q in (2.11):

$$\|Tf\|_{L^q_{ud\mu}(\Omega)}\leq 4\left[\sum_{k\geq 0}\alpha_k^q\left(\int_{s_{k+2}}f^pvd\nu\right)^{q/p}\right]^{1/p}\leq 4\left(\sum_{k\geq 0}\alpha_k^r\right)^{1/r}\left(\sum_{k\geq 0}\int_{s_{k+2}}f^pvd\nu\right)^{1/p}.$$

We want to bound

$$\begin{aligned}\alpha_k^r &= \left(\int_{s_{k+1}} ud\mu\right)^{r/q}\left(\int_{s_{k+2}} v^{-p'/p}d\nu\right)^{r/p'}\\ &= \int_{s_{k+1}} ud\mu\left(\int_{s_{k+1}} ud\mu\right)^{r/p}\left(\int_{s_{k+2}} v^{-p'/p}d\nu\right)^{r/p'}\end{aligned}$$

by an integral. Select $t'_k \in (t_{k+1}, t_k)$ so that

$$\int_{s'_{k+1}} ud\mu = \int_{s''_{k+1}} ud\mu = \frac{1}{2}\int_{s_{k+1}} ud\mu, \text{ where } s'_{k+1} = \Omega(t_k)\backslash\Omega(t'_k),\ s''_{k+1} = \Omega(t'_k)\backslash\Omega(t_{k+1}).$$

First, we replace integrals and explicitly write out the domains of integration:

$$\begin{aligned}\alpha_k^r &\equiv \int_{s_{k+1}} ud\mu\left(\int_{s_{k+1}} ud\mu\right)^{r/p}\left(\int_{s_{k+2}} v^{-p'/p}d\nu\right)^{r/p'}\\ &= 2^{r/q}\int_{s''_{k+1}} ud\mu\left(\int_{s'_{k+1}} ud\mu\right)^{r/p}\left(\int_{s_{k+2}} v^{-p'/p}d\nu\right)^{r/p'}\\ &= 2^{r/q}\int_{t_{k+1}\le\tau(y)<t'_k} u(y)d\mu(y)\left(\int_{t'_k\le\tau(z)<t_k} u(z)d\mu(z)\right)^{r/p}\\ &\quad\times\left(\int_{t_{k+2}\le\tau(z)<t_{k+1}} v^{-p'/p}(z)d\nu(z)\right)^{r/p'}.\end{aligned}$$

Next, we increase the domains of integration in the last two integrals:

$$\begin{aligned}\alpha_k^r &\le 2^{r/q}\int_{t_{k+1}\le\tau(y)<t'_k} u(y)d\mu(y)\left(\int_{\tau(y)\le\tau(z)<\infty} u(z)d\mu(z)\right)^{r/p}\\ &\quad\times\left(\int_{\tau(z)<\tau(y)} v^{-p'/p}(z)d\nu(z)\right)^{r/p'}.\end{aligned}$$

Finally, we increase the domain of integration in the outer integral:

$$\begin{aligned}\alpha_k^r &\le 2^{r/q}\int_{s_{k+1}} u(y)d\mu(y)\left(\int_{\Omega\backslash\Omega(\tau(y))} ud\mu)\right)^{r/p}\left(\int_{\Omega(\tau(y))} v^{-p'/p}d\nu\right)^{r/p'}\\ &= 2^{r/q}\int_{s_{k+1}}\Phi^r ud\mu.\end{aligned}$$

Thus,

$$\|Tf\|_{L^q_{ud\mu}(\Omega)} \le 2^{2+1/q}\|f\|_{L^p_{vd\nu}(\Omega)}\left(\sum_{k\ge0}\int_{s_{k+1}}\Phi^r ud\mu\right)^{1/r} = 2^{2+1/q}\|f\|_{L^p_{vd\nu}(\Omega)}\|\Phi\|_{L^r_{ud\mu}(\Omega)}.$$

Lower bound. Inspired by [33] we define

$$f(t) = \left(\int_{\Omega\backslash\Omega(\tau(t))} u_0 d\mu\right)^{r/(pq)}\left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t).$$

Then

$$\int_{\Omega(\tau(x))} f d\nu \geq \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/(pq)} \left[\int_{\Omega(\tau(x))} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t)d\nu(t)\right]. \tag{2.14}$$

Let $k = \min\{j : t_j \leq \tau(x)\}$, for which $t_k \leq \tau(x) < t_{k-1}$, and consider the integral in the square brackets:

$$\begin{aligned} I &\equiv \int_{\Omega(\tau(x))} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t)d\nu(t) \\ &\geq \int_{\Omega(t_k)} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t)d\nu(t) \\ &= \sum_{j\geq k} \int_{s_{j+1}} \left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/(p'q)-1} v_0(t)d\nu(t) \\ &\geq \sum_{j\geq k} \int_{s_{j+1}} \left(\int_{\Omega(t_{j+1})} v_0 d\nu\right)^{r/(p'q)-1} v_0(t)d\nu(t) = \sum_{j\geq k}\left(\sum_{i\geq j+1} V_i\right)^{r/(p'q)-1} V_j. \end{aligned}$$

By the sliding property and Lemma 2.1 a) with $a = \dfrac{r}{p'q}$

$$I \geq 2\sum_{j\geq k}\left(\sum_{i\geq j+1} V_i\right)^{r/(p'q)-1} V_{j+1} \geq \frac{2p'q}{r}\left(\sum_{i\geq k+1} V_i\right)^{r/(p'q)}.$$

(2.14), (2.12) and this bound give

$$\begin{aligned} \int_{\Omega(\tau(x))} f d\nu &\geq \frac{2p'q}{r}\left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/(pq)} \left(\int_{\Omega(t_{k+1})} v_0 d\nu\right)^{r/(p'q)} \\ &= \frac{2p'q}{r}4^{-r/(p'q)}\left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/(pq)} \left(\int_{\Omega(t_{k-1})} v_0 d\nu\right)^{r/(p'q)} \\ &\geq c_1\left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/(pq)} \left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{r/(p'q)} = c_1\Phi_0^{r/q}(x), \end{aligned}$$

where $c_1 = \dfrac{p'q}{r}2^{1-2r/(p'q)}$ and we have denoted

$$\Phi_0(x) = \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{1/p} \left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{1/p'}.$$

Assuming that $\|T\| < \infty$ we have

$$\begin{aligned} c_1^q\int_\Omega \Phi_0^r u_0 d\mu &\leq \int_\Omega\left(\int_{\Omega(\tau(x))} f d\nu\right)^q u(x)d\mu(x) \leq \|T\|^q\left(\int_\Omega f^p v d\nu\right)^{q/p} \\ &\leq \|T\|^q\left[\int_\Omega\left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/q}\left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{r/q'} v_0(x)d\nu(x)\right]^{q/p}. \end{aligned} \tag{2.15}$$

We have applied the inequality $v_0^p v \le v_0^{p+1/(1-p')} = v_0^{p-p/p'} = v_0$. Further, we need to bound

$$\begin{aligned} I &\equiv \int_\Omega \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/q} \left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{r/q'} v_0(x) d\nu(x) \\ &= \sum_{k\ge 0} \int_{s_{k+1}} \left(\int_{\Omega\setminus\Omega(\tau(x))} u_0 d\mu\right)^{r/q} \left(\int_{\Omega(\tau(x))} v_0 d\nu\right)^{r/q'} v_0(x) d\nu(x) \\ &\le \sum_{k\ge 0} \left(\int_{\Omega\setminus\Omega(t_{k+1})} u_0 d\mu\right)^{r/q} \left(\int_{\Omega(t_k)} v_0 d\nu\right)^{r/q'} \int_{s_{k+1}} v_0 d\nu. \end{aligned} \tag{2.16}$$

Using the third identity in (2.13) and Lemma 2.1 b) with $a = r/p'$ we can write

$$\begin{aligned} \left(\int_{\Omega(t_k)} v_0 d\nu\right)^{r/q'} \int_{s_{k+1}} v_0 d\nu &= \left(\sum_{j\ge k} V_j\right)^{r/p'-1} V_k = \frac{1}{2}\left(\sum_{j\ge k} V_j\right)^{r/p'-1} V_{k-1} \\ &\le \frac{p'}{2r}\left[\left(\sum_{i\ge k-1} V_i\right)^{r/p'} - \left(\sum_{i\ge k} V_i\right)^{r/p'}\right] \\ &= \frac{p'}{2r}\left(x_{k-1}^{r/p'} - x_k^{r/p'}\right). \end{aligned} \tag{2.17}$$

Next combine (2.16) and (2.17), denoting $c_2 = \dfrac{p'}{2r}$ and keeping in mind that $x_{-1} = x_0$:

$$\begin{aligned} I/c_2 &\le \sum_{k\ge 0} y_k^{r/q}\left(x_{k-1}^{r/p'} - x_k^{r/p'}\right) = \sum_{k\ge 1} y_k^{r/q}\left(x_{k-1}^{r/p'} - x_k^{r/p'}\right) + y_0^{r/q}\left({x_0}^{r/p'} - {x_0}^{r/p'}\right) \\ &= y_0^{r/q}{x_0}^{r/p'} + \sum_{k\ge 0}\left(y_{k+1}^{r/q} - y_k^{r/q}\right) x_k^{r/p'}. \end{aligned}$$

By Lemma 2.1 c)

$$I/c_2 \le y_0^{r/q}{x_0}^{r/p'} + \frac{r}{q}\sum_{k\ge 0}\left(\int_{\Omega\setminus\Omega(t_{k+2})} u_0 d\mu\right)^{r/p} \left(\int_{\Omega(t_k)} v_0 d\nu\right)^{r/p'} \int_{s_{k+2}} u_0 d\mu. \tag{2.18}$$

Let $t'_{k+1} \in (t_{k+2}, t_{k+1})$ satisfy the following equality

$$\int_{\Omega(t'_{k+1})\setminus\Omega(t_{k+2})} u_0 d\mu = \int_{\Omega(t_{k+1})\setminus\Omega(t'_{k+1})} u_0 d\mu.$$

Then

$$\begin{aligned} &\left(\int_{\Omega\setminus\Omega(t_{k+2})} u_0 d\mu\right)^{r/p} \int_{s_{k+2}} u_0 d\mu \\ &= 2\left(2\int_{\Omega(t_{k+1})\setminus\Omega(t'_{k+1})} u_0 d\mu + \int_{\Omega\setminus\Omega(t_{k+1})} u_0 d\mu\right)^{r/p} \int_{\Omega(t'_{k+1})\setminus\Omega(t_{k+2})} u_0 d\mu \\ &\le 2^{1+r/p}\left(\int_{\Omega\setminus\Omega(t'_{k+1})} u_0 d\mu\right)^{r/p} \int_{\Omega(t'_{k+1})\setminus\Omega(t_{k+2})} u_0 d\mu \\ &\le 2^{1+r/p}\int_{s_{k+2}}\left(\int_{\Omega\setminus\Omega(\tau(t))} u_0 d\mu\right)^{r/p} u_0(t) d\mu(t). \end{aligned}$$

Besides,

$$\int_{\Omega(t_k)} v_0 d\nu = 2^{-k}\int_\Omega v_0 d\nu = 4\cdot 2^{-(k+2)}\int_\Omega v_0 d\nu = 4\int_{\Omega(t_{k+2})} v_0 d\nu.$$

Hence, the big sum in (2.18) is bounded since

$$\begin{aligned}
&\sum_{k\geq 0}\left(\int_{\Omega\setminus\Omega(t_{k+2})} u_0 d\mu\right)^{r/p}\left(\int_{\Omega(t_k)} v_0 d\nu\right)^{r/p'}\int_{s_{k+2}} u_0 d\mu\\
\leq\ & 2^{1+r+r/p'}\sum_{k\geq 0}\left(\int_{\Omega(t_{k+2})} v_0 d\nu\right)^{r/p'}\int_{s_{k+2}}\left(\int_{\Omega\setminus\Omega(\tau(t))} u_0 d\mu\right)^{r/p} u_0(t)d\mu(t)\\
\leq\ & 2^{1+r+r/p'}\sum_{k\geq 0}\int_{s_{k+2}}\left(\int_{\Omega\setminus\Omega(\tau(t))} u_0 d\mu\right)^{r/p}\left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/p'} u_0(t)d\mu(t)\\
\leq\ & 2^{1+r+r/p'}\int_\Omega \Phi_0^r u_0 d\mu.
\end{aligned} \tag{2.19}$$

The first term in (2.18) has to be treated separatel. Note that

$$y_0^{r/q}{x_0}^{r/p'} = \left(\int_{\Omega\setminus\Omega(t_1)} u_0 d\mu\right)^{r/p}\left(\int_\Omega v_0 d\nu\right)^{r/p'}\int_{\Omega\setminus\Omega(t_1)} u_0 d\mu.$$

Let $t'\in[t_1,\infty)$ satisfy $\int_{\Omega(t')\setminus\Omega(t_1)} u_0 d\mu = \int_{\Omega\setminus\Omega(t')} u_0 d\mu$. Then,

$$\begin{aligned}
y_0^{r/q}{x_0}^{r/p'} &= 2^{r/p+r/p'+1}\left(\int_{\Omega\setminus\Omega(t')} u_0 d\mu\right)^{r/p}\left(\int_{\Omega(t_1)} v_0 d\nu\right)^{r/p'}\int_{\Omega(t')\setminus\Omega(t_1)} u_0 d\mu\\
&\leq 2^{1+r}\int_{\Omega(t')\setminus\Omega(t_1)}\left(\int_{\Omega\setminus\Omega(\tau(t))} u_0 d\mu\right)^{r/p}\left(\int_{\Omega(\tau(t))} v_0 d\nu\right)^{r/p'} u_0(t)d\mu(t)\\
&\leq 2^{1+r}\int_\Omega \Phi_0^r u_0 d\mu.
\end{aligned} \tag{2.20}$$

Equations (2.18), (2.19) and (2.20) are summarized to give

$$I \leq \frac{p'}{r}\left(1+\frac{r}{q}\right)2^{r+r/p'}\int_\Omega \Phi_0^r u_0 d\mu.$$

Recalling also (2.15) we get

$$c_1\left(\int_\Omega \Phi_0^r u_0 d\mu\right)^{1/q} \leq \|T\|\, I^{1/p} \leq \|T\|\left(\frac{p'}{r}\left(1+\frac{r}{q}\right)2^{r+r/p'}\right)^{1/p}\left(\int_\Omega \Phi_0^r u_0 d\mu\right)^{1/p}.$$

It remains to divide both sides by $\left(\int_\Omega \Phi_0^r u_0 d\mu\right)^{1/p}$. Finally, choose u_0, v_0 monotonously approaching to $u, v^{1-p'}$, respectively, and apply Fatou's lemma, which holds on a general measure space (no topology is needed) [10]. □

With simple changes in the proofs, analogues of Theorems 2.1 and 2.2 hold for the adjoint operator $T^*: L^p_{vd\nu}(\Omega)\to L^q_{ud\mu}(\Omega)$,

$$T^*f\,(y) = \int_{\Omega\setminus\Omega(y)} f d\nu.$$

Denote

$$\begin{aligned}\Psi^*(t) &= \left(\int_{\Omega(t)} u d\mu\right)^{1/q} \left(\int_{\Omega\setminus\Omega(t)} v^{-p'/p} d\nu\right)^{1/p'}, \ t>0,\\ \Phi^*(y) &= \left(\int_{\Omega(\tau(y))} u d\mu\right)^{1/p} \left(\int_{\Omega\setminus\Omega(\tau(y))} v^{-p'/p} d\nu\right)^{1/p'}, \ y\in\Omega,\end{aligned}$$

and consider the inequality

$$\left[\int_\Omega \left|\int_{\Omega\setminus\Omega(\tau(x))} f d\nu\right|^q u(x) d\mu(x)\right]^{1/q} \le \|T^*\| \left(\int_\Omega |f|^p v d\nu\right)^{1/p}.$$

Theorem 2.3. *1) If $1 < p \le q < \infty$, then T^* is bounded if and only if $A^* < \infty$ where $A^* = \sup_{t>0} \Psi^*(t)$, and $A^* \le \|T^*\| \le 4A^*$.*

2) If $0 < q < p$, $1 < p < \infty$, then T^* is bounded if and only if $B^* < \infty$ where $B^* = \left(\int_\Omega (\Phi^*)^r u d\mu\right)^{1/r}$, $1/r = 1/q - 1/p$. Moreover,

$$\frac{q\,(p'/r)^{1/p'}\,2^{1-2r/p'q}}{\left(\left(1+\frac{r}{q}\right)2^{r+r/p'}\right)^{1/p}} B^* \le \|T^*\| \le 2^{2+1/q} B^*.$$

Remark 1. Instead of being parameterized by $t \in (0,\infty)$ the family $\{\Omega(t)\}$ can be parameterized by $t \in [a,b]$ with $-\infty \le a < b \le \infty$. The resulting bounds will be applied below.

3 Hardy operator compactness

The next subject is the compactness of T. The notation below allows one to trace similarity with [32]. Denote

$$\begin{aligned} a(x) &= \int_{\Omega\setminus\Omega(x)} u d\mu, \ b(x) = \int_{\Omega(x)} v^{-p'/p} d\nu, \ 0 < x < \infty,\\ l_i &= \limsup_{x\to i} a(x)^{1/q} b(x)^{1/p'}, \text{ for } i = 0,\infty, \ l = \max\{l_0, l_\infty\}.\end{aligned}$$

Lemma 3.1. *Suppose that $a(x) < \infty$, $b(x) < \infty$ on $(0,\infty)$. If $l > \varepsilon > 0$, then there exists a sequence $\{g_n\}$ such that $\|g_n\|_{L^p_{vd\nu}(\Omega)} = 1$, $\|Tg_n - Tg_m\|_{L^q_{ud\mu}(\Omega)} > \varepsilon$.*

Proof. Let $f_x(z) = b(x)^{-1/p} \chi_{\Omega(x)}(z) v^{-p'/p}(z)$, $x \in \Omega$. Then

$$\|f_x\|_{p,vd\nu} = b(x)^{-1/p} \left(\int_{\Omega(x)} v^{1-p'} d\nu\right)^{1/p} = 1 \tag{3.1}$$

and

$$Tf_x(y) = b(x)^{-1/p} \int_{\Omega(\tau(y))} \chi_{\Omega(x)} v^{-p'/p} d\nu = b(x)^{-1/p} \int_{\Omega(\min\{\tau(y),x\})} v^{-p'/p} d\nu. \tag{3.2}$$

Case $i = 0$. Suppose that $l_0 > \varepsilon > 0$. Choose $x_1 > 0$ for which

$$a(x_1)^{1/q} b(x_1)^{1/p'} > \varepsilon. \tag{3.3}$$

Since $b(x) \to 0$ as $x \to 0$, we can select $x_2 < x_1$ so that

$$a(x_2)^{1/q} b(x_2)^{1/p'} > \varepsilon \text{ and } a(x_1)^{1/q} \left(b(x_1)^{1/p'} - b(x_2)^{1/p'} \right) > \varepsilon.$$

Similarly, we can choose $x_1 > x_2 > \ldots$ recursively so that for each n

$$a(x_n)^{1/q} b(x_n)^{1/p'} > \varepsilon \text{ and } a(x_n)^{1/q} \left(b(x_n)^{1/p'} - b(x_{n+1})^{1/p'} \right) > \varepsilon. \tag{3.4}$$

If $m > n$ then $x_n > x_{n+1} \geq x_m$ and

$$b(x_n) > b(x_{n+1}) \geq b(x_m). \tag{3.5}$$

If $\tau(y) \geq x_n$, then by (3.2) and (3.5)

$$T f_{x_n} - T f_{x_m} \geq b(x_n)^{1/p'} - b(x_m)^{1/p'} \geq 0.$$

Hence, by (3.4)

$$\begin{aligned} \|T f_{x_n} - T f_{x_m}\|_{L^q_{ud\mu}(\Omega)} &\geq \left(b(x_n)^{1/p'} - b(x_m)^{1/p'} \right) \left(\int_{\{y:\tau(y)\geq x_n\}} ud\mu \right)^{1/q} \\ &\geq a(x_n)^{1/q} \left(b(x_n)^{1/p'} - b(x_{n+1})^{1/p'} \right) > \varepsilon. \end{aligned}$$

This and (3.1) show that the functions $g_n = f_{x_n}$ possess the required properties.

Case $i = \infty$. Suppose $l_\infty > \varepsilon$. Choose x_1 satisfying (3.3). Obviously, $a(x) \to 0$ as $x \to \infty$ and, therefore, $b(x) \to \infty$. Using (3.3) we can choose $z_1 > x_1$ such that

$$(a(x_1) - a(z_1))^{1/q} b(x_1)^{1/p'} > \varepsilon. \tag{3.6}$$

The inequality $l_\infty > \varepsilon$ and (3.6) imply that we can select $x_2 > z_1$ with $a(x_2)^{1/q} b(x_2)^{1/p'} > \varepsilon$ and

$$(a(x_1) - a(z_1))^{1/q} \left(b(x_1)^{1/p'} - b(z_1) b(x_2)^{-1/p} \right) > \varepsilon.$$

Continuing in this way, we obtain points $x_1 < z_1 < x_2 < z_2 < \ldots$ such that for each n

$$\begin{gathered} a(x_n)^{1/q} b(x_n)^{1/p'} > \varepsilon, \ (a(x_n) - a(z_n))^{1/q} b(x_n)^{1/p'} > \varepsilon, \\ (a(x_n) - a(z_n))^{1/q} \left(b(x_n)^{1/p'} - b(z_n) b(x_{n+1})^{-1/p} \right) > \varepsilon. \end{gathered} \tag{3.7}$$

If $m > n$, then $x_n < z_n < x_{n+1} \leq x_m$, leading to the inequlities

$$b(x_n) \leq b(z_n) \leq b(x_{n+1}) \leq b(x_m). \tag{3.8}$$

Let $x_n \leq \tau(y) \leq z_n$. Then by (3.2)

$$T f_{x_n}(y) = b(x_n)^{1/p'}, \ T f_{x_m}(y) = b(x_m)^{-1/p} b(\tau(y)).$$

Because of (3.7) and (3.8) this implies

$$\begin{aligned} T f_{x_n}(y) - T f_{x_m}(y) &= b(x_n)^{1/p'} - b(x_m)^{-1/p} b(\tau(y)) \\ &\geq b(x_n)^{1/p'} - b(x_m)^{-1/p} b(z_n) \\ &\geq b(x_n)^{1/p'} - b(x_{n+1})^{-1/p} b(z_n) > 0. \end{aligned}$$

Now we apply (3.7) to get

$$\begin{aligned}\|Tf_{x_n}-Tf_{x_m}\|_{L^q_{ud\mu}(\Omega)} &\geq \left(\int_{\{y:x_n\leq\tau(y)\leq z_n\}}(Tf_{x_n}(y)-Tf_{x_m}(y))^q\,u(y)\,d\mu(y)\right)^{1/q}\\ &\geq (a(x_n)-a(z_n))^{1/q}\left(b(x_n)^{1/p'}-b(z_n)\,b(x_{n+1})^{-1/p}\right)>\varepsilon.\end{aligned}$$

Denoting $g_n=f_{x_n}$ we see that this bound and (3.1) complete the proof of the lemma. □

Theorem 3.1. *a) If $1<p\leq q<\infty$, then T is compact if and only if $l=0$. b) If $1<q<p$ and T is bounded, then T is compact.*

Proof. Approximation to T. The points $t_k=k2^{-n}$, $k=0,...,n2^n$, $t_{n2^n+1}=\infty$, lead to two partitions: one of $(0,\infty)$, consisting of the intervals

$$\Delta_k=(t_k,t_{k+1}),\ k=0,...,n2^n-1,\quad \Delta_{n2^n}=(t_{n2^n},t_{n2^n+1})=(n,\infty),$$

and the another one of Ω, consisting of the sets

$$\Omega_k=\Omega(t_{k+1})\backslash\Omega(t_k),\ k=0,...,n2^n.$$

Define $\kappa_n(t)=\sum_{k=0}^{n2^n}t_k\chi_{\Delta_k}(t)$, $t>0$. Since $x\in\Omega_k$ is equivalent to $\tau(x)\in\Delta_k$ we have

$$\begin{aligned}\kappa_n(\tau(x)) &= t_k<\tau(x)<t_k+2^{-n}\quad\text{for } x\in\Omega_k,\quad k=0,...,n2^n-1,\\ \kappa_n(\tau(x)) &= n<\tau(x)\quad\text{for } x\in\Omega_{n2^n}.\end{aligned}$$

Put

$$T_nf(y)=\int_{\Omega(\kappa_n(\tau(y)))}fd\nu=\int_{\Omega\left(\sum_{k=0}^{n2^n}t_k\chi_{\Omega_k}(y)\right)}fd\nu=\sum_{k=0}^{n2^n}\int_{\Omega(t_k)}fd\nu\chi_{\Omega_k}(y).$$

Obviously T_n is a finite-rank operator.

For the difference $T-T_n$ we have the representation

$$\begin{aligned}Tf(y)-T_nf(y) &= \sum_{k=0}^{n2^n}\left(\int_{\Omega(\tau(y))}fd\nu-\int_{\Omega(t_k)}fd\nu\right)\chi_{\Omega_k}(y)\\ &= \sum_{k=0}^{n2^n}\int_{\Omega(\tau(y))\backslash\Omega(t_k)}fd\nu\chi_{\Omega_k}(y).\end{aligned}\tag{3.9}$$

Case $p\leq q$. *Sufficiency.* By Theorem 2.1

$$\left[\int_{\Omega_k}\left|\int_{\Omega(\tau(y))\backslash\Omega(t_k)}fd\nu\right|^q u(y)d\mu(y)\right]^{1/q}\leq ca_k\left(\int_{\Omega_k}|f|^p\,vd\nu\right)^{1/p}\tag{3.10}$$

where

$$a_k=\sup_{t_k<t<t_{k+1}}\left(\int_{\Omega(t_{k+1})\backslash\Omega(t)}ud\mu\right)^{1/q}\left(\int_{\Omega(t)\backslash\Omega(t_k)}v^{-p'/p}d\nu\right)^{1/p'}.$$

Since $p \le q$ we see by (3.9)-(3.10) that

$$\left(\int_{\Omega}|Tf-T_nf|^q\,ud\mu\right)^{1/q} = \left[\sum_{k=0}^{n2^n}\int_{\Omega_k}|Tf-T_nf|^q\,ud\mu\right]^{1/q}$$
$$\le c\left[\sum_{k=0}^{n2^n}a_k^q\left(\int_{\Omega_k}|f^p|\,vd\nu\right)^{q/p}ud\mu\right]^{1/q} \le c\sup_k a_k\,\|f\|_{L^p_{vd\nu}(\Omega)}\,. \tag{3.11}$$

Let us prove the compactness of T assuming that $l=0$. For any $\varepsilon>0$ we can choose $x_1<x_2$ such that

$$a\,(x)^{1/q}\,b\,(x)^{1/p'}<\varepsilon,\quad x\in(0,x_1]\cup[x_2,\infty). \tag{3.12}$$

Since $v^{-p'/p}$ and u are positive almost everywhere, this implies that $a\,(x)$ and $b\,(x)$ are positive in the range in (3.12) and then $a\,(x)\le a\,(x_1)<\infty$, $b\,(x)\le b\,(x_2)<\infty$ for $x\in[x_1,x_2]$. This justifies the calculations that yielded (3.11).

We want to evaluate the sets

$$\widetilde{\Omega}_1=\bigcup_{t_{k+1}<x_1}\Omega_k,\quad \widetilde{\Omega}_2=\bigcup_{x_1\le t_{k+1}\le 2x_2}\Omega_k,\quad \widetilde{\Omega}_3=\bigcup_{t_{k+1}>2x_2}\Omega_k.$$

Obviously, $\widetilde{\Omega}_1\subseteq\Omega\,(x_1)$. Assuming that $2^{-n}\le x_2$ we see that $t_{k+1}>2x_2$ implies that $t_k>x_2$ and $\widetilde{\Omega}_3\subseteq\Omega\backslash\Omega\,(x_2)$. Further, provided that $2^{-n}\le x_1/2$ from $x_1\le t_{k+1}\le 2x_2$ we have $t_k\ge x_1/2$ and $\widetilde{\Omega}_2\subseteq\Omega\,(2x_2)\backslash\Omega\,(x_1/2)$. We have shown that

$$\widetilde{\Omega}_1\subseteq\Omega\,(x_1)\,,\quad \widetilde{\Omega}_2\subseteq\Omega\,(2x_2)\backslash\Omega\,(x_1/2)\,,\quad \widetilde{\Omega}_3\subseteq\Omega\backslash\Omega\,(x_2)\,. \tag{3.13}$$

Since $a_k\le\sup\limits_{t_k<t<t_{k+1}}\Psi\,(t)\,$, the inclusions in (3.13) and (3.12) give

$$\sup_{t_{k+1}<x_1}a_k<\varepsilon,\quad \sup_{t_{k+1}>2x_2}a_k<\varepsilon. \tag{3.14}$$

For Δ_k with $x_1\le t_{k+1}\le 2x_2$ we have

$$a_k\le\left(\int_{\Omega(t_k+2^{-n})\backslash\Omega(t_k)}ud\mu\right)^{1/q}\left(\int_{\Omega(t_k+2^{-n})\backslash\Omega(t_k)}v^{-p'/p}d\nu\right)^{1/p'}=\psi\left(t_k,2^{-n}\right) \tag{3.15}$$

where ψ is defined as

$$\begin{aligned}\psi\,(x,\delta)&=\left(\int_{\Omega(x+\delta)\backslash\Omega(x)}ud\mu\right)^{1/q}\left(\int_{\Omega(x+\delta)\backslash\Omega(x)}v^{-p'/p}d\nu\right)^{1/p'},\\ (x,\delta)&\in\left[\frac{x_1}{2},2x_2\right]\times[0,\delta_0]\end{aligned}$$

for some $\delta_0>0$. This function is continuous on a compact domain and has the property that $\lim\limits_{\delta\to0}\psi\,(x,\delta)=0$ for any $x\in[x_1/2,2x_2]\,$. Hence, there exists $\delta_1\in(0,\delta_0]$ such that

$$a_k\le\sup_{(x,\delta)\in[x_1/2,2x_2]\times(0,\delta_1]}\psi\,(x,\delta)<\varepsilon.$$

If we choose n satisfying the inequalities $2^{-n}\le x_1/2<x_2$ and $2^{-n}\le\delta_1$ then (3.11), (3.14), (3.15) give the desired bound from above: $\|T-T_n\|\le c\varepsilon$ and T is compact.

Necessity. Suppose that T is compact, which implies that $\|T\| < \infty$ and $A < \infty$ by Theorem 2.1. As above, it follows that a and b are finite on $(0, \infty)$ and we can use Lemma 3.1. Suppose $l > 0$. Taking $\varepsilon = l/2$ in Lemma 3.1 we obtain a sequence $\{g_n\}$ such that $\|g_n\|_{L^p_{vd\nu}(\Omega)} = 1$, $\|Tg_n - Tg_m\|_{L^q_{ud\mu}(\Omega)} > \varepsilon$. This shows that T cannot be compact and that the condition $l = 0$ is necessary.

Case $q < p$. Let $\|T\| < \infty$. By Theorem 2.2, instead of (3.11) we have

$$\left(\int_\Omega |Tf - T_n f|^q \, ud\mu\right)^{1/q} \leq c \left(\sum_{k=0}^{n2^n} b_k^r\right)^{1/r} \|f\|_{L^p_{vd\nu}(\Omega)}, \tag{3.16}$$

where

$$b_k = \left[\int_{\Omega_k} \left(\int_{\Omega(t_{k+1})\backslash\Omega(\tau(y))} ud\mu\right)^{r/p} \left(\int_{\Omega(\tau(y))\backslash\Omega(t_k)} v^{-p'/p} d\nu\right)^{r/p'} u(y)\, d\mu(y)\right]^{1/r}.$$

Also $B < \infty$. Therefore, we can select $x_1 < x_2$ so that $\left(\int_{\widetilde{\Omega}} \Phi^r ud\mu\right)^{1/r} < \varepsilon$ for both $\widetilde{\Omega} = \Omega(x_1)$ and $\widetilde{\Omega} = \Omega\backslash\Omega(x_2)$. This implies that

$$\sum_{t_{k+1} < x_1} b_k^r \leq \sum_{t_{k+1} < x_1} \int_{\Omega_k} \Phi^r ud\mu \leq \int_{\Omega(x_1)} \Phi^r ud\mu < \varepsilon. \tag{3.17}$$

Assuming that $2^{-n} \leq x_2$ we can use (3.13) to obtain

$$\sum_{\{k: t_{k+1} > 2x_2\}} b_k^r \leq \int_{\Omega\backslash\Omega(x_2)} \Phi^r ud\mu < \varepsilon. \tag{3.18}$$

Again using (3.13) with $2^{-n} \leq x_1/2$ we get

$$\begin{aligned} \sum_{\{k: x_1 \leq t_{k+1} \leq 2x_2\}} b_k^r &\leq \sum_{\{k: x_1 \leq t_{k+1} \leq 2x_2\}} \int_{\Omega_k} ud\mu \; \psi\left(t_k, 2^{-n}\right)^r \\ &\leq \int_{\Omega(2x_2)\backslash\Omega(x_1/2)} ud\mu \sup_{x_1/2 \leq x \leq 2x_2} \psi\left(x, 2^{-n}\right)^r. \end{aligned} \tag{3.19}$$

The function Φ is integrable and the choice of x_1 can be subject to one more condition: $\Phi(x_1/2) < \infty$. As $b(x_1/2) > 0$, by the inequality

$$\int_{\Omega(2x_2)\backslash\Omega(x_1/2)} ud\mu \leq \int_{\Omega\backslash\Omega(x_1/2)} ud\mu = \frac{\Phi(x_1/2)^p}{b(x_1/2)^{p/p'}} < \infty$$

we see that the right-hand side in (3.19) tends to zero as $n \to \infty$. Bounds (3.16)-(3.19) imply that T can be approximated arbitrarily well with finite-rank operators and thus is compact. □

4 Bounds for approximation numbers

Our next task is to obtain bounds for the approximation numbers (*a-numbers*) of operator (2.5). Let X, Y be two Banach spaces. For a bounded linear operator $T : X \to Y$ its *n-th a-number*, $n \in N$, is defined by

$$a_n(T) = \inf\left\{\|T - P\| : P : X \to Y \text{ is a bounded linear operator and } \operatorname{rank} P < n\right\}.$$

For $[a,b]\subseteq[0,\infty)$ we initially consider the problem of how well the operator $\chi_{[a,b]}T$ is approximated by averages. To this end, successively define

$$\begin{aligned}\mu_u\left(\Omega\left[a,b\right]\right) &= \int_{\Omega[a,b]} ud\mu,\ \bar{T}_{[a,b]}f=\frac{1}{\mu_u\left(\Omega\left[a,b\right]\right)}\int_{\Omega[a,b]}(Tf)ud\mu, \\ T_{[a,b]}f\left(x\right) &= \chi_{\Omega[a,b]}\left(x\right)\left(Tf\left(x\right)-\bar{T}_{[a,b]}f\right).\end{aligned} \tag{4.1}$$

Theorem 4.1. *Choose the point c so that*

$$\mu_u\left(\Omega\left[a,c\right]\right)=\mu_u\left(\Omega\left[c,b\right]\right)=\frac{1}{2}\mu_u\left(\Omega\left[a,b\right]\right).$$

a) Let $1<p\leq q<\infty$,

$$\begin{aligned}A^*\left[a,c\right] &= \sup_{a<\tau(x)<c}\left(\int_{\Omega[a,\tau(x)]}ud\mu\right)^{1/q}\left(\int_{\Omega[\tau(x),c]}v^{-p'/p}d\nu\right)^{1/p'}, \\ A\left[c,b\right] &= \sup_{c<\tau(x)<b}\left(\int_{\Omega[\tau(x),b]}ud\mu\right)^{1/q}\left(\int_{\Omega[c,\tau(x)]}v^{-p'/p}d\nu\right)^{1/p'}\end{aligned}$$

and $\mathbf{A}[a,b]=\max\{A^*[a,c],A[c,b]\}$. *Then*

$$\left(1-2^{-1/q}\right)\mathbf{A}[a,b]\leq\left\|T_{[a,b]}\right\|\leq 8\mathbf{A}[a,b].$$

b) Let $1<q<p<\infty$, $1/r=1/q-1/p$,

$$\begin{aligned}B^*\left[a,c\right] &= \left[\int_{\Omega[a,c]}\left(\int_{\Omega[a,\tau(x)]}ud\mu\right)^{r/p}\left(\int_{\Omega[\tau(x),c]}v^{-p'/p}d\nu\right)^{r/p'}u\left(x\right)d\mu\left(x\right)\right]^{1/r}, \\ B\left[c,b\right] &= \left[\int_{\Omega[c,b]}\left(\int_{\Omega[\tau(x),b]}ud\mu\right)^{r/p}\left(\int_{\Omega[c,\tau(x)]}v^{-p'/p}d\nu\right)^{r/p'}u\left(x\right)d\mu\left(x\right)\right]^{1/r}\end{aligned}$$

and $\mathbf{B}\left[a,b\right]=\max\left\{B^*\left[a,c\right],B\left[c,b\right]\right\}$. *Then*

$$\frac{q\left(p'/r\right)^{1/p'}2^{1-2r/p'q}\left(1-2^{-1/q}\right)}{\left(\left(1+\frac{r}{q}\right)2^{r+r/p'}\right)^{1/p}}\mathbf{B}\left[a,b\right]\leq\left\|T_{[a,b]}\right\|\leq 2^{4+1/q}\,\mathbf{B}\left[a,b\right].$$

Proof. Define

$$F_{[a,b]}f\left(x\right)=\begin{cases}-\displaystyle\int_{\Omega[\tau(x),c]}fd\nu, & a<\tau\left(x\right)<c,\\ \displaystyle\int_{\Omega[c,\tau(x)]}fd\nu, & c<\tau\left(x\right)<b,\end{cases}$$

$$\bar{F}_{[a,b]}f=\frac{1}{\mu_u\left(\Omega\left[a,b\right]\right)}\int_{\Omega[a,b]}(F_{[a,b]}f)ud\mu.$$

With this notation, we have the following identity

$$Tf\left(x\right)-\bar{T}_{[a,b]}f=F_{[a,b]}f(x)-\bar{F}_{[a,b]}f,\ a<\tau\left(x\right)<b. \tag{4.2}$$

To prove it, we start with adding and subtracting terms in

$$\begin{aligned}
&\int_{\Omega[a,b]}(Tf)ud\mu \\
= &\left[\int_{\Omega[a,c]}\left(\int_{\Omega[0,\tau(x)]}fd\nu\right)u(x)\,d\mu(x)+\int_{\Omega[c,b]}\left(\int_{\Omega[0,\tau(x)]}fd\nu\right)u(x)\,d\mu(x)\right] \\
&-\left[\int_{\Omega[a,c]}\left(\int_{\Omega[0,c]}fd\nu\right)u(x)\,d\mu(x)+\int_{\Omega[c,b]}\left(\int_{\Omega[0,c]}fd\nu\right)u(x)\,d\mu(x)\right] \\
&+\int_{\Omega[a,b]}\left(\int_{\Omega[0,c]}fd\nu\right)u(x)\,d\mu(x)
\end{aligned}$$

(joining similar terms in the square brackets and then using the definition of F)

$$\begin{aligned}
= &\left[\int_{\Omega[a,c]}\left(-\int_{\Omega[\tau(x),c]}fd\nu\right)u(x)\,d\mu(x)+\int_{\Omega[c,b]}\left(\int_{\Omega[c,\tau(x)]}fd\nu\right)u(x)\,d\mu(x)\right] \\
&+\mu_u(\Omega[a,b])\int_{\Omega[0,c]}fd\nu \\
= &\int_{\Omega[a,b]}(F_{[a,b]}f)ud\mu+\mu_u(\Omega[a,b])\begin{cases}\displaystyle\int_{\Omega[0,\tau(x)]}fd\nu+\int_{\Omega[\tau(x),c]}fd\nu, & a<\tau(x)<c\\ \displaystyle\int_{\Omega[0,\tau(x)]}fd\nu-\int_{\Omega[c,\tau(x)]}fd\nu, & c<\tau(x)<b\end{cases} \\
= &\int_{\Omega[a,b]}(F_{[a,b]}f)ud\mu+\mu_u(\Omega[a,b])\left[Tf(x)-F_{[a,b]}f(x)\right].
\end{aligned}$$

Rearranging this gives (4.2).

Upper bound. a) Equation (4.2) implies

$$\begin{aligned}
&\left(\int_{\Omega[a,b]}\left|T_{[a,b]}f\right|^q ud\mu\right)^{1/q} \\
\leq\ &\left\|F_{[a,b]}f\right\|_{L^q_{ud\mu}(a,b)}+\left|\bar{F}_{[a,b]}f\right|\left(\int_{\Omega[a,b]}ud\mu\right)^{1/q} \\
\leq\ &\left\|F_{[a,b]}f\right\|_{L^q_{ud\mu}(a,b)}+\frac{1}{\mu_u(\Omega[a,b])}\int_{\Omega[a,b]}\left|F_{[a,b]}f\right|ud\mu\left(\int_{\Omega[a,b]}ud\mu\right)^{1/q} \\
&\text{(applying Hölder's inequality)} \\
\leq\ &2\left\|F_{[a,b]}f\right\|_{L^q_{ud\mu}(a,b)}.
\end{aligned}$$

Next we apply the definition of $F_{[a,b]}$ and Theorems 2.1, 2.3:

$$\begin{aligned}
&\left\|T_{[a,b]}f\right\|_{L^q_{ud\mu}(a,b)} \\
\leq\ &2\left(\int_{\Omega[a,c]}\left|\int_{\Omega[\tau(x),c]}fd\nu\right|^q u(x)\,d\mu(x)+\int_{\Omega[c,b]}\left|\int_{\Omega[c,\tau(x)]}fd\nu\right|^q u(x)\,d\mu(x)\right)^{1/q} \\
\leq\ &2\left[\left((4A^*[a,c])^p\int_{\Omega[a,c]}|f|^p vd\nu\right)^{q/p}+\left((4A[c,b])^p\int_{\Omega[c,b]}|f|^p vd\nu\right)^{q/p}\right]^{1/q} \\
\leq\ &8\,\mathbf{A}[a,b]\left(\int_{\Omega[a,b]}|f|^p vd\nu\right)^{1/p}.
\end{aligned}$$

This proves the upper bound.

The upper bound in the case b) is proved similarly, using Theorems 2.2, 2.3.

Lower bound. a) For any $f \geq 0$ supported in $[a,c]$ we have

$$\begin{aligned}\left|\int_{\Omega[a,b]} \left(F_{[a,b]}f\right) ud\mu\right| &\leq \left(\int_{\Omega[a,b]} \left|F_{[a,b]}f\right|^q ud\mu\right)^{1/q} \left(\int_{\Omega[a,b]} ud\mu\right)^{1/q'} \\ &= \left(\int_{\Omega[a,c]} \left|F_{[a,b]}f\right|^q ud\mu\right)^{1/q} \left(2\mu_u\left(\Omega\left[a,c\right]\right)\right)^{1/q'}.\end{aligned}$$

Therefore, by (4.2)

$$\begin{aligned}&\left\|T_{[a,b]}\right\| \left\|f\right\|_{L^p_{vd\nu}(a,c)} \geq \left\|T_{[a,b]}f\right\|_{L^q_{ud\mu}(a,b)} \geq \left\|Tf - \bar{T}_{[a,b]}f\right\|_{L^q_{ud\mu}(a,c)} \\ =\ &\left\|F_{[a,b]}f - \bar{F}_{[a,b]}f\right\|_{L^q_{ud\mu}(a,c)} \geq \left\|F_{[a,b]}f\right\|_{L^q_{ud\mu}(a,c)} - \left|\bar{F}_{[a,b]}f\right| \left(\int_{\Omega[a,c]} ud\mu\right)^{1/q} \\ =\ &\left\|F_{[a,b]}f\right\|_{L^q_{ud\mu}(a,c)} - \frac{1}{\mu_u\left(\Omega\left[a,b\right]\right)} \left|\int_{\Omega[a,b]} (F_{[a,b]}f) ud\mu\right| \left(\int_{\Omega[a,c]} ud\mu\right)^{1/q} \\ \geq\ &\left\|F_{[a,b]}f\right\|_{L^q_{ud\mu}(a,c)} \left(1 - \frac{1}{\mu_u\left(\Omega\left[a,b\right]\right)} 2^{1/q'} \mu_u\left(\Omega\left[a,c\right]\right)\right) \\ =\ &\left\|F_{[a,b]}f\right\|_{L^q_{ud\mu}(a,c)} \left(1 - 2^{-1/q}\right).\end{aligned}$$

The conclusion is that

$$\begin{aligned}\left\|T_{[a,b]}\right\| &\geq \frac{\left\|F_{[a,b]}f\right\|_{L^q_{ud\mu}(a,c)}}{\left\|f\right\|_{L^p_{vd\nu}(a,c)}} \left(1 - 2^{-1/q}\right) \\ &= \frac{\left[\int_{\Omega[a,c]} \left|\int_{\Omega[\tau(x),c]} fd\nu\right|^q u(x)\, d\mu(x)\right]^{1/q}}{\left\|f\right\|_{L^p_{vd\nu}(a,c)}} \left(1 - 2^{-1/q}\right)\end{aligned}$$

for any non-negative f with supp $\subseteq [a,c]$ and by Theorem 2.3

$$\left\|T_{[a,b]}\right\| \geq \left(1 - 2^{-1/q}\right) A^*\left[a,c\right].$$

Selecting f supported in $[c,b]$ similarly yields

$$\left\|T_{[a,b]}\right\| \geq \left(1 - 2^{-1/q}\right) A\left[c,b\right].$$

b) The lower bound in this case is obtained similarly using Theorems 2.2, 2.3. □

Obviously, for any $0 < x < \infty$ we have

$$\mathbf{A}\left[a,b\right] \to 0 \text{ if } a,b \to x; \ \ \mathbf{A}\left[a,b\right] > 0 \text{ if } a < b.$$

Everywhere below we assume that T is a compact operator.

Lemma 4.1. *Let $1 < p \leq q < \infty$ and $0 < \varepsilon < \max\Psi$. There exist points $0 = t_0 < t_1 < ... < t_N < t_{N+1} = \infty$ such that with the notation $\Delta_k = [t_k, t_{k+1})$, $k = 0, ..., N$ one has*

$$\sup_{t\in\Delta_0} \Psi\left(t\right) = \varepsilon, \ \max_{k=1,...,N-2} \mathbf{A}\left(\Delta_k\right) = \varepsilon, \ \ \mathbf{A}\left(\Delta_{N-1}\right) \leq \varepsilon, \ \ \sup_{t\in\Delta_N} \Psi\left(t\right) = \varepsilon. \tag{4.3}$$

Proof. Let $t_0 = 0$. Since Ψ is continuous and $\Psi(t) \to 0$ when $t \to 0$ or $t \to \infty$, we can define $t' = \min\{t > 0 : \Psi(t) \geq \varepsilon\}$ and $t'' = \max\{t > 0 : \Psi(t) \geq \varepsilon\}$. Then

$$\max_{t \leq t'} \Psi(t) = \varepsilon,\ \max_{t \geq t''} \Psi(t) = \varepsilon,$$
$$c' = \int_{\Omega[t',\infty]} u d\mu < \infty,\ c'' = \int_{\Omega(t'')} v^{-p'/p} d\nu < \infty. \tag{4.4}$$

and we put $t_1 = t'$. On the n-th step, if $\sup\limits_{t \in [t_n, t'']} \mathbf{A}[t_n, t] \leq \varepsilon$, we set $t_{n+1} = t''$, $t_{n+2} = \infty$. If, on the other hand, $\sup\limits_{t \in [t_n, t'']} \mathbf{A}[t_n, t] > \varepsilon$, then we put $t_{n+1} = \min\{t > t_n : \mathbf{A}[t_n, t] \geq \varepsilon\}$ so that by continuity $\mathbf{A}[t_n, t_{n+1}] = \varepsilon$. Thus, we have disjoint segments $[t_n, t_{n+1}] \subseteq [t', \infty)$.

We want to show that this process stops in a finite number of steps. Suppose it does not and $t_n \to t \leq t''$. Then $\mathbf{A}[t_n, t_{n+1}] = \varepsilon$ for $n = 1, 2, \ldots$ Obviously,

$$\mathbf{A}^*[t_n, t_{n+1}] \leq (c'')^{1/p'} \left(\int_{\Omega[t_n, \tau(x_n)]} u d\mu \right)^{1/q},$$
$$\mathbf{A}[t_n, t_{n+1}] \leq (c')^{1/p'} \left(\int_{\Omega[\tau(x_n), t_{n+1}]} u d\mu \right)^{1/q}.$$

Hence,

$$\varepsilon^q = \mathbf{A}[t_n, t_{n+1}]^q \leq \max\left\{ (c')^{1/p'}, (c'')^{1/p'} \right\}^q \int_{\Omega[t_n, t_{n+1}]} u d\mu \text{ for all } n.$$

This contradicts the fact that in (4.4) $c' < \infty$. $\square$

With $\Omega_k = \Omega(\Delta_k)$ put for $k = 1, \ldots, N-1$

$$T_k f(x) = \int_{\Omega(\tau(x)) \setminus \Omega(t_k)} f d\nu,\ \mu_u(\Omega_k) = \int_{\Omega_k} u d\mu,\ \bar{T}_k f = \frac{1}{\mu_u(\Omega_k)} \int_{\Omega_k} (Tf) u d\mu,$$

$$P_k f(x) = \chi_{\Omega_k}(x) \left\{ Tf(x) - \left[T_k f(x) - \bar{T}_k f \right] \right\} = \chi_{\Omega_k}(x) \left\{ \int_{\Omega(t_k)} f d\nu + \bar{T}_k f \right\}, \tag{4.5}$$

$$P_0 f = 0,\ P_N f(x) = \chi_{\Omega_N}(x) \int_{\Omega(t_N)} f d\nu.$$

Each of P_k is one-dimensional, so $P = \sum\limits_{k=1}^{N}$ has $\operatorname{rank} P \leq N$.

We use the approach developed in [5].

Theorem 4.2. *Let $1 < p \leq q < \infty$ and suppose the covering $\{\Omega_k : k = 0, \ldots, N\}$ satisfies (4.3). Then*

$$\frac{\left(2^{1/q} - 1\right)}{\left(2^{1/q+1}\right)} \varepsilon (N-2)^{1/q - 1/p} \leq a_{N-1}(T),\ a_{N+1}(T) \leq 8\varepsilon. \tag{4.6}$$

Proof. Upper bound. By Theorem 2.1

$$\left(\int_{\Omega_0} |Tf|^q u d\mu \right)^{1/q} \leq 4A[0, t_1] \left(\int_{\Omega_0} |f|^p v d\nu \right)^{1/p} \leq 4 \sup_{t \in \Delta_0} \Psi(t) \left(\int_{\Omega_0} |f|^p v d\nu \right)^{1/p},$$

$$\begin{aligned}\left(\int_{\Omega_N}|Tf-P_Nf|^q\,ud\mu\right)^{1/q} &\le 4A\,[t_N,t_{N+1}]\left(\int_{\Omega_N}|f|^p\,vd\nu\right)^{1/p}\\ &\le 4\sup_{t\in\Delta_N}\Psi(t)\left(\int_{\Omega_N}|f|^p\,vd\nu\right)^{1/p}.\end{aligned}$$

By Theorem 4.1

$$\begin{aligned}\left(\int_{\Omega_k}|Tf-P_kf|^q\,ud\mu\right)^{1/q} &= \left(\int_{\Omega_k}\left|T_kf-\bar{T}_kf\right|^q\,ud\mu\right)^{1/q}\\ &\le 8\mathbf{A}(\Omega_k)\left(\int_{\Omega_k}|f|^p\,vd\nu\right)^{1/p},\ k=1,...,N-1.\end{aligned}$$

Summing these bounds and remembering (4.3) we get

$$\left(\int_{\Omega}|Tf-Pf|^q\,ud\mu\right)^{1/q}\le 8\varepsilon\left(\int_{\Omega}|f|^p\,vd\nu\right)^{1/p}$$

which implies the upper bound in (4.6).

Lower bound. By Theorem 4.1 we can choose functions f_k satisfying $\operatorname{supp} f_k\subseteq\Omega_k$ and

$$\left(\int_{\Omega_k}\left|T_kf_k-\bar{T}_kf_k\right|^q\,ud\mu\right)^{1/q}\ge\left(1-2^{-1/q}\right)\mathbf{A}(\Omega_k)\left(\int_{\Omega_k}|f|^p\,vd\nu\right)^{1/p},\ k=1,...,N-1. \tag{4.7}$$

Let $P:L^p_{vd\nu}\to L^q_{ud\mu}$ be an arbitrary bounded linear operator, $\operatorname{rank}P<N-1$. Then because of the linear independence of Pf_k, $k=1,...,N-1$, there are constants $\alpha_1,...,\alpha_{N-1}$ such that $P\left(\sum_{k=1}^{N-1}\alpha_kf_k\right)=0$. Denote $f=\sum_{k=1}^{N-1}\alpha_kf_k$. For $\tau(x)\in\Delta_k$, $k=1,...,N-1$, we have

$$Tf(x)=\int_{\Omega(t_k)}fd\nu+\alpha_k\int_{\Omega(\tau(x))\setminus\Omega(t_k)}f_kd\nu=\beta_k+\alpha_kT_kf_k,$$

where the value of the constant $\beta_k=\int_{\Omega(t_k)}fd\nu$ does not matter, as we will see shortly. We need a well-known property that in L_p spaces the average of a function is a good approximation to it in the sense that

$$\left(\int_{\Omega_k}\left|T_kf_k-\bar{T}_kf_k\right|^q\,ud\mu\right)^{1/q}\le 2\inf_c\left(\int_{\Omega_k}|T_kf_k-c|^q\,ud\mu\right)^{1/q}. \tag{4.8}$$

Now using (4.8) and Theorem 4.1 we can proceed with the following estimate:

$$\begin{aligned}\int_{\Omega_k}|\beta_k+\alpha_kT_kf_k|^q\,ud\mu &\ge \left(\frac{1}{2}\right)^q\int_{\Omega_k}\left|\alpha_kT_kf_k-\alpha_k\bar{T}_kf_k\right|^q\,ud\mu\\ &\ge \left(\frac{1}{2}\left(1-2^{-1/q}\right)\mathbf{A}(\Omega_k)\right)^q|\alpha_k|^q\left(\int_{\Omega_k}|f_k|^p\,vd\nu\right)^{q/p}.\end{aligned}$$

Therefore, by (4.3) and discrete Hölder's inequality

$$\begin{aligned}
\int_\Omega |Tf - Pf|^q \, ud\mu &\geq \left(\frac{\left(2^{1/q}-1\right)\varepsilon}{\left(2^{1/q+1}\right)}\right)^q \sum_{k=1}^{N-2} |\alpha_k|^q \left(\int_{\Omega_k} |f_k|^p \, vd\nu\right)^{q/p} \\
&= \left(\frac{\left(2^{1/q}-1\right)\varepsilon}{\left(2^{1/q+1}\right)}\right)^q \sum_{k=1}^{N-2} \left(\int_{\Omega_k} |\alpha_k f_k|^p \, vd\nu\right)^{q/p} \\
&\geq \left(\frac{\left(2^{1/q}-1\right)\varepsilon}{\left(2^{1/q+1}\right)}\right)^q (N-2)^{1-q/p} \left(\int_\Omega |f|^p \, vd\nu\right)^{q/p}.
\end{aligned}$$

In the first line the term with $k = N-1$ was omitted because $\mathbf{A}(\Delta_{N-1})$ may be less than ε. The last inequality proves the lower bound. $\square$

Remark 2. Obviously, when $p = q$, (4.6) gives a same-order two-sided bound for a-numbers. Besides, the upper bound on a-numbers gives an upper bound for the Gelfand, Kolmogorov and entropy numbers because the a-numbers are the largest among s-numbers of linear operators [23].

To consider the case $1 < q < p < \infty$ we assume that $\|T\| < \infty$ and therefore $B < \infty$ by Theorem 2.2. Denote

$$\begin{aligned}
\Phi^*_{[a,b]}(x) &= \left(\int_{\Omega[a,\tau(x)]} ud\mu\right)^{1/p} \left(\int_{\Omega[\tau(x),b]} v^{-p'/p} d\nu\right)^{1/p'}, \\
\Phi_{[a,b]}(x) &= \left(\int_{\Omega[\tau(x),b]} ud\mu\right)^{1/p} \left(\int_{\Omega[a,\tau(x)]} v^{-p'/p} d\nu\right)^{1/p'} \\
\mathbf{\Phi}[a,b] &= \left[\int_{\Omega[a,b]} \left(\Phi^*_{[a,c]}\chi_{[a,c]} + \Phi_{[c,b]}\chi_{[c,b]}\right)^r ud\mu\right]^{1/r}
\end{aligned}$$

where $c = c(a,b)$ is the constant defined in Theorem 4.1.

Theorem 4.3. *Suppose that* $1 < q < p < \infty$, $1/r = 1/q - 1/p$, T *is bounded and* $0 < \varepsilon < B$. *Then*

$$a_{N+1}(T) \leq 32^{1/q}\varepsilon.$$

Proof. Upper bound. Let $0 < \varepsilon < B$. Select t', t'' to satisfy

$$\left(\int_{\Omega(t')} \Phi^r ud\mu\right)^{1/r} = \varepsilon, \; \left(\int_{\Omega[t'',\infty]} \Phi^r ud\mu\right)^{1/r} = \varepsilon. \tag{4.9}$$

This implies, in particular, (4.4). Let $\{\Delta_k : k = 1, ..., N\}$ be a uniform (and finite) partition of $[t', t'']$ into segments Δ_k of length m. From the bound

$$\sum_{k=1}^{N} \mathbf{\Phi}(\Delta_k)^r \leq \max_k \sup_{\tau(x)\in\Delta_k} \left(\Phi^*_{\Delta_k}(x) + \Phi_{\Delta_k}(x)\right) \int_{\Omega[t',t'']} ud\mu$$

we see that m can be chosen so that

$$\left(\sum_{k=1}^{N} \mathbf{\Phi}(\Delta_k)^r\right)^{1/r} = \varepsilon. \tag{4.10}$$

With definitions (4.5) and putting $\Omega_k = \Omega(\Delta_k)$ we have for each k by Theorem 4.1

$$\left(\int_{\Omega_k} |Tf - P_k f|^q \, ud\mu\right)^{1/q} \le 2^{4+1/q}\,\mathbf{\Phi}(\Delta_k)\left(\int_{\Omega_k} |f|^p \, vd\nu\right)^{1/p}.$$

By Theorem 2.2 (4.9) implies

$$\begin{aligned}\left(\int_{\Omega(t')} |Tf|^q \, ud\mu\right)^{1/q} &\le 2^{2+1/q}\varepsilon\left(\int_{\Omega(t')} |f|^p \, vd\nu\right)^{1/p}, \\ \left(\int_{\Omega[t'',\infty]} |Tf|^q \, ud\mu\right)^{1/q} &\le 2^{2+1/q}\varepsilon\left(\int_{\Omega[t'',\infty]} |f|^p \, vd\nu\right)^{1/p}.\end{aligned}$$

We use definitions of $P_0, ..., P_N$ from Theorem 4.2. Put $P = \sum_{k=1}^{N} P_k$. The last three estimates and (4.10) give

$$\left(\int_{\Omega} |Tf - Pf|^q \, ud\mu\right)^{1/q} \le 2^{4+1/q}\varepsilon\left(\int_{\Omega} |f|^p \, vd\nu\right)^{1/p}.$$

Since $\mathrm{rank}P \le N$ this proves that $a_{N+1}(T) \le 2^{4+1/q}\varepsilon$. $\square$

Lower bound. Let t', t'' be chosen as in (4.9) and put $t_0 = 0$, $t_1 = t'$. On the n-th step, if $\sup\limits_{t>t_n} \mathbf{\Phi}(t_n, t) \ge \varepsilon$ then we put $t_{n+1} = \min\{t > t_n : \mathbf{\Phi}(t_n, t) = \varepsilon\}$. If $\sup\limits_{t>t_n} \mathbf{\Phi}(t_n, t) < \varepsilon$ we put $t_{n+1} = \infty$. This process stops in a finite number of steps. Suppose that it does not and that $t_n \to t \le \infty$. From (4.4) we conclude that

$$\begin{aligned}\max\left\{\Phi^*_{[a,b]}(x), \Phi_{[a,b]}(x)\right\} &\le \left(\int_{\Omega[t',\infty]} ud\mu\right)^{1/p}\left(\int_{\Omega(t'')} v^{-p'/p} d\nu\right)^{1/p'} = c \\ \text{for } t' &\le a < b \le t''.\end{aligned}$$

Hence, for each k, $\varepsilon^r = \mathbf{\Phi}(\Delta_k)^r \le (2c)^r \int_{\Omega(\Delta_k)} ud\mu$, $\sum_k \int_{\Omega(\Delta_k)} ud\mu = \infty$, which contradicts (4.4).

Denoting N the total number of segments, for an arbitrary bounded linear operator $P : L^p_{vd\nu} \to L^q_{ud\mu}$, $\mathrm{rank}P < N - 1$, instead of (4.7) we have

$$\left(\int_{\Omega_k} \left|T_k f_k - \bar{T}_k f_k\right|^q ud\mu\right)^{1/q} \ge c\varepsilon\left(\int_{\Omega_k} |f|^p \, vd\nu\right)^{1/p}, \ k = 1, ..., N-1,$$

where c is defined in Theorem 4.1 b). Repeating the argument based on (4.8) we get

$$\begin{aligned}\int_{\Omega} |Tf - Pf|^q \, ud\mu &\ge \left(\frac{q\,(p'/r)^{1/p'}\, 2^{1-2r/p'q}\left(1 - 2^{-1/q}\right)}{2\left(\left(1 + \frac{r}{q}\right) 2^{r+r/p'}\right)^{1/p}}\varepsilon\right)^q \sum_{k=1}^{N-2}\left(\int_{\Omega_k} |\alpha_k f_k|^p \, vd\nu\right)^{q/p} \\ &\ge \left(\frac{q\,(p'/r)^{1/p'}\, 2^{1-2r/p'q}\left(1 - 2^{-1/q}\right)}{2\left(\left(1 + \frac{r}{q}\right) 2^{r+r/p'}\right)^{1/p}}\varepsilon\right)^q \left(\int_{\Omega} |f|^p \, vd\nu\right)^{q/p}\end{aligned}$$

where the last transition is by Jensen's inequality with the exponent $0 < q/p < 1$. Thus, $a_{N-1}(T) \ge c\varepsilon$ with the partition we have defined here and the constant c that depends only on p and q. We have proved the following statement.

Theorem 4.4. *Suppose that* $1 < q < p < \infty$, T *is bounded and* $0 < \varepsilon < B$. *Then*

$$a_{N-1}(T) \geq \frac{q\left(p'/r\right)^{1/p'} 2^{1-2r/p'q}\left(1-2^{-1/q}\right)}{2\left(\left(1+\frac{r}{q}\right)2^{r+r/p'}\right)^{1/p}}\varepsilon.$$

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ONE-DIMENSIONAL INTEGRAL RELLICH TYPE INEQUALITIES

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Abstract. The motive of this note is twofold. Inspired by the recent development of a new kind of Hardy inequality, here we discuss the corresponding Hardy–Rellich and Rellich inequality versions in the integral form. The obtained sharp Hardy–Rellich type inequality improves the previously known result. Meanwhile, the established sharp Rellich type integral inequality seems new.

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1 Introduction

In the celebrated paper, [9], Godfrey H. Hardy first stated the famous inequality. The result reads as follows. For any $1 < p < \infty$ and f be a p-integrable function on $(0, \infty)$, then the function $r \longmapsto \frac{1}{r}\int_0^r f(t)\,\mathrm{d}t$ is p-integrable over $(0, \infty)$ and there holds

$$\int_0^\infty \left|\frac{1}{r}\int_0^r f(t)\,\mathrm{d}t\right|^p \mathrm{d}r \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(r)|^p\,\mathrm{d}r. \tag{1.1}$$

The constant on the right-hand side of (1.1) is sharp. The development of the famous Hardy inequality (1.1) during the period 1906–1928 has its own history and we refer to [12] (also, see the preface of [22]). Recent progress by Frank–Laptev–Weidl [10] presents a novel one-dimensional inequality with the same sharp constant, which improves the classical Hardy inequality (1.1).

This new version looks as follows. For any $1 < p < \infty$ and for any $f \in L^p(0, \infty)$, which vanishes at zero, there holds

$$\int_0^\infty \sup_{0<s<\infty}\left|\min\left\{\frac{1}{r}, \frac{1}{s}\right\}\int_0^s f(t)\,\mathrm{d}t\right|^p \mathrm{d}r \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(r)|^p\,\mathrm{d}r. \tag{1.2}$$

Certainly, (1.2) gives an improvement of (1.1). Recently, the multidimensional version in the supercritical case and the discrete version of (1.2) have been established in [20] and [19], respectively. In the same spirit, one may ask about the possible structure of Hardy–Rellich and Rellich type inequalities. In this short note, we obtain the possible form of these two types of inequalities.

Let us recall the one-dimensional Hardy–Rellich inequality. For $f \in C^1[0, \infty)$ with $f(0) = 0$, there holds

$$\int_0^\infty \frac{|f(r)|^2}{r^2}\,\mathrm{d}r \leq 4\int_0^\infty |f'(r)|^2\,\mathrm{d}r. \tag{1.3}$$

Starting from it, there have been several articles in which the authors studied many improvements in inequality (1.3). Here we mention only a few of them [3, 6, 7, 11, 13, 16, 17, 24, 23] and references therein. Now let us write (1.3) in the integral form. Note that it can be derived from the weighted one-dimensional classical Hardy inequality. This reads as follows. Let $f \in C^1(0,\infty)$, then there holds

$$\int_0^\infty \frac{|\int_0^r f'(t)\,\mathrm{d}t|^2}{r^2}\,\mathrm{d}r \le 4\int_0^\infty |f'(r)|^2\,\mathrm{d}r. \tag{1.4}$$

Here the constant 4 is sharp. We give an improved version of this inequality in Theorem 2.1.

Let us briefly mention another important function inequality the so-called Rellich inequality which was first introduced in [18]. It is worth recalling the one-dimensional Rellich inequality. The classical one-dimensional Rellich inequality states that for $f \in C^2[0,\infty)$ with $f(0) = 0$ and $f'(0) = 0$, there holds

$$\int_0^\infty \frac{|f(r)|^2}{r^4}\,\mathrm{d}r \le C\int_0^\infty |f''(r)|^2\,\mathrm{d}r, \tag{1.5}$$

where $C > 0$ is independent of f. Over the past few decades, there has been a constant effort to improve (1.5). Here are some closely related papers [8, 14, 1, 15, 4, 21, 5]. In this short contribution, we also obtain another type of Rellich inequality (see Theorem 2.2 with $p = 2$). To the best of our knowledge, the most recent progress in this direction was made in [4]. However, a one-dimensional study is still missing. As far as we know, a sharp constant in this inequality was not found. Thus, trying to fill this gap is another motivation for the present paper. Taking inspiration from there we obtain the following version of Rellich inequality. For any $f \in L^2(0,\infty)$ there holds

$$\begin{aligned}
&\int_0^\infty \frac{1}{r^4}\left(\int_0^r\int_0^\tau |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^2\mathrm{d}r\\
&\le \int_0^\infty \frac{1}{r^4}\left(\int_0^r \sup_{0<s<\infty}\min\left\{1,\frac{\tau}{s}\right\}\int_0^s |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^2\mathrm{d}r\\
&\le \frac{16}{9}\int_0^\infty |f(r)|^2\,\mathrm{d}r.
\end{aligned} \tag{1.6}$$

Moreover, we will show that the constant 16/9 is a sharp constant. Therefore, (1.6) can be compared with (1.5). Note that we have mentioned only the $L^2(0,\infty)$ case but we will discuss the result for the general $L^p(0,\infty)$ case.

2 Preliminaries and main results

Let us begin this section with basic facts about *a decreasing rearrangement*. For more details, we refer to [2, Section 2.1]. The decreasing rearrangement of f is the function f^* defined on $[0,\infty)$ by

$$f^*(x) = \inf\{\lambda : \mu_f(\lambda) \le x\}, \quad x \ge 0,$$

where $\mu_f(\lambda) = |\{x \in \mathbb{R} : |f(x)| > \lambda\}|, \quad \lambda \ge 0$. Here $|J|$ is the Lebesgue measure of the set $J \subset \mathbb{R}$. It is well known that f^* is a nonnegative and nonincreasing function. Irrespective of several properties of f^*, the useful property in our context is the equimeasurability property, i.e.

$$|\{|f| > \tau\}| = |\{f^* > \tau\}| \text{ for all } \tau \ge 0. \tag{2.1}$$

By using the *layer cake representation* and the above property, we have the following helpful identity:

$$\int_0^\infty |f(t)|^p\,\mathrm{d}t = \int_0^\infty |f^*(t)|^p\,\mathrm{d}t \quad \text{for all } p \ge 1. \tag{2.2}$$

Also, for any $s>0$ there holds

$$\int_0^s |f(t)|\,\mathrm{d}t \leq \int_0^s f^*(t)\,\mathrm{d}t. \tag{2.3}$$

These relations will be valuable in the proofs.

Now, we are ready to state the following important observation.

Lemma 2.1. *For any $r>0$ and $f\in L^1(0,r)$, the following identity holds:*

$$\sup_{0<s<\infty} \min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t = \int_0^r f^*(t)\,\mathrm{d}t. \tag{2.4}$$

Proof. We wish to calculate the supremum by using the monotonicity of f^*. For any fixed $r>0$, we consider the following two cases:

Case 1. Let $0<s\leq r$. Then we obtain

$$\min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t = \int_0^s f^*(t)\,\mathrm{d}t \leq \int_0^r f^*(t)\,\mathrm{d}t.$$

Case 2. Let $r\leq s<\infty$. Then we have by change of variable

$$\min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t = \frac{r}{s}\int_0^s f^*(t)\,\mathrm{d}t \leq \frac{r}{s}\int_0^s f^*(tr/s)\,\mathrm{d}t = \int_0^r f^*(v)\,\mathrm{d}v.$$

In both cases, we get

$$\min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t \leq \int_0^r f^*(t)\,\mathrm{d}t.$$

Hence, the supremum is attained at $s=r$ and we arrive at

$$\sup_{0<s<\infty} \min\left\{1, \frac{r}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t = \int_0^r f^*(t)\,\mathrm{d}t.$$

$\square$

Now, we are ready to present an improvement of (1.4). That is, this gives a natural improvement of the Hardy–Rellich inequality in the integral form. Below we will describe the corresponding differential form which improves the original Hardy–Rellich inequality (1.3) in a simple form.

Theorem 2.1. *Let $f\in L^2(0,\infty)$, then there holds*

$$\int_0^\infty \sup_{0<s<\infty}\left|\min\left\{\frac{1}{r}, \frac{1}{s}\right\} \int_0^s f(t)\,\mathrm{d}t\right|^2 \mathrm{d}r \leq 4\int_0^\infty |f(r)|^2\,\mathrm{d}r. \tag{2.5}$$

Moreover, the constant 4 in the above inequality is sharp in the sense that no inequality of the form

$$\int_0^\infty \sup_{0<s<\infty}\left|\min\left\{\frac{1}{r}, \frac{1}{s}\right\} \int_0^s f(t)\,\mathrm{d}t\right|^2 \mathrm{d}r \leq C\int_0^\infty |f(r)|^2\,\mathrm{d}r$$

holds, for $f\in L^2(0,\infty)$ such that $f\not\sim 0$ on $(0,\infty)$, when $C<4$.

Now, we are going to discuss the second main result of this note. Before presenting the statement first let us recall the classical one-dimensional L^p-Rellich inequality (see, e.g. [1]). This reads as follows. Let $p > 1$, $f \in C^2[0, \infty)$ with $f(0) = 0$ and $f'(0) = 0$ there holds

$$\int_0^\infty \frac{|f(r)|^p}{r^{2p}} \, \mathrm{d}r \le \frac{p^{2p}}{(p-1)^p(2p-1)^p} \int_0^\infty |f''(r)|^p \, \mathrm{d}r. \tag{2.6}$$

Now, we are ready to demonstrate the one-dimensional Rellich-type inequality in the following integral form.

Theorem 2.2. *Let $f \in L^p(0, \infty)$, $p > 1$. Then we have*

$$\begin{aligned} &\int_0^\infty \frac{1}{r^{2p}} \left(\int_0^r \int_0^\tau |f(t)| \, \mathrm{d}t \, \mathrm{d}\tau \right)^p \mathrm{d}r \\ &\le \int_0^\infty \frac{1}{r^{2p}} \left(\int_0^r \sup_{0<s<\infty} \min\left\{1, \frac{\tau}{s}\right\} \int_0^s |f(t)| \, \mathrm{d}t \, \mathrm{d}\tau \right)^p \mathrm{d}r \\ &\le \frac{p^{2p}}{(p-1)^p(2p-1)^p} \int_0^\infty |f(r)|^p \, \mathrm{d}r. \end{aligned} \tag{2.7}$$

Moreover, the constant $\frac{p^{2p}}{(p-1)^p(2p-1)^p}$ in the above inequality turns out to be sharp in the sense that no inequality of the form

$$\int_0^\infty \frac{1}{r^{2p}} \left(\int_0^r \int_0^\tau |f(t)| \, \mathrm{d}t \, \mathrm{d}\tau \right)^p \mathrm{d}r \le C \int_0^\infty |f(r)|^p \, \mathrm{d}r.$$

for all $f \in L^p(0, \infty)$ such that $f \not\sim 0$ on $(0, \infty)$, when $C < \frac{p^{2p}}{(p-1)^p(2p-1)^p}$.

3 Proofs of Theorems 2.1 and 2.2

This section is concerned with the proofs of Theorems 2.1 and 2.2. Before going further let us recall the following lemma.

Lemma 3.1. [20, Lemma 3.1] *Let $1 < p < \infty$. Let w be any nonnegative measurable function on $(0, \infty)$. Assume h is a strictly positive non-decreasing function on $(0, \infty)$ such that $sh(r) \le rh(s)$ for any $r, s \in (0, \infty)$ with $r \le s$. Let $f \in L^1(0, r)$ for any $r > 0$. Then we have*

$$\int_0^\infty w(r) \sup_{0<s<\infty} \left| \min\left\{ \frac{1}{h(r)}, \frac{1}{h(s)} \right\} \int_0^s f(t) \, \mathrm{d}t \right|^p \mathrm{d}r \le \int_0^\infty w(r) \left| \frac{1}{h(r)} \int_0^r f^*(t) \, \mathrm{d}t \right|^p \mathrm{d}r.$$

Now, as a direct corollary of Lemma 3.1, we derive the proof of Theorem 2.1.

Proof of Theorem 2.1. Let us consider $w(r) = 1$ and $h(r) = r$ to be functions on $(0, \infty)$ and substitute these in Lemma 3.1 with $p = 2$, then we have

$$\int_0^\infty \sup_{0<s<\infty} \left| \min\left\{ \frac{1}{r}, \frac{1}{s} \right\} \int_0^s f(t) \, \mathrm{d}t \right|^2 \mathrm{d}r \le \int_0^\infty \frac{1}{r^2} \left| \int_0^r f^*(t) \, \mathrm{d}t \right|^2 \mathrm{d}r.$$

By using the Hardy–Rellich inequality in form (1.4) for the function f^*, we obtain

$$\begin{aligned} \int_0^\infty \sup_{0<s<\infty} \left| \min\left\{ \frac{1}{r}, \frac{1}{s} \right\} \int_0^s f(t) \, \mathrm{d}t \right|^2 \mathrm{d}r &\le 4 \int_0^\infty |f^*(r)|^2 \, \mathrm{d}r \\ &= 4 \int_0^\infty |f(r)|^2 \, \mathrm{d}r. \end{aligned}$$

In the last step, we have used norm preserving property (2.2). The sharpness follows from the optimality of the constant in (1.4). This completes the proof.

Proof of Theorem 2.2. The first inequality follows from the property of the supremum. Now taking the integral of (2.4) from 0 to r we have

$$\int_0^r \sup_{0<s<\infty} \min\left\{1, \frac{\tau}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t\,\mathrm{d}\tau = \int_0^r \int_0^\tau f^*(t)\,\mathrm{d}t\,\mathrm{d}\tau. \tag{3.1}$$

Then

$$\begin{aligned}
&\int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r \sup_{0<s<\infty} \min\left\{1, \frac{\tau}{s}\right\} \int_0^s |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r \\
&\overset{(2.3)}{\leq} \int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r \sup_{0<s<\infty} \min\left\{1, \frac{\tau}{s}\right\} \int_0^s f^*(t)\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r \\
&\overset{(3.1)}{=} \int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r \int_0^\tau f^*(t)\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r \\
&\overset{(2.6)}{\leq} \frac{p^{2p}}{(p-1)^p(2p-1)^p} \int_0^\infty |f^*(r)|^p\,\mathrm{d}r \\
&\overset{(2.2)}{=} \frac{p^{2p}}{(p-1)^p(2p-1)^p} \int_0^\infty |f(r)|^p\,\mathrm{d}r.
\end{aligned}$$

Optimality. We set

$$C_p := \sup_{f\in L^p(0,\infty)\setminus\{0\}} \frac{\int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r \int_0^\tau |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r}{\int_0^\infty |f(r)|^p\,\mathrm{d}r}. \tag{3.2}$$

The validity of (2.7) immediately implies

$$C_p \leq \frac{p^{2p}}{(p-1)^p(2p-1)^p}.$$

So, it remains to show the reverse inequality and this will be done by giving a proper minimizing sequence. We divide the proof into some steps.

Step 1. Let us start with a cut-off function $\chi : [0,\infty) \to \mathbb{R}$ with the following properties:

1. $\chi(r) \in [0,1]$ for all $r \in [0,\infty)$ and χ is smooth;
2. χ satisfies the following
$$\chi(r) = \begin{cases} 1, & 0 \leq r \leq 1, \\ 0, & 2 \leq r < \infty \end{cases}$$
3. χ is decreasing function, i.e. $\chi'(r) \leq 0$ for all $r \in (0,\infty)$.

Now for a small $\epsilon > 0$, let us define the minimizing functions $\{f_\epsilon\}$ as follows:

$$f_\epsilon(r) := r^{\frac{\epsilon-1}{p}}\chi(r).$$

Step 2. In this step we will estimate the right-hand side of (2.7). The denominator of (3.2) gives

$$\begin{aligned}
\int_0^\infty |f_\epsilon(r)|^p\,\mathrm{d}r &= \int_0^\infty r^{\epsilon-1}\chi^p(r)\,\mathrm{d}r \\
&= \int_0^1 r^{\epsilon-1}\,\mathrm{d}r + \int_1^2 r^{\epsilon-1}\chi^p(r)\,\mathrm{d}r \\
&= \frac{1}{\epsilon} + O(1).
\end{aligned} \tag{3.3}$$

Therefore, for a fixed positive ϵ, we have $f_\epsilon \in L^p(0,\infty)$.

Step 3. In this part we will evaluate the numerator of (3.2). Using the integration by parts, we have

$$
\begin{aligned}
&\int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r\int_0^\tau |f_\epsilon(t)|\,\mathrm{d}t\mathrm{d}\tau\right)^p \mathrm{d}r\\
&= \int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r\int_0^\tau t^{\frac{\epsilon-1}{p}}\chi(t)\,\mathrm{d}t\mathrm{d}\tau\right)^p \mathrm{d}r\\
&= \left(\frac{p}{\epsilon-1+p}\right)^p\int_0^\infty \frac{1}{r^{2p}}\left[\int_0^r \chi(\tau)\tau^{\frac{\epsilon-1+p}{p}}\,\mathrm{d}\tau - \int_0^r\int_0^\tau t^{\frac{\epsilon-1+p}{p}}\chi'(t)\,\mathrm{d}t\,\mathrm{d}\tau\right]^p \mathrm{d}r\\
&\geq \left(\frac{p}{\epsilon-1+p}\right)^p\int_0^\infty \frac{1}{r^{2p}}\left[\int_0^r \chi(\tau)\tau^{\frac{\epsilon-1+p}{p}}\,\mathrm{d}\tau\right]^p \mathrm{d}r\\
&= \left(\frac{p}{\epsilon-1+p}\right)^p\left(\frac{p}{\epsilon-1+2p}\right)^p\int_0^\infty \frac{1}{r^{2p}}\left[\chi(r)r^{\frac{\epsilon-1+2p}{p}} - \int_0^r \tau^{\frac{\epsilon-1+2p}{p}}\chi'(\tau)\,\mathrm{d}\tau\right]^p \mathrm{d}r\\
&\geq \left(\frac{p}{\epsilon-1+p}\right)^p\left(\frac{p}{\epsilon-1+2p}\right)^p\int_0^\infty r^{\epsilon-1}\chi^p(r)\,\mathrm{d}r\\
&= \left(\frac{p}{\epsilon-1+p}\right)^p\left(\frac{p}{\epsilon-1+2p}\right)^p\left[\int_0^1 r^{\epsilon-1}\,\mathrm{d}r + \int_1^2 r^{\epsilon-1}\chi^p(r)\,\mathrm{d}r\right]\\
&= \frac{1}{\epsilon}\left(\frac{p}{\epsilon-1+p}\right)^p\left(\frac{p}{\epsilon-1+2p}\right)^p + O(1). \qquad (3.4)
\end{aligned}
$$

In between, exploiting $\chi' \leq 0$, we used an obvious inequality $(a+b)^p \geq a^p$ twice, for nonnegative real numbers a and b.

Step 4. Finally, by using (3.3) and (3.4) we estimate the ratio

$$
\begin{aligned}
&\frac{\int_0^\infty \frac{1}{r^{2p}}\left(\int_0^r\int_0^\tau |f(t)|\,\mathrm{d}t\,\mathrm{d}\tau\right)^p \mathrm{d}r}{\int_0^\infty |f(r)|^p\,\mathrm{d}r}\\
&\geq \frac{\frac{1}{\epsilon}\left(\frac{p}{\epsilon-1+p}\right)^p\left(\frac{p}{\epsilon-1+2p}\right)^p + O(1)}{\frac{1}{\epsilon}+O(1)} \to \frac{p^{2p}}{(p-1)^p(2p-1)^p} \quad \text{for } \epsilon \to 0.
\end{aligned}
$$

Hence $\{f_\epsilon\}$ is a required minimizing sequence and, in turn, we have

$$
C_p = \frac{p^{2p}}{(p-1)^p(2p-1)^p}.
$$

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