EURASIAN MATHEMATICAL JOURNAL
volume 15 , number 4, 2024

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EURASIAN NATIONAL UNIVERSITY

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

EURASIAN MATHEMATICAL JOURNAL

ISSN 2077-9879

VOLUME 15, NUMBER 4 2024

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2024, Volume 15, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

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The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Republic of Kazakhstan

The Moscow Editorial Office
The Patrice Lumumba Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
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EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 4 (2024), 08 – 32

ORDER-SHARP ESTIMATES FOR DECREASING REARRANGEMENTS OF CONVOLUTIONS

E.G. Bakhtigareeva, M.L. Goldman

Communicated by V.D. Stepanov

Key words: convolutions, radial symmetrical functions, order-sharp estimates, decreasing rearrangements, symmetrical rearrangements, mean values of rearrangements.

AMS Mathematics Subject Classification: 46E30, 47A15, 47A30.

Abstract. In this paper, we study estimates for convolutions on some classes of measurable, positive and radial symmetrical functions. On this base we prove then order-sharp estimates for decreasing and symmetrical rearrangements of convolutions and for weighted mean values of rearrangements. These estimates give, in particular, a reversal of the well-known inequalities for convolutions proved by R. O'Neil.

DOI: https://doi.org/10.32523/2077-9879-2024-15-4-08-32

1 Introduction

In this paper, we consider estimates for decreasing rearrangements of convolutions. The books by S.G. Krein, Yu.I. Petunin and E.M. Semenov [12], C. Bennett and R. Sharpley [3] contain main definitions and basic facts related to this topic. The properties of the classical Bessel and Riesz potentials are described in the books by V.G. Maz'ya [13], S.M. Nikol'skii [14], E.M. Stein [17].

In Section 2 of the paper, we obtain two–sided estimates for convolutions for some classes of radial symmetrical functions. The case of functions that are positive on $\mathbb{R}^n$ is considered here. In Section 3 we consider the case, where one of the convolved function has support contained in the finite ball $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$ for some $R \in (0, \infty)$. Such consideration will be useful for application of these results to generalized Bessel potentials. In that case the kernel of the convolution is splitted into two parts, and one part is supported in B_R.

We apply these estimates in Section 4 to obtaining two–sided estimates for symmetrical and decreasing rearrangements of convolutions. These estimates give, in particular, a reversal of the well-known inequality for convolutions proved by R. O'Neil [16]. They develop and refine the estimates obtained in our papers [5]–[6], [8]–[10]. We will use these results to justify pointwise and integral coverings for cones of decreasing rearrangements for generalized Bessel-Riesz potentials. As a result, exact descriptions of equivalent cones for cones of decreasing rearrangements of potentials will be obtained. They develop the results of our works [9], [10]. Note that E. Nursultanov and S. Tikhonov [15] obtained some further developments of O'Neil's results. For researches related to the topic, see [2, 4, 11].

In Section 5 we prove a lemma which may be useful in many considerations related to the subject of this paper. The proof of this lemma is related to the proofs of Theorems in Sections 2–4.

2 Two–sided estimates for convolutions. The case $R = \infty$

Let $\alpha \in (1, \infty)$, $R \in (0, \infty]$.

Definition 1. As $J_\alpha(\infty)$ we denote the class of all measurable functions $F : (0, \infty) \to (0, \infty)$, such that for all $\xi \in (0, \infty)$

$$\tau \in [\xi, 2\xi] \quad \text{implies} \quad \alpha^{-1} F(\xi) \le F(\tau) \le \alpha F(\xi). \tag{2.1}$$

Remark 1. Let $\alpha \in (1, \infty)$, $F \in J_\alpha(\infty)$, $m \in \mathbb{N}$, $\xi \in (0, \infty)$. Then, the following estimate holds

$$\eta \in [\xi, 2^m \xi] \Rightarrow \alpha^{-m} F(\xi) \le F(\eta) \le \alpha^m F(\xi). \tag{2.2}$$

Proof. Let us use the method of induction.

For $m = 1$ estimate (2.2) for $F \in J_\alpha(\infty)$ follows from the definition.

Assumption of induction: assume that estimate (2.2) holds for all numbers from 1 to m. Step of induction: let us prove that then it is true for the number $m + 1$.

For $\eta \in [\xi, 2^{m+1}\xi] = [\xi, 2^m\xi] \bigcup [2^m\xi, 2^{m+1}\xi]$ we have on $[\xi, 2^m\xi]$ estimate (2.2), and for $\eta \in [2^m\xi, 2^{m+1}\xi]$ the estimate holds for $F \in J_\alpha(\infty)$

$$\alpha^{-1} F(2^m\xi) \le F(\eta) \le \alpha F(2^m\xi).$$

For $\eta = 2^m\xi$, according to (2.2), $\alpha^{-m} F(\xi) \le F(2^m\xi) \le \alpha^m F(\xi)$, so that we obtain

$$\alpha^{-(m+1)} F(\xi) \le F(\eta) \le \alpha^{(m+1)} F(\xi), \quad \eta \in [2^m\xi, 2^{m+1}\xi].$$

Recall that $\alpha > 1$, so that (2.2) implies, in particular, that

$$\alpha^{-(m+1)} F(\xi) \le F(\eta) \le \alpha^{(m+1)} F(\xi), \quad \eta \in [\xi, 2^m\xi].$$

These estimates give the desired inequality:

$$\alpha^{-(m+1)} F(\xi) \le F(\eta) \le \alpha^{(m+1)} F(\xi), \quad \eta \in [\xi, 2^{m+1}\xi].$$

□

Definition 2. As $J_\alpha(R)$ with $R \in (0, \infty)$ we denote the class of all measurable functions $F : (0, \infty) \to [0, \infty)$, such that $F(\xi) > 0$, $\xi \in (0, R]$, $F(\xi) = 0$ for $\xi > R$ and

$$\xi \in (0, R),\, \tau \in [\xi, \min\{2\xi, R\}] \Rightarrow \alpha^{-1} F(\xi) \le F(\tau) \le \alpha F(\xi).$$

For a function $F \in J_\alpha(R)$, $R \in (0, \infty)$ we have an analogue of (2.2):

$$\xi \in (0, R), \tau \in [\xi, \min\{2^m\xi, R\}] \Rightarrow \alpha^{-m} F(\xi) \le F(\tau) \le \alpha^m F(\xi). \tag{2.3}$$

The following remark shows the link of two–sided estimates for the left and the right ends of the segment $[\xi, 2^m\xi]$.

Remark 2. 1. Let $\alpha \in (1, \infty)$. From (2.2) it follows easily that for $\beta = \alpha^2$

$$\tau \in [\xi, 2^m\xi] \Rightarrow \beta^{-m} F(2^m\xi) \le F(\tau) \le \beta^m F(2^m\xi). \tag{2.4}$$

2. Let $\beta \in (1, \infty)$. From (2.2) it follows easily that for $\alpha = \beta^2$

$$\tau \in [\xi, 2^m\xi] \Rightarrow \alpha^{-m} F(\xi) \le F(\tau) \le \alpha^m F(\xi). \tag{2.5}$$

The next remark shows the link of two–sided estimates for any two points of the segment $[\xi, 2^m\xi]$.

Remark 3. Let $\alpha \in (1,\infty)$, $m \in \mathbb{N}$, $F \in J_\alpha(\infty)$, so that estimate (2.2) holds. Then, it follows easily that for any two points $t, \tau \in [\xi, 2^m\xi]$ the following estimate holds:

$$\alpha^{-2m}F(t) \le F(\tau) \le \alpha^{2m}F(t).$$

Remark 4. Let $\alpha \in (1,\infty)$, $m \in \mathbb{N}$, $R \in (0,\infty)$, $F \in J_\alpha(R)$, so that estimate (2.3) holds. Then, it follows easily that for any $t, \tau \in [\xi, \min\{2^m\xi, R\}]$ the following estimate holds:

$$\alpha^{-2m}F(t) \le F(\tau) \le \alpha^{2m}F(t).$$

Theorem 2.1. *Let $\alpha, \beta \in (1,\infty)$; $F \in J_\alpha(\infty)$, $G \in J_\beta(\infty)$, $x \in \dot{\mathbb{R}}^n = \{x \in \mathbb{R}^n, x \ne 0\}$,*

$$f(x) = F(|x|),\ g(x) = G(|x|); \tag{2.6}$$

$$u(x) = (f * g)(x) = (g * f)(x) = \int\limits_{\mathbb{R}^n} f(y)g(x-y)dy; \tag{2.7}$$

$$\tilde{u}(x) = \int\limits_0^\infty \left[F(\tau)G(|x|+\tau) + F(|x|+\tau)G(\tau)\right]\tau^{n-1}d\tau. \tag{2.8}$$

Then, there exist constants $c_i = c_i(\alpha, \beta, n)$, $i = 1, 2$, such that $0 < c_1 \le c_2 < \infty$ and

$$c_1 u(x) \le \tilde{u}(x) \le c_2 u(x), \quad x \in \dot{\mathbb{R}}^n. \tag{2.9}$$

Proof. 1. Let $S^{n-1} = \{\omega \in \mathbb{R}^n : |\omega| = 1\}$ be the unit sphere in $\mathbb{R}^n$, $C_n = \int\limits_{S^{n-1}} d\omega = 2\pi^{n/2}\Gamma(n/2)^{-1}$ be the integral over all angles in S^{n-1}.

For $x \in \dot{\mathbb{R}}^n$ we introduce the spherical system of coordinates with the center at the point 0 and the polar axis L_0 such that $x \in L_0$. In the spherical coordinates for $y \in \dot{\mathbb{R}}^n$ we have

$$y = (\tau, \omega),\ \tau = |y| > 0,\ \omega \in S^{n-1};$$

and we obtain that

$$\begin{aligned}\int\limits_{\dot{\mathbb{R}}^n} F(|y|)G(|x|+|y|)dy &= \int\limits_0^\infty F(\tau)G(|x|+\tau)\left(\int\limits_{S^{n-1}} d\omega\right)\tau^{n-1}d\tau \\ &= C_n \int\limits_0^\infty F(\tau)G(|x|+\tau)\tau^{n-1}d\tau. \end{aligned} \tag{2.10}$$

Let $\Omega = B(x, |x|/2)$ be the ball with the center x and the radius $r = |x|/2$. It follows from (2.6) and (2.7) that

$$u(x) = \int\limits_{\dot{\mathbb{R}}^n} F(|y|)G(|x-y|)dy = I_1 + I_2, \quad x \in \dot{\mathbb{R}}^n, \tag{2.11}$$

where

$$I_1 = \int\limits_{\dot{\mathbb{R}}^n \setminus \Omega} F(|y|)G(|x-y|)dy, \quad I_2 = \int\limits_{\Omega} F(|y|)G(|x-y|)dy. \tag{2.12}$$

For $y \in \dot{\mathbb{R}}^n \setminus \Omega$ we have $|x| \le 2|x-y|$, so

$$|y| = |y - x + x| \le |y-x| + |x| \le 3|y-x|.$$

Then,

$$|x-y| \le |x| + |y| \le 5|x-y| < 2^3|x-y|, \quad y \in \dot{\mathbb{R}}^n \setminus \Omega,$$

and for $G \in J_\beta(\infty)$ it follows from (2.2) with $m = 3$, $\alpha = \beta$ that

$$\beta^{-3} \le G(|x| + |y|)/G(|x-y|) \le \beta^3, \quad y \in \dot{\mathbb{R}}^n \setminus \Omega.$$

It means that

$$\beta^{-3} I_1 \le \int\limits_{\dot{\mathbb{R}}^n \setminus \Omega} F(|y|)G(|x| + |y|)dy \le \beta^3 I_1. \tag{2.13}$$

The left–hand–side inequality in (2.13) shows that

$$I_1 \le \beta^3 \int\limits_{\dot{\mathbb{R}}^n} F(|y|)G(|x| + |y|)dy.$$

Therefore, analogously to (2.10) we obtain in the spherical coordinates

$$I_1 \le \beta^3 C_n \int\limits_0^\infty F(\tau)G(|x| + \tau)\tau^{n-1}d\tau. \tag{2.14}$$

Moreover, let K_Ω be a minimal cone with the cone apex at the origin, such that $\Omega \subset K_\Omega$. Denote

$$\Sigma_\Omega = \left\{\omega \in S^{n-1} : \omega \notin K_\Omega\right\}, \ \sigma_n = \int\limits_{\Sigma_\Omega} d\omega;$$

$$\Delta_\Omega = \left\{\omega \in S^{n-1} : \omega \in K_\Omega\right\}, \ \delta_n = \int\limits_{\Delta_\Omega} d\omega.$$

Our construction is such that the sets K_Ω, Σ_Ω, Δ_Ω are the same for all $x \in L_0$, they depend only on dimension n. Moreover, $\Sigma_\Omega \cap \Delta_\Omega = \{\emptyset\}$, $\Sigma_\Omega \cup \Delta_\Omega = S^{n-1}$. Then, $0 < \sigma_n, \delta_n$, $\sigma_n + \delta_n = \int\limits_{S^{n-1}} d\omega = C_n$, so that, in particular, $0 < \sigma_n < C_n$.

Note that $\Omega \subset K_\Omega \Rightarrow \mathbb{R}^n \setminus K_\Omega \subset \mathbb{R}^n \setminus \Omega$. Thus, the right–hand–side estimate in (2.13) implies

$$I_1 \ge \beta^{-3} \int\limits_{\mathbb{R}^n \setminus \Omega} F(|y|)G(|x| + |y|)dy \ge \beta^{-3} \int\limits_{\mathbb{R}^n \setminus K_\Omega} F(|y|)G(|x| + |y|)dy.$$

Like in (2.10), we obtain in the spherical coordinates that

$$\int\limits_{\mathbb{R}^n\setminus K_\Omega} F(|y|)G(|x|+|y|)dy = \int\limits_0^\infty F(\tau)G(|x|+\tau)\left(\int\limits_{\Sigma_\Omega} d\omega\right)\tau^{n-1}d\tau$$
$$= \sigma_n \int\limits_0^\infty F(\tau)G(|x|+\tau)\tau^{n-1}d\tau.$$

As a result,

$$I_1 \geq \beta^{-3}\sigma_n \int\limits_0^\infty F(\tau)G(|x|+\tau)\tau^{n-1}d\tau. \tag{2.15}$$

Estimates (2.14) and (2.15) give the two–sided inequality:

$$\beta^{-3}C_n^{-1}I_1 \leq \int\limits_0^\infty F(\tau)G(|x|+\tau)\tau^{n-1}d\tau \leq \beta^3\sigma_n^{-1}I_1. \tag{2.16}$$

2. We move on to the estimates for $I_2 = \int\limits_\Omega F(|y|)G(|x-y|)dy$. For $y \in \Omega$ we have

$$y \in \Omega \Rightarrow \begin{cases} |y| \leq |x| + |y-x|; \\ 3|y| \geq \frac{3}{2}|x| = |x| + \frac{1}{2}|x| \geq |x| + |y-x|. \end{cases}$$

Thus, $y \in \Omega \Rightarrow 2^{-2}(|x|+|y-x|) \leq |y| \leq |x|+|y-x|$.

For $F \in J_\alpha(\infty)$ it follows from here and from Remark 2 (see (2.4)) that

$$\alpha^{-2}F(|x|+|y-x|) \leq F(|y|) \leq \alpha^2 F(|x|+|y-x|), \quad y \in \Omega.$$

Therefore,

$$\alpha^{-2}I_2 \leq \int\limits_\Omega F(|x|+|y-x|)G(|y-x|)dy \leq \alpha^2 I_2.$$

We introduce the spherical system of coordinates with the center at the point x and the spherical radius $\lambda = |y-x|$. Then,

$$y \in \Omega,\, y \neq x \Leftrightarrow y - x = (\lambda, \omega),\, 0 < \lambda = |y-x| \leq |x|/2, \omega \in S^{n-1},$$

and we obtain the following equality with $C_n = \int\limits_{S^{n-1}} d\omega = 2\pi^{n/2}\Gamma(n/2)^{-1}$:

$$\int\limits_\Omega F(|x|+|y-x|)G(|y-x|)dy = C_n \int\limits_0^{|x|/2} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda,$$

so

$$\alpha^{-2}C_n^{-1}I_2 \leq \int\limits_0^{|x|/2} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda \leq \alpha^2 C_n^{-1}I_2. \tag{2.17}$$

3. For the further consideration it is convenient to use the following notation: let $A(x), B(x), C(x), D(x), E(x) \geq 0,\ x \in \dot{\mathbb{R}}^n$. We write $D(x) \cong E(x)$ if there exist constants $c_i = c_i(\alpha, \beta, n), i = 1, 2$, such that $0 < c_1 \leq c_2 < \infty$ and

$$c_1 D(x) \leq E(x) \leq c_2 D(x), x \in \dot{\mathbb{R}}^n. \tag{2.18}$$

Let us note that if

$$0 \leq C(x) \leq c_3 A(x), x \in \dot{\mathbb{R}}^n, \tag{2.19}$$

with $0 \leq c_3 = c_3(\alpha, \beta, n) < \infty$, then

$$A(x) + B(x) \cong A(x) + B(x) + C(x), x \in \dot{\mathbb{R}}^n. \tag{2.20}$$

Indeed, according to (2.19)

$$A(x) + B(x) \leq A(x) + B(x) + C(x) \leq (1 + c_3)(A(x) + B(x)), x \in \dot{\mathbb{R}}^n.$$

Let here (see estimates (2.16), (2.17))

$$A(x) := I_1 \cong \int_0^\infty F(\tau) G(|x| + \tau) \tau^{n-1} d\tau,$$

$$B(x) := I_2 \cong \int_0^{|x|/2} F(|x| + \tau) G(\tau) \tau^{n-1} d\tau,$$

$$C(x) := \int_{|x|/2}^\infty F(|x| + \tau) G(\tau) \tau^{n-1} d\tau.$$

For $\tau \geq |x|/2$ we have $|x| \leq 2\tau$, so that

$$\tau \leq |x| + \tau \leq 3\tau \Rightarrow |x| + \tau \in [\tau, 2^2\tau].$$

Therefore, for $F \in J_\alpha(\infty)$, $G \in J_\beta(\infty)$ we have estimates like in (2.2):

$$F(|x| + \tau) \leq \alpha^2 F(\tau), \quad G(\tau) \leq \beta^2 G(|x| + \tau),$$

so that

$$0 \leq C(x) \leq \alpha^2 \beta^2 \int_{|x|/2}^\infty F(\tau) G(|x| + \tau) \tau^{n-1} d\tau \leq \alpha^2 \beta^2 \int_0^\infty F(\tau) G(|x| + \tau) \tau^{n-1} d\tau,$$

that is

$$0 \leq C(x) \leq c_3 A(x), x \in \dot{\mathbb{R}}^n. \tag{2.21}$$

Let us consider

$$\tilde{u}(x) = \int_0^\infty \left[F(\tau) G(|x| + \tau) + F(|x| + \tau) G(\tau) \right] \tau^{n-1} d\tau \cong A(x) + B(x) + C(x).$$

Estimates (2.19) -(2.21) show that here

$$A(x) + B(x) + C(x) \cong A(x) + B(x).$$

Therefore,

$$\tilde{u}(x) \cong A(x) + B(x) = I_1 + I_2 = u(x).$$

This completes the proof of estimate (2.9).

□

Corollary 2.1. *Under the assumptions of Theorem 2.1 the following two–sided estimate holds:*

$$u(x) \cong F(|x|) \int_0^{|x|} G(\tau)\tau^{n-1}d\tau + G(|x|) \int_0^{|x|} F(\tau)\tau^{n-1}d\tau + \int_{|x|}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau \tag{2.22}$$

with positive constants depending only on α, β, n (as in (2.18)).

Proof. Indeed, for functions $F \in J_\alpha(\infty)$, $G \in J_\beta(\infty)$ we have

$$F(|x| + \tau) \cong F(|x|), \quad G(|x| + \tau) \cong G(|x|), \quad \tau \in (0, |x|];$$

$$F(|x| + \tau) \cong F(\tau), \quad G(|x| + \tau) \cong G(\tau), \quad \tau > |x|;$$

and estimate (2.9) implies (2.22). □

Remark 5. Under notation (2.6)- (2.8) let functions F and G be nonnegative and decreasing. Then,

$$u(x) \geq 2^{-1}C_n\tilde{u}(x), x \in \dot{\mathbb{R}}^n, \quad C_n = 2\pi^{n/2}\Gamma(n/2)^{-1}. \tag{2.23}$$

Proof. For decreasing functions F and G we have:

$$|y - x| \leq |x| + |y| \Rightarrow F(|y - x|) \geq F(|x| + |y|),\ G(|y - x|) \geq G(|x| + |y|).$$

Then,

$$u(x) = \int_{\mathbb{R}^n} F(|y|)G(|y - x|)dy \geq \int_{\mathbb{R}^n} F(|y|)G(|x| + |y|)dy.$$

Thus, in the spherical coordinates we have

$$u(x) \geq C_n \int_0^{\infty} F(\tau)G(|x| + \tau)\tau^{n-1}d\tau. \tag{2.24}$$

But $u = f * g = g * f$, so

$$u(x) = \int_{\mathbb{R}^n} F(|x - y|)G(|y|)dy \geq \int_{\mathbb{R}^n} F(|x| + |y)G(|y|)dy.$$

In the spherical coordinates we have

$$u(x) \geq C_n \int_0^{\infty} F(|x| + \tau)G(\tau)\tau^{n-1}d\tau. \tag{2.25}$$

We add estimates (2.24), (2.25) and obtain that

$$2u(x) \geq C_n \int_0^{\infty} \left[F(\tau)G(|x| + \tau) + F(|x| + \tau)G(\tau)\right]\tau^{n-1}d\tau = C_n\tilde{u}(x). \tag{2.26}$$

This implies estimate (2.23). □

Corollary 2.2. *Under the assumptions of Remark 5 the following estimate holds for the symmetrical rearrangement of convolution*

$$u^{\#}(\rho) \geq 2^{-1} C_n \int_0^{\infty} \left[F(\tau)G(\rho+\tau) + F(\rho+\tau)G(\tau)\right] \tau^{n-1} d\tau, \; \rho \in (0,\infty). \tag{2.27}$$

Indeed, estimate (2.23) implies the related estimate for symmetrical rearrangements:

$$u^{\#}(\rho) \geq 2^{-1} C_n \tilde{u}^{\#}(\rho), \; \rho \in (0,\infty).$$

But, under the assumptions of Remark 5, function $\tilde{u}$ (2.8) is nonnegative, radial symmetrical and decreasing as the function of $\rho = |x|$. Therefore, its symmetrical rearrangement $u^{\#}$ coincides with the integral in the right–hand side of (2.27).

3 Two–sided estimates for convolutions. The case $R < \infty$

First, we formulate a useful technical result.

Lemma 3.1. *1. Let $G \in J_\beta(\infty)$, $\xi \in (0,\infty)$. Then,*

$$\int_0^{\xi/2} G(\lambda)\lambda^{n-1} d\lambda \leq \int_0^{\xi} G(\lambda)\lambda^{n-1} d\lambda \leq (1 + 2^n \beta^3) \int_0^{\xi/2} G(\lambda)\lambda^{n-1} d\lambda. \tag{3.1}$$

2. Let $F \in J_\alpha(R)$, $\xi \in (0,R]$. Then,

$$\int_0^{\xi/2} F(\lambda)\lambda^{n-1} d\lambda \leq \int_0^{\xi} F(\lambda)\lambda^{n-1} d\lambda \leq (1 + 2^n \alpha^3) \int_0^{\xi/2} F(\lambda)\lambda^{n-1} d\lambda. \tag{3.2}$$

Proof. We will prove (3.1) (for (3.2) the proof is analogous). For $G \in J_\beta(\infty)$ we have

$$G(\lambda) \geq 0 \Rightarrow \int_0^{\xi/2} G(\lambda)\lambda^{n-1} d\lambda \leq \int_0^{\xi} G(\lambda)\lambda^{n-1} d\lambda. \tag{3.3}$$

Thus, the left part in estimate (3.1) holds. Let us prove the right part in estimate (3.1). Note that for $G \in J_\beta(\infty)$, $\xi \in (0,\infty)$ we have inequalities

$$\beta^{-1} G(\xi/2) \leq G(\lambda) \leq \beta G(\xi/2), \; \lambda \in [\xi/2, \xi].$$

Therefore,

$$\beta^{-1} G(\xi/2) \int_{\xi/2}^{\xi} \lambda^{n-1} d\lambda \leq \int_{\xi/2}^{\xi} G(\lambda)\lambda^{n-1} d\lambda \leq \beta G(\xi/2) \int_{\xi/2}^{\xi} \lambda^{n-1} d\lambda,$$

and we obtain by calculation of integrals

$$\beta^{-1} n^{-1} (1 - 2^{-n}) \xi^n G(\xi/2) \leq \int_{\xi/2}^{\xi} G(\lambda)\lambda^{n-1} d\lambda \leq \beta n^{-1} (1 - 2^{-n}) \xi^n G(\xi/2). \tag{3.4}$$

Moreover, by application of Remark 3, we have for $G \in J_\beta(\infty)$

$$\beta^{-2}G(\xi/2) \leq G(\lambda) \leq \beta G(\xi/2), \quad \lambda \in [\xi/4, \xi/2]\,,$$

and, therefore,

$$\int_{\xi/4}^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda \geq \beta^{-2}\left(\int_{\xi/4}^{\xi/2} \lambda^{n-1}d\lambda\right) G(\xi/2) = \beta^{-2}n^{-1}2^{-n}(1-2^{-n})\xi^n G(\xi/2).$$

Thus,

$$\int_0^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda \geq \int_{\xi/4}^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda \geq \beta^{-2}n^{-1}2^{-n}(1-2^{-n})\xi^n G(\xi/2).$$

Together with the right estimate in (3.4) this shows that

$$\int_{\xi/2}^{\xi} G(\lambda)\lambda^{n-1}d\lambda \leq 2^n\beta^3 \int_0^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda,$$

and we obtain

$$\int_0^{\xi} G(\lambda)\lambda^{n-1}d\lambda = \int_0^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda + \int_{\xi/2}^{\xi} G(\lambda)\lambda^{n-1}d\lambda \leq (1+2^n\beta^3)\int_0^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda.$$

Thus, we arrive at the right estimate in (3.1). □

Corollary 3.1. *Let $0 < \rho < 1$, $m \in \mathbb{N}$ be such that $2^{-m} \leq \rho \leq 2^{-m+1}$. Then, the following estimates hold.*

1. *For $1 < \beta < \infty$, $G \in J_\beta(\infty)$, $\xi \in (0,\infty)$ we have*

$$\int_0^{\rho\xi} G(\lambda)\lambda^{n-1}d\lambda \leq \int_0^{\xi} G(\lambda)\lambda^{n-1}d\lambda \leq \left(1+2^n\beta^3\right)^m \int_0^{\rho\xi} G(\lambda)\lambda^{n-1}d\lambda. \tag{3.5}$$

2. *For $1 < \alpha < \infty$, $0 < R < \infty$, $F \in J_\alpha(R)$, $\xi \in (0,R]$ we have*

$$\int_0^{\rho\xi} F(\lambda)\lambda^{n-1}d\lambda \leq \int_0^{\xi} F(\lambda)\lambda^{n-1}d\lambda \leq \left(1+2^n\alpha^3\right)^m \int_0^{\rho\xi} F(\lambda)\lambda^{n-1}d\lambda. \tag{3.6}$$

Proof. We will prove (3.5) (for (3.6) the proof is analogous). The left estimate in (3.5) is evident. By induction we can easily prove that for $m \in \mathbb{N}$ the following estimate holds

$$\int_0^{\xi} G(\lambda)\lambda^{n-1}d\lambda \leq \left(1+2^n\beta^3\right)^m \int_0^{2^{-m}\xi} G(\lambda)\lambda^{n-1}d\lambda. \tag{3.7}$$

Indeed, for $m = 1$ it coincides with (3.1). Assumption of induction is that it holds for all numbers from 1 to m. Then, for the number $m+1$ we have by application of (3.1) with $2^{-m}\xi$ instead of $\xi \in (0,\infty)$:

$$\int_0^{2^{-m}\xi} G(\lambda)\lambda^{n-1}d\lambda \le \left(1+2^n\beta^3\right) \int_0^{2^{-(m+1)}\xi} G(\lambda)\lambda^{n-1}d\lambda.$$

Therefore, application of (3.7) shows that

$$\int_0^{\xi} G(\lambda)\lambda^{n-1}d\lambda \le \left(1+2^n\beta^3\right)^{m+1} \int_0^{2^{-(m+1)}\xi} G(\lambda)\lambda^{n-1}d\lambda.$$

Thus, (3.7) holds for any $m \in \mathbb{N}$. Therefore, for $2^{-m} \le \rho \le 2^{-m+1}$ we have

$$\int_0^{\xi} G(\lambda)\lambda^{n-1}d\lambda \le \left(1+2^n\beta^3\right)^{m} \int_0^{2^{-m}\xi} G(\lambda)\lambda^{n-1}d\lambda \le \left(1+2^n\beta^3\right)^{m} \int_0^{\rho\xi} G(\lambda)\lambda^{n-1}d\lambda.$$

This is the right estimate in (3.5). □

Theorem 3.1. *Let*

$$\alpha,\beta \in (1,\infty),\ R \in (0,\infty),\ F \in J_\alpha(R),\ G \in J_\beta(\infty); \tag{3.8}$$

$$f(x) = F(|x|), g(x) = G(|x|), x \in \dot{\mathbb{R}}^n; \tag{3.9}$$

$$u(x) = (f*g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy,\ x \in \dot{\mathbb{R}}^n. \tag{3.10}$$

For $x \in \dot{\mathbb{R}}^n$ we define $\tilde{u}(x)$ by the following formulas:

1. *If $|x| < 2R/3$, then*

$$\tilde{u}(x) = \int_0^{R-|x|} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda + \int_0^{R} F(\lambda)G(|x|+\lambda)\lambda^{n-1}d\lambda. \tag{3.11}$$

2. *If $2R/3 \le |x| \le 4R/3$, then*

$$\tilde{u}(x) = F(R)\int_0^{R} G(\lambda)\lambda^{n-1}d\lambda + G(R)\int_0^{R} F(\lambda)\lambda^{n-1}d\lambda. \tag{3.12}$$

3. *If $4R/3 < |x| < \infty$, then*

$$\tilde{u}(x) = G(|x|)\int_0^{R} F(\lambda)\lambda^{n-1}d\lambda. \tag{3.13}$$

Then, there exist constants $c_i = c_i(\alpha,\beta,n)$, $i = 1,2$, $0 < c_1 \le c_2 < \infty$, such that

$$c_1 u(x) \le \tilde{u}(x) \le c_2 u(x),\ x \in \dot{\mathbb{R}}^n. \tag{3.14}$$

Proof. 1. We consider the case $|x| < 2R/3$. In this case the proof is similar to the proof of Theorem 2.1. Let $\Omega = B(x, |x|/2)$ be the ball with the center $x \in \dot{\mathbb{R}}^n$ and the radius $r = |x|/2$. Note that for $0 < |x| < 2R/3$ we have $\Omega \subset B_R = B(0, R)$. Let $\dot{B}_R = B(0, R) \setminus \{0\}$.

We will take into account that $F(|y|) = 0$ for $|y| > R$ and obtain

$$u(x) = \int\limits_{B_R} F(|y|)G(|x-y|)dy = u_1 + u_2, \quad x \in \dot{\mathbb{R}}^n, \tag{3.15}$$

where

$$u_1 = \int\limits_{\dot{B}_R \setminus \Omega} F(|y|)G(|x-y|)dy, \quad u_2 = \int\limits_{\Omega} F(|y|)G(|x-y|)dy. \tag{3.16}$$

For $y \in \dot{B}_R \setminus \Omega$ we have $|x| \le 2|x-y|$, so

$$|y| = |y - x + x| \le |y - x| + |x| \le 3|y - x|.$$

Then,

$$|x - y| \le |x| + |y| \le 5|x - y|, \quad y \in \dot{B}_R \setminus \Omega,$$

that is

$$|x-y|/\left(|x|+|y|\right) \in \left[5^{-1}, 1\right] \subset \left[2^{-3}, 1\right],$$

and for $G \in J_\beta(\infty)$, $y \in \dot{B}_R \setminus \Omega$ we obtain from (2.2) (with $m = 3$, $\xi = 2^{-3}$) that

$$|x-y|/\left(|x|+|y|\right) \in \left[\xi, 2^3\xi\right] \Rightarrow \beta^{-3} \le G(|x-y|)/G(|x|+|y|) \le \beta^3. \tag{3.17}$$

It follows from (3.17) that

$$\beta^{-3}u_1 \le \int\limits_{\dot{B}_R \setminus \Omega} F(|y|)G(|x|+|y|)dy \le \beta^3 u_1. \tag{3.18}$$

The left–hand–side inequality in (3.18) shows that

$$u_1 \le \beta^3 \int\limits_{B_R} F(|y|)G(|x|+|y|)dy.$$

For $x \in \dot{\mathbb{R}}^n$ we introduce the spherical system of coordinates with the center at the point 0 and the polar axis L_0 such that $x \in L_0$. In the spherical coordinates for $y \in \dot{B}_R$ we have

$$y = (\tau, \omega), \; 0 < \tau = |y| \le R, \; \omega \in S^{n-1}.$$

Analogously to (2.10), we obtain that

$$u_1 \le \beta^3 C_n \int\limits_0^R F(\tau)G(|x|+\tau)\tau^{n-1}d\tau. \tag{3.19}$$

Here $C_n = 2\pi^{n/2}\Gamma(n/2)^{-1}$. As in Theorem 2.1, we introduce the minimal cone K_Ω with the cone apex at the origin, such that $\Omega \subset K_\Omega$, and define

$$\Sigma_\Omega = \left\{\omega \in S^{n-1} : \omega \notin K_\Omega\right\}, \; \sigma_n = \int\limits_{\Sigma_\Omega} d\omega; \; \Delta_\Omega = \left\{\omega \in S^{n-1} : \omega \in K_\Omega\right\}, \; \delta_n = \int\limits_{\Delta_\Omega} d\omega.$$

We have $y = |y|\omega \in B_R \setminus K_\Omega$ for $\omega \in \Sigma_\Omega$ and for any $0 < |y| \le R$. Note that our construction is such that the cone K_Ω and σ_n, δ_n do not depend on $x \in L_0$ with $0 < |x| < 2R/3$,they depend only on the dimension n. Moreover, $\Sigma_\Omega \cap \Delta_\Omega = \{\emptyset\}$, $\Sigma_\Omega \cup \Delta_\Omega = S^{n-1}$. Then, $0 < \sigma_n, \delta_n$; $\sigma_n + \delta_n = \int\limits_{S^{n-1}} d\omega = C_n$, so that, in particular, $0 < \delta_n < C_n$. The right–hand–side estimate in (3.18) shows that

$$u_1 \ge \beta^{-3} \int\limits_{B_R \setminus K_\Omega} F(|y|)G(|x| + |y|)dy.$$

As in (2.10) we obtain in the spherical coordinates that

$$\int\limits_{B_R \setminus K_\Omega} F(|y|)G(|x| + |y|)dy = \sigma_n \int\limits_0^R F(\tau)G(|x| + \tau)\tau^{n-1}d\tau.$$

As a result,

$$u_1 \ge \beta^{-3}\sigma_n \int\limits_0^R F(\tau)G(|x| + \tau)\tau^{n-1}d\tau. \tag{3.20}$$

Estimates (3.19) and (3.20) give the two–sided inequality:

$$\beta^{-3}C_n^{-1}u_1 \le \int\limits_0^R F(\tau)G(|x| + \tau)\tau^{n-1}d\tau \le \beta^3\sigma_n^{-1}u_1. \tag{3.21}$$

We move on to the estimates for $u_2 = \int\limits_\Omega F(|y|)G(|x - y|)dy$. Note that

$$y \in \Omega \Rightarrow \begin{cases} |y| \le |x| + |x - y| \le \frac{3}{2}|x| \le R; \\ 3|y| \ge \frac{3}{2}|x| = |x| + \frac{1}{2}|x| \ge |x| + |x - y|. \end{cases}$$

Therefore, for $y \in \Omega$ we have

$$|y| \le |x| + |x - y| \le \min\left\{2^2|y|, R\right\}.$$

For $F \in J_\alpha(R)$ it follows from here and from (2.3) with $m = 2$ that

$$\alpha^{-2}F(|x| + |x - y|) \le F(|y|) \le \alpha^2 F(|x| + |x - y|), \quad y \in \Omega.$$

Therefore,

$$\alpha^{-2}u_2 \le \int\limits_\Omega F(|x| + |x - y|)G(|x - y|)dy \le \alpha^2 u_2.$$

In Ω we introduce the spherical system of coordinates with the center at the point x and the spherical radius $\lambda = |y - x|$. Then,

$$y \in \Omega,\, y \ne x \Leftrightarrow y - x = (\lambda, \omega),\, \lambda = |y - x| = |x - y| \in (0, |x|/2],\, \omega \in S^{n-1},$$

and we obtain the equality

$$\int\limits_\Omega F(|x| + |x - y|)G(|x - y|)dy = C_n \int\limits_0^{|x|/2} F(|x| + \lambda)G(\lambda)\lambda^{n-1}d\lambda,$$

with $C_n = \left(\int\limits_{S^{n-1}} d\omega\right) = 2\pi^{n/2}\Gamma(n/2)^{-1}$. These estimates show that

$$\alpha^{-2}C_n^{-1}u_2 \leq \int\limits_0^{|x|/2} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda \leq \alpha^{-2}C_n^{-1}u_2. \tag{3.22}$$

For the further consideration, let us recall the notation and properties (2.19)–(2.21). We consider here (see estimates (2.18)– (2.21))

$$A(x) := u_1 \cong \int\limits_0^R F(\lambda)G(|x|+\lambda)\lambda^{n-1}d\lambda, \tag{3.23}$$

$$B(x) := u_2 \cong \int\limits_0^{|x|/2} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda, \tag{3.24}$$

$$C(x) := \int\limits_{|x|/2}^{R-|x|} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda. \tag{3.25}$$

For $|x|/2 \leq \lambda \leq R-|x|$ we have $|x| \leq 2\lambda$, so that

$$\lambda \leq |x|+\lambda \leq \min\{3\lambda, R\} \leq \min\left\{2^2\lambda, R\right\}.$$

Now, for $F \in J_\alpha(R)$, $\alpha \in (1,\infty)$ we can apply estimate (2.3) with $\xi = \lambda$, $m = 2$. Then

$$F(|x|+\lambda) \leq \alpha^2 F(\lambda).$$

For $G \in J_\beta(\infty)$, $\beta \in (1,\infty)$ we will apply analogue of Remark 3 with $\xi = \lambda$, $m = 2$ and β instead of α, and obtain:

$$G(\lambda) \leq \beta^4 G(|x|+\lambda).$$

Therefore,

$$\begin{aligned} 0 \leq C(x) &\leq \alpha^2\beta^4 \int\limits_{|x|/2}^{R-|x|} F(\lambda)G(|x|+\lambda)\lambda^{n-1}d\lambda \\ &\leq \alpha^2\beta^4 \int\limits_0^R F(\lambda)G(|x|+\lambda)\lambda^{n-1}d\lambda \leq c_3 A(x). \end{aligned} \tag{3.26}$$

Let us consider $\tilde{u}(x)$ defined in (3.11) We see from (3.23) -(3.25) that

$$\tilde{u}(x) \cong A(x) + B(x) + C(x).$$

Estimates (3.20) –(3.21), (3.26) show that here

$$A(x) + B(x) + C(x) \cong A(x) + B(x).$$

Therefore,

$$\tilde{u}(x) \cong A(x) + B(x) = u_1 + u_2 = u(x).$$

This completes the proof of estimate (3.14) in the case $|x| < 2R/3$.

2. Now we consider the case $2R/3 \le |x| \le 4R/3$. Introduce the ball $\Omega_0 = B(x/2, |x|/4)$ with the center $x/2$ and the radius $r = |x|/4$. Note that $\Omega_0 \subset B_R = B(0, R)$. As in (3.15), (3.16) we have

$$u(x) = \int_{B_R} F(|y|)G(|x-y|)dy = u_{1,0}(x) + u_{2,0}(x), \quad x \in \dot{\mathbb{R}}^n, \tag{3.27}$$

where

$$u_{1,0}(x) = \int_{B_R \setminus \Omega_0} F(|y|)G(|x-y|)dy, \quad u_{2,0}(x) = \int_{\Omega_0} F(|y|)G(|x-y|)dy.$$

For $y \in B_R \setminus \Omega_0$ we have $|x| < 4|x-y|$, $|y| \le |x| + |x-y| < 5|x-y|$; so

$$|x-y| \le |x| + |y| \le 4|x-y| + 5|x-y| = 9|x-y|.$$

For $G \in J_\beta(\infty)$ this implies that

$$G(|x-y|) \cong G(|x| + |y|), \quad y \in B_R \setminus \Omega_0,$$

and, therefore,

$$u_{1,0}(x) \cong \int_{B_R \setminus \Omega_0} F(|y|)G(|x| + |y|)dy.$$

As in (3.16) - (3.21) we obtain from here that

$$u_{1,0} \cong \int_0^R F(\tau)G(|x| + \tau)\tau^{n-1}d\tau. \tag{3.28}$$

But, for $2R/3 \le |x| \le 4R/3$, $0 < \tau \le R$ we have $2R/3 \le |x| + \tau \le 7R/3 < 2^2(2R/3)$, and for $G \in J_\beta(\infty)$ according to the analogue of Remark 3 with $\xi = 2R/3$, $m = 2$ and β instead of α we obtain $G(|x| + \tau) \cong G(R)$. Therefore,

$$u_{1,0}(x) \cong G(R) \int_0^R F(\tau)\tau^{n-1}d\tau. \tag{3.29}$$

For $y \in \Omega_0$ we have $|x/2 - y| \le r = |x|/4$, so that $|y| \le |x|/2 + r = 3|x|/4 \le R$, $|y| \ge |x|/2 - r = |x|/4$. Thus, we have

$$|x|/4 \le |y| \le 3|x|/4; \quad 2R/3 \le |x| \le 4R/3.$$

For $F \in J_\alpha(R)$ it implies that

$$F(|y|) \cong F(|x|/4) \cong F(R), \quad y \in \Omega_0.$$

Therefore,

$$u_{2,0}(x) \cong F(R) \int_{\Omega_0} G(|x-y|)dy.$$

In Ω_0 we introduce the spherical system of coordinates with the center at the point $x/2$ and the spherical radius $\lambda = |x-y|$. Then,

$$u_{2,0}(x) \cong F(R) \int_0^{|x|/4} G(\lambda)\lambda^{n-1}d\lambda. \tag{3.30}$$

For $2R/3 \le |x| \le 4R/3$ we apply several times estimate (3.5) with related choose of $\xi \in \mathbb{R}_+$, and obtain that

$$\int_0^{|x|/4} G(\lambda)\lambda^{n-1}d\lambda \cong \int_0^R G(\lambda)\lambda^{n-1}d\lambda. \tag{3.31}$$

The constants in estimate (3.31) depend only on β, n (estimates of such type were proved in Lemma 3.1). Together with (3.27) and (3.29) this gives desired estimates (3.12), (3.14).

Remark 6. Under the assumptions of Theorem 3.1 let $2R/3 \le |x| \le R$. Then, we have the equivalence

$$\tilde{u}(x) \cong F(|x|)\int_0^{|x|} G(\lambda)\lambda^{n-1}d\lambda + G(|x|)\int_0^{|x|} F(\lambda)\lambda^{n-1}d\lambda. \tag{3.32}$$

To show this let us note that for $2R/3 \le |x| \le R$ and for functions $F \in J_\alpha(R)$, $G \in J_\beta(\infty)$ we have $F(R) \cong F(|x|)$, $G(R) \cong G(|x|)$. Moreover, an application of Corollary of Lemma 3.1 gives

$$\int_0^{|x|} G(\lambda)\lambda^{n-1}d\lambda \cong \int_0^R G(\lambda)\lambda^{n-1}d\lambda, \quad \int_0^{|x|} F(\lambda)\lambda^{n-1}d\lambda \cong \int_0^R F(\lambda)\lambda^{n-1}d\lambda.$$

This means that estimates (3.12), (3.14) imply estimate (3.32).

3. Consider the case $|x| > 4R/3$. We have the equality

$$u(x) = \int_{B_R} F(|y|)G(|x-y|)dy.$$

Note that $|y| \le R$, $|x| > 4R/3 \Rightarrow |x|/4 \le |x-y| \le 7|x|/4$, and for $G \in J_\beta(\infty)$ we obtain $G(|x-y|) \cong G(|x|)$, $y \in B_R$. Therefore,

$$u(x) \cong G(|x|)\int_{B_R} F(|y|)dy = C_n G(|x|)\int_0^R F(\tau)\tau^{n-1}d\tau.$$

□

4 Two–sided estimates for decreasing rearrangements of convolutions

4.1 Estimates for decreasing and symmetrical rearrangements

Here we consider estimates for decreasing and symmetrical rearrangements of convolutions. The books by S.G. Krein, Yu.I. Petunin and E.M. Semenov [12], C. Bennett and R. Sharpley [3] contain the main definitions and basic facts related to this topic. We recall some formulas.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function such that its distribution function

$$\lambda_h(y) = \mu_n \left\{ x \in \dot{\mathbb{R}}^n : |h(x)| > y \right\}, \quad y \in [0, \infty),$$

is not identically equal to infinity. Then, $0 \le \lambda_h(y) \downarrow$ on $[0, \infty)$. The decreasing rearrangement of the function h is defined by the formula

$$h^*(\tau) = \inf \left\{ y \in [0, \infty) : \lambda_h(y) \le \tau \right\}, \tau \in (0, \infty). \tag{4.1}$$

Note that $0 \leq h^* \downarrow$ on $(0, \infty)$. The symmetrical rearrangement $h^\#$ is a radially symmetrical function related to the decreasing rearrangement by the formulas

$$h^\#(\rho) = h^*(V_n \rho^n), \quad h^*(\tau) = h^\#((\tau/V_n)^{1/n}); \ \rho, \tau \in (0, \infty). \tag{4.2}$$

Here V_n is the volume of the unit ball in $\mathbb{R}^n$.

Moreover,

$$h(x) = H(|x|), \ 0 \leq H \downarrow \text{on} \ (0, \infty) \Rightarrow h^\#(\rho) = H(\rho), \ \rho \in (0, \infty). \tag{4.3}$$

Theorem 4.1. *Under the assumptions of Theorem 2.1 let additionally F, G be decreasing. Then, there exist constants $c_i = c_i(\alpha, \beta, n)$, $i = 1, 2$, such that $0 < c_1 \leq c_2 < \infty$ and for the symmetrical rearrangement of convolution (2.7) the following estimates hold*

$$c_1 u^\#(\rho) \leq \int_0^\infty \left[F(\rho + \tau)G(\tau) + F(\tau)G(\rho + \tau)\right] \tau^{n-1} d\tau \leq c_2 u^\#(\rho), \ \rho \in (0, \infty). \tag{4.4}$$

Moreover,

$$u^\#(\rho) \cong F(\rho) \int_0^\rho G(\tau)\tau^{n-1} d\tau + G(\rho) \int_0^\rho F(\tau)\tau^{n-1} d\tau + \int_\rho^\infty F(\tau)G(\tau)\tau^{n-1} d\tau \tag{4.5}$$

with understanding $\cong$ as in (2.18).

Proof. From (2.9) it follows that

$$c_1 u^\#(\rho) \leq \tilde{u}^\#(\rho) \leq c_2 u^\#(\rho), \ \rho \in (0, \infty).$$

Note that the function $\tilde{u}$ defined by (2.8) is radially symmetrical and decreases as a function of $\rho = |x|$. Thus, according to (4.3) it coincides with its symmetrical rearrangement, and we can apply definition (2.8) with $\rho = |x|$. By Theorem 2.1 this proves estimate (4.4).

Let us deduce (4.5) from (4.4). We have

$$\int_0^\infty F(\rho + \tau)G(\tau)\tau^{n-1} d\tau = \int_0^\rho F(\rho + \tau)G(\tau)\tau^{n-1} d\tau + \int_\rho^\infty F(\rho + \tau)G(\tau)\tau^{n-1} d\tau.$$

For $\tau \in [0, \rho]$ we have $\rho + \tau \in [\rho, 2\rho]$, so that for the function $F \in J_\alpha(\infty)$ there is the estimate:

$$\alpha^{-1} F(\rho + \tau) \leq F(\rho) \leq \alpha F(\rho + \tau).$$

Therefore,

$$\alpha^{-1} \int_0^\rho F(\rho + \tau)G(\tau)\tau^{n-1} d\tau \leq F(\rho) \int_0^\rho G(\tau)\tau^{n-1} d\tau \leq \alpha \int_0^\rho F(\rho + \tau)G(\tau)\tau^{n-1} d\tau.$$

For $\tau > \rho$ we have $\rho + \tau \in [\tau, 2\tau]$, so that for the function $F \in J_\alpha(\infty)$ there is the estimate:

$$\alpha^{-1} F(\rho + \tau) \leq F(\tau) \leq \alpha F(\rho + \tau).$$

Therefore,

$$\alpha^{-1}\int_{\rho}^{\infty} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau \le \int_{\rho}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau \le \alpha\int_{\rho}^{\infty} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau.$$

So, we have the two–sided estimate

$$\int_{0}^{\infty} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau \cong F(\rho)\int_{0}^{\rho} G(\tau)\tau^{n-1}d\tau + \int_{\rho}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau.$$

Analogously, for $G\in J_{\beta}(\infty)$, we obtain

$$\int_{0}^{\infty} F(\tau)G(\rho+\tau)\tau^{n-1}d\tau \cong G(\rho)\int_{0}^{\rho} F(\tau)\tau^{n-1}d\tau + \int_{\rho}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau.$$

As a result,

$$\int_{0}^{\infty}\left[F(\rho+\tau)G(\tau)+F(\tau)G(\rho+\tau)\right]\tau^{n-1}d\tau$$
$$\cong F(\rho)\int_{0}^{\rho} G(\tau)\tau^{n-1}d\tau + G(\rho)\int_{0}^{\rho} F(\tau)\tau^{n-1}d\tau + \int_{\rho}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau.$$

We put this estimate into (4.4) and obtain (4.5). □

Remark 7. Note that the right–hand–side inequality in (4.4) follows immediately from Remark 5 and Corollary 2.2 (see estimate (2.27)) without restrictions $F\in J_{\alpha}$, $G\in J_{\beta}$.

Corollary 4.1. *Under the assumptions of Theorem 4.1 we define*

$$\varphi(\lambda)=F\left((\lambda/V_n)^{1/n}\right),\ \psi(\lambda)=G\left((\lambda/V_n)^{1/n}\right),\ \lambda\in(0,\infty). \tag{4.6}$$

Then, the following estimate holds for the decreasing rearrangement of the convolution u:

$$u^{*}(t)\cong\varphi(t)\int_{0}^{t}\psi(\lambda)d\lambda+\psi(t)\int_{0}^{t}\varphi(\lambda)d\lambda+\int_{t}^{\infty}\varphi(\lambda)\psi(\lambda)d\lambda,\ t\in(0,\infty), \tag{4.7}$$

with understanding $\cong$ *as in (2.18).*

Proof. We introduce the new variable $\lambda=V_n\tau^n$ for integrals in (4.5). Then,

$$\tau=(\lambda/V_n)^{1/n},\ \tau^{n-1}d\tau=d\lambda/(nV_n),$$

and we obtain from (4.5)–(4.6)

$$u^{\#}(\rho)\cong F(\rho)\int_{0}^{V_n\rho^n}\psi(\lambda)d\lambda+G(\rho)\int_{0}^{V_n\rho^n}\varphi(\lambda)d\lambda+\int_{V_n\rho^n}^{\infty}\varphi(\lambda)\psi(\lambda)d\lambda.$$

We put here $\rho=(t/V_n)^{1/n}$ and take into account notation (4.6) and the equality: $u^{\#}\left((t/V_n)^{1/n}\right)=u^{*}(t)$ (see (4.2)). Thus, we come to (4.7). □

Corollary 4.2. *Under the assumptions of Theorem 4.1 the following estimate holds for the decreasing rearrangement of the convolution:*

$$u^*(t) \cong f^*(t)\int_0^t g^*(\lambda)d\lambda + g^*(t)\int_0^t f^*(\lambda)d\lambda + \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda,\ t \in (0,\infty), \tag{4.8}$$

with understanding $\cong$ *as in* (2.18).

Proof. Indeed, formula (4.8) follows from (4.7) and from the equalities

$$f(x) = F(|x|),\ 0 \le F \downarrow \Rightarrow f^\#(\rho) = F(\rho) \Rightarrow f^*(t) = F\left((t/V_n)^{1/n}\right) = \varphi(t),$$

$$g(x) = G(|x|),\ 0 \le G \downarrow \Rightarrow g^\#(\rho) = G(\rho) \Rightarrow g^*(t) = G\left((t/V_n)^{1/n}\right) = \psi(t).$$

□

Remark 8. Note that under the assumptions of Theorem 4.1

$$2^{-n}t_1 \le t_2 \le t_1 \Rightarrow f^*(t_1) \le f^*(t_2) \le \alpha^2 f^*(t_1). \tag{4.9}$$

Indeed,

$$\frac{1}{2}\left(\frac{t_1}{V_n}\right)^{1/n} \le \left(\frac{t_2}{V_n}\right)^{1/n} \le \left(\frac{t_1}{V_n}\right)^{1/n},$$

and for $F \in J_\alpha(\infty)$ we have by application of Remark 3

$$\alpha^{-2}F\left(\left(\frac{t_1}{V_n}\right)^{1/n}\right) \le F\left(\left(\frac{t_2}{V_n}\right)^{1/n}\right) \le \alpha^2 F\left(\left(\frac{t_1}{V_n}\right)^{1/n}\right).$$

Moreover, the function F decreases and for $t_2 \le t_1$ in the left–hand–side of this estimate we can replace $\alpha^{-2} < 1$ by 1. Therefore, for the function $f^*(t) = F\left(\left(\frac{t}{V_n}\right)^{1/n}\right)$ we obtain (4.9).

Analogously,

$$2^{-n}t_1 \le t_2 \le t_1 \Rightarrow g^*(t_1) \le g^*(t_2) \le \beta^2 g^*(t_1). \tag{4.10}$$

Corollary 4.3. *Under the assumptions of Theorem 4.1 for* $\xi \in (0,\infty)$ *the following estimates hold for the decreasing rearrangement of a function* f *(see (2.7)):*

$$\xi \le \eta \le 2\xi \Rightarrow f^*(2\xi) \le f^*(\eta) \le \alpha^2 f^*(2\xi); \tag{4.11}$$

$$\xi \le \eta \le 2\xi \Rightarrow g^*(2\xi) \le g^*(\eta) \le \beta^2 g^*(2\xi); \tag{4.12}$$

Proof. Indeed, we put $t_1 = 2\xi$ in (4.9) and obtain

$$\xi \le \eta \le 2\xi \Leftrightarrow 2^{-1}t_1 \le \eta \le t_1 \Rightarrow 2^{-n}t_1 \le \eta \le t_1 \Rightarrow f^*(2\xi) \le f^*(\eta) \le \alpha^2 f^*(2\xi).$$

Analogously, we obtain (4.12) from (4.10). □

Corollary 4.4. *Under the assumptions of Theorem 4.1 the following estimate holds for the decreasing rearrangement of the convolution u:*

$$u^*(t) \cong \int_0^\infty \left[f^*(t+\lambda)g^*(\lambda) + f^*(\lambda)g^*(t+\lambda)\right] d\lambda,\ t \in (0,\infty), \tag{4.13}$$

with understanding $\cong$ *as in* (2.18).

Proof. We must show that estimate (4.13) is equivalent to (4.8). We have the equality

$$I := \int_0^\infty [f^*(t+\lambda)g^*(\lambda) + f^*(\lambda)g^*(t+\lambda)]\,d\lambda = \int_0^t [\ldots]\,d\lambda + \int_t^\infty [\ldots]\,d\lambda.$$

Note that, according to (4.11), (4.12), the following estimates hold:

$$0 < \lambda \le t \Rightarrow t < t+\lambda \le 2t \Rightarrow f^*(t+\lambda) \cong f^*(t);\ g^*(t+\lambda) \cong g^*(t);$$

$$\lambda > t \Rightarrow \lambda < t+\lambda \le 2\lambda \Rightarrow f^*(t+\lambda) \cong f^*(\lambda);\ g^*(t+\lambda) \cong g^*(\lambda).$$

Therefore,

$$\int_0^t [f^*(t+\lambda)g^*(\lambda) + f^*(\lambda)g^*(t+\lambda)]\,d\lambda \cong f^*(t)\int_0^t g^*(\lambda)d\lambda + g^*(t)\int_0^t f^*(\lambda)d\lambda;$$

$$\int_t^\infty [f^*(t+\lambda)g^*(\lambda) + f^*(\lambda)g^*(t+\lambda)]\,d\lambda \cong \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda.$$

This shows that

$$I \cong f^*(t)\int_0^t g^*(\lambda)d\lambda + g^*(t)\int_0^t f^*(\lambda)d\lambda + \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda.$$

Now, we apply (4.8) and obtain (4.13). □

4.2 Estimates for integral mean values of rearrangements

We move on to estimating the integral mean value for the decreasing rearrangement of the convolution. Let

$$0 < \nu(\tau),\ \tau \in (0,\infty);\ 0 < V(t) := \int_0^t \nu(\tau)d\tau < \infty,\ t \in (0,\infty);$$

$$u_\nu^{**}(t) = \frac{1}{V(t)}\int_0^t u^*(\tau)\nu(\tau)d\tau,\ t \in (0,\infty). \tag{4.14}$$

Such variant of the mean value for the decreasing rearrangement was introduced in [1].

Theorem 4.2. *Under the assumptions of Theorem 4.1 the following estimate holds*

$$u_\nu^{**}(t) \cong I_1(t) + I_2(t) + I_3(t),\ t \in (0,\infty); \tag{4.15}$$

$$I_1(t) = V(t)^{-1}\int_0^t \left[f^*(\tau)\int_0^\tau g^*(\lambda)d\lambda + g^*(\tau)\int_0^\tau f^*(\lambda)d\lambda\right]\nu(\tau)d\tau; \tag{4.16}$$

$$I_2(t) = V(t)^{-1}\int_0^t f^*(\lambda)g^*(\lambda)V(\lambda)d\lambda; \quad I_3(t) = \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda. \tag{4.17}$$

Proof. By (4.13) we have

$$u_\nu^{**}(t) \cong \hat{I}_1(t) + \hat{I}_2(t) + \hat{I}_3(t),\ t \in (0,\infty); \tag{4.18}$$

$$\hat{I}_1(t) = V(t)^{-1} \int_0^t \left(\int_0^\tau \left[f^*(\tau+\lambda) g^*(\lambda) + f^*(\lambda) g^*(\tau+\lambda) \right] d\lambda \right) \nu(\tau) d\tau;$$

$$\hat{I}_2(t) = V(t)^{-1} \int_0^t \left(\int_\tau^t \left[f^*(\tau+\lambda) g^*(\lambda) + f^*(\lambda) g^*(\tau+\lambda) \right] d\lambda \right) \nu(\tau) d\tau;$$

$$\hat{I}_3(t) = V(t)^{-1} \int_0^t \left(\int_t^\infty \left[f^*(\tau+\lambda) g^*(\lambda) + f^*(\lambda) g^*(\tau+\lambda) \right] d\lambda \right) \nu(\tau) d\tau.$$

Let us recall inequalities (4.11), (4.12). Thus, we have estimates

$$0 < \lambda \le \tau \Rightarrow \tau < \tau + \lambda \le 2\tau \Rightarrow f^*(\tau+\lambda) \cong f^*(\tau);\ g^*(\tau+\lambda) \cong g^*(\tau);$$

$$\lambda > \tau \Rightarrow \lambda < \tau + \lambda \le 2\lambda \Rightarrow f^*(\tau+\lambda) \cong f^*(\lambda);\ g^*(\tau+\lambda) \cong g^*(\lambda).$$

Therefore, for $t \in (0,\infty)$

$$\hat{I}_1(t) \cong V(t)^{-1} \int_0^t \left[f^*(\tau) \int_0^\tau g^*(\lambda) d\lambda + g^*(\tau) \int_0^\tau f^*(\lambda) d\lambda \right] \nu(\tau) d\tau = I_1(t);$$

$$\hat{I}_2(t) \cong V(t)^{-1} \int_0^t \left[\int_\tau^t f^*(\lambda) g^*(\lambda) d\lambda \right] \nu(\tau) d\tau = V(t)^{-1} \int_0^t f^*(\lambda) g^*(\lambda) \int_0^\lambda \nu(\tau) d\tau d\lambda = I_2(t);$$

$$\hat{I}_3(t) \cong V(t)^{-1} \int_0^t \left[\int_t^\infty f^*(\lambda) g^*(\lambda) d\lambda \right] \nu(\tau) d\tau = V(t)^{-1} \int_t^\infty f^*(\lambda) g^*(\lambda) d\lambda \int_0^t \nu(\tau) d\tau = I_3(t).$$

Thus, (4.18) implies (4.15) - (4.17). □

In some special cases we can simplify the general answer.

Remark 9. Under the assumptions of Theorem 4.1 we assume additionally that there exists a constant $c_0 \in (0,\infty)$, such that

$$\nu(\tau)\tau \ge c_0 V(\tau),\ \tau \in (0,\infty). \tag{4.19}$$

Then,

$$u_\nu^{**}(t) \cong I_1(t) + I_3(t),\ t \in (0,\infty). \tag{4.20}$$

Moreover, here

$$I_1(t) \ge 2c_0 V(t)^{-1} \int_0^t f^*(\tau) g^*(\tau) V(\tau) d\tau. \tag{4.21}$$

Proof. We put estimate (4.19) into (4.16) and obtain

$$I_1(t) \geq 2c_0 V(t)^{-1} \int_0^t \left[f^*(\tau)\frac{1}{\tau}\int_0^\tau g^*(\lambda)d\lambda + g^*(\tau)\frac{1}{\tau}\int_0^\tau f^*(\lambda)d\lambda \right] V(\tau)d\tau.$$

Functions f^*, g^* decrease, so that we have inequalities

$$\frac{1}{\tau}\int_0^\tau g^*(\lambda)d\lambda \geq g^*(\tau),\ \frac{1}{\tau}\int_0^\tau f^*(\lambda)d\lambda \geq f^*(\tau).$$

Therefore,

$$I_1(t) \geq 2c_0 V(t)^{-1} \int_0^t \left[f^*(\tau)g^*(\tau)\right] V(\tau)d\tau = 2c_0 I_2(t).$$

This means that in the right–hand side of (4.15) the second term is covered by the first one, and we come to estimates (4.20), (4.21).

Note that inequality (4.19) holds with the constant $c_0 = 1$ in the case of the increasing weight ν. □

Remark 10. The non-weighted case, where $\nu(\tau) \equiv 1$, is of special interest. Thus,

$$\nu(\tau) \equiv 1 \Rightarrow V(\tau) = \tau \Rightarrow u_\nu^{**}(t) = u^{**}(t) := \frac{1}{t}\int_0^t u^*(\tau)d\tau,\ t \in (0,\infty). \tag{4.22}$$

In this case we have the estimate

$$u^{**}(t) \cong t^{-1}\int_0^t f^*(\lambda)d\lambda \int_0^t g^*(\lambda)d\lambda + \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda. \tag{4.23}$$

Indeed, in the non–weighted case we have

$$I_1(t) = t^{-1}\int_0^t \left[f^*(\tau)\int_0^\tau g^*(\lambda)d\lambda + g^*(\tau)\int_0^\tau f^*(\lambda)d\lambda \right] d\tau$$

$$= t^{-1}\int_0^t \frac{d}{d\tau}\left[\int_0^\tau f^*(\lambda)d\lambda \int_0^\tau g^*(\lambda)d\lambda\right] d\tau = t^{-1}\int_0^t f^*(\lambda)d\lambda \int_0^t g^*(\lambda)d\lambda. \tag{4.24}$$

We put this equality into (4.20), take into account equality (4.17) for $I_3(t)$ and obtain (4.23).

5 One useful lemma

The following lemma may be useful in many considerations related to the subject of this paper. The proof of this lemma is related to the proofs of Theorems in Sections 2–4.

Lemma 5.1. *Let functions $F, G \geq 0$ be measurable on $(0, \infty)$, let*

$$G \in J_\beta(\infty), \tag{5.1}$$

$$R \in (0, \infty],\ F \in J_\alpha(R), \tag{5.2}$$

Denote

$$D_\infty(\rho) = \int_0^\infty [F(\rho+\tau)G(\tau) + F(\tau)G(\rho+\tau)]\,\tau^{n-1}d\tau,\ \rho \in (0,\infty); \tag{5.3}$$

$$D_R(\rho) = \int_0^{R-\rho} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau + \int_0^R F(\tau)G(\rho+\tau)\tau^{n-1}d\tau,\ R<\infty,\ \rho \in (0,R]; \tag{5.4}$$

$$D_R(\rho) = \int_0^R F(\tau)G(\rho+\tau)\tau^{n-1}d\tau,\ R<\infty,\ \rho > R. \tag{5.5}$$

1. Then, for $R = \infty$ we have the estimate:

$$D_\infty(\rho) \cong F(\rho)\int_0^\rho G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^\rho F(\tau)\tau^{n-1}d\tau + \int_\rho^\infty F(\tau)G(\tau)\tau^{n-1}d\tau,\ \rho \in (0,\infty). \tag{5.6}$$

2. For $R < \infty$ we have the estimates:

(a) if $\rho \in (0, R/2]$, then

$$D_R(\rho) \cong F(\rho)\int_0^\rho G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^\rho F(\tau)\tau^{n-1}d\tau + \int_\rho^R F(\tau)G(\tau)\tau^{n-1}d\tau; \tag{5.7}$$

(b) if $\rho \in (R/2, R]$, then

$$D_R(\rho) \cong F(\rho)\int_0^{R-\rho} G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^R F(\tau)\tau^{n-1}d\tau; \tag{5.8}$$

(c) if $\rho > R$, then

$$D_R(\rho) \cong G(\rho)\int_0^R F(\tau)\tau^{n-1}d\tau. \tag{5.9}$$

In these formulas $A \cong B$ means that for each formula there exist constants $0 < d_1 \leq d_2 < \infty$, depending only on α, β, such that $d_1 \leq A/B \leq d_2$.

Proof. 1. For $R = \infty$ we have

$$D_\infty(\rho) = A_1(\rho) + A_2(\rho); \tag{5.10}$$

$$A_1(\rho) = \int_0^\rho F(\rho+\tau)G(\tau)\tau^{n-1}d\tau + \int_\rho^\infty F(\rho+\tau)G(\tau)\tau^{n-1}d\tau,$$

$$A_2(\rho) = \int\limits_0^\rho F(\tau)G(\rho+\tau)\tau^{n-1}d\tau + \int\limits_\rho^\infty F(\tau)G(\rho+\tau)\tau^{n-1}d\tau.$$

For $0 \le \tau \le \rho$ we have $\rho+\tau \in [\rho, 2\rho] \Rightarrow \alpha^{-1}F(\rho) \le F(\rho+\tau) \le \alpha F(\rho)$.

For $\rho \le \tau$ we have $\rho+\tau \in [\tau, 2\tau] \Rightarrow \alpha^{-1}F(\tau) \le F(\rho+\tau) \le \alpha F(\tau)$, (see (5.2)). Therefore,

$$A_1(\rho) \cong F(\rho)\int\limits_0^\rho G(\tau)\tau^{n-1}d\tau + \int\limits_\rho^\infty F(\tau)G(\tau)\tau^{n-1}d\tau.$$

Analogously, for $0 \le \tau \le \rho$ we have $\beta^{-1}G(\rho) \le G(\rho+\tau) \le \beta G(\rho)$; for $\rho \le \tau$ we have $\beta^{-1}G(\tau) \le G(\rho+\tau) \le \beta G(\tau)$, (see (5.1)). Thus,

$$A_2(\rho) \cong G(\rho)\int\limits_0^\rho F(\tau)\tau^{n-1}d\tau + \int\limits_\rho^\infty F(\tau)G(\tau)\tau^{n-1}d\tau.$$

As a result, we come to estimate (5.6).

2. Let $R < \infty$, $\rho \in (0, R/2]$. Then, $\rho \le R-\rho$ and

$$D_R(\rho) = B_1(\rho) + B_2(\rho);$$

$$B_1(\rho) = \int\limits_0^\rho F(\rho+\tau)G(\tau)\tau^{n-1}d\tau + \int\limits_\rho^{R-\rho} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau,$$

$$B_2(\rho) = \int\limits_0^\rho F(\tau)G(\rho+\tau)\tau^{n-1}d\tau + \int\limits_\rho^R F(\tau)G(\rho+\tau)\tau^{n-1}d\tau.$$

As in Step 1 we have

$$F(\rho+\tau) \cong F(\rho),\ G(\rho+\tau) \cong G(\rho), \quad 0 \le \tau \le \rho;$$

$$F(\rho+\tau) \cong F(\tau) \text{ for } \rho < \tau \le R-\rho,\ G(\rho+\tau) \cong G(\tau) \text{ for } \tau > \rho,$$

so that

$$B_1(\rho) \cong F(\rho)\int\limits_0^\rho G(\tau)\tau^{n-1}d\tau + \int\limits_\rho^{R-\rho} F(\tau)G(\tau)\tau^{n-1}d\tau; \tag{5.11}$$

$$B_2(\rho) \cong G(\rho)\int\limits_0^\rho F(\tau)\tau^{n-1}d\tau + \int\limits_\rho^R F(\tau)G(\tau)\tau^{n-1}d\tau. \tag{5.12}$$

We take into account that the second term in (5.11) is majored by the second term in (5.12) and obtain

$$D_R(\rho) = B_1(\rho) + B_2(\rho) \cong F(\rho)\int\limits_0^\rho G(\tau)\tau^{n-1}d\tau + G(\rho)\int\limits_0^\rho F(\tau)\tau^{n-1}d\tau + \int\limits_\rho^R F(\tau)G(\tau)\tau^{n-1}d\tau.$$

It gives estimate (5.7).

3. Now, let $R \in (0,\infty)$, $\rho \in (R/2, R]$. Then, $R-\rho < R/2 < \rho$, and

$$D_R(\rho) = E_1(\rho) + E_2(\rho),$$

$$E_1(\rho) = \int_0^{R-\rho} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau,\ E_2(\rho) = \int_0^{R} F(\tau)G(\rho+\tau)\tau^{n-1}d\tau.$$

For $0 \le \tau \le R-\rho$ we have $\rho < \rho+\tau \le R < 2\rho$, so that

$$F(\rho+\tau) \cong F(\rho) \Rightarrow \int_0^{R-\rho} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau \cong F(\rho)\int_0^{R-\rho} G(\tau)\tau^{n-1}d\tau.$$

For $0 < \tau \le R$ we have $\rho < \rho+\tau \le \rho+R \le \rho+2\rho = 3\rho$, so that

$$G(\rho+\tau) \cong G(\rho) \Rightarrow \int_0^{R} F(\tau)G(\rho+\tau)\tau^{n-1}d\tau \cong G(\rho)\int_0^{R} F(\tau)\tau^{n-1}d\tau.$$

As a result,

$$D_R(\rho) = E_1(\rho) + E_2(\rho) \cong F(\rho)\int_0^{R-\rho} G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^{R} F(\tau)\tau^{n-1}d\tau.$$

4. It remains to consider the case $R \in (0,\infty)$, $\rho > R$. Then, $G(\rho+\tau) \cong G(\rho)$ for $0 < \tau \le R$, so that (see (5.5))

$$D_R(\rho) = \int_0^{R} F(\tau)G(\rho+\tau)\tau^{n-1}d\tau \cong G(\rho)\int_0^{R} F(\tau)\tau^{n-1}d\tau.$$

This estimate coincides with (5.9).

□

Acknowledgments

The results of Sections 2, 4.1, 5 were obtained by E.G. Bakhtigareeva at the Steklov Mathematical Institute of the Russian Academy of Sciences and her work was financially supported by the Russian Science Foundation under grant no. 24-11-00170, https://rscf.ru/project/24-11-00170/. The results of Sections 1, 3, 4.2 were obtained by M.L. Goldman.

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Elza Gizarovna Bakhtigareeva
Department of Function Theory
Steklov Mathematical Institute of the Russian Academy of Sciences
8 Gubkin St,
119991 Moscow, Russian Federation
E-mail: bakhtigareeva-eg@rudn.ru.

Mikhail L'vovich Goldman
is now retired
E-mail: seulydia@yandex.ru.

Received: 03.06.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 4 (2024), 33 – 53

ON DIRECT AND INVERSE PROBLEMS FOR SYSTEMS OF ODD-ORDER QUASILINEAR EVOLUTION EQUATIONS

O.S. Balashov, A.V. Faminskii

Communicated by V.I. Burenkov

Key words: initial-boundary value problem, inverse problem, system of quasilinear evolution equations, odd order.

AMS Mathematics Subject Classification: 93B05, 35Q53, 35Q55.

Abstract. Direct and inverse initial-boundary problems on a bounded interval for systems of odd-order quasilinear evolution equations with general nonlinearities are considered. In the case of inverse problems conditions of integral overdetermination are introduced and right-hand sides of equations of special types are chosen as controls. Results on well-posedness of such problems are established. Assumptions on smallness of the input data or smallness of a time interval are required.

DOI: https://doi.org/10.32523/2077-9879-2024-15-4-33-53

1 Introduction. Notation. Description of main results

Consider the following system of odd-order quasilinear equations

$$u_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}u + a_{2l}\partial_x^{2l}u) - \sum_{j=0}^{l-1}(-1)^j\partial_x^j\big[a_{2j+1}(t,x)\partial_x^{j+1}u + a_{2j}(t,x)\partial_x^j u\big] + \sum_{j=0}^{l}(-1)^j\partial_x^j\big[g_j(t,x,u,\dots,\partial_x^{l-1}u)\big] = f(t,x), \quad l\in\mathbb{N}, \quad (1.1)$$

posed on an interval $I=(0,R)$ ($R>0$ is arbitrary). Here $u=u(t,x)=(u_1,\dots,u_n)^T$, $n\in\mathbb{N}$, is the unknown vector-function, $f=(f_1,\dots,f_n)^T$, $g_j=(g_{j1},\dots,g_{jn})^T$ are also vector-functions, $a_{2l+1}=diag(a_{(2l+1)i})$, $a_{2l}=diag(a_{(2l)i})$, $i=1,\dots,n$, are constant diagonal $n\times n$ matrices, $a_j(t,x)=\big(a_{jim}(t,x)\big)$, $i,m=1,\dots,n$, for $j=0,\dots,2l-1$, are $n\times n$ matrices.

In a rectangle $Q_T=(0,T)\times I$ for certain $T>0$ consider an initial-boundary value problem for system (1.1) with the initial condition

$$u(0,x)=u_0(x), \quad x\in[0,R], \quad (1.2)$$

and the boundary conditions

$$\partial_x^j u(t,0)=\mu_j(t),\ j=0,\dots,l-1, \quad \partial_x^j u(t,R)=\nu_j(t),\ j=0,\dots,l, \quad t\in[0,T], \quad (1.3)$$

where $u_0=(u_{01},\dots,u_{0n})^T$, $\mu_j=(\mu_{j1},\dots,\mu_{jn})^T$, $\nu_j=(\nu_{j1},\dots,\nu_{jn})^T$.

Besides this direct problem consider the following inverse problem: let for any $i = 1, \dots, n$ the function f_i be represented in the form

$$f_i(t,x) \equiv h_{0i}(t,x) + \sum_{k=1}^{m_i} F_{ki}(t) h_{ki}(t,x) \tag{1.4}$$

for a certain non-negative integer number m_i (if $m_i = 0$ then $f_i = h_{0i}$), where the functions h_{ki} are given and the functions F_{ki} are unknown. Then problem (1.1)–(1.3) is supplemented with overdetermination conditions in an integral form: if $m_i > 0$ for certain i, then

$$\int_I u_i(t,x)\omega_{ki}(x)\,dx = \varphi_{ki}(t), \quad t \in [0,T], \quad k = 1, \dots, m_i, \tag{1.5}$$

for certain given functions ω_{ki} and φ_{ki}. In particular, for certain i the overdetermination conditions on u_i can be absent, but in the case of the inverse problem we always assume that

$$M = \sum_{i=1}^{n} m_i > 0. \tag{1.6}$$

Then the aim is to find the functions F_{ki} such that the corresponding solution u to problem (1.1)–(1.3) satisfies conditions (1.5).

In the case of a single equation $n = 1$ equations of type (1.1) were considered in [9] (direct problem) and [10] (inverse problems). In particular, in these articles one can found examples of physical models, which can be described by such equations: the Korteweg–de Vries (KdV) and Kawahara equations with generalizations, the Kortewes–de Vries–Burgers and Benney-Lin equations, the Kaup–Kupershmidt equation and others (see also [1], [14]). However, besides the single equations, systems of odd-order quasilinear evolution equations also arise in real physical situations. Among such systems on can mention the Majda–Biello system (see [17])

$$\begin{cases} u_t + u_{xxx} + vv_x = 0, \\ v_t + \alpha v_{xxx} + (uv)_x = 0, \quad \alpha > 0, \end{cases}$$

and more general systems of KdV-type equations with coupled nonlinearities ([5]).

The KdV-type Boussinesq system ([6, 23, 25])

$$\begin{cases} u_t + v_x + v_{xxx} + (uv)_x = 0, \\ v_t + u_x + u_{xxx} + vv_x = 0 \end{cases}$$

and the coupled system of two KdV equations, derived in [13] and studied in [3, 4, 7, 15, 18, 19, 20, 21, 22] (also with more general nonlinearities)

$$\begin{cases} u_t + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2(uv)_x = 0, \\ b_1 v_t + rv_x + vv_x + b_2 a_3 u_{xxx} + v_{xxx} + b_2 a_2 uu_x = 0, \quad b_1 > 0, b_2 > 0, \end{cases}$$

are not directly written in form (1.1), but can be transformed to it by a linear change of unknown functions (see [3, 6, 23]).

In paper [9] initial-boundary value problem (1.1)–(1.3) was considered in the scalar case and a result on global well-posedness in the class of weak solutions under small input data was established. For simplicity it was assumed there that $\mu_j(t) = \nu_j(t) \equiv 0$ for $j \le l-1$. Note that the general case of (1.3) can be reduced to the homogeneous one by the simple substitution $v(t,x) = u(t,x) - \psi(t,x)$, where the sufficiently smooth function ψ satisfies (1.3) for $j \le l-1$, while the form of equation (1.1) is invariant under the corresponding transformation.

In the present paper a result on global well-posedness of problem (1.1)–(1.3) itself is obtained in the class of weak solutions under small input data. Note that in the aforementioned articles in the case of systems such a problem was not studied. The assumptions on system (1.1) are similar to the ones in [9, 10] in the case of single equations.

The significance of integral overdermination conditions in inverse problems is discussed in [24]. The study of inverse problems for the KdV-type equation with integral overdetermination was started in [8]. In paper [10] for problem (1.1)–(1.3) in the scalar case two inverse problems with one integral overdetemination condition of type (1.5) were considered. In the first one the right-hand side of the equation of a type similar to (1.4) was chosen as the control, in the second one — the boundary data ν_l. Results on well-posedness either for small input data or small time interval were established. In paper [12] an initial-boundary value problem on a bounded interval for the higher order nonlinear Schrödinger equation

$$iu_t + au_{xx} + ibu_x + iu_{xxx} + \lambda|u|^p u + i\beta\big(|u|^p u\big)_x + i\gamma\big(|u|^p\big)_x u = 0$$

(u is a complex-valued function) with initial and boundary conditions similar to (1.2), (1.3) was considered and three inverse problems were studied. The first two of them were similar to the problems considered in [10] with similar results. In the third problem two overdetermination conditions of (1.5) type were introduced and both the right-hand side of the equation and the boundary function were chosen as controls. The results were similar to the first two cases.

Note that the inverse problem with two integral overdetermination conditions for the Korteweg–de Vries type equation

$$u_t + u_{xxx} + uu_x + \alpha(t)u = F(t)g(t)$$

in the periodic case, where the functions α and F were unknown, was considered in [16] and the existence and uniqueness results were obtained for a small time interval.

In paper [21] an inverse problem on a bounded interval with the terminal overdetermination condition

$$u(T,x) = u_T(x)$$

for a given function u_T (such problems are called controllability ones) was studied for the aforementioned coupled system of two KdV equations. Results on existence of solutions under small input data were established.

In the present paper, results on well-posedness of inverse problem (1.1)–(1.6) are obtained either for small input data or small time interval. Note that since the amount of integral overdetermination conditions is arbitrary, the result is new even in the case of one equation.

Solutions of the considered problems are constructed in the special function space $\big(X(Q_T)\big)^n$ of all vector-functions $u = (u_1,\dots,u_n)^T$ such that such that for every $i = 1,\dots,n$

$$u_i(t,x) \in X(Q_T) = C([0,T]; L_2(I)) \cap L_2(0,T; H^l(I)),$$

endowed with the norm

$$\|u\|_{(X(Q_T))^n} = \sum_{i=1}^n \Big(\sup_{t\in(0,T)} \|u_i(t,\cdot)\|_{L_2(I)} + \|\partial_x^l u_i\|_{L_2(Q_T)}\Big).$$

For $r > 0$ let $\overline{X}_{rn}(Q_T)$ denote the closed ball $\{u \in \big(X(Q_T)\big)^n : \|u\|_{(X(Q_T))^n} \le r\}$.

Introduce the notion of a weak solution of problem (1.1)–(1.3).

Definition 1. Let $u_0 \in \big(L_2(I)\big)^n$, $\mu_j, \nu_j \in \big(L_2(0,T)\big)^n$ $\forall j$, $f \in \big(L_1(Q_T)\big)^n$, $a_j \in \big(C(\overline{Q}_T)\big)^{n^2}$ $\forall j$. A function $u \in \big(X(Q_T)\big)^n$ is called a weak solution of problem (1.1)–(1.3) if $\partial_x^j u(t,0) \equiv \mu_j(t)$, $\partial_x^j u(t,R) \equiv \nu_j(t)$, $j = 0,\dots,l-1$, and for all test functions $\phi(t,x)$, such that $\phi \in \big(L_2(0,T;H^{l+1}(I))\big)^n$, $\phi_t \in \big(L_2(Q_T)\big)^n$, $\phi\big|_{t=T} \equiv 0$, $\partial_x^j\phi\big|_{x=0} = \partial_x^j\phi\big|_{x=R} \equiv 0$, $j = 0,\dots,l-1$, $\partial_x^l\phi\big|_{x=0} \equiv 0$, the functions $\big(g_j(t,x,u,\dots,\partial_x^{l-1}u), \partial_x^j\phi\big) \in L_1(Q_T)$, $j = 0,\dots,l$, and the following integral identity holds:

$$\iint_{Q_T} \Big[(u,\phi_t) - (a_{2l+1}\partial_x^l u, \partial_x^{l+1}\phi) + (a_{2l}\partial_x^l u, \partial_x^l\phi) + \sum_{j=0}^{l-1}\big((a_{2j+1}\partial_x^{j+1}u + a_{2j}\partial_x^j u), \partial_x^j\phi\big) - \sum_{j=0}^{l}\big(g_j(t,x,u,\dots,\partial_x^{l-1}u), \partial_x^j\phi\big) + (f,\phi)\Big]\,dxdt + \int_I (u_0, \phi\big|_{t=0})\,dx + \int_0^T (a_{2l+1}\nu_l, \partial_x^l\phi\big|_{x=R})\,dt = 0, \quad (1.7)$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$.

Let $\widehat{f}(\xi) \equiv \mathcal{F}[f](\xi)$ and $\mathcal{F}^{-1}[f](\xi)$ be the direct and inverse Fourier transforms of a function f, respectively . In particular, for $f \in \mathcal{S}(\mathbb{R})$

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\,dx, \qquad \mathcal{F}^{-1}[f](x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} f(\xi)\,d\xi.$$

For $s \in \mathbb{R}$ define the fractional order Sobolev space

$$H^s(\mathbb{R}) = \big\{f : \mathcal{F}^{-1}[(1+|\xi|^s)\widehat{f}(\xi)] \in L_2(\mathbb{R})\big\}$$

and for certain $T > 0$ let $H^s(0,T)$ be the space of restrictions on $(0,T)$ of functions from $H^s(\mathbb{R})$. To describe properties of boundary functions μ_j, ν_j we use the following function spaces. Let $m = l-1$ or $m = l$, define

$$\big(\mathcal{B}^m(0,T)\big)^n = \Big(\prod_{j=0}^{m} H^{(l-j)/(2l+1)}(0,T)\Big)^n,$$

endowed with the natural norm.

The coefficients of the linear part of the system further are always assumed to verify the following conditions:

$$a_{(2l+1)i} > 0, \quad a_{(2l)i} \le 0, \quad i = 1,\dots,n, \quad (1.8)$$

and for any $0 \le j \le l-1$, $i,m = 1,\dots n$

$$\partial_x^k a_{(2j+1)im} \in C(\overline{Q}_T),\ k = 0,\dots,j+1, \quad \partial_x^k a_{(2j)im} \in C(\overline{Q}_T),\ k = 0,\dots,j. \quad (1.9)$$

Let $y_m = (y_{m1},\dots,y_{mn})$ for $m = 0,\dots,l-1$. The functions $g_j(t,x,y_0,\dots,y_{l-1})$ for any $0 \le j \le l$ are always subjected to the following assumptions: for $i = 1,\dots,n$

$$g_{ji}, grad_{y_k} g_{ji} \in C(\overline{Q}_T \times \mathbb{R}^{ln}),\ j = 0,\dots,l-1, \quad g_{ji}(t,x,0,\dots,0) \equiv 0, \quad (1.10)$$

$$\big|grad_{y_k} g_{ji}(t,x,y_0,\dots,y_{l-1})\big| \le c\sum_{m=0}^{l-1}\big(|y_m|^{b_1(j,k,m)} + |y_m|^{b_2(j,k,m)}\big), k = 0,\dots,l-1, \quad \forall (t,x,y_0,\dots,y_{l-1}) \in Q_T \times \mathbb{R}^{ln}, \quad (1.11)$$

where $0 < b_1(j,k,m) \le b_2(j,k,m)$, $|y_m| = (y_m,y_m)^{1/2}$.

Regarding the functions ω_{ki} we always need the following conditions:

$$\omega \in H^{2l+1}(I), \quad \omega^{(m)}(0) = 0,\ m = 0,\dots,l, \quad \omega^{(m)}(R) = 0,\ m = 0,\dots,l-1, \tag{1.12}$$

for all ω_{ki} (where here ω stands for ω_{ki}).

Now we can pass to the main results and begin with the direct problem.

Theorem 1.1. *Let the coefficients a_j, $j = 0,\dots,2l+1$, satisfy conditions (1.8), (1.9). Let the functions g_j satisfy conditions (1.10), (1.11), where*

$$b_2(j,k,m) \le \frac{4l-2j-2k}{2m+1} \quad \forall\ j,k,m. \tag{1.13}$$

Let $u_0 \in \big(L_2(I)\big)^n$, $(\mu_0,\dots,\mu_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0,\dots,\nu_l) \in \big(\mathcal{B}^{l}(0,T)\big)^n$, $f \in \big(L_1(0,T;L_2(I))\big)^n$ for an arbitrary $T > 0$. Denote

$$c_0 = \|u_0\|_{(L_2(I))^n} + \|(\mu_0,\dots,\mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} + \|(\nu_0,\dots,\nu_l)\|_{(\mathcal{B}^{l}(0,T))^n} + \|f\|_{(L_1(0,T;L_2(I)))^n}. \tag{1.14}$$

Then there exists $\delta > 0$ such that under the assumption $c_0 \le \delta$ there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ of problem (1.1)–(1.3). Moreover, the map

$$\big(u_0, (\mu_0,\dots,\mu_{l-1}), (\nu_0,\dots,\nu_l), f\big) \to u \tag{1.15}$$

is Lipschitz continuous on the closed ball of the radius δ in the space $\big(L_2(I)\big)^n \times \big(\mathcal{B}^{l-1}(0,T)\big)^n \times \big(\mathcal{B}^{l}(0,T)\big)^n \times \big(L_1(0,T;L_2(I))\big)^n$ into the space $\big(X(Q_T)\big)^n$.

Theorem 1.2. *Let the hypotheses of Theorem 1.1 be satisfied except inequalities (1.13) which are substituted by the following ones:*

$$b_2(j,k,m) < \frac{4l-2j-2k}{2m+1} \quad \forall\ j,k,m. \tag{1.16}$$

Let c_0 is given by formula (1.14).

Then for a fixed arbitrary $\delta > 0$ there exists $T_0 > 0$ such that if $c_0 \le \delta$ and $T \in (0,T_0]$ there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ of problem (1.1)–(1.3). Moreover, the map (1.15) is Lipschitz continuous on the closed ball of the radius δ similarly to Theorem 1.1.

For the inverse problem the results are as follows.

Theorem 1.3. *Let the coefficients a_j, $j = 0,\dots,2l+1$, satisfy conditions (1.8), (1.9) and the functions g_j satisfy conditions (1.10), (1.11), (1.13). Let $u_0 \in \big(L_2(I)\big)^n$, $(\mu_0,\dots,\mu_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0,\dots,\nu_l) \in \big(\mathcal{B}^{l}(0,T)\big)^n$, $h_0 = (h_{01},\dots,h_{0n})^T \in \big(L_1(0,T;L_2(I))\big)^n$ for an arbitrary $T > 0$. Assume that condition (1.6) holds and for any $i = 1,\dots,n$, satisfying $m_i > 0$, for $k = 1,\dots m_i$ the functions ω_{ki} satisfy condition (1.12); $\varphi_{ki} \in W_1^1(0,T)$ and*

$$\varphi_{ki}(0) = \int_I u_{0i}(x)\omega_{ki}(x)\,dx; \tag{1.17}$$

$h_{ki} \in C([0,T];L_2(I))$ for $k = 1,\dots,m_i$. Let

$$\psi_{kji}(t) \equiv \int_I h_{ji}(t,x)\omega_{ki}(x)\,dx, \quad k,j = 1,\dots,m_i, \tag{1.18}$$

and assume that

$$\Delta_i(t) \equiv \det\bigl(\psi_{kji}(t)\bigr) \neq 0 \quad \forall\, t \in [0,T]. \tag{1.19}$$

Denote

$$c_0 = \|u_0\|_{(L_2(I))^n} + \|(\mu_0,\dots,\mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} + \|(\nu_0,\dots,\nu_l)\|_{(\mathcal{B}^l(0,T))^n} + \|h_0\|_{(L_1(0,T;L_2(I)))^n} + \sum_{i:m_i>0}\sum_{k=1}^{m_i} \|\varphi'_{ki}\|_{L_1(0,T)}. \tag{1.20}$$

Then there exists $\delta > 0$ such that under the assumption $c_0 \le \delta$ there exist functions $F_{ki} \in L_1(0,T)$, $i : m_i > 0$, $k = 1,\dots,m_i$, and the corresponding weak solution $u \in \bigl(X(Q_T)\bigr)^n$ of problem (1.1)–(1.3) satisfying (1.5), where the function f is given by formula (1.4). Moreover, there exists $r > 0$ such that this solution u is unique in the ball $\overline{X}_{rn}(Q_T)$ with the corresponding unique functions $F_{ki} \in L_1(0,T)$ and the map

$$\bigl(u_0, (\mu_0,\dots,\mu_{l-1}), (\nu_0,\dots,\nu_l), h_0, \{\varphi'_{ki}\}\bigr) \to (u, \{F_{ki}\}) \tag{1.21}$$

is Lipschitz continuous on the closed ball of the radius δ in the space $\bigl(L_2(I)\bigr)^n \times \bigl(\mathcal{B}^{l-1}(0,T)\bigr)^n \times \bigl(\mathcal{B}^l(0,T)\bigr)^n \times \bigl(L_1(0,T;L_2(I))\bigr)^n \times \bigl(L_1(0,T)\bigr)^M$ into the space $\bigl(X(Q_T)\bigr)^n \times \bigl(L_1(0,T)\bigr)^M$.

Theorem 1.4. *Let the hypotheses of Theorem 1.3 be satisfied except inequalities (1.13) which are substituted by inequalities (1.16). Let c_0 be given by formula (1.20). Then two assertions are valid.*

1. For a fixed arbitrary $\delta > 0$ there exists $T_0 > 0$ such that if $c_0 \le \delta$ and $T \in (0, T_0]$, there exist unique functions $F_{ki} \in L_1(0,T)$, $i : m_i > 0$, $k = 1,\dots,m_i$, and the corresponding unique weak solution $u \in \bigl(X(Q_T)\bigr)^n$ of problem (1.1)–(1.3) satisfying (1.5), where the function f is given by formula (1.4).

2. For a fixed arbitrary $T > 0$ there exists $\delta > 0$ such that under the assumption $c_0 \le \delta$ there exist unique functions $F_{ki} \in L_1(0,T)$, $i : m_i > 0$, $k = 1,\dots,m_i$, and the corresponding unique weak solution $u \in \bigl(X(Q_T)\bigr)^n$ of problem (1.1)–(1.3) satisfying (1.5), where the function f is given by formula (1.4).

Moreover, map (1.21) is Lipschitz continuous on the closed ball of the radius δ similarly to Theorem 1.3.

Remark 1. Theorems 1.2 and 1.4 are valid for the aforementioned Majda–Biello system. In the case of such a system with more general nonlinearities

$$\begin{cases} u_t + u_{xxx} + \bigl(g_1(u,v)\bigr)_x = f_1, \\ v_t + \alpha v_{xxx} + \bigl(g_2(u,v)\bigr)_x = f_2, \quad \alpha > 0, \end{cases}$$

Theorems 1.1 and 1.3 are valid if

$$|\partial_{y_k} g_j(y_1,y_2)| \le c\bigl(|y_1|^{b_1} + |y_2|^{b_1} + |y_1|^{b_2} + |y_2|^{b_2}\bigr), \quad k,j = 1,2,$$

where $0 < b_1 \le b_2 \le 2$, for example, if $g_1(y_1,y_2) = c y_2^3$, $g_2(y_1,y_2) = c_1 y_1^2 y_2 + c_2 y_1 y_2^2$.

The paper is organized as follows. Section 2 contains certain auxiliary results on the corresponding linear initial-boundary value problem and interpolating inequalities. Section 3 is devoted to the direct problem, Section 4 — to the inverse one.

2 Preliminaries

Further we use the following interpolating inequality (see, for example, [2]): there exists a constant $c = c(R, l, p)$ such that for any $\varphi \in H^l(I)$, integer $m \in [0, l)$ and $p \in [2, +\infty]$

$$\|\varphi^{(m)}\|_{L_p(I)} \le c\|\varphi^{(l)}\|^{2s}_{L_2(I)}\|\varphi\|^{1-2s}_{L_2(I)} + c\|\varphi\|_{L_2(I)}, \quad s = s(p, l, m) = \frac{2m+1}{4l} - \frac{1}{2lp}. \tag{2.1}$$

On the basis of (2.1) in [10, Lemma 3.3] the following inequality was proved: let $j \in [0, l]$, $k, m \in [0, l-1]$, $b \in (0, (4l - 2j - 2k)/(2m + 1)]$, then for any functions $v, w \in X(Q_T)$

$$\big\||\partial_x^m v|^b \partial_x^k w\big\|_{L_{2l/(2l-j)}(0,T;L_2(I))} \le c\big(T^{((4l-2j-2k)-(2m+1)b)/(4l)} + T^{(2l-j)/(2l)}\big)\|v\|^b_{X(Q_T)}\|w\|_{X(Q_T)}. \tag{2.2}$$

Besides nonlinear system (1.1) consider its linear analogue

$$u_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}u + a_{2l}\partial_x^{2l}u) - \sum_{j=0}^{l-1}(-1)^j\partial_x^j\big[a_{2j+1}(t,x)\partial_x^{j+1}u + a_{2j}(t,x)\partial_x^j u\big] = f(t,x) + \sum_{j=0}^{l}(-1)^j\partial_x^j G_j(t,x), \tag{2.3}$$

$G_j = (G_{j1}, \dots, G_{jn})^T$. The notion of a weak solution to the corresponding initial-boundary value problem is similar to Definition 1. In particular, the corresponding integral identity (for the same test functions as in Definition 1) is as follows:

$$\iint_{Q_T}\Big[(u, \phi_t) - (a_{2l+1}\partial_x^l u, \partial_x^{l+1}\phi) + (a_{2l}\partial_x^l u, \partial_x^l\phi) + \sum_{j=0}^{l-1}\big((a_{2j+1}\partial_x^{j+1}u + a_{2j}\partial_x^j u), \partial_x^j\phi\big) + \big(f(t,x), \phi\big) + \sum_{j=0}^{l}\big(G_j(t,x), \partial_x^j\phi\big)\Big]\,dxdt + \int_I (u_0, \phi\big|_{t=0})\,dx + \int_0^T (a_{2l+1}\nu_l, \partial_x^l\phi\big|_{x=R})\,dt = 0. \tag{2.4}$$

First consider the case $a_j \equiv 0$ for $j \le 2l - 1$. Then system (2.3) is obviously splitted into the set of separate equations and we can use the corresponding results from [11] and [9] for single equations.

Lemma 2.1. *Let the coefficients a_{2l+1} and a_{2l} satisfy condition (1.8), $a_j \equiv 0$ for $j \le 2l-1$, $u_0 \in \big(L_2(I)\big)^n$, $(\mu_0, \dots, \mu_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0, \dots, \nu_l) \in \big(\mathcal{B}^l(0,T)\big)^n$, $f = G_j \equiv 0\ \forall j$.*

Then there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ of problem (2.3), (1.2), (1.3) and for any $t \in (0, T]$

$$\|u\|_{(X(Q_t))^n} \le c(T)\Big[\|u_0\|_{(L_2(I))^n} + \|(\mu_0, \dots, \mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,t))^n} + \|(\nu_0, \dots, \nu_l)\|_{(\mathcal{B}^l(0,t))^n}\Big]. \tag{2.5}$$

Proof. This assertion succeeds from [11, Lemma 4.3]. □

Lemma 2.2. *Let the coefficients a_{2l+1} and a_{2l} satisfy condition (1.8), $a_j \equiv 0$ for $j \le 2l-1$, $u_0 \equiv 0$, $\mu_j \equiv 0$ for $j = 0, \dots, l-1$, $\nu_j \equiv 0$ for $j = 0, \dots, l$, $f \in \big(L_1(0,T; L_2(I))\big)^n$, $G_j \in \big(L_{2l/(2l-j)}(0,T; L_2(I))\big)^n$, $j = 0, \dots, l$.*

Then there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ *of problem* (2.3), (1.2), (1.3) *and for any* $t \in [0,T]$

$$\|u\|_{(X(Q_t))^n} \leq c(T)\Big[\|f\|_{(L_1(0,t;L_2(I)))^n} + \sum_{j=0}^{l} \|G_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n}\Big]; \tag{2.6}$$

moreover, for $i = 1, \dots, n$ *and* $\rho(x) \equiv 1 + x$

$$\int_I u_i^2(t,x)\rho(x)\,dx + \iint_{Q_t} \big((2l+1)a_{(2l+1)i} - 2a_{(2l)i}\rho(x)\big)\big(\partial_x^l u_i(\tau,x)\big)^2\,dxd\tau$$
$$\leq 2\iint_{Q_t} f_i u_i \rho\,dxd\tau + 2\sum_{j=0}^{l}\iint_{Q_t} G_{ji}(\partial_x^j u_i\rho + j\partial_x^{j-1}u_i)\,dxd\tau. \tag{2.7}$$

Proof. This assertion succeeds from [9, Lemma 4]. □

Theorem 2.1. *Let the coefficients* a_j *satisfy conditions* (1.8), (1.9), $u_0 \in \big(L_2(I)\big)^n$, $(\mu_0, \dots, \mu_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0, \dots, \nu_l) \in \big(\mathcal{B}^l(0,T)\big)^n$, $f \in \big(L_1(0,T;L_2(I))\big)^n$, $G_j \in \big(L_{2l/(2l-j)}(0,T;L_2(I))\big)^n$, $j = 0, \dots, l$.

Then there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ *of problem* (2.3), (1.2), (1.3) *and for any* $t \in (0,T]$

$$\|u\|_{(X(Q_t))^n} \leq c(T)\Big[\|u_0\|_{(L_2(I))^n} + \|(\mu_0, \dots, \mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,t))^n}$$
$$+ \|(\nu_0, \dots, \nu_l)\|_{(\mathcal{B}^l(0,t))^n} + \|f\|_{(L_1(0,t;L_2(I)))^n} + \sum_{j=0}^{l}\|G_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n}\Big]. \tag{2.8}$$

Proof. Denote by $w = (w_1, \dots, w_n)^T$ the solution of problem (2.3), (1.2), (1.3) constructed in Lemma 2.1 Let $U(t,x) \equiv u(t,x) - w(t,x)$. Consider an initial-boundary value problem for the function U:

$$U_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}U + a_{2l}\partial_x^{2l}U) - \sum_{j=0}^{l-1}(-1)^j\partial_x^j\big[a_{2j+1}(t,x)\partial_x^{j+1}U + a_{2j}(t,x)\partial_x^j U\big]$$
$$= f(t,x) + \sum_{j=0}^{l}(-1)^j\partial_x^j\widetilde{G}_j(t,x), \tag{2.9}$$

where $\widetilde{G}_l \equiv G_l$, while $\widetilde{G}_j \equiv G_j + a_{2j+1}\partial_x^{j+1}w + a_{2j}\partial_x^j w$ for $j < l$, and zero initial and boundary conditions (1.2), (1.3). Note that by virtue of (2.1) for $m = 0$ or $m = 1$, $j < l$ and $i = 1, \dots, n$

$$\|\partial_x^{j+m}w_i\|_{L_2(I)} \leq c\|\partial_x^l w_i\|_{L_2(I)}^{(j+m)/l}\|w_i\|_{L_2(I)}^{(l-j-m)/l} + c\|w_i\|_{L_2(I)}.$$

Therefore, $\widetilde{G}_j \in \big(L_{2l/(2l-j)}(0,T;L_2(I))\big)^n$ with

$$\|\widetilde{G}_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n} \leq \|G_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n} + c(T)\|w\|_{(X(Q_t))^n}. \tag{2.10}$$

In order to obtain the solution to the initial-value problem for system (2.9) we apply the contraction principle and first construct it on a small time interval $[0,t_0]$ as the fixed point of a map

$U = \Lambda V$, where for $V \in \big(X(Q_{t_0})\big)^n$ the function $U \in \big(X(Q_{t_0})\big)^n$ is a solution to an initial-boundary value problem for the system

$$U_t - (-1)^l (a_{2l+1}\partial_x^{2l+1} U + a_{2l}\partial_x^{2l} U) = \sum_{j=0}^{l-1} (-1)^j \partial_x^j \big[a_{2j+1}(t,x)\partial_x^{j+1} V + a_{2j}(t,x)\partial_x^j V\big] + f(t,x) + \sum_{j=0}^{l} (-1)^j \partial_x^j \widetilde{G}_j(t,x), \quad (2.11)$$

with zero initial and boundary conditions (1.2), (1.3). Note that similarly to (2.10) the hypothesis of Lemma 2.2 is verified and such a map is defined for any $t_0 \in (0,T]$. Moreover, according to (2.6)

$$\|U\|_{(X(Q_{t_0}))^n} \le c(T)\Big[\|f\|_{(L_1(0,t_0;L_2(I)))^n} + \sum_{j=0}^{l} \|\widetilde{G}_j\|_{(L_{2l/(2l-j)}(0,t_0;L_2(I)))^n} + \sum_{j=0}^{l-1} \big(\|\partial_x^{j+1} V\|_{(L_{2l/(2l-j)}(0,t_0;L_2(I)))^n} + (\|\partial_x^j V\|_{(L_{2l/(2l-j)}(0,t_0;L_2(I)))^n}\big)\Big]. \quad (2.12)$$

By virtue of (2.1) if $j+m \le 2l-1$ for $i=1,\dots,n$

$$\begin{aligned}\|\partial_x^m V_i\|_{L_{2l/(2l-j)}(0,t_0;L_2(I))} &\le c\Big(\int_0^{t_0} \big(\|\partial_x^l V_i\|_{L_2(I)}^{2m/(2l-j)} \|V_i\|_{L_2(I)}^{2(l-m)/(2l-j)} + \|V_i\|_{L_2(I)}^{2l/(2l-j)}\big)\,dt\Big)^{(2l-j)/(2l)} \\ &\le c t_0^{(2l-j-m)/(2l)} \|V_i\|_{C([0,t_0];L_2(I))}^{(l-m)/l} \|\partial_x^l V_i\|_{L_2(Q_{t_0})}^{m/l} + c t_0^{(2l-j)/(2l)} \|V_i\|_{C([0,t_0];L_2(I))} \\ &\le c(T) t_0^{1/(2l)} \|V_i\|_{X(Q_{t_0})}. \end{aligned} \quad (2.13)$$

Therefore, it follows from (2.12) that

$$\|U\|_{(X(Q_{t_0}))^n} \le c(T)\Big[\|f\|_{(L_1(0,t_0;L_2(I)))^n} + \sum_{j=0}^{l} \|\widetilde{G}_j\|_{(L_{2l/(2l-j)}(0,t_0;L_2(I)))^n} + t_0^{1/(2l)} \|V\|_{(X(Q_{t_0}))^n}\Big]. \quad (2.14)$$

Similarly to (2.14) for $\widetilde{V} \in \big(X(Q_{t_0})\big)^n$, $\widetilde{U} = \Lambda \widetilde{V}$

$$\|U - \widetilde{U}\|_{(X(Q_{t_0}))^n} \le c(T) t_0^{1/(2l)} \|V - \widetilde{V}\|_{(X(Q_{t_0}))^n}. \quad (2.15)$$

Inequalities (2.14), (2.15) provide existence of a unique solution $U \in \big(X(Q_{t_0})\big)^n$ to the considered problem if, for example, $c(T) t_0^{1/(2l)} \le 1/2$. Then since the value of t_0 depends only on T step by step this solution can be extended to the whole time segment $[0,T]$, moreover,

$$\|U\|_{(X(Q_t))^n} \le c(T)\Big[\|f\|_{(L_1(0,t;L_2(I)))^n} \sum_{j=0}^{l} \|\widetilde{G}_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n}\Big]. \quad (2.16)$$

Combining (2.5) (applied to the function w), (2.10) and (2.16), for $u \equiv U + w$ we complete the proof. $\square$

Introduce certain additional notation. Let

$$u = S(u_0, (\mu_0, \dots, \mu_{l-1}), (\nu_0, \dots, \nu_l), f, (G_0, \dots, G_l))$$

be a weak solution of problem (2.3), (1.2), (1.3) from the space $\bigl(X(Q_T)\bigr)^n$ under the hypotheses of Theorem 2.1. Define also

$$W = (u_0, (\mu_0, \dots, \mu_{l-1}), (\nu_0, \dots, \nu_l)),$$

$$\widetilde{S}W = S(W, 0, (0, \dots, 0)), \quad \widetilde{S} : \bigl(L_2(I) \times \mathcal{B}^{l-1}(0,T) \times \mathcal{B}^l(0,T)\bigr)^n \to \bigl(X(Q_T)\bigr)^n,$$

$$S_0 f = S(0, (0, \dots, 0), (0, \dots, 0), f, (0, \dots, 0)), S_0 : \bigl(L_1(0,T; L_2(I))\bigr)^n \to \bigl(X(Q_T)\bigr)^n,$$

$$\widetilde{S}_j G_j = S(0, (0, \dots, 0), (0, \dots, 0), 0, (0, \dots, G_j, \dots, 0)),$$
$$S_j : \bigl(L_{2l/(2l-j)}(0,T; L_2(I))\bigr)^n \to \bigl(X(Q_T)\bigr)^n, \quad j = 0, \dots, l.$$

Let $\widetilde{W_1^1}(0,T) = \{\varphi \in W_1^1(0,T) : \varphi(0) = 0\}$. Obviously, the equivalent norm in this space is $\|\varphi'\|_{L_1(0,T)}$.

Let a function $\omega \in C(\bar{I})$. On the space of functions $u(t,x)$, lying in $L_1(I)$ for all $t \in [0,T]$, define a linear operator $Q(\omega)$ by a formula $(Q(\omega)u)(t) = q(t; u, \omega)$, where

$$q(t; u, \omega) \equiv \int_I u(t,x)\omega(x)\, dx, \quad t \in [0,T].$$

Lemma 2.3. *Let the hypotheses of Theorem 2.1 be satisfied. Let the function ω satisfy condition (1.12).*

Then for the function $u = (u_1 \dots, u_n)^T = S(u_0, (\mu_0, \dots, \mu_{l-1}), (\nu_0, \dots, \nu_l), f, (G_0, \dots, G_l))$ the functions $q(\cdot; u_i, \omega) = Q(\omega)u_i \in W_1^1(0,T)$, i=1,…,n, and for almost every $t \in (0,T)$

$$\begin{aligned}
q'(t; u_i, \omega) = r(t; u_i, \omega) &\equiv \nu_{li}(t) a_{(2l+1)i} \omega^{(l)}(R) \\
&+ \sum_{k=0}^{l-1} (-1)^{l+k} \Bigl[\nu_{ki}(t) \bigl(a_{(2l+1)i}\omega^{(2l-k)}(R) - a_{(2l)i}\omega^{(2l-k-1)}(R)\bigr) \\
&\qquad - \mu_{ki}(t) \bigl(a_{(2l+1)i}\omega^{(2l-k)}(0) - a_{(2l)i}\omega^{(2l-k-1)}(0)\bigr)\Bigr] \\
&+ \sum_{m=1}^{n} \sum_{j=0}^{l-1} \sum_{k=0}^{j-1} (-1)^{j+k} \Bigl[\nu_{km}(t) \bigl((a_{(2j+1)im}\omega^{(j)})^{(j-k)}(R) - (a_{(2j)im}\omega^{(j)})^{(j-k-1)}(R)\bigr) \\
&\qquad - \mu_{km}(t) \bigl((a_{(2j+1)im}\omega^{(j)})^{(j-k)}(0) - (a_{(2j)im}\omega^{(j)})^{(j-k-1)}(0)\bigr)\Bigr] \\
&+ (-1)^{l+1} \int_I u_i(t,x) \bigl(a_{(2l+1)i}\omega^{(2l+1)} - a_{(2l)i}\omega^{(2l)}\bigr)\, dx \\
&+ \sum_{m=1}^{n} \sum_{j=0}^{l-1} (-1)^{j+1} \int_I u_m(t,x) \bigl[(a_{(2j+1)im}\omega^{(j)})^{(j+1)} - (a_{(2j)im}\omega^{(j)})^{(j)}\bigr]\, dx \\
&+ \int_I f_i(t,x)\omega\, dx + \sum_{j=0}^{l} \int_I G_{ji}(t,x)\omega^{(j)}\, dx, \qquad (2.17)
\end{aligned}$$

$$\begin{aligned}
\|q'(\cdot; u_i, \omega)\|_{L_1(0,T)} &\le c(T)\Bigl[\|u_0\|_{(L_2(I))^n} + \|(\mu_0, \dots, \mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} \\
&+ \|(\nu_0, \dots, \nu_l)\|_{(\mathcal{B}^l(0,T))^n} + \|f\|_{(L_1(0,T;L_2(I)))^n} + \sum_{j=0}^{l} \bigl(\|G_j\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n}\bigr)\Bigr], \qquad (2.18)
\end{aligned}$$

where the constant c does not decrease in T.

Proof. For an arbitrary function $\psi \in C_0^\infty(0,T)$ let $\phi_i(t,x) \equiv \psi(t)\omega(x)$ for certain i, $\phi_m(x) \equiv 0$ when $m \neq i$. This function ϕ satisfies the assumption on a test function in Definition 1 and then equality (2.4) after integration by parts yields that

$$\int_0^T \psi'(t) q(t; u_i, \omega)\, dt = -\int_0^T \psi(t) r(t; u_i, \omega)\, dt. \tag{2.19}$$

Since $r \in L_1(0,T)$ it follows from (2.19) that there exists the weak derivative $q'(t; u_i, \omega) = r(t; u_i, \omega) \in L_1(0,T)$ and

$$\|q'\|_{L_1(0,T)} \le c\Big[\sum_{j=0}^{l-1} \|\mu_j\|_{(L_1(0,T))^n} + \sum_{j=0}^{l} \|\nu_j\|_{(L_1(0,T))^n} + \|f\|_{(L_1(0,T;L_2(I)))^n} \\ + \sum_{j=0}^{l} \|G_j\|_{(L_1(0,T;L_1(I)))^n} + \|u\|_{(L_1(0,T;L_2(I)))^n}\Big].$$

Since $\|u\|_{(L_1(0,T;L_2(I))^n} \le T\|u\|_{(C([0,T];L_2(I)))^n} \le T\|u\|_{(X(Q_T))^n}$, application of inequality (2.8) completes the proof. □

3 The direct problem

Proof of the existence part of Theorem 1.1. On the space $\big(X(Q_T)\big)^n$ consider the map Θ

$$u = \Theta v \equiv \widetilde{S}W + S_0 f - \sum_{j=0}^{l} \widetilde{S}_j g_j(t,x,v,\dots,\partial_x^{l-1}v). \tag{3.1}$$

Note that according to conditions (1.10), (1.11) for $i = 1,\dots,n$

$$|g_{ji}(t,x,v,\dots,\partial_x^{l-1}v)| \le c\sum_{k=0}^{l-1}\sum_{m=0}^{l-1}\big(|\partial_x^m v|^{b_1(j,k,m)} + |\partial_x^m v|^{b_2(j,k,m)}\big)|\partial_x^k v| \tag{3.2}$$

In particular, conditions (1.13) and inequality (2.2) yield that $g_{ji}(t,x,v,\dots,\partial_x^{l-1}v) \in L_{2l/(2l-j)}(0,T;L_2(I))$, moreover,

$$\|g_j(t,x,v,\dots,\partial_x^{l-1}v)\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n} \\ \le c\sum_{k=0}^{l-1}\sum_{m=0}^{l-1}\sum_{i=1}^{2}(T^{((4l-2j-2k)-(2m+1)b_i(j,k,m))/(4l)} + T^{(2l-j)/(2l)})\|v\|^{b_i(j,k,m)+1}_{(X(Q_T))^n}. \tag{3.3}$$

In particular, Theorem 2.1 ensures that the map Θ exists. Let

$$b_1 = \min_{j,k,m}(b_1(j,k,m)), \quad b_2 = \max_{j,k,m}(b_2(j,k,m)), \quad 0 < b_1 \le b_2, \tag{3.4}$$

then it follows from (3.3) that

$$\|g_j(t,x,v,\dots,\partial_x^{l-1}v)\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n} \le c(T)\left(\|v\|^{b_1+1}_{(X(Q_T))^n} + \|v\|^{b_2+1}_{(X(Q_T))^n}\right), \tag{3.5}$$

therefore, inequality (2.8) implies that

$$\|\Theta v\|_{(X(Q_T))^n} \le c(T)c_0 + c(T)\left(\|v\|^{b_1+1}_{(X(Q_T))^n} + \|v\|^{b_2+1}_{(X(Q_T))^n}\right). \tag{3.6}$$

Next, for any functions $v_1, v_2 \in \big(X(Q_T)\big)^n$

$$|g_{ji}(t,x,v_1,\dots,\partial_x^{l-1}v_1) - g_{ji}(t,x,v_2,\dots,\partial_x^{l-1}v_2)| \\ \le c\sum_{k=0}^{l-1}\sum_{m=0}^{l-1}\big(|\partial_x^m v_1|^{b_1(j,k,m)} + |\partial_x^m v_2|^{b_1(j,k,m)} + |\partial_x^m v_1|^{b_2(j,k,m)} + |\partial_x^m v_2|^{b_2(j,k,m)}\big) \\ \times |\partial_x^k(v_1 - v_2)|, \quad (3.7)$$

therefore, similarly to (3.5)

$$\|g_j(t,x,v_1,\dots,\partial_x^{l-1}v_1) - g_j(t,x,v_2,\dots,\partial_x^{l-1}v_2)\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n} \\ \le c(T)\Big(\|v_1\|^{b_1}_{(X(Q_T))^n} + \|v_2\|^{b_1}_{(X(Q_T))^n} + \|v_1\|^{b_2}_{(X(Q_T))^n} + \|v_2\|^{b_2}_{(X(Q_T))^n}\Big)\|v_1 - v_2\|_{(X(Q_T))^n}. \quad (3.8)$$

and similarly to (3.6)

$$\|\Theta v_1 - \Theta v_2\|_{(X(Q_T))^n} \\ \le c(T)\Big(\|v_1\|^{b_1}_{(X(Q_T))^n} + \|v_2\|^{b_1}_{(X(Q_T))^n} + \|v_1\|^{b_2}_{(X(Q_T))^n} + \|v_2\|^{b_2}_{(X(Q_T))^n}\Big)\|v_1 - v_2\|_{(X(Q_T))^n}. \quad (3.9)$$

Now, choose $r > 0$ such that

$$r^{b_1} + r^{b_2} \le \frac{1}{4c(T)} \quad (3.10)$$

and then $\delta > 0$ such that

$$\delta \le \frac{r}{2c(T)}. \quad (3.11)$$

Then it follows from (3.6) and (3.9) that on the ball $\overline{X}_{rn}(Q_T)$ the map Θ is a contraction. Its unique fixed point $u \in \big(X(Q_T)\big)^n$ is the desired solution. Moreover,

$$\|u\|_{(X(Q_T))^n} \le c(c_0). \quad (3.12)$$

□

Note that the above argument ensures uniqueness only in a certain ball. In order to establish uniqueness and continuous dependence in the whole space we apply another approach. Then the rest part of Theorem 1.1 succeeds from (3.12) and the theorem below.

Theorem 3.1. *Let the assumptions on the functions a_j and g_j from the hypotheses of Theorem 1.1 be satisfied. Let $u_0, \widetilde{u}_0 \in \big(L_2(I)\big)^n$, $(\mu_0,\dots,\mu_{l-1}), (\widetilde{\mu}_0,\dots\widetilde{\mu}_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0,\dots,\nu_l), (\widetilde{\nu}_0,\dots,\widetilde{\nu}_l) \in \big(\mathcal{B}^l(0,T)\big)^n$, $f, \widetilde{f} \in \big(L_1(0,T;L_2(I))\big)^n$ and let $u, \widetilde{u}$ be two weak solutions to corresponding problems (1.1)–(1.3) in the space $\big(X(Q_T)\big)^n$ with $\|u\|_{(X(Q_T))^n}, \|\widetilde{u}\|_{(X(Q_T))^n} \le K$ for a certain positive K.*

Then there exists a positive constant $c = c(T,K)$ such that

$$\|u - \widetilde{u}\|_{(X(Q_T))^n} \le c\big(\|u_0 - \widetilde{u}_0\|_{(L_2(I))^n} + \|(\mu_0 - \widetilde{\mu}_0,\dots,\mu_{l-1} - \widetilde{\mu}_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} \\ + \|(\nu_0 - \widetilde{\nu}_0,\dots,\nu_l - \widetilde{\nu}_l)\|_{(\mathcal{B}^l(0,T))^n} + \|f - \widetilde{f}\|_{(L_1(0,T;L_2(I)))^n}\big). \quad (3.13)$$

Proof. Let $w \in \big(X(Q_T)\big)^n$ be a solution to the linear problem

$$w_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}w + a_{2l}\partial_x^{2l}w) = 0, \quad (3.14)$$

$$w(0,x) = u_0(x) - \widetilde{u}_0(x), \tag{3.15}$$

$$\partial_x^j w(t,0) = \mu_j(t) - \widetilde{\mu}_j(t),\ j = 0,\dots,l-1, \quad \partial_x^j w(t,R) = \nu_j(t) - \widetilde{\nu}_j(t),\ j=0,\dots,l. \tag{3.16}$$

Lemma 2.1 ensures that such a function exists and according to (2.5)

$$\|w\|_{(X(Q_T))^n} \le c(T)\big(\|u_0 - \widetilde{u}_0\|_{(L_2(I))^n} + \|(\mu_0 - \widetilde{\mu}_0,\dots,\mu_{l-1} - \widetilde{\mu}_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} \\ + \|(\nu_0 - \widetilde{\nu}_0,\dots,\nu_l - \widetilde{\nu}_l)\|_{(\mathcal{B}^{l}(0,T))^n}\big). \tag{3.17}$$

Let $v(t,x) \equiv u(t,x) - \widetilde{u}(t,x) - w(t,x)$, Then $v \in \big(X(Q_T)\big)^n$ is a solution to the initial-boundary problem in Q_T for the system

$$v_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}v + a_{2l}\partial_x^{2l}v) = (f - \widetilde{f}) \\ + \sum_{j=0}^{l-1}(-1)^j\partial_x^j\big[a_{2j+1}(t,x)\partial_x^{j+1}(u-\widetilde{u}) + a_{2j}(t,x)\partial_x^j(u-\widetilde{u})\big] \\ - \sum_{j=0}^{l}(-1)^j\partial_x^j\big[g_j(t,x,u,\dots,\partial_x^{l-1}u) - g_j(t,x,\widetilde{u},\dots,\partial_x^{l-1}\widetilde{u})\big] \tag{3.18}$$

with zero initial and boundary conditions of (1.2), (1.3) type. Similarly to (2.11)–(2.13) $a_{2j+1}(t,x)\partial_x^{j+1}u + a_{2j}(t,x)\partial_x^j u \in \big(L_{2l/(2l-j)}(0,T;L_2(I))\big)^n$, similarly to (3.2), (3.3) $g_j(t,x,u,\dots,\partial_x^{l-1}u) \in \big(L_{2l/(2l-j)}(0,T;L_2(I))\big)^n$. The same properties hold in the case of the function $\widetilde{u}$. Therefore, the hypothesis of Lemma 2.2 is satisfied and for $i = 1,\dots,n$ according to (2.7)

$$\int_I v_i^2(t,x)\rho\,dx + \iint_{Q_t}\big((2l+1)a_{(2l+1)i} - 2a_{(2l)i}\rho\big)\big(\partial_x^l v_i(\tau,x)\big)^2\,dxd\tau \\ \le 2\iint_{Q_t}(f_i - \widetilde{f}_i)v_i\rho\,dxd\tau \\ + 2\sum_{m=1}^{n}\sum_{j=0}^{l-1}\iint_{Q_t}\Big(a_{(2j+1)im}(t,x)\partial_x^{j+1}(v_m + w_m) + a_{(2j)im}(t,x)\partial_x^j(v_m + w_m)\Big) \\ \times(\partial_x^j v_i\rho + j\partial_x^{j-1}v_i)\,dxd\tau \\ - 2\sum_{j=0}^{l}\iint_{Q_t}\Big(g_{ji}(t,x,u,\dots,\partial_x^{l-1}u) - g_{ji}(t,x,\widetilde{u},\dots,\partial_x^{l-1}\widetilde{u})\Big) \\ \times(\partial_x^j v_i\rho + j\partial_x^{j-1}v_i)\,dxd\tau. \tag{3.19}$$

Note that by virtue of (1.8) uniformly in i and x

$$(2l+1)a_{(2l+1)i} - 2a_{(2l)i}\rho(x) \ge \alpha_0 > 0. \tag{3.20}$$

It follows from (2.1) for $p = 2$ that if $j \le l-1$

$$\iint_{Q_t}|\partial_x^{j+1}v_m|\cdot|\partial_x^j v_i|\,dxd\tau \le c\int_0^t\Big[\|\partial_x^l v\|_{(L_2(I))^n}^{(2l-1)/l}\|v\|_{(L_2(I))^n}^{1/l} + \|v\|_{(L_2(I))^n}^2\Big]\,d\tau \\ \le \varepsilon\iint_{Q_t}|\partial_x^l v|^2\,dxd\tau + c(\varepsilon)\iint_{Q_t}|v|^2\rho\,dxd\tau, \tag{3.21}$$

where $\varepsilon > 0$ can be chosen arbitrarily small;

$$\iint_{Q_t} |\partial_x^{j+1} w_m| \cdot |\partial_x^j v_i| \, dx d\tau \le \Bigl(\iint_{Q_t} (\partial_x^j v_i)^2 \, dx d\tau \iint_{Q_t} (\partial_x^{j+1} w_m)^2 \, dx d\tau \Bigr)^{1/2}$$
$$\le \varepsilon \iint_{Q_t} |\partial_x^l v|^2 \, dx d\tau + c(\varepsilon) \iint_{Q_t} |v|^2 \rho \, dx d\tau + c \|w\|^2_{(X(Q_T))^n}. \quad (3.22)$$

Next, similarly to (3.7)

$$|g_{ji}(t,x,u,\ldots,\partial_x^{l-1} u) - g_{ji}(t,x,\widetilde{u},\ldots,\partial_x^{l-1}\widetilde{u})|$$
$$\le c \sum_{k=0}^{l-1} \sum_{m=0}^{l-1} \bigl(|\partial_x^m u|^{b_1(j,k,m)} + |\partial_x^m \widetilde{u}|^{b_1(j,k,m)} + |\partial_x^m u|^{b_2(j,k,m)} + |\partial_x^m \widetilde{u}|^{b_2(j,k,m)} \bigr)$$
$$\times |\partial_x^k (v+w)|. \quad (3.23)$$

Note that, for example, for $j \le l$, $k,m \le l-1$ if $0 \le b \le (4l-2j-2k)/(2m+1)$

$$\int_I |\partial_x^m u|^b |\partial_x^k v| \cdot |\partial_x^j v| \, dx \le \sup_{x\in I} |\partial_x^m u|^b \Bigl(\int_I |\partial_x^k v|^2 \, dx \int_I |\partial_x^j v|^2 \, dx \Bigr)^{1/2}$$
$$\le c \sup_{x\in I} |\partial_x^m u|^b \Bigl[\Bigl(\int_I |\partial_x^l v|^2 \, dx \Bigr)^{(k+j)/(2l)} \Bigl(\int_I |v|^2 \, dx \Bigr)^{(2l-j-k)/(2l)} + \int_I |v|^2 \, dx \Bigr]$$
$$\le \varepsilon \int_I |\partial_x^l v|^2 \, dx + c(\varepsilon) \Bigl[\sup_{x\in I} |\partial_x^m u|^{2lb/(2l-j-k)} + \sup_{x\in I} |\partial_x^m u|^b \Bigr] \int_I |v|^2 \rho \, dx, \quad (3.24)$$

where

$$\int_0^T \sup_{x\in I} |\partial_x^m u|^{2lb/(2l-j-k)} \, dt$$
$$\le \sup_{t\in(0,T)} \Bigl(\int_I |u|^2 \, dx \Bigr)^{(2l-2m-1)b/(4l-2j-2k)} \int_0^T \Bigl(\int_I |\partial_x^l u|^2 \, dx \Bigr)^{(2m+1)b/(4l-2j-2k)} dt$$
$$\le c(T) \|u\|^{2lb/(2l-j-k)}_{(X(Q_T))^n} \, dt; \quad (3.25)$$

also split b into two parts: $b = b' + b''$, where $0 \le b' \le (2l-2j)/(2m+1)$, $0 \le b'' \le (2l-2k)/(2m+1)$, then similarly to (3.24)

$$\int_I |\partial_x^m u|^b |\partial_x^k w| \cdot |\partial_x^j v| \, dx \le \sup_{x\in I} |\partial_x^m u|^{b'+b''} \Bigl(\int_I |\partial_x^j v|^2 \, dx \int_I |\partial_x^k w|^2 \, dx \Bigr)^{1/2}$$
$$\le \varepsilon \int_I |\partial_x^l v|^2 \, dx + c(\varepsilon) \Bigl[\sup_{x\in I} |\partial_x^m u|^{2lb'/(l-j)} + \sup_{x\in I} |\partial_x^m u|^{2b'} \Bigr] \int_I |v|^2 \rho \, dx$$
$$+ c \int_I |\partial_x^l w|^2 \, dx + c \Bigl[\sup_{x\in I} |\partial_x^m u|^{2lb''/(l-k)} + \sup_{x\in I} |\partial_x^m u|^{2b''} \Bigr] \int_I |w|^2 \, dx, \quad (3.26)$$

where similarly to (3.25)

$$\int_0^T \sup_{x\in I} |\partial_x^m u|^{2lb'/(l-j)} \, dt, \int_0^T \sup_{x\in I} |\partial_x^m u|^{2lb''/(l-k)} \, dt \le c(T,K). \quad (3.27)$$

Gathering (3.20)–(3.27) we deduce from inequality (3.19) that

$$\int_I v_i^2(t,x) \rho \, dx + \alpha_0 \iint_{Q_t} (\partial_x^l v_i)^2 \, dx d\tau \le \frac{\alpha_0}{2n} \iint_{Q_t} |\partial_x^l v|^2 \, dx d\tau$$
$$+ \int_0^t \gamma(\tau) \int_I |v|^2 \rho \, dx d\tau + 2 \int_0^t \|f - \widetilde{f}\|_{(L_2(I))^n} \|v_i\|_{L_2(I)} \, d\tau + c(T,K) \|w\|^2_{(X(Q_T))^n}, \quad (3.28)$$

where $\|\gamma\|_{L_1(0,T)} \le c(T,K)$. Summing inequalities (3.28) with respect to i, using estimate (3.17) and applying Gronwall lemma we complete the proof. $\square$

In this section it remains to prove Theorem 1.2.

Proof of Theorem 1.2. Overall, the proof repeats the proof of the existence part of Theorem 1.1. The desired solution is constructed as a fixed point of the map Θ from (3.1). In comparison with (3.3) here we obtain the following estimate: let

$$\sigma = \frac{\min\limits_{j,k,m}(4l - 2j - 2k - (2m+1)b_2(j,k,m))}{4l} \tag{3.29}$$

(note that $\sigma > 0$ because of (1.16)), then

$$\|g_j(t,x,v,\dots,\partial_x^{l-1}v)\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n} \le c(T)T^\sigma \sum_{k=0}^{l-1}\sum_{m=0}^{l-1}\sum_{i=1}^{2} \|v\|^{b_i(j,k,m)+1}_{(X(Q_T))^n}. \tag{3.30}$$

and similarly to (3.6), (3.9)

$$\|\Theta v\|_{(X(Q_T))^n} \le c(T)c_0 + c(T)T^\sigma \left(\|v\|^{b_1+1}_{(X(Q_T))^n} + \|v\|^{b_2+1}_{(X(Q_T))^n}\right). \tag{3.31}$$

$$\begin{aligned}\|\Theta v_1 - \Theta v_2\|_{(X(Q_T))^n} \\ \le c(T)T^\sigma \left(\|v_1\|^{b_1}_{(X(Q_T))^n} + \|v_2\|^{b_1}_{(X(Q_T))^n} + \|v_1\|^{b_2}_{(X(Q_T))^n} + \|v_2\|^{b_2}_{(X(Q_T))^n}\right) \\ \times \|v_1 - v_2\|_{(X(Q_T))^n}.\end{aligned} \tag{3.32}$$

Now for a fixed δ choose $T_0 > 0$ such that

$$4c(T_0)T_0^\sigma\left((2c(T_0)\delta)^{b_1} + (2c(T_0)\delta)^{b_2}\right) \le 1 \tag{3.33}$$

(it is possible since $c(T)$ does not decrease in T) and then for every $T \in (0,T_0]$ choose an arbitrary r such that

$$r \ge 2c(T)\delta, \quad 4c(T)T^\sigma(r^{b_1} + r^{b_2}) \le 1 \tag{3.34}$$

(this set is not empty because of (3.33)). Then the map Θ is a contraction on the ball $\overline{X}_{rn}(Q_T)$.

In order to prove uniqueness in the whole space note that for an arbitrarily large r the value of T_0 can be chosen sufficiently small such that the solution of the considered problem $u \in (X(Q_{T_0})^n)$ is the unique fixed point of the contraction Θ in $\overline{X}_{rn}(Q_{T_0})$. $\square$

4 The inverse problem

We start with the linear case. The following lemma is the crucial part of the study.

Lemma 4.1. *Let the assumptions on the functions a_j from the hypotheses of Theorem 1.3 be satisfied. Let condition (1.6) be valid and for any $i = 1,\dots,n$, satisfying $m_i > 0$, for $k = 1,\dots,m_i$ the functions ω_{ki} satisfy condition (1.12), $\varphi_{ki} \in \widetilde{W}_1^1(0,T)$, $h_{ki} \in C([0,T];L_2(I))$ and for the corresponding functions ψ_{kji} conditions (1.19) be satisfied.*

Then there exists a unique set of M functions

$$\begin{aligned}F = \{F_{ki}(t), i : m_i > 0, k = 1,\dots,m_i\} \\ = \Gamma\{\varphi_{ki}, i : m_i > 0, k = 1,\dots,m_i\} \in (L_1(0,T))^M\end{aligned}$$

such that for $f=(f_1,\dots,f_n)^T\equiv HF$, *where for any* $i=1,\dots,n$ *the function* $f_i(t,x)$ *is presented by formula* (1.4), *where* $h_{0i}\equiv 0$ ($f_i\equiv 0$ *if* $m_i=0$), *the corresponding function*

$$u=S_0 f=(S_0\circ H)F, \tag{4.1}$$

satisfies all conditions (1.5). *Moreover, the linear operator* $\Gamma:\big(\widetilde{W}_1^1(0,T)\big)^M\to\big(L_1(0,T)\big)^M$ *is bounded and its norm does not decrease in* T.

Proof. First of all note that by virtue of (1.18), (1.19)

$$|\Delta_i(t)|\ge\Delta_0>0,\ |\psi_{kji}(t)|\le\psi_0,\quad t\in[0,T]. \tag{4.2}$$

On the space $\big(L_1(0,T)\big)^M$ introduce M linear operators $\Lambda_{ki}=Q(\omega_{ki})\circ S_0\circ H$. Let $\Lambda=\{\Lambda_{ki}\}$. Then since $HF\in\big(L_1(0,T;L_2(I))\big)^n$ by Theorem 2.1 and Lemma 2.1 the operator Λ acts from the space $\big(L_1(0,T)\big)^M$ into the space $\big(\widetilde{W}_1^1(0,T)\big)^M$ and is bounded.

Note that the set of equalities $\varphi_{ki}=\Lambda_{ki}F$, $i:m_i>0, k=1,\dots,m_i$, for $F\in\big(L_1(0,T)\big)^M$ obviously means that the set of functions F is the desired one.

Let for i verifying $m_i>0$

$$\begin{aligned}\widetilde{r}(t;u_i,\omega_{ki})\equiv(-1)^{l+1}\int_I u_i(t,x)\left(a_{(2l+1)i}\omega_{ki}^{(2l+1)}-a_{2l}\omega_{ki}^{(2l)}\right)dx\\ +\sum_{m=1}^n\sum_{j=0}^{l-1}(-1)^{j+1}\int_I u_m(t,x)\left[(a_{(2j+1)im}\omega_{ki}^{(j)})^{(j+1)}-(a_{(2j)im}\omega_{ki}^{(j)})^{(j)}\right]dx,\end{aligned} \tag{4.3}$$

where $u=(u_1,\dots,u_n)^T=(S_0\circ H)F$. Then from (2.17) it follows that for $q(t;u_i,\omega_{ki})=\big(\Lambda_{ki}F\big)(t)$

$$q'(t;u_i,\omega_{ki})=\widetilde{r}(t;u_i,\omega_{ki})+\sum_{j=1}^{m_i}F_{ji}(t)\psi_{kji}(t), \tag{4.4}$$

where the functions ψ_{kji} are given by formula (1.18). Let

$$y_{ki}(t)\equiv q'(t;u_i,\omega_{ki})-\widetilde{r}(t;u_i,\omega_{ki}),\quad k=1,\dots,m_i. \tag{4.5}$$

and $\widetilde{\Delta}_{ki}(t)$ be the determinant of the $m_i\times m_i$-matrix, where in comparison with the matrix $\big(\psi_{kji}(t)\big)$ the k-th column is substituted by the column $\big(y_{1i}(t),\dots,y_{m_i i}(t)\big)^T$. Then (4.4) implies

$$F_{ki}(t)=\frac{\widetilde{\Delta}_{ki}(t)}{\Delta_i(t)},\quad k=1,\dots,m_i. \tag{4.6}$$

Let

$$z_{ki}(t)\equiv\varphi'_{ki}(t)-\widetilde{r}(t;u_i,\omega_{ki}),\quad k=1,\dots,m_i, \tag{4.7}$$

and $\Delta_{ki}(t)$ be the determinant of the $m_i\times m_i$-matrix, where in comparison with $\widetilde{\Delta}_{ki}(t)$ the k-th column $\big(y_{1i}(t),\dots,y_{m_i i}(t)\big)^T$ is substituted by the column $\big(z_{1i}(t),\dots,z_{m_i i}(t)\big)^T$.

Introduce operators $A_{ki}:L_1(0,T)\to L_1(0,T)$ by

$$(A_{ki}F)(t)\equiv\frac{\Delta_{ki}(t)}{\Delta_i(t)} \tag{4.8}$$

and let $AF=\{A_{ki}F\}$, $A:\big(L_1(0,T)\big)^M\to\big(L_1(0,T)\big)^M$.

Note that $\varphi_{ki} = \Lambda_{ki} F$, for all $i : m_i > 0, k = 1, \dots, m_i$ if and only if $AF = F$.

Indeed, if $\varphi_{ki} = \Lambda_{ki} F$, then $\varphi'_{ki}(t) \equiv q'(t; u_i, \omega_{ki})$ for the function $q(t; u_i, \omega_{ki}) \equiv \big(\Lambda_{ki} F\big)(t)$ and equalities (4.5), (4.7) yield $\Delta_{ki}(t) \equiv \widetilde{\Delta}_{ki}(t)$. Hence, $AF = F$.

Vice versa, if $AF = F$, then $\Delta_{ki}(t) \equiv \widetilde{\Delta}_{ki}(t)$ and the condition $\Delta_i(t) \neq 0$ implies $z_{ki}(t) \equiv y_{ki}(t)$ and so $\varphi'_{ki}(t) \equiv q'(t; u_i, \omega_{ki})$. Since $\varphi_{ki}(0) = q(0; u_i, \omega_{ki}) = 0$, we have $q(t; u_i, \omega_{ki}) \equiv \varphi_{ki}(t)$.

Next, we show that the operator A is a contraction under the choice of a special norm in the space $\big(L_1(0,T)\big)^M$.

Let $F_1, F_2 \in \big(L_1(0,T)\big)^M$, $u_m \equiv (S_0 \circ H) F_m$, $m = 1,\ 2$, and let $\Delta^*_{ki}(t)$ be the determinant of the $m_i \times m_i$-matrix, where in comparison with the matrix $\big(\psi_{kji}(t)\big)$ the k-th column is substituted by the column, where on the j-th line stands the element $\widetilde{r}(t; u_{1i}, \omega_{ji}) - \widetilde{r}(t; u_{2i}, \omega_{ji}) = \widetilde{r}(t; u_{1i} - u_{2i}, \omega_{ji})$. Then

$$\big(A_{ki} F_1\big)(t) - \big(A_{ki} F_2\big)(t) = -\frac{\Delta^*_{ki}(t)}{\Delta_i(t)}. \tag{4.9}$$

By (2.8) for $t \in [0,T]$

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{(L_2(I))^n} \le c(T) \sum_{i:m_i>0} \sum_{j=1}^{m_i} \|h_{ji}\|_{C([0,T];L_2(I))} \|F_{1ji} - F_{2ji}\|_{L_1(0,t)}. \tag{4.10}$$

Let $\gamma > 0$, then by virtue of (4.2), (4.3), (4.9) and (4.10)

$$\begin{aligned} &\|e^{-\gamma t}(AF_1 - AF_2)\|_{(L_1(0,T))^M} \\ &\qquad\le \frac{c\big(\{\|\omega_{ji}\|_{H^{2l+1}(I)}\}, \psi_0\big)}{\Delta_0} \int_0^T e^{-\gamma t} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{(L_2(I))^n}\, dt \\ &\qquad\le c\big(T, \big(\{\|\omega_{ji}\|_{H^{2l+1}(I)}\}, \psi_0, \{\|h_{ji}\|_{C([0,T];L_2(I))}\}\big) \\ &\qquad\times \int_0^T e^{-\gamma t} \int_0^t \sum_{i:m_i>0} \sum_{j=1}^{m_i} |F_{1ji}(\tau) - F_{2ji}(\tau)|\, d\tau dt \\ &= c \int_0^T \sum_{i:m_i>0} \sum_{j=1}^{m_i} |F_{1ji}(\tau) - F_{2ji}(\tau)| \int_\tau^T e^{-\gamma t}\, dt d\tau \le \frac{c}{\gamma} \|e^{-\gamma\tau}(F_1 - F_2)\|_{(L_1(0,T))^M}. \end{aligned} \tag{4.11}$$

It remains to choose sufficiently large γ.

As a result, for any set of functions $\varphi_{ki} \in \big(\widetilde{W}_1^1(0,T)\big)^M$ there exists a unique set of functions $F \in \big(L_1(0,T)\big)^M$ satisfying $AF = F$, that is $\varphi_{ki} = \Lambda_{ki} F$. This means that the operator Λ is invertible and so the Banach theorem implies that the inverse operator $\Gamma = \Lambda^{-1} : \big(\widetilde{W}_1^1(0,T)\big)^M \to \big(L_1(0,T)\big)^M$ is continuous. In particular,

$$\|\Gamma\{\varphi_{ki}\}\|_{(L_1(0,T))^M} \le c(T) \|\{\varphi_{ki}\}\|_{(\widetilde{W}_1^1(0,T))^M}. \tag{4.12}$$

For an arbitrary $T_1 > T$ extend the functions φ_{ki} by the constant $\varphi_{ki}(T)$ to the interval (T, T_1). Then the analogue of inequality (4.12) on the interval $(0, T_1)$ for such a function evidently holds with $c(T) \le c(T_1)$. This means that the norm of the operator Γ is non-decreasing in T. $\square$

The next result is the solution of the corresponding inverse problem for the full linear problem.

Theorem 4.1. *Let the function f be given by formula (1.4) and condition (1.6) be satisfied. Let the functions a_i, u_0, $(\mu_0, \dots, \mu_{l-1})$, $(\nu_0, \dots, \nu_l)$, h_0, φ_{ki}, ω_{ki}, h_{ki} satisfy the hypothesis of Theorem 1.3 and the functions G_j satisfy the hypothesis of Theorem 2.1.*

Then there exists a unique set of M functions

$$F = \{F_{ki}(t), i : m_i > 0, k = 1, \dots, m_i\} \in (L_1(0,T))^M$$

such that the corresponding unique weak solution $u \in \big(X(Q_T)\big)^n$ of problem (2.3), (1.2), (1.3) satisfies all conditions (1.5). Moreover, the functions F and u are given by formulas

$$F = \Gamma\Big\{\varphi_{ki} - Q(\omega_{ki})\big(\widetilde{S}W + S_0 h_0 + \sum_{j=0}^{l} \widetilde{S}_j G_j\big)_i\Big\}, \tag{4.13}$$

$$u = \widetilde{S}W + S_0 h_0 + \sum_{j=0}^{l} S_j G_j + (S_0 \circ H)F. \tag{4.14}$$

Proof. Set

$$v \equiv S(u_0, (\mu_0, \dots, \mu_{l-1}), (\nu_0, \dots, \nu_l), h_0, (G_0, \dots, G_l)) = \widetilde{S}W + S_0 h_0 + \sum_{j=0}^{l} \widetilde{S}_j G_j.$$

Lemma 2.1 implies $Q(\omega_{ki})v_i \in W_1^1(0,T)$. Moreover, by virtue of (1.17) $Q(\omega_{ki})v_i\big|_{t=0} = \varphi_{ki}(0)$. Set

$$\widetilde{\varphi}_{ki} \equiv \varphi_{ki} - Q(\omega_{ki})v_i,$$

then $\widetilde{\varphi}_{ki} \in \widetilde{W_1^1}(0,T)$. In turn, Lemma 4.1 implies that the functions $F \equiv \Gamma\{\widetilde{\varphi}_{ki}\}$ and $u \equiv v + (S_0 \circ H)F$ provide the desired result. Uniqueness also follows from Lemma 4.1. □

Now we pass to the nonlinear equation.

Proof of Theorem 1.3. On the space $\big(X(Q_T)\big)^n$ consider a map Θ

$$u = \Theta v \equiv \widetilde{S}W + S_0 h_0 - \sum_{j=0}^{l} \widetilde{S}_j g_j(t,x,v,\dots,\partial_x^{l-1} v) + (S_0 \circ H)F, \tag{4.15}$$

$$F \equiv \Gamma\Big\{\varphi_{ki} - Q(\omega_{ki})\big(\widetilde{S}W + S_0 h_0 - \sum_{j=0}^{l} \widetilde{S}_j g_j(t,x,v,\dots,\partial_x^{l-1} v)\big)_i\Big\}. \tag{4.16}$$

Then estimate (3.5) and Theorem 4.1 applied to $G_j(t,x) \equiv g_j(t,x,v,\dots,\partial_x^{l-1} v)$ ensure that the map Θ exists.

Apply Lemmas 2.3 and 4.1, then the function F from (4.16) is estimated as follows:

$$\begin{aligned} \|F\|_{(L_1(0,T))^M} \le c(T)\Big[&\|u_0\|_{(L_2(I))^n} + \|(\mu_0,\dots,\mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} + \|(\nu_0,\dots,\nu_l)\|_{(\mathcal{B}^l(0,T))^n} \\ &+ \|h_0\|_{(L_1(0,T;L_2(I)))^n} + \|\{\varphi'_{ki}\}\|_{(L_1(0,T))^M} + \|v\|^{b_1+1}_{(X(Q_T))^n} + \|v\|^{b_2+1}_{(X(Q_T))^n}\Big]; \end{aligned} \tag{4.17}$$

therefore, since also

$$\|HF\|_{(L_1(0,T;L_2(I)))^n} \le \max_{i:m_i>0,k=1,\dots,m_i} \big(\|h_{ki}\|_{C([0,T];L_2(I))}\big)\|F\|_{(L_1(0,T))^M},$$

Theorem 2.1 provides for the map Θ estimate (3.6).

Next, for any functions $v_1, v_2 \in \big(X(Q_T)\big)^n$ since

$$\Theta v_1 - \Theta v_2 = -\sum_{j=0}^{l} \widetilde{S}_j \left[g_j(t,x,v_1,\dots,\partial_x^{l-1}v_1) - g_j(t,x,v_2,\dots,\partial_x^{l-1}v_2)\right]$$
$$+ (S_0 \circ H \circ \Gamma)\Big\{Q(\omega_{ki})\Big(\sum_{j=0}^{l} \widetilde{S}_j \left[g_j(t,x,v_1,\dots,\partial_x^{l-1}v_1) - g_j(t,x,v_2,\dots,\partial_x^{l-1}v_2)\right]\Big)_i\Big\}, \quad (4.18)$$

using (3.8) we derive estimate (3.9).

Now choose $r > 0$ and $\delta > 0$ as in (3.10), (3.11). Then it follows from (3.6) and (3.9) that on the ball $\overline{X}_{rn}(Q_T)$ the map Θ is a contraction. Its unique fixed point $u \in \big(X(Q_T)\big)^n$ is the desired solution. Moreover, Theorem 4.1 implies that the function F in (4.16) (for $v \equiv u$) is determined in a unique way.

Continuous dependence is obtained similarly to (3.6), (3.9). □

Proof of Theorem 1.4. In general, the proof repeats the previous argument. The desired solution is constructed as a fixed point of the map Θ from (4.15), (4.16). In comparison with (3.6), (3.9) here (also with the use of (4.18)) we obtain estimates (3.31) and (3.32), where σ is defined in (3.29).

The end of the proof is the same as in Theorem 1.2 (with the corresponding supplements as in Theorem 1.3). □

Acknowledgments

This paper was written under the financial support of the Russian Science Foundation, grant 23-21-00101.

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Oleg Sergeevich Balashov, Andrei Vadimovich Faminskii
S.M. Nikol'skii Mathematical Institute
RUDN University
6 Miklukho-Maklaya St
117198 Moscow, Russian Federation
E-mails: balashovos@s1238.ru, faminskiy-av@pfur.ru

Received: 20.06.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 4 (2024), 54 – 65

NEW WEIGHTED HARDY-TYPE INEQUALITIES FOR MONOTONE FUNCTIONS

A.A. Kalybay, A.M. Temirkhanova

Communicated by Ya.T. Sultanaev

Key words: integral operator, Hardy-type inequality, weight, non-increasing function, non-decreasing function.

AMS Mathematics Subject Classification: 47G10, 47B38.

Abstract. The famous Hardy inequality was formulated in 1920 and finally proved in 1925. Since then, this inequality has been greatly developed. The first development was related to the consideration of more general weights. The next step was to use more general operators with different kernels instead of the Hardy operator. At present, there are many works devoted to Hardy-type inequalities with iterated operators. Motivated by important applications, all these generalizations of the Hardy inequality are studied not only on the cone of non-negative functions but also on the cone of monotone non-negative functions. In this paper, new Hardy-type inequalities are proved for operators with kernels that satisfy less restrictive conditions than those considered earlier. The presented inequalities have already been characterized for non-negative functions. In this paper, we continue this study but for monotone non-negative functions.

DOI: https://doi.org/10.32523/2077-9879-2024-15-4-54-65

1 Introduction

Let $I = (0, \infty)$, $1 < p, q < \infty$ and $p' = \frac{p}{p-1}$. Suppose that v, u and $v^{1-p'}$ are positive functions locally integrable on I.

We consider the following Hardy-type inequality

$$\left(\int_0^\infty u(x) \left| \int_0^x K(x,t) f(t) dt \right|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty v(x) |f(x)|^p dx \right)^{\frac{1}{p}}, \tag{1.1}$$

for all functions $f \in L_{p,v}(I)$, where $C > 0$ is independent of f and $L_{p,v}(I)$ is the weighted Lebesgue space of all functions f, Lebesgue measurable on I, such that $\|f\|_{p,v} = \left(\int_0^\infty v(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty$. Here

$$Kf(x) = \int_0^x K(x,t) f(t) dt, \; x > 0, \tag{1.2}$$

is an integral operator with a non-negative kernel $K(x,t)$.

Inequality (1.1) has been completely characterized for the kernel $K(x,t) \equiv 1$ (for more details see [8, 9]) and the kernel $K(x,t) \equiv (x-t)^{\alpha-1}$, $\alpha > 1$ (see [17, 18, 19, 20] and for more details see [6]).

In works [3] and [7, 10, 11, 12], inequality (1.1) was studied for kernels $K(x,t)$ satisfying the Oinarov condition stating that there exists a number $d \geq 1$ such that

$$d^{-1}\left(K(x,s)+K(s,t)\right) \leq K(x,t) \leq d\left(K(x,s)+K(s,t)\right) \tag{1.3}$$

for all $x,s,t : x \geq s \geq t > 0$. A further development of this problem was the introduction of the classes $\mathcal{O}_n^{\pm}$, $n \geq 0$, which are less restrictive for kernels $K(x,t)$ than the Oinarov condition. We will refer to $\mathcal{O}_n^{\pm}$, $n \geq 0$, as *the Oinarov classes (the definitions of these classes are given in Section 2)*. In paper [13], inequality (1.1) was studied in the case $1 < p \leq q < \infty$. The case $1 < q < p < \infty$ was considered in the paper [1], but for kernels belonging to the Oinarov classes $\mathcal{O}_1^{\pm}$. In the recent paper [14] the case $1 < q < p < \infty$ is also discussed, but now kernels are from $\mathcal{O}_2^{\pm}$. For operators with kernels from the classes $\mathcal{O}_1^{\pm}$ in paper [14] an alternative criterion for the validity of (1.1) is presented.

If, in addition, f is a monotone function, characterizations of the Hardy-type inequalities help to find boundedness of certain operators in Lorentz spaces. Moreover, the Hardy-type inequalities restricted to monotone functions are used for the weighted Marcinkiewicz interpolation results. For more applications, we refer to monograph [9, Chapter 8] (see also [16]).

Motivated by the applications, in this paper, we find necessary and sufficient conditions for the validity of inequality (1.1) for operator (1.2) with kernels from the Oinarov classes $\mathcal{O}_2^{\pm}$ on the cone of monotone functions in the case $1 < q < p < \infty$. The case $1 < p \leq q < \infty$ was discussed in paper [2] for kernels from $\mathcal{O}_n^{-}$, $n \geq 0$. We note that the case when kernels belong to the classes $\mathcal{O}_n^{+}$, $n \geq 0$, has been left in [2] as an open question. The presented paper covers the class $\mathcal{O}_2^{+}$. As soon as inequality (1.1) is established for kernels from the general classes $\mathcal{O}_n^{\pm}$, $n \geq 0$, on the cone of non-negative functions in the case $1 < q < p < \infty$, it can be established on the cone of monotone functions in the same way as here. Moreover, in paper [2], the authors also considered the conjugate operator $K^*f(x) = \int\limits_x^\infty K(t,x)f(t)dt$, $x > 0$, but kernels were from $\mathcal{O}_n^{+}$, $n \geq 0$. Since the conjugate operator K^*f needs a different approach than operator (1.2), so this is one more topic for a separate paper.

This paper is organized as follows. Section 2 contains all the auxiliary statements required to prove the main results. In Section 3, the validity of inequality (1.1) is established on the cone of non-increasing functions for operator (1.2) with kernels from the Oinarov class $\mathcal{O}_2^{+}$. In Section 4, we present a similar result but for the operator (1.2) with kernels from the class $\mathcal{O}_2^{-}$. Section 5 is devoted to the case $1 < p \leq q < \infty$ when kernels belong to the class $\mathcal{O}_2^{+}$, which has not been considered in [2].

2 Auxiliary statements

Throughout the paper, the symbol $A \ll B$ means that $A \leq cB$ with some constant $c > 0$. The symbol $A \approx B$ stands for $A \ll B \ll A$. Moreover, $f \uparrow$ and $f \downarrow$ mean non-decreasing or non-increasing non-negative functions, respectively.

Let us give the definitions of the classes $\mathcal{O}_1^{\pm}$ and $\mathcal{O}_2^{\pm}$. Let $\Omega = \{(x,t) \in I \times I : x \geq t\}$.

Definition 1. A measurable function $K_1(\cdot,\cdot) \geq 0$ defined on the set Ω belongs to the class $\mathcal{O}_1^{+}$, if it does not decrease in the first argument and there exists a non-negative function $K_{1,0}(\cdot,\cdot)$ measurable on Ω and a number $d_1 \geq 1$ such that

$$d_1^{-1}\left(K_{1,0}(x,s)+K_1(s,t)\right) \leq K_1(x,t) \leq d_1\left(K_{1,0}(x,s)+K_1(s,t)\right) \tag{2.1}$$

for all $x,s,t : x \geq s \geq t > 0$.

Definition 2. A measurable function $K_1(\cdot,\cdot) \geq 0$ defined on the set Ω belongs to the class $\mathcal{O}_1^-$, if it does not increase in the second argument and there exists a non-negative function $K_{0,1}(\cdot,\cdot)$ measurable on Ω and a number $\bar{d}_1 \geq 1$ such that

$$\bar{d}_1^{-1}\left(K_1(x,s) + K_{0,1}(s,t)\right) \leq K_1(x,t) \leq \bar{d}_1\left(K_1(x,s) + K_{0,1}(s,t)\right)$$

for all $x, s, t : x \geq s \geq t > 0$.

Definition 3. A measurable function $K_2(\cdot,\cdot) \geq 0$ defined on the set Ω belongs to the class $\mathcal{O}_2^+$, if it does not decrease in the first argument and there exist non-negative functions $K_{2,0}(\cdot,\cdot)$, $K_{2,1}(\cdot,\cdot)$ and $K_1(\cdot,\cdot)$ measurable on Ω and a number $d_2 \geq 1$ such that $K_1(\cdot,\cdot) \in \mathcal{O}_1^+$ and

$$\begin{aligned} d_2^{-1}\left(K_{2,0}(x,s) + K_{2,1}(x,s)K_1(s,t) + K_2(s,t)\right) &\leq K_2(x,t) \\ &\leq d_2\left(K_{2,0}(x,s) + K_{2,1}(x,s)K_1(s,t) + K_2(s,t)\right) \end{aligned} \quad (2.2)$$

for all $x, s, t : x \geq s \geq t > 0$.

Definition 4. A measurable function $K_2(\cdot,\cdot) \geq 0$ defined on the set Ω belongs to the class $\mathcal{O}_2^-$, if it does not increase in the second argument and there exist non-negative functions $K_{0,2}(\cdot,\cdot)$, $K_{1,2}(\cdot,\cdot)$ and $K_1(\cdot,\cdot)$ measurable on Ω and a number $\bar{d}_2 \geq 1$ such that $K_1(\cdot,\cdot) \in \mathcal{O}_1^-$ and

$$\begin{aligned} \bar{d}_2^{-1}\left(K_2(x,s) + K_1(x,s)K_{1,2}(s,t) + K_{0,2}(s,t)\right) &\leq K_2(x,t) \\ &\leq \bar{d}_2\left(K_2(x,s) + K_1(x,s)K_{1,2}(s,t) + K_{0,2}(s,t)\right) \end{aligned} \quad (2.3)$$

for all $x, s, t : x \geq s \geq t > 0$.

Note that since the classes $\mathcal{O}_2^{\pm}$ are wider than the classes of operators satisfying condition (1.3), many recent publications have been devoted to them (see, e.g., [5, 14]). Examples of kernels that belong to the classes $\mathcal{O}_1^{\pm}$ and $\mathcal{O}_2^{\pm}$ can be found in [14].

To prove our main results we use the following theorems established in [14].

Theorem A. *Let $1 < q < p < \infty$ and $K(\cdot,\cdot) \equiv K_2(\cdot,\cdot) \in \mathcal{O}_2^+$. Then inequality (1.1) holds if and only if $B_2 = \max\{B_{2,0}, B_{2,1}, B_{2,2}\} < \infty$. Moreover, $C \approx B_2$, where C is best constant in inequality (1.1) and*

$$B_{2,0} = \left(\int_0^\infty \left(\int_z^\infty K_{2,0}^q(x,z)u(x)dx\right)^{\frac{p}{p-q}} \left(\int_0^z v^{1-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}} v^{1-p'}(z)dz\right)^{\frac{p-q}{pq}},$$

$$B_{2,1} = \left(\int_0^\infty \left(\int_z^\infty K_{2,1}^q(x,z)u(x)dx\right)^{\frac{p}{p-q}} \left(\int_0^z K_1^{p'}(z,s)v^{1-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}} \times d\left(\int_0^z K_1^{p'}(z,t)v^{1-p'}(t)dt\right)\right)^{\frac{p-q}{pq}},$$

$$B_{2,2} = \left(\int_0^\infty \left(\int_z^\infty u(t)dt\right)^{\frac{p}{p-q}} \left(\int_0^z K_2^{p'}(z,s)v^{1-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}} d\left(\int_0^z K_2^{p'}(z,s)v^{1-p'}(s)ds\right)\right)^{\frac{p-q}{pq}}.$$

Theorem B. *Let $1 < q < p < \infty$ and $K(\cdot,\cdot) \equiv K_1(\cdot,\cdot) \in \mathcal{O}_1^-$. Then inequality (1.1) holds if and only if $\mathcal{B}_1 = \max\{\mathcal{B}_{0,1}, \mathcal{B}_{1,1}\} < \infty$. Moreover, $C \approx \mathcal{B}_1$, where where C is the best constant in inequality (1.1) and*

$$\mathcal{B}_{0,1} = \left(\int_0^\infty \left(\int_0^t K_{0,1}^{p'}(t,x) v^{1-p'}(x) dx \right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty u(s) ds \right)^{\frac{q}{p-q}} u(t) dt \right)^{\frac{p-q}{pq}},$$

$$\mathcal{B}_{1,1} \quad = \quad \left(\int_0^\infty \left(\int_0^t v^{1-p'}(x) dx \right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty K_1^q(s,t) u(s) ds \right)^{\frac{q}{p-q}} d\left(- \int_t^\infty K_1^q(s,t) u(s) ds \right) \right)^{\frac{p-q}{pq}}.$$

Theorem C. *Let $1 < q < p < \infty$ and $K(\cdot,\cdot) \equiv K_2(\cdot,\cdot) \in \mathcal{O}_2^-$. Then inequality (1.1) holds if and only if $\mathcal{B}_2 = \max\{\mathcal{B}_{0,2,}, \mathcal{B}_{1,2}, \mathcal{B}_{2,2}\} < \infty$. Moreover, $C \approx \mathcal{B}_2$, where C is the best constant in inequality (1.1) and*

$$\mathcal{B}_{0,2} = \left(\int_0^\infty \left(\int_0^z K_{0,2}^{p'}(z,s) v^{1-p'}(s) ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_z^\infty u(s) ds \right)^{\frac{q}{p-q}} u(z) dz \right)^{\frac{p-q}{pq}},$$

$$\mathcal{B}_{1,2} = \left(\int_0^\infty \left(\int_0^z K_{1,2}^{p'}(z,s) v^{1-p'}(s) ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_z^\infty K_1^q(x,z) u(x) dx \right)^{\frac{q}{p-q}} \right.$$
$$\left. \times\, d\left(- \int_z^\infty K_1^q(x,z) u(x) dx \right) \right)^{\frac{p-q}{pq}},$$

$$\mathcal{B}_{2,2} = \left(\int_0^\infty \left(\int_0^z v^{1-p'}(t) dt \right)^{\frac{p(q-1)}{p-q}} \left(\int_z^\infty K_2^q(x,z) u(x) dx \right)^{\frac{p}{p-q}} v^{1-p'}(z) dz \right)^{\frac{p-q}{pq}}.$$

In paper [15], there is a formula that gives the equivalence between inequality (1.1) for all non-increasing non-negative functions and a certain inequality, but for arbitrary non-negative functions. This equivalence is now called the Sawyer duality principle and has the form:

$$\sup_{0 \le f \downarrow} \frac{\int_0^\infty g(x) f(x) dx}{\left(\int_0^\infty v(x) f^p(x) dx \right)^{\frac{1}{p}}} \approx \left(\int_0^\infty v(x) \left(\frac{\int_0^x g(t) dt}{\int_0^x v(t) dt} \right)^{p'} dx \right)^{\frac{1}{p'}} + \frac{\int_0^\infty g(x) dx}{\left(\int_0^\infty v(x) dx \right)^{\frac{1}{p}}}. \tag{2.4}$$

Equivalence (2.4) can be transformed into the following statement (see, e.g., [4]). The inequality

$$\left(\int_0^\infty u(x) (Kf(x))^q dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty v(x) f^p(x) dx \right)^{\frac{1}{p}} \tag{2.5}$$

holds for a non-increasing function $f \geq 0$ if and only if the following two inequalities

$$\left(\int_0^\infty u\left(K\left(\int_x^\infty h\right)\right)^q\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v^{1-p}V^p h^p\right)^{\frac{1}{p}}, \tag{2.6}$$

$$\left(\int_0^\infty u(K\mathbf{1})^q\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v\right)^{\frac{1}{p}} \tag{2.7}$$

hold for any function $h \geq 0$ and $V(\infty) < \infty$, where $V(t) := \int_0^t v(x)dx$ and $\mathbf{1}$ is a function identically equal to 1 on I. From (2.4) it is obvious that in the case $V(\infty) = \infty$ for inequality (2.5) to hold we need only the validity of inequality (2.6).

3 Main result for the class $\mathcal{O}_2^+$

Assume that

$$M_1^\pm = \left(\int_0^\infty u(x)\left(\int_0^x K(x,t)dt\right)^q dx\right)^{\frac{1}{q}}\left(\int_0^\infty v(x)dx\right)^{-\frac{1}{p}},$$

$$M_2^\pm = \left(\int_0^\infty\left(\int_0^t\left(\int_0^x K(x,z)dz\right)^q u(x)dx\right)^{\frac{p}{p-q}}\left(\int_t^\infty V^{-p'}(s)v(s)ds\right)^{\frac{p(q-1)}{p-q}} V^{-p'}(t)v(t)dt\right)^{\frac{p-q}{pq}},$$

$$M_3^+ = \left(\int_0^\infty\left(\int_t^\infty K_{2,0}^q(x,z)u(x)dx\right)^{\frac{p}{p-q}}\left(\int_0^t s^{p'}V^{-p'}(s)v(s)ds\right)^{\frac{p(q-1)}{p-q}} t^{p'}V^{-p'}(t)v(t)dt\right)^{\frac{p-q}{pq}},$$

$$M_4^+ = \left(\int_0^\infty\left(\int_t^\infty K_{2,1}^q(x,z)u(x)dx\right)^{\frac{p}{p-q}}\left(\int_0^t\left(\int_0^s K_1(t,z)dz\right)^{p'} V^{-p'}(s)v(s)ds\right)^{\frac{p(q-1)}{p-q}}\right.$$
$$\left.\times\, d\left(\int_0^t\left(\int_0^s K_1(t,z)dz\right)^{p'} V^{-p'}(s)v(s)ds\right)\right)^{\frac{p-q}{pq}},$$

$$M_5^+ = \left(\int_0^\infty\left(\int_t^\infty u(x)dx\right)^{\frac{p}{p-q}}\left(\int_0^t\left(\int_0^s K(t,z)dz\right)^{p'} V^{-p'}(s)v(s)ds\right)^{\frac{p(q-1)}{p-q}}\right.$$
$$\left.\times\, d\left(\int_0^t\left(\int_0^s K(t,z)dz\right)^{p'} V^{-p'}(s)v(s)ds\right)\right)^{\frac{p-q}{pq}}.$$

$$M^+ = \max\{M_1^\pm, M_2^\pm, M_3^+, M_4^+, M_5^+\} \text{ and } \widetilde{M}^+ = \max\{M_2^\pm, M_3^+, M_4^+, M_5^+\}.$$

Our main result of this section reads as follows.

Theorem 3.1. *Let $1 < q < p < \infty$ and $K(\cdot,\cdot) \in \mathcal{O}_2^+$. Then inequality (1.1) holds for any non-increasing $f \geq 0$ if and only if $M^+ < \infty$ for $V(\infty) < \infty$ and $\widetilde{M}^+ < \infty$ for $V(\infty) = \infty$.*

Proof. Since $K\mathbf{1} = \int\limits_0^x K(x,t)dt$, inequality (2.7) has the form

$$\left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)dt\right)^q dx\right)^{\frac{1}{q}} \leq C\left(\int\limits_0^\infty v(x)dx\right)^{\frac{1}{p}},$$

which is equivalent to the condition $M_1^\pm < \infty$. As we mentioned above, in the case of $V(\infty) = \infty$, inequality (2.7) is not required, so the condition $M_1^\pm < \infty$ is also not required.

Let us turn to inequality (2.6) for non-negative functions, the validity of which is necessary and sufficient for the validity of (2.5) for non-increasing functions for the both cases $V(\infty) < \infty$ and $V(\infty) = \infty$. Inequality (2.6) can be rewritten as follows:

$$\left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)\left(\int\limits_t^\infty h(s)ds\right)dt\right)^q dx\right)^{\frac{1}{q}} \leq C\left(\int\limits_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}. \tag{3.1}$$

Our aim is to characterize inequality (3.1) for any non-negative function $h \geq 0$. Let us transform the left-hand side S of (3.1). We split the inner integral in (3.1) and get

$$S \approx \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)\left(\int\limits_t^x h(s)ds\right)dt\right)^q dx\right)^{\frac{1}{q}} + \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)\left(\int\limits_x^\infty h(s)ds\right)dt\right)^q dx\right)^{\frac{1}{q}}. \tag{3.2}$$

The change of the order of integration in the first term of (3.2) gives

$$S \approx \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x \left(\int\limits_0^s K(x,t)dt\right)h(s)ds\right)^q dx\right)^{\frac{1}{q}} + \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)dt\right)^q \left(\int\limits_x^\infty h(s)ds\right)^q dx\right)^{\frac{1}{q}}.$$

Therefore, the validity of inequality (3.1) is equivalent to the validity of the following two inequalities:

$$\left(\int\limits_0^\infty u(x)\left(\int\limits_0^x \left(\int\limits_0^s K(x,t)dt\right)h(s)ds\right)^q dx\right)^{\frac{1}{q}} \leq C_1\left(\int\limits_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}, \tag{3.3}$$

$$\left(\int_0^\infty u(x)\left(\int_0^x K(x,t)dt\right)^q\left(\int_x^\infty h(s)ds\right)^q dx\right)^{\frac{1}{q}} \le C_2\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}. \tag{3.4}$$

The inequality (3.4) is the standard weighted Hardy inequality, which holds if and only if $M_2^{\pm}<\infty$ (see, e.g., [9]).

Inequality (3.3) can be rewritten in the form:

$$\left(\int_0^\infty u(x)\left(\int_0^x \overline{K}(x,s)s\,h(s)ds\right)^q dx\right)^{\frac{1}{q}} \le C_1\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}.$$

where $\overline{K}(x,s)=\frac{1}{s}\int\limits_0^s K(x,t)dt$ with $K(x,t)$ from $\mathcal{O}_2^+$. Using relation (2.2), for $x\ge z\ge t$ we get

$$\overline{K}(x,s)\approx\frac{1}{s}\int_0^s(K_{2,0}(x,z)+K_{2,1}(x,z)K_1(z,t)+K(z,t))dt$$

$$=\frac{1}{s}K_{2,0}(x,z)\,s+K_{2,1}(x,z)\frac{1}{s}\int_0^s K_1(z,t)dt+\frac{1}{s}\int_0^s K(z,t)dt$$

$$=K_{2,0}(x,z)+K_{2,1}(x,z)\overline{K}_1(z,s)+\overline{K}(z,s), \tag{3.5}$$

where $\overline{K}_1(z,s)=\frac{1}{s}\int\limits_0^s K_1(z,t)dt$. If we prove that $\overline{K}_1(z,s)\in\mathcal{O}_1^+$, we prove that $\overline{K}(x,s)\in\mathcal{O}_2^+$. By the definition $K_1(z,t)\in\mathcal{O}_1^+$, therefore from (2.1) for $z\ge\tau\ge t$ we have that $K_1(z,t)\approx K_{1,0}(z,\tau)+K_1(\tau,t)$. Hence,

$$\overline{K}_1(z,s)\approx\frac{1}{s}\int_0^s(K_{1,0}(z,\tau)+K_1(\tau,t))dt$$

$$=\frac{1}{s}K_{1,0}(z,\tau)\,s+\frac{1}{s}\int_0^s K_1(\tau,t)dt=K_{1,0}(z,\tau)+\overline{K}_1(\tau,s).$$

Then $\overline{K}_1(z,s)$ belongs to the class $\mathcal{O}_1^+$. Consequently, from (3.5) we obtain that $\overline{K}(x,s)$ belongs to the class $\mathcal{O}_2^+$. Thus, replacing $s\,h(s)$ by $g_1(s)$, by Theorem A inequality (3.3) holds for $g_1(s)$ if and only if $M_3^+<\infty$, $M_4^+<\infty$ and $M_5^+<\infty$. □

4 Main result for the class $\mathcal{O}_2^-$

Assume that

$$M_3^-=\left(\int_0^\infty\left(\int_0^t K_{0,2}^{p'}(t,s)\,s^{p'}V^{-p'}(s)v(s)ds\right)^{\frac{q(p-1)}{p-q}}\left(\int_t^\infty u(x)dx\right)^{\frac{q}{p-q}}u(t)dt\right)^{\frac{p-q}{pq}}<\infty,$$

$$M_4^- = \left(\int_0^\infty \left(\int_0^t K_{1,2}^{p'}(t,s)\, s^{p'} V^{-p'}(s) v(s) ds\right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty K_1^q(x,t) u(x) dx\right)^{\frac{q}{p-q}} \times d\left(-\int_t^\infty K_1^q(x,t)u(x)dx\right)\right)^{\frac{p-q}{pq}} < \infty,$$

$$M_5^- = \left(\int_0^\infty \left(\int_0^t s^{p'} V^{-p'}(s) v(s) ds\right)^{\frac{p(q-1)}{p-q}} \left(\int_t^\infty K^q(x,t) u(x) dx\right)^{\frac{p}{p-q}} t^{p'} V^{-p'}(t) v(t) dt\right)^{\frac{p-q}{pq}},$$

$$M_6^- = \left(\int_0^\infty \left(\int_0^t K_{0,1}^{p'}(t,s) V^{-p'}(s) v(s) \left(\int_0^s K_{1,2}(s,z)dz\right)^{p'} ds\right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty u(x)dx\right)^{\frac{q}{p-q}} u(t)dt\right)^{\frac{p-q}{pq}},$$

$$M_7^- = \left(\int_0^\infty \left(\int_0^t V^{-p'}(s) v(s) \left(\int_0^s K_{1,2}(s,z)dz\right)^{p'} ds\right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty K_1^q(x,t)u(x)dx\right)^{\frac{q}{p-q}} \times d\left(-\int_t^\infty K_1^q(x,t)u(x)dx\right)\right)^{\frac{p-q}{pq}} < \infty,$$

$$M_8^- = \left(\int_0^\infty \left(\int_0^t V^{-p'}(s) v(s) \left(\int_0^s K_{0,2}(s,z)dz\right)^{p'} ds\right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty u(x)dx\right)^{\frac{q}{p-q}} u(t)dt\right)^{\frac{p-q}{pq}} < \infty,$$

$$M^- = \max\{M_1^{\pm}, M_2^{\pm}, M_3^-, M_4^-, M_5^-, M_6^-, M_7^-, M_8^-\},$$
$$\widetilde{M}^- = \max\{M_2^{\pm}, M_3^-, M_4^-, M_5^-, M_6^-, M_7^-, M_8^-\}.$$

Our main result of this section reads as follows.

Theorem 4.1. *Let $1 < q < p < \infty$ and $K(\cdot,\cdot) \in \mathcal{O}_2^-$. Then inequality (1.1) holds for any non-increasing $f \geq 0$ if and only if $M^- < \infty$ for $V(\infty) < \infty$ and $\widetilde{M}^- < \infty$ for $V(\infty) = \infty$.*

Proof. The beginning of the proof of Theorem 4.1 is the same as the beginning of the proof of Theorem 3.1, i.e., for the validity of (1.1) we need the condition $M_1^{\pm} < \infty$ for $V(\infty) < \infty$ and the condition $M_2^{\pm} < \infty$ for both $V(\infty) = \infty$ and $V(\infty) < \infty$.

Let us turn to inequality (3.3). Using relation (2.3) in inequality (3.3), it is equivalent to the inequality

$$\left(\int_0^\infty u(x) \left(\int_0^x \left(\int_0^s (K(x,s) + K_1(x,s)K_{1,2}(s,t) + K_{0,2}(s,t))dt\right) h(s)ds\right)^q dx\right)^{\frac{1}{q}} \leq C_1 \left(\int_0^\infty v^{1-p}(x) V^p(x) h^p(x) dx\right)^{\frac{1}{p}}.$$

Thus, the validity of inequality (3.3) is equivalent to the validity of the following three inequalities:

$$\left(\int_0^\infty u(x)\left(\int_0^x K(x,s)\,s\,h(s)ds\right)^q dx\right)^{\frac{1}{q}} \le C_{11}\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}, \tag{4.1}$$

$$\left(\int_0^\infty u(x)\left(\int_0^x K_1(x,s)\left(\int_0^s K_{1,2}(s,t)dt\right)h(s)ds\right)^q dx\right)^{\frac{1}{q}} \le C_{12}\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}, \tag{4.2}$$

$$\left(\int_0^\infty u(x)\left(\int_0^x\left(\int_0^s K_{0,2}(s,t)dt\right)h(s)ds\right)^q dx\right)^{\frac{1}{q}} \le C_{13}\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}. \tag{4.3}$$

If we replace $s\,h(s)$ by $g_1(s)$, then by Theorem C inequality (4.1) holds for $g_1(s)$ if and only if $M_3^- < \infty$, $M_4^- < \infty$ and $M_5^- < \infty$.

If we replace $\left(\int_0^s K_{1,2}(s,t)dt\right)h(s)$ by $g_2(s)$, then by Theorem B inequality (4.2) holds for $g_2(s)$ if and only if $M_6^- < \infty$ and $M_7^- < \infty$.

If we replace $\left(\int_0^s K_{0,2}(s,t)dt\right)h(s)$ by $g_3(s)$, then (4.3) is the standard weighted Hardy inequality for $g_3(s)$, which holds if and only if $M_8^- < \infty$ (see, e.g., [9]).

□

Remark 1. Let us note that the proofs of Theorems 3.1 and 4.1 need different approaches because the kernel $\overline{K}(x,s) = \frac{1}{s}\int_0^s K(x,t)dt$ belongs to the class $\mathcal{O}_2^+$ if the kernel $K(x,t)$ belongs to the class $\mathcal{O}_2^+$ but it does not belong to the class $\mathcal{O}_2^-$ if the kernel $K(x,t)$ belongs to the class $\mathcal{O}_2^-$.

5 Supplementary results

In the paper [13], it was proved that if $1 < p \le q < \infty$ and $K(\cdot,\cdot) \in \mathcal{O}_2^+$, then inequality (1.1) holds for any $f \ge 0$ if and only if one of the following conditions

$$A_1^+ = \sup_{0<z<\infty}\left(\int_z^\infty u(x)\left(\int_0^z K^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}} < \infty,$$

$$A_2^+ = \sup_{0<z<\infty}\left(\int_0^z v^{1-p'}(s)\left(\int_z^\infty K^q(x,s)u(x)dx\right)^{\frac{p'}{q}} ds\right)^{\frac{1}{p'}} < \infty$$

holds, in addition, $C \approx A_1^+ \approx A_2^+$, where C is the best constant in inequality (1.1).

Using the above result and following the same steps as in the proof of Theorem 3.1, we can present the statement on the cone of non-increasing functions for the case $1 < p \le q < \infty$ when kernels $K(\cdot,\cdot)$ belong to the class $\mathcal{O}_2^+$, which was not considered in [2].

Theorem 5.1. *Let $1 < p \leq q < \infty$ and $K(\cdot,\cdot) \in \mathcal{O}_2^+$. Then inequality (1.1) holds for any non-increasing $f \geq 0$ if and only if one of the conditions max$\{M_1^{\pm}, \mathcal{M}_2^+, \mathcal{M}_3^+\} < \infty$ and max$\{M_1^{\pm}, \mathcal{M}_2^+, \mathcal{M}_4^+\} < \infty$ holds for $V(\infty) < \infty$ and one of the conditions max$\{\mathcal{M}_2^+, \mathcal{M}_3^+\} < \infty$ and max$\{\mathcal{M}_2^+, \mathcal{M}_4^+\} < \infty$ holds for $V(\infty) = \infty$, where*

$$\mathcal{M}_2^+ = \sup_{0<z<\infty} \left(\int_0^z \left(\int_0^x K(x,t)dt \right)^q u(x)dx \right)^{\frac{1}{q}} \left(\int_z^\infty V^{-p'}(s)v(s)ds \right)^{\frac{1}{p'}},$$

$$\mathcal{M}_3^+ = \sup_{0<z<\infty} \left(\int_z^\infty u(x) \left(\int_0^z \left(\int_0^s K(x,t)dt \right)^{p'} V^{-p'}(s)v(s)ds \right)^{\frac{q}{p'}} dx \right)^{\frac{1}{q}},$$

$$\mathcal{M}_4^+ = \sup_{0<z<\infty} \left(\int_0^z V^{-p'}(s)v(s) \left(\int_z^\infty \left(\int_0^s K(x,t)dt \right)^q u(x)dx \right)^{\frac{p'}{q}} ds \right)^{\frac{1}{p'}}.$$

Remark 2. On the basis of the duality principle for a non-decreasing function $f \geq 0$:

$$\sup_{0\leq f\uparrow} \frac{\int_0^\infty g(x)f(x)dx}{\left(\int_0^\infty v(x)f^p(x)dx\right)^{\frac{1}{p}}} \approx \left(\int_0^\infty v(x) \left(\frac{\int_x^\infty g(t)dt}{\int_x^\infty v(t)dt} \right)^{p'} dx \right)^{\frac{1}{p'}} + \frac{\int_0^\infty g(x)dx}{\left(\int_0^\infty v(x)dx\right)^{\frac{1}{p}}},$$

where $g \geq 0$ is any function, we can characterize inequality (1.1) on the cone of non-decreasing functions for operator (1.2) with kernels from the Oinarov classes O_2^+ and O_2^-. However, we omit both statements and their proofs here, since they are similar. Let us only present as an example that the value $M_2^{\pm}$ turns to

$$\mathbb{M}_2^{\pm} = \left(\int_0^\infty \left(\int_t^\infty \left(\int_x^\infty K(z,x)dz \right)^q u(x)dx \right)^{\frac{p}{p-q}} \left(\int_0^t V_*^{-p'}(s)v(s)ds \right)^{\frac{p(q-1)}{p-q}} V_*^{-p'}(t)v(t)dt \right)^{\frac{p-q}{pq}},$$

where $V_*(t) := \int_t^\infty v(x)dx$. All other quantities in M^+ and M^- can be rewritten similarly.

Acknowledgments

The authors would like to thank Professor Ryskul Oinarov for his suggestions, which have improved this paper.

The paper was written under the financial support of the Ministry of Science and Higher Education of the Republic of Kazakhstan, grant no. AP23488579.

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Aigerim Aisultankyzy Kalybay
Department of Economics
KIMEP University
4 Abay Ave,
050010 Almaty, Kazakhstan
E-mails: kalybay@kimep.kz

Ainur Maralkyzy Temirkhanova
Department of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
13 Kazhymukan St
010008 Astana, Kazakhstan
E-mails: ainura-t@yandex.kz

Received: 27.08.2023

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 4 (2024), 66 – 81

A DISCRETE MODEL OF A TRANSMISSION LINE AND THE FABER POLYNOMIALS

V.G. Kurbatov

Communicated by T.V. Tararykova

Key words: discrete transmission line model, interconnection, matrix exponential, Faber polynomial, spectrum.

AMS Mathematics Subject Classification: 65F60, 30E10, 97N50.

Abstract. The spectrum of the matrix coefficient A corresponding to a discrete model of a transmission line often has the shape of a cross. The paper suggests to use the Faber series instead of the Taylor series when calculating the matrix exponential of A. This method can enlarge the accuracy and speed up calculations.

DOI: https://doi.org/10.32523/2077-9879-2024-15-4-66-81

1 Introduction

Transmission lines play the role of interconnections in electrical circuits. Discrete transmission line models (see an example in Figure 5) are often used in circuit theory, see, e. g., [6, 7, 11, 22, 25]. Discrete modelling of a transmission line may be more convenient than a more accurate partial differential equation description because, together with equations of other circuit elements, we obtain a system of ordinary differential (and possibly algebraic) equations only. Furthermore, approximate solving of partial differential equations usually also involves passing to a discrete model, which leads to a similar loss of accuracy.

A linear discrete stationary circuit is described (after eliminating algebraic equations) by an ordinary differential equation of the form $x'(t) = Ax(t) + f(t)$ with a matrix coefficient A. Its solving is reduced to finding the matrix exponential e^{At}, see Section 2. In turn, approximate calculation of e^{At} is usually based [10, 15, 20, 21] on approximation of the function $\exp_t(\lambda) = e^{\lambda t}$ by a polynomial (or a rational function) p_t on the spectrum $\sigma(A)$ of A and subsequent substitution of A into p_t.

An approximation of $\exp_t$ on a set wider than $\sigma(A)$ is not necessary. Moreover, it usually decreases the accuracy of approximation (by a polynomial of the same degree); an example of this phenomena is demonstrated in Figures 6-8. Using the Faber polynomials (see the definition in Section 3) allows us to restrict the set of approximation to (almost) $\sigma(A)$, i. e., the minimal possible. The idea of using the Faber polynomials to calculate matrix functions has been employed by many authors, see, e. g., [3, 4, 5, 14, 16, 26, 27, 28, 29, 31].

We propose to apply the Faber polynomials for approximate solving equations (Section 6) of a discrete model of a transmission line. In this case, the spectrum $\sigma(A)$ has a cross shape, see Figure 1. Our numerical experiments (Section 7) demonstrate that using the Faber expansion instead of the Taylor expansion can increase the accuracy by a factor of 100–1000. The main results of the paper are the exact formulas for the Faber functions Ψ and Φ for the cross (Section 4), and the algorithm (Section 5) that calculates the coefficients for expansion (3.6) of the exponential function in the Faber series.

For numerical calculations we use 'Wolfram Mathematica' [34].

2 Functions of matrices

In this section, we recall the definition of a matrix function and its main application.

Let A be a complex square matrix. The spectrum of A is the set $\sigma(A) \subseteq \mathbb{C}$ of all its eigenvalues. Let f be a complex-valued holomorphic function defined in a neighbourhood of $\sigma(A)$. The function f applied to A is [10, 15, 20, 21] the matrix

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)\,(\lambda\mathbf{1} - A)^{-1}\,d\lambda,$$

where the contour Γ surrounds $\sigma(A)$ and $\mathbf{1}$ is the identity matrix.

The exponential function $\exp_t(\lambda) = e^{\lambda t}$ is the most important for applications. It depends on the parameter $t \in \mathbb{R}$. Its importance is explained by the fact that the solution of the initial value problem

$$\begin{aligned} x'(t) &= Ax(t) + f(t),\\ x(t_0) &= x_0 \end{aligned}$$

can be represented in the form

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}f(t)\,d\tau.$$

More generally, let the relation between the input vector function u and the output vector function y be described by the relations

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\\ x(t_0) &= x_0, \end{aligned}$$

where A, B, C, D — are matrices of compatible sizes. Then the dependence of y on u can [1, p. 65] be expressed as

$$y(t) = C\left(e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-r)}Bu(r)\,dr\right) + Du(t).$$

Usually, the matrix exponential e^{At} can be calculated only numerically. There is vast literature on approximate calculation of e^{At}, see, e. g., [10, 15, 20, 21]. The main goal is fast and accurate calculations; it is clear that these two goals are contradictory. Most methods for approximate calculation of $f(A)$ are based on approximating f by a polynomial or a rational function and substituting A into it. In this paper, we consider a special case when the differential equation $x'(t) = Ax(t) + f(t)$ describes a discrete model of a transmission line (Section 6). In this case, the set $\sigma(A)$ has the shape of a cross, see Figure 1. We use the Faber polynomials generated by the cross to reduce the order of the approximating polynomial.

3 Faber polynomials

A detailed exposition of the theory of Faber series and expansions can be found in [13, 19, 24, 30, 32]. Here we only recall the facts that are necessary for our aims.

Let $K\subset\mathbb{C}$ be a compact simply connected set containing more than one point. We denote by G the complement $\mathbb{C}\setminus K$. Let

$$D=\{\,w\in\mathbb{C}:\;|w|>1\,\}.$$

It is known [24, p. 104] that there exists a unique mapping $\Phi:\,G\to D$ such that (i) Φ is bijective, (ii) Φ has a complex derivative at all points $z\in G$ with $\Phi'(z)\neq 0$, and (iii) there exists a number $\gamma>0$ such that

$$\lim_{z\to\infty}\Phi(\infty)=\infty \qquad \text{and} \qquad \lim_{z\to\infty}\frac{\Phi(z)}{z}=\gamma. \tag{3.1}$$

The number γ is called the *capacity* of K. Evidently, in a neighborhood of infinity, the function Φ possesses the Laurent expansion

$$\Phi(z)=\gamma z+\gamma_0+\frac{\gamma_1}{z}+\frac{\gamma_2}{z^2}+\frac{\gamma_3}{z^3}+\dots, \tag{3.2}$$

where $\gamma>0$ is the same as in (3.1). The general theory of Laurent series states that series (3.2) converges absolutely for all z such that $z\in G_0$, where G_0 is the outer part of the smallest circle with center at zero containing K:

$$G_0=\{\,z\in\mathbb{C}:\;|z|>|\zeta|\text{ for all }\zeta\in K\,\}.$$

For $z\in G_0$, we have the representation

$$\Phi^n(z)=\Bigl(\gamma z+\gamma_0+\frac{\gamma_1}{z}+\frac{\gamma_2}{z^2}+\frac{\gamma_3}{z^3}+\dots\Bigr)^n.$$

Due to absolute convergence, the Laurent series in the parentheses can be multiplied and summed in any order. As a result we obtain the Laurent expansion of the function Φ^n. Removing the parentheses we see that the Laurent series of Φ^n has the form

$$\Phi^n(z)=\gamma^n z^n+a_{n-1}^{(n)}z^{n-1}+\dots+a_1^{(n)}z+a_0^{(n)}+\frac{b_1^{(n)}}{z}+\frac{b_2^{(n)}}{z^2}+\frac{b_3^{(n)}}{z^3}+\dots. \tag{3.3}$$

The polynomials

$$\Phi_n(z)=\gamma^n z^n+a_{n-1}^{(n)}z^{n-1}+\dots+a_1^{(n)}z^1+a_0^{(n)} \tag{3.4}$$

containing the terms with nonnegative powers of z in Laurent expansions (3.3) of Φ^n are called [24, p. 105], [32, p. 33] the *Faber polynomials generated by* K. By definition, $\Phi_0(z)=1$.

We denote by $\Psi:\,D\to G$ the inverse of $\Phi:\,G\to D$. It is easy to show that Ψ has the Laurent expansion of the form

$$\Psi(w)=\beta w+\beta_0+\frac{\beta_1}{w}+\frac{\beta_2}{w^2}+\frac{\beta_3}{w^3}+\dots, \tag{3.5}$$

with $\beta=1/\gamma$. Series (3.5) converges absolutely for all $w\in D$.

Often a holomorphic functions f can be represented as the *Faber series*

$$f(z)=\sum_{n=0}^{\infty}c_n\Phi_n(z),$$

and such an expansion is unique. For our aims, it is important that the Faber series converges faster [13, 19, 24, 30, 32] than the Taylor series. An accurate formulation of the existence of the Faber series expansion is presented in the following theorem.

Theorem 3.1 ([32, Chapter III, § 2]). *Let f be a holomorphic function defined on an open neighbourhood U of K. Let the function Ψ possess a continuous extension to the closure*

$$\overline{D} = \{\, w \in \mathbb{C} : \ |w| \geq 1 \,\}.$$

Then the function f can be expanded into the Faber series

$$f(z) = \sum_{n=0}^{\infty} c_n \Phi_n(z), \tag{3.6}$$

which uniformly converges on compact subsets of U. The coefficients c_n can be found by the formula

$$c_n = \frac{1}{2\pi i} \int_{|w|=1} \frac{f\big(\Psi(w)\big)}{w^{n+1}}\, dw, \qquad n = 0, 1, \dots. \tag{3.7}$$

It is known [19, § 18.2.V] that the approximation of f by partial sums of series (3.6) is close to the best uniform approximation on K by polynomials. This fact explains the efficiency of the transition from the Taylor approximation to the Faber one.

For our goal, it is important that expansion (3.6) extends to functions of matrices.

Corollary 3.1 ([16, Theorem 3.1]). *Let assumptions of Theorem 3.1 be satisfied. Let A be a square complex matrix with $\sigma(A) \subseteq K$. Then*

$$f(A) = \sum_{n=0}^{\infty} c_n \Phi_n(A).$$

4 The functions Ψ and Φ for the cross

For some sets K, the Faber polynomials can be calculated explicitly. Examples can be found in [2, 8, 9, 17, 24, 32]. In this section, we restrict ourselves to the case, which is related to our problem.

Let $a, b > 0$ and $c \in \mathbb{R}$ be some numbers. We consider the set $K \subseteq \mathbb{C}$ shown in left Figure 1 and having the shape of a cross. It consists of two segments intersecting at the point c on the real axis. The endpoints of one segment are the points $c - a$ and $c + a$, the endpoints of the second segment are the points $c - ib$ and $c + ib$. In our situation, K contains the spectrum of our matrix A, see Section 7.

Theorem 4.1. *For the cross shown in the left Figure 1 with parameters $a > 0$, $b > 0$ and $c \in \mathbb{R}$, the function Ψ has the form*

$$\Psi(w) = c + w\sqrt{\frac{a^2+b^2}{2}} \sqrt{\frac{a^2-b^2}{a^2+b^2}\,\frac{1}{w^2} + \frac{1}{2}\Big(1 + \frac{1}{w^4}\Big)}, \tag{4.1}$$

where the square root means the principal value, i. e. $\sqrt{\cdot}$ takes values in the right complex plane

$$\mathbb{C}_r = \{\, z \in \mathbb{C} : \ \operatorname{Re} z > 0 \,\} \cup \{0\}.$$

The function Ψ is bijective and holomorphic on D, and is continuous on the closure

$$\overline{D} = \{\, w \in \mathbb{C} : \ |w| \geq 1 \,\}.$$

The conditions

$$\lim_{w\to\infty} \Psi(w) = \infty \qquad \text{and} \qquad \lim_{w\to\infty} \frac{\Psi(w)}{w} = \frac{\sqrt{a^2+b^2}}{2}. \tag{4.2}$$

are satisfied.

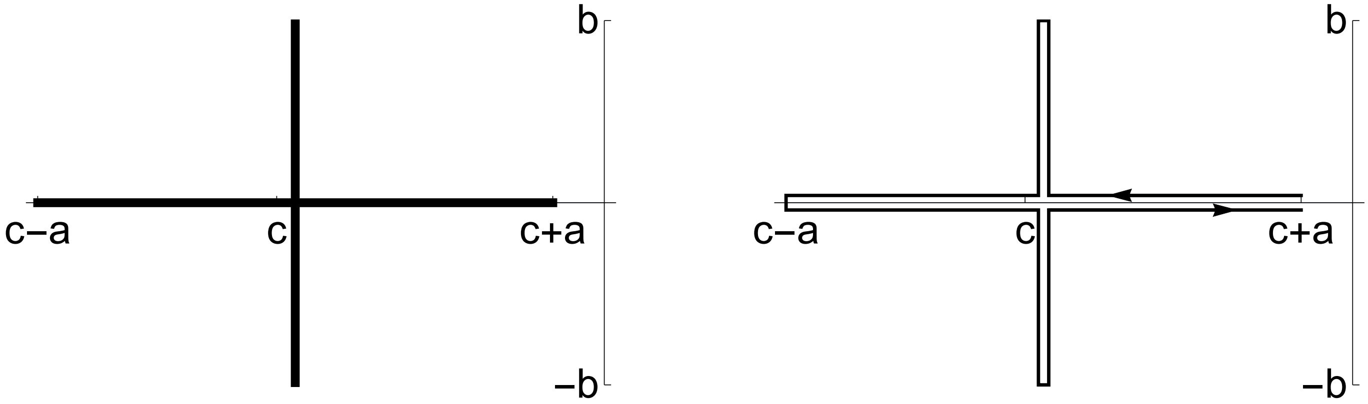

Figure 1: Left: set K having the cross shape; right: the image of the set $\partial D = \{\, w \in \mathbb{C} :\ |w| = 1 \,\}$ under the action of the function Ψ

Proof. The considered principal value of the square root function $\sqrt{\cdot}$ is defined and continuous on the complement of the open half-line

$$L = \{\, z \in \mathbb{C} :\ z \in \mathbb{R} \text{ and } z < 0 \,\}$$

and holomorphic on the complement of the closed half-line

$$\overline{L} = \{\, z \in \mathbb{C} :\ z \in \mathbb{R} \text{ and } z \le 0 \,\}.$$

Therefore, Ψ is (defined and) holomorphic at $w \in \mathbb{C}$ as long as $\zeta(w) \notin \overline{L}$, where

$$\zeta(w) = \frac{a^2 - b^2}{a^2 + b^2}\frac{1}{w^2} + \frac{1}{2}\Bigl(1 + \frac{1}{w^4}\Bigr),$$

and Ψ is (defined and) continuous at $w_0 \in \mathbb{C}$ if $w \notin L$ for all w in a neighbourhood of w_0.

Let us find out when $\zeta(w) \in L$ and $\zeta(w) \in \overline{L}$. For brevity we set $g = \frac{a^2-b^2}{a^2+b^2}$; obviously, g can take any value from $(-1, 1)$. We represent w in the form $w = r(\cos t + i \sin t)$, where $r > 0$ and $t \in (-\pi/2, \pi/2]$, and substitute it into the definition of ζ:

$$\begin{aligned}
\zeta(w) &= g\frac{1}{w^2} + \frac{1}{2}\Bigl(1 + \frac{1}{w^4}\Bigr)\\
&= \frac{2gw^2 + w^4 + 1}{2w^4}\\
&= \frac{2gr^2(\cos t + i\sin t)^2 + r^4(\cos t + i\sin t)^4 + 1}{2r^4(\cos t + i\sin t)^4}\\
&= \frac{\bigl(2gr^2(\cos t + i\sin t)^2 + r^4(\cos t + i\sin t)^4 + 1\bigr)(\cos t - i\sin t)^4}{2r^4}\\
&= \frac{2gr^2(\cos t - i\sin t)^2 + r^4 + (\cos t - i\sin t)^4}{2r^4}\\
&= \frac{-2gr^2\sin^2 t + 2gr^2\cos^2 t + r^4 + \sin^4 t + \cos^4 t - 6\sin^2 t\cos^2 t}{2r^4}\\
&\quad + i\frac{-4gr^2\sin t\cos t - 4\sin t\cos^3 t + 4\sin^3 t\cos t}{2r^4}\\
&= \frac{2gr^2\cos 2t + r^4 + 2\cos^2 2t - 1}{2r^4} - i\frac{\bigl(gr^2 + \cos 2t\bigr)\sin 2t}{r^4}.
\end{aligned} \tag{4.3}$$

We observe that $\zeta(w) \in \overline{L}$ (respectively, $\zeta(w) \in L$) if and only if (i) $\operatorname{Im}\zeta(w) = 0$ and (ii) $\operatorname{Re}\zeta(w) \le 0$ (respectively, $\operatorname{Re}\zeta(w) < 0$). According to representation (4.3), conditions (i) and (ii) mean that

$$\begin{gathered}\big(gr^2 + \cos 2t\big)\sin 2t = 0,\\ 2gr^2\cos 2t + r^4 + 2\cos^2 2t - 1 \le 0 \quad (< 0).\end{gathered}$$

The first condition is satisfied if and only if $t = 0, \pm\pi/2, \pi$ or $\cos 2t = -gr^2$ (provided $|gr^2| \le 1$).

After substituting $t = 0, \pm\pi/2, \pi$, the second condition turns into

$$\pm 2gr^2 + r^4 + 2 - 1 \le 0 \quad (< 0)$$

or

$$(r^2 - 1)^2 + 2r^2(1 \pm g) \le 0 \quad (< 0),$$

which is never true, because $|g| < 1$ and $r > 0$. Thus, in this case $\zeta(w) \notin \overline{L}$ and, moreover, $\zeta(w) \notin L$.

After substituting $\cos 2t = -gr^2$, the second condition turns into

$$r^4 - 1 \le 0 \quad (< 0).$$

If $w \in D$, i. e. $r > 1$, then $r^4 - 1 > 0$ and $r^4 - 1 \le 0$ does not hold; thus $\zeta(w) \notin \overline{L}$ for all $w \in D$. Therefore, the function ζ is holomorphic on D.

However, if $w \in \overline{D}$, i. e. $r \ge 1$, then $r^4 - 1 \ge 0$ and only $r^4 - 1 < 0$ does not hold; thus $\zeta(w) \notin L$ for all $w \in \overline{D}$. Therefore, the function ζ is only continuous on $\overline{D}$.

For a curious reader, the entire set of points w at which $\zeta(w) \in \overline{L}$ (not only its intersection with $\overline{D}$) is shown in the left Figure 2.

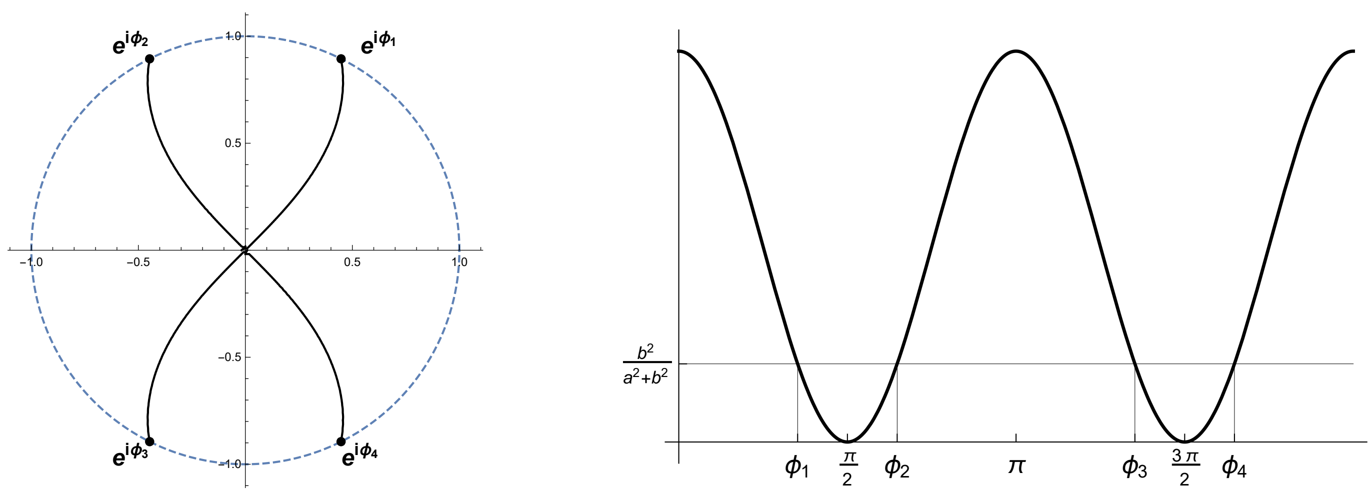

Figure 2: Left: the bold curves constitute the set of points w such that $\zeta(w) \in \overline{L}$; right: the solutions $\varphi_{1,2,3,4}$ of the equation $\cos^2 t - \frac{b^2}{a^2+b^2} = 0$ on $[0, 2\pi]$

Now let us move w along the boundary ∂D. From (4.3) we have

$$\begin{aligned}\zeta(e^{it}) &= \frac{1}{2}\big(2g\cos 2t + 2\cos^2 2t\big) - i\sin 2t(g + \cos 2t)\\ &= \cos 2t(g + \cos 2t) - i\sin 2t(g + \cos 2t)\\ &= (g + \cos 2t)e^{-2it}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Psi(e^{it}) &= c + e^{it}\frac{\sqrt{a^2+b^2}}{\sqrt{2}}\sqrt{(g+\cos 2t)e^{-2it}}\\ &= c + e^{it}\frac{\sqrt{a^2+b^2}}{\sqrt{2}}\sqrt{\Big(\frac{a^2-b^2}{a^2+b^2}+\cos 2t\Big)e^{-2it}}\\ &= c + e^{it}\frac{\sqrt{a^2+b^2}}{\sqrt{2}}\sqrt{\Big(\frac{a^2-b^2}{a^2+b^2}+2\cos^2 t - 1\Big)e^{-2it}}\\ &= c + e^{it}\sqrt{a^2+b^2}\sqrt{\Big(\cos^2 t - \frac{b^2}{a^2+b^2}\Big)e^{-2it}}.\end{aligned}$$

We denote by $\varphi_{1,2,3,4}$ the solutions of the equation $\cos^2 t - \frac{b^2}{a^2+b^2} = 0$ (here the unknown is t) on $[0, 2\pi]$, see the right Figure 2. Then $\Psi(e^{it})$ can be represented as

$$\Psi(e^{it}) = c + e^{it}\sqrt{a^2+b^2}\sqrt{(\cos^2 t - \cos^2\varphi_1)e^{-2it}},$$

where

$$\varphi_1 = \arccos\frac{b}{\sqrt{a^2+b^2}}.$$

In Figure 3, the real and imaginary parts of the function $t \mapsto \Psi(e^{it}) - c$ are presented; the main features are the values (and signs!) at points of extremums.

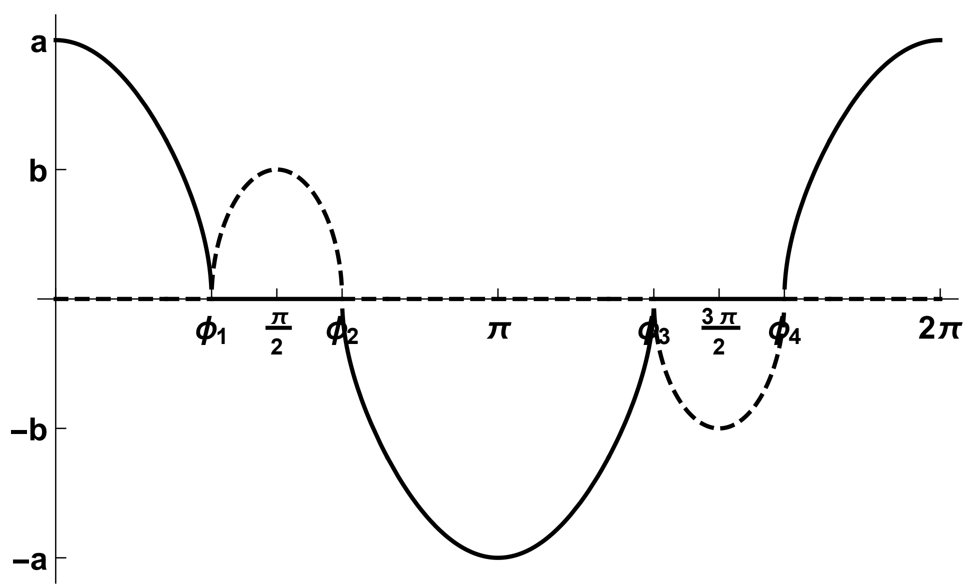

Figure 3: The real (solid line) and imaginary (dashed line) parts of the function $t \mapsto e^{it}\sqrt{a^2+b^2}\sqrt{(\cos^2 t - \cos^2\varphi_1)e^{-2it}}$

Let us describe the curve $z(t) = \Psi(e^{it})$, $t \in [0, 2\pi]$, see the right Figure 1. When $t \in [0, \varphi_1]$, the number $z(t)$ is real and moves from $c + a$ to c. When $t \in [\varphi_1, \pi/2]$, the number $z(t)$ is imaginary and varies form c to $c + ib$. When $t \in [\pi/2, \varphi_2]$, the number $z(t)$ remains imaginary and varies back form $c + ib$ to c. When $t \in [\varphi_2, \pi]$, the number $z(t)$ becomes real and varies form c to $c - a$. When $t \in [\pi, \varphi_3]$, the number $z(t)$ remains real and varies form $c - a$ to c. When $t \in [\varphi_3, 3\pi/2]$, the number $z(t)$ varies form c to $c - ib$. When $t \in [3\pi/2, \varphi_4]$, the number $z(t)$ varies form $c - ib$ to c. When $t \in [\varphi_4, 2\pi]$, the number $z(t)$ varies form c to $c + a$ and thus returns back.

The fulfillment of (4.2) immediately follows from (4.1).

It remains to prove that $\Psi : D \to G$ is bijective. Let us take an arbitrary $z \in \mathbb{C} \setminus K$ and consider the equation $\Psi(w) = z$. We have to prove that the equation $\Psi(w) = z$ has exactly one solution $w \in D$ for any $z \in G$.

We take an arbitrary point $z \in G$. We denote by S_R the circle $\{ w \in \mathbb{C} : |w| = R \}$ of large radius R centered at 0 and oriented counterclockwise, and we denote by $-S_1$ the circle $\{ w \in \mathbb{C} : |w| = 1 \}$ of the radius 1 centered at 0 and oriented clockwise, see Figure 4. We denote by $S_R - S_1$ the contour consisting of S_R and $-S_1$. It is clear that $S_R - S_1$ is the oriented boundary of the annulus $A_R = \{ w \in \mathbb{C} : 1 \le |w| \le R \}$. From (4.2) it follows that the point z lies inside the image $\Psi(S_R)$ of the circle S_R under the action of Ψ if R is large enough.

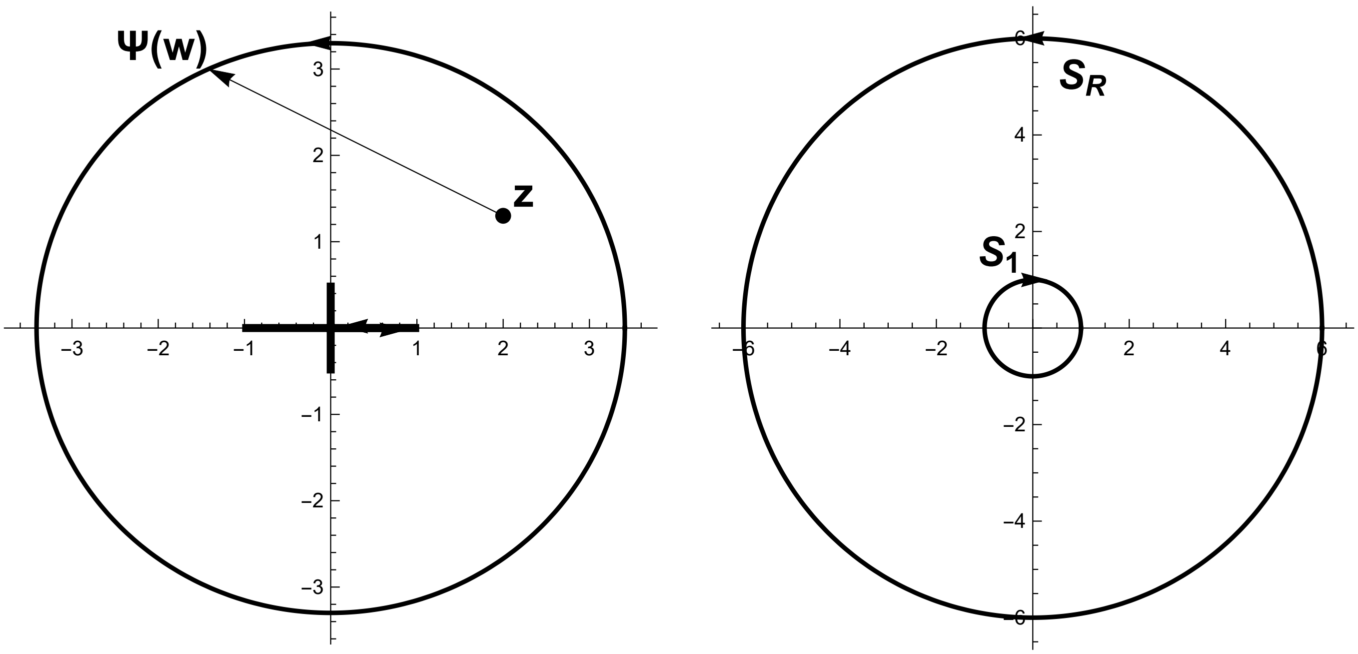

Figure 4: The circles S_R and S_1 (right) and their images (left) under the action of Ψ

We make use of the argument principle [23, p. 48, Theorem 2.3], [18, p. 278, Theorem 4.10a]: the number of solutions w of the equation $\Psi(w) = z$ in the annulus A_R is equal to the increment of the argument of the complex number $\Psi(w) - z$ along the oriented boundary $S_R - S_1$ divided by 2π. Since z lies outside $K = \Psi(S_1)$, the increment of the argument of $\Psi(w) - z$ along S_1 equals zero. On the other hand, from formula (4.1) and the Rouche theorem [23, p. 48, Theorem 2.4], [18, p. 280, Theorem 4.10b] (more correctly, from the proof of the Rouche theorem) it is seen that the increment of the argument of $\Psi(w) - z$ along S_R equals the increment of the argument of $\Psi_1(w) - z$ along S_R, where

$$\Psi_1(w) = c + w\frac{\sqrt{a^2 + b^2}}{2},$$

provided R is large enough. But the increment of the argument of $\Psi_1(w) - z$ along S_R is obviously equal to 2π. Thus, for all R large enough, there is exactly one solution w of the equation $\Psi(w) = z$ in the annulus A_R. Hence, there is exactly one solution in D. □

Corollary 4.1. *Let the assumptions of Theorem 4.1 be satisfied. Then for the function $\Phi : G \to D$, inverse to Ψ, conditions (3.1) are satisfied with $\gamma = \frac{2}{\sqrt{a^2+b^2}}$.*

Proof. The Laurent series of Ψ in a neighborhood of infinity has the form (3.5) and converges at all points of D. Hence, the series

$$h(w)=\beta_0+\frac{\beta_1}{w}+\frac{\beta_2}{w^2}+\frac{\beta_3}{w^3}+\dots$$

also converges in D and is bounded in $\overline{D_2}=\{\,w\in\mathbb{C}:\,|w|\geq 2\,\}$. At the same time, by Theorem 4.1, Ψ and, consequently, h are bounded in the annulus $A_2=\{\,w\in\mathbb{C}:\,1\leq|w|\leq 2\,\}$. Therefore h is bounded in D. Then from (4.2) it follows that $\Psi(w)\to\infty$, $w\in D$, implies that $w\to\infty$.

Let us calculate $\lim_{z\to\infty}\Phi(z)$. We set $w=\Phi(z)$ or $z=\Psi(w)$. By the proved, when $z=\Psi(w)\to\infty$, we also have $\Phi(z)=w\to\infty$. This shows that $\lim_{z\to\infty}\Phi(z)=\infty$.

Now, with the same change $w=\Phi(z)$ or $z=\Psi(w)$, we have $\lim_{z\to\infty}\frac{\Phi(z)}{z}=\lim_{w\to\infty}\frac{w}{\Psi(w)}=\frac{2}{\sqrt{a^2+b^2}}$. □

Corollary 4.2. *Let the assumptions of Theorem 4.1 be satisfied. Then the function $\Phi: G\to D$, inverse to Ψ, possesses the representation*

$$\Phi(z)=(z-c)\sqrt{\frac{\dfrac{b^2-a^2}{(z-c)^2}+2\sqrt{\Big(1-\dfrac{a^2}{(z-c)^2}\Big)\Big(\dfrac{b^2}{(z-c)^2}+1\Big)}+2}{a^2+b^2}}.$$

Proof. For brevity, we temporary set $g=\frac{a^2-b^2}{a^2+b^2}$ and $h=\frac{a^2+b^2}{2}$. To find Φ we solve the equation $z=\Psi(w)$ (from Theorem 4.1 we know that the solution exists and unique):

$$z=c+w\sqrt{h}\sqrt{g\frac{1}{w^2}+\frac{1}{2}\Big(1+\frac{1}{w^4}\Big)},$$

$$\frac{z-c}{w\sqrt{h}}=\sqrt{g\frac{1}{w^2}+\frac{1}{2}\Big(1+\frac{1}{w^4}\Big)},$$

$$\frac{(z-c)^2}{w^2h}=g\frac{1}{w^2}+\frac{1}{2}\Big(1+\frac{1}{w^4}\Big),$$

$$0=\frac{1}{2}+\Big(g-\frac{(z-c)^2}{h}\Big)\frac{1}{w^2}+\frac{1}{2}\frac{1}{w^4},$$

$$0=\frac{1}{2}w^4+\Big(g-\frac{(z-c)^2}{h}\Big)w^2+\frac{1}{2},$$

$$w^2=-g+\frac{(z-c)^2}{h}\pm\sqrt{\Big(-g+\frac{(z-c)^2}{h}\Big)^2-1},$$

$$w^2=\frac{b^2-a^2+2(z-c)^2}{a^2+b^2}\pm\sqrt{\Big(\frac{b^2-a^2+2(z-c)^2}{a^2+b^2}\Big)^2-1},$$

$$w^2=\frac{b^2-a^2+2(z-c)^2}{a^2+b^2}\pm\frac{1}{a^2+b^2}\sqrt{\big(b^2-a^2+2(z-c)^2\big)^2-\big(a^2+b^2\big)^2},$$

$$w^2=\frac{b^2-a^2+2(z-c)^2}{a^2+b^2}\pm\frac{2}{a^2+b^2}\sqrt{((z-c)^2-a^2)((z-c)^2+b^2)},$$

$$w^2=\frac{b^2-a^2+2(z-c)^2\pm 2\sqrt{((z-c)^2-a^2)((z-c)^2+b^2)}}{a^2+b^2},$$

$$w^2 = (z-c)^2 \frac{\frac{b^2-a^2}{(z-c)^2} + 2 \pm 2\sqrt{\left(1-\frac{a^2}{(z-c)^2}\right)\left(1+\frac{b^2}{(z-c)^2}\right)}}{a^2+b^2},$$

$$w = \pm(z-c)\sqrt{\frac{\frac{b^2-a^2}{(z-c)^2} + 2 \pm 2\sqrt{\left(1-\frac{a^2}{(z-c)^2}\right)\left(1+\frac{b^2}{(z-c)^2}\right)}}{a^2+b^2}}.$$

We choose the signs + in the both ± because $\lim_{z\to\infty} \frac{\Phi(z)}{z} = \frac{2}{\sqrt{a^2+b^2}}$. □

Now we can easily calculate the Faber polynomials Φ_n for the cross. According to definition (3.4) we calculate the initial terms of the Laurent series of the function $z \mapsto \Phi^n(z)$ and take its polynomial part. Since we have an exact representation for Φ (Corollary 4.2), the calculations can be performed symbolically and thus Φ_n can be found explicitly. For example,

$$\begin{aligned}\Phi_{11}(z) = 2\left(\frac{1}{a^2+b^2}\right)^{11/2} &(z-c)\Big(-11\,a^{10} + 55\,a^8\left(5\,b^2 + 4(z-c)^2\right)\\
&- 44\,a^6\left(25\,b^4 + 50\,b^2(z-c)^2 + 28(z-c)^4\right)\\
&+ 44\,a^4\left(25\,b^6 + 100\,b^4(z-c)^2 + 140\,b^2(z-c)^4 + 64(z-c)^6\right)\\
&- 11\,a^2\left(5\,b^4 + 20\,b^2(z-c)^2 + 16(z-c)^4\right)^2\\
&+ 11\,b^{10} + 220\,b^8(z-c)^2 + 1232\,b^6(z-c)^4 + 2816\,b^4(z-c)^6\\
&+ 2816\,b^2(z-c)^8 + 1024\,(z-c)^{10}\Big).\end{aligned}$$

5 Calculating the Faber coefficients of the exponential function

We begin with the presentation of a simple algorithm for calculating the Faber coefficients c_m in the expansion

$$e^z = \sum_{m=0}^{\infty} c_m \Phi_m(z).$$

For doing it we use formula (3.7):

$$c_m = \frac{1}{2\pi i}\int_{|w|=1} \frac{f\big(\Psi(w)\big)}{w^{m+1}}\,dw = \frac{1}{2\pi}\int_0^{\pi} \exp\big(\Psi(e^{it})\big)e^{-imt}\,dt. \tag{5.1}$$

Since the function Φ has breaks at the points $\varphi_{1,2,3,4}$ (see Figure 3), it is reasonable to represent the integral as the sum of four ones:

$$\int_0^{\pi} = \int_{-\varphi_1}^{\varphi_1} + \int_{\varphi_1}^{\varphi_2} + \int_{\varphi_2}^{\varphi_3} + \int_{\varphi_3}^{\varphi_4}$$

and use for each integral the Gauss quadrature rule with the Chebyshev weight. Since we are going to substitute a matrix A instead of z, a high accuracy in c_m is desirable. The high accuracy of integral values can be archived by calculating the integrals with an increased number of significant digits (this will not lead to the significant loss of time compared to matrix operations to come later).

Remark 1. A useful idea is proposed in paper [12]. According to formula (5.1), the numbers c_m can be interpreted as the Fourier coefficients of the function $w \mapsto \exp\big(\Psi(e^{it})\big)$. This observation makes it possible to use the fast Fourier transform to calculate integrals of kind (5.1), which speeds up calculations.

The above algorithm for calculating c_m has a drawback: it calculates e^{At} only at one point $t = 1$. Nevertheless, it is often important to have the resulting matrix e^{At} in the form of an expression depending on t. Now we present another algorithm that is free from this shortcoming.

By formula (3.5), the function Ψ has the expansion

$$\Psi(w) = \beta w + \beta_0 + \frac{\beta_1}{w} + \frac{\beta_2}{w^2} + \frac{\beta_3}{w^3} + \dots,$$

which converges in the open exterior D of the unit circle. We consider the Laurent expansions for the powers Ψ^n of Ψ:

$$\Psi^n(w) = \Bigl(\beta w + \beta_0 + \frac{\beta_1}{w} + \frac{\beta_2}{w^2} + \frac{\beta_3}{w^3} + \dots\Bigr)^n,$$

and in analogy with the Faber polynomials Φ_n define Ψ_n as the polynomial part of Ψ^n:

$$\Psi_n(w) = b_n^{(n)} w^n + b_{n-1}^{(n)} w^{n-1} + \dots + b_1^{(n)} w^1 + b_0^{(n)}.$$

The polynomials Ψ_n and their coefficients $b_k^{(n)}$ can be calculated symbolically (and therefore explicitly) in the same way as was done for Φ_n.

We set

$$M = \max_{|w|=1} |\Psi(w)|.$$

Obviously,

$$|\Psi^n(w)| \le M^n, \qquad |w| = 1. \tag{5.2}$$

From the formula for the Laurent coefficients [23, p. 6, Theorem 1.2] we have

$$b_k^{(n)} = \frac{1}{2\pi i} \int_{|w|=1} \frac{\Psi_n(w)}{w^{k+1}}\, dw = \frac{1}{2\pi i} \int_{|w|=1} \frac{\Psi^n(w)}{w^{k+1}}\, dw, \qquad 0 \le k \le n.$$

which implies

$$|b_k^{(n)}| \le M^n. \tag{5.3}$$

We consider the function $\exp_t(z) = e^{tz}$. For it, expansion (3.6) looks like

$$\exp_t(z) = \sum_{m=0}^{\infty} c_m(t) \Phi_m(z).$$

For the coefficients $c_m(t)$, from formula (3.7) we have (due to estimates (5.2) and (5.3), all series converges absolutely):

$$\begin{aligned}
\exp_t\bigl(\Psi(w)\bigr) &= \sum_{n=0}^{\infty} \frac{t^n \Psi^n(w)}{n!},\\
c_m(t) &= \frac{1}{2\pi i} \int_{|w|=1} \frac{\sum_{n=0}^{\infty} \frac{t^n \Psi^n(w)}{n!}}{w^{m+1}}\, dw\\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{2\pi i} \int_{|w|=1} \frac{\Psi^n(w)}{w^{m+1}}\, dw\\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} b_m^{(n)} = \sum_{n=m}^{\infty} \frac{t^n}{n!} b_m^{(n)}.
\end{aligned}$$

The series $\sum_{n=m}^{\infty} \frac{t^n}{n!} b_m^{(n)}$ converges quickly. Therefore, we can use the approximate formula

$$c_m(t) \approx \sum_{n=m}^{N} \frac{t^n}{n!} b_m^{(n)},$$

where N is a large number; in our numerical examples we take $N = 20$. Numerical experiments show that for $t = 1$ the both algorithms give practically the same result.

6 A discrete model of a transmission line

We consider the circuit shown in Figure 5 consisting of $n = 150$ sections. It is a discrete transmission line model. We take the following parameters: $C = C_0/n$, $L = L_0/n$, $R = R_0/n$, $G = G_0/n$ (specific values of the constants C_0, L_0, R_0, G_0 are given in Figures 6-8). We use the state variable formulation [33] of the circuit to derive its equations in the form $\dot{x}(t) = Ax(t) + f(t)$ with the matrix A of the size 300×300. The chosen directions of voltages and currents are shown in Figure 5.

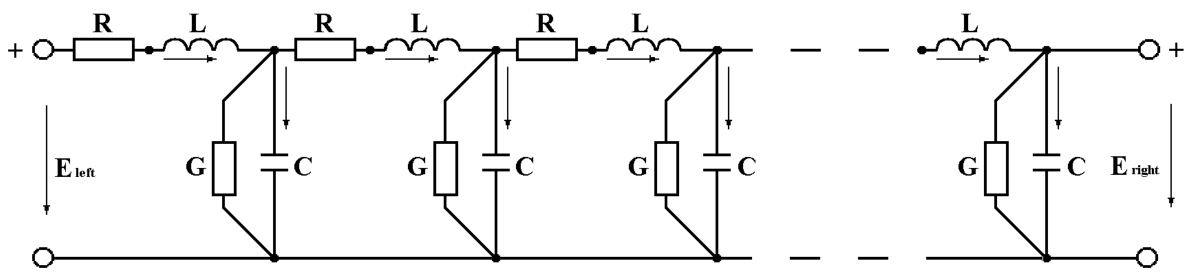

Figure 5: A discrete model of a transmission line

Let us assume that an independent voltage source $E_{\mathrm{left}}(t)$ is connected to the left side, while the right side is open (the right contacts are disconnected). We take as unknowns the vector U_C of voltages across the inductors and the vector I_L of currents through the capacitors. Skipping dull calculations, we present the final differential equation that describe the considered circuit:

$$\begin{pmatrix} \dot{U}_C(t) \\ \dot{I}_L(t) \end{pmatrix} = -\begin{pmatrix} \frac{R_0}{C_0}\mathbf{1} & \frac{n}{C_0}(N - \mathbf{1}) \\ \frac{n}{L_0}(\mathbf{1} - N^T) & \frac{G_0}{L_0}\mathbf{1} \end{pmatrix} \begin{pmatrix} U_C(t) \\ I_L(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \dots \\ -\frac{n}{L_0} E_{left}(t) \\ \dots \\ 0 \end{pmatrix},$$

where the nonzero coordinate $-\frac{n}{L_0}E_{left}(t)$ in the free term corresponds to the first coordinate of I_L, $\mathbf{1}$ is the identity matrix of the size $n \times n$, and

$$N = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Thus the block matrix A has the form

$$A = -\begin{pmatrix} \frac{R_0}{C_0}\mathbf{1} & \frac{n}{C_0}(N - \mathbf{1}) \\ \frac{n}{L_0}(\mathbf{1} - N^T) & \frac{G_0}{L_0}\mathbf{1} \end{pmatrix}. \tag{6.1}$$

7 Numerical experiments

Example 1. We compare the approximation of the function $z \mapsto e^z$ by the 10-th Faber polynomial generated by the cross with different parameters and the 10-th Taylor polynomial. We take a discrete model of transmission line (Figure 5) consisting of 150 sections with parameters C_0, L_0, R_0, and G_0 shown at the tops of Figures 6–8. We calculate the spectrum of corresponding matrix (6.1) and the parameters a, b, and c of the corresponding cross that contains the spectrum. We graph the level curves of the functions

$$F(z) = \Big|e^z - \sum_{k=0}^{10} c_k \Phi_k(z)\Big|, \qquad T(z) = \Big|e^z - \sum_{k=0}^{10} \frac{e^c}{k!}(z-c)^k\Big|. \tag{7.1}$$

The results are shown in Figures 6–8. We present two level curves: the inner level curve corresponds to the minimal value C_F at which the F level curve surrounds the spectrum $\sigma(A)$; the outer level curve corresponds to the minimal value C_T at which the T level curve surrounds the spectrum $\sigma(A)$. The points of $\sigma(A)$ are shown by dots.

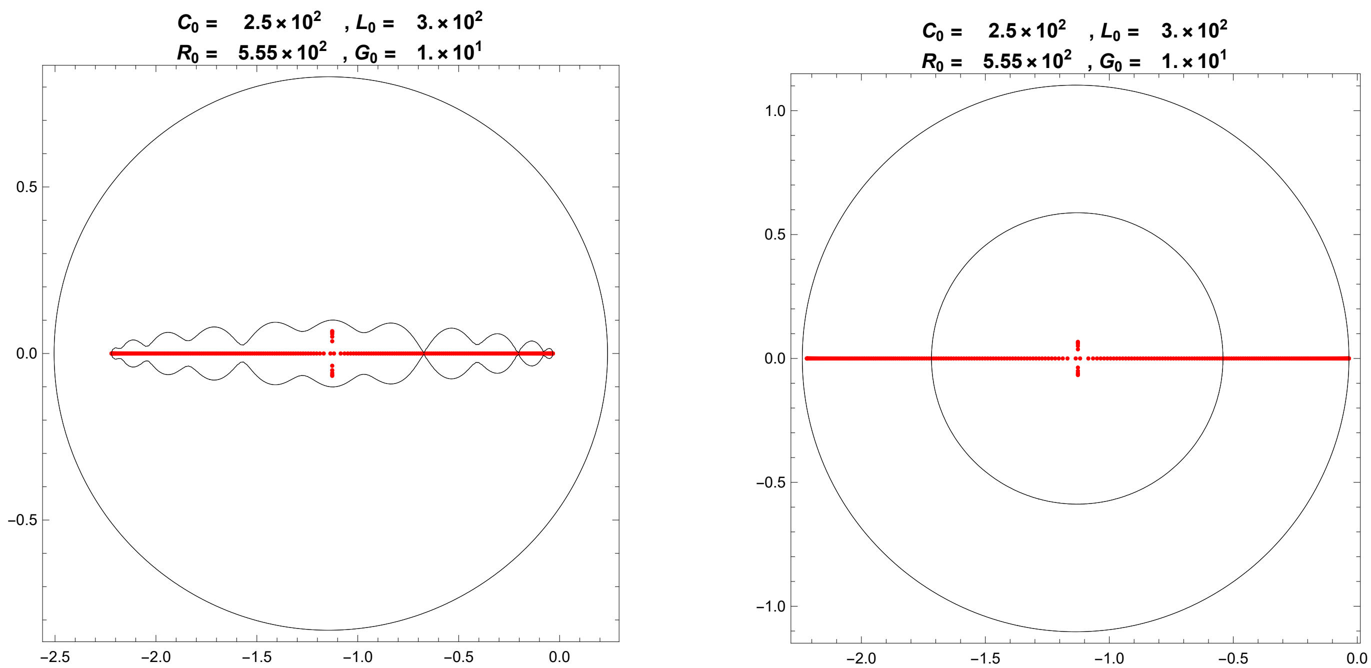

Figure 6: The eigenvalues of matrix (6.1) and the level curves of the functions F (left) and T (right) corresponding to the levels $C_F = 2.362 \cdot 10^{-11}$ and $C_T = 2.382 \cdot 10^{-8}$; $C_T/C_F = 1008.36$

Example 2. We consider the circuit with parameters shown in Figure 6 and the corresponding matrix A. We substitute A into the 10-th Faber polynomial and the 10-th Taylor polynomial, i. e. we calculate the matrices

$$E_F = \sum_{k=0}^{10} c_k \Phi_k(A), \qquad E_T = \sum_{k=0}^{10} \frac{e^c}{k!}(A - c\mathbf{1})^k.$$

We also calculate the precise matrix e^A using the `MatrixExp` command from 'Wolfram Mathematica' [34]. The comparison of accuracy gives

$$\|e^A - E_F\| = 4.7 \cdot 10^{-10}, \qquad \|e^A - E_T\| = 2.4 \cdot 10^{-8}.$$

For matrices, we use the norm induced by the Euclidean norm in $\mathbb{C}^{2n}$.

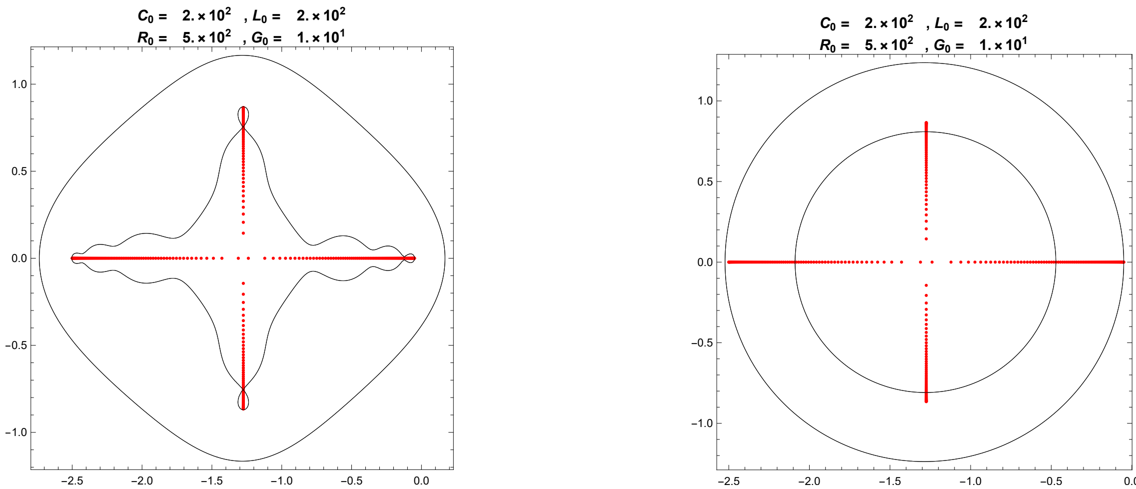

Figure 7: The eigenvalues of matrix (6.1) and the level curves of the functions F (left) and T (right) corresponding to the levels $C_F = 6.795 \cdot 10^{-10}$ and $C_T = 7.259 \cdot 10^{-8}$; $C_T/C_F = 106.8$

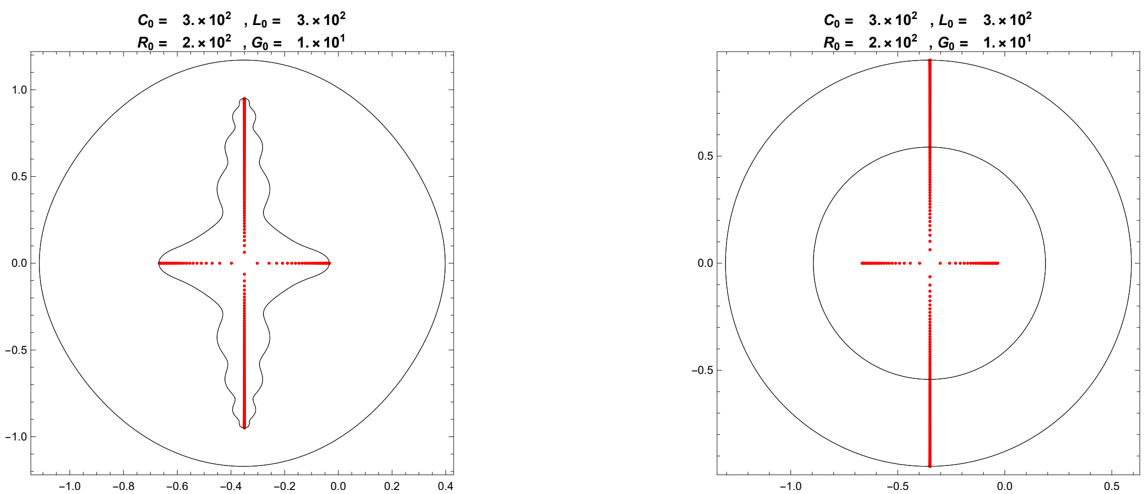

Figure 8: The eigenvalues of matrix (6.1) and the level curves of the functions F (left) and T (right) corresponding to the levels $C_F = 2.114 \cdot 10^{-11}$ and $C_T = 9.84 \cdot 10^{-9}$; $C_T/C_F = 465.484$

So, we have seen that the Faber polynomials can give higher accuracy than the Taylor ones of the same order. Of course, the calculation of the Faber polynomials takes more time. But this loss of time is insignificant compared to subsequent matrix operations.

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Vitalii Gennad'evich Kurbatov
Department of System Analysis and Control,
Voronezh State University,
1 Universitetskaya Square,
Voronezh 394018, Russian Federation
E-mail: kv51@inbox.ru

Received: 01.06.2024

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 4 (2024), 82 – 95

WEAK CONTINUITY OF JACOBIANS OF W^1_ν-HOMEOMORPHISMS ON CARNOT GROUPS

S.V. Pavlov, S.K. Vodop'yanov

Communicated by V.I. Burenkov

Key words: Carnot group, Sobolev mapping, Jacobian, continuity property.

AMS Mathematics Subject Classification: 30C65, 31C45, 58G03.

Abstract. The limit of a locally uniformly converging sequence of analytic functions is an analytic function. Yu.G. Reshetnyak obtained a natural generalization of that in the theory of mappings with bounded distortion: the limit of every locally uniformly converging sequence of mappings with bounded distortion is a mapping with bounded distortion, and established the weak continuity of the Jacobians.

In this article, similar problems are studied for a sequence of Sobolev-class homeomorphisms defined on a domain in a two-step Carnot group. We show that if such a sequence converges to some homeomorphism locally uniformly, the sequence of horizontal differentials of its terms is bounded in $L_{\nu,\mathrm{loc}}$, and the Jacobians of the terms of the sequence are nonnegative almost everywhere, then the sequence of Jacobians converges to the Jacobian of the limit homeomorphism weakly in $L_{1,\mathrm{loc}}$; here ν is the Hausdorff dimension of the group.

DOI: https://doi.org/10.32523/2077-9879-2024-15-4-82-95

1 Introduction

Consider a mapping $f = (f^1, \dots, f^n)$ of class $W^1_{1,\mathrm{loc}}(\Omega; \mathbb{R}^n)$, where Ω is a domain in $\mathbb{R}^n$. Given two multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ of length $k \le n$ with $i_1 < i_2 < \dots < i_k$ and $j_1 < j_2 < \dots < j_k$, denote the (I, J)-minor of the generalized differential Df of f by

$$\frac{\partial f^I}{\partial x_J} = \det \left(\frac{\partial f^{i_\alpha}}{\partial x_{j_\beta}} \right)^k_{\alpha,\beta=1}.$$

The following nontrivial property holds for the differentials of Sobolev-class mappings: the $*$-weak continuity of their minors.

Theorem 1.1. *Given a positive integer $k \le n$ and some domain $\Omega \subset \mathbb{R}^n$, consider a sequence $\{f_m : \Omega \to \mathbb{R}^n\}$ of mappings of class $W^1_{p,\mathrm{loc}}(\Omega; \mathbb{R}^n)$ bounded in $W^1_{p,\mathrm{loc}}(\Omega; \mathbb{R}^n)$, where $p \ge k$. If the sequence $\{f_m\}$ converges in $L_{1,\mathrm{loc}}(\Omega; \mathbb{R}^n)$ to some mapping f_0, then for every pair of multi-indices (I, J) of length k the sequence $\{\frac{\partial f^I_m}{\partial x_J}\}$ converges in the sense of distributions to the (I, J)-minor of the generalized differential of f_0, that is*

$$\lim_{m\to\infty} \int\limits_\Omega \frac{\partial f^I_m}{\partial x_J}(x)\theta(x)\,dx = \int\limits_\Omega \frac{\partial f^I_0}{\partial x_J}(x)\theta(x)\,dx \tag{1.1}$$

for all functions $\theta \in C^\infty_0(\Omega)$.

This property was obtained in the case $n = 2$ in [2], while in the form presented above it was established in [15, Chapter II, Lemma 4.9] and [12]. Note that in [15] Theorem 1.1 appears as a corollary to a claim about the convergence of transported exterior differential k-forms. See [9, Theorem 8.2.1] as well.

Recall that for $1 < q < \infty$ the dual space to the Lebesgue space $L_q(D)$ is the Lebesgue space $L_{q'}(D)$, where the Hölder exponent q' dual to q is determined by the condition $\frac{1}{q'} + \frac{1}{q} = 1$, while the dual space to $L_1(D)$ is the space $L_\infty(D)$ of essentially bounded functions.

Since the space of C_0^∞ functions on a domain $D \Subset \Omega$ is dense in $L_r(D)$ for each $1 \leq r < \infty$, and the hypotheses of Theorem 1.1 imply that the sequence of minors $\left\{\frac{\partial f_m^I}{\partial x_J}\right\}$ is bounded in $L_{p/k,\mathrm{loc}}(\Omega)$, we conclude that for $p > k$ it is not difficult to extend (1.1) to all functions $\theta \in L_{(p/k)'}(\Omega)$ with compact supports in Ω. The latter means that for $p > k$ the sequence of (I, J)-minors of the differentials of f_m converges *weakly* in the space $L_{p/k,\mathrm{loc}}(\Omega)$ to the (I, J)-minor of the differential of the limit mapping f_0.

At the same time, continuous functions do not constitute a dense subspace in $L_\infty(D)$. Therefore, for $p = k = n$ the transition in (1.1) from C_0^∞ functions to all functions $\theta \in L_\infty(\Omega)$ with compact supports in Ω is not obvious. However, that turns out feasible if we assume in addition that the Jacobians are nonnegative: $\det Df_m \geq 0$ almost everywhere. In this case the local uniform integrability of the sequence $\{\det Df_m\}$ established in [13] plays a key role.

Note that the conditions imposed on the sequence $\{f_m\}$ in Theorem 1.1 are equivalent to the weak convergence of $\{f_m\}$ to f_0 in the space $W^1_{p,\mathrm{loc}}(\Omega; \mathbb{R}^n)$.

The main result of this article is the following generalization of Theorem 1.1 to the case of Carnot groups, where ν stands for the homogeneous dimension of the group $\mathbb{G}$; see also [20], where a similar result on Carnot groups is established for sequences of mappings with bounded distortion.

Theorem 1.2. *Consider domains Ω, Ω'_0, $\Omega'_1, \dots$ in a two-step Carnot group $\mathbb{G}$ and a sequence $\{\varphi_k : \Omega \to \Omega'_k\}_{k=1}^\infty$ of homeomorphisms of class $W^1_{\nu,\mathrm{loc}}(\Omega; \mathbb{G})$. Suppose that $\{\varphi_k\}$ converges to some homeomorphism $\varphi_0 : \Omega \to \Omega'_0$ locally uniformly in Ω, the sequence $\{|D_h\varphi_k|\}_{k=1}^\infty$ is bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, and $\det \widehat{D}\varphi_k \geq 0$ almost everywhere, for $k = 1, 2, \dots$.*

Then the sequence of Jacobians $\{\det \widehat{D}\varphi_k\}$ converges to $\det \widehat{D}\varphi_0$ weakly in $L_{1,\mathrm{loc}}(\Omega)$, that is,

$$\lim_{k\to\infty} \int\limits_\Omega \theta(x) \det \widehat{D}\varphi_k(x)\, dx = \int\limits_\Omega \theta(x) \det \widehat{D}\varphi_0(x)\, dx$$

for each function $\theta \in L_\infty(\Omega)$ vanishing almost everywhere outside some compact set $K \subset \Omega$.

In the case of H-type Carnot groups the local uniform convergence of a sequence $\{\varphi_k\}$ of homeomorphisms of class $W^1_{\nu,\mathrm{loc}}(\Omega; \mathbb{G})$, the horizontal differentials of whose terms are bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, to some mapping φ_0 is equivalent to the convergence of $\{\varphi_k\}$ to φ_0 in $L_{1,\mathrm{loc}}(\Omega; \mathbb{G})$ because this sequence possesses a common local continuity modulus [22].

The weak continuity of minors of the differentials of Sobolev-class mappings is one of the main arguments when studying the existence of solutions to nonlinear elasticity problems. Namely, it is related to the possibility of applying Mazur's lemma to establish the semicontinuity of the functionals satisfying the polyconvexity condition, which is a generalized convexity condition, see [1], [4], [13], and [11] for instance.

Even though Theorem 1.2 assumes that the limit mapping φ_0 is bijective, this variation of Theorem 1.1 turns out suitable for deriving theorems about the existence of solutions to the model problems of elasticity on Carnot groups which will be considered by the authors in forthcoming articles.

2 Preliminaries

CARNOT GROUPS. Recall that a *stratified graded nilpotent group* or a *Carnot group*, see [5, Chapter 1] for instance, is a *connected simply-connected* Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ of left-invariant vector fields decomposes as a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m$ of subspaces $\mathfrak{g}_i$ satisfying the conditions $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for $i = 1, \ldots, m-1$ and $[\mathfrak{g}_1, \mathfrak{g}_m] = \{0\}$. A Carnot group $\mathbb{G}$ is called *two-step* whenever $m = 2$.

Fix some inner product in $\mathfrak{g}$. The subspace $\mathfrak{g}_1 \subset \mathfrak{g}$ is called the *horizontal space* of the algebra $\mathfrak{g}$, and its elements are *horizontal vector fields*. Put $N = \dim \mathfrak{g}$ and $n_i = \dim \mathfrak{g}_i$ for $i = 1, \ldots, m$. For convenience also put $n = n_1$. Fix an orthonormal basis $X_{i1}, \ldots, X_{in_i}$ of $\mathfrak{g}_i$. Since the exponential mapping

$$g = \exp\Big(\sum_{i=1}^{m}\sum_{j=1}^{n_i} x_{ij}X_{ij}\Big)(e),$$

where e is the neutral element of $\mathbb{G}$, is a global diffeomorphism of $\mathfrak{g}$ onto $\mathbb{G}$ [5, Proposition 1.2], we can identify the point $g \in \mathbb{G}$ with the point $x = (x_{ij}) \in \mathbb{R}^N$. Then $e = 0$ and $x^{-1} = -x$. The dilation δ_λ specified as $\delta_\lambda(x_{ij}) = (\lambda^i x_{ij})$ is an automorphism of the group for all $\lambda > 0$.

A *homogeneous norm* on $\mathbb{G}$ is a continuous function $\rho : \mathbb{G} \to [0, +\infty)$ of class $C^\infty(\mathbb{G} \setminus \{0\})$ such that

(a) $\rho(x) = 0$ if and only if $x = 0$;

(b) $\rho(x^{-1}) = \rho(x)$ and $\rho(\delta_\lambda x) = \lambda\rho(x)$.

This definition also implies [5, Proposition 1.6] the following properties:

(c) there exists a number $c > 0$ such that $\rho(xy) \le c\big(\rho(x) + \rho(y)\big)$ for all $x, y \in \mathbb{G}$;

(d) two arbitrary homogeneous norms are equivalent, that is, given two homogeneous norms ρ_1 and ρ_2, there are numbers $0 < \alpha \le \beta < \infty$ such that $\alpha\rho_1(x) \le \rho_2(x) \le \beta\rho_1(x)$ for all $x \in \mathbb{G}$.

Example 1. Given some point $x = (x_{ij}) \in \mathbb{G}$ and some index $i = 1, \ldots, m$, define $X^{(i)} \in \mathfrak{g}_i$ as $\sum\limits_{j=1}^{n_i} x_{ij}X_{ij}$. The equality

$$\rho(x) = \Big(\sum_{i=1}^{m} |X^{(i)}|^{2m!/i}\Big)^{\frac{1}{2m!}}, \tag{2.1}$$

where $|X^{(i)}|$ is the Euclidean norm in $\mathfrak{g}_i$, defines a homogeneous norm $\rho : \mathbb{G} \to [0; +\infty)$.

A piecewise smooth curve $\gamma : [a; b] \to \mathbb{G}$ is called *horizontal* whenever $\dot\gamma(t) \in \mathfrak{g}_1(\gamma(t))$ for almost all t. The *Carnot–Carathéodory distance* $d_{cc}(x, y)$ between two points x, $y \in \mathbb{G}$ is the greatest lower bound of the lengths $\int_a^b |\dot\gamma(t)|\, dt$ of horizontal curves with endpoints x and y. According to the Rashevskiĭ–Chow theorem, see [7, §0.4, §1.1] for instance, we can connect two arbitrary points with a piecewise smooth horizontal curve of finite length. The metric d_{cc} and every homogeneous norm ρ are equivalent: there exist positive constants α and β such that

$$\alpha d_{cc}(x, y) \le \rho(y^{-1}x) \le \beta d_{cc}(x, y). \tag{2.2}$$

The Lebesgue measure dx on $\mathbb{R}^N$ is a bi-invariant Haar measure on $\mathbb{G}$ and $d(\delta_\lambda x) = \lambda^\nu dx$, where $\nu = \sum\limits_{i=1}^{m} i\, n_i$ is the *homogeneous dimension* of the group $\mathbb{G}$. The measure is normalized by choosing its value on the unit ball: $|B(0, 1)| = 1$. Here $B(x, r) = \{y \in \mathbb{G} \mid d_{cc}(x, y) < r\}$ is a ball with respect to the Carnot–Carathéodory metric. We denote balls and spheres in the homogeneous norm ρ by $B_\rho(x, r) = \{y \in \mathbb{G} \mid \rho(y^{-1}x) < r\}$ and $S_\rho(x, r) = \{y \in \mathbb{G} \mid \rho(y^{-1}x) = r\}$ respectively.

Example 2. The Heisenberg group $\mathbb{H}^k = (\mathbb{R}^{2k+1}, *)$ with the group operation

$$(x,y,z)*(x',y',z') = \big(x+x', y+y', z+z'+\tfrac{x\cdot y'-x'\cdot y}{2}\big), \quad x,x',y,y' \in \mathbb{R}^k,\ \ z,z' \in \mathbb{R},$$

is the classical example of a nonabelian Carnot group. Its Lie algebra $\mathfrak{h}^k$ is formed by the vector fields

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial z}, \quad i = 1,\dots,k, \quad Z = \frac{\partial}{\partial z}.$$

Here $\mathfrak{h}^k_1 = \operatorname{span}\{X_i, Y_i \mid i = 1,\dots,k\}$ and $\mathfrak{h}^k_2 = \operatorname{span}\{Z\}$, while the only nontrivial Lie brackets are $[X_i, Y_i] = Z$ for $i = 1,\dots,k$. The homogeneous dimension of $\mathbb{H}^k$ equals $\nu = 2k+2$.

Sobolev-class mappings. Consider a domain $\Omega \subset \mathbb{G}$, which is a nonempty connected open subset of $\mathbb{G}$. The space $L_p(\Omega)$, where $p \in [1;\infty)$, consists of all measurable functions $u : \Omega \to \mathbb{R}$ integrable to power p. The norm on $L_p(\Omega)$ is defined as

$$\|u \mid L_p(\Omega)\| = \Big(\int\limits_\Omega |u(x)|^p\,dx\Big)^{\frac{1}{p}}.$$

The space $L_\infty(\Omega)$ consists of all measurable essentially bounded functions $u : \Omega \to \mathbb{R}$. The norm on $L_\infty(\Omega)$ is defined as

$$\|u \mid L_\infty(\Omega)\| = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|,$$

where $\operatorname*{ess\,sup}_{x\in\Omega} |u(x)|$ is the essential supremum of u. A function u belongs to $L_{p,\mathrm{loc}}(\Omega)$, where $p \in [1;\infty]$, whenever $u \in L_p(K)$ for every compact set $K \subset \Omega$.

Take some basis $X_1 = X_{11},\dots,X_n = X_{1n}$ of the horizontal space $\mathfrak{g}_1$. Denote by Π_j the hyperplane $\{x \in \mathbb{G} \mid x_j = 0\}$, for $j = 1,\dots,n$, where $x_j = x_{1j}$ is the horizontal coordinate of the point $x = (x_{ij})$. The measure $d\mu_j = \imath(X_j)dx$ on Π_j is determined by the contraction of X_j with the volume form. Associated to each $y \in \Pi_j$ there is the integral line $\gamma_j(t) = \exp(tX_j)(y)$. A mapping $\varphi : \Omega \to M$ from some domain $\Omega \subset \mathbb{G}$ into some metric space M is *absolutely continuous on almost all lines*, briefly $\varphi \in \mathrm{ACL}(\Omega; M)$, if we can modify it on a measure zero set so that for each $j = 1,\dots,n$ it becomes absolutely continuous on the integral line $\{\exp(tX_j)(y) \mid t \in \mathbb{R}\} \cap \Omega$ of the vector field X_j for μ_j-almost all $y \in \Pi_j$. Put $\mathrm{ACL}(\Omega) = \mathrm{ACL}(\Omega;\mathbb{R})$.

The space $L^1_p(\Omega)$, where $p \in [1;\infty]$, consists of all functions $u \in L_{1,\mathrm{loc}}(\Omega) \cap \mathrm{ACL}(\Omega)$ with the classical derivatives[1] $X_j u$ lying in $L_p(\Omega)$ for all $j = 1,\dots,n$. The seminorm of the function $u \in L^1_p(\Omega)$ equals $\|u \mid L^1_p(\Omega)\| = \big\| |\nabla_h u| \mid L_p(\Omega)\big\|$, where $\nabla_h u = (X_1 u,\dots,X_n u) = \sum\limits_{j=1}^n (X_j u)X_j$ is the *horizontal gradient* of u. Henceforth, instead of $\big\| |\nabla_h u| \mid L_p(\Omega)\big\|$ we write $\|\nabla_h u \mid L_p(\Omega)\|$.

An equivalent definition of the space $L^1_p(\Omega)$ relies on the concept of generalized derivative in the sense of Sobolev: a locally summable function $u_j : \Omega \to \mathbb{R}$ is called the *generalized derivative of the function* $u \in L_{1,\mathrm{loc}}(\Omega)$ *along the vector field* X_j, for $j = 1,\dots,n$, whenever

$$\int\limits_\Omega u_j(x)v(x)\,dx = -\int\limits_\Omega u(x)X_j v(x)\,dx$$

for every test function $v \in C_0^\infty(\Omega)$. A locally summable function $u : \Omega \to \mathbb{R}$ belongs to $L^1_p(\Omega)$ if and only if its generalized derivatives $u_j \in L_p(\Omega)$ exist for $j = 1\dots,n$. Moreover, $u_j = X_j u$ almost

[1] More exactly, the derivatives of a representative of the function u which is absolutely continuous on almost all integral lines of $X_1,\dots,X_n$. The classical derivatives of this representative exist almost everywhere.

everywhere, where $X_j u$ are the classical derivatives of the function[2] $u \in \mathrm{ACL}(\Omega)$, which exist almost everywhere.

The Sobolev space $W_p^1(\Omega)$ consists of all functions $u \in L_p(\Omega) \cap L_p^1(\Omega)$ and is equipped with the norm

$$\|u \mid W_p^1(\Omega)\| = \|u \mid L_p(\Omega)\| + \|u \mid L_p^1(\Omega)\|.$$

Given two Carnot groups $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ and a domain $\Omega \subset \mathbb{G}$, consider $\varphi \in \mathrm{ACL}(\Omega; \widetilde{\mathbb{G}})$. Then $X_j\varphi(x) \in \tilde{\mathfrak{g}}_1(\varphi(x))$ for almost all $x \in \Omega$ [14, Proposition 4.1]. The matrix $D_h\varphi(x) = (X_i\varphi_j)$, where $i = 1, \dots, n$ and $j = 1, \dots, \widetilde{n}$, determines the linear operator $D_h\varphi(x) : \mathfrak{g}_1 \to \tilde{\mathfrak{g}}_1$ called the *horizontal differential* of φ. It is known [18, Theorem 1.2] that for almost all $x \in \Omega$ the linear operator $D_h\varphi(x)$ is defined and extends to a Lie algebra homomorphism $\widehat{D}\varphi(x) : \mathfrak{g} \to \widetilde{\mathfrak{g}}$, which we can also consider as a linear operator $\widehat{D}\varphi(x) : T_x\mathbb{G} \to T_{\varphi(x)}\widetilde{\mathbb{G}}$. The norms of both operators satisfy

$$\big|D_h\varphi(x)\big| \le \big|\widehat{D}\varphi(x)\big| \le C\big|D_h\varphi(x)\big|, \tag{2.3}$$

where C depends only on the group structures. Here the norm of $\widehat{D}\varphi(x)$ is defined as

$$\sup\big\{\widetilde{\rho}\big(\widehat{D}\varphi(x)\langle X\rangle\big) \mid X \in \mathfrak{g},\ \rho(X) \le 1\big\}, \tag{2.4}$$

where we put $\rho(X) = \rho(\exp(X))$ and $\widetilde{\rho}(\widetilde{X}) = \widetilde{\rho}(\widetilde{\exp}(\widetilde{X}))$ for $X \in \mathfrak{g}$ and $\widetilde{X} \in \widetilde{\mathfrak{g}}$ for brevity. Corresponding to $\widehat{D}\varphi(x)$, there is the group homomorphism

$$D_{\mathcal{P}}\varphi(x) = \widetilde{\exp} \circ \widehat{D}\varphi(x) \circ \exp^{-1}$$

known as the *Pansu differential*, which is the approximative differential of φ with respect to the group structure [18].

Definition 1. The class $W_p^1(\Omega; \widetilde{\mathbb{G}})$ of Sobolev mappings consists of all measurable mappings $\varphi \in \mathrm{ACL}(\Omega; \widetilde{\mathbb{G}})$ for which

$$\big\|\varphi \mid W_p^1(\Omega)\big\| = \|\rho \circ \varphi \mid L_p(\Omega)\| + \|\, |D_h\varphi| \mid L_p(\Omega)\|$$

is finite. A mapping φ belongs to $W^1_{p,\mathrm{loc}}(\Omega; \widetilde{\mathbb{G}})$ whenever $\varphi \in W_p^1(U; \widetilde{\mathbb{G}})$ for every compactly embedded domain $U \Subset \Omega$. Henceforth we write $\|D_h\varphi \mid L_p(\Omega)\|$ instead of $\|\, |D_h\varphi| \mid L_p(\Omega)\|$.

Some equivalent descriptions of Sobolev-class mappings of Carnot groups appeared in [18, Proposition 4.2]. If $\varphi \in W_p^1(\Omega; \widetilde{\mathbb{G}})$ then all coordinate functions φ_i for $i = 1, \dots, \widetilde{N}$ belong to $W_p^1(\Omega)$.

3 Uniform integrability and weak continuity of the determinant of the Pansu differential

First, we establish the uniform integrability of the Jacobians of a sequence of orientation-preserving mappings whose horizontal differentials are bounded in $L_{\nu,\mathrm{loc}}$. In connection with that we generalize to the case of Carnot groups the results of paper [16], in which the $L\log L$-norm of an arbitrary summable function f is estimated via the L_1-norm of its maximal function Mf, as well as the results of paper [13], in which the L_1-norm of the maximal function of the Jacobian of an arbitrary orientation-preserving mapping of class W_n^1 is estimated.

[2] Namely, the derivatives of a representative of the function u which is absolutely continuous on almost all lines.

In order to reproduce the arguments of paper [16], we extend the widely known Calderon–Zygmund lemma [3, Lemma 1] to Carnot groups by replacing a system of binary cubes with a suitable system of Borel sets adequate for the geometry of Carnot groups.

Since in a Carnot group equipped with the Carnot–Carathéodory metric each open ball can be covered with a finite number, independent of the ball, of open balls of half the radius, [8, Theorem 2.2] directly implies the following lemma[3].

Lemma 3.1. *Given an arbitrary Carnot group $\mathbb{G}$, there exist collections $\{x_{k,i} \in \mathbb{G}\}_{i\in\mathbb{N}}$, for $k \in \mathbb{Z}$, of points and $\{Q_{k,i} \subset \mathbb{G}\}_{i\in\mathbb{N}}$, for $k \in \mathbb{Z}$, of Borel sets with the following properties:*

(1) *for all $k \in \mathbb{Z}$ the collection $\{Q_{k,i}\}_{i\in\mathbb{N}}$ is disjoint and $\mathbb{G} = \bigcup\limits_{i\in\mathbb{N}} Q_{k,i}$;*

(2) *if $m \geq k$ then either $Q_{m,j} \subset Q_{k,i}$ or $Q_{m,j} \cap Q_{k,i} = \emptyset$;*

(3) *for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$ we have the inclusions*

$$B\Big(x_{k,i}, c\frac{1}{24^k}\Big) \subset Q_{k,i} \subset B\Big(x_{k,i}, C\frac{1}{24^k}\Big),$$

where $c = \frac{1}{3}$ and $C = 4$.

Proposition 3.1. *Given an arbitrary Carnot group $\mathbb{G}$ and a nonnegative function $f \in L_1(\mathbb{G})$, for every $\alpha > 0$ the collection $\{Q_{k,i} \mid i \in \mathbb{N}, k \in \mathbb{Z}\}$ of Lemma 3.1 includes a disjoint subcollection[4] $\mathcal{Q} = \{Q_j\}$ of Borel sets such that*

$$\alpha \leq \frac{1}{|Q_j|}\int\limits_{Q_j} f(x)\,dx \leq 288^\nu\alpha \quad \textit{for all } j, \tag{3.1}$$

and $f(x) \leq \alpha$ for almost all $x \notin \bigcup_j Q_j$.

Proof. Put $\mathcal{Q}_k = \{Q_{k,i}\}_{i\in\mathbb{N}}$ for $k \in \mathbb{Z}$. Since f is an integrable function and each $Q_{k,i}$ contains a ball of radius $\frac{1}{3}\cdot\frac{1}{24^k}$, there is $k_0 \in \mathbb{Z}$ such that

$$\frac{1}{|Q_{k_0,i}|}\int\limits_{Q_{k_0,i}} f(x)\,dx < \alpha$$

for all $i \in \mathbb{N}$. Fix such $k_0 \in \mathbb{Z}$ and an arbitrary $i \in \mathbb{N}$. Add to $\mathcal{Q}$ the sets $Q \in \mathcal{Q}_{k_0+1}$ included into $Q_{k_0,i}$ with

$$\frac{1}{|Q|}\int\limits_{Q} f(x)\,dx \geq \alpha.$$

For these Q claim (3) of Lemma 3.1 yields

$$\frac{1}{|Q|}\int\limits_{Q} f(x)\,dx \leq \Big(\frac{24C}{c}\Big)^\nu \frac{1}{|Q_{k_0,i}|}\int\limits_{Q_{k_0,i}} f(x)\,dx \leq 288^\nu\alpha.$$

Repeat this procedure taking instead of $Q_{k_0,i}$ each set $Q \in \mathcal{Q}_{k_0+1}$ with $Q \subset Q_{k_0,i}$ still not in $\mathcal{Q}$ while they exist. Continue this process by induction and take the union of the resulting collections over $i \in \mathbb{N}$.

[3] In Theorem 2.2 of [8] it suffices to put $A_0 = 1$, $c_0 = 1$, $C_0 = 2$, and $\delta = \frac{1}{24}$ and choose the families $\{x_{k,i}\}_i$ as maximal δ^k-sparse sets in $(\mathbb{G}, d_{cc})$. Each of these collections is obviously countable.

[4] The collection in question can be countable, finite, or empty.

The construction of the family $\mathcal{Q}$ immediately implies that (3.1) holds. It is clear also that if some set $Q \in \{Q_{k,i} \mid k \in \mathbb{Z},\, i \in \mathbb{N}\}$ is disjoint from all sets in $\mathcal{Q}$ then $\frac{1}{|Q|}\int\limits_Q f(x)\,dx < \alpha$.

Assuming now that $y \notin \bigcup \mathcal{Q}$ is a Lebesgue point of the functions $f\chi_{(\bigcup \mathcal{Q})^c}$ and $\chi_{(\bigcup \mathcal{Q})^c}$, verify that $f(y) \le \alpha$. Since every point $z \notin \bigcup \mathcal{Q}$ lies in some $Q_{k,i}$ for all sufficiently large k, it follows that for arbitrary $r > 0$ we can express the complement $Q(r) = B(y,r) \setminus \bigcup \mathcal{Q}$ as the union of a countable collection of disjoint sets $Q_{k,i}$ with

$$\frac{1}{|Q_{k,i}|}\int\limits_{Q_{k,i}} f(x)\,dx < \alpha.$$

Their union $Q(r) = \bigcup Q_{k,i}$ also satisfies

$$\frac{1}{|Q(r)|}\int\limits_{Q(r)} f(x)\,dx < \alpha.$$

Since $\lim\limits_{r\to 0} \frac{|Q(r)|}{|B(y,r)|} = 1$ and

$$\frac{1}{|B(y,r)|}\int\limits_{Q(r)} f(x)\,dx = \frac{1}{|B(y,r)|}\int\limits_{B(y,r)} (f\chi_{(\bigcup \mathcal{Q})^c})(x)\,dx \to f(y)$$

as $r \to 0$, we infer that $f(y) \le \alpha$. □

In the following statement we consider the maximal function in the sense of balls $B(x,r) = \{y \in \mathbb{G} \mid d_{cc}(x,y) < r\}$ with respect to the Carnot–Carathéodory metric:

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|\,dy.$$

Theorem 3.1. *Given an arbitrary Carnot group $\mathbb{G}$, let a function $f \in L_1(\mathbb{G})$ vanish almost everywhere outside the ball $B = B(0,R)$. If $Mf \in L_1(2B)$, then $|f|\log^+|f| \in L_1(B)$. Moreover, we have*

$$\big\||f|\log^+|f| \mid L_1(B)\big\| \le C_L \cdot \Big(\|Mf \mid L_1(2B)\| + \|f \mid L_1(B)\| + R^{-\nu}\|f \mid L_1(B)\|^2\Big),$$

where C_L depends only on the homogeneous dimension ν of $\mathbb{G}$.

Proof. We may assume that $f \ge 0$. Fix $\alpha > 0$. Choose $\{Q_j\}$ according to Proposition 3.1. Take $z \in Q_j$. By claim (3) of Lemma 3.1, we can choose $r > 0$ such that the ball $B(z,r)$ includes Q_j and $|B(z,r)| < c_0|Q_j|$, where the constant c_0 depends only on ν. The definition of the maximal function and the choice of $\{Q_j\}$ show that

$$Mf(z) \ge \frac{1}{|B(z,r)|}\int\limits_{B(z,r)} f(x)\,dx > \frac{c_0^{-1}}{|Q_j|}\int\limits_{Q_j} f(x)\,dx \ge c_0^{-1}\alpha.$$

This means that $\bigcup_j Q_j \subset \{z \in \mathbb{G} \mid Mf(z) > c_0^{-1}\alpha\}$. Since the collection $\{Q_j\}$ is disjoint, we infer that[5]

$$|\{z \in \mathbb{G} \mid Mf(z) > c_0^{-1}\alpha\}| \ge \sum_j |Q_j| \ge \frac{288^{-\nu}}{\alpha}\sum_j \int\limits_{Q_j} f(x)\,dx \ge \frac{288^{-\nu}}{\alpha}\int\limits_{f>\alpha} f(x)\,dx.$$

[5] If for this $\alpha > 0$ the collection $\{Q_j\}$ is empty then the required inequality is obvious because in this case $f \le \alpha$ almost everywhere.

Replacing α with $c_0\alpha$, we obtain

$$|\{z \in \mathbb{G} \mid Mf(z) > \alpha\}| \geq \frac{288^{-\nu} c_0^{-1}}{\alpha} \int\limits_{f > c_0\alpha} f(x)\,dx.$$

Integrate this over $\alpha \in (c_0^{-1}; \infty)$. Thanks to the Cavalieri–Lebesgue formula, the left-hand side equals $\int_{Mf > c_0^{-1}} Mf(x)\,dx$. Rearranging the integral in the right-hand side,

$$\int\limits_{c_0^{-1}}^{\infty} \int\limits_{f > c_0\alpha} \frac{1}{\alpha} f(x)\,dx d\alpha = \int\limits_{1}^{\infty} \int\limits_{f > \beta} \frac{1}{\beta} f(x)\,dx d\beta = \int\limits_{f > 1} \int\limits_{1}^{f(x)} \frac{1}{\beta} f(x)\,d\beta dx$$

$$= \int\limits_{f > 1} f(x) \log f(x)\,dx = \int\limits_{\mathbb{G}} f(x) \log^+ f(x)\,dx,$$

we arrive at the inequality

$$\int\limits_{\mathbb{G}} f(x) \log^+ f(x)\,dx \leq c_1 \int\limits_{Mf > c_2} Mf(x)\,dx,$$

where $c_1 = 288^\nu c_0$ and $c_2 = c_0^{-1}$.

In order to estimate $\int\limits_{Mf > c_2} Mf(x)\,dx$, take $z \notin \overline{B}$; therefore, $d_{cc}(z, 0) > R$. For $t < (d_{cc}(z, 0) - R)$ the intersection $B(z, t) \cap B$ is empty. Since the function f vanishes almost everywhere outside B, it follows that

$$Mf(z) \leq \frac{\|f \mid L_1(B)\|}{\big(d_{cc}(z, 0) - R\big)^\nu}. \tag{3.2}$$

Hence, for $d_{cc}(z, 0) \geq R_f = c_3 \|f \mid L_1(B)\|^{1/\nu} + R$, where $c_3 = c_2^{-1/\nu}$, the value of the maximal function Mf at z is at most c_2, and so the set $\{x \in \mathbb{G} \mid Mf(x) > c_2\}$ lies in the ball $B(0, R_f)$. Using (3.2), we infer that

$$\int\limits_{Mf > c_2} Mf(x)\,dx \leq \int\limits_{2B} Mf(x)\,dx + \int\limits_{B(0, R_f) \setminus 2B} Mf(x)\,dx$$

$$\leq \int\limits_{2B} Mf(x)\,dx + \frac{\|f \mid L_1(B)\|}{R^\nu} R_f^\nu$$

$$= \int\limits_{2B} Mf(x)\,dx + \frac{\|f \mid L_1(B)\|}{R^\nu} \big(c_3 \|f \mid L_1(B)\|^{1/\nu} + R\big)^\nu.$$

□

Consider the maximal function defined with respect to some homogeneous norm ρ:

$$M^\rho f(x) = \sup_{r > 0} \frac{1}{B_\rho(x, r)} \int\limits_{B_\rho(x, r)} |f(y)|\,dy.$$

Since the Carnot–Carathéodory metric d_{cc} is equivalent to every homogeneous norm, we have

$$aMf(x) \leq M^\rho f(x) \leq bMf(x) \quad \text{for all } x,$$

where the constants a and b depend only on ρ. This implies that Theorem 3.1 remains valid when we replace Mf by $M^\rho f$.

Definition 2. A sequence of integrable functions $\{f_k\}$ defined on a measurable space X endowed with some measure μ is called *uniformly integrable* whenever the sequence of integrals $\int\limits_X |f_k|\,d\mu$ is bounded and, given a positive number ε, there is positive δ such that

$$\int\limits_E |f_k|\,d\mu < \varepsilon$$

for all k and all measurable sets $E \subset X$ with $\mu(E) < \delta$.

By the commensurability of the maximal functions Mf and $M^\rho f$, Theorem 3.1 and the de la Vallée Poussin theorem directly imply the following corollary.

Corollary 3.1. *Given a domain Ω in an arbitrary Carnot group $\mathbb{G}$ and an arbitrary homogeneous norm ρ on $\mathbb{G}$, if $\{f_k \in L_{1,\mathrm{loc}}(\Omega)\}$ is a sequence of functions such that for each compact set $K \Subset \Omega$ the sequence $\{M^\rho(f_k\chi_K)\}$ is bounded in $L_{1,\mathrm{loc}}(\Omega)$, then the sequence $\{f_k\}$ is uniformly integrable on every compact subset of Ω.*

In the following two statements we denote by ρ homogeneous norm (2.1), while $\mathcal{H}^{\nu-1}$ stands for the spherical Hausdorff measure defined with respect to ρ. The adjoint operator $\mathrm{adj}\widehat{D}\varphi(y) : \mathfrak{g} \to \mathfrak{g}$ is determined by the condition

$$\widehat{D}\varphi(y) \cdot \mathrm{adj}\widehat{D}\varphi(y) = \det \widehat{D}\varphi(y) \cdot \mathrm{Id}$$

provided that the determinant of the $N \times N$ matrix $\widehat{D}\varphi(y)$ is nonzero and extended by continuity in the topology of $\mathbb{R}^{N\times N}$ otherwise. Its norm $|\mathrm{adj}\widehat{D}\varphi(y)|$ is defined by analogy with (2.4).

Lemma 3.2 ([21, Theorem 3.1]). *Given a two-step Carnot group $\mathbb{G}$ and a bounded domain $\Omega \subset \mathbb{G}$, consider $\varphi \in W^1_\nu(\Omega;\mathbb{G})$.*

Then for almost all $x \in \Omega$ and almost all $r \in (0; \mathrm{dist}_\rho(x, \partial\Omega))$ we have

$$\Big|\int\limits_{B_\rho(x,r)} \det \widehat{D}\varphi(y)\,dy\Big|^{\frac{\nu-1}{\nu}} \le C_I \int\limits_{S_\rho(x,r)} |\mathrm{adj}\widehat{D}\varphi(y)|\,d\mathcal{H}^{\nu-1}(y),$$

where the constant C_I is independent of φ.

Proposition 3.2. *Given a two-step Carnot group $\mathbb{G}$ and a bounded domain $\Omega \subset \mathbb{G}$, consider $\varphi \in W^1_\nu(\Omega;\mathbb{G})$ with $\det \widehat{D}\varphi \ge 0$ almost everywhere. Then for every measurable set $K \Subset \Omega$ there is a constant $C(K)$ independent of φ such that*

$$\|M^\rho(\chi_K \det \widehat{D}\varphi) \mid L_1(\Omega)\| \le C(K)\big(\|D_h\varphi \mid L_\nu(\Omega)\|^{\frac{\nu}{\nu-1}} + |\Omega| \cdot \|D_h\varphi \mid L_\nu(\Omega)\|\big).$$

Proof. Fix a measurable set $K \Subset \Omega$. Put $g = \chi_K \det \widehat{D}\varphi$ and $d = \mathrm{dist}_\rho(K, \partial\Omega)$, as well as $\alpha = \frac{1}{2c}$, and $\beta = \frac{1}{6c^2}$, where c is the constant involved in the generalized triangle inequality. To estimate $\frac{1}{|B_\rho(x,R)|}\int\limits_{B_\rho(x,R)} g(y)\,dy$, make a brute-force search of the cases.

If $x \in \Omega$ is an arbitrary point and $R > \beta d$, then

$$\frac{1}{|B_\rho(x,R)|}\int\limits_{B_\rho(x,R)} g(y)\,dy \le c_1(\beta d)^{-\nu}\int\limits_\Omega \det \widehat{D}\varphi(y)\,dy.$$

However, if $\operatorname{dist}_\rho(x, K) > \alpha d$ and $R \le \beta d$, then by the choice of α and β the intersection $K \cap B_\rho(x, R)$ is empty, and so $\frac{1}{|B_\rho(x,R)|} \int\limits_{B_\rho(x,R)} g(y)\, dy = 0$.

Assume now that $\operatorname{dist}_\rho(x, K) \le \alpha d$ and $R \le \beta d$. In this case the ball $B_\rho(x, 2R)$ lies in Ω, and so for almost all x with $\operatorname{dist}_\rho(x, K) \le \alpha d$ and almost all $r \in (R; 2R)$ we have

$$\Big(\int\limits_{B_\rho(x,R)} g(y)\, dy\Big)^{\frac{\nu-1}{\nu}} \le \Big(\int\limits_{B_\rho(x,r)} \det \widehat{D}\varphi(y)\, dy\Big)^{\frac{\nu-1}{\nu}} \le C_I \int\limits_{S_\rho(x,r)} |\operatorname{adj} \widehat{D}\varphi(y)|\, d\mathcal{H}^{\nu-1}(y).$$

Integrate the last inequality over $r \in (R; 2R)$, see the coarea formula [10, Theorem 6.1], and divide by $|B_\rho(x, R)|$. Taking into account the local boundedness of the horizontal gradient of the function ρ and making some easy rearrangements, we obtain[6]

$$\Big(\frac{1}{|B_\rho(x, r)|} \int\limits_{B_\rho(x,R)} g(y)\, dy\Big)^{\frac{\nu-1}{\nu}} \le \frac{C}{|B_\rho(x, 2R)|} \int\limits_{B_\rho(x,2R)} |\operatorname{adj} \widehat{D}\varphi(y)|\, dy \le C M^\rho f(x),$$

where $f = |\operatorname{adj} \widehat{D}\varphi| \in L_{\frac{\nu}{\nu-1}}(\Omega)$, while C is a constant independent of φ, x and r. Adding the resulting estimates, we see that

$$M^\rho g(x) \le C(K)\big((M^\rho f(x))^{\frac{\nu}{\nu-1}} + \| \det \widehat{D}\varphi \mid L_1(\Omega)\|\big)$$

for almost all $x \in \Omega$. Integrating this over $x \in \Omega$, we obtain

$$\|M^\rho g \mid L_1(\Omega)\| \le C(K)\big(\|M^\rho f \mid L_{\frac{\nu}{\nu-1}}(\Omega)\|^{\frac{\nu}{\nu-1}} + |\Omega| \cdot \| \det \widehat{D}\varphi \mid L_1(\Omega)\|\big).$$

It remains to observe that the Hardy–Littlewood theorem [17, Chapter I.3, Theorem 1], Hölder's inequality, and (2.3) yield

$$\|M^\rho f \mid L_{\frac{\nu}{\nu-1}}(\Omega)\| \le C\|\operatorname{adj} \widehat{D}\varphi \mid L_{\frac{\nu}{\nu-1}}(\Omega)\| \le C\|D_h\varphi \mid L_\nu(\Omega)\|,$$

$$\| \det \widehat{D}\varphi \mid L_1(\Omega)\| \le C\|D_h\varphi \mid L_\nu(\Omega)\|.$$

□

Corollary 3.1 and Proposition 3.2 directly imply the following statement.

Theorem 3.2. *Given a domain Ω in a two-step Carnot group $\mathbb{G}$, if $\{\varphi_k : \Omega \to \mathbb{G}\}$ is a sequence of mappings of class $W^1_{\nu,\mathrm{loc}}(\Omega; \mathbb{G})$ such that $\det \widehat{D}\varphi_k \ge 0$ almost everywhere and the sequence $\{|D_h\varphi_k|\}$ is bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, then the sequence $\{\det \widehat{D}\varphi_k\}$ of Jacobians is uniformly integrable on every compact set $K \Subset \Omega$.*

Let us use the following particular case of Theorem 1 of [6].

Lemma 3.3. *Consider domains Ω, Ω'_0, $\Omega'_1, \dots$ in $\mathbb{R}^N$ and a sequence of homeomorphisms $\{\varphi_k : \Omega \to \Omega'_k\}_{k=1}^\infty$ converging locally uniformly in Ω to some homeomorphism $\varphi_0 : \Omega \to \Omega'_0$.*

Then every compact set $K \Subset \Omega'_0$ lies in Ω'_k for all sufficiently large k, while the sequence $\{\varphi_k^{-1}\}$ of the inverse homeomorphisms converges to φ_0^{-1} locally uniformly on Ω'_0.

[6] In these estimates we use the property that the full-dimensional Hausdorff ν-measure and the Hausdorff measure on the level lines of the function $\varphi(y) = \rho(y^{-1}x)$ considered in [10, Theorem 6.1] are equivalent respectively to the Lebesgue measure and the Hausdorff measure defined with respect to homogeneous norm (2.1).

Recall that the space $C_0(\Omega)$ consists of all continuous functions $\theta : \Omega \to \mathbb{R}$ with compact support in Ω.

Lemma 3.4. *Consider domains Ω, Ω'_0, $\Omega'_1, \dots$ in some Carnot group $\mathbb{G}$ and a sequence $\{\varphi_k : \Omega \to \Omega'_k\}_{k=1}^{\infty}$ of homeomorphisms of class $W^1_{\nu,\mathrm{loc}}(\Omega;\mathbb{G})$ such that $\{\varphi_k\}$ converges to some homeomorphism $\varphi_0 : \Omega \to \Omega'_0$ locally uniformly in Ω, the sequence $\{|D_h\varphi_k|\}_{k=1}^{\infty}$ is bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, and $\det \widehat{D}\varphi_k \geq 0$ almost everywhere, for $k = 1, 2, \dots$.*

Then the sequence $\{\det \widehat{D}\varphi_k\}$ of Jacobians converges $*$-weakly in $L_{1,\mathrm{loc}}(\Omega)$ to $\det \widehat{D}\varphi_0$, that is

$$\lim_{k\to\infty} \int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_k(x)\, dx = \int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_0(x)\, dx$$

for all functions $\theta \in C_0(\Omega)$.

Proof. Since the sequence $\{|D_h\varphi_k|\}$ is bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, the limit homeomorphism φ_0 is also of class $W^1_{\nu,\mathrm{loc}}(\Omega;\mathbb{G})$ [22, Proposition 3.3].

For all quasi-monotone mappings $\varphi \in W^1_{\nu,\mathrm{loc}}(\Omega;\mathbb{G})$, in particular for all homeomorphisms, we have the following change-of-variables formula [19, Theorem 4]:

$$\int\limits_{D} (u \circ \varphi)(x) \det \widehat{D}\varphi(x)\, dx = \int\limits_{\mathbb{G}} u(y)\mu(y,\varphi,D)\, dy, \tag{3.3}$$

where $D \Subset \Omega$ is a compactly embedded subdomain such that $|\varphi(\partial D)| = 0$, while $\mu(y,\varphi,D)$ is the topological degree of the mapping φ at $y \notin \varphi(\partial D)$ defined with respect to the domain D, while u is an arbitrary measurable function such that the function $y \mapsto u(y)\mu(y,\varphi,D)$ is integrable on $\mathbb{G}$.

According to [19, Theorem 3], all quasi-monotone mappings of class $W^1_{\nu,\mathrm{loc}}(\Omega;\mathbb{G})$ have Luzin's $\mathcal{N}$-property. Hence, for every finite collection of balls $B_j \Subset \Omega$ and arbitrary $k = 0, 1, \dots$ the measure of the set $\varphi_k(\partial \bigcup_j B_j)$ vanishes. Consequently, we can put $D = \bigcup_j B_j$ in (3.3).

The degree $\mu(\cdot,\varphi,D)$ of each homeomorphism $\varphi : \overline{D} \to \mathbb{G}$ is a constant on the image $\varphi(D)$ and equals either 1 or -1. Since $\det \widehat{D}\varphi_k \geq 0$ almost everywhere on Ω, for $k = 1, 2, \dots$, we find that (3.3) applied to the mapping $\varphi = \varphi_k$ and the functions $u = \chi_{\varphi_k(D)}$ and $D = \bigcup_j B_j$ for $k = 1, 2, \dots$ implies that $\mu(y,\varphi_k,D) = 1$ for $y \in \varphi_k(D)$.

Furthermore, the continuity of the degree of a mapping under uniform convergence also implies that $\mu(y,\varphi_0,D) = 1$ for $y \in \varphi_0(D)$. Now put $\varphi = \varphi_0$ and $u = \chi_U$ in (3.3), where $U \subset \varphi_0(D)$ is an arbitrary open set. This yields

$$\int\limits_{\varphi_0^{-1}(U)} \det \widehat{D}\varphi_0(x)\, dx = \int\limits_{U} \mu(y,\varphi_0,D)\, dy = |U| > 0. \tag{3.4}$$

Since φ_0 is a homeomorphism, while the open set $U \subset \varphi_0(D)$ and the balls $B_j \Subset \Omega$ which constitute the subdomain D are arbitrary, (3.4) implies that $\det \widehat{D}\varphi_0$ is nonnegative almost everywhere on Ω.

Put $\psi_k = \varphi_k^{-1}$. For $\theta \in C_0(\Omega)$ and $k = 0, 1, 2, \dots$ the change-of-variables formula (3.3) yields[7]

$$\int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_k(x)\, dx = \int\limits_{\Omega'_k} \theta(\psi_k(y))\, dy.$$

[7] As D we should consider a finite union of compactly embedded balls in Ω covering the support of the function θ.

Since $\{\varphi_k\}_{k=1}^{\infty}$ converges uniformly to φ_0 on the support of θ, according to Lemma 3.3 the supports of the functions $\theta \circ \psi_k$ for all sufficiently large k lie in some compact set $K \Subset \Omega'_0$. The uniform convergence of $\{\psi_k\}$ to ψ_0 on K implies that

$$\lim_{k\to\infty} \int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_k(x)\, dx = \lim_{k\to\infty} \int\limits_{\Omega'_k} \theta(\psi_k(y))\, dy = \lim_{k\to\infty} \int\limits_{K} \theta(\psi_k(y))\, dy$$
$$= \int\limits_{K} \theta(\psi_0(y))\, dy = \int\limits_{\Omega'_0} \theta(\psi_0(y))\, dy = \int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_0(x)\, dx.$$

Finally, Theorem 3.2 and Lemma 3.4 imply Theorem 1.2 in the standard fashion. □

Acknowledgments

The authors thank the unknown referee for usefull remarks.

The work was financially supported by the Russian Science Foundation, project no. 23-21-00359.

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Stepan Valerievich Pavlov, Sergey Konstantinovich Vodop'yanov
Department of Mechanics and Mathematics
Novosibirsk State University
1 Pirogov St,
630090 Novosibirsk, Russian Federation
E-mails: s.pavlov4254@gmail.com, vodopis@math.nsc.ru

Received: 04.11.2024

Events

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 4 (2025), 96 – 101

THE 15th CONGRESS OF THE INTERNATIONAL SOCIETY FOR ANALYSIS, ITS APPLICATIONS AND COMPUTATION (ISAAC)

July 21-25, 2025
Nazarbayev University, Astana, Kazakhstan

SECOND INFORMATION LETTER

Dear Colleagues,

We are pleased to share the second information letter for the upcoming 15th International ISAAC Congress, which will take place from July 21 to 25, 2025, at the Nazarbayev University, Astana, Kazakhstan. Below you will find important updates and additional details to help you plan your participation.

Plenary Speakers

- Manuel Del Pino (University of Bath, United Kingdom)
- Hongjie Dong (Brown University, United States)
- Yoshikazu Giga (University of Tokyo, Japan)
- Martin Hairer (EPFL, Switzerland)
- Carlos Kenig (University of Chicago, United States)
- Ari Laptev (Imperial College London, United Kingdom)
- Alexei A. Mailybaev (IMPA, Brazil)
- Nader Masmoudi (Courant Institute at NYU, United States)
- Mete Soner (Princeton University, US)
- Carola-Bibiane Schönlieb (University of Cambridge, United Kingdom)
- Susanna Terracini (University of Turin, Italy)
- Gang Tian (Peking University, China)
- Xiangyu Zhou (Institute of Mathematics, Chinese Academy of Sciences, China)

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Sessions

Each session operates independently. ISAAC recommends allocating 25-30 minutes for invited speakers and 15-20 minutes for contributed speakers. Participants are encouraged to present no more than two presentations per session.

List of Sessions

1. Advances in Nonlinear Analysis
2. Analysis and Mathematical Finance
3. Application of Dynamical Systems Theory in Biology
4. Beyond Nodes and Edges: The Power of Complex Networks in Modeling Real-World Systems
5. Complex Analysis and Partial Differential Equations
6. Complex Variables and Potential Theory
7. Computational and Applied Mathematics
8. Constructive Methods in Boundary Value Problems and Applications
9. Function Spaces and their Applications to Nonlinear Evolutional Equations
10. Fractional Calculus and Fractional Differential Equations
11. Generalized Functions and Applications
12. Harmonic Analysis and Partial Differential Equations
13. Integral Transforms and Reproducing Kernels
14. Inverse Problems and Artificial Intelligence
15. Methods of Analysis in the Research of the Geometry of Pseudo-Euclidean Spaces and their Generalization
16. Modern Problems of Fractional Order Differential Equations
17. Partial Differential Equations on Curved Spacetimes
18. PDEs, Dynamical Systems and Nonlinear Analysis
19. Pseudo Differential Operators
20. Quaternionic and Clifford Analysis
21. Recent Advances in Analysis of Nonlinear Dispersive PDEs
22. Recent Progress in Evolution Equations
23. Recent Progress in Functional Inequalities and Applications
24. Several Complex Analysis
25. Wavelet Theory and its Related Topics
26. Contributed Talks (Open Session)

Registration

Those wishing to participate please follow the steps below:

1. Register at the conference web page: https://isaac2025.org
2. Transfer the registration fee via cashless payment. The registration fees for the event are structured as follows:

Registration Fees

Table 1: Registration Fees

Category	**Time Period**	**EUR**	**KZT**
ISAAC Members	Before April 30, 2025	150	82,500
	From May 1, 2025	200	110,000
Non-Members	Before April 30, 2025	200	110,000
	From May 1, 2025	250	137,500
Students	Before April 30, 2025	80	44,000
	From May 1, 2025	130	71,500
Participants from Developing Countries	Before April 30, 2025	80	44,000
	From May 1, 2025	130	71,500

The registration fee covers all usual expenses related to the conference, including conference materials, schedules of talks, and coffee breaks.

Deadlines

To ensure your participation in the event, please take note of the following important deadlines:

- **May 31, 2025**: Plenary lecturers and thematic session lecturers should submit or update their title and abstract through the registration form.
- **June 5, 2025**: Acceptance notifications for Contributed Lectures will be sent out.
- **June 23, 2025**: Deadline to provide booking information.

Accommodation

Hotels Nearby the Nazarbayev University in Astana

1. Wyndham Garden Astana
2. Hampton by Hilton Astana Triumphal Arch
3. Hilton Garden Inn Astana
4. Eco Apart Hotel Astana
5. E-lite Meatador

Satbayev Travel Grants for Young Mathematicians

Eligibility Criteria

- Age limit: 35 years or younger at the time of application.
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- Ability to articulate the potential benefits of attending the ISAAC Congress.

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- Grants will be awarded to qualified mathematicians based on their submitted applications.
- Each successful applicant will receive financial support to cover travel costs and a per diem of up to 500 USD.
- Registration fees will be covered by ISAAC. Accommodation will be provided by the local organizers.

Application Process

- Prepare a well-crafted curriculum vitae (CV) highlighting your educational background, academic achievements, publications, and relevant experience.
- Prepare a letter of support: a brief letter from your supervisor (for graduate students) or from the applicant (for PhD holders) outlining the importance of attending the congress.
- Submit your CV and letter of support via email to isaac2025.grants@math.kz by March 31, 2025.

Contact Information

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School of Science and Humanities, Nazarbayev University
Qabanbay Batyr Ave 53, Astana 010000, Kazakhstan
E-mail: info@isaac2025.org
Website: https://isaac2025.org

We encourage you to share this information with interested colleagues. We look forward to welcoming you to Astana for an engaging and fruitful congress.

Warm regards,

Prof. Durvudkhan Suragan
Chairman of the Organizing Committee
Nazarbayev University

Dr. Bolys Sabitbek
Member of the Organizing Committee
Queen Mary University of London

EURASIAN MATHEMATICAL JOURNAL

2024 – Том 15, № 4 – Астана: ЕНУ. – 102 с.

Подписано в печать 30.12.2024 г. Тираж – 40 экз.

Адрес редакции: 010008, Астана, ул. Кажымукан, 13,
Евразийский национальный университет имени Л.Н. Гумилева,
корпус № 3, каб. 306а.
Тел.: +7-7172-709500, добавочный 33312.

Дизайн: К. Булан

Отпечатано в типографии ЕНУ имени Л.Н. Гумилева

Зарегистрировано
Министерством информации и общественного развития Республики Казахстан.
Свидетельство о постановке на переучет печатного издания
№ KZ30VPY00032663 от 19.02.2021 г.
(Дата и номер первичной постановки на учет: № 10330 – Ж от 25.09.2009 г.)