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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

Doctor of physical and mathematical sciences, Professor Smailov Esmuhanbet Saidakhmetovich passed away on May 24, 2024, at the age of 78 years.

Esmuhanbet Saidakhmetovich was well known to the scientific community as a high qualified specialist in science and education, and an outstanding organizer. Fundamental scientific articles and textbooks written in various fields of the theory of functions of several variables and functional analysis, the theory of approximation of functions, embedding theorems, and harmonic analysis are a significant contribution to the development of mathematics.

E.S. Smailov was born on October 18, 1946, in the village of Kyzyl Kesik, Aksuat district, Semipalatinsk region. In 1963, he graduated from high school with a silver medal, and in the same year he entered the Faculty of Mechanics and Mathematics of the Kazakh State University (Almaty) named after Kirov (now named after Al-Farabi). In 1971 he graduated from graduate school at the Institute of Mathematics and Mechanics.

He defended his PhD thesis in 1973 (supervisor was K.Zh. Nauryzbaev) and defended his doctoral thesis “Fourier multipliers, embedding theorems and related topics” in 1997. In 1993 he was awarded the academic title of professor.

E.S. Smailov since 1972 worked at the Karaganda State University named after E.A. Buketov as an associate professor (1972-1978), the head of the department of mathematical analysis (1978-1986, 1990-2000), the dean of the Faculty of Mathematics (1983-1987) and was the director of the Institute of Applied Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda (2004 -2018).

Professor Smailov was one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the Karaganda State University named after E.A. Buketov and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He published more than 150 scientific papers and 2 monographs. Under his scientific advice, 4 doctoral and 10 candidate theses were defended.

In 1999 the American Biographical Institute declared professor Smailov “Man of the Year” and published his biography in the “Biographical encyclopedia of professional leaders of the Millennium”.

For his contribution to science and education, he was awarded the Order of “Kurmet” (=“Honour”).

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Esmuhanbet Saidakhmetovich Smailov.

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INVARIANT SUBSPACES IN NON-QUASIANALYTIC SPACES OF Ω-ULTRADIFFERENTIABLE FUNCTIONS ON AN INTERVAL

N.F. Abuzyarova, Z.Yu. Fazullin

Communicated by E.D. Nursultanov

Key words: Ω-ultradifferentiable function, Ω-ultradistribution, Fourier-Laplace transform, invariant subspace, spectral synthesis.

AMS Mathematics Subject Classification: 30D15, 42A38, 46F05.

Abstract. We consider and solve a weakened version of the classical spectral synthesis problem for differentiation operator in non-quasianalytic spaces of ultradifferentiable functions (UDF). Moreover, we deal with the widest class of UDF among all known ones. Namely, we study the spaces of Ω-ultradifferentiable functions introduced by Alexander Abanin in 2007-08. For subspaces of these spaces which are invariant under the differentiation operator we establish general conditions of weak spectral synthesis.

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1 Introduction

Let X be a locally convex space of infinitely differentiable functions on an interval $(a;b) \subseteq \mathbb{R}$ and $X \subset C^\infty(a;b)$ be a continuous embedding. Set $D = \frac{\mathrm{d}}{\mathrm{d}t}$,

$$e_{k,\lambda}(t) = t^k e^{\mathrm{i}\lambda t}, \quad t \in (a;b), \ k \in \mathbb{N} \bigcup \{0\}, \ \lambda \in \mathbb{C}.$$

We assume that
1) D acts continuously in X;
2) X contains all functions of the form $e_{0,\lambda}, \quad \lambda \in \mathbb{C}$;
3) X is a non-quasianalytic function class.

Let $W \subset X$ be a closed subspace of X which is invariant under the differentiation operator: $D(W) \subset W$. Briefly, W is *D-invariant subspace*. By $\operatorname{Exp} W$ we denote the set of all exponential monomials $e_{k,\lambda}$ contained in W. Clearly, for any D-invariant subspace W we have the following implication:

$$e_{k,\lambda} \in W, \ k \geq 1 \Longrightarrow e_{j,\lambda} \in W, \quad j = 0, \dots, k-1.$$

A *classical spectral synthesis problem* for the differentiation operator in X is to find under what conditions D-invariant subspaces $W \subsetneq X$ are spanned by their sets $\operatorname{Exp} W$:

$$W = \overline{\operatorname{span} \operatorname{Exp} W}? \tag{1.1}$$

There are well-known results on this topic for D-invariant subspaces of holomorphic functions on a convex domain in $\mathbb{C}$ (see [15]–[17]), and for translation invariant subspaces in $C^\infty(\mathbb{R})$ (see [21]–[22]). However, as it has been noticed in [9], generally speaking, the classical spectral synthesis does not suit for description of D-invariant subspaces in $C^\infty(a;b)$. The matter is that there are non-trivial D-invariant subspaces in $C^\infty(a;b)$. containing no exponential functions. These are of the form

$$W_I = \{f \in C^\infty(a;b): \quad f = 0 \text{ on } I\}, \tag{1.2}$$

where I is any non-empty relatively closed subinterval of $(a;b)$.

In [9, Theorem 4.1], the authors also show that any non-trivial D-invariant subspace $W \subset C^\infty(a;b)$ contains maximal "residual subspace" of form (1.2). It implies that W has *residual interval* I_W defined as the smallest relatively closed subinterval of $(a;b)$ among all $I \subset (a;b)$ with the property $W_I \subset W$. We see that for D-invariant subspaces in $C^\infty(a;b)$ it is not enough to consider classical spectral synthesis (1.1). In [9], the authors have proposed another form of spectral synthesis problem. We call it the *problem of weak spectral synthesis*. The question is to know which non-trivial D-invariant subspaces W in $C^\infty(a;b)$ admit the representation

$$W = \overline{W_{I_W} + \operatorname{span} \operatorname{Exp} W}? \tag{1.3}$$

It is easy to see that the weakened form of spectral synthesis problem contains the classical one as a particular case. It corresponds to the case $I_W = (a;b)$. The problem of weak spectral synthesis (1.3) in $C^\infty(a;b)$ has been studied in papers [3]–[7].

Any non-quasianalytic function space $X \subsetneq C^\infty(a;b)$ contains D-invariant subspaces of form (1.2). For example,

$$W_c = \{f \in X: \ f^{(k)}(c) = 0: \ k = 0, 1, 2 \ldots\}, \quad c \in (a;b).$$

It means that the problem of spectral synthesis in X should also be considered in its weakened form (1.3). Recently, we have studied this problem in the Beurling space of ultradifferentiable functions of normal type (see [8]). The dual approach we applied earlier in [3]–[4] for D-invariant subspaces in $C^\infty(a;b)$ turns out to be useful and effective in the space of ultradifferentiable functions. $X \subsetneq C^\infty(a;b)$.

In this paper, we study weak spectral synthesis problem (1.3) for a wide class of spaces of Ω-ultradifferentiable functions (briefly, Ω-UDF). General theory of Ω-UDF and Ω-ultradistributions is constructed in [1], [2] by Abanin. In particular, this theory includes all well-known spaces of UDF (Beurling-Börck spaces, Roumier-Komatsu ones, etc.) And we obtain new results on weak spectral synthesis in these general spaces of Ω-UDF.

2 Spectral synthesis

Let X be the space of Ω-UDF on an interval $(a;b)$ of the real line, that is $X = \mathcal{U}_\Omega(a;b)$, where $\Omega = \{\omega_n\}_{n=1}^\infty$ is a *regular* increasing (or decreasing) sequence of non-quasianalytic weights. For the explicit definition and main properties of such spaces see [1], [2].

Given a sequence of complex numbers Λ, we denote by exp^Λ the set of exponential monomials generated by this sequence. It means that for any λ, contained in Λ with the multiplicity $k \in \mathbb{N}$, set exp^Λ contains all functions $e^{-\mathrm{i}\lambda t}$, ..., $t^{k-1}e^{-\mathrm{i}\lambda t}$.

Recall that *completeness radius* $r(\Lambda)$ of Λ equals the infimum of the set of all $r > 0$ such that the system exp^Λ is not complete in $C^\infty(-r;r)$ (or, equivalently, in each of spaces $C(-r;r)$, $L^2(-r;r)$).

By the well-known Beurling-Malliavin theorems (see, e.g., [13, Chapters X, XI]) Paley-Wiener-Schwartz-type theorem on strong dual space $\mathcal{U}'_\Omega(a;b)$ due to Abanin [1, Chapter 5], [2]), taking into account the property of non-quasianalyticity of weights ω_n, we get that the *function system* exp^Λ *is not complete in* $\mathcal{U}_\Omega(a;b)$ *if and only if* $r(\Lambda) < \frac{b-a}{2}$.

Let $I \subseteq (a;b)$ be a relatively closed interval, $|I|$ denote its length, and

$$W_I = \{f \in \mathcal{U}_\Omega(a;b): \quad f = 0 \text{ на } I\}. \tag{2.1}$$

To apply the dual scheme for studying weak spectral synthesis problem in $\mathcal{U}_\Omega(a;b)$, first, we sholud assure that every non-trivial D-invariant subspace W has a *residual interval* $I_W \subseteq (a;b)$ and a *residual* subspace W_{I_W}. In fact, we establish more general assertion.

Proposition 2.1. *For any closed subspace $L \subset \mathcal{U}_\Omega(a;b)$, there exists a relatively closed interval $I_L \subseteq (a;b)$ such that*

$$W_{I_L} \subset L, \quad W_I \setminus L \neq \emptyset \quad \forall I \subsetneq I_L.$$

Proof of this proposition is contained in Remark 3.

Consider a D-invariant subspace $W \subset \mathcal{U}_\Omega(a;b)$ with the residual interval $I_W \subseteq (a;b)$ and the supply of exponential monomials $\operatorname{Exp} W$. Let $\Lambda_W \subset \mathbb{C}$ be the sequence of exponents generating $\operatorname{Exp} W$, that is $\operatorname{Exp} W = exp^{\Lambda_W}$.

The spectrum of the restricted operator $D : W \to W$ is called a *spectrum of* W. We denote it by σ_W.

Proposition 2.2. 1) *For the spectrum of any non-trivial D-invariant subspace W, we have either $\sigma_W = \mathbb{C}$, or $\sigma_W = (-\mathrm{i}\Lambda_W)$.*
2) $r(\Lambda_W) > \frac{|I_W|}{2}$ *implies that $W = \mathcal{U}_\Omega(a;b)$.*

Remark 1. 1. It is not difficult to check that the spectrum of

$$\widetilde{W} = \overline{W_{I_W} + \operatorname{span} \operatorname{Exp} W}$$

equals $(-\mathrm{i}\Lambda_W)$. Particularly, it means that the relation $\sigma_W = (-\mathrm{i}\Lambda_W)$ is a necessary condition of the weak spectral synthesis for W.

2. Let $c,\ d \in (a;b)$, $W_{c,d} = \{f \in \mathcal{U}_\Omega(a;b) :\ f^{(k)}(c) = f^{(k)}(d) = 0,\ k = 0,1,2,\dots\}$, $W_{[c;d]}$ be defined by (2.1) with $I = [c;d]$. By the argument similar to one used in [9, §2], we get that $\sigma_{W_{c,d}} = \mathbb{C}$ and $\sigma_{W_{[c;d]}} = \emptyset$. At the same time, $\operatorname{Exp} W_{c,d} = \operatorname{Exp} W_{[c;d]} = \emptyset$. There may also be constructed generalisations with non-empty $\operatorname{Exp} W$.

Now, we formulate conditions of the weak spectral synthesis in $\mathcal{U}_\Omega(a;b)$.

Theorem 2.1. *Let $W \subsetneq \mathcal{U}_\Omega(a;b)$ be D-invariant subspace and $\sigma_W = -\mathrm{i}\Lambda_W$.*
If $r(\Lambda_W) < \frac{|I_W|}{2}$, then

$$W = \overline{W_{I_W} + \operatorname{span} exp^{\Lambda_W}}.$$

Corollary 2.1. *Let $W \subsetneq \mathcal{U}_\Omega(a;b)$ be D-invariant subspace and $\sigma_W = -\mathrm{i}\Lambda_W$.*
1) *If the residual interval I_W is not compact in $(a;b)$ then W admits weak spectral synthesis* (1.3).
2) *W admits classical spectral synthesis* (1.1) *if and only if $I_W = (a;b)$.*

Remark 2. It turns out that the sufficient condition in Theorem 2.1 coincides with the condition of admitting of weak spectral synthesis by D-invariant subspace in $C^\infty(a;b)$ (see [3, Theorem 2, Corollaries 2, 3, Remark 3] or [4, Theorem 5, Corollary 2], and [10, Theorems 1.1, 1.3]).

3 Preliminaries. Dual scheme

3.1 Spaces $\mathcal{U}_\Omega(a;b)$, $\mathcal{U}'_\Omega(a;b)$ and $\mathcal{P}$

Any Ω-UDF space is defined by a weight sequence $\Omega = \{\omega_n\}$ that may be increasing or decreasing:

$$\omega_n \le \omega_{n+1} \quad \forall n \in \mathbb{N} \quad \text{or} \quad \omega_n \le \omega_{n+1} \quad \forall n \in \mathbb{N}.$$

An element of Ω is a *weight function* $\omega_n : \mathbb{R} \to [0;\infty)$, which is Lebesque measurable and locally bounded in $\mathbb{R}$. Additionally, it must subject the requirements

$$\int_{\mathbb{R}} e^{\omega(t)} \mathrm{d}t < \infty, \tag{3.1}$$

$$\int_1^\infty \frac{\overline{\omega}(t)}{t^2} \mathrm{d}t < \infty, \tag{3.2}$$

where $\overline{\omega}(t) := \sup\{\omega(s) : \ |s| \le t\}$.

It should also be assumed that all weights $\omega_n \in \Omega$ and the sequence Ω itself satisfy some additional restrictions in order to guarantee that $\mathcal{U}_\Omega(a;b)$ is continuousy embedded into $C^\infty(a;b)$ and invariant under the differentiation operator. In this case, $\mathcal{U}_\Omega(a;b)$ is a locally convex space of (M^*)-type if Ω is increasing or, respectovely is a locally convex space of (LN^*)-type if Ω is decreasing. Particularly, in both cases, $\mathcal{U}_\Omega(a;b)$ is a complete reflexive Hausdorff space, the open mapping theorem and the closed graph theorem are true in this space. Moreover, $\mathcal{U}_\Omega(a;b)$ contains all polynomials, all exponential functions e^{-itz}, $z \in \mathbb{C}$, and it is a toplogical module over the ring $\mathbb{C}[t]$. The differentiation $D = \frac{\mathrm{d}}{\mathrm{d}t}$ is a continuous operator in $\mathcal{U}_\Omega(a;b)$.

Recall that given a sequence Ω, by $\mathcal{D}_\Omega(a;b)$ we denote the space of all *test Ω-UDF*, that are compactly supported in $(a;b)$. Because of (3.2), this space is non-trivial. Ω-ultradistributions are defined to be elements of the strong dual space $\mathcal{D}'_\Omega := \mathcal{D}'_\Omega(\mathbb{R})$ (see [1, Chapter 2,3]). It is known that any classical distribution also is an Ω-ultradistributions, that is $\mathcal{D}' \subsetneq \mathcal{D}_\Omega(\mathbb{R})$.

All basic notions of the classical distribution theory are extended to Ω-ultradistributions. In particular, it is true for the notion of "support" of an Ω-ultradistribution and the meaning of the phrase "Ω-ultradistribution equals zero on an open set". If supports of an Ω-ultradistribution S and a test Ω-UDF f have no common point then $S(f) = 0$. For $S \in \mathcal{D}'_\Omega \bigcap \mathcal{D}'_{\widetilde{\Omega}}$, where Ω and $\widetilde{\Omega}$ are two different weight sequences, the support of S as an Ω-ultradistribution equals its support if we think of S as an $\widetilde{\Omega}$-ultradistribution.

According to Theorem 5.2.2 in [1], *the strong dual space $\mathcal{U}'_\Omega(a;b)$ is formed by all Ω-ultradistributions that are compactly supported in $(a;b)$.*

For the technical convinience, now we consider a symmetric interval $(-a;a)$ instead of an arbitrary one $(a;b)$.

Given a weight ω, we recall that its continuation to $\mathbb{C}$ is defined by the formula

$$H_\omega(x + \mathrm{i}y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(x + \xi y)}{1 + \xi^2} \mathrm{d}\xi$$

(see [1, 1.4]). This is a non-negative function in $\mathbb{C}$, $H_\omega(z) = H_\omega(\bar{z})$, $z \in \mathbb{C}$. It is also harmonic in the upper half-plane and in the lower one.

For every $\omega_n \in \Omega$, and every $k \in \mathbb{N}$, we set

$$\mathcal{P}_{n,k} = \left\{ \varphi \in H(\mathbb{C}) : \ \|\varphi\|_{n,k} := \sup_{z \in \mathbb{C}} \frac{|\varphi(z)|}{\exp\left(c_k |\operatorname{Im} z| + H_{\omega_n}(-z)\right)} < \infty \right\}, \tag{3.3}$$

where $0 < c_k \nearrow a$. Clearly, $\mathcal{P}_{n,k}$ is a Banach space.

Set

$$\mathcal{P}_{(\Omega),a} = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{P}_{n,k}$$

if Ω is increasing, and

$$\mathcal{P}_{\{\Omega\},a} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \mathcal{P}_{n,k}$$

if Ω is decreasing.

By $\mathcal{P}$ we denote an arbitrary one of these two spaces, $\mathcal{P}_{(\Omega),a}$ or $\mathcal{P}_{\{\Omega\},a}$. Locally convex space $\mathcal{P}$ is a complete Hausdorff reflexive and bornological. It is also important to notice that $\mathcal{P}$ is a topological module over the ring of all polynomials $\mathbb{C}[z]$.

Recall that for every $S \in \mathcal{U}'_\Omega(-a;a)$, its Fourier-Laplace transform is defined by formula

$$S \mapsto \mathcal{F}(S)(z) := S(e^{-\mathrm{i}tz}), \quad z \in \mathbb{C},$$

and

$$\mathcal{F} : \mathcal{U}'_\Omega(-a;a) \to H(\mathbb{C}).$$

Theorem A. *Fourier-Laplace transform $\mathcal{F}$ is a linear and topological isomorphism between spaces $\mathcal{U}'_\Omega(-a;a)$ and $\mathcal{P}$.*

For the regular weight sequences Ω, Theorem A was established in [1, Theorem 5.4.2], [2]. The author also proved in [1] that the norm $\|\varphi\|_{n,k}$ defined by (3.3) may be replaced by the following one:

$$\|\varphi\|_{n,k} = \sup_{z\in\mathbb{C}} \frac{|\varphi(z)|}{\exp\left(c_k|\operatorname{Im} z| + \omega_n(-\operatorname{Re} z)\right)}.$$

This change leads to the same locally convex space $\mathcal{P}$.

Notice that all above defintions and facts are true for an arbitrary interval $(a;b) \subseteq \mathbb{R}$, not only for the symmetric one. In further presentation, we denote by $\mathcal{P}$ the space $\mathcal{F}(\mathcal{U}'_\Omega(a;b))$.

For a closed subspace $W \subset \mathcal{U}_\Omega(a;b)$, its *annihilator subspace* W^0 is defined to be

$$W^0 = \{S \in \mathcal{U}'_\Omega(a;b) : \ S(f) = 0 \ \forall f \in W\}.$$

Because of $\mathcal{U}_\Omega(a;b)$ is reflexive, by Khan-Banach theorem and Theorem A, we obtain

Proposition 3.1. (**General duality principle.**) *There is one-to-one correspondence between the set $\{W\}$ of all closed subspaces of $\mathcal{U}_\Omega(a;b)$ and the set $\{\mathcal{J}\}$ of all closed subspaces of $\mathcal{P}$:*

$$W \longleftrightarrow \mathcal{J} \Longleftrightarrow \mathcal{J} = \mathcal{F}(W^0).$$

Now, we list some properties of elements of $\mathcal{P}$.

Because of (3.2), all functions in $\mathcal{P}$ belong to the Cartwright class C of entire functions. In particular, any $\varphi \in \mathcal{P}$ is an entire function of completely regular growth with respect to the order 1 having exponential type less that $\frac{b-a}{2}$. Multiplying φ by a suitable function of the form $e^{-\mathrm{t}_\varphi z}$, $t_\varphi \in \mathbb{R}$, we get an entire function ψ with the indicator function

$$h_\psi(\theta) = \pi\Delta_\varphi|\sin\theta|, \qquad \Delta_\varphi < \frac{b-a}{2},$$

where $2\Delta_\varphi$ denotes the density of zero sequence $\mathcal{Z}_\varphi$ of φ. An indicator diagram of ψ equals

$$\mathrm{i}[-h_\psi(-\pi/2); h_\psi(\pi/2)].$$

Finally, because of the well-known Beurlig-Malliavin results (cf. [13]), from relation (3.2) we derive that given a complex sequence Λ, $r(\Lambda) < \frac{b-a}{2}$ is equivalent to $\Lambda \subset \mathcal{Z}_\varphi$ for some $\varphi \in \mathcal{P}$. In this case, the inequality $r(\Lambda) \le \pi\Delta_\varphi$ is also true. And $\Lambda = \mathcal{Z}_\varphi$ implies $r(\Lambda) = \pi\Delta_\varphi$.

3.2 Dual scheme

According to **the general duality principle** (Proposition 3.1), there is a one-to-one correspondence between closed subspaces $W \subset \mathcal{U}_\Omega(a;b)$ and closed subspaces $\mathcal{J} \subset \mathcal{P}$. It is not difficult to check that W is D-invariant if and only if $z\mathcal{J} \subset \mathcal{J}$, that is $\mathcal{J}$ is a *closed submodule* in $\mathcal{P}$ (over the ring $\mathbb{C}[z]$). In further presentation, we consider only *closed* submodules in $\mathcal{P}$ and write "submodule" instead of "closed submodule".

For an arbitrary submodule $\mathcal{J} \subset \mathcal{P}$ its *zero set* $\mathcal{Z}_\mathcal{J}$ is defined by

$$\mathcal{Z}_\mathcal{J} = \bigcap_{\varphi\in\mathcal{J}} \mathcal{Z}_\varphi,$$

where $\mathcal{Z}_\varphi$ is zero set of φ.

Indicator segment of $\mathcal{J}$ is denoted by

$$[c_\mathcal{J}; d_\mathcal{J}] \subset \overline{\mathbb{R}}, \tag{3.4}$$

where $c_\mathcal{J} = -\sup\limits_{\varphi\in\mathcal{J}} h_\varphi(-\pi/2)$, $d_\mathcal{J} = \sup\limits_{\varphi\in\mathcal{J}} h_\varphi(\pi/2) \in \overline{\mathbb{R}}$, and h_φ is the indicator function of φ.

Proposition 3.2. (**Special duality principle.**) *There is one-to-one correspondence between the set $\{W\}$ of all D-invariant subspaces in $\mathcal{U}_\Omega(a;b)$ and the set $\{\mathcal{J}\}$ of all submodules in $\mathcal{P}$:*

$$W \longleftrightarrow \mathcal{J} \Longleftrightarrow \mathcal{J} = \mathcal{F}(W^0),$$

where $W^0 = \{S \in \mathcal{U}'_\Omega(a;b) :\ S(f) = 0\ \forall f \in W\}$. In addition,

$$I_W = [c_\mathcal{J}; d_\mathcal{J}] \bigcap (a;b), \quad \operatorname{Exp} W = Exp^{\mathcal{Z}_\mathcal{J}}. \tag{3.5}$$

Proof. We need only to check the first relation in (3.5).

Set $I_0 = (a;b) \bigcap [c_\mathcal{J}; d_\mathcal{J}]$. Notice that

$$f \mapsto f(\cdot + y), \quad f \mapsto f(\cdot - y), \quad y > 0,$$

are continuous operators acting in $\mathcal{U}_\Omega(a;+\infty)$ and $\mathcal{U}_\Omega(-\infty;b)$, respectively.

For a function $f \in W_{I_0} \subset \mathcal{U}_\Omega(a;b)$, we can write

$$f = f_- + f_+, \quad f_- \in W_{I_-}, \quad f_+ \in W_{I_+},$$

where $I_- = (-\infty; d_\mathcal{J}]$, $I_+ = [c_\mathcal{J}; +\infty)$, $W_{I_-} \subset \mathcal{U}_\Omega(a;+\infty)$, $W_{I_+} \subset \mathcal{U}_\Omega(-\infty;b)$.

Further, for $S \in \mathcal{F}^{-1}(\mathcal{J})$ we have

$$\operatorname{supp} g(\cdot - y) \bigcap \operatorname{supp} S = \emptyset, \quad g \in W_{I_-},\ y > 0,$$

$$\operatorname{supp} \tilde{g}(\cdot + y) \bigcap \operatorname{supp} S = \emptyset, \quad \forall\, \tilde{g} \in W_{I_+},\ y > 0.$$

It follows that

$$S(f) = S(f_- + f_+) = \lim_{y \to 0+} \left(S(f_-(x-y)) + S(f_+(x+y))\right) = 0$$

for any Ω-ultradistribution $S \in \mathcal{F}^{-1}(\mathcal{J})$. By the general duality principle, we get that $W_{I_0} \subset W$.

Now, let us consider an arbitrary interval $I' \subsetneq I_0$ respectively closed in $(a;b)$. From the definition of $c_{\mathcal{J}}$ and $d_{\mathcal{J}}$ and general theory of Ω-UDF and Ω-ultradistributions, we derive that for every $c' \in (c_{\mathcal{J}}; d_{\mathcal{J}}) \setminus I'$ there exist $S \in \mathcal{F}^{-1}(\mathcal{J})$, $f \in \mathcal{U}_\Omega(a;b)$ and $\delta > 0$ such that

$$S(f) \neq 0, \quad \operatorname{supp} f \subset (c' - \delta; c' + \delta) \subset (c_{\mathcal{J}}; d_{\mathcal{J}}) \setminus I'.$$

Hence, by the duality principle, $f \notin W$. On the other hand, we have $f \in W_{I'}$. It means that interval I_0 is the smallest one among all respectively closed in $(a;b)$ intervals I for which $W_I \subset W$.

So, we get the relation $I_W = I_0$ and finish the proof.

□

Remark 3. The notion of the indicator segment may be defined for an arbitrary closed subspace $\mathcal{J} \subset \mathcal{P}$. Applying the argument used in the proof of the first relation in (3.5) to an arbitrary closed subspace $W \subset \mathcal{U}_\Omega(a;b)$ and $\mathcal{J} = \mathcal{F}(W^0)$, we easily get Proposition 2.1.

We call a submodule $\mathcal{J} \subset \mathcal{P}$ *weakly localisable* if it contains *all* functions $\varphi \in \mathcal{P}$ satisfying the conditions

$$\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_\varphi \quad \text{and} \quad [-h_\varphi(-\pi/2); h_\varphi(\pi/2)] \subset [c_{\mathcal{J}}; d_{\mathcal{J}}]$$

Submodule $\mathcal{J} \subset \mathcal{P}$ is called *localisable* (*ample*) if it contains *all* functions $\varphi \in \mathcal{P}$ with the property $\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_\varphi$. In other words, the localisable submodule is a weakly localisable one with the indicator segment equaled to $[a;b] \subset \overline{\mathbb{R}}$.

Weakly localisable submodule $\mathcal{J}$ is the biggest one among all submodules $\widetilde{\mathcal{J}}$ such that

$$\mathcal{Z}_{\widetilde{\mathcal{J}}} = \mathcal{Z}_{\mathcal{J}} \quad \text{and} \quad [c_{\widetilde{\mathcal{J}}}; d_{\widetilde{\mathcal{J}}}] = [c_{\mathcal{J}}; d_{\mathcal{J}}].$$

By **special duality principle**, we obtain

Proposition 3.3. *D-invatiant subspace $W \subset \mathcal{U}_\Omega(a;b)$ admits weak spectral synthesis if and only if its annihilator submodule $\mathcal{J} = \mathcal{F}(W^0)$ is weakly localisable.*

Proposition 3.3 is the basis of the *dual scheme*: it reduces spectral synthesis problem to the equivalent dual one dealing with the question of local description of submodules of entire functions. This dual scheme goes back to I.F. Krasichkov-Ternovskii and L. Ehrenpreis.

4 Stability, saturation and weak localisability

As it was shown in [18]–[20] due to Krasichkov-Ternovskii, studying of (weak) localisability of submodules is equivalent to exploring their *stability* and *saturation* properties. We use notions and notations from [18]–[20].

From the definition and topological properties of $\mathcal{P}$, it follows that $\mathcal{P}$ is *b-stable*, that is *for any bounded set $B \subset \mathcal{P}$, the set defined by*

$$B' := \left\{ \frac{\varphi}{z-\lambda} : \ \varphi \in B, \ \lambda \in \mathbb{C}, \ \varphi(\lambda) = 0 \right\}$$

is also bounded in $\mathcal{P}$.

Notice that $\mathcal{P}$ is b-stable bornological space. It implies that $\mathcal{P}$ is *pointwise stable*:
for every $\lambda \in \mathbb{C}$ and any neighbourhood of zero $U \subset \mathcal{P}$, there exists a neighbourhood of zero U'_λ such that

$$\varphi \in U'_\lambda, \quad \varphi(\lambda) = 0 \Longrightarrow \frac{\varphi}{z-\lambda} \in U$$

(see [19, § 4]).

Submodule $\mathcal{J} \subset \mathcal{P}$ is *stable at a point* $\lambda \in \mathbb{C}$ if for any $\varphi \in \mathcal{J}$ vanishing at λ with the multiplicity exceeding the multiplicity of λ as a zero of $\mathcal{J}$ implies that $\frac{\varphi}{z-\lambda} \in \mathcal{J}$. Submodule $\mathcal{J}$ is *stable* if it is stable at every point $\lambda \in \mathbb{C}$.

From Propositions 4.2–4.6 in [19] and pointwise stability of $\mathcal{P}$, it follows that stability of $\mathcal{J}$ at one point implies its stability at all points in $\mathbb{C}$.

Because of pointwise stability of $\mathcal{P}$, a weak localizable submodule is necessarily stable. However, in general, the inverse is not true (see [18], [19]).

Recall some notions and facts from these papers that will be required in further presentation. We cite all them for a space of scalar entire functions.

A separable locally convex space $\mathcal{P} \subset H(\mathbb{C})$ is called *b-stable* if for any bounded set $B \subset \mathcal{P}$, the set of all *entire* functions ψ of the form

$$\psi = \frac{\varphi}{z-\lambda}, \ \lambda \in \mathbb{C}, \ \varphi \in B,$$

is contained and bounded in $\mathcal{P}$.

The space $\mathcal{P}$ is *analytically condensed* if for any finite set of functions $\varphi_1, \dots, \varphi_m \in \mathcal{P}$, the set

$$B = \{\psi \in H(\mathbb{C}) : \ |\psi(z)| \le |\varphi_1(z)| + \dots + |\varphi_m(z)|, \ z \in \mathbb{C}\}$$

is contained and bounded in $\mathcal{P}$.

A subset $\mathcal{J} \subset \mathcal{P}$ is *b-saturated with respect to* $\varphi \in \mathcal{P}$ if there exists a bounded set $B \subset \mathcal{P}$ for which the following implication holds: if an entire function ν satisfies the inequality

$$|\nu(z)\psi(z)| \le |\psi(z)| + |\varphi(z)|, \quad z \in \mathbb{C},$$

for every $\psi \in B \bigcap \mathcal{J}$, then $\nu = const$.

A closed subspace $\mathcal{J} \subset \mathcal{P}$ is called a *submodule* in $\mathcal{P}$ if the implication

$$\varphi \in \mathcal{J}, \ p \in \mathbb{C}[z], \ p\varphi \in \mathcal{P} \Longrightarrow p\varphi \in \mathcal{J}$$

holds. Notice that in this definition the space $\mathcal{P}$ must not be a module over $\mathbb{C}[z]$. *Stability* and *zero set* $\mathcal{Z}_\mathcal{J}$ for $\mathcal{J}$ are also defined in the same way in this case.

For bornological b-stable spaces, the following assertion holds.

Theorem C. [18] (**Bornological version of individual theorem.**) *Let $\mathcal{J}$ be a stable submodule in a Hausdorff bornological b-stable space $\mathcal{P}$, $\psi \in \mathcal{P}$ and $\mathcal{Z}_\mathcal{J} \subset \mathcal{Z}_\psi$.*

Then, $\psi \in \mathcal{J}$ if and only if $\mathcal{J}$ be b-saturated with respect to ψ.

Now, we obtain a sufficient condition of b-saturation suitable for applications.

Proposition 4.1. *Let $\mathcal{P}$ be an analytically condensed Hausdorff b-stable locally convex space of entire functions, $\mathcal{J} \subset \mathcal{P}$, $\varphi \in \mathcal{P}$. For a function $\psi \in \mathcal{J}$, we set*

$$B_{\varphi,\psi} := \left\{ \Psi \in \mathcal{P} : \ \frac{\Psi}{\psi} \in H(\mathbb{C}), \ |\Psi(z)| \le |\varphi(z)| + |\psi(z)|, \ z \in \mathbb{C} \right\}. \tag{4.1}$$

If $B_{\varphi,\psi} \subset \mathcal{J}$, then $\mathcal{J}$ is b-saturated with respect to φ.

Proof. Define

$$B = \{\Phi \in H(\mathbb{C}) : \ |\Phi(z)| \le |\varphi(z)| + |\psi(z)|, \ z \in \mathbb{C}\}.$$

This set is bounded in $\mathcal{P}$.

Let ν be an entire function satisfying the inequality

$$|\nu(z)\Phi(z)| \le |\varphi(z)| + |\Phi(z)|, \quad \forall z \in \mathbb{C}, \tag{4.2}$$

with any $\Phi \in B \bigcap \mathcal{J}$.

Setting $\Phi = \psi \in \mathcal{J} \bigcap B$, we get

$$|\nu(z)\psi(z)| \le |\varphi(z)| + |\psi(z)|, \quad z \in \mathbb{C}.$$

In what folows that

$$\Phi_1 = \nu\psi \in \mathcal{J} \bigcap B,$$

and

$$|\nu(z)\Phi_1(z)| \le |\varphi(z)| + |\Phi_1(z)|, \ z \in \mathbb{C}.$$

This leads us to the inequality

$$\left|\frac{1}{2}\nu^2(z)\psi(z)\right| \le |\varphi(z)| + |\psi(z)|, \ z \in \mathbb{C},$$

which means that $\frac{1}{2}\nu^2\psi \in B \bigcap \mathcal{J}$.

Continuing to argue in a similar way, we obtain that

$$\frac{1}{2^{k-1}}\nu^k\psi \in B \bigcap \mathcal{J}, \quad k = 2, 3, \ldots$$

Hence, we have

$$\frac{|\nu(z)|^k}{2^{k+1}} \le \frac{|\varphi(z)|}{|\psi(z)|} + 1, \ z \in \mathbb{C}, \ k = 2, 3, \ldots$$

These inequalities imply that $\nu = \text{const}$. Because ν is an *arbitrary* entire function satisfying (4.2), we conclude that $\mathcal{J}$ b-saturated with respect to φ. □

Remark 4. If it is additionaly known that submodule $\mathcal{J}$ is stable and $\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_{\varphi}$, then the sufficient condition in Proposition 4.1 is also necessary one. Indeed, because of Theorem C, we have $\varphi \in \mathcal{J}$. Setting $\psi = \varphi$, we obtain the required assertion.

Given a function $\varphi \in \mathcal{P}$, we denote by $\mathcal{J}(\varphi)$ the submodule consisting of all functions $\psi \in \mathcal{P}$ of the form $\psi = \omega\varphi$, where ω is an entire function of minimal type with respect to the order 1. Clearly, $\mathcal{J}(\varphi)$ is weakly localisable submodule.

Proposition 4.2. *Let $\mathcal{J} \subset \mathcal{P}$ be a stable submodule.*
If $\varphi \in \mathcal{P}_a$ satisfies the conditions

$$\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_{\varphi}, \quad [h_\varphi(-\pi/2); h_\varphi(\pi/2)] \subset (c_{\mathcal{J}}; d_{\mathcal{J}}),$$

then $\mathcal{J}(\varphi) \subset \mathcal{J}$.

Proof. Consider an arbitrary function $\psi \in \mathcal{J}(\varphi)$. Because $c_{\mathcal{J}} < c_\varphi$ and $d_{\mathcal{J}} > d_\varphi$, taking into account the definitions of $c_{\mathcal{J}}$ and $d_{\mathcal{J}}$ (see (3.4)), we derive that there exist φ_1, $\varphi_2 \in \mathcal{J}$ for which

$$c_{\mathcal{J}} \le c_{\varphi_1} < c_\varphi, \quad d_\varphi < d_{\varphi_2} \le d_{\mathcal{J}}.$$

Set $\varphi_B = \varphi_1 + \varphi_2$. This function has completely regular growth with respect to the order 1. Notice that the indicator diagram of $\psi \in \mathcal{J}(\varphi)$ equals $\mathrm{i}[c_\varphi; d_\varphi]$. Hence, it is a compact subset of the indicaor diagram of φ_B, that implies

$$\frac{\psi(z)}{\varphi_B(z)} \to 0, \quad z = re^{i\theta} \tag{4.3}$$

as $r \to \infty$ outside some subset of $(0; +\infty)$ of zero relative measure.

Moreover, relation (4.3) holds unformly with respect to all

$$\theta \in \{|\pi/2 - \theta| < \delta\} \bigcup \{|-\pi/2 - \theta| < \delta\},$$

where $\delta > 0$ is small enough.

Show that $\mathcal{J}$ is b-saturated with respect to ψ. For this purpose, we set $B = \{\varphi_B\}$ and consider an entire function ρ satisfying

$$|\rho(z)\varphi_B(z)| \le |\psi(z)| + |\varphi_B(z)|, \quad z \in \mathbb{C}. \tag{4.4}$$

By the theorem on summation of indicator functions, we derive that ρ has minimal type with respect to the order 1. Moreover, by the maximum modulus principle, from (4.3) we get that ρ is bounded along the imaginary axis. Hence, $\rho = \mathit{const}$. So, we conclude that the stable submodule $\mathcal{J}$ is b-saturated with respect to ψ. Finally, by the bornologiacal version of individual theorem (Theorem C), we obtain that $\psi \in \mathcal{J}$. □

Now, we can prove the criterion of weak localizability for stable submodules in $\mathcal{P}$.

Theorem 4.1. *A stable submodule $\mathcal{J} \subset \mathcal{P}$ is weakly localizable if and only if there exists $\varphi \in \mathcal{J}$ such that*

$$\mathcal{J}(\varphi) \subset \mathcal{J}.$$

Proof. Clearly, we need to prove only the assertion on sufficiency.

1) First, we assume that $\mathcal{J}(\varphi) \subset \mathcal{J}$ and the indicator diagram of φ equals $\mathrm{i}\,[c_{\mathcal{J}}; d_{\mathcal{J}}]$. Notice that the case when $c_{\mathcal{J}} = d_{\mathcal{J}}$ is also non-trivial.

Let $\psi \in \mathcal{P}$ be such that

$$\mathcal{Z}_\psi \supset \mathcal{Z}_{\mathcal{J}}, \quad \mathrm{i}\,[c_\psi; d_\psi] \subset \mathrm{i}\,[c_{\mathcal{J}}; d_{\mathcal{J}}].$$

For $\mathcal{P}$, all conditions of Proposition 4.1 are satisfied. From $\mathcal{J}(\varphi) \subset \mathcal{J}$ and conditions on the indicator diagrams of φ and ψ, it follows that the set defined by them in (4.1) is a subset of $\mathcal{J}$ (it equals $\mathcal{J}(\varphi)$). By Proposition 4.1, we derive that $\mathcal{J}$ is b-saturated with respect to ψ. By Theorem C, it means that $\psi \in \mathcal{J}$. Because ψ is an arbitrary function, we conclude that $\mathcal{J}$ is weakly localisable.

2) Now, assume that

$$\mathcal{J}(\varphi) \subset \mathcal{J}, \quad [c_\varphi; d_\varphi] \subsetneq [c_{\mathcal{J}}; d_{\mathcal{J}}] \subset (a; b).$$

Then, value of at least one of the expressions

$$\delta_1 := c_\varphi - c_{\mathcal{J}} \quad \text{or} \quad \delta_2 := d_{\mathcal{J}} - d_\varphi$$

is strictly positive. Consider in detail the case, when $\delta_1 > 0$ and $\delta_2 > 0$.

By Proposition 4.2, for all $\delta' \in [0;\delta_1)$ and $\delta'' \in [0;\delta_2)$, we have

$$\mathcal{J}(e^{i\delta' z}\varphi) \subset \mathcal{J}, \quad \mathcal{J}(e^{-i\delta'' z}\varphi) \subset \mathcal{J}.$$

Particularly,

$$e^{i\delta' z}\varphi, \quad e^{-i\delta'' z}\varphi \in \mathcal{J}, \quad \delta' \in [0;\delta_1),\ \delta'' \in [0;\delta_2). \tag{4.5}$$

Set $\Phi = (e^{i\delta_1 z} + e^{-i\delta_2 z})\varphi$. Because the relations

$$\lim_{\delta'\to\delta_1} e^{i\delta' z}\varphi = e^{i\delta_1 z}\varphi, \quad \lim_{\delta''\to\delta_2} e^{-i\delta'' z}\varphi = e^{-i\delta_2 z}\varphi$$

hold with respect to the topology of $\mathcal{P}$, taking into account (4.5), we obtain that $\Phi \in \mathcal{J}$.

Any function $\Psi \in \mathcal{J}(\Phi)$ can be represented as

$$\Psi = \rho\Phi = \rho(e^{i\delta_1 z} + e^{-i\delta_2 z})\varphi,$$

where ρ is an entire function of zero exponential type.

It is not difficult to check that $\rho\varphi \in \mathcal{P}$. By Proposition 4.2, we get

$$\rho\varphi \in \mathcal{J}, \quad \Psi_{\delta'} = e^{i\delta' z}\rho\varphi \in \mathcal{J}, \quad \forall\, \delta' \in (0;\delta_1),$$

$$\Psi_{\delta''} = e^{-i\delta'' z}\rho\varphi \in \mathcal{J}, \quad \forall\, \delta'' \in (0;\delta_2).$$

From

$$\Psi = \lim\,(\Psi_{\delta'} + \Psi_{\delta''}) \quad \text{as} \quad \delta' \to \delta_1, \quad \delta'' \to \delta_2,$$

it follows that $\Psi \in \mathcal{J}$. Because Ψ is an arbitrary function in $\mathcal{J}(\Phi)$, the relation $\mathcal{J}(\Phi) \subset \mathcal{J}$ holds.

We have established that our submodule $\mathcal{J}$ contains the submodule $\mathcal{J}(\Phi)$ generated by the function Φ which indicator diagram equals $\mathrm{i}[c_{\mathcal{J}};d_{\mathcal{J}}]$. Together with the first part of the proof, this leads us to the conclusion that $\mathcal{J}$ is a weakly localizable submodule.

3) It remains to consider the case, in which $c_{\mathcal{J}} = a$ or (and) $d_{\mathcal{J}} = b$.

Let $\Psi \in \mathcal{P}_a$ and $\mathrm{i}[c_\Psi;d_\Psi] \subset \mathrm{i}[c_{\mathcal{J}};d_{\mathcal{J}}]$, $\mathcal{Z}_\Psi \supset \mathcal{Z}_{\mathcal{J}}$. To check that $\Psi \in \mathcal{J}$ we fix a segment $[c';d']$ satisfying the relations

$$[c';d'] \subset (a;b)\bigcap[c_{\mathcal{J}};d_{\mathcal{J}}], \quad [c_\Psi;d_\Psi] \subset [c';d'], \quad [c_\varphi;d_\varphi] \subset [c';d']. \tag{4.6}$$

Denote by $\mathcal{J}'$ a weakly localizable submodule with the indicator segment $[c';d']$ and $\mathcal{Z}_{\mathcal{J}'} = \mathcal{Z}_{\mathcal{J}}$. It is easy to see that $\widetilde{\mathcal{J}} = \mathcal{J}\bigcap\mathcal{J}'$ is a closed stable submodule with the indicator segment $[c';d']$ and $\mathcal{Z}_{\widetilde{\mathcal{J}}} = \mathcal{Z}_{\mathcal{J}}$.

By (4.6) it follows that $\mathcal{J}(\varphi) \subset \widetilde{\mathcal{J}}$. Further, by two previous parts of the proof, we get $\widetilde{\mathcal{J}} = \mathcal{J}'$. Taking into account (4.6) one more time, we obtain that

$$\Psi \in \widetilde{\mathcal{J}} \subset \mathcal{J}.$$

□

By the Beurling-Malliavin radius of completeness and multiplier theorems (see, e.g., [13, X-XI]), we derive

Proposition 4.3. *Submodule $\mathcal{J} \subset \mathcal{P}$ contains non-zero functions if and only if the relation*

$$d_{\mathcal{J}} - c_{\mathcal{J}} \geq 2(\mathcal{Z}_{\mathcal{J}}) \tag{4.7}$$

holds.

By Proposition 4.3, we see that the weak localizability problem is non-trivial only for submodules satisfying (4.7). It turns out that there may be two essentially different cases:

$$d_{\mathcal{J}} - c_{\mathcal{J}} = 2r(\mathcal{Z}_{\mathcal{J}})$$

and

$$d_{\mathcal{J}} - c_{\mathcal{J}} > 2r(\mathcal{Z}_{\mathcal{J}}). \tag{4.8}$$

In the first case, there exist stable submodules, that are not weakly localizable. There also exist weakly localizable ones (cf. [4], [6]). And vice versa, any stable submodule satisfying (4.8) is weakly localizable.

Theorem 4.2. *Let $\mathcal{J} \subset \mathcal{P}$ be a stable submodule.*
If

$$d_{\mathcal{J}} - c_{\mathcal{J}} > 2r(\mathcal{Z}_{\mathcal{J}}) \tag{4.9}$$

then $\mathcal{J}$ is non-trivial and weakly localizable.

Proof. By the Beurling-Malliavin theorems and Theorem A, taking into account the properties of weights in Ω, we obtain that there exists non-zero function $\varphi_0 \in \mathcal{P}$ such that

$$\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_{\varphi_0}, \quad [h_{\varphi_0}(-\pi/2); h_{\varphi_0}(\pi/2)] \subset (c_{\mathcal{J}}; d_{\mathcal{J}}).$$

According to Proposition 4.2, the inclusion $\mathcal{J}(\varphi_0) \subset \mathcal{J}$ holds. By Theorem 4.1, we get the required assertion.

□

Corollary 4.1. *Let $\mathcal{J} \subset \mathcal{P}$ be a stable submodule and its indicator segment be not compact in $(a; b)$. Then, $\mathcal{J}$ is a weakly localizable submodule. In particular, the stable submodule $\mathcal{J} \subset \mathcal{P}_a$ is localizable if and only if*

$$c_{\mathcal{J}} = -a, \quad d_{\mathcal{J}} = a.$$

Remark 5. Notice that we work with dual scheme using two famous Beurling-Malliavin theorems. This is one more example of applying them for solving problems which concern with completeness of exponential systems and exponential bases (cf. [3], [4], [7], [9],[10], [12]).

5 Solving weak spectral synthesis problem in $\mathcal{U}_\Omega(a; b)$

5.1 Spectrum of D-invariant subspace

From the previous section, taking into account Proposition 2.1 we see that a necessary condition of weak spectral synthesis for a D-invariant subspace $W \subset \mathcal{U}_\Omega(a; b)$ is the property of stability of its annihilator submodule. It follows that we need to know an equivalent dual requirement for the subspace itself.

The first step on this way belongs to A. Aleman and B. Korenblum. In [9], the notion of *spectrum* of D-invariant subspace in $C^\infty(a; b)$ was introduced. Namely, the spectrum σ_W was defined as a complement of $\mathbb{C}$ to the set of all *regular points* of the restricted operator $D : W \to W$. Here, $\mu \in \mathbb{C}$ is *regular* if $(D - \mu\,\mathbf{id})$ is a bijective map in W. For any regular point $\mu \in \mathbb{C}$, there exists a linear and continuous inverse operator

$$(D - \mu\,\mathbf{id})^{-1} : W \to W.$$

A. Aleman and B. Korenblum proved the following two assertions (see [9, Theorem 2.1, Proposition 3.1]):
1) the spectrum of D-invariant subspace $W \subset C^\infty(a;b)$ is either equal to the whole complex plane, or equal to a finite or denumerable (may be, empty) set of multiple points in $\mathbb{C}$ with the unique possible limit point at infinity;
2) the relation $\sigma_W \neq \mathbb{C}$ implies that $\mathcal{J} = \mathcal{F}(W^0)$ is stable at any point $\lambda \notin \mathrm{i}\sigma_W$ (hence, as we have noticed above, the annihilator submodule is stable).

We should mention that the initial form of the second assertion in [9] is a different one, because its authors used other techniques, not the dual scheme.

Our purposes are to establish the same assertion like the first cited one for D-invariant subspaces $W \subset \mathcal{U}_\Omega(a;b)$ and to prove that σ_W is discrete if and only if the corresponding annihilator submodule $\mathcal{J} = \mathcal{F}(W^0)$ is stable.

Proposition 5.1. *Let $W \subset \mathcal{U}_\Omega(a;b)$ be a D-invariant subspace, $\mathcal{J}$ be its annihilator submodule.*

A point $\mu \in \mathbb{C}$ is regular for the restricted operator $D : W \to W$ if and only if both following conditions hold: 1) $\mathrm{i}\mu \notin \mathcal{Z}_\mathcal{J}$; 2) *submodule $\mathcal{J}$ is stable at $\lambda = \mathrm{i}\mu$.*

Proof. Necessity. 1) Because $\mu \notin \sigma_W$ implies that $e^{\mu t} \notin W$, according to the duality principle, we get $\mathrm{i}\mu \notin \mathcal{Z}_\mathcal{J}$.
2) Without loss of generality, assume that $\mu = 0$.

Let $\varphi \in \mathcal{J}$ be such that $\varphi(0) = 0$, and set

$$S = \mathcal{F}^{-1}(\varphi), \quad \widetilde{S} = \mathrm{i}\mathcal{F}^{-1}\Big(\frac{\varphi}{z}\Big).$$

Denote by D^* a generalized differentiation operator acting in $\mathcal{U}'_\Omega(a;b)$. This is an adjoint operator to D. It is not difficult to check that

$$\mathcal{F}(D^*(\widetilde{S})) = \varphi.$$

This is equivalent to the relation

$$D^*(\widetilde{S}) = S.$$

For any $f \in W$, there exists $g \in W$ such that $Dg = f$. Therefore, it follows that

$$\widetilde{S}(f) = \widetilde{S}(Dg) = D^*(\widetilde{S})(g) = S(g) = 0.$$

Hence, we conclude that

$$\widetilde{S} \in W^0, \qquad \frac{\varphi}{z} \in \mathcal{J}.$$

Sufficiency. Without loss of generality, we assume that $\mu = 0$.

Let A be an inverse-shift operator acting in $\mathcal{P}$, that is

$$A(\psi)(z) = \frac{\psi(z) - \psi(0)}{z}.$$

The space $\mathcal{U}_\Omega(a;b)$ may be considered as strong dual space to $\mathcal{P}'$. Then, we see that the "lifting" $\widehat{A}$ of A^* acts in $\mathcal{U}_\Omega(a;b)$ and satisfies the relation

$$D\widehat{A}(f) = -\mathrm{i}f, \quad f \in \mathcal{U}_\Omega(a;b). \tag{5.1}$$

Similarly, for the "lifting" $\widehat{D}$ of D^*, we have

$$A\widehat{D}(\varphi) = -\mathrm{i}\varphi, \tag{5.2}$$

and $\widehat{D}$ acts in $\mathcal{P}$.

Let $\mathcal{J}$ be a stable submodule, $0 \notin \mathcal{Z}_{\mathcal{J}}$.

First, we consider the case $\mathcal{J} = \mathcal{J}_\varphi$, $\varphi(0) = 1$. Setting $S = \mathcal{F}^{-1}(\varphi)$, we write

$$W = W_S = \{f \in \mathcal{E}_a : \ S(D^k f) = 0, \ k = 0, 1, 2, \dots\}. \tag{5.3}$$

For any $g \in W_S$, set

$$f = \mathrm{i}\widehat{A}(g) - S(\mathrm{i}\widehat{A}(g)). \tag{5.4}$$

Clearly, $S(f) = 0$. Because of (5.1), we have

$$Df = g,$$

Hence,

$$S(D^k f) = 0, \ k = 0, 1, 2, \dots,$$

and $f \in W_S$.

We have shown that $D : W_S \to W_S$ is a surjective operator. Further, because of $\varphi(0) = 1$, the only solution of equation $Df = 0$ in W_S is zero. It follows that $D : W_S \to W_S$ is a bijection.

Now, consider an arbitrary D-invariant subspace W. Let $\mathcal{J}$ be its annihilator submodule. There exists $\varphi \in \mathcal{J}$ such that $\varphi(0) = 1$. For an arbitrary $\psi \in \mathcal{J}$, we have

$$\psi = z\frac{\psi - \psi(0)\varphi}{z} + \psi(0)\varphi.$$

This relation and the stability of $\mathcal{J}$ imply that

$$\mathcal{J} = z\mathcal{J} + \mathcal{J}_\varphi.$$

By the duality principle, $W = W_1 \bigcap W_S$, where W_1 is the D-invariant subspace whose annihilator submodule equals $z\mathcal{J}$, and W_S is defined by formula (5.3) for $S = \mathcal{F}^{-1}(\varphi)$.

For any $g \in W$, we define function f by formula (5.4). Then, as above, we have $f \in W_S$. Taking into account (5.1) and (5.2), we also get $f \in W_1$. Finally, we obtain $f \in W$ and conclude that $D : W \to W$ is a surjection. Clearly, this operator is also injective. □

Corollary 5.1. *For the spectrum of a D-invariant subspace $W \subset \mathcal{E}_a$ we have either $\sigma_W = -\mathrm{i}\mathcal{Z}_{\mathcal{J}}$, where $\mathcal{J} = \mathcal{F}(W^0)$ if the annihilator submodule is stable,*
or $\sigma_W = \mathbb{C}$ if the annihilator submodule is not stable.

Now we are ready to prove all new propositions formulated in this paper.

Assertions of Proposition 2.2 follow from Proposition 5.1 and Proposition 4.3 by the duality argument.

Theorem 2.1 and Corollary 2.1 are dual propositions to Theorem 4.2 and Corollary 4.1, respectively.

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OPTIMAL CUBATURE FORMULAS FOR MORREY TYPE FUNCTION CLASSES ON MULTIDIMENSIONAL TORUS

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Abstract. In the paper, we establish estimates, sharp in order, for the error of optimal cubature formulas for the smoothness spaces $B^{s\,\tau}_{p\,q}(\mathbb{T}^m)$ of Nikol'skii – Besov type and $F^{s\,\tau}_{p\,q}(\mathbb{T}^m)$ of Lizorkin – Triebel type, both related to Morrey spaces, on multidimensional torus, for some range of the parameters s, p, q, τ ($0 < s < \infty$, $1 \le p$, $q \le \infty$, $0 \le \tau \le 1/p$). In particular, we obtain those estimates for the isotropic Lizorkin – Triebel function spaces $F^{s}_{\infty\,q}(\mathbb{T}^m)$.

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1 Introduction

Let Ω be a compact set in $\mathbb{R}^m$ ($m \ge 2$) (with nonempty interior), F a set (class) of complex–valued continuous functions with domain Ω. In numerical integration, for the approximation of the integral

$$\int_\Omega f(x)dx, \quad f \in \mathrm{F},$$

expressions of the form (cubature formulas)

$$\mathscr{Q}(f, C_N, \Lambda_N) := \sum_{k=1}^{N} c(k) f(\lambda(k)), \tag{1.1}$$

are used; here $C_N := (c(1), \dots, c(N)) \in \mathbb{C}^N$ is the collection of weights and $\Lambda_N := (\lambda(1), \dots, \lambda(N)) \subset \Omega^N$ is the grid of nodes of the cubature formula, and

$$\mathscr{R}(f, \Omega, C_N, \Lambda_N) := \int_\Omega f(x)dx - \mathscr{Q}(f, C_N, \Lambda_N)$$

is its error on a function f. Denote

$$\mathscr{R}(\mathrm{F}, \Omega, C_N, \Lambda_N) := \sup\{|\mathscr{R}(f, \Omega, C_N, \Lambda_N)| \ \mid \ f \in \mathrm{F}\}.$$

The problem of optimal numerical integration under consideration here consists in determining the exact order (in N) of the quantity

$$\mathscr{R}_N(\mathrm{F}, \Omega) := \inf\{\mathscr{R}(\mathrm{F}, \Omega, C_N, \Lambda_N) \,|\, C_N, \Lambda_N\} \tag{1.2}$$

(which is N-th optimal error of numerical integration on the class F) and constructing a sequence $(C_N^*, \Lambda_N^* \mid N \in \mathbb{N})$ of weights and nodes such that the errors $\mathscr{R}(\mathrm{F}, \Omega, C_N^*, \Lambda_N^*)$ of cubature formulas (1.1) realize the order of optimal error (1.2). Cubature formulas $\mathscr{Q}(f, C_N^*, \Lambda_N^*)$ are called optimal in order.

A lot of works are devoted to the study of different formulations of problems of optimal numerical integration for various classes of smooth functions in several variables, see, for example, monographs [17], [19], [20, Chapter 6] and survey [7, Chapter 8] and the bibliographies therein. Comprehensive survey [7], monograph [20], papers [21], [11], [6], [3] show that the interest to problem of optimal numerical integration we will study here is unabated; a fairly detailed history of the issue and an extensive bibliographies can be found there as well.

In this paper, we give exact (in the sense of the order) estimates for quantity (1.2) in the case in which $\Omega = \mathbb{T}^m$ is the m-dimensional torus, F is the function class $\mathrm{B}_{pq}^{s\,\tau}(\mathbb{T}^m)$ of Nikol'skii – Besov type or $\mathrm{L}_{pq}^{s\,\tau}(\mathbb{T}^m)$ of Lizorkin – Triebel type, for some range of the parameters of these classes.

Let us introduce the notation that we will use throughout this article. Let $m \in \mathbb{N}$, $m \geq 2$, $\mathrm{z}_m = \{1, \ldots, k\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, +\infty)$. For $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, put $xy = x_1y_1 + \ldots + x_my_m$, $|x| = |x_1| + \ldots + |x_m|$, $|x|_\infty = \max(|x_\mu| : \mu \in \mathrm{z}_m)$; $x \leq y$ $(x < y) \Leftrightarrow x_\mu \leq y_\mu$ $(x_\mu < y_\mu)$ for all $\mu \in \mathrm{z}_m$. For $t \in \mathbb{R}$, $t_+ := \max\{0, t\}$.

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^m)$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^m)$ be the Schwartz spaces of test functions and tempered distributions, respectively; $\widehat{f} \equiv \mathcal{F}_m(f)$ and $\mathcal{F}_m^{-1}(f)$ direct and inverse Fourier transforms of $f \in \mathcal{S}'(\mathbb{R}^m)$; in particular, for $\varphi \in \mathcal{S}$,

$$\widehat{\varphi}(\xi) = \mathcal{F}_m(\varphi)(\xi) = \int_{\mathbb{R}^m} \varphi(x) e^{-2\pi i\,\xi x} dx, \quad \mathcal{F}_m^{-1}(\varphi)(\xi) = \int_{\mathbb{R}^m} \varphi(x) e^{2\pi i\,\xi x} dx, \quad \xi \in \mathbb{R}^m,$$

where $\xi x = \xi_1 x_1 + \ldots = \xi_m x_m$.

Let $\mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$ be the m-dimensional torus; sometimes it will be convenient for us to identify $\mathbb{T}^m$ with the cube $Q_0 := [0,1)^m$ in $\mathbb{R}^m$. Further, we denote by $\widetilde{\mathcal{S}}' \equiv \mathcal{S}'(\mathbb{T}^m)$ the space of all distributions f from $\mathcal{S}'$ which are 1-periodic in each variable (i.e. such that $\langle f, \varphi(\cdot + \xi)\rangle = \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{S}$ and any $\xi \in \mathbb{Z}^m$) and by $\widetilde{\mathcal{S}} := \widetilde{\mathcal{S}} := \mathcal{S}(\mathbb{T}^m)$ the space of all infinitely continuously differentiable functions on $\mathbb{T}^m$ endowed with the topology of uniform convergence of all derivatives over $\mathbb{T}^m$. Then the space $\mathcal{S}'(\mathbb{T}^m)$ is naturally identified with the space that is topologically dual to $\mathcal{S}(\mathbb{T}^m)$. It is well known that $f \in \widetilde{\mathcal{S}}'$ if and only if $\operatorname{supp} \widehat{f} \subset \mathbb{Z}^m$, i.e. distribution $\widehat{f}$ vanishes on the open set $\mathbb{R}^m \backslash \mathbb{Z}^m$.

For $0 < p \leq \infty$ and a measurable set $G \subset \mathbb{R}^m$, as usual, let $L_p(G)$ be the space of measurable functions $f : G \to \mathbb{C}$, which are Lebesgue integrable in p-th power (when $p = \infty$ essentially bounded) over G, endowed with the standard quasi-norm (norm if $p \geq 1$)

$$\|\, f \mid L_p(G) \,\| = \Big(\int_G |\, f(x) \,|^p dx \Big)^{\frac{1}{p}} \ (p < \infty), \ \|\, f \mid L_\infty(G) \,\| = \operatorname{ess\,sup}(|\, f(x) \,| : x \in G).$$

For $0 < q \leq \infty$, let $\ell_q := \ell_q(\mathbb{N}_0)$ be the space of all (complex) number sequences $(c_j) = (c_j : j \in \mathbb{N}_0)$ with finite standard quasi-norm (norm if $q \geq 1$) $\|(c_j) \mid \ell_q\|$.

Further, let $\ell_q(L_p(G))$ (respectively, $L_p(G; \ell_q)$) be the space of all function sequences $(g_j(x)) = (g_j(x) : k \in \mathbb{N}_0)$ $(x \in G)$ with finite standard quasi-norm (norm if $p, q \geq 1$)

$$\|\, (g_j(x)) \mid \ell_q(L_p(G)) \,\| = \|\, (\, \|\, g_j \mid L_p(G) \|) \mid \ell_q \,\|$$

(respectively,

$$\|\, (g_j(x)) \mid L_p(G; \ell_q) \,\| = \|\, \|\, (g_j(\cdot)) \mid \ell_q \,\| \mid L_p(G) \|).$$

In what follows we will often use the abbreviation $L_p := L_p(\mathbb{R}^m)$, $\widetilde{L}_p := L_p(\mathbb{T}^m)$, $\ell_q(L_p) := \ell_q(L_p(\mathbb{R}^m))$, $\ell_q(\widetilde{L}_p) := \ell_q(L_p(\mathbb{T}^m))$, $L_p(\ell_q) = L_p(\mathbb{R}^m; \ell_q)$, $\widetilde{L}_p(\ell_q) = L_p(\mathbb{T}^m; \ell_q)$.

Let $\mathcal{Q}$ be the set of all half-open dyadic cubes in $\mathbb{R}^m$ of the form

$$Q = Q_{j\xi} = \{\, x \in \mathbb{R}^m \;:\; 2^j x - \xi \in [0,1)^m \,\} \; (j \in \mathbb{Z}, \xi \in \mathbb{Z}^m).$$

For a cube $Q = Q_{j\xi}$, we denote by $x_Q := 2^{-j}\cdot\xi$, $l(Q)(=2^{-j})$, $j(Q) := j$ and $|Q|(=2^{-jm})$ its "lower left" corner, side length, level and volume, respectively.

2 Definition of function spaces $\widetilde{B}^{s\,\tau}_{p\,q}$ and $\widetilde{F}^{s\,\tau}_{p\,q}$

First we choose a test function $\eta_0 \in \mathcal{S}$ such that

$$0 \le \widehat{\eta}_0(\xi) \le 1, \quad \xi \in \mathbb{R}^m; \quad \widehat{\eta}_0(\xi) = 1 \;\text{ if } |\xi|_\infty \le 1; \quad \operatorname{supp}\widehat{\eta}_0 = \{\xi \in \mathbb{R}^m \mid |\xi|_\infty \le 2\}.$$

Put $\widehat{\eta}(\xi) = \widehat{\eta}_0(2^{-1}\xi) - \widehat{\eta}_0(\xi)$, $\;\widehat{\eta}_j(\xi) := \widehat{\eta}_j(\xi) = \widehat{\eta}(2^{1-j}\xi)$, $\;j \in \mathbb{N}$. Then

$$\sum_{j=0}^{\infty} \widehat{\eta}_j(\xi) \equiv 1, \quad \xi \in \mathbb{R}^m,$$

i.e. $\{\widehat{\eta}_j(\xi) \,|\, j \in \mathbb{N}_0\}$ is a resolution of unity (by corridors) on $\mathbb{R}^m$. It is clear that

$$\eta(x) = 2^m\eta_0(2x) - \eta_0(x),\; \eta_j(x) := 2^{(j-1)m}\eta(2^{j-1}x),\; j \in \mathbb{N}. \tag{2.1}$$

Next we denote by Δ^η_j operators on $\mathcal{S}'$ defined as follows: for $f \in \mathcal{S}'$

$$\Delta^\eta_j(f,x) = f * \eta_j(x) = \langle f, \eta_j(x - \cdot)\rangle; \tag{2.2}$$

for the sake of convenience we put $\Delta^\eta_j(f,x) \equiv 0$ if $j < 0$.

We recall the definitions of two scales of the (inhomogeneous) smoothness spaces (on the whole Euclidean space) related to Morrey spaces.

Definition 1. Let $s, \tau \in \mathbb{R}$, $0 < p, q \le \infty$. Then

I. the Nikol'skii – Besov type space $B^{s\,\tau}_{p\,q} := B^{s\,\tau}_{p\,q}(\mathbb{R}^m)$ consists of all distributions $f \in \mathcal{S}'$, for which the quasi-norm

$$\|\, f \,|\, B^{s\,\tau}_{p\,q} \,\| \;= \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \|(2^{sj}\Delta^\eta_j(f,x)\mathrm{sign}((j+1-j(Q))_+)) \,|\, \ell_q(L_p(Q))\|$$

is finite,

II. the Lizorkin – Triebel type space $F^{s\,\tau}_{p\,q} := F^{s\,\tau}_{p\,q}(\mathbb{R}^m)$ $(p < \infty)$ consists of all distributions $f \in \mathcal{S}'$, for which the quasi-norm

$$\|\, f \,|\, F^{s\,\tau}_{p\,q} \,\| \;= \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \|(2^{sj}\Delta^\eta_j(f,x)\mathrm{sign}((j+1-j(Q))_+)) \,|\, L_p(Q;\ell_q)\|$$

is finite.

Remark 1. The inhomogeneous spaces $B^{s\,\tau}_{p\,q}$ and $F^{s\,\tau}_{p\,q}$ are introduced in [24] and thoroughly studied in [24], [15], [16], [22], [23]. We also note that (local) Morrey spaces and Nikol'skii – Besov – Morrey and Lizorkin – Triebel – Morrey spaces have been attracted a lot of attention, see, for instance, [24], [15], [16], [22], [23], [10], [9], [14] and bibliographies therein.

Let $g : \mathbb{R}^m \to \mathbb{C}$ be an arbitrary function, its periodization $\widetilde{g} : \mathbb{T}^m \to \mathbb{C}$ is defined as the (formal) sum of the series $\sum\limits_{\xi \in \mathbb{Z}^m} g(x+\xi)$.

By the Poisson summation formula (see, for example, [18, Chapter VII, Theorem 2.4]) it is easy to see that if $\varphi \in \mathcal{S}$ then $\widetilde{\varphi} \in \widetilde{\mathcal{S}}$, and, moreover, $\widetilde{\varphi}(x) = \sum_{\xi \in \mathbb{Z}^m} \widehat{\varphi}(\xi) e^{2\pi i \xi x}$.

Let

$$\widetilde{\mathcal{Q}} = \{Q \in \mathcal{Q} \,|\, Q \subset Q_0 = [0,1)^m\} = \{Q_{j\xi} \,|\, j \in \mathbb{N}_0,\, \xi \in \mathbb{Z}^m : \mathbf{0} \le \xi < 2^j \mathbf{1}\} \quad (\mathbf{0}, \mathbf{1} \in \mathbb{R}^m).$$

Next we denote by $\widetilde{\Delta}_j^\eta$ the operators defined on $\widetilde{\mathcal{S}}'$ $(j \in \mathbb{N}_0)$, as follows: for $f \in \widetilde{\mathcal{S}}'$

$$\widetilde{\Delta}_j^\eta(f,x) = f * \widetilde{\eta}_j(x) = \langle f, \widetilde{\eta}_j(x-\cdot)\rangle = \sum_{\xi \in \mathbb{Z}^m} \widehat{\eta}_j(\xi)\widehat{f}(\xi) e^{2\pi i\, \xi x}. \tag{2.3}$$

Again, for the sake of convenience we put $\widetilde{\Delta}_j^\eta(f,x) \equiv 0$ if $j < 0$.

In next definition we introduce two scales of the smoothness spaces (over $m-$dimensional torus) related to Morrey spaces.

Definition 2. $s, \tau \in \mathbb{R}$, $0 < p, q \le \infty$. Then

I. the Nikol'skii – Besov type space $\widetilde{B}_{pq}^{s\,\tau} := B_{pq}^{s\,\tau}(\mathbb{T}^m)$ consists of all distributions $f \in \widetilde{\mathcal{S}}'$, for which the quasi-norm

$$\|\, f \,|\, \widetilde{B}_{pq}^{s\,\tau} \,\| \;= \sup_{Q \in \widetilde{\mathcal{Q}}} \frac{1}{|Q|^\tau} \|(2^{sj}\widetilde{\Delta}_j^\eta(f,x)\mathrm{sign}((j+1-j(Q))_+)) \,|\, \ell_q(L_p(Q))\|$$

is finite,

II. the Lizorkin – Triebel type space $\widetilde{F}_{pq}^{s\,\tau} := F_{pq}^{s\,\tau}(\mathbb{T}^m)$ $(p < \infty)$ consists of all distributions $f \in \widetilde{\mathcal{S}}'$, for which the quasi-norm

$$\|\, f \,|\, \widetilde{F}_{pq}^{s\,\tau} \,\| \;= \sup_{Q \in \widetilde{\mathcal{Q}}} \frac{1}{|Q|^\tau} \|(2^{sj}\widetilde{\Delta}_j^\eta(f,x)\mathrm{sign}((j+1-j(Q))_+)) \,|\, L_p(Q;\ell_q)\|$$

is finite.

We will call the unit balls $\widetilde{\mathrm{B}}_{pq}^{s\,\tau} := \mathrm{B}_{pq}^{s\,\tau}(\mathbb{T}^m)$ and $\widetilde{\mathrm{F}}_{pq}^{s\,\tau} := \mathrm{F}_{pq}^{s\,\tau}(\mathbb{T}^m)$ of those spaces the Nikol'skii-Besov and Lizorkin-Triebel classes, respectively.

Remark 2. Evidently the spaces $\widetilde{B}_{pq}^{s\,0}$ and $\widetilde{F}_{pq}^{s\,0}$ coincide with the well–known isotropic periodic Nikol'skii-Besov spaces $\widetilde{B}_{pq}^{s}$ and Lizorkin-Triebel spaces $\widetilde{F}_{pq}^{s}$ respectively (see, for instance, [13]). Furthermore, it is not hard to see that for any $\tau \le 0$ $\widetilde{B}_{pq}^{s\,\tau} = \widetilde{B}_{pq}^{s}$ and $F_{pq}^{s\,\tau} = \widetilde{F}_{pq}^{s}$ in the sense of equivalent quasi-norms unlike the spaces $B_{pq}^{s\,\tau}$ and $F_{pq}^{s\,\tau}$: as well known, $B_{pq}^{s\,\tau} = \{0\}$ and $F_{pq}^{s\,\tau} = \{0\}$ when $\tau < 0$ (see [24, Chapter 2]).

We note that periodic Morrey spaces and Nikol'skii – Besov – Morrey and Lizorkin – Triebel – Morrey spaces (over $\mathbb{T}^m$) have been attracted increasing attention as well, see, for instance, [1], [12], [5] and bibliographies therein.

We will need φ – transform characterization for the spaces $\widetilde{B}_{pq}^{s\,\tau}$ and $\widetilde{F}_{pq}^{s\,\tau}$.

We choose test functions $\phi_0, \phi \in \mathcal{S}$ satisfying the following conditions :

$$\mathrm{supp}\,\widehat{\phi}_0 \subset \{\xi \,:\, |\xi|_\infty \le 2\}, \quad \mathrm{supp}\,\widehat{\phi} \subset \{\xi \,:\, 1/2 \le |\xi|_\infty \le 2\}, \tag{2.4}$$

$$|\widehat{\phi}_0(\xi)| \ge c > 0 \text{ when } |\xi|_\infty \le \frac{5}{3}, \quad |\widehat{\phi}(\xi)| \ge c > 0 \text{ when } \frac{3}{5} \le |\xi|_\infty \le \frac{5}{3}. \tag{2.5}$$

Next we choose test functions $\psi_0, \psi \in \mathcal{S}$ satisfying conditions (2.4), (2.5) (with ψ instead of ϕ) and such that

$$\widehat{\breve{\varphi}}_0(\xi)\widehat{\psi}_0(\xi) + \sum_{j\in\mathbb{N}} \widehat{\breve{\phi}}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^m \tag{2.6}$$

($\breve{g}(x) \equiv \bar{g}(-x)$, $\bar{z}$ is the number complex conjugate to $z \in \mathbb{C}$). For $Q = Q_{j\lambda} \in \widetilde{\mathcal{Q}}$, we set (functions $\widetilde{\phi}_j$ are defined via (2.1) and the periodization)

$$\widetilde{\phi}_Q(x) \equiv |Q|^{1/2}\widetilde{\phi}_{j(Q)}(x - x_Q) = 2^{-jm/2}\widetilde{\phi}_j(x - 2^{-j}\lambda),$$

functions $\widetilde{\psi}_Q$ are defined analogously. Then in view of (2.6) it is not hard to show that for any $f \in \widetilde{\mathcal{S}}'$ we have the following decomposition (the convergence in the sense of $\widetilde{\mathcal{S}}'$)

$$f = \sum_{Q\in\widetilde{\mathcal{Q}}} \langle f, \widetilde{\phi}_Q\rangle\, \widetilde{\psi}_Q = \sum_{j\in\mathbb{N}_0}\sum_{j_Q=j} \langle f, \widetilde{\phi}_Q\rangle\, \widetilde{\psi}_Q. \tag{2.7}$$

Let us introduce (direct) φ – transform $\widetilde{S}_\varphi$ on $\widetilde{\mathcal{S}}'$ as follows

$$\widetilde{S}_\varphi \ : \ \widetilde{\mathcal{S}}' \ni f \mapsto \widetilde{S}_\varphi(f) \equiv (\langle f, \widetilde{\phi}_Q\rangle \,|\, Q \in \widetilde{\mathcal{Q}}),$$

and φ – transform $\widetilde{T}_\psi$ (formal left inverse to $\widetilde{S}_\varphi$) as follows

$$\widetilde{T}_\psi \ : \ (c_Q) = (c_Q \,|\, Q \in \widetilde{\mathcal{Q}}) \mapsto \widetilde{T}_\psi((c_Q)) = \sum_{Q\in\widetilde{\mathcal{Q}}} c_Q\, \widetilde{\psi}_Q.$$

Equality (2.7) means that the composition $\widetilde{T}_\psi \circ \widetilde{S}_\varphi$ is the identity on $\widetilde{\mathrm{S}}$.

Definition 3. Let $0 < p, q \le \infty; s, \tau \in \mathbb{R}$. A number sequence $(c_Q) = (c_Q \,|\, Q \in \widetilde{\mathcal{Q}})$ belongs to the space $\widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}$, if $\|(c_Q) \,|\, \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}\| < \infty$, where $\mathtt{A} \in \{\mathtt{B}, \mathtt{F}\}$ and

$$\|(c_Q) \,|\, \widetilde{\mathtt{B}}^{s\,\tau}_{p\,q}\| = \sup_{P\in\widetilde{\mathcal{Q}}} \frac{1}{|P|^\tau}\Big\{ \sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})q}\Big[\sum_{Q\in\widetilde{\mathcal{Q}}:Q\subset P, j(Q)=j} |c_Q|^p\Big]^{q/p}\Big\}^{1/q},$$

$$\|(c_Q) \,|\, \widetilde{\mathtt{F}}^{s\,\tau}_{p\,q}\| = \sup_{P\in\widetilde{\mathcal{Q}}} \frac{1}{|P|^\tau}\Big\{ \int_P \Big[\sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2})q} \sum_{Q\in\widetilde{\mathcal{Q}}:Q, j(Q)=j} |c_Q|^q \chi_Q(x)\Big]^{p/q}\Big\}^{1/p} \quad (p < \infty).$$

(natural modification if $p = \infty$ and/or $q = \infty$)

(Here χ_Q is the characteristic function of Q.)

Theorem 2.1. *Let $(A, \mathtt{A}) \in \{(B, \mathtt{B}), (F, \mathtt{F})\}$, $0 < p, q \le \infty$, ($p < \infty$ if $(A, \mathtt{A}) = (F, \mathtt{F})$), $s \in \mathbb{R}, \tau \ge 0$. Then a distribution $f \in \widetilde{\mathcal{S}}'$ belongs to $\widetilde{A}^{s\,\tau}_{p\,q}$, if and only if the sequence $(\langle f, \widetilde{\phi}_Q\rangle \,|\, Q \in \widetilde{\mathcal{Q}})$ belongs to $\widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}$, moreover,*

$$\|\,(\langle f, \widetilde{\phi}_Q\rangle) \,|\, \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}\,\| \ \approx^1 \ \|\, f \,|\, \widetilde{A}^{s\,\tau}_{p\,q}\,\|.$$

Furthermore, the operators $\widetilde{S}_\varphi \ : \ \widetilde{A}^{s\,\tau}_{p\,q} \to \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}$ and $\widetilde{T}_\psi \ : \ \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q} \to \widetilde{A}^{s\,\tau}_{p\,q}$ are bounded and their composition $\widetilde{T}_\psi \circ \widetilde{S}_\varphi$ is the identity on $\widetilde{A}^{s\,\tau}_{p\,q}$.

[1] sign "$\approx$" means that there exist positive constants C_1, C_2 independent of $f \in \widetilde{A}^{s\,\tau}_{p\,q}$ such that $C_1\|\, f \,|\, \widetilde{A}^{s\,\tau}_{p\,q}\,\| \ \le \ \|\,(\langle f, \widetilde{\phi}_Q\rangle) \,|\, \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}\,\| \ \le C_2\|\, f \,|\, \widetilde{A}^{s\,\tau}_{p\,q}\,\|$.

Remark 3. The notion of φ – transform was invented by M. Frazier and B. Jawerth [8]. This theorem is the periodic analogue of Theorem 2.1 in [24] for the spaces $B^{s\,\tau}_{p\,q}$ and $F^{s\,\tau}_{p\,q}$. A special case of Theorem 2.1 for the isotropic spaces $\widetilde{B}^{s}_{p\,q}$ and $\widetilde{F}^{s}_{p\,q}$ was established in [4].

Theorem 2.2. *Let $A \in \{B, F\}$, $0 < p, q \le \infty$, ($p < \infty$ when $A = F$), $s \in \mathbb{R}, \tau \ge 0$. Then we have the following continuous embedding*

$$\widetilde{A}^{s\,\tau}_{p\,q} \hookrightarrow \widetilde{B}^{s+\tau m-\frac{m}{p}}_{\infty\infty}.$$

Moreover, if $\tau > \frac{1}{p}, 0 < q < \infty$ or $\tau \ge \frac{1}{p}, q = \infty$ we have

$$\widetilde{A}^{s\,\tau}_{p\,q} = \widetilde{B}^{s+\tau m-\frac{m}{p}}_{\infty\infty}$$

in the sense of equivalent quasi-norms.

Remark 4. The first statement of this theorem is an analogue of the results on the embedding of the spaces $A^{s\,\tau}_{p\,q}(\mathbb{R}^m)$ into the space $C_{ub}(\mathbb{R}^d)$ of uniformly continuous and bounded functions, see [24, Chapter 2, Section 2.2] and [16, Theorem 4.4]. Second statement is a direct periodic analogue of Theorem 2 in [22]. Note that for $s > 0$ the space $\widetilde{B}^{s}_{\infty\infty}$ coincides with the well-known Zygmund spaces $\mathcal{Z}^s(\mathbb{T}^m)$ (see details in [13, Chapter 3]).

3 Optimal error of numerical integration on classes $\widetilde{\mathrm{B}}^{s\,\tau}_{p\,q}$ and $\widetilde{\mathrm{L}}^{s\,\tau}_{p\,q}$

In this section, we formulate and discuss the main result of the paper on estimates exact in order for optimal errors of numerical integration on the Nikol'skii – Besov and Lizorkin – Triebel classes $\widetilde{\mathrm{B}}^{s\,\tau}_{p\,q} = \mathrm{B}^{s\,\tau}_{p\,q}(\mathbb{T}^m)$ and $\widetilde{\mathrm{F}}^{s\,\tau}_{p\,q} = \mathrm{F}^{s\,\tau}_{p\,q}(\mathbb{T}^m)$ under some condition on parameters s, p, q, τ, m ($s \in \mathbb{R}_+$, $1 \le p, q \le \infty$, $\tau \in [0, 1/p]$).

In what follows, we will use the signs $\ll$ and $\asymp$ of the ordinal inequality and equality: for functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ we write $F(u) \ll H(u)$ as $u \to \infty$, if there exists a constant $C = C(F, H) > 0$ such that the inequality $F(u) \le CH(u)$ holds true for $u \ge u_0 > 0$; $F(u) \asymp H(u)$ if $F(u) \ll H(u)$ and $H(u) \ll F(u)$ simultaneously.

Main result of the paper is the following

Theorem 3.1. *Assume that $\mathrm{A} \in \{\mathrm{B}, \mathrm{F}\}$, $1 \le p, q \le \infty$, $s > 0$, $\tau \ge 0$ ($p < \infty$ if $\mathrm{A} = \mathrm{F}$). Then the relation*

$$\mathscr{R}_N(\widetilde{\mathrm{A}}^{s\,\tau}_{p\,q}) \asymp N^{-\frac{s}{m}-(\tau-\frac{1}{p})_+} \quad \text{as} \quad N \to \infty$$

holds true.

Remark 5. By Theorem 2.2 the hypotheses of Theorem 3.1 guarantee the continuous embedding $\widetilde{A}^{s\,\mathtt{m}}_{p\,q} \hookrightarrow C(\mathbb{T}^m)$, which is required in problems of numerical integration ($A \in \{B, F\}$).

Remark 6. As mentioned in Introduction, there is an extensive literature devoted to optimal cubature formulas for classes of functions of several variables. Here we discuss results directly related to Theorem 3.1, namely, results on function classes on the torus included in the Nikol'skii – Besov and Lizorkin – Triebel scales from Definition 1.

For $s > m/p, 1 \le p \le \infty$, the estimates of $\mathscr{R}_N(\widetilde{\mathrm{F}})$, exact in order, for the isotropic Sobolev and Nikol'skii classes ($\widetilde{\mathrm{F}} = \widetilde{\mathrm{W}}^s_p$ and $\widetilde{\mathrm{F}} = \widetilde{\mathrm{H}}^s_p \equiv \widetilde{\mathrm{B}}^s_{p\infty}$) are given in [20, Chapter 3] (in fact, the anisotropic case is also considered there):

$$\mathscr{R}_N(\widetilde{\mathrm{W}}^s_p) \asymp \mathscr{R}_N(\widetilde{\mathrm{H}}^s_p) \asymp N^{-\frac{s}{m}} \quad \text{as} \quad N \to \infty$$

and the simplest sequence of "parallelepipedal" cubature formulas

$$\mathscr{Q}_N^\star(f) := \sum_{\xi\in\mathbb{Z}^m: 0\le\xi_\mu<M(N), \mu\in z_m} \frac{1}{M(N)^m} f\Big(\frac{\xi}{M(N)}\Big),$$

$$M(N)\in\mathbb{N} : M(N)^m \le N < (M(N)+1)^m,$$

can be taken as optimal one.

We recall that $\widetilde{H}^s_\infty = \mathcal{Z}^s(\mathbb{T}^m)$ and for $1<p<\infty$ we have $\widetilde{W}^s_p = \widetilde{F}^s_{p2}$ in the sense of equivalent norms (see details in [13, Chapter 3]).

Further, in [3] the following sharp estimates are obtained: for $A\in\{B,F\}$, $1\le p,q\le\infty$ ($p<\infty$ if $A=F$), $s>m/p$ if $A=B$ and $s>\max\{m/p, m/q\}$ if $A=F$ we have

$$\mathscr{R}_N(\widetilde{\mathrm{A}}^s_{pq}) \asymp N^{-\frac{s}{m}} \quad \text{as} \quad N\to\infty.$$

In [3], to prove upper estimates, the well-known Frolov's cubature formulas are used because there it is studied general case of the function spaces of product type, in particular, the function spaces with mixed smoothness. But it is easy to see that for isotropic classes $\widetilde{\mathrm{A}}^s_{pq}$ the sequence of "parallelepipedal" cubature formulas $\mathscr{Q}_N^\star(f)$ can be taken as optimal one as well.

Thus, in view of Theorem 2.2 it remains to prove the theorem for the case $0<\tau\le 1/p$.

4 Proof of Theorem 3.1

By Definition 1 it is evident that the quasi-norms of both scales $\widetilde{B}^{s\,\tau}_{pq}$ and $\widetilde{F}^{s\,\tau}_{pq}$ are monotonic with respect to parameter τ: for any $\tau_1<\tau_2$ we have $\|\cdot\mid \widetilde{A}^{s\,\tau_1}_{pq}\| \le \|\cdot\mid\widetilde{A}^{s\,\tau_2}_{pq}\|$. Hence, the elementary embedding $\widetilde{A}^{s\,\tau_2}_{pq} \hookrightarrow \widetilde{A}^{s\,\tau_1}_{pq}$ holds ($A\in\{B,F\}$). From here and Remark 6, it follows that the upper estimates

$$\mathscr{R}_N(\widetilde{\mathrm{A}}^{s\,\tau}_{pq}) \ll \mathscr{R}_N(\widetilde{\mathrm{A}}^s_{pq}) \asymp N^{-\frac{s}{m}} \quad \text{as} \quad N\to\infty$$

hold for any $\tau>0$.

Now we turn to proving the matching lower estimates.

Taking into account the monotonicity of norms $\|\cdot\mid\widetilde{A}^{s\,\tau}_{pq}\|$ (with respect to τ) as well as Jensen's inequality ($\|\cdot\mid\ell_{q_1}\| \ge \|\cdot\mid\ell_{q_2}\|$ if $1\le q_1<q_2\le\infty$), we get the following simple inclusions $\widetilde{\mathrm{B}}^{s\,\tau}_{p\,q} \supset \widetilde{\mathrm{B}}^{s\,\frac{1}{p}}_{p\,1}$ and $\widetilde{\mathrm{F}}^{s\,\tau}_{p\,q} \supset \widetilde{\mathrm{F}}^{s\,\frac{1}{p}}_{p\,1}$ if $1\le q\le\infty$ and $\tau\le 1/p$.

Since the estimates in Theorem 3.1 do not depend on p, q and $\tau\le 1/p$, in view of inclusions mentioned above, it suffices to prove the required lower estimates for the classes $\widetilde{\mathrm{B}}^{s\,\frac{1}{p}}_{p\,1}$ and $\widetilde{\mathrm{F}}^{s\,\frac{1}{p}}_{p\,1}$. Moreover, for $\widetilde{\mathrm{B}}^{s\,\frac{1}{p}}_{p\,1}$, we can restrict ourselves to the case $1\le p<\infty$ because the required estimate for $\widetilde{\mathrm{B}}^{s\,0}_{\infty\,1} \equiv \widetilde{\mathrm{B}}^{s}_{\infty\,1}$ is known (see Remark 6).

To this end, we apply Bakhvalov's method to obtain those lower bounds for optimal error $\mathscr{R}_N(\mathrm{F},\Omega)$. This method was proposed by N.S. Bakhvalov [2]. Its idea is for a given N and any cubature formula (1.1) to construct a "bad" function g_{Λ_N}, $\|g_{\Lambda_N}\mid F\|=1$, vanishing at all nodes, in the form of a sum with positive coefficients of special shifted dilations, a suitable fixed smooth bump function for which

$$\mathscr{R}(g_{\Lambda_N},\Omega,C_N,\Lambda_N) = \int_\Omega g_{\Lambda_N}(x)dx = \|g_{\Lambda_N}\mid\widetilde{L}_1\|$$

has the required order.

To construct those "bad" functions, we will use the so-called atomic decomposition of the spaces $\widetilde{A}^{s\,\tau}_{pq}$.

We need some notions and notation.

For $s,t \in \mathbb{R}$, $0 < p,q \le \infty$, we define the numbers: $\lfloor t \rfloor$ (the integer part of t), $t_* = t - \lfloor t \rfloor$, $p \wedge q = \min\{p,q\}$, $\sigma_p = m(1/p-1)_+$, $\sigma_{pq} = m(1/(p\wedge q)-1)_+$. Further, $\tau_{sp} = 1/p+(1-(\sigma_p+m-s)_*)/m$ if $s \le \sigma_p$, and $\tau_{sp} = 1/p+(s-\sigma_p)/m$ if $s > \sigma_p$, $\tau_{spq} = 1/p+(1-(\sigma_{pq}+m-s)_*)/m$ if $s \le \sigma_{pq}$, and $\tau_{spq} = 1/p+(s-\sigma_{pq})/m$ if $s > \sigma_p$.

Let $Q \in \mathcal{Q}$. A function $a_Q : \mathbb{T}^m \to \mathbb{C}$ is called a smooth atom ("with a support close to Q") if the following conditions are satisfied:

$$\operatorname{supp}(a_Q) \subset \widetilde{3Q},\ |\partial^\alpha a_Q(x)| \le |Q|^{-1/2-|\alpha|/m},\ |\alpha| \le \max\{\lfloor s+\tau m+1 \rfloor, 0\}.$$

(Here $3Q$ is the dilation of Q with the same center, $\widetilde{D}$ is " the periodic continuation" of a set $D \subset Q_0$, i.e.

$$\widetilde{D} = \mathbb{Z}^m + D = \cup_{\xi\in\mathbb{Z}^m}(\xi+D), \quad \xi + D = \{\xi + x \,|\, x \in D\}.)$$

Then we call the sequence $(a_Q|Q \in \mathcal{Q})$ a family of (smooth) atoms for $\widetilde{A}^{s,\tau}_{p,q}$.

Theorem 4.1. *Let* $(A, \mathtt{A}) \in \{(B,\mathtt{B})(F,\mathtt{F})\}$, $s \in \mathbb{R}$, $0 < p,q \le \infty$. *Assume that* $0 \le \tau < \tau_{sp}$ *if* $A = B$ *and* $0 \le \tau < \tau_{spq}$, $p < \infty$ *if* $A = F$. *Then* $f \in \widetilde{A}^{s\,\tau}_{p\,q}$ *if and only if there exist* $(a_Q \,|\, Q \in \widetilde{\mathcal{Q}})$, *a family of atoms for* $\widetilde{A}^{s\,\tau}_{p\,q}$, *and a sequence* $(c_Q \,|\, Q \in \widetilde{\mathcal{Q}}) \in \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}$ *such that*

$$f = \sum_{Q\in\widetilde{\mathcal{Q}}} c_Q\, a_Q \qquad (\text{convergence in } \widetilde{\mathcal{S}}') \tag{4.1}$$

and

$$\| f \,|\, \widetilde{A}^{s\,\tau}_{p\,q}\| \approx \inf \|(c_Q \,|\, Q \in \widetilde{\mathcal{Q}})) \,|\, \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}\|, \tag{4.2}$$

where inf *is taken over all representations* (4.1).

Remark 7. This theorem is a direct periodic analog of Theorem 3.3 from [24] for the spaces $\widetilde{A}^{s\,\tau}_{p\,q}$. Notice that in [3] we use an analog of Theorem 4.1 for product spaces, which includes as special case atomic characterizations for isotropic function spaces $\widetilde{B}^{s\,\tau}_{p\,q}$ and $\widetilde{F}^{s\,\tau}_{p\,q}$ (with the restriction $p < \infty$ in the case of $F-$spaces). Up to now for function spaces $\widetilde{F}^{s}_{\infty\,q}(0 < q < \infty)$, atomic decomposition remained unproven. Theorem 4.1 completes this gap because we have the coincidence $\widetilde{F}^{s}_{\infty\,q} = \widetilde{F}^{s\,1/p}_{p\,q}(0 < p < \infty, 0 < q \le \infty)$ in the sense of equivalent quasi-norms. In non-trivial case $0 < p,q < \infty$, the coincidence $F^{s}_{\infty\,q}(\mathbb{R}^m) = F^{s\,\frac{1}{p}}_{p\,q}(\mathbb{R}^m)$ is shown in [24, Chapter 2], arguing in periodic settings is the same.

Remark 8. Here we recall a very important (correct and constructive) definition of the Lizorkin – Triebel spaces $F^{s}_{\infty\,q}(\mathbb{R}^m)(0 < q < \infty)$ invented by M. Frazier and B. Jawerth [8] : for $s \in \mathbb{R}$, $0 < q \le \infty$, the Lizorkin – Triebel space $F^{s}_{\infty\,q} := F^{s}_{\infty\,q}(\mathbb{R}^m)$ consists of all distributions $f \in \mathcal{S}'$, for which the quasi-norm

$$\| f \,|\, F^{s}_{\infty\,q}\| \;=\; \|\Delta^{\eta}_0(f) \,|\, L_\infty\| + \Big(\sup_{Q\in\mathcal{Q}:j(Q)\ge 1} \frac{1}{|Q|}\int_Q \sum_{j=j(Q)}^{\infty} |2^{sj}\Delta^{\eta}_j(f,x)|^q dx\Big)^{1/q}$$

is finite.

Moreover, in [8] the following quasi-norm

$$\| f \,|\, F^{s}_{\infty\,q}\|_\star \;=\; \Big(\sup_{Q\in\mathcal{Q}:j(Q)\ge 0} \frac{1}{|Q|}\int_Q \sum_{j=j(Q)}^{\infty} |2^{sj}\Delta^{\eta}_j(f,x)|^q dx\Big)^{1/q}$$

is defined which is equivalent to the original one.

In [4], we studied the spaces (there different notation was used) $\widetilde{F}^s_{\infty q}$ which are defined as follows : for $s \in \mathbb{R}, 0 < q < \infty$, the Lizorkin – Triebel space $\widetilde{F}^s_{\infty q} := F^s_{\infty q}(\mathbb{T}^m)$ consists of all distributions $f \in \widetilde{\mathcal{S}}'$, for which the quasi-norm

$$\| f \,|\, \widetilde{F}^s_{\infty q} \| = \Big(\sup_{Q \in \mathcal{Q}: j(Q) \geq 0} \frac{1}{|Q|} \int_Q \sum_{j=j(Q)}^{\infty} |2^{sj} \Delta_j^\eta (f,x)|^q dx \Big)^{1/q}$$

is finite.

Proof of Theorem 3.1. Now we turn directly to constructing the "bad" functions mentioned above.

We pick a function $h \in \mathcal{S}$ such that

$$\mathrm{supp}\,(h) = [0,1]^m, \ \widehat{h}(0) > 0, \ \max\{|\partial^\alpha h(x)| : x \in [0,1]^m, \alpha \leq \lfloor s + \tau m + 1 \rfloor\} = 1.$$

For $Q \in \widetilde{\mathcal{Q}}$, we define

$$h_Q(x) := |Q|^{-1/2} h(2^{j(Q)} \cdot (x - x_Q)) := 2^{j(Q)m/2} h(2^{j(Q)} \cdot (x - x_Q))$$

and their periodizations $\widetilde{h}_Q(x)$. It is clear that the sequence $(\widetilde{h}_Q \,|\, Q \in \widetilde{\mathcal{Q}})$ is a family of atoms for all $\widetilde{A}^{s\,\tau}_{pq}$.

For a sequence $\mathrm{c} := (c_Q \,|\, Q \in \widetilde{\mathcal{Q}})$ (which will be specified later), we consider a function

$$\widetilde{H}_{\mathrm{c}}(x) := \sum_{Q \in \widetilde{\mathcal{Q}}} c_Q \widetilde{h}_Q(x).$$

First we evaluate the integral $\int_{Q_0} \widetilde{H}_{\mathrm{c}}(x) dx$:

$$\int_{Q_0} \widetilde{H}_{\mathrm{c}}(x) dx = \widehat{h}(0) \sum_{Q \in \widetilde{\mathcal{Q}}} c_Q |Q|^{1/2}. \tag{4.3}$$

In view of Theorem 4.1 (see (4.2)) we get the inequality

$$\|\widetilde{H}_{\mathrm{c}} \,|\, \widetilde{A}^{s\,\tau}_{pq}\| \ll \|\mathrm{c} \,|\, \widetilde{\mathtt{A}}^{s\,\tau}_{pq}\|. \tag{4.4}$$

Next we write down the norms $\|\mathrm{c} \,|\, \widetilde{\mathtt{B}}^{s\,\frac{1}{p}}_{p1}\|$ and $\|\mathrm{c} \,|\, \widetilde{\mathtt{F}}^{s\,1}_{11}\|$ (in view of Remark 7 and Theorem 2.1 the last norm is equivalent to $\|\mathrm{c} \,|\, \widetilde{\mathtt{F}}^{s\,\frac{1}{p}}_{p1}\|$) :

$$\|\mathrm{c} \,|\, \widetilde{\mathtt{B}}^{s\,\frac{1}{p}}_{p1}\| = \sup_{P \subset \widetilde{\mathcal{Q}}} \frac{1}{|P|^{1/p}} \sum_{j=j(P)}^{\infty} 2^{j(s + \frac{m}{2} - \frac{m}{p})} \Big(\sum_{Q \subset P: j(Q) = j} |c_Q|^p \Big)^{1/p} =: \sup_{P \subset \widetilde{\mathcal{Q}}} J(P) \tag{4.5}$$

and from the coincidence of the spaces $\widetilde{\mathtt{B}}^{s\,1}_{11}$ and $\widetilde{\mathtt{F}}^{s\,1}_{11}$ and the equality of their norms $\|\cdot \,|\, \widetilde{\mathtt{B}}^{s\,1}_{11}\| = \|\cdot \,|\, \widetilde{\mathtt{F}}^{s\,1}_{11}\|$ we get

$$\|\mathrm{c} \,|\, \widetilde{\mathtt{F}}^{s\,1}_{11}\| = \|\mathrm{c} \,|\, \widetilde{\mathtt{B}}^{s\,1}_{11}\| = \sup_{P \subset \widetilde{\mathcal{Q}}} \frac{1}{|P|} \sum_{j=j(P)}^{\infty} 2^{j(s - \frac{m}{2})} \sum_{Q \subset P: j(Q) = j} |c_Q|. \tag{4.6}$$

Let $N \in \mathbb{N}$ be an arbitrary number and $\mathscr{Q}(\,\cdot\,, C_N, \Lambda_N)$ be an arbitrary cubature formula of form (1.1), $\Lambda_N := (\lambda(1), \ldots, \lambda(N)) \subset \Omega^N$ its grid of nodes. We choose the natural number j_N such that $2^{(j_N - 2)m} \leq N < 2^{(j_N - 1)m}$.

Further, we denote $\widetilde{\mathscr{Q}}^j := \{Q \in \widetilde{\mathscr{Q}} \,|\, j(Q) = j\}$. It is clear that in the collection $\widetilde{\mathscr{Q}}^{j_N}$ consisting of $2^{j_N m}$ cubes there exist at least $2^{(j_N-1)m}$ cubes $Q(1), \dots, Q(2^{(j_N-1)m})$ which are free of nodes belonging to Λ_N. We put $\bar{Q}(\Lambda_N) = Q(1) \cup \dots \cup Q(2^{(j_N-1)m})$.

Now we are in position to define the required sequence of coefficients $\mathrm{c}^\star = (c^\star_Q \,|\, Q \in \widetilde{\mathscr{Q}})$:

$$c^\star_Q = 0 \;\text{ if }\; Q \cap \bar{Q}(\Lambda_N) = \varnothing, \quad c^\star_Q = c_j = 2^{-jt} \;\text{ if }\; Q \in \widetilde{\mathscr{Q}}^j \;\text{ and }\; Q \subset \bar{Q}(\Lambda_N),$$

here the real number $t > s + m/2$ is fixed. Then, it is not hard to verify that for any $\lambda \in \Lambda_N$ we have $\widetilde{H}_{\mathrm{c}^\star}(\lambda) = 0$. Therefore,

$$\mathscr{Q}(\widetilde{H}_{\mathrm{c}^\star}, C_N, \Lambda_N) = 0, \quad \mathscr{R}(\widetilde{H}_{\mathrm{c}^\star}, Q_0, C_N, \Lambda_N) = \int_{Q_0} \widetilde{H}_{\mathrm{c}^\star}(x)dx. \tag{4.7}$$

From (4.5) and the definition of $\mathrm{c}^\star$ it follows that for any P with $j(P) < j_N$

$$J(P) = \frac{1}{|P|^{1/p}} \sum_{j=j_N}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})} c_j \Big(\sum_{Q \subset P \cap \bar{Q}(\Lambda_N): j(Q)=j} 1 \Big)^{1/p} \le$$

$$\le 2^{j(P)\frac{m}{p}} \sum_{j=j_N}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})} c_j 2^{(j-j(P))\frac{m}{p}} = \sum_{j=j_N}^{\infty} 2^{j(s+\frac{m}{2})} c_j \ll 2^{j_N(s+\frac{m}{2}-t)},$$

further, for any P with $j(P) \ge j_N$ such that $P \cap \bar{Q}(\Lambda_N) = \varnothing$ obviously we have $J(P) = 0$. Finally, for any P with $j(P) \ge j_N$ such that $P \subset \bar{Q}(\Lambda_N) = \varnothing$ we get

$$J(P) = \frac{1}{|P|^{1/p}} \sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})} c_j \Big(\sum_{Q \subset P: j(Q)=j} 1 \Big)^{1/p} =$$

$$= 2^{j(P)\frac{m}{p}} \sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})} c_j 2^{(j-j(P))\frac{m}{p}} = \sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2})} c_j \ll 2^{j(P)(s+\frac{m}{2}-t)} \le 2^{j_N(s+\frac{m}{2}-t)},$$

Hence, taking into account (4.4) we obtain

$$\|\widetilde{H}_{\mathrm{c}^\star} \,|\, \widetilde{B}^{s\,\frac{1}{p}}_{p1}\| \ll \|\mathrm{c}^\star \,|\, \widetilde{\mathrm{B}}^{s\,\frac{1}{p}}_{p1}\| \ll 2^{j_N(s+\frac{m}{2}-t)},$$

in particular,

$$\|\widetilde{H}_{\mathrm{c}^\star} \,|\, \widetilde{F}^{s\,1}_{11}\| \ll \|\mathrm{c}^\star \,|\, \widetilde{\mathrm{F}}^{s\,1}_{11}\| \ll 2^{j_N(s+\frac{m}{2}-t)},$$

From (4.3) and the definition of $\mathrm{c}^\star$ it follows that

$$\int_{Q_0} \widetilde{H}_{\mathrm{c}^\star}(x)dx = \widehat{h}(0) \sum_{Q \in \widetilde{\mathscr{Q}}} c^\star_Q |Q|^{1/2} = \widehat{h}(0) 2^{(j_N-1)m} \sum_{Q \in Q(1)} c^\star_Q |Q|^{1/2} = \widehat{h}(0) 2^{(j_N-1)m} \times$$

$$\times \sum_{j=j_N}^{\infty} c_j 2^{-jm/2} \sum_{Q \in Q(1): j(Q)=j} 1 = \widehat{h}(0) 2^{(j_N-1)m} \sum_{j=j_N}^{\infty} c_j 2^{-jm/2} 2^{(j-j_N)m} \asymp 2^{j_N(m/2-t)}.$$

Therefore, for an arbitrary cubature formula $\mathscr{Q}(\,\cdot\,, C_N, \Lambda_N)$ and functions

$$\widetilde{h}^{\mathrm{c}^\star} := \frac{\widetilde{H}_{\mathrm{c}^\star}}{\|\widetilde{H}_{\mathrm{c}^\star} \,|\, \widetilde{B}^{s\,1/p}_{p1}\|} \in \widetilde{\mathrm{B}}^{s\,1/p}_{p1}, \quad \widetilde{g}^{\mathrm{c}^\star} := \frac{\widetilde{H}_{\mathrm{c}^\star}}{\|\widetilde{H}_{\mathrm{c}^\star} \,|\, \widetilde{F}^{s\,1}_{11}\|} \in \widetilde{\mathrm{F}}^{s\,1/p}_{p1}$$

we get

$$\mathscr{R}(\widetilde{\mathrm{B}}^{s\,1/p}_{p1}, C_N, \Lambda_N) \geq \mathscr{R}(\widetilde{h}^{\mathrm{C}^\star}, C_N, \Lambda_N) \gg \int_{Q_0} \widetilde{H}_{\mathrm{C}^\star}(x)dx/\|\mathrm{C}^\star \,|\, \widetilde{\mathtt{B}}^{s\,\frac{1}{p}}_{p1}\| \gg 2^{-sj_N} \asymp N^{-\frac{s}{m}}$$

and

$$\mathscr{R}(\widetilde{\mathrm{F}}^{s\,1/p}_{p1}, C_N, \Lambda_N) \geq \mathscr{R}(\widetilde{g}^{\mathrm{C}^\star}, C_N, \Lambda_N) \gg \int_{Q_0} \widetilde{H}_{\mathrm{C}^\star}(x)dx/\|\mathrm{C}^\star \,|\, \widetilde{\mathtt{F}}^{s\,1}_{11}\| \gg 2^{-sj_N} \asymp N^{-\frac{s}{m}}$$

From the last two inequalities it follows that

$$\mathscr{R}_N(\widetilde{\mathrm{F}}^{s\,1/p}_{p1}) \gg N^{-\frac{s}{m}}, \qquad \mathscr{R}_N(\widetilde{\mathrm{B}}^{s\,1/p}_{p1}) \gg N^{-\frac{s}{m}} \qquad \text{as} \quad N \to \infty.$$

Thus, the required lower estimates

$$\mathscr{R}_N(\widetilde{\mathrm{A}}^{s\,\tau}_{pq}) \gg N^{-\frac{s}{m}} \qquad \text{as} \quad N \to \infty.$$

are established, which completes the proof of Theorem 3.1. □

Remark 9. Here we emphasize the most important special case of Theorem 3.1 ($1 \leq q < \infty$)

$$\mathscr{R}_N(\widetilde{\mathrm{F}}^{s}_{\infty q}) \asymp N^{-\frac{s}{m}} \qquad \text{as} \quad N \to \infty,$$

which completes investigation of optimal numerical integration on isotropic function spaces of both Nikol'skii – Besov and Lizorkin – Triebel scales.

Remark 10. Proofs of Theorem 2.1, Theorem 2.2 and Theorem 4.1 will be published elsewhere.

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WEAK VERSION OF SYMMETRIC SPACE

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Abstract. In this paper, we defined weak versions of symmetric spaces and established Hölder and Chebyshev type inequalities for noncommutative spaces associated with these spaces.

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1 Introduction

In the realm of the classical analysis, the utilization of weak L_p spaces in both harmonic analysis and martingale theory has received significant scholarly attention. These spaces have proven instrumental in various areas such as interpolation theory, rearrangement-invariant function spaces, weighted inequalities, singular integral operators, and beyond, playing pivotal roles in advancing theoretical frameworks and facilitating analytical investigations. For example, using the weak L_p norm, Ledoux and Talagrand [16] conducted an investigation into the integrability properties and tail probability behavior of p-stable random variables. Soria [19] delved into the discussion of weak-type Lorentz space $\Lambda_{p,\infty}(\omega)$ for $0 < p < \infty$. Fefferman and Soria [11] also addressed various properties of the weak Hardy space H_1. Weisz [23, 24] dedicated his studies to the weak atom decompositions of martingales and martingale inequalities within weak Hardy spaces. Furthermore, Cwikel and other scholars extensively examined the dual of weak L_p spaces (cf. [6, 7]).

Liu/Hou/Wang [17] introduced the weak version of Orlicz spaces and proved the Burkholder-Gundy inequalities for martingales in these weak Orlicz spaces. The noncommutative version of the weak Orlicz spaces was investigated in [1] and was utilized in the theory of noncommutative martingales. In [3], Raikhan and the author considered the weak noncommutative Orlicz space cases associated with arbitrary faithful normal locally finite weights on a semi-finite von Neumann algebra $\mathcal{M}$, and characterized the dual spaces of the noncommutative weak Orlicz-Hardy spaces.

Since the weak versions of L_p spaces and Orlicz spaces have opened new research avenues in (noncommutative) harmonic analysis and (noncommutative) martingale theory, we are investigating a weak version of symmetric spaces. We will apply them in the study of (noncommutative) harmonic analysis and (noncommutative) martingale theory. Notice that for a symmetric (quasi-) Banach space E, we define the weak version of E as the usual Marcinkiewicz space M_{φ_E} associated with the fundamental function φ_E of E. In the rearrangement-invariant Banach space case, it is the space $M(E)$ ([4, Definition 2.5.2]).

The purpose of this paper is to investigate a weak version of symmetric spaces and to study some properties of noncommutative spaces associated with the weak version of symmetric spaces.

2 Preliminaries

Let $L_0(0,1)$ be the set of all Lebesgue measurable almost everywhere finite real-valued functions on $(0,1)$. For $f \in L_0(0,1)$ we define the distribution function $\lambda(f)$ of f by

$$\lambda_s(f) = m(\{\omega \in (0,1) : |f(\omega)| > s\}), \quad s > 0$$

and its decreasing rearrangement $\mu(f)$ by

$$\mu_t(f) = \inf\{s > 0 \ : \ \lambda_s(f) \le t\}, \quad t > 0.$$

If $f, g \in L_0(0,1)$ and

$$\int_0^t \mu_s(f) ds \le \int_0^t \mu_s(g) ds, \qquad \text{for all} \quad t > 0,$$

we say f is *majorized* by g, and write $f \preccurlyeq g$.

If E is a (quasi-)Banach lattice of measurable functions on $(0,1)$ (with the Lebesgue measure) and satisfies the following properties:

if $f \in E$, $g \in L_0(0,1)$ and $\mu(g) \le \mu(f)$ implies that $g \in E$ and $\|g\|_E \le \|f\|_E$,

then E is called a symmetric (quasi-)Banach space on $(0,1)$. E is called fully symmetric if, in addition,

for $x \in L_0(I)$ and $y \in E$ with $x \preceq y$ it follows that $x \in E$ and $\|x\|_E \le \|y\|_E$.

For $0 < p < \infty$, $E^{(p)}$ will denote the quasi-Banach lattice defined by

$$E^{(p)} = \{f : \ |f|^p \in E\},$$

equipped with the quasi-norm

$$\|f\|_{E^{(p)}} = \||f|^p\|_E^{\frac{1}{p}}.$$

Observe that, if $0 < p,\ q < \infty$, then $(E^{(p)})^{(q)} = E^{(pq)}$. It is to be noted that, if E is a Banach space and $p > 1$, then the space $E^{(p)}$ is a Banach space and is usually called the p-convexification of E.

Let $0 < \alpha, \beta < \infty$. If there a constant $C > 0$ such that for all finite sequences $(f_n)_{n\ge1}$ in E

$$\|(\sum |f_n|^\alpha)^{\frac{1}{\alpha}}\|_E \le C(\sum \|f_n\|_E^\alpha)^{\frac{1}{\alpha}}$$
$$(\text{respectively, } \|(\sum |f_n|^\beta)^{\frac{1}{\beta}}\|_E \ge C^{-1}(\sum \|f_n\|_E^\beta)^{\frac{1}{\beta}}),$$

then E is called α-convex (respectively, β-concave). The least such constant C is called the α-convexity (respectively, β-concavity) constant of E and is denoted by $M^{(\alpha)}(E)$ (respectively, $M_{(\beta)}(E)$). If E is α-convex and β-concave, then $E^{(p)}$ is $p\alpha$-convex and $p\beta$-concave with $M^{(p\alpha)}(E^{(p)}) = M^{(\alpha)}(E)^{\frac{1}{p}}$ and $M_{(p\beta)}(E^{(p)}) = M_{(\beta)}(E)^{\frac{1}{p}}$ (see [9, Proposition 3.1]). Therefore, if E is α-convex then $E^{(\frac{1}{\alpha})}$ is 1-convex, so it can be renormed as a Banach lattice (see [15, Proposition 1.d.8] and [22, p. 544]).

A symmetric (quasi-)Banach space E on $(0,1)$ is said to have the Fatou property if for every net $(x_i)_{i\in I}$ in E satisfying $0 \le x_i \uparrow$ and $\sup_{i\in I} \|x_i\|_E < \infty$ the supremum $x = \sup_{i\in I} x_i$ exists in E and $\|x_i\|_E \uparrow \|x\|_E$; We say that E has order continuous norm, if for every net $(f_i)_{i\in I}$ in E such that $f_i \downarrow 0$, $\|f_i\|_E \downarrow 0$ holds; E is called a rearrangement invariant space if it has order continuous (quasi-)norm or the Fatou property.

Let E_i be a symmetric (quasi-)Banach space on $(0,1)$, $i = 1,2$. We define the pointwise product space $E_1 \odot E_2$ as

$$E_1 \odot E_2 = \{f : \ f = f_1 f_2, \ f_i \in E_i, \ i = 1, \ 2\} \tag{2.1}$$

with a functional $\|f\|_{E_1\odot E_2}$ defined by

$$\|f\|_{E_1\odot E_2}=\inf\{\|f_1\|_{E_1}\|f_2\|_{E_2}:\ f=f_1f_2,\ f_i\in E_i,\ i=1,\ 2\}.$$

If E_i is a symmetric quasi-Banach space on $(0,1)$, $i=1,2$, then by [3, Corollary 1], there is an equivalent quasi-norm $\|\cdot\|$ such that $(E_1\odot E_2,\ \|\cdot\|)$ is a symmetric quasi-Banach space on $(0,1)$.

It is clear that if E is a symmetric (quasi-)Banach space on $(0,1)$, then for different Lebesgue measurable subsets A of $(0,1)$ with the same measure $m(A)=t$, the value of $\|\chi_A\|$ remains constant, where χ_A is the characteristic function of A.

Definition 1. Let E be a symmetric (quasi-)Banach space on $(0,1)$. The fundamental function φ_E is defined by $\varphi_E(t)=\|\chi_A\|$, where $t\in[0,1)$ and A is a Lebesgue measurable subset of $(0,1)$ with $m(A)=t$.

Note that $\varphi_{L_1(0,1)}=t$ (see [4, p. 65]). Let $0<p<\infty$. If $A\subset(0,1)$ with $m(A)=t$ $(0\le t<1)$, then

$$\varphi_{L_p(0,1)}(t)=\|\chi_A\|_p=\|\chi_A\|_1^{\frac{1}{p}}=t^{\frac{1}{p}}.$$

Let $M_{\varphi_E}(0,1)$ be the usual Marcinkiewicz space:

$$M_{\varphi_E}(0,1)=\{f\in L_0(0,1):\ \|f\|_{M_{\varphi_E}}=\sup_{t>0}\frac{\varphi_E(t)}{t}\int_0^t\mu_s(f)ds<\infty\}.$$

Definition 2. Let E be a symmetric (quasi-)Banach space on $(0,1)$. We call $M_{\varphi_E}(0,1)$ is a weak version of E and denote it by E_∞.

The classical weak L_p space $L_{p,\infty}(0,1)$ $(1\le p<\infty)$ is defined as the set of all measurable functions f on $(0,1)$ such that

$$\|f\|_{L_{p,\infty}}=\sup_{t>0}t^{\frac{1}{p}}\mu_t(f)<\infty.$$

For $p>1$, $L_{p,\infty}(0,1)$ can be renormed into a Banach space. More precisely,

$$f\mapsto\sup_{t>0}t^{-1+\frac{1}{p}}\int_0^t\mu_s(f)ds$$

gives an equivalent norm on $L_{p,\infty}(0,1)$. We refer to [12] for more information about weak L_p spaces.

If $E=L_p(0,1)$ $(1<p<\infty)$, then $E_\infty=L_{p,\infty}(0,1)$. But for $0<p\le1$, if $f\in(L_p(0,1))_\infty$, then

$$\|f\|_{(L_p(0,1))_\infty}=\sup_{t>0}t^{\frac{1}{p}-1}\int_0^t\mu_s(f)ds=\int_0^1\mu_s(f)ds=\|f\|_1.$$

Hence, $(L_p(0,1))_\infty=L_1(0,1)$ and it is different from the classical weak L_p space.

Let Φ be an N-function, we define

$$a_\Phi=\inf_{t>0}\frac{t\Phi'(t)}{\Phi(t)}\quad\text{and}\quad b_\Phi=\sup_{t>0}\frac{t\Phi'(t)}{\Phi(t)}.$$

If $b_\Phi<\infty$, then the fundamental function of Orlicz space $L_\Phi(0,1)$ on $(0,1)$ equipped with the Luxemburg norm, is the following

$$\varphi_{L_\Phi(\Omega)}(t)=1/\Phi^{-1}(\frac{1}{t}),\quad t>0,$$

where the Luxemburg norm is defined by

$$\|x\|_\Phi = \inf\{\lambda > 0 : \int_0^1 \Phi(\frac{|x|}{\lambda})dx \leq 1\}.$$

Hence, if $E = L_\Phi(0,1)$ and $1 < a_\Phi \leq b_\Phi < \infty$, then $E_\infty = L_{\Phi,\infty}(0,1)$.

For more details on symmetric (quasi-)Banach space and Orlicz spaces we refer to [4, 5, 9, 14, 15, 18, 21, 25].

Let $\mathcal{M}$ be a finite von Neumann algebra with a normal finite faithful trace τ ($\tau(1) = 1$) and $L_0(\mathcal{M})$ be the topological $*$-algebra of measurable operators with respect to $(\mathcal{M}, \tau)$. For $x \in L_0(\mathcal{M})$, we define the distribution function $\lambda(x)$ of x as follows:

$$\lambda_t(x) = \tau(e_{(t,\infty)}(|x|)) \quad \text{for} \quad t > 0,$$

where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ in the interval (t, ∞). We also define the generalized singular numbers $\mu(x)$ of x as

$$\mu_t(x) = \inf\{s > 0 : \ \lambda_s(x) \leq t\} \quad \text{for} \quad t > 0.$$

Recall that both functions $\lambda(x)$ and $\mu(x)$ are decreasing and continuous from the right on $(0, \infty)$ (for further information, see [10]).

For a symmetric quasi-Banach function space E on $(0, 1)$, set

$$E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \ \mu(x) \in E\};$$

$$\|x\|_E = \|\mu(x)\|_E, \qquad x \in E(\mathcal{M}).$$

Recall that $(E(\mathcal{M}), \|.\|_E)$ is a Banach space and we call $(E(\mathcal{M}), \|.\|_E)$ a noncommutative symmetric Banach space (see for reference [8, 20]).

3 Properties

If E_1 and E_2 are symmetric Banach spaces on $(0, 1)$, then by [13, Theorem 2], we know that

$$\varphi_{E_1 \odot E_2}(t) = \varphi_{E_1}(t)\varphi_{E_2}(t), \quad t \geq 0. \tag{3.1}$$

We claim that if E is a symmetric (quasi-)Banach space on $(0, 1)$ and $0 < p < \infty$, then

$$\varphi_{E^{(p)}}(t) = \varphi_E(t)^{\frac{1}{p}}, \quad t \geq 0. \tag{3.2}$$

Indeed, if $A \subset (0, 1)$ with $m(A) = t$ $(0 \leq t < 1)$, then

$$\varphi_{E^{(p)}}(t) = \||\chi_A|^p\|_E^{\frac{1}{p}} = \|\chi_A\|_E^{\frac{1}{p}} = \varphi_E(t)^{\frac{1}{p}}.$$

Proposition 3.1. *Let E_i be a symmetric (quasi-)Banach space on $(0, 1)$ which is α_i-convex for some $0 < \alpha_i < \infty$ $(i = 1,\ 2)$. Then E_1 and E_2 can be equipped with equivalent quasi norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, so that $\varphi_{E_1 \odot E_2}(t) = \varphi_{E_1}(t)\varphi_{E_2}(t)$, for any $t \geq 0$.*

Proof. Let $n \in \mathbb{N}$ such that $n\alpha_i \geq 1$ $(i = 1,\ 2)$. Then $E_i^{(n)} = (E_i^{\alpha_1})^{(n\alpha_i)}$ can be renormed as a symmetric Banach space $(i = 1,\ 2)$. In the following, we consider $E_j^{(n)}$ with this new symmetric norm $(j = 1,\ 2)$. Using [13, Theorem 1 (iii)], we get that $(E_1 \odot E_2)^{(n)} = E_1^{(n)} \odot E_2^{(n)}$. Applying (3.1), we get $\varphi_{(E_1 \odot E_2)^{(n)}}(t) = \varphi_{E_1^{(n)}}(t)\varphi_{E_2^{(n)}}(t)$, for each $t \geq 0$. Hence, by (3.2), we have that

$$\varphi_{E_1 \odot E_2}^{\frac{1}{n}}(t) = \varphi_{E_1}^{\frac{1}{n}}(t)\varphi_{E_2}^{\frac{1}{n}}(t), \quad t \geq 0.$$

Thus, we obtain the desired result. □

In rest of this paper, $\mathcal{M}$ will always denote a finite von Neumann algebra with a normal finite faithful trace τ ($\tau(1)=1$).

Theorem 3.1. *Let E_i be a symmetric (quasi-)Banach space on $(0,1)$ which is α_i-convex for some $0<\alpha_i<\infty$ ($i=1,\ 2$) and $0<a<1$. If $x\in ((E_1^{(a)})_\infty)^{(\frac{1}{a})}(\mathcal{M})$ and $y\in ((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}(\mathcal{M})$, then $xy\in (E_1\odot E_2)_\infty(\mathcal{M})$ and the following Hölder type inequality holds*

$$\|xy\|_{(E_1\odot E_2)_\infty}\leq \|x\|_{((E_1^{(a)})_\infty)^{(\frac{1}{a})}}\|y\|_{((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}}.$$

Proof. Let $x\in ((E_1^{(a)})_\infty)^{(\frac{1}{a})}(\mathcal{M})$ and $y\in ((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}(\mathcal{M})$. By Proposition 3.1, [10, Theorem 4.2, Lemma 2.3(iv)], classical Hölder inequality and (3.2), we have that

$$\begin{aligned}
\|xy\|_{(E_1\odot E_2)_\infty} &= \sup_{t>0}\frac{\varphi_{E_1\odot E_2}(t)}{t}\int_0^t\mu_s(xy)ds\\
&= \sup_{t>0}\frac{\varphi_{E_1}(t)\varphi_{E_2}(t)}{t}\int_0^t\mu_s(xy)ds\\
&\leq \sup_{t>0}\frac{\varphi_{E_1}(t)\varphi_{E_2}(t)}{t}\int_0^t\mu_s(x)\mu_s(y)ds\\
&\leq \sup_{t>0}\frac{\varphi_{E_1}(t)\varphi_{E_2}(t)}{t}\Big(\int_0^t\mu_s(x)^{\frac{1}{a}}ds\Big)^a\Big(\int_0^t\mu_s(y)^{\frac{1}{1-a}}ds\Big)^{1-a}\\
&\leq \sup_{t>0}\frac{\varphi_{E_1}(t)\varphi_{E_2}(t)}{t}\Big(\int_0^t\mu_s(|x|^{\frac{1}{a}})ds\Big)^a\Big(\int_0^t\mu_s(|y|^{\frac{1}{1-a}})ds\Big)^{1-a}\\
&= \sup_{t>0}\Big(\frac{\varphi_{E_1}(t)^{\frac{1}{a}}}{t}\int_0^t\mu_s(|x|^{\frac{1}{a}})ds\Big)^a\Big(\frac{\varphi_{E_2}(t)^{\frac{1}{1-a}}}{t}\int_0^t\mu_s(|y|^{\frac{1}{1-a}})ds\Big)^{1-a}\\
&= \sup_{t>0}\Big(\frac{\varphi_{E_1^{(a)}}(t)}{t}\int_0^t\mu_s(|x|^{\frac{1}{a}})ds\Big)^a\Big(\frac{\varphi_{E_2^{(1-a)}}(t)}{t}\int_0^t\mu_s(|y|^{\frac{1}{1-a}})ds\Big)^{1-a}\\
&\leq \sup_{t>0}\Big(\frac{\varphi_{E_1^{(a)}}(t)}{t}\int_0^t\mu_s(|x|^{\frac{1}{a}})ds\Big)^a\sup_{t>0}\Big(\frac{\varphi_{E_2^{(1-a)}}(t)}{t}\int_0^t\mu_s(|y|^{\frac{1}{1-a}})ds\Big)^{1-a}\\
&= \||x|^{\frac{1}{a}}\|^a_{(E_1^{(a)})_\infty}\||y|^{\frac{1}{1-a}}\|^{(1-a)}_{(E_2^{(1-a)})_\infty}=\|x\|_{((E_1^{(a)})_\infty)^{(\frac{1}{a})}}\|y\|_{((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}}.
\end{aligned}$$

□

Proposition 3.2. *Let E be a symmetric (quasi-)Banach space on $(0,1)$.*

(i) *If $1\leq p<\infty$, then $(E_\infty)^{(p)}(\mathcal{M})\hookrightarrow (E^{(p)})_\infty(\mathcal{M})$.*

(ii) *If $0<p\leq 1$, then $(E^{(p)})_\infty(\mathcal{M})\hookrightarrow (E_\infty)^{(p)}(\mathcal{M})$.*

Proof. (i) Let $x\in (E_\infty)^{(p)}(\mathcal{M})$. Using Jensen's inequality and [10, Lemma 2.3(iv)], we obtain that

$$\begin{aligned}
\|x\|_{(E^{(p)})_\infty} &= \sup_{t>0}\frac{\varphi_{E^{(p)}}(t)}{t}\int_0^t\mu_s(x)ds\\
&= \sup_{t>0}\frac{\varphi_E(t)^{\frac{1}{p}}}{t}\int_0^t\mu_s(x)ds\\
&= \Big(\sup_{t>0}\varphi_E(t)(\tfrac{1}{t}\int_0^t\mu_s(x)ds)^p\Big)^{\frac{1}{p}}\\
&\leq \Big(\sup_{t>0}\frac{\varphi_E(t)}{t}\int_0^t\mu_s(x)^pds\Big)^{\frac{1}{p}}\\
&= \Big(\sup_{t>0}\frac{\varphi_E(t)}{t}\int_0^t\mu_s(|x|^p)ds\Big)^{\frac{1}{p}}\\
&= \||x|^p\|^{\frac{1}{p}}_{E_\infty}=\|x\|_{(E_\infty)^{(p)}}.
\end{aligned}$$

The proof of (ii) is similar to the proof of (i). □

In general, $(E_\infty)^{(p)}(\mathcal{M})\neq (E^{(p)})_\infty(\mathcal{M})$. For example, let $E=L_1(\mathcal{M})$. If $1<p<\infty$, then $(E_\infty)^{(p)}(\mathcal{M})=L_p(\mathcal{M})$ and $(E^{(p)})_\infty(\mathcal{M})=L_{p,\infty}(\mathcal{M})$. If $0<p<1$, then $(E_\infty)^{(p)}(\mathcal{M})=L_p(\mathcal{M})$ and $(E^{(p)})_\infty(\mathcal{M})=L_1(\mathcal{M})$.

Theorem 3.2. *Let E be a symmetric (quasi-)Banach space on $(0,1)$. Then we have the following Chebyshev type inequality*

$$t\varphi_E(\tau(e_{(t,\infty)}(|x|))) \leq \|x\|_{E\infty}, \quad \forall x \in E_\infty(\mathcal{M}).$$

Proof. It is clear that for $s \geq 0$,

$$\mu_s(e_{(t,\infty)}(|x|)) = \chi_{[0,\tau(e_{(t,\infty)}(|x|))}.$$

Since $|x|e_{(t,\infty)}(|x|) \geq te_{(t,\infty)}(|x|)$,

$$\begin{aligned} \varphi_E(\tau(e_{(t,\infty)}(|x|))) &\leq \sup_{s>0} \tfrac{\varphi_E(s)}{s} \int_0^s \mu_\nu(e_{(t,\infty)}(|x|))d\nu \\ &= \|e_{(t,\infty)}(|x|)\|_{E_\infty} \leq \|\tfrac{1}{t}|x|e_{(t,\infty)}(|x|)\|_{E_\infty} \\ &= \tfrac{1}{t}\||x|e_{(t,\infty)}(|x|)\|_{E_\infty} \leq \tfrac{1}{t}\||x|\|_{E_\infty} = \tfrac{1}{t}\|x\|_{E_\infty}. \end{aligned}$$

□

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SYMMETRY GROUPS OF PFAFFIANS OF SYMMETRIC MATRICES

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Abstract. We prove that the symmetry group of the pfaffian polynomial of a symmetric matrix is a dihedral group.

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1 Introduction

At a first glance the title of the paper is not correct. Pfaffians usually are connected with determinants of skew-symmetric matrices. If $a_{i,j} = -a_{j,i}$, for any $1 \le i, j \le 2n$, then the determinant of a skew-symmetric matrix $A = (a_{i,j})$ is a complete square and the square root of the determinant is a pfaffian, so

$$det\, A = (pf_{2n}A)^2.$$

In fact, the pfaffian polynomial is defined by using not the whole matrix A. To construct pfaffians it suffices to know the upper triangular part of A.

The connection between determinants of skew-symmetric matrices and pfaffians was first noted in [2]. For details of pfaffian constructions see also [1] and [3].

Let S_{2n} be set the of all permutations of the set $[2n] = \{1, 2, \dots, 2n\}$ and $S_{2n,pf}$ its subset of all permutations called *Pfaff permutations*,

$$S_{2n,pf} = \{\sigma = (i_1, j_1, \dots, i_n, j_n) \in S_{2n} | i_1 < i_2 < \cdots < i_n, i_s < j_s, 1 \le s \le n\}.$$

For any $\sigma \in S_{2n,pf}$ we define *Pfaff aggregates* a_σ by

$$a_\sigma = a_{i_1,j_1} \cdots a_{i_n,j_n}.$$

We see that the Pfaff aggregates are defined for any triangular array $\bar{A} = (a_{i,j})_{1 \le i < j \le 2n}$. Then the pfaffian of order $2n$ is the polynomial defined as the alternating sum of Pfaff aggregates

$$pf_n = \sum_{\sigma \in S_{2n,pf}} sign\ \sigma\ a_\sigma.$$

Here $sign\, \sigma$ is the signature of the permutation σ,

$$sign\, \sigma = (-1)^{k(\sigma)},$$

where $k(\sigma)$ is the number of inversions

$$k(\sigma) = |\{\ (i, j) \mid \sigma(i) > \sigma(j),\ 1 \le i < j \le n\ \}|.$$

Suppose now that $\{a_{i,j}, 1 \le i,j \le 2n\}$ are n^2 generators and endow the space of polynomials $K[a_{i,j} | 1 \le i,j \le n]$ with the structure of S_{2n}-module by the following action on generators

$$\sigma a_{i,j} = a_{\sigma^{-1}(i),\sigma^{-1}(j)}.$$

In particular, if $A = (a_{i,j})$ has a skew-symmetric set of generators, $a_{i,j} = -a_{j,i}$ then this action induces the structure of S_{2n}-module on the space of polynomials with $\binom{n}{2}$ generators $K[a_{i,j} | 1 \le i < j \le n]$. Similarly, we obtain one more structure of S_{2n}-module on this space if generators are symmetric, $a_{i,j} = a_{j,i}$. In both cases natural questions appear about invariants under these actions of permutation groups. In particular, we can ask about symmetry and skew-symmetry groups of a given polynomial $f \in K[a_{i,j}]$,

$$Sym\ f = \{\sigma \in S_n | \sigma\ f = f\},$$

$$SSym\ f = \{\sigma \in S_n | \sigma\ f = sign\,\sigma\ f\}.$$

For example, the determinant polynomial $det\,A$ for $A = (a_{ij})_{1 \le i,j \le n}$ is a polynomial of degree n and its symmetry group is isomorphic to S_n.

Another example: if a matrix A is skew-symmetric, then the pfaffian polynomial $pf_{2n} = pf_{2n}A$ is a polynomial of degree n and

$$SSym\ pf_{2n} \cong S_{2n}.$$

Let the characteristic of the main field be $p \ne 2$ and

$$g_{2n}(x_1, \ldots, x_{2n}) = (x_1 - x_2)(x_2 - x_3) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_1).$$

Theorem 1.1. *Let* $\bar{A} = (a_{i,j})_{1 \le i < j \le 2n}$ *be the triangular array with components* $a_{i,j} = (x_i - x_j)^2$ *for* $1 \le i < j \le 2n$. *Then*

$$pf_{2n}\ \bar{A} = -(-2)^{n-1}\ g_{2n}.$$

Theorem 1.2. *The symmetry group of the polynomial* g_{2n} *is isomorphic to the dihedral group* D_{2n}.

Based on these two results our main result is as follows.

Theorem 1.3. *If generators* $a_{i,j}$ *are symmetric,* $a_{i,j} = a_{j,i}$, *then the symmetry group of the pfaffian polynomial* $pf_{2n} = pf_{2n}\bar{A}$ *is isomorphic to the dihedral group*

$$Sym\ pf_{2n} \cong D_{2n}.$$

Recall that the dihedral group D_n is the symmetry group of a regular n-gon. It can be generated by n rotations and n reflections,

$$D_n = \langle a, b \mid a^n = e, b^2 = e,\ bab = a^{n-1} \rangle.$$

In our paper we use the following notation for permutations. The standard notation for a permutation is a two row notation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \in S_n.$$

The one row notation of σ is $i_1 i_2 \cdots i_n$. If σ is a cycle on the set $i_1, i_2, \ldots, i_k$, i.e., $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_{k-1}) = i_k, \sigma(i_k) = i_1$, then we will write $\sigma = (i_1, i_2, \ldots, i_k)$. For example,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 4 & 1 & 6 & 2 & 8 & 7 \end{pmatrix} \in S_8 \Rightarrow \sigma = 35416287 = (1,3,4)(2,5,6)(7,8).$$

2 Pfaffian of $(x_i - x_j)^2$

If $A = (a_{i,j})$ is skew-symmetric, then

$$det\, A = (pf_{2n}\bar{A})^2.$$

If a matrix A is not skew-symmetric, say if A is symmetric, then the determinant polynomial $det\, A$ and the pfaffian polynomial $pf_{2n}(\bar{A})$ have no such connection. For example, if $A_n = ((x_i - x_j)^2)_{1\le i,j\le n}$, then

$$det\, A_n = \begin{cases} -(x_1 - x_2)^4, & \text{if } n = 2, \\ 2((x_2 - x_2)(x_2 - x_3)(x_3 - x_1))^2, & \text{if } n = 3, \\ 0, & \text{otherwise,} \end{cases}$$

while, by Theorem 1.1, pfaffians are non-trivial for any even n.

Proof of Theorem 1.1. Let $\psi(x, y) = (x - y)^2$ and $a_{i,j} = \psi(x_i, x_j) = (x_i - x_j)^2$. Note that the pfaffian $pf_{2n}A$ is a polynomial in the variables $x_1, \ldots, x_{2n}$. We have to prove that

$$pf_{2n} = -(-2)^{n-1} g_{2n}.$$

Since $\psi(x, x) = 0$, the polynomial $pf_{2n}(x_1, \ldots, x_s, x_{s+1}, \ldots, x_{2n})$ is divisable by $x_s - x_{s+1}$ for any $1 \le s \le 2n$. Here we set $x_{2n+1} = x_1$. Note that the degree of the polynomial $g_{2n}(x_1, \ldots, x_{2n})$ is $2n$ and the degree of $pf_{2n}((x_i - x_j)^2)$ is also $2n$. Therefore,

$$pf_{2n}(x_1, x_2, \ldots, x_{2n}) = c\, g_{2n}(x_1, x_2, \ldots, x_{2n}),$$

for some constant c. Take $x_i = i$. It is easy to see that

$$g_{2n}(1, 2, \ldots, 2n) = (1 - 2)(2 - 3)\cdots(2n - 1 - 2n)(2n - 1) = -(2n - 1).$$

It remains to prove that

$$pf_{2n}(1, 2, \ldots, 2n) = (-2)^{n-1}(2n - 1) \tag{2.1}$$

to obtain that $c = -(-2)^{n-1}$.

By induction on n we will prove that

$$pf_{2n}(x_1, x_2, \ldots, x_{2n}) = -(-2)^{n-1} g_{2n}(x_1, x_2, \ldots, x_{2n}).$$

For $n = 1$ our statement is evident:

$$pf_2\bar{A} = a_{1,2} = -(x_1 - x_2)(x_2 - x_1).$$

Suppose that our statement is true for $n - 1$,

$$pf_{2n-2}\bar{A} = -(-2)^{n-2}(x_1 - x_2)(x_2 - x_3)\cdots(x_{2n-3} - x_{2n-2})(x_{2n-2} - x_1).$$

Let us prove it for n.

Let us decompose the pfaffian along the first row

$$pf_{2n}\bar{A} = \sum_{i=2}^{2n}(-1)^i a_{1,i}\, pf_{2n-2}A_{\hat{1},\hat{i}}.$$

We see that

$$pf_{2n}\bar{A} = R_1 + R_2 + R_3,$$

where

$$R_1 = a_{1,2}\ pf_{2n}\bar{A}_{\hat{1}\hat{2}},$$

$$R_2 = \sum_{i=3}^{2n-1}(-1)^i a_{1,i}\ pf_{2n-2}\bar{A}_{\hat{1},\hat{i}},$$

$$R_3 = a_{1,2n}\ pf_{2n-2}\bar{A}_{\hat{1},\hat{2n}}.$$

By the inductive suggestion

$$R_1 = -(-2)^{n-2}(x_1-x_2)^2(x_3-x_4)\cdots(x_{2n-1}-x_{2n})(x_{2n}-x_3).$$

Hence,

$$R_1|_{x_i\to i} = -(-2)^{n-2}(1-2)^2(3-4)\cdots(2n-1-2n)(2n-3) = (-2)^{n-2}(2n-3),$$

$$R_3|_{x_i\to i} = -(-2)^{n-2}(1-2n)^2(2-3)(3-4)\cdots(2n-2-2n+1)(2n-1-2) =$$

$$-(-2)^{n-2}(2n-1)^2(-1)^{2n-3}(2n-3) = (-2)^{n-2}(2n-3)(2n-1)^2.$$

Further, if $2 < i < 2n$, then

$$(-1)^i a_{1,i}\ pf_{2n}\bar{A}_{\hat{1},\hat{i}}|_{x_j\to j} =$$

$$(-1)^i(-(-2)^{n-2})(x_1-x_i)^2(x_2-x_3)(x_3-x_4)\cdots(x_{i-1}-x_{i+1})(x_{i+1}-x_{i+2})\times\cdots$$
$$\times(x_{2n-1}-x_{2n})(x_{2n}-x_2)|_{x_j\to j} =$$

$$(-1)^i(-(-2)^{n-2})(i-1)^2(-2)(2n-2) = (-1)^i(i-1)^2(-2)^{n-2}4(n-1).$$

Hence,

$$R_2|_{x_i\to i} = -(-2)^{n-2}\sum_{i=3}^{2n-1}-(-1)^i(i-1)^2 4(n-1) = (-2)^{n-2}4(n-1)(2n^2-3n+2).$$

So, we see that (2.1) is true for n,

$$f_{2n}(1,2,\ldots,2n) = R_1+R_2+R_3 =$$

$$(-2)^{n-2}[(2n-3)-4(n-1)(2n^2-3n+2)+(2n-3)(2n-1)^2] =$$

$$-(-2)^{n-2}2(2n-1) = (-2)^{n-1}(2n-1).$$

□

3 Symmetry group of the polynomial g_{2n}

Proof of Theorem 1.2. First, we check that any dihedral permutation $\sigma \in D_{2n}$ is a symmetry of the polynomial g_{2n}.

Let us take the realization of a dihedral group as the symmetry group of the regular n-gon whose vertices are clockwise labelled by $1, 2, \ldots, 2n$. Elements of a dihedral group might have:

I. one up-run: $\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2n \\ 1 & 2 & \cdots & 2n \end{pmatrix}$,

II. one down-run: $\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2n \\ 2n & 2n-1 & \cdots & 1 \end{pmatrix}$,

III. two up-run

$$\sigma(1) = s < \sigma(2) = s+1 < \cdots < \sigma(2n-s+1) = 2n, \quad \sigma(2n-s+2) = 1 < \cdots < \sigma(2n) = s-1,$$

for some $1 < s \leq 2n$,

IV. or two down-run

$$\sigma(1) = s > \sigma(2) = s-1 > \cdots > \sigma(s) = 1, \quad \sigma(s+1) = 2n > \cdots > \sigma(2n) = s+1.$$

for some $1 \leq s < 2n$.

In cases **I** and **II** our statement is evident.

In case **III** we have

$$g_{2n}(x_{\sigma(1)}, \ldots, x_{\sigma(2n)}) =$$

$$(x_s - x_{s+1})(x_{s+1} - x_{s+2}) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_1)\ (x_1 - x_2) \cdots (x_{s-2} - x_{s-1})(x_{s-1} - x_s) =$$

$$(x_1 - x_2) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_1) = g_{2n}(x_1, \ldots, x_{2n}).$$

In case **IV**

$$g_{2n}(x_{\sigma(1)}, \ldots, x_{\sigma(2n)}) =$$

$$(x_s - x_{s-1})(x_{s-1} - x_{s-2}) \cdots (x_2 - x_1)(x_1 - x_{2n})\ (x_{2n} - x_{2n-1}) \cdots (x_{s+1} - x_s) =$$

$$(-1)^s (x_{s-1} - x_s)(x_{s-2} - x_{s-1}) \cdots (x_1 - x_2)(x_{2n} - x_1)\ (-1)^{2n-s} (x_{2n-1} - x_{2n}) \cdots (x_s - x_{s+1}) =$$

$$(x_1 - x_2) \cdots (x_{2n-1} - x_{2n}) = g_{2n}(x_1, \ldots, x_{2n}).$$

So,

$$D_{2n} \subseteq Sym(g_{2n}).$$

Now we will prove that any $\sigma \in Sym(g_{2n})$ is a dihedral permutation.

Let $M_{2n} = \{1, 2, \ldots, 2n\}$. For $i, j \in M_{2n}$ we say that they are connected, if $|i - j| = 1$ or $|i - j| = 2n - 1$. So, if $i < j < 2n$, then i, j are connected iff $j = i + 1$. If $j = 2n$, and i, j are connected, then $i = 2n - 1$ or $i = 1$. It is clear that this relation is symmetric: i, j are connected iff j, i are connected. So, $i, j \in M_{2n}$ are connected, if $|i - j| = 1$ or $(i, j) = (1, 2n)$ or $(i, j) = (2n, 1)$.

Note that the polynomial $g_{2n}(x_1, \ldots, x_{2n})$ is a product of polynomials $x_i - x_j$, $i < j$, where i and j are connected. Therefore, any symmetry $\sigma \in Sym(g_{2n})$ has the following property: if i and j are connected, then $\sigma(i)$ and $\sigma(j)$ are also connected.

Let $\sigma \in Sym(g_{2n})$ and $\sigma(1) = i_1$. The following possibilities may arise.

Case A. Suppose that $\sigma(1) = i_1 < \sigma(2)$. Take $k > 1$, such that $\sigma(k-1) < \sigma(k)$ and $\sigma(k+1) < \sigma(k)$. Since $\sigma(1)$ and $\sigma(2)$ are connected and $\sigma(2) > \sigma(1)$, then $\sigma(2) = i_1 + 1$. By similar arguments,

$$\sigma(3) = i_1 + 2, \ldots, \sigma(k) = i_1 + k - 1,$$

but $\sigma(k+1) \neq i_1 + k$. Such situation is possible only in one case: $i_1 = 2n - k + 1$ and $\sigma(k+1) = 1$. So,

$$\sigma(k+1) = 1, \sigma(k+2) = 2, \ldots, \sigma(2n) = i_1 - 1.$$

In other words,

$$\sigma = i_1\ (i_1+1)\ \ldots\ (2n)\ 1\ 2\ \ldots (i_1 - 1).$$

We obtained a permutation σ that has exactly one up-run if $i_1 = 1$, or two up-runs if $i_1 > 1$. So, we obtain permutations of type **I** or **III**. Therefore, $\sigma \in D_{2n}$.

Case B. Now consider the case $\sigma(1) = i_1 > \sigma(2)$. Take $k > 1$, such that $\sigma(k-1) > \sigma(k)$ and $\sigma(k+1) > \sigma(k)$.

Since $\sigma(1)$ and $\sigma(2)$ are connected and $\sigma(2) < \sigma(1)$, then $\sigma(2) = i_1 - 1$. By similar arguments,

$$\sigma(3) = i_1 - 2, \ldots, \sigma(k) = i_1 - k + 1.$$

but $\sigma(k+1) \neq i_1 - k$. Such situation is possible only in one case: $i_1 = k, \sigma(k+1) = 2n$. So,

$$\sigma(k+1) = 2n, \sigma(k+2) = 2n - 1, \ldots, \sigma(2n) = i_1 + 1.$$

In other words,

$$\sigma = i_1\ (i_1 - 1)\ \ldots\ 1\ 2n\ (2n-1)\ \ldots (i_1 + 1).$$

We obtained a permutation σ that has exactly one down-run if $i_1 = 2n$ or two down-runs if $i_1 < 2n$. In other words we obtained a permutations of type **II** or **IV**. Thus, $\sigma \in D_{2n}$. □

4 Proof of Theorem 1.3

First we prove that $D_{2n} \subseteq Sym\, pf_{2n}$.

Lemma 4.1. *If $A = (a_{i,j})$ is symmetric, then the pfaffian is invariant under action of the dihedral group D_{2n},*

$$\mu(pf_{2n}) = pf_{2n}$$

for any $\mu \in D_{2n}$.

Proof. The dihedral group D_{2n} has order $4n$ and is generated by the cyclic permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & 2n-1 & 2n \\ 2 & 3 & 4 & \cdots & 2n & 1 \end{pmatrix}$$

and reflection

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n & n+1 & n+2 & \cdots & 2n-1 & 2n \\ 1 & 2n & 2n-1 & \cdots & n+2 & n+1 & n & \cdots & 3 & 2 \end{pmatrix}.$$

To prove our lemma it suffices to establish that

$$\sigma(pf_{2n}) = pf_{2n},$$

$$\tau(pf_{2n}) = pf_{2n},$$

if $a_{i,j} = a_{j,i}$, for any $1 \le i < j \le 2n$.

Recall that $\alpha = (i_1, i_2, \ldots, i_{2n-1}, i_{2n})$ is a Pfaff permutation, if

$$i_1 < i_3 < i_5 < \cdots < i_{2n-1},$$

$$i_1 < i_2, i_3 < i_4, \ldots, i_{2n-1} < i_{2n}.$$

Let $S_{2n,pf}$ be set of all Pfaff permutations. Below we use the one-line notation for permutations. We write $\alpha = (i_1, i_2, \ldots, i_{2n-1}, i_{2n})$ instead of

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & i_2 & \ldots & i_{2n-1} & i_{2n} \end{pmatrix}.$$

Note that

$$\tau(i) + i = \begin{cases} 2n+2, & \text{if } 1 < i \le 2n, \\ 2 & \text{if } i = 1. \end{cases}$$

Set

$$\bar{i} = 2n + 2 - i,$$

if $i > 1$.

Now we study the action of the generator σ on pfaffian polynomials, when generators are symmetric, $a_{i,j} = a_{j,i}$, for any $1 \le i, j \le 2n$. Let $\alpha = (1, i_2, i_3, \ldots, i_{2n}) \in S_{2n,pf}$, and $l = \alpha^{-1}(2n)$. Then l is even, $l = 2k$, and

$$\sigma(a_\alpha) = \sigma(a_{i_1,i_2} \cdots a_{i_{2n-1} i_{2n}}) =$$

$$a_{i_1+1,i_2+1} \cdots a_{i_{2k-3}+1,i_{2k-2}+1} a_{i_{2k-1}+1,1} a_{i_{2k+1}+1,i_{2k+2}+1} \cdots a_{i_{2n-1}+1,i_{2n}+1} = a_{\tilde{\alpha}},$$

where

$$\tilde{\alpha} = (1, i_{2k-1}+1, i_1+1, i_2+1, \ldots, i_{2k-3}+1, i_{2k-2}+1, i_{2k+1}+1, i_{2k+2}+1, \ldots, i_{2n-1}+1, i_{2n}+1).$$

Here we replace $a_{i_{2k-1}+1,1}$ by $a_{1,i_{2k-1}+1}$. We see that the map

$$S_{2n,pf} \to S_{2n,pf}, \quad \alpha \mapsto \tilde{\alpha}$$

is a bijection and

$$sign\ \tilde{\alpha} = sign\ \alpha.$$

Hence,

$$\sigma(pf_{2n}) = \sum_{\alpha \in S_{2n,pf}} sign\, \alpha\, \sigma(a_\alpha) = \sum_{\alpha \in S_{2n,pf}} sign\, \tilde{\alpha}\, a_{\tilde{\alpha}} = pf_{2n}.$$

So, we have estabilshed that the pfaffian is invariant under action $\sigma \in D_{2n}$. □

Let us study the action of the generator τ on pfaffian polynomials.

We have

$$\tau : a_\alpha \mapsto a_{1,\,\overline{i_2}} a_{\overline{i_3},\,\overline{i_4}} \cdots a_{\overline{i_{2n-1}},\,\overline{i_{2n}}}.$$

Since

$$\overline{i_{2k-1}} > \overline{i_{2k}}, \quad 1 < k \le n,$$

we have to replace $a_{\overline{i_{2k-1}},\,\overline{i_{2k}}}$ by $a_{\overline{i_{2k}},\,\overline{i_{2k-1}}}$. Further,

$$\overline{i_3} > \overline{i_5} > \cdots > \overline{i_{2n-1}} > 1.$$

Therefore,

$$\tau : a_\alpha \mapsto a_{\bar{\alpha}},$$

where

$$a_{\bar{\alpha}} = a_{1,\,\overline{i_2}}\, a_{\overline{i_{2n}},\,\overline{i_{2n-1}}}\, a_{\overline{i_{2n-2}},\,\overline{i_{2n-3}}} \cdots a_{\overline{i_4},\,\overline{i_3}}.$$

We see that

$$sign\ \alpha = sign\ \bar{\alpha}.$$

Note that the map

$$S_{2n,pf} \to S_{2n,pf}, \quad \alpha \mapsto \bar{\alpha},$$

is a bijection. Therefore,

$$\tau(pf_{2n}) = \sum_{\alpha \in S_{2n,pf}} sign\, \alpha\ \tau(a_\alpha) = \sum_{\alpha \in S_{2n,pf}} sign\, \bar{\alpha}\ a_{\bar{\alpha}} = pf_{2n}.$$

So, we have proved that the pfafian pf_{2n} is invariant under the action of the dihedral group D_{2n} of order $4n$, if the matrix $(a_{i,j})_{1\le i,j\le 2n}$ is symmetric.

Example. Let

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Then

$$\tau(pf_4) = \tau(a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}) =$$
$$a_{1,4}a_{3,2} - a_{1,3}a_{4,2} + a_{1,2}a_{4,3} =$$
$$a_{1,4}a_{2,3} - a_{1,3}a_{2,4} + a_{1,2}a_{3,4} = pf_4,$$

$$\mu(pf_4) = \mu(a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}) =$$
$$a_{4,3}a_{2,1} - a_{4,2}a_{3,1} + a_{4,1}a_{3,2} =$$
$$a_{3,4}a_{1,2} - a_{2,4}a_{1,3} + a_{1,4}a_{2,3} = pf_4.$$

Proof of Theorem 1.3. . Let $\sigma \in Sym\, pf_{2n}$ i.e.,

$$\sigma\,(pf_{2n}) = pf_{2n}$$

for any $a_{i,j}$, such that $a_{i,j} = a_{j,i}$. In particular, σ is a symmetry of the pfaffian polynomial $pf_{2n}((x_i - x_j)^2)_{1\le i<j\le 2n}$. By Theorems 1.1 and 1.2 and Lemma 4.1 our theorem is valid. □

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THE SPECTRUM AND PRINCIPAL FUNCTIONS OF A NONSELF-ADJOINT STURM–LIOUVILLE OPERATOR WITH DISCONTINUITY CONDITIONS

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Abstract. This paper deals with the nonself-adjoint Sturm–Liouville operator (or one-dimensional time-independent Schrödinger operator) with discontinuity conditions on the positive half line. In this study, the spectral singularities and the eigenvalues are investigated and it is proved that this problem has a finite number of spectral singularities and eigenvalues with finite multiplicities under two additional conditions. Moreover, we determine the principal functions with respect to the eigenvalues and the spectral singularities of this operator.

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1 Introduction

In mathematical physics, when we investigate solutions of partial differential equations under given initial and boundary conditions using the Fourier method, we encounter the following types of problems: to determine the eigenvalues and eigenfunctions of differential operators and to expand an arbitrary function as a series of eigenfunctions. Therefore, since it is interesting to study these types of problems, many works have been done on such problems and continue to be done. An important and interesting problem is that of the examination of the spectrum and expanding a given function via eigenfunctions of a differential operator which is not self-adjoint.

In the present paper, we examine the spectrum and the principal functions of a nonself-adjoint Sturm–Liouville operator with discontinuity conditions on the positive half plane. That is, we deal with in the following nonself-adjoint problem for the Sturm–Liouville equation

$$-\omega'' + q(x)\omega = \mu^2\omega, \quad x \in (0,a) \cup (a,\infty), \tag{1.1}$$

with the discontinuity conditions

$$\omega(a-0) = \alpha\omega(a+0), \qquad \omega'(a-0) = \alpha^{-1}\omega'(a+0) \tag{1.2}$$

and the boundary condition

$$\omega(0) = 0, \tag{1.3}$$

where $a > 0$, $0 < \alpha \neq 1$, μ is a complex parameter, $q(x)$ is a complex-valued function which satisfies the condition

$$\int_0^\infty x|q(x)|dx < \infty. \tag{1.4}$$

The spectral theory of nonself-adjoint operator in the classical case (i.e., $\alpha = 1$) was studied by Naimark [17, 18], he showed that some poles of the resolvent kernel are not the eigenvalues of the operator and belong to the continuous spectrum, moreover, these poles are called spectral singularities and were first introduced by Schwartz [22]. In the self-adjoint case (i.e. $\text{Im}q(x) \equiv 0$), the number of the eigenvalues of the operator is finite under condition (1.4) (see [15]).

In the nonself-adjoint case, Naimark demonstrated that the number of eigenvalues is finite under the condition (see [17, 18])

$$\int_0^\infty \exp(\epsilon x)|q(x)|dx < \infty, \quad \epsilon > 0.$$

This condition is too strict and Pavlov weakened this condition as follows (see [19]):

$$\sup_{0\le x<\infty} \left\{|q(x)| \exp(\epsilon\sqrt{x})\right\} < \infty, \quad \epsilon > 0$$

and he proved that if $q(x)$ satisfies the above condition, then there is a finite number of eigenvalues of the operator.

In the spectral analysis of nonself-adjoint operators, the spectral singularities have an essential role and the influence of the spectral singularities in the spectral expansion with respect to the principal functions of the operator is investigated by Lyantse [12, 13]. The investigations on the spectrum, principal functions and the spectral expansion by the principal functions of the nonself-adjoint operator are very attractive and there are many works on the nonself-adjoint operator under different boundary conditions (see [2, 4, 5, 6, 8, 10, 11, 14, 16, 20, 23, 24, 25] and the references therein). Moreover, the nonself-adjoint operator with discontinuous coefficient is studied in [1], some spectral properties of the Sturm–Liouville operator with impulsive condition is worked in [3].

The distinction between this work and other studies is that the nonself-adjoint boundary value problem (1.1)-(1.3) has discontinuity conditions at $x = a \in (0,\infty)$. The presence of discontinuity condition (1.2) in problem (1.1)-(1.3) seriously affects the structure of a Jost solution to (1.1), i.e., a Jost solution is not expressed as a transformation operator, it has the integral representation which is obtained by Huseynov and Osmanova [9] and in this work. It is seen from this representation that the triangular property of a Jost solution is lost and the kernel function has a discontinuity along the line $s = 2a - x$ for $x \in (0, a)$. In this paper, we will obtain our results using this integral representation.

The conclusions drawn from this paper are as follows: in Section 2, we give an estimate of the kernel $k(x, s)$ of a Jost solution to equation (1.1) with discontinuity conditions (1.2) and examine the spectrum and the resolvent of problem (1.1)-(1.3). Moreover, it is demonstrated that under additional conditions, the number of the eigenvalues and the spectral singularities of this problem is finite. In Section 3, the principal functions are determined and their convergence properties are examined.

2 The spectrum and resolvent of L

Assume that a function $e(x, \mu)$ satisfies equation (1.1), discontinuity conditions (1.2) and the following condition at infinity

$$\lim_{x\to\infty} e^{-i\mu x}e(x, \mu) = 1.$$

Then, the function $e(x, \mu)$ is called a Jost solution to equation (1.1). When $q(x) \equiv 0$ in (1.1), the Jost solution has the form:

$$e_0(x, \mu) = \begin{cases} e^{i\mu x}, & x > a \\ \alpha^+ e^{i\mu x} + \alpha^- e^{i\mu(2a-x)}, & 0 < x < a \end{cases}$$

where $\alpha^\pm = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right)$.

Theorem 2.1. [9] *Let a complex-valued function $q(x)$ satisfy (1.4). Then for any μ from the closed upper half-plane, there exists a Jost solution $e(x,\mu)$ to equation (1.1) with discontinuity conditions (1.2), it is unique and representable in the form*

$$e(x,\mu) = e_0(x,\mu) + \int_x^{\infty} k(x,s)e^{i\mu s}ds, \tag{2.1}$$

where for every fixed $x \in (0,a) \cup (a,\infty)$, the kernel $k(x,.) \in L_1(x,\infty)$ and satisfies the inequality

$$\int_x^{\infty} |k(x,s)|ds \le e^{c\sigma_1(x)} - 1, \qquad \sigma_1(x) = \int_x^{\infty} t|q(t)|dt, \qquad c = \alpha^+ + |\alpha^-|.$$

Moreover, the function $k(x,s)$ is continuous for $s \neq 2a - x$.

Remark 1. The following estimate holds:

$$|k(x,s)| \le \frac{c}{2}\sigma\left(\frac{x+s}{2}\right) e^{(c+1)\sigma_1(x)} \tag{2.2}$$

with $\sigma(x) = \int_x^{\infty} |q(u)|du$ and $c = \alpha^+ + |\alpha^-|$. This estimate is obtained as follows.

The function $k(x,s)$ is of the form for $0 < x < a$ (see [9]):

$$\begin{aligned} k(x,s) \;&=\; k_0(x,s) + \frac{1}{2}\int_x^a q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k(\zeta,u)dud\zeta \\ &+\frac{\alpha^+}{2}\int_a^{\infty} q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k(\zeta,u)dud\zeta \\ &-\frac{\alpha^-}{2}\int_a^{2a-x} q(\zeta)\int_{s+\zeta-2a+x}^{s-\zeta+2a-x} k(\zeta,u)dud\zeta \\ &+\frac{\alpha^-}{2}\int_{2a-x}^{\infty} q(\zeta)\int_{s-\zeta+2a-x}^{s+\zeta-2a+x} k(\zeta,u)dud\zeta, \end{aligned}$$

where

$$\begin{aligned} k_0(x,s) \;&=\; \frac{\alpha^+}{2}\int_{\frac{x+s}{2}}^{\infty} q(\zeta)d\zeta + \frac{\alpha^-}{2}\int_{\frac{2a+x-s}{2}}^{a} q(\zeta)d\zeta \\ &-\frac{\alpha^-}{2}\int_a^{\frac{s+2a-x}{2}} q(\zeta)d\zeta, \qquad x < s < 2a - x, \end{aligned} \tag{2.3}$$

$$k_0(x,s) = \frac{\alpha^+}{2}\int_{\frac{x+s}{2}}^{\infty} q(\zeta)d\zeta + \frac{\alpha^-}{2}\int_{\frac{s+2a-x}{2}}^{\infty} q(\zeta)d\zeta, \qquad s > 2a - x \tag{2.4}$$

and for $x > a$

$$k(x,s) = k_0(x,s) + \frac{1}{2}\int_x^{\infty} q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k(\zeta,u)dud\zeta,$$

where

$$k_0(x,s) = \frac{1}{2}\int_{\frac{x+s}{2}}^{\infty} q(\zeta)d\zeta.$$

When $x > a$, we face the classical case (see [18]). In this case, we have

$$|k(x,s)| \le \frac{1}{2}e^{\sigma_1(x)}\sigma\left(\frac{x+s}{2}\right).$$

Now, let us examine the case $0 < x < a$. Set, for $n \in \mathbb{N}$

$$\begin{aligned} k_n(x,s) &= \frac{1}{2}\int_x^a q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k_{n-1}(\zeta,u)dud\zeta \\ &+\frac{\alpha^+}{2}\int_a^\infty q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k_{n-1}(\zeta,u)dud\zeta \\ &-\frac{\alpha^-}{2}\int_a^{2a-x} q(\zeta)\int_{s+\zeta-2a+x}^{s-\zeta+2a-x} k_{n-1}(\zeta,u)dud\zeta \\ &+\frac{\alpha^-}{2}\int_{2a-x}^\infty q(\zeta)\int_{s-\zeta+2a-x}^{s+\zeta-2a+x} k_{n-1}(\zeta,u)dud\zeta \end{aligned}$$

and $k_0(x,s)$ is specified by relations (2.3) and (2.4). Then, we obtain

$$|k_0(x,s)| \le \frac{c}{2}\sigma\left(\frac{x+s}{2}\right), \qquad |k_n(x,s)| \le \frac{c}{2}\sigma\left(\frac{x+s}{2}\right)\frac{(c+1)^n(\sigma_1(x))^n}{n!}.$$

This implies that the series $\sum_{n=0}^{\infty} k_n(x,s)$ converges and its sum $k(x,s)$ satisfies inequality (2.2). Consequently, for $x \in (0,a) \cup (a,\infty)$ inequality (2.2) is valid.

Now, we define $\hat{e}(x,\mu)$ as a solution to equation (1.1) with discontinuity conditions (1.2) and the following condition at infinity

$$\lim_{x\to\infty} e^{i\mu x}\hat{e}(x,\mu) = 1.$$

When $q(x) \equiv 0$ in equation (1.1), the solution has the form:

$$\hat{e}_0(x,\mu) = \begin{cases} e^{-i\mu x}, & x > a, \\ \alpha^+ e^{-i\mu x} + \alpha^- e^{-i\mu(2a-x)}, & 0 < x < a \end{cases}$$

for $\mathrm{Im}\mu \ge 0$. The Wronskian of the solutions $e(x,\mu)$ and $\hat{e}(x,\mu)$ is

$$W[e(x,\mu),\hat{e}(x,\mu)] = -2i\mu, \quad \mathrm{Im}\mu \ge 0.$$

Now, we consider problem (1.1)-(1.3) as an operator L operating on the Hilbert space $L_2(0,\infty)$. The values $\lambda = \mu^2$ for which L has a non-zero solution are called eigenvalues and the corresponding solutions are called eigenfunctions.

Consider $\tilde{e}(x,\mu) = e(x,-\mu)$ with $\mathrm{Im}\mu \le 0$ and the expression of the Wronskian of $e(x,\mu)$ and $\tilde{e}(x,\mu)$ is

$$W[e(x,\mu),\tilde{e}(x,\mu)] = -2i\mu, \quad \mathrm{Im}\mu = 0. \tag{2.5}$$

Lemma 2.1. *The nonself-adjoint operator L does not have positive eigenvalues.*

Proof. It follows from (2.5) that for $\lambda > 0$, the general solution to (1.1) is of the form $\omega = c_1 e(x,\mu) + c_2\tilde{e}(x,\mu)$ and as $x \to \infty$, $\omega = c_1 e^{i\mu x} + c_2 e^{-i\mu x} + o(1)$. This function does not belong to $L_2(0,\infty)$ if both c_1 and c_2 are not equal to zero. □

Let us define $s(x,\mu)$ as a solution to (1.1) under discontinuity conditions (1.2) and the initial conditions

$$s(0,\mu) = 0, \qquad s'(0,\mu) = 1.$$

Now, consider non-positive or complex λ. Since the general solution to (1.1) satisfying the initial condition $\omega(0) = 0$ has the form $\omega(x) = cs(x,\mu)$, it follows that $\lambda = \mu^2$ is an eigenvalue of the operator L if and only if $s(.,\mu) \in L_2(0,\infty)$. Moreover,

$$s(x,\mu) = \frac{\hat{e}(0,\mu)e(x,\mu) - e(0,\mu)\hat{e}(x,\mu)}{2i\mu}, \quad \mathrm{Im}\mu > 0. \tag{2.6}$$

Lemma 2.2. *The necessary and sufficient conditions for $\lambda \neq 0$ to be an eigenvalue of L are*

$$e(0,\mu) = 0, \quad \lambda = \mu^2, \quad \mathrm{Im}\mu > 0.$$

Proof. It follows from the representations of $e(x,\mu)$ and $\hat{e}_0(x,\mu)$ that $e(.,\mu) \in L_2(0,\infty)$ and $\hat{e}(x,\mu) \notin L_2(0,\infty.)$ Then, from (2.6), $s(.,\mu) \in L_2(0,\infty)$ if and only if $e(0,\mu) = 0$. □

Lemma 2.3. *The set of eigenvalues of L is bounded, is no more than countable and its limit points can lie only on the half-axis $\lambda \geq 0$.*

Proof. Using the representation of solution $e(x,\mu)$ given by (2.1), as $|\mu| \to \infty$, we have $e(0,\mu) \to \alpha^+$ for $\mathrm{Im}\mu > 0$. Therefore, the set of the zeros of $e(0,\mu)$ is bounded in the half plane $\mathrm{Im}\mu > 0$. Since $e(0,\mu)$ is holomorphic in the half plane $\mathrm{Im}\mu > 0$, the set of its zeros is no more than countable and can have limit points only on the real axis. □

All numbers λ of the form $\lambda = \mu^2$, $\mathrm{Im}\mu > 0$, $e(0,\mu) \neq 0$ belong to the resolvent set of L. The resolvent operator $R_{\mu^2} = (L - \mu^2 I)^{-1}$ exists and has the following form:

$$\omega(x,\mu) =: R_{\mu^2}(L)f(x) = \int_0^\infty g(x,s;\mu^2)f(s)ds,$$

where

$$g(x,s;\mu^2) = \begin{cases} \frac{\hat{e}(0,\mu)e(x,\mu)e(s,\mu)}{2i\mu e(0,\mu)} - \frac{\hat{e}(x,\mu)e(s,\mu)}{2i\mu}, & x < s < \infty, \\ \frac{\hat{e}(0,\mu)e(x,\mu)e(s,\mu)}{2i\mu e(0,\mu)} - \frac{e(x,\mu)\hat{e}(s,\mu)}{2i\mu}, & 0 < s < x \end{cases}$$

and $\omega(x,\mu)$ is a solution to the following nonhomogeneous problem:

$$\begin{gathered} -\omega'' + q(x)\omega = \mu^2\omega + f(x), \\ \omega(a-0) = \alpha\omega(a+0), \quad \omega'(a-0) = \alpha^{-1}\omega'(a+0), \\ \omega(0) = 0. \end{gathered}$$

Note that all numbers $\lambda \geq 0$ belong to the continuous spectrum of L (see [18]). Moreover, the spectral singularities defined as the poles of the kernel function of the resolvent operator belong to the continuous spectrum. The set of spectral singularities of L is closed and its Lebesgue measure is zero which can be seen from the boundary uniqueness theorem for analytic functions [21] (also, see [1]).

Now, let us use the notation $\sigma_d(L)$ and $\sigma_{ss}(L)$ for the eigenvalues and spectral singularities of L, respectively.

$$\sigma_d(L) = \left\{\lambda : \lambda = \mu^2,\ \mathrm{Im}\mu > 0,\ e(0,\mu) = 0\right\},$$

$$\sigma_{ss}(L) = \left\{\lambda : \lambda = \mu^2,\ \mathrm{Im}\mu = 0,\ \mu \neq 0,\ e(0,\mu) = 0\right\}.$$

Moreover, the multiplicity m_k of a root μ_k of the equation $e(0,\mu)$ is called the multiplicity of μ_k.

Now, we will show that the nonself-adjoint operator L has a finite number of eigenvalues and spectral singularities under the following additional restrictions

$$\int_0^\infty e^{\epsilon x}|q(x)|dx < \infty, \quad \epsilon > 0, \tag{2.7}$$

$$\sup_{0 \leq x < \infty} \left\{\exp(\epsilon\sqrt{x})|q(x)|\right\} < \infty, \quad \epsilon > 0. \tag{2.8}$$

First, assume that condition (2.7) introduced by M.A. Naimark holds. This condition implies that

$$\sigma(x) = \int_x^\infty |q(t)|dt \leq C_\epsilon e^{-\epsilon x},$$

$$\sigma_1(x) = \int_x^\infty t|q(t)|dt \leq C_{\epsilon'} e^{-\epsilon' x},$$

where $C_\epsilon > 0$, $C_{\epsilon'} > 0$ and $0 < \epsilon' < \epsilon$ (see [18]). Using these relations and estimate (2.2), we have

$$|k(x,s)| \leq C \exp\left\{-\epsilon\left(\frac{x+s}{2}\right)\right\}, \tag{2.9}$$

where $C = cc_\epsilon e^{(c+1)d_\epsilon}$, $c = \alpha^+ + |\alpha^-|$, $c_\epsilon > 0$ and $d_\epsilon > 0$.

Theorem 2.2. *Suppose that condition* (2.7) *is valid. Then, the operator* L *has finite number of eigenvalues and spectral singularities with finite multiplicity.*

Proof. It is obtained from (2.9) that the function $e(0,\mu)$ has an analytic continuation from the real axis to the half plane $\text{Im}\mu > -\frac{\epsilon}{2}$. Then, there are no limit points of the sets of eigenvalues $\sigma_d(L)$ and spectral singularities $\sigma_{ss}(L)$ on the positive real line. Since $\sigma_d(L)$ and $\sigma_{ss}(L)$ are bounded and $e(0,\mu)$ is holomorphic in the half plane $\text{Im}\mu > -\frac{\epsilon}{2}$, L has finite number of eigenvalues and spectral singularities with finite multiplicity. □

Now, let condition (2.8) be satisfied. We need to show that the numbers of the spectral singularities and the eigenvalues under condition (2.8) are finite. First, we define the set of zeros of $e(0,\mu)$ in the closed upper half plane $\text{Im}\mu \geq 0$:

$$S_1 := \{\mu : \ \mu \in \mathbb{C}_+, \ e(0,\mu) = 0\}, \quad S_2 := \{\mu : \ \mu \in \mathbb{R}, \ \mu \neq 0, \ e(0,\mu) = 0\}.$$

Moreover, let us take into account that the sets S_3 and S_4 contain all limit points of S_1 and S_2 respectively, and the set S_5 has all infinite multiple zeros of $e(0,\mu)$. We can write

$$S_1 \cap S_5 = \emptyset, \quad S_3 \subset S_2, \quad S_4 \subset S_2, \quad S_5 \subset S_2$$

from the uniqueness theorem of analytic functions (see [7]) and

$$S_3 \subset S_5, \quad S_4 \subset S_5 \tag{2.10}$$

from the continuity of all derivatives of $e(0,\mu)$ up to the real axis.

Lemma 2.4. *Assume that condition* (2.8) *is satisfied, then* $S_5 = \emptyset$.

Proof. To prove this lemma, we use the following theorem (see [19], also [1, 2]): Suppose that the function φ is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis, and there exists $M > 0$ such that

$$|\varphi^{(v)}(z)| \leq K_v, \quad v = 0, 1, ... \ z \in \mathbb{C}_+, \ |z| < 2M, \tag{2.11}$$

and

$$\left|\int_{-\infty}^{-M} \frac{\ln|\varphi(x)|}{1+x^2}dx\right| < \infty, \quad \left|\int_M^\infty \frac{\ln|\varphi(x)|}{1+x^2}dx\right| < \infty. \tag{2.12}$$

If the set Q with the one-dimensional Lebesgue measure zero is the set of all zeros of the function φ with infinite multiplicity and the relation

$$\int_0^u \ln H(s) d\mu(Q_s) = -\infty \tag{2.13}$$

holds, then $\varphi(z) \equiv 0$, where u is an arbitrary positive constant, $H(s) = \inf_\upsilon \frac{K_\upsilon s^\upsilon}{\upsilon!}$, $\upsilon = 0, 1, \ldots$ and $\mu(Q_s)$ is the Lebesgue measure of s-neighborhood of Q.

Now, it follows from relation (2.2) and condition (2.8) that

$$|k(x,s)| \le \widetilde{C} \exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\}, \quad \widetilde{C} = cc_\epsilon e^{(c+1)c_\epsilon}, \quad c = \alpha^+ + |\alpha^-| > 0.$$

Then, the function $e(0,\mu)$ is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis and we have

$$\left|\frac{d^\upsilon e(0,\mu)}{d\mu^\upsilon}\right| \le K_\upsilon, \ \ \mu \in \overline{\mathbb{C}}_+, \ \ \upsilon = 1, 2, \ldots, \tag{2.14}$$

where

$$K_\upsilon = \widetilde{C}(2a)^\upsilon \left(1 + \int_0^\infty s^\upsilon \exp\left\{-\epsilon\left(\frac{s}{2}\right)^\delta\right\} ds\right), \quad \upsilon = 1, 2, \ldots$$

Moreover, since the set of zeros of $e(0,\mu)$ is bounded, for sufficiently large M the function $e(0,\mu)$ satisfies condition (2.12). Thus, it follows from this fact and relation (2.14) that $e(0,\mu)$ provides conditions (2.11) and (2.12). Since the function $e(0,\mu) \ne 0$, we have from (2.13)

$$\int_0^u \ln H(s) d\mu(S_{5,s}) > -\infty, \tag{2.15}$$

where $H(s) = \inf_\upsilon \frac{K_\upsilon s^\upsilon}{\upsilon!}$ and $\mu(S_{5,s})$ is the Lebesgue measure of the s-neighborhood of S_5. The following estimate holds

$$K_\upsilon \le \left(\widetilde{C}(2a)^\upsilon + Dd^\upsilon\right) \upsilon^\upsilon \upsilon!, \tag{2.16}$$

where $D = 4\frac{\widetilde{C}e}{\delta}\epsilon^{-\frac{1}{\delta}}(\upsilon+1)$ and $d = 8a\epsilon^{-\frac{1}{\delta}}$. In fact, we can write

$$\begin{aligned} K_\upsilon &= \widetilde{C}(2a)^\upsilon \left(1 + \int_0^\infty s^\upsilon \exp\left\{-\epsilon\left(\frac{s}{2}\right)^\delta\right\} ds\right) \\ &\le \widetilde{C}(2a)^\upsilon \left(1 + \frac{2^{(\upsilon+1)}}{\delta}\epsilon^{-\frac{(\upsilon+1)}{\delta}}(2\upsilon+2)^{\upsilon+1}\upsilon!\right) \\ &\le \widetilde{C}(2a)^\upsilon \left(1 + \frac{2^{2(\upsilon+1)}}{\delta}\epsilon^{-\frac{(\upsilon+1)}{\delta}}\left(1+\frac{1}{\upsilon}\right)^\upsilon (\upsilon+1)\upsilon^\upsilon\upsilon!\right) \\ &\le \left(\widetilde{C}(2a)^\upsilon + Dd^\upsilon\right)\upsilon^\upsilon\upsilon!. \end{aligned}$$

Putting estimate (2.16) into $H(s)$, we get

$$\begin{aligned} H(s) &\le \widetilde{C}\inf_\upsilon\{(2a)^\upsilon\upsilon^\upsilon\upsilon!\} + D\inf_\upsilon\{d^\upsilon\upsilon^\upsilon s^\upsilon\} \\ &\le \widetilde{C}\exp\left\{-(2a)^{-1}s^{-1}e^{-1}\right\} + D\exp\left\{-d^{-1}s^{-1}e^{-1}\right\}. \end{aligned} \tag{2.17}$$

Then, taking into account (2.15) and (2.17), we have

$$\int_0^u \frac{1}{s} d\mu(S_{5,s}) < \infty.$$

This inequality is valid for an arbitrary s if and only if $d\mu(S_{5,s}) = 0$ or $S_5 = \emptyset$. □

Theorem 2.3. *If condition (2.8) is satisfied, then L has finite number of eigenvalues and spectral singularities with finite multiplicity.*

Proof. It follows from (2.10) and Lemma 2.4 that $S_3 = \emptyset$ and $S_4 = \emptyset$. For this reason, the bounded sets S_1 and S_2 do not have limit points. Thus, the finiteness of the sets of $\sigma_d(L)$ and $\sigma_{ss}(L)$ are established. Moreover, due to $S_5 = \emptyset$, the eigenvalues and spectral singularities have finite multiplicities. □

3 Principal functions

Now, we examine the principal functions of L. Assume that condition (2.8) is satisfied.

Denote $\mu_1, \mu_2, ..., \mu_\ell$ by the zeros of $e(0,\mu)$ in $\mathbb{C}_+$ with multiplicities $m_1, m_2, ..., m_\ell$ respectively (note that $\mu_1^2, \mu_2^2, ..., \mu_\ell^2$ are the eigenvalues of L). We can write

$$\left\{\frac{d^\nu}{d\mu^\nu} W[e(x,\mu), s(x,\mu)]\right\}_{\mu=\mu_\eta} = \left\{\frac{d^\nu}{d\mu^\nu} e(0,\mu)\right\}_{\mu=\mu_\eta} = 0 \tag{3.1}$$

for $\nu = \overline{0, m_\eta - 1}$, $\eta = \overline{1,\ell}$. In case of $\nu = 0$, we have

$$e(x,\mu_\eta) = \kappa_0(\mu_\eta) s(x,\mu_\eta), \quad \kappa_0(\mu_\eta) \neq 0, \quad \eta = \overline{1,\ell}. \tag{3.2}$$

Lemma 3.1. *The following relation*

$$\left\{\frac{\partial^\nu}{\partial\mu^\nu} e(x,\mu)\right\}_{\mu=\mu_\eta} = \sum_{i=0}^{\nu} \binom{\nu}{i} \kappa_{\nu-i} \left\{\frac{\partial^i}{\partial\mu^i} s(x,\mu)\right\}_{\mu=\mu_\eta} \tag{3.3}$$

is valid for $\nu = \overline{0, m_\eta - 1}$, $\eta = \overline{1,\ell}$ *and here* $\kappa_0, \kappa_1 ..., \kappa_\nu$ *depend on* μ_η.

Proof. To prove of this lemma, we use the mathematical induction. Consider $\nu = 0$. It follows from relation (3.2) that the proof is trivial. Now, suppose that formula (3.3) is valid for ν_0 such that $0 < \nu_0 \leq m_\eta - 2$. That is,

$$\left\{\frac{\partial^{\nu_0}}{\partial\mu^{\nu_0}} e(x,\mu)\right\}_{\mu=\mu_\eta} = \sum_{i=0}^{\nu_0} \binom{\nu_0}{i} \kappa_{\nu_0-i} \left\{\frac{\partial^i}{\partial\mu^i} s(x,\mu)\right\}_{\mu=\mu_\eta}. \tag{3.4}$$

Then, we will show that formula (3.3) is satisfied for $\nu_0 + 1$. If $\omega(x,\mu)$ is a solution to (1.1), then we find

$$\left\{-\frac{d^2}{dx^2} + q(x) - \mu^2\right\} \frac{\partial^\nu}{\partial\mu^\nu}\omega(x,\mu) = 2\mu\nu \frac{\partial^{\nu-1}}{\partial\mu^{\nu-1}}\omega(x,\mu) + \nu(\nu-1)\frac{\partial^{\nu-2}}{\partial\mu^{\nu-2}}\omega(x,\mu). \tag{3.5}$$

Since $e(x,\mu)$ and $s(x,\mu)$ are solutions to equation (1.1), using (3.4) and (3.5) we calculate

$$\left\{-\frac{d^2}{dx^2} + q(x) - \mu_\eta^2\right\} h_{\nu_0+1}(x,\mu_\eta) = 0,$$

where

$$h_{\nu_0+1}(x,\mu_\eta) = \left\{\frac{\partial^{\nu_0+1}}{\partial\mu^{\nu_0+1}} e(x,\mu)\right\}_{\mu=\mu_\eta} - \sum_{i=0}^{\nu_0+1} \binom{\nu_0+1}{i} \kappa_{\nu_0+1-i} \left\{\frac{\partial^i}{\partial\mu^i} s(x,\mu)\right\}_{\mu=\mu_\eta}.$$

It follows from (3.1) that

$$W[h_{\nu_0+1}(x,\mu_\eta), s(x,\mu_\eta)] = \left\{\frac{d^{\nu_0+1}}{d\mu^{\nu_0+1}} W[e(x,\mu), s(x,\mu)]\right\}_{\mu=\mu_\eta} = 0.$$

Then, this shows that

$$h_{\nu_0+1}(x,\mu_\eta) = \kappa_{\nu_0+1}(\mu_\eta) s(x,\mu_\eta), \quad \eta = \overline{1,\ell}.$$

Consequently, we obtain that formula (3.3) is satisfied for $\nu = \nu_0 + 1$. □

Define the functions

$$\psi_\nu(x,\lambda_\eta)=\left\{\frac{\partial^\nu}{\partial\mu^\nu}e(x,\mu)\right\}_{\mu=\mu_\eta}=\sum_{i=0}^{\nu}\left(\begin{array}{c}\nu\\ i\end{array}\right)\kappa_{\nu-i}\left\{\frac{\partial^i}{\partial\mu^i}s(x,\mu)\right\}_{\mu=\mu_\eta} \tag{3.6}$$

for $\nu=\overline{0,m_\eta-1}$, $\eta=\overline{1,\ell}$ and $\lambda_\eta=\mu_\eta^2$.

Theorem 3.1. $\psi_\nu(x,\lambda_\eta)\in L_2(0,\infty)$ *for* $\nu=\overline{0,m_\eta-1},\ \ \eta=\overline{1,\ell}$.

Proof. Since

$$|k(x,s)|\le\widetilde{C}\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\},\quad \widetilde{C}=cc_\epsilon e^{(c+1)c_\epsilon},\quad c=\alpha^++|\alpha^-|>0,$$

using integral representation (2.1) we have for $0<x<a$

$$\begin{aligned}\left|\left\{\frac{\partial^\nu}{\partial\mu^\nu}e(x,\mu)\right\}_{\mu=\mu_\eta}\right| &\le x^\nu\alpha^+e^{-\mathrm{Im}\mu_\eta x}+(2a-x)^\nu|\alpha^-|e^{-\mathrm{Im}\mu_\eta(2a-x)}\\ &\quad+\widetilde{C}\int_x^\infty s^\nu\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\}e^{-\mathrm{Im}\mu_\eta}ds\end{aligned} \tag{3.7}$$

and for $a<x<\infty$

$$\left|\left\{\frac{\partial^\nu}{\partial\mu^\nu}e(x,\mu)\right\}_{\mu=\mu_\eta}\right|\le x^\nu e^{-\mathrm{Im}\mu_\eta x}+\widetilde{C}\int_x^\infty s^\nu\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\}e^{-\mathrm{Im}\mu_\eta}ds. \tag{3.8}$$

Since $\lambda_\eta=\mu_\eta^2$, $\eta=\overline{1,\ell}$ are eigenvalues of operator L, it follows from (3.7) and (3.8) for $\mathrm{Im}\mu_\eta>0$ that

$$\left\{\frac{\partial^\nu}{\partial\mu^\nu}e(x,\mu)\right\}_{\mu=\mu_\eta}\in L_2(0,\infty),\quad \nu=\overline{0,m_\eta-1},\quad \eta=\overline{1,\ell}.$$

Consequently, from (3.6) we have $\psi_\nu(x,\lambda_\eta)\in L_2(0,\infty)$, $\nu=\overline{0,m_\eta-1}$, $\eta=\overline{1,\ell}$. □

Definition 1. Functions $\psi_0(x,\lambda_\eta)$, $\psi_1(x,\lambda_\eta)$,...,$\psi_{m_\eta-1}(x,\lambda_\eta)$ are called principle functions associated with the eigenvalues $\lambda_\eta=\mu_\eta^2$, $\eta=\overline{1,\ell}$ of L. The function $\psi_0(x,\lambda_\eta)$ is the eigenfunction, $\psi_1(x,\lambda_\eta)$, $\psi_2(x,\lambda_\eta)$,..., $\psi_{m_\eta-1}(x,\lambda_\eta)$ are the associated functions of $\psi_0(x,\lambda_\eta)$ corresponding to the eigenvalue λ_η.

Now, we define the spectral singularities of L: $\mu_{\ell+1},\mu_{\ell+2},...,\mu_\beta$ are the zeros of $e(0,\mu)$ in $\mathbb{R}-\{0\}$ with multiplicities $m_{\ell+1},m_{\ell+2},...,m_\beta$, respectively. Then, similarly to the proof of Lemma 3.1, we obtain

$$\left\{\frac{\partial^v}{\partial\mu^v}e(x,\mu)\right\}_{\mu=\mu_\gamma}=\sum_{j=0}^{v}\left(\begin{array}{c}v\\ j\end{array}\right)\tau_{v-j}(\mu_\gamma)\left\{\frac{\partial^j}{\partial\mu^j}s(x,\mu)\right\}_{\mu=\mu_\gamma}$$

for $v=\overline{0,m_\gamma-1}$, $\gamma=\ell+1,\ell+2,...,\beta$. Consider the functions

$$\psi_v(x,\lambda_\gamma)=\left\{\frac{\partial^v}{\partial\mu^v}e(x,\mu)\right\}_{\mu=\mu_\gamma}=\sum_{j=0}^{v}\left(\begin{array}{c}v\\ j\end{array}\right)\tau_{v-j}(\mu_\gamma)\left\{\frac{\partial^j}{\partial\mu^j}s(x,\mu)\right\}_{\mu=\mu_\gamma} \tag{3.9}$$

for $v=\overline{0,m_\gamma-1}$, $\gamma=\ell+1,\ell+2,...,\beta$ and $\lambda_j=\mu_j^2$.

Theorem 3.2. *The functions $\psi_v(x,\lambda_\gamma)$ do not belong to $L_2(0,\infty)$ for $v=\overline{0,m_\gamma-1}$, $\gamma=\ell+1,\ell+2,...,\beta$.*

Proof. Take into account relations (3.7) and (3.8) for $\mu=\mu_\gamma$, $\gamma=\ell+1,\ell+2,...,\beta$ and since $\mathrm{Im}\mu_\gamma=0$ for the spectral singularities, we have

$$\left\{\frac{\partial^v}{\partial\mu^v}e(x,\mu)\right\}_{\mu=\mu_\gamma}\notin L_2(0,\infty),\ v=\overline{0,m_\gamma-1},\ \gamma=\overline{\ell+1,\beta}.$$

As a result, from the definition of functions (3.9), we find $\psi_v(x,\lambda_\gamma)\notin L_2(0,\infty)$ for $v=\overline{0,m_\gamma-1}$, $\gamma=\ell+1,\ell+2,...,\beta$. □

Now, we introduce the Hilbert spaces

$$H_\rho=\left\{f:\|f\|_\rho<\infty\right\},\quad H_{-\rho}=\left\{f:\|f\|_{-\rho}<\infty\right\},\quad \rho=1,2,...$$

with the norms

$$\|f\|_\rho^2=\int_0^\infty(1+s)^{2\rho}|f(s)|^2ds,\quad \|f\|_{-\rho}^2=\int_0^\infty(1+s)^{-2\rho}|f(s)|^2ds$$

respectively and evidently, $H_0=L_2(0,\infty)$.

Let m_0 denote the greatest of the multiplicities of the spectral singularities of L:

$$m_0=\max\left\{m_{\ell+1},m_{\ell+2},...,m_\beta\right\}.$$

We put

$$H_+=H_{m_0+1},\quad H_-=H_{-(m_0+1)}$$

Then, we have

$$H_+\subset L_2(0,\infty)\subset H_-$$

and for all $f\in H_+$, $\|f\|_+\geq\|f\|\geq\|f\|_-$, where $\|.\|_\pm=\|.\|_{\pm(m_0+1)}$, $\|.\|=\|.\|_0$ (see [18]). We are particularly interested in the space H_- because the space H_- contains the principal functions for the spectral singularities. Now, we will prove the above claim by using the following lemma.

Lemma 3.2. *The following relations hold:*

$$\sup_{0\leq x<\infty}\frac{\left|e^{(m)}(x,\mu)\right|}{(1+x)^m}<\infty,\quad e^{(m)}=\left(\frac{d}{d\mu}\right)^m e,\quad \mathrm{Im}\mu=0,\quad m=0,1,2,... \tag{3.10}$$

Proof. Using integral representation (2.1), we obtain for $\mathrm{Im}\mu=0$

$$\begin{aligned}\left|e^{(m)}(x,\mu)\right| &\leq x^m\alpha^++(2a-x)^m|\alpha^-|\\ &\quad+\widetilde{C}\int_x^\infty s^m\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\}ds,\quad 0<x<a\end{aligned} \tag{3.11}$$

and

$$\left|e^{(m)}(x,\mu)\right|\leq x^m+\widetilde{C}\int_x^\infty s^m\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\}ds,\quad a<x<\infty. \tag{3.12}$$

Then, taking into account (3.11) and (3.12), we find $\sup_{0\leq x<\infty}\frac{\left|e^{(m)}(x,\mu)\right|}{(1+x)^m}<\infty$. □

Theorem 3.3. *$\psi_v(x,\lambda_\gamma) \in H_{-(v+1)}$ for $v = 0,1,...,m_\gamma - 1$, $\gamma = \ell+1, \ell+2, ..., \beta$.*

Proof. Using relation (3.10), we have

$$\left\| e^{(v)}(x,\mu) \right\|_{-(v+1)}^2 = \int_0^\infty \left| \frac{e^{(v)}(x,\mu)}{(1+x)^{v+1}} \right|^2 dx < \infty.$$

That is, the functions $e^{(v)}(x,\mu) = \frac{\partial^v}{\partial \mu^v} e(x,\mu) \in H_{-(v+1)}$ for $\mathrm{Im}\mu = 0$ and $v = 0,1,2,....$ Then, we get

$$\left\{ \frac{\partial^v}{\partial \mu^v} e(x,\mu) \right\}_{\mu=\mu_\gamma} \in H_{-(v+1)}$$

for $\mathrm{Im}\mu_\gamma = 0$, $v = 0,1,...,m_\gamma - 1$ and $\gamma = \ell+1, \ell+2, ..., \beta$. Consequently, it follows from formula (3.9) that $\psi_v(x,\lambda_\gamma) \in H_{-(v+1)}$ for $v = 0,1,...,m_\gamma - 1$, $\gamma = \ell+1, \ell+2, ..., \beta$. □

Definition 2. The functions $\psi_0(x,\lambda_\gamma), \psi_1(x,\lambda_\gamma), ..., \psi_{m_\gamma - 1}(x,\lambda_\gamma)$ are called the principal functions associated with the spectral singularities $\lambda_\gamma = \mu_\gamma^2$, $\gamma = \ell+1, \ell+2, ..., \beta$ of operator L. The function $\psi_0(x,\lambda_\gamma)$ is the generalized eigenfunction, $\psi_1(x,\lambda_\gamma), ..., \psi_{m_\gamma - 1}(x,\lambda_\gamma)$ are the generalized associated functions of $\psi_0(x,\lambda_\gamma)$ corresponding to the spectral singularity λ_γ.

4 Conclusion

In this paper, we examine the spectrum and principal functions of a nonself-adjoint Sturm-Liouville operator with discontinuity conditions at the point $x = a \in (0,\infty)$. When examining the spectrum of problem (1.1)-(1.3), we use the Jost solution to equation (1.1) with discontinuity condition (1.2) which is obtained by Huseynov and Osmanova [9] and in this work. The triangular property of the Jost solution is lost and the kernel function has a discontinuity along the line $s = 2a - x$ for $x \in (0,a)$. Under two different additional conditions, it is proved that problem (1.1)-(1.3) has finite number of eigenvalues and spectral singularities with finite multiplicity. Finally, since restriction (2.8) is weaker than restriction (2.7), we determine the principal functions corresponding to the eigenvalues and spectral singularities of problem (1.1)-(1.3) under additional restriction (2.8).

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BARRIER COMPOSED OF PERFORATED RESONATORS AND BOUNDARY CONDITIONS

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Abstract. We consider the Laplace operator with the Neumann boundary condition in a two-dimensional domain divided by a barrier composed of many small Helmholtz resonators coupled with the both parts of the domain through small windows of diameter $2a$. The main terms of the asymptotic expansions in a of the eigenvalues and eigenfunctions are considered in the case in which the number of the Helmholtz resonators tends to infinity. It is shown that such a homogenization procedure leads to some energy-dependent boundary condition in the limit. We use the method of matching the asymptotic expansions of boundary value problem solutions.

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1 Introduction

Construction of unusual boundary conditions, particularly, energy-dependent (see, e.g., [2]), is important for many physical applications. A possible way for doing this is as follows. Let us take a resonator with the boundary composed of many small resonators coupling to the main cavity. After performing the limiting procedure when the number of these small resonators tends to infinity one obtains the system with a boundary condition as a result of homogenization. The problem of such type was considered in [7]. Later it was investigated in the framework of the model of point-like windows [16]. A number of works were devoted to the problem of homogenization [3, 4, 5, 6, 11, 20, 23]. As for physical applications, the additional interest was recently excited by the problem of metamaterials creation [13, 20, 12, 15].

In the present paper, we consider the two-dimensional system of two resonators Ω^+ and Ω^- separated by a barrier composed of N identical small resonators Ω^N coupled to each big resonator through small windows of width $2a$. The geometry of the domain is shown in Fig. 1a.

We consider the asymptotics of an eigenvalue and an eigenfunction close to some eigenvalue of Ω^-. We use the method of matching the asymptotic expansions of boundary value problems [10, 9, 14, 22, 18, 8]. Briefly speaking, the scheme is as follows. We construct to circles of radii $\sqrt{a}$, $\sqrt{2a}$ centered at the center of each window (see Fig. 1b). One constructs the external asymptotic expansion in the exterior of each small circle and the internal asymptotic expansion in the interior of each larger circle. In the domain between the two circles we make matching of the asymptotic expansions. We obtain the main terms of the asymptotic expansion of the eigenvalue and the corresponding eigenfunction. We construct formal asymptotic expansion. The justification of the matching method is, e.g., in [9].

The next step is performing the limiting procedure as $N \to \infty$. We show that the limit of the eigenfunction satisfies the integral equation at the boundary which corresponds to some energy-dependent boundary condition analogous to delta-potential supported by line in $\mathbb{R}$.

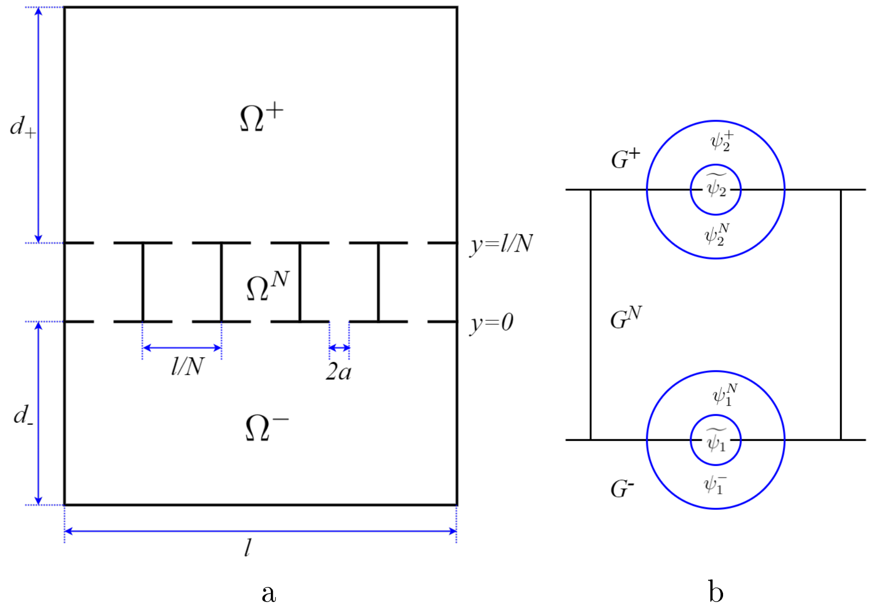

Figure 1: The geometry of the system: a – the whole system; b – construction of the domains for matching the asymptotic expansions. The Green function (G^+, G^N or G^-) which is used in the corresponding domain is indicated.

2 Matching of asymptotic expansions

First, let us consider the spectral problem for the Neumann Laplacian in $\Omega = \Omega^+ \cup \Omega^N \cup \Omega^-$, i.e. we deal with the following boundary problem

$$\Delta u + k^2 u = 0, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \tag{2.1}$$

where $\partial\Omega$ is the domain boundary.

The Green functions for the upper (G^+), intermediate (G^N) and the lower (G^-) resonators with the Neumann boundary condition can be expressed using the corresponding eigenfunctions:

$$G^+(X, X', k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{4}{ld_+} \frac{\cos(\frac{\pi n x}{l})\cos(\frac{\pi n x'}{l})\cos(\frac{\pi m y}{d_+})\cos(\frac{\pi m y'}{d_+})}{\left(k^2 - \frac{\pi^2 n^2}{l^2} - \frac{\pi^2 m^2}{d_+^2}\right)(\delta_{nm}+1)}, \tag{2.2}$$

$$G^N(X, X', k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{4N^2}{l^2} \frac{\cos(\frac{\pi n N x}{l})\cos(\frac{\pi n N x'}{l})\cos(\frac{\pi m N y}{l})\cos(\frac{\pi m N y'}{l})}{\left(k^2 - \frac{\pi^2 N^2}{l^2}(n^2+m^2)\right)(\delta_{nm}+1)}, \tag{2.3}$$

$$G^-(X, X', k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{4}{ld_-} \frac{\cos(\frac{\pi n x}{l})\cos(\frac{\pi n x'}{l})\cos(\frac{\pi m y}{d_-})\cos(\frac{\pi m y'}{d_-})}{\left(k^2 - \frac{\pi^2 n^2}{l^2} - \frac{\pi^2 m^2}{d_-^2}\right)(\delta_{nm}+1)}, \tag{2.4}$$

where $X = (x, y)$, $X' = (x', y')$, δ_{nm} is the Kronecker symbol, ($\delta_{nm} = 1$ for $n^2 + m^2 = 0$, otherwise $\delta_{nm} = 0$).

We will investigate the asymptotics in the small parameter a of the eigenvalues of the corresponding operator in the case in which there are small coupling windows (of width $2a$) connecting Ω^N with Ω^- and Ω^+. Naturally, we pose the Meixner condition at the windows edges. Let the considered eigenvalue k^2 be close to the following eigenvalue (λ_{11}) of the lower resonator: $k^2 \approx \lambda_{11} = \frac{\pi^2}{l^2} + \frac{\pi^2}{d_-^2}$.

We assume the following ansatz for the asymptotics:

$$\gamma_a := k_a^2 - \frac{\pi^2}{l^2} - \frac{\pi^2}{d_-^2} = \tau_1 \ln^{-1} a + \tau_2 \ln^{-2} a + o(\ln^{-2} a), \quad a \to 0+. \tag{2.5}$$

In such a case, the term $n = m = 1$ in (2.4) has a singularity.

To find the coefficients of asymptotic expansion (2.5), we use the conventional scheme of matching the asymptotic expansions of boundary value problems (see, e.g., [10, 9, 14, 22, 18, 8, 17]. Briefly speaking, it is as follows. Let the coupling windows be centered at points $(\frac{2jl+1}{2N}, 0), (\frac{2jl+1}{2N}, \frac{l}{N}), j = 0, ...N-1$. For each point, let us form two circles of radii $\sqrt{a}$ and $\sqrt{2a}$ centered at these points (see Fig. 1b). One constructs the internal asymptotic expansion of the solution inside the larger circle and the external asymptotic expansion of the solution outside the smaller circle. Correspondingly, in the ring between the two circles one has two asymptotic expansions. The proper expansion is obtained by matching of these two expansions in each ring.

We search a solution to (2.1) near the i-th coupling window in the following form:

Near the line $y = l/N$ we have $\psi_2(x)$:

$$\begin{cases} \psi_2^+(x, \frac{l}{N}) = -\gamma_a \sum_{j=1}^N \beta_j G^+((x, \frac{l}{N}), (x_j, \frac{l}{N}), k); & x \in \Omega^+, \\ \tilde{\psi}_2(x), & \text{matching domain}, \\ \psi_2^N = \gamma_a \Big[\alpha_i G^N((x, \frac{l}{N}), (x_i, 0), k) + \beta_i G^N((x, \frac{l}{N}), (x_i, \frac{l}{N}), k)\Big], & x \in \Omega^N; \end{cases} \tag{2.6}$$

near the line $y = 0$, $\psi_1(x)$:

$$\begin{cases} \psi_1^N = \gamma_a \Big[\alpha_i G^N((x, 0), (x_i, 0), k) + \beta_i G^N((x, 0), (x_i, \frac{l}{N}), k)\Big], & x \in \Omega^N, \\ \tilde{\psi}_1(x) & \text{matching domain}, \\ \psi_1^+(x, 0) = -\gamma_a \sum_{j=1}^N \alpha_j G^-((x, 0), (x_j, 0), k); & x \in \Omega^-. \end{cases} \tag{2.7}$$

The asymptotics of the Green functions near the coupling windows are as follows (here $\xi = \frac{x - x_i}{a}$). We have two small parameters: the width of window a and the distance between the neighbor windows centers ε. Correspondingly, one has the following asymptotics.

Near the line $y = l/N$:

$$G^+\Big((x, \frac{l}{N}), (x_i, \frac{l}{N}), k_a\Big) = -\frac{1}{\pi} \ln a + g_1^+(x) - \frac{1}{\pi} \ln |\xi|, \quad a \to 0; \tag{2.8}$$

$$G^+\Big((x, \frac{l}{N}), (x_i, \frac{l}{N}), k_a\Big) = -\frac{1}{\pi} \ln \varepsilon + h^+(x) - \frac{1}{\pi} \ln |\xi|, \quad \varepsilon \to 0; \tag{2.9}$$

$$G^N\Big((x, \frac{l}{N}), (x_i, \frac{l}{N}), k_a\Big) = -\frac{1}{\pi} \ln a + g_2^N(x) - \frac{1}{\pi} \ln |\xi|, \quad a \to 0; \tag{2.10}$$

$$G^N\Big((x, \frac{l}{N}), (x_i, 0), k_a\Big) = -\frac{1}{\pi} \ln \varepsilon + g_3^N(x) - \frac{1}{\pi} \ln |\xi|, \quad \varepsilon \to 0; \tag{2.11}$$

near the line $y = 0$:

$$G^-\Big((x, 0), (x_i, 0), k_a\Big) = -\frac{1}{\pi} \ln a + \frac{4}{ld_-} \frac{\cos(\frac{\pi x}{l}) \cos(\frac{\pi x_i}{l})}{k^2 - \frac{\pi^2}{l^2} - \frac{\pi^2}{d_-^2}} + g_4^-(x) - \frac{1}{\pi} \ln |\xi|, \quad a \to 0; \tag{2.12}$$

$$G^-\Big((x, 0), (x_i, 0), k_a\Big) = -\frac{1}{\pi} \ln \varepsilon + \frac{4}{ld_-} \frac{\cos(\frac{\pi x}{l}) \cos(\frac{\pi x_i}{l})}{k^2 - \frac{\pi^2}{l^2} - \frac{\pi^2}{d_-^2}} + h^-(x) - \frac{1}{\pi} \ln |\xi|, \quad \varepsilon \to 0; \tag{2.13}$$

$$G^N\Big((x,0),(x_i,0),k_a\Big) = -\frac{1}{\pi}\ln a + g_5^N(x) - \frac{1}{\pi}\ln|\xi|, \quad a \to 0; \tag{2.14}$$

$$G^N\Big((x,0),(x_i,\frac{l}{N}),k_a\Big) = -\frac{1}{\pi}\ln\varepsilon + g_6^N(x) - \frac{1}{\pi}\ln|\xi|, \quad \varepsilon \to 0. \tag{2.15}$$

Here g_1^+, h^+, $g_{2,3}^N$, g_4^-, h^-, $g_{5,6}^N$ are regular functions in the corresponding domains.

Let us assume that there is a relation between the small parameters:

$$\varepsilon = ma^\delta, \quad \ln\varepsilon = \delta\ln a + \text{const}, \quad \delta \in (0,1). \tag{2.16}$$

Taking equal terms of zero order of $\psi_2(x)$ on the line $y = l/N$ near the i-th window, one obtains the following relation

$$-\Big[\beta_i\big(-\frac{\tau_1}{\pi}\big) + \sum_{j\neq i}\beta_j\big(-\frac{\delta\tau_1}{\pi}\big)\Big] = \alpha_i\big(-\frac{\delta\tau_1}{\pi}\big) + \beta_i\big(-\frac{\tau_1}{\pi}\big), \tag{2.17}$$

or

$$\delta\alpha_i + 2\beta_i + \delta\sum_{j\neq i}\beta_j = 0. \tag{2.18}$$

The analogous operation with zero order terms of $\psi_1(x)$ on the line $y = 0$ gives one the relation:

$$\begin{gathered}\alpha_i\big(-\tfrac{\tau_1}{\pi}\big) + \beta_i\big(-\tfrac{\delta\tau_1}{\pi}\big) = -\alpha_i\Big[-\tfrac{\tau_1}{\pi} + \tfrac{4}{ld_-}\cos^2\big(\tfrac{\pi x_i}{l}\big)\Big] - \\ \textstyle\sum_{j\neq i}\alpha_j\Big[-\tfrac{\delta\tau_1}{\pi} + \tfrac{4}{ld_-}\cos\big(\tfrac{\pi x_j}{l}\big)\cos\big(\tfrac{\pi x_i}{l}\big)\Big]\end{gathered} \tag{2.19}$$

or

$$\begin{gathered}\alpha_i\Big[\cos^2\big(\tfrac{\pi x_i}{l}\big) - \tfrac{\tau_1 l d_-}{2\pi}\Big] + \\ \textstyle\sum_{j\neq i}\alpha_j\Big[\cos\big(\tfrac{\pi x_i}{l}\big)\cos\big(\tfrac{\pi x_j}{l}\big) - \tfrac{\delta\tau_1 l d_-}{4\pi}\Big] - \beta_i\cdot\tfrac{\tau_1 l d_-}{4\pi} = 0.\end{gathered} \tag{2.20}$$

Let us denote for brevity

$$b = \frac{\tau_1 l d_-}{4\pi}, \quad \tilde{x}_i = \frac{\pi x_i}{l}. \tag{2.21}$$

Then

$$\alpha_i\Big[\cos^2(\tilde{x}_i) - 2b] + \sum_{j\neq i}\alpha_j\Big[\cos(\tilde{x}_i)\cos(\tilde{x}_j) - \delta b\Big] - \delta b\beta_i = 0. \tag{2.22}$$

Incorporating (2.18) and (2.22), we come to the following theorem.

Theorem 2.1. *If $\varepsilon = ma^\delta$, $\delta \in (0,1)$, then matching of terms of zero order in asymptotic expansion of a solution to* (2.1) *leads to the following system of equations for the coefficients α_i, β_i of representations* (2.7), (2.6)*:*

$$\begin{pmatrix} \cos^2\tilde{x}_1 - 2b & \cos\tilde{x}_1\cos\tilde{x}_2 - \delta b & \dots & \cos\tilde{x}_1\cos\tilde{x}_N - \delta b & -\delta b & 0 & \dots & 0 \\ \cos\tilde{x}_2\cos\tilde{x}_1 - \delta b & \cos^2\tilde{x}_2 - 2b & \dots & \cos\tilde{x}_2\cos\tilde{x}_N - \delta b & 0 & -\delta b & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 & \dots & \dots & 0 \\ \cos\tilde{x}_N\cos\tilde{x}_1 - \delta b & \cos\tilde{x}_N\cos\tilde{x}_2 - \delta b & \dots & \cos^2\tilde{x}_N - 2b & 0 & 0 & \dots & -\delta b \\ \delta & 0 & \dots & 0 & 2 & \delta & \dots & \delta \\ 0 & \delta & \dots & 0 & \delta & 2 & \dots & \delta \\ \dots & \dots & \dots & \dots & \delta & \delta & \dots & \delta \\ 0 & 0 & \dots & \delta & \delta & \delta & \dots & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix} = 0. \tag{2.23}$$

A necessary and sufficient condition for the existence of non-trivial solutions to (2.23) is vanishing of the system determinant. The determinant can be exactly calculated [19]. This gives one the expression for b:

$$b = \frac{N}{8} \cdot \frac{2-\delta}{1-\delta}. \tag{2.24}$$

Taking into account (2.21), one obtains the expression for τ_1. This is resulted in the following theorem.

Theorem 2.2. *If* $\varepsilon = ma^\delta$, $\delta \in (0,1)$, *then the coefficient* τ_1 *of the main term of asymptotic expansion* (2.5) *of the eigenvalue close to* $\frac{\pi^2}{l^2} + \frac{\pi^2}{d_-^2}$ *is as follows:*

$$\tau_1 = \frac{\pi N}{2ld_-} \cdot \frac{2-\delta}{1-\delta}. \tag{2.25}$$

3 Integral equation

Let us consider the limiting case $N \to \infty$. Our goal is obtaining the integral equation for the limit of the eigenfunction corresponding to eigenvalue k_a^2.

Using asymptotics of the Green functions (2.8), (2.9), (2.12), (2.13) and expansion (2.5), one can obtain the asymptotic expansion for the eigenfunction at the upper and lower sides of the barrier:

$$\psi_1^-(x_i,0) = \alpha_i\Big(\frac{\tau_1}{\pi} - \frac{4}{ld_-}\cos^2 x_i\Big) + \sum_{j\neq i}\alpha_j\Big(\frac{\tau_1}{\pi}\delta - \frac{4}{ld_-}\cos x_i \cos x_j\Big) + o(1), \tag{3.1}$$

$$\psi_2^+(x_i,\frac{l}{N}) = \frac{\tau_1}{\pi}\beta_i + \frac{\tau_1}{\pi}\delta\sum_{j\neq i}\beta_j + o(1). \tag{3.2}$$

It is convenient to return from τ_1 to b: $\tau_1 = \frac{4\pi}{ld_-}b$. Then, one rewrites the equations:

$$-\frac{ld_-}{4}\psi_1^-(x_i,0) = \alpha_i(\cos^2 x_i - b) + \sum_{j\neq i}\alpha_j(\cos x_i\cos x_j - \delta b) + o(1), \tag{3.3}$$

$$\frac{ld_-}{4b}\psi_2^+(x_i,\frac{l}{N}) = \beta_i + \delta\sum_{j\neq i}\beta_j + o(1). \tag{3.4}$$

Let us join (2.18), (2.22), (3.3) and (3.4) into the system:

$$\begin{cases} \delta\alpha_i + 2\beta_i + \delta\sum_{j\neq i}\beta_j = 0, \\ \alpha_i(\cos^2(x_i) - 2b) - \delta b\beta_i + \sum_{j\neq i}\alpha_j(\cos x_i\cos x_j - \delta b) = 0, \\ \alpha_i(\cos^2 x_i - b) + \sum_{j\neq i}\alpha_j(\cos x_i \cos x_j - \delta b) = -\frac{ld_-}{4}\psi_1^-(x_i,0), \\ \beta_i + \delta\sum_{j\neq i}\beta_j = \frac{ld_-}{4b}\psi_2^+(x_i,\frac{l}{N}). \end{cases} \tag{3.5}$$

Solving (3.5) with respect to α_i and β_i, one obtains:

$$\alpha_i = \frac{ld_-}{4(1-\delta^2)b}\Big[\delta\psi_2^+(x_i,\frac{l}{N}) - \psi_1^-(x_i,0)\Big], \tag{3.6}$$

$$\beta_i = \frac{ld_-}{4(1-\delta^2)b}\Big[\delta\psi_1^-(x_i,0) - \psi_2^+(x_i,\frac{l}{N})\Big]. \tag{3.7}$$

Note that if one poses $\psi_1^-(x_i,0) = \psi_2^+(x_i,\frac{l}{N})$ then $\alpha_i = \beta_i$.

Let us substitute the obtained expressions in (2.6) and (2.7), summarize them and use expression (2.24) for b. Then one obtains

$$\psi_1^-(x,0)+\psi_2^+(x,\frac{l}{N})=\frac{2d_-}{(1+\delta)(2-\delta)}\gamma_a\times$$

$$\sum_{i=1}^{N}\frac{l}{N}\Big[G^-(x,x_i)\psi_1^-(x_i,0)+G^+(x,x_i)\psi_2^+(x_i,\frac{l}{N})\Big]+$$

$$\frac{l}{N}\delta\Big[G^-(x,x_i)\psi_2^+(x_i,\frac{l}{N})+G^+(x,x_i)\psi_1^-(x_i,0)\Big]. \quad (3.8)$$

We assume that $\psi_1^-(x,0)=\psi_2^+(x,\frac{l}{N})=\psi(x)$. It leads to simplifying the expression:

$$2\psi(x)=\frac{2d_-(1-\delta)}{(1+\delta)(2-\delta)}\gamma_a\sum_{i=1}^{N}\frac{l}{N}\Big(G^-(x,x_i)+G^+(x,x_i)\Big)\psi(x_i). \quad (3.9)$$

One can see that the sum in the right hand side is the integral sum which turns in the integral over Γ, i.e. over segment $[0,l]$ in our case. Correspondingly, we come to the integral equation. As a result, we obtain the following theorem.

Theorem 3.1. *If* $\varepsilon=ma^\delta, \quad \delta\in(0,1)$, *then the eigenfunction corresponding to the eigenvalue close to* $\frac{\pi^2}{l^2}+\frac{\pi^2}{d_-^2}$ *tends as* $N\to\infty$ *to the function satisfying the following integral equation:*

$$\psi(x)=\frac{d_-(1-\delta)}{(1+\delta)(2-\delta)}\gamma_a\int_0^l\Big(G^-(x,x',k_a)+G^+(x,x',k_a)\Big)\psi(x')dx'. \quad (3.10)$$

4 Boundary condition and integral equation

After performing the limiting procedure $N\to\infty$, one comes to a rectangle $\Omega^+\cup\Omega^-$ divided by line $\Gamma=\{(x,y),y=0\}$. Let us consider the following boundary problem for $u=u(x,y)$:

$$\begin{gathered}\Delta u+k^2u=0,(x,y)\in\Omega^+\cup\Omega^-,\\ \frac{\partial u}{\partial n}\Big|_+=\frac{\alpha}{2}u\Big|_\Gamma,\quad \frac{\partial u}{\partial n}\Big|_-=\frac{\alpha}{2}u\Big|_\Gamma,\quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega^\pm\setminus\Gamma}=0,\\ u(x,0+)=u(x,0-).\end{gathered} \quad (4.1)$$

Here n is the inward normal for the corresponding domain.
Remark. Note that boundary conditions (4.1) on Γ correspond to the jump of the derivative at Γ:

$$\frac{\partial u}{\partial y}\Big|_+-\frac{\partial u}{\partial y}\Big|_-=\alpha u\Big|_\Gamma,$$

i.e. to 1D delta-potential at $y=0$ in the transverse direction.

Theorem 4.1. *A solution* u *to boundary problem* (4.1) *satisfies the following integral equation:*

$$u(x)=\frac{\alpha}{2}\int_\Gamma\Big(G^-(x,x',k_a)+G^+(x,x',k_a)\Big)u(x')dx',\quad x'\in\Gamma. \quad (4.2)$$

Proof. Let us multiply the normal derivative of u by the Green function for the Neumann Laplacian $G^{\pm}((x,y),(x',y'),k)$ in $\Omega^{\pm}$ and integrate over the boundary of $\partial\Omega^{-}\cup\partial\Omega^{+}$. Due to the properties of the Green function, one obtains the value of u in $\Omega^{\pm}$

$$u(x)=\int\limits_{\partial\Omega^{\pm}} G^{\pm}(x,x',k_a)\frac{\partial u}{\partial n}(x')dx', \quad x\in\Omega^{\pm}. \tag{4.3}$$

Note that $\frac{\partial u}{\partial n}\Big|_{\partial\Omega^{\pm}\setminus\Gamma}=0$. Correspondingly, only the integral over Γ is non-zero in the right hand side of (4.3). Considering the limiting value of u at $\partial\Omega^{\pm}$ and taking into account boundary conditions (4.1), one comes to the following integral equation:

$$u(x)=\frac{\alpha}{2}\int\limits_{\Gamma}\Big(G^{-}(x,x',k_a)+G^{+}(x,x',k_a)\Big)u(x')dx', \quad x'\in\Gamma. \tag{4.4}$$

One can see that equation (4.4) coincides with equation (3.10) if

$$\alpha=2\frac{d_{-}(1-\delta)}{(1+\delta)(2-\delta)} \quad \gamma_a.$$

□

Remark. Numerical results for resonator with a boundary composed of many Helmholtz resonators [1] show that the ratio of the eigenfunction normal derivative and the eigenfunction value stabilizes near a value depending on energy but independent of the point position at the boundary. It is in agreement with the results obtained above.

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WEAVING FRAMES LINKED WITH FRACTAL CONVOLUTIONS

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Abstract. Weaving frames have been introduced to deal with some problems in signal processing and wireless sensor networks. More recently, the notion of fractal operator and fractal convolutions have been linked with perturbation theory of Schauder bases and frames. However, the existing literature has established limited connections between the theory of fractals and frame expansions. In this paper we define weaving frames generated via fractal operators combined with fractal convolutions. The aim is to demonstrate how partial fractal convolutions are associated to Riesz bases, frames and the concept of weaving frames in a Hilbert space. The context deals with ones-sided convolutions i.e both left and right partial fractal convolution operators on Lebesgue space L^p $(1 \leq p \leq \infty)$. Some applications using partial fractal convolutions with null function have been obtained for the perturbation theory of bases and weaving frames.

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1 Introduction

Transmission of signals using frames, a redundant set of vectors in a Hilbert space are preferred over orthonormal or Reisz basis as they minimize the chance of losses and errors in the process of signal transmission. Due to the useful applications in the characterization of function spaces, signal processing, and many other fields of applications, the theory of frames has developed rather rapidly in recent years. One amongst many recent generalisation of frames given by Bemrose et al. in [2] is introduction of a new concept of *weaving frames* in separable Hilbert spaces. Because of some potential applications such as in wireless sensor networks and distributed signal processing, frames and weaving frames have attracted many researchers attention [1, 4, 9, 11]. Furthermore, some variations of woven frames were also considered [10, 12, 25, 26].

This paper highlights the connection of weaving frames with fractal interpolation functions. Many authors have studies that the framework of fractal interpolation which makes it possible to enlarge and improve the classical methods of approximation theory. In previous papers, the authors defined fractal functions constructed by means of iterated function systems, see, e.g, [3, 19]. These maps are fractal perturbations of arbitrary continuous functions defined on compact intervals. Recently, in [18] the fractal convolution, an internal binary operation, has been treated as an operation between two functions, namely the germ function f and the base function b (aside from other elements such as partition and scale factors). Navascues [19, 20, 21] introduced fractal versions of functions, in spaces associated fractal operator and some related notions. The current literature [23, 24] contains some interesting developments in which new bases and frames are obtained from the old ones by using the algebraic operation of fractal convolution allowing construction of frames and bases consisting of self-referential functions.

In this piece of research, an interested reader will further notice the properties of a fractal operator and its relationship with perturbation of woven frames. The motivation is derived from an examination of one-sided fractal convolutions, which we call the left and right partial fractal convolutions, with a different perspective. To be particular, the objective is to extend the link between fractal convolutions with the perturbation theory of bases and frames for Lebesgue space L^p $(1 \leq p \leq \infty)$ encouraged by developments in [21, 28]. The current source of information is a sequel to study frames linked to fractal convolutions. Not only this, weaving properties of frames lay the primary foundation of the extension.

In Section 2, we collect some definitions, and recall known results that are available in the theory of frames [5, 6, 8] and for the concept of fractal interpolation functions [3, 18, 19], which will be used in following sections. Main results are contained in Section 3 and Section 4. Future plans are discussed in Section 5.

2 Background on frames and fractals

2.1 Preliminaries for frames

In this section, we provide an overview of the technical background on *frames* and related work relevant to our research.

Definition 1. Let $\mathcal{H}$ be a real (or complex) separable Hilbert space with inner product $\langle .,\rangle$. A countable sequence $\{f_k\} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist numbers $A,\ B > 0$ such that,

$$A\|f\|^2 \leq \|\{\langle f, f_k\rangle\}\|_{\ell^2}^2 \leq B\|f\|^2 \ \ \forall\ f \in \mathcal{H}.$$

The inequality is called the *frame inequality* of the frame.

The operator $V : \ell^2 \to \mathcal{H}$ defined as

$$V(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k,\ \ \{c_k\} \in \ell^2,$$

is called the *pre-frame operator* (or *synthesis operator*) and its adjoint operator $V^* : \mathcal{H} \to \ell^2$, which is called the *analysis operator* is given by

$$V^*(f) = \{\langle f, f_k\rangle\} \ \ \forall\ f \in \mathcal{H}.$$

By composing V and V^* we obtain the *frame operator* $S = VV^* : \mathcal{H} \to \mathcal{H}$ defined by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k,\ \ \forall\ f \in \mathcal{H}.$$

The frame operator S is a positive, self-adjoint and invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* [6] for all $f \in \mathcal{H}$:

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k\rangle f_k \quad \left(= \sum_{k=1}^{\infty} \langle f, S^{-1}f_k\rangle f_k \right).$$

The scalars $\{\langle S^{-1}f, f_k\rangle\}$ are called *frame coefficients* of the vector $f \in \mathcal{H}$.

Definition 2 (Bessel sequence)**.** A family $\{\phi_m\} \subset \mathcal{H}$ is a *Bessel sequence for* $\mathcal{H}$ *if there exist a positive constant* K *such that,*

$$\sum_{m=0}^{\infty} |\langle f, \phi_m \rangle|^2 \leq K\|f\|^2, \ \forall \ f \in \mathcal{H}.$$

Definition 3 (Riesz sequence)**.** [5] A family $\{\phi_m\}$ in a separable Hilbert space $\mathcal{H}$ is a *Riesz sequence* for $\mathcal{H}$ if for all $\{c_m\} \in l^2(\mathbb{N}_0)$, there exists constants $0 < A \leq B < \infty$ such that,

$$A \sum_{m=0}^{\infty} |c_m|^2 \leq \sum_{m=0}^{\infty} \|c_m \phi_m\|^2 \leq B \sum_{m=0}^{\infty} |c_m|^2.$$

Definition 4 (Riesz basis)**.** [5] A collection of vectors $\{\phi_m\}$ in a Hilbert space $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if it is the image of an orthonormal basis for H under an invertible linear transformation. In other words, if there is an orthonormal basis $\{e_k\}$ for $\mathcal{H}$ and an invertible transformation $\mathcal{T}$ such that $\mathcal{T} e_k = \phi_k$.

Let Ξ be a finite or countable index set.

Definition 5 (Weaving frames)**.** [2] Two frames $\{\phi_i\}_{i\in\Xi}$ and $\{\psi_i\}_{i\in\Xi}$ in a separable Hilbert space $\mathcal{H}$ are said to be woven, if there are universal positive constants A and B such that for every subset $\sigma \subset \Xi$, the family $\{\phi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively.

Definition 6. [2] A family of frames $\{\phi_{ij}\}_{j=1, i\in\Xi}^{M}$ in $\mathcal{H}$ is said to be woven if there are universal constants A and B so that for every partition $\{\sigma_j\}_{j=1}^{M}$ of Ξ, the family $\{\phi_{ij}\}_{j=1, i\in\sigma_j}^{M}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively. Each such family $\{\phi_{ij}\}_{j=1, i\in\sigma_j}^{M}$ is called a weaving.

Remark 1. If a family of frames $\{\phi_{ij}\}_{j=1, i\in\Xi}^{M}$ is a *Bessel sequence* for $\mathcal{H}$ with bound B_j, then every weaving is a *Bessel sequence* with the Bessel bound $\sum_{j=1}^{M} B_j$.

Remark 2. If $\{\phi_i\}_{i\in\Xi}$ is a Riesz basis with Riesz bounds A, B and π is a permutation of Ξ, then for every $\sigma \subset \Xi$, the family $\{\phi_i\}_{i\in\sigma} \bigcup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ is a frame sequence with bounds A and $2B$. However, $\{\phi_i\}_{i\in\Xi}$ and $\{\phi_{\pi(i)}\}_{i\in\Xi}$ are woven if and only if $\pi = \Xi_d$.

2.2 Preliminaries for fractal functions

In this section, we provide an overview of the technical background and related work relevant to fractal interpolation functions (FIFs). The notion of FIF can be used to associate a parameterized family of fractal functions with a prescribed function that belongs to a standard function space. We observe that the Read-Bajraktarević operator leads the way to address the technical details concerning this in the setting of $\mathcal{L}^p$-spaces, see [18, 20, 28].

Let $\Delta := \{x_0, x_1, \ldots, x_T\}$, where $T \in \mathbb{N}, T > 1$, $x_0 < x_1 \ldots, x_T$ and $I = [x_0, x_T]$. We denote by $\mathcal{L}^p(I)$, the Banach space of all real-valued Lebesgue integrable functions defined on I, equipped with $\mathcal{L}^p$-norm $\|.\|$ for $1 \leq p < \infty$. The *germ function* will be denoted by f. It is a prescribed function belonging to $\mathcal{L}^p(I)$. Next we shall define a family of fractal functions which are self-referential functions associated to f.

For $T \in \mathbb{N}$ denote by $\mathbb{N}_T$ the subset of $\mathbb{N}$ consisting of the first T natural numbers. For $t \in \mathbb{N}_{T-1}$, let $I_t = [x_{t-1}, x_t)$ and $I_T = [x_{T-1}, x_T]$. Note that $I = \cup_{t\in\mathbb{N}_t} I_t$ and each point in the partition is exactly in one of the subintervals I_t. For $t \in \mathbb{N}_T$, suppose that $L_t : [x_0, x_T] \to [x_{t-1}, x_t]$ are affine maps of the form $L_t(x) = a_t x + b_t$, where a_t and b_t are determined such that $L_t(x_0) = x_{t-1}$ and $L_t(x_T) = x_t$.

For a fixed base function $\beta \in \mathcal{L}^p(I)$, we pick a *scale factor* and *scale vector* (or a *scaling vector*) respectively, defined as $\alpha_t \in (-1, 1)$ for $t \in \mathbb{N}_T$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_T) \in (-1, 1)^T$.

To each $\rho \in \mathcal{L}^p(I)$, we associate a Read-Bajraktarević type operator $B^{\alpha,\beta}_{f,\Delta} : \mathcal{L}^p(I) \to \mathcal{L}^p(I)$ in contrast with the germ function f and the parameters Δ, β and α, and is as follows,

$$B^{\alpha,\beta}_{f,\Delta}\rho(x) = f(x) + \alpha_t(\rho - \beta) \circ L_t^{-1}(x)$$

for $x \in I_t, t \in \mathbb{N}_T$.

The steps to show that $B^{\alpha,\beta}_{f,\Delta}$ is a contraction map are fairly direct and, hence, by the Banach fixed point theorem, $B^{\alpha,\beta}_{f,\Delta}$ admits a unique fixed point $f^{\alpha,\beta}_{\Delta}$. Correspondingly, we have: $f^{\alpha,\beta}_{\Delta}$ satisfies the self-referential equation:

$$f^{\alpha,\beta}_{\Delta} = f(x) + \alpha_t(f^{\alpha,\beta}_{\Delta} - \beta) \circ L_t^{-1}(x). \tag{2.1}$$

Let us define

$$\Lambda = \max\{|\alpha_t| : t \in \mathbb{N}_T < 1\}$$

The following inequality can be easily obtained using equation (2.1):

$$\|f^{\alpha,\beta}_{\Delta} - f\| \leq \frac{\Lambda}{1-\Lambda}\|f - \beta\|. \tag{2.2}$$

Remark 3. Note that, in particular, $f^{\alpha,\beta}_{\Delta} = f$, at $\alpha = 0$. It is clear that $f^{\alpha,\beta}_{\Delta}$ may agree with f in specified subintervals, taking the corresponding zero scale factors.

We call, $f^{\alpha,\beta}_{\Delta}$ an α-fractal function or "fractalization" of f. In fact, with different choices of the parameters as specified above, we obtain a family of fractal functions $\{f^{\alpha,\beta}_{\Delta}\}$ corresponding to f.

In the remaining part of the work, for any $\rho \in \mathcal{L}^p(I)$ and a sequence $(a_m) \in L^p(\mathrm{I})$, the existence of $k := \sum_{m=0}^{\infty}\|\rho - a_m\| < +\infty$, will be termed as that the sequence (ρ, a_m) satisfies the *k-condition.*

2.3 Fractal convolutions

Navascues and Massopust have defined and studied the fractal convolution operator in detail for L^p spaces in [18, 22]. More recently, an extensive study of perturbation of bases and frames have been presented via fractal convolution in [24]. The underlying rule considers "fractalization" of f as convolution. The fractal convolution associated with an Read-Bajraktarević operator B acting on the Lebesgue spaces $L^p(I)$. With respect to a given partition Δ, and a fixed scaling vector α, it is simply a binary operation on $\mathcal{L}^p(I) \times \mathcal{L}^p(I)$ given by

Definition 7 (fractal convolution)**.** For a germ f and a base β in $L^p(\mathrm{I})$

$$\digamma_{\Delta,\alpha} : L^p(\mathrm{I}) \longrightarrow L^p(\mathrm{I})$$
$$\digamma_{\Delta,\alpha}(f, \beta) = f *_{\Delta,\alpha} \beta$$

where $f *_{\Delta,\alpha} \beta$ is the α-fractal function $f^{\alpha,\beta}_{\Delta}$.

For the rest of the paper, for brevity, let us denote

$$\mathcal{F}_{\Delta,\alpha} = \mathcal{F}$$
$$f *_{\Delta,\alpha} \beta = f * \beta$$

It is easy to check that for any $f, \beta \in L^p(\mathrm{I})$, the fractal convolution operator $\mathcal{F}$ is linear and bounded.

The goal of the paper is to link weaving frames and fractal convolutions. The conceptual approach focuses on the core of convolutions but in a one-sided way. We shall now look at the partial convolution or one-sided convolutions via left or via right as defined with the help of operator $\mathcal{F}$.

Definition 8 (left partial fractal convolution)**.** For a fixed f in $L^p(\mathrm{I})$,

$$\mathcal{F}^l_f(\beta) = f * \beta, \;\; \beta \in L^p(\mathrm{I})$$

This is called f-left partial fractal convolution.

Definition 9 (right partial fractal convolution)**.** For a fixed β,

$$\mathcal{F}^r_\beta(f) = f * \beta, \;\; f \in L^p(\mathrm{I})$$

This is called β-*right* partial fractal convolution.

The authors in [24] listed several straight forward properties of fractal convolution operator $\mathcal{F}$ and the partial fractal convolutions $\mathcal{F}^l_f$, $\mathcal{F}^r_\beta$ respectively. Let us now review these properties for our later use in woven.

1. The fractal convolution $f * \beta$ satisfies a fixed point equation analogous to equation (2.1)
$$(f * b)(x) = f(x) + \alpha_n((f * b) - b) \circ L_n^{-1}(x), \forall\; x \in I_n, \tag{2.3}$$
where $f_\Delta^{\alpha,\beta}$ has been replaced by $f * \beta$.

2. The following inequalities are direct corollaries of above equation (2.3), algebraically observed in [22]
$$\|f * \beta - f * \beta'\| = \|\mathcal{F}^l_f(\beta) - \mathcal{F}^l_f(\beta')\| \leq \frac{\Lambda}{1-\Lambda}\|\beta - \beta'\|, \tag{2.4}$$
$$\|f * \beta - f' * \beta\| = \|\mathcal{F}^r_\beta(f) - \mathcal{F}^r_\beta(f')\| \leq \frac{1}{1-\Lambda}\|f - f'\|. \tag{2.5}$$

3. The f-left partial convolution $\mathcal{F}^l_f$ is nonlinear and Lipschitz continuous. Furthermore, if $\Lambda < \frac{1}{2}$, then it is a contraction whose unique fixed point is f. Similarly, the operator $\mathcal{F}^r_b$ is nonlinear and Lipschitz continuous

4. The closeness of $f * \beta$ to f and β are transferred in the form of inequalities as a result of the above first two items in this list in conjunction with (a): the uniqueness of the fixed point of the operator $B^{\alpha,\beta}_{f,\Delta}$ and (b): establishing that the operator $\mathcal{F}$ is idempotent, that is, $b * b = b$, for any b. Therefore,
$$\|f * \beta - f\| = \|\mathcal{F}^l_f(\beta) - \mathcal{F}^l_f(f)\| \leq \frac{\Lambda}{1-\Lambda}\|f - \beta\|, \tag{2.6}$$
$$\|f * \beta - \beta\| = \|\mathcal{F}^r_\beta(f) - \mathcal{F}^r_\beta(\beta)\| \leq \frac{1}{1-\Lambda}\|f - \beta\|. \tag{2.7}$$

In the current work, we start by linking weaving frames with left partial fractal convolutions.

3 Background on frames and fractals

3 Main results

So far, we have not seen a mechanism to construct weaving frames linked to fractal convolutions. The results presented in the mentioned literature in Section 2.3 are a stepping stone to derive links between fractal convolutions and perturbation theory of bases and frames for Banach spaces and Hilbert spaces. The primary objective in this section is to provide interesting connections between fractal convolutions and cases in which weaving frames can be obtained. In the presence of left-partial fractal convolutions, our approach is well aligned to prove the following theorems on obtaining sufficient conditions for existence of weaving frames. Note that these characterisations of weaving of left convolved frames is studied in a Hilbert space setting.

Theorem 3.1. *In a Hilbert space $\mathcal{H} = L^2(I)$, where (δ_m) is an orthonormal basis, and a given convolved Riesz basis $(\rho * \{\delta_m\}_{m\in I})$ with bounds M_l and M_u respectively, where,*

$$M_l = \frac{1-\lambda-k}{1-\lambda} \quad \& \quad M_u = \frac{1-\lambda+k}{1-\lambda}$$

- *λ is as defined in Section 2.2.*
- *(ρ, δ_m) satisfies the k-*condition *which is bounded by $(1-\lambda)$.*

then for every $\sigma \subset I$, the family $(\rho * \{\delta_m\}_{m\in\sigma}) \bigcup (\rho * \{\delta_{\pi(m)}\}_{m\in\sigma^c})$ is a frame sequence with bounds M_l and $2M_u$.

Moreover, $(\rho * \{\delta_m\}_{m\in I})$ and $(\rho * \{\delta_{\pi(m)}\}_{m\in I})$ are woven if and only if $\pi = I_d$, π is a permutation of I.

We shall present the proof of Theorem 3.1 a little later after we have discussed the existence of the frame $(\rho * \phi_m)$ as a frame for $\mathcal{H} = L^2(\mathrm{I})$ (left convolved fractal frame), for any given frame (ϕ_m), see Proposition 2. But even before that, let us observe some sufficient conditions for the existence of a Schauder basis $(\rho * \beta_m)$ for $\mathcal{L}^p(I)$ (left convolved fractal basis) for any given Schauder basis (β_m) for $\mathcal{L}^p(I)$.

For a given Banach space $\mathcal{L}^p(I)$, we begin by considering a Schauder basis (β_m) of $\mathcal{L}^p(I), (1 \le p < \infty, m = 0, 1, \ldots)$, and a fixed $\rho \in \mathcal{L}^p(I)$. It is obvious that, for any f, we have $f = \sum_{m=0}^{\infty} c_m(f)\beta_m$, where for M^{th} partial sum operator R_M, the coefficients $c_m(f)$ satisfy

$$R_M(f) = \sum_{m=0}^{M} c_m(f)\beta_m,$$

Let us write, $\Re = \sup_M \parallel R_M \parallel$

Proposition 3.1. *For an arbitrary $\rho \in \mathcal{L}^p(I)$ and a normalised Scahuder basis (β_m) of $\mathcal{L}^p(I), (1 \le p < \infty, m = 0, 1, \ldots)$, if (ρ, δ_m) satisfies the k-*condition*, then the operator*

$$\Theta_\rho{}^l(f) = \sum_{m=0}^{\infty} c_m(f)(\rho * \beta_m)$$

is well defined, linear and bounded, where $c_m(f)$ are the coefficients of the expansion of f along the basis (β_m) as before.

A related phenomenon is that of obtaining left convolved Schauder basis of type $(\rho * \beta_m)$ for $\mathcal{L}^p(I)$, where it is mathematically observed that the process demands to drop the condition of normality in the given Schauder basis (β_m).

It is also worth noticing that in a Hilbert space $L^2(\mathrm{I})$, for any given orthonormal basis (δ_m), $(\rho * \delta_m), \rho \in L^2(\mathrm{I})$ is a Bessel sequence, provided (ρ, δ_m) satisfies the *k-condition.*

We now provide a proof of Theorem 3.1 in which the approach encompasses the use of Proposition 1 and the boundedness property of the Schauder basis (β_m) plus the norm of c_m, given in [24].

Proof of Theorem 3.1. Since any subsequence of a Reisz basis is a Reisz sequence with the same bounds, for every $\sigma \subset I$, we have that for any $f \in \overline{\text{span}}(\rho * \{\delta_m\}_{m\in\sigma}) \bigcup (\rho * \{\delta_{\pi(m)}\}_{m\in\sigma^c})$,

$$\sum_{m\in\sigma} |\langle f, \rho * \{\delta_m\}\rangle|^2 + \sum_{m\in\sigma^c} |\langle f, \rho * \{\delta_{\pi(m)}\}\rangle|^2 \\ \geq \sum_{\sigma\cup(\sigma^c\cap\pi(\sigma^c))} |\langle f, \rho * \{\delta_m\}\rangle|^2 \geq M_l \| f \|^2.$$

Clearly, we note that the upper bound is $2 * M_u$. Let us prove the woven part. If we assume that $\pi \neq I_d$, π is a permutation of I, which implies that $\pi(m_0) = r_0 \neq s_0$ for some $r_0, s_0 \in I$ $(\rho * \{\delta_m\}_{m\in I})$ and $(\rho * \{\delta_{\pi(m)}\}_{m\in I})$. Excluding s_0 from I, we can obtain a set in which δ_{r_0} appears twice and δ_{s_0} is absent. This is a contradiction to the fact that the closure of the span is the entire space. □

Theorem 3.2. *If a family $\{\phi_m\}$ is a frame for $\mathcal{H}$ with bounds $T, D > 0$, and $\rho \in L^2(I)$ is such that*

$$\sum_{m=0}^{\infty} \| (\rho - \phi_m) \|^2 \leq T(1-\lambda)^2, \tag{3.1}$$

$0 < \lambda < 1$, then $\gamma = \sum_{m=0}^{\infty} \| (\rho * \phi_m - \phi_m) \|^2 < T$ and $(\rho * \phi_m)$ constitutes a frame with the following frame bounds,

$$T\left(1 - \sqrt{\frac{\gamma}{T}}\right)^2 \quad \& \quad D\left(1 + \sqrt{\frac{\gamma}{D}}\right)^2$$

Proof. Using equation (2.7) and equation (3.1), it is an easy algebra to drive that,

$$\gamma = \sum_{m=0}^{\infty} \| (\rho * \phi_m - \phi_m) \|^2 < \frac{1}{(1-\lambda)^2} \sum_{m=0}^{\infty} \| (\rho - \phi_m) \|^2 < T.$$

The remaining algebra is given by the authors in Theorem 1 of [8]. □

Corollary 3.1. *Let families $\{\phi_m\}$ and $\{\psi_m\}$ be frames with bounds $T, D > 0$, and $T^*, D^* > 0$ respectively. Let $\rho \in L^2(I)$, be such that*

$$\sum_{m=0}^{\infty} \| (\rho - \phi_m) \|^2 \leq T_1(1-\lambda)^2 \quad \& \quad \sum_{m=0}^{\infty} \| (\rho - \psi_m) \|^2 \leq T_2(1-\lambda)^2.$$

Then for $\gamma_1 = \sum_{m=0}^{\infty} \parallel (\rho * \phi_m - \phi_m) \parallel^2$, $(\rho * \phi_m)$ *and* $(\rho * \psi_m)$ *constitute frames for* $\mathcal{H}$ *with the bounds*

$$P_1 = T\left(1 - \sqrt{\frac{\gamma_1}{T}}\right)^2 \quad \& \quad Q_1 = D\left(1 + \sqrt{\frac{\gamma_1}{D}}\right)^2.$$

Analogously for $\gamma_2 = \sum_{m=0}^{\infty} \parallel (\rho * \psi_m - \psi_m) \parallel^2$

$$P_2 = T^*\left(1 - \sqrt{\frac{\gamma_2}{T^*}}\right)^2 \quad \& \quad Q_2 = D^*\left(1 + \sqrt{\frac{\gamma_2}{D^*}}\right)^2.$$

Corollary 1 enables the interest in studying the perturbation of woven family of frames in a Hilbert space.

4 Perturbation of woven frames linked with fractal convolutions

We are interested to explore the connections of weaving frames linked with fractal convolutions with the classical perturbation result by Paley and Wiener [7], stating that a sequence that is sufficiently close to an (orthonormal) basis in a Hilbert space automatically forms a basis. Some basic results in perturbation of frames are provided in [5, 15, 16, 17].

The following theorem tells us that perturbations can be linked with woven convoluted frames by considering the closeness property of a family of frames in a special manner as you will see.

Theorem 4.1. *Let* $(\rho * \{\phi_m\}_{m\in\Xi})$ *and* $(\rho * \{\psi_m\}_{m\in\Xi})$ *be frames for* $\mathcal{H}$ *with the bounds* P_1, Q_1, *and* P_2, Q_2 *respectively. Suppose that there exists* $\mu \in (0,1)$ *such that*

$$\mu\left(\sqrt{Q_1} + \sqrt{Q_2}\right) \leq \frac{P_1}{2}. \tag{4.1}$$

Then, for all sequences $\{\xi_m\}_{m\in\Xi}$, *we have*

$$\|\sum_{m\in\Xi} \xi_m(\rho * \phi_m - \rho * \psi_m)\| \leq \mu \|\{\xi_m\}_{m\in\Xi}\|. \tag{4.2}$$

Moreover, for every $\sigma \subset \Xi$, *the family* $(\rho * \{\phi_m\}_{m\in\sigma^c}) \bigcup (\rho * \{\psi_m\}_{m\in\sigma})$ *is a frame for* $\mathcal{H}$ *with frame bounds* $\frac{P_1}{2}, Q_1 + Q_2$.

Proof. The goal is to obtain the lower and upper frame bounds of the weaving frame which are worked out separately below. The upper frame bound of the weaving can be obtained by the claim that for Bessel sequence $(\rho * \{\phi_m\}_{m\in\Xi})$ and $(\rho * \{\psi_m\}_{m\in\Xi})$, every weaving is bounded above by sum of their respective upper bounds, i.e $Q_1 + Q_2$ in this proof, see [2].

The algebra for calculating lower frame bound is more complex and we shall begin by introducing synthesis operator given by χ and ω respectively for the two frames linked with fractal convolutions $(\rho * \{\phi_m\}_{m\in\Xi})$ and $(\rho * \{\psi_m\}_{m\in\Xi})$. For any $\sigma \subset \Xi$ and any given orthogonal projection $\acute{P}$ onto span of standard orthonormal basis of $l^2(\Xi)$, let

$$\chi_\sigma(\{\xi_m\}_{m\in\Xi}) = \chi \acute{P}_\sigma(\{\xi_m\}_{m\in\Xi}) = \sum_{m\in\sigma} \xi_m \rho * \phi_m \tag{4.3}$$

Combining equation (4.3) with its analogous equation using ω and $\acute{P}$, we obtain a restatement of inequality (4.2) to be written as $\parallel \chi - \omega \parallel < \mu$.
Now for any $f \in \mathcal{H}$, consider

$$\| \sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m + \sum_{m\in\sigma^c} \langle f, \rho * \phi_m \rangle \rho * \phi_m \| . \tag{4.4}$$

Taking into account that $\sigma^c = \Xi - \sigma$, we can obtain

$$\| \sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m + \sum_{m\in\sigma^c} \langle f, \rho * \phi_m \rangle \rho * \phi_m \|$$

$$\begin{aligned} &\geq \| \sum_{m\in\Xi} \langle f, \rho * \phi_m \rangle \rho * \phi_m + \left(\sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m - \sum_{m\in\sigma} \langle f, \rho * \phi_m \rangle \rho * \phi_m \right) \| \\ &\geq P_1 \| f \| - \| \sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m - \sum_{m\in\sigma} \langle f, \rho * \phi_m \rangle \rho * \phi_m \| . \end{aligned}$$

Applying the fact that synthesis operator for the two frames linked with fractal convolutions $(\rho * \{\phi_m\}_{m\in\Xi})$ and $(\rho * \{\psi_m\}_{m\in\Xi})$, are given by χ and ω respectively, combining equation (4.1) and equation (4.2), with the help of basic algebra over properties of norms, we can compute,

$$\begin{aligned} &\| \sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m - \sum_{m\in\sigma} \langle f, \rho * \phi_m \rangle \rho * \phi_m \| \\ &= \| \chi_\sigma {\chi_\sigma}^* f - \omega_\sigma {\omega_\sigma}^* f \| \\ &\leq \frac{P_1}{2} \| f \| . \end{aligned}$$

This leads us to prove that expression (4.4) $\geq \frac{P_1}{2} \| f \|$. □

In Remark 4, we observe the perturbation of weaving frames linked with fractal convolutions as image of bounded invertible operator for a given frame.

Remark 4. Let $(\rho * \{\phi_m\}_{m\in\Xi})$ be a frame for $\mathcal{H}$ with bounds P and Q respectively. For any bounded operator τ, such that

$$\|I_d - \tau\|^2 < \frac{P}{Q},$$

we have $(\rho * \{\phi_m\}_{m\in\Xi})$ and $(\rho * \{\tau\phi_m\}_{m\in\Xi})$ are woven.

The famous Minkowski's inequality implies that τ is invertible, and the fact that $(\rho * \{\tau\phi_m\}, m \in \Xi)$ is automatically a frame underpin the remark.

5 Conclusion and future plan

To conclude, we remind that in this article, we developed weaving properties of frames linked with fractal convolutions. One of the many problems concerned with diagnosis of woven frames, which are actually Reisz bases for Hilbert spaces have been studied. The finding in this paper have created broader impact in planning the future work in the direction of obtaining links of fractal convolutions and a variety of other frames. Not only this, there is an ample scope of solving problems that connect fractals with frames and bases.

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KOLMOGOROV WIDTHS OF ANISOTROPIC FUNCTION CLASSES AND FINITE-DIMENSIONAL BALLS

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Abstract. In this paper, we obtain order estimates for the Kolmogorov widths of anisotropic periodic Sobolev and Nikol'skii classes, as well as anisotropic finite-dimensional balls.

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Let $1 \le p_1, \ldots, p_d \le \infty$, $\overline{p} = (p_1, \ldots, p_d)$, $\mathbb{T} = [0, 2\pi]$. The anisotropic norm $\|\cdot\|_{L_{\overline{p}}(\mathbb{T}^d)}$ is defined by

$$\|f\|_{L_{\overline{p}}(\mathbb{T}^d)} = \Big(\int\limits_{\mathbb{T}} \ldots \Big(\int\limits_{\mathbb{T}} \Big(\int\limits_{\mathbb{T}} |f(x_1, x_2, \ldots, x_d)|^{p_1}\, dx_1\Big)^{p_2/p_1} dx_2\Big)^{p_3/p_2} \ldots dx_d\Big)^{1/p_d}$$

for finite p_j; if $p_j = \infty$ for some j, the corresponding jth integral norm is replaced by an essential supremum.

The space $L_{\overline{p}}(\mathbb{T}^d)$ consists of the equivalence classes of measurable functions satisfying $\|f\|_{L_{\overline{p}}(\mathbb{T}^d)} < \infty$.

We define the anisotropic Sobolev and Nikol'skii classes as in [30]. For different definitions of generalized smoothness, see, e.g., [4].

Given $r > 0$, $\alpha \in \mathbb{R}$, we set

$$F_r(x, \alpha) = 1 + 2\sum_{k=1}^{\infty} k^{-r} \cos(kx - \alpha\pi/2), \quad x \in \mathbb{R}.$$

Let $\overline{r} = (r_1, \ldots, r_d)$, $\overline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $\overline{p} = (p_1, \ldots, p_d)$, $r_j > 0$, $\alpha_j \in \mathbb{R}$, $1 \le p_j \le \infty$, $j = 1, \ldots, d$.

Definition 1. The Sobolev class $W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d)$ consists of functions f on $\mathbb{T}^d$ such that for all $j \in \{1, \ldots, d\}$ the integral representation

$$f(x_1, \ldots, x_d) = \frac{1}{2\pi} \int\limits_{\mathbb{T}} \varphi_j(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_d) F_{r_j}(x_j - y, \alpha_j)\, dy$$

holds with $\|\varphi_j\|_{L_{\overline{p}}(\mathbb{T}^d)} \le 1$.

Let $h \in \mathbb{R}$, $f \in L_{\overline{p}}(\mathbb{T}^d)$. We continue f periodically to $\mathbb{R}^d$ and set

$$\Delta_h^{1,j} f(x_1, \ldots, x_d) = f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_d) -$$

$$-f(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d).$$

For $l \in \mathbb{N}$, $l \ge 2$, the operator $\Delta_h^{l,j}$ is defined by the equality $\Delta_h^{l,j} = \Delta_h^{1,j} \circ \Delta_h^{l-1,j}$.

Definition 2. The Nikol'skii class $H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d)$ consists of all functions $f \in L_{\overline{p}}(\mathbb{T}^d)$ such that

$$\|f\|_{L_{\overline{p}}(\mathbb{T}^d)} \le 1, \quad \|\Delta_h^{l_j,j} f\|_{L_{\overline{p}}(\mathbb{T}^d)} \le |h|^{r_j}, \quad h \in \mathbb{R}, \quad 1 \le j \le d,$$

where $l_j = \lfloor r_j \rfloor + 1$.

It is well-known [30, Theorem 3.4.6] that $W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d) \subset C(\overline{r}) H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d)$, where $C(\overline{r})$ is a positive number depending only on $\overline{r}$.

Definition 3. Let X be a normed space, $M \subset X$, $n \in \mathbb{Z}_+$. The Kolmogorov n-width of M in X is defined by

$$d_n(M, X) = \inf_{L \in \mathcal{L}_n(X)} \sup_{x \in M} \inf_{y \in L} \|x - y\|;$$

here, $\mathcal{L}_n(X)$ is the family of all subspaces in X of dimension at most n.

Estimates for the widths of Sobolev classes in L_q on one-dimensional domains were obtained in [19, 15, 20, 21, 5]. In [25, 26, 27, 28], the problem of estimating the widths of Sobolev and Nikol'skii classes in $L_q(\mathbb{T}^d)$ was studied (see Definitions 1 and 2 for $p_1 = \dots = p_d = p$); in [7, 8, 29], a similar problem was considered for periodic Sobolev and Nikol'skii classes with dominating mixed smoothness. For details, see [11, 31, 30]. In [7], the anisotropic norms were also considered, but only for the following cases: 1) $1 < q_j \le p_j < \infty$, $1 \le j \le d$, 2) $1 < p_j \le q_j \le 2$, $1 \le j \le d$, 3) $2 \le p_j \le q_j < \infty$, $1 \le j \le d$, 4) $1 < p_j \le 2 \le q_j < \infty$, $1 \le j \le d$, $p_1 = \dots = p_d$. (Actually, from the proof we can see that $p_1 = \dots = p_d$, $q_1 = \dots = q_d$ in case 3).) Moreover, in cases 3), 4), estimates of the widths were obtained only for the "large smoothness". The case of "small smoothness" was considered in [8] for multivariate functions and isotropic norms. In [9, 33, 34], estimates for the widths of intersections of Sobolev classes were obtained.

In [1, 2], estimates for the widths of Nikol'skii–Besov–Amanov classes in Lorentz spaces with anisotropic norms were obtained; the parameters p_j, q_j satisfied the above conditions 1)–4) (see also [3]).

In the present paper, we obtain order estimates for the Kolmogorov widths of $W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d)$ and $H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d)$ in $L_{\overline{q}}(\mathbb{T}^d)$ for $2 \le q_j < \infty$, $j = 1, \ldots, d$. The parameters $p_j \in [1, +\infty]$ are arbitrary, except some limit cases (see the condition $\theta_{j_*} < \min_{j \in J \setminus \{j_*\}} \theta_j$ in Theorem 1 below). In addition, the estimate for the widths is obtained in the case $1 \le p_j \le q_j \le 2$ for $1 \le j \le \nu$, $1 \le q_j \le p_j \le \infty$ for $\nu + 1 \le j \le d$ (see Theorem 2).

For $2 \le q < \infty$, $1 \le p \le \infty$, we set

$$\omega_{p,q} = \begin{cases} 0 & \text{for } p > q, \\ \frac{1/p - 1/q}{1/2 - 1/q} & \text{for } 2 < p \le q, \\ 1 & \text{for } 1 \le p \le 2. \end{cases} \tag{1}$$

Let $I \subset \{1, \ldots, d\}$ be a nonempty set, $\overline{p} = (p_1, \ldots, p_d)$. We define the number $\langle \overline{p} \rangle_I$ by the equation $\frac{1}{\langle \overline{p} \rangle_I} = \frac{1}{|I|} \sum\limits_{j \in I} \frac{1}{p_j}$. We also write $\langle \overline{p} \rangle := \langle \overline{p} \rangle_{\{1, \ldots, d\}}$. For $I = \varnothing$, we set $\langle \overline{p} \rangle_I = 1$.

Let σ be a permutation of d elements such that

$$\omega_{p_{\sigma(1)},q_{\sigma(1)}} \le \omega_{p_{\sigma(2)},q_{\sigma(2)}} \le \dots \le \omega_{p_{\sigma(d)},q_{\sigma(d)}}. \tag{2}$$

The numbers μ, $\nu \in \{0, \dots, d\}$ are defined by the equations

$$\{1, \dots, \mu\} = \{j : \omega_{p_{\sigma(j)},q_{\sigma(j)}} = 0\}, \ \{1, \dots, \nu\} = \{j : \omega_{p_{\sigma(j)},q_{\sigma(j)}} < 1\}. \tag{3}$$

By (5), the condition $j \in \{1, \dots, \mu\}$ is equivalent to the equation $p_{\sigma(j)} \ge q_{\sigma(j)}$ for $q_{\sigma(j)} > 2$, and to $p_{\sigma(j)} > q_{\sigma(j)}$, for $q_{\sigma(j)} = 2$; the condition $j \in \{1, \dots, \nu\}$ is equivalent to the equation $p_{\sigma(j)} > 2$.

Let

$$I(t, s) = \{\sigma(t), \sigma(t+1), \dots, \sigma(s-1), \sigma(s)\}, \quad 1 \le t \le s \le d.$$

For $\overline{a} = (a_1, \dots, a_d)$, $\overline{b} = (b_1, \dots, b_d)$, we set $\overline{a} \circ \overline{b} = (a_1b_1, \dots, a_db_d)$.

Now we give notation for order equalities. Let X, Y be sets, $f_1, f_2 : X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \underset{y}{\asymp} f_2(x, y)$ if, for each $y \in Y$, there is $c(y) \ge 1$ such that $[c(y)]^{-1} f_2(x, y) \le f_1(x, y) \le c(y) f_2(x, y)$ for all $x \in X$.

Theorem 1. *Let $d \in \mathbb{N}$, $r_j > 0$, $\alpha_j \in \mathbb{R}$, $1 \le p_j \le \infty$, $2 \le q_j < \infty$, $j = 1, \dots, d$. Suppose that*

$$1 + \frac{d-\mu}{\langle \overline{r} \circ \overline{q} \rangle_{I(\mu+1, d)}} - \frac{d-\mu}{\langle \overline{r} \circ \overline{p} \rangle_{I(\mu+1, d)}} > 0.$$

We set

$$\theta_t = \frac{1}{\frac{t}{\langle \overline{r} \rangle_{I(1, t)}} + \frac{2(d-t)}{\langle \overline{r} \circ \overline{q} \rangle_{I(t+1, d)}}} \left(1 + (d-t)\left(\frac{1}{\langle \overline{r} \circ \overline{q} \rangle_{I(t+1, d)}} - \frac{1}{\langle \overline{r} \circ \overline{p} \rangle_{I(t+1, d)}}\right)\right),$$

$\mu \le t \le \nu$; if $\nu < d$ and there is $j \in \{\nu+1, \dots, d\}$ such that $q_{\sigma(j)} > 2$, we also write

$$\theta_d = \frac{\langle \overline{r} \rangle}{d}\left(1 + \frac{d-\nu}{2\langle \overline{r} \rangle_{I(\nu+1, d)}} - \frac{d-\nu}{\langle \overline{r} \circ \overline{p} \rangle_{I(\nu+1, d)}}\right).$$

Let $J = \{\mu, \mu+1, \dots, \nu\} \cup \{d\}$ if $\nu < d$ and there is $j \in \{\nu+1, \dots, d\}$ such that $q_{\sigma(j)} > 2$; otherwise, we set $J = \{\mu, \mu+1, \dots, \nu\}$. Suppose that there is $j_* \in J$ such that

$$\theta_{j_*} < \min_{j \in J \setminus \{j_*\}} \theta_j.$$

Then

$$d_n(W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d), L_{\overline{q}}(\mathbb{T}^d)) \underset{\overline{p},\overline{q},\overline{r},d}{\asymp} d_n(H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d), L_{\overline{q}}(\mathbb{T}^d)) \underset{\overline{p},\overline{q},\overline{r},d}{\asymp} n^{-\theta_{j_*}}.$$

Theorem 2. *Let $r_j > 0$, $\alpha_j \in \mathbb{R}$, $1 \le j \le d$, $\nu \in \{0, \dots, d\}$, $1 \le p_j \le q_j \le 2$ for $1 \le j \le \nu$, $1 \le q_j \le p_j \le \infty$ for $\nu+1 \le j \le d$. Suppose that*

$$\theta := \frac{\langle \overline{r} \rangle}{d}\left(1 + \frac{\nu}{\langle \overline{r} \circ \overline{q} \rangle_{\{1, \dots, \nu\}}} - \frac{\nu}{\langle \overline{r} \circ \overline{p} \rangle_{\{1, \dots, \nu\}}}\right) > 0.$$

Then

$$d_n(W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d), L_{\overline{q}}(\mathbb{T}^d)) \underset{\overline{p},\overline{q},\overline{r},d}{\asymp} d_n(H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d), L_{\overline{q}}(\mathbb{T}^d)) \underset{\overline{p},\overline{q},\overline{r},d}{\asymp} n^{-\theta}.$$

In order to prove Theorems 1, 2, we obtain order estimates for the widths of finite-dimensional balls (see Theorem 3 and Proposition 1 below). After that we apply the standard discretization method following [30].

Given $N \in \mathbb{N}$, $1 \le s \le \infty$, $(x_i)_{i=1}^N \in \mathbb{R}^N$, we set $\|(x_i)_{i=1}^N\|_{l_s^N} = \left(\sum_{i=1}^N |x_i|^s\right)^{1/s}$ for $s < \infty$, $\|(x_i)_{i=1}^N\|_{l_s^N} = \max_{1\le i\le N} |x_i|$ for $s = \infty$.

Let $k_1, \ldots, k_d \in \mathbb{N}$, $1 \le p_1, \ldots, p_d \le \infty$. By $l^{k_1,\ldots,k_d}_{p_1,\ldots,p_d}$ we denote the space $\mathbb{R}^{k_1\ldots k_d} = \{(x_{j_1,\ldots,j_d})_{1\le j_s\le k_s,\,1\le s\le d} : \; x_{j_1,\ldots,j_d} \in \mathbb{R}\}$ with norm defined by induction: for $d = 1$ it is $\|\cdot\|_{l^{k_1}_{p_1}}$; for $d \ge 2$,

$$\|(x_{j_1,\ldots,j_d})_{1\le j_s\le k_s,\,1\le s\le d}\|_{l^{k_1,\ldots,k_d}_{p_1,\ldots,p_d}} = \left\|\left(\|(x_{j_1,\ldots,\,j_{d-1},\,j_d})_{1\le j_s\le k_s,\,1\le s\le d-1}\|_{l^{k_1,\ldots,\,k_{d-1}}_{p_1,\ldots,\,p_{d-1}}}\right)_{j_d=1}^{k_d}\right\|_{l^{k_d}_{p_d}}.$$

By $B^{k_1,\ldots,k_d}_{p_1,\ldots,p_d}$ we denote the unit ball of the space $l^{k_1,\ldots,k_d}_{p_1,\ldots,p_d}$.

For $d = 1$, estimates for the widths of these balls were obtained in [23, 24, 18, 19, 13, 14, 12]. The case $d = 2$ was studied in [9, 10, 16, 17, 22, 32, 6]; for details, see, e.g., [35].

Theorem 3. *Let $d \in \mathbb{N}$, $k_1, \ldots, k_d \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $n \le \frac{k_1\ldots k_d}{2}$, $2 \le q_j < \infty$, $1 \le p_j \le \infty$, $j = 1, \ldots, d$. Let σ be a permutation of $\{1, \ldots, d\}$ such that (5) holds. The numbers $\mu \in \{0, \ldots, d\}$ and $\nu \in \{0, \ldots, d\}$ are defined by (5). We denote $p_j^* = \max\{p_j, 2\}$, $1 \le j \le d$. Then*

$$d_n(B^{k_1,\ldots,k_d}_{p_1,\ldots,p_d}, l^{k_1,\ldots,k_d}_{q_1,\ldots,q_d}) \underset{\overline{q}}{\asymp} \Phi(k_1, \ldots, k_d, n) :=$$

$$:= \prod_{j=1}^{\mu} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p_{\sigma(j)}} \cdot \min\Big\{1, \min_{\mu+1\le t\le d} \prod_{j=\mu+1}^{t-1} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p^*_{\sigma(j)}} \times$$
$$\times (n^{-1/2} k_{\sigma(1)}^{1/2} \ldots k_{\sigma(t-1)}^{1/2} k_{\sigma(t)}^{1/q_{\sigma(t)}} \ldots k_{\sigma(d)}^{1/q_{\sigma(d)}})^{\omega_{p_{\sigma(t)},q_{\sigma(t)}}}\Big\};$$

in addition, for $\nu < d$,

$$\Phi(k_1, \ldots, k_d, n) = \prod_{j=1}^{\mu} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p_{\sigma(j)}} \cdot \min\Big\{1, \min_{\mu+1\le t\le \nu} \prod_{j=\mu+1}^{t-1} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p_{\sigma(j)}} \times$$
$$\times (n^{-1/2} k_{\sigma(1)}^{1/2} \ldots k_{\sigma(t-1)}^{1/2} k_{\sigma(t)}^{1/q_{\sigma(t)}} \ldots k_{\sigma(d)}^{1/q_{\sigma(d)}})^{\omega_{p_{\sigma(t)},q_{\sigma(t)}}},$$
$$\prod_{j=\mu+1}^{\nu} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p_{\sigma(j)}} \cdot n^{-1/2} k_{\sigma(1)}^{1/2} \ldots k_{\sigma(\nu)}^{1/2} k_{\sigma(\nu+1)}^{1/q_{\sigma(\nu+1)}} \ldots k_{\sigma(d)}^{1/q_{\sigma(d)}}\Big\}.$$

The proof generalizes the arguments from [13, 32].

Proposition 1. *Let $\nu \in \{0, \ldots, d\}$, $1 \le p_j \le q_j \le 2$ for $1 \le j \le \nu$, $1 \le q_j \le p_j \le \infty$ for $\nu + 1 \le j \le d$, $n \le \frac{k_1\ldots k_d}{2}$. Then*

$$d_n(B^{k_1,\ldots,k_d}_{p_1,\ldots,p_d}, l^{k_1,\ldots,k_d}_{q_1,\ldots,q_d}) \asymp k_{\nu+1}^{1/q_{\nu+1}-1/p_{\nu+1}} \ldots k_d^{1/q_d-1/p_d}.$$

This estimate is a simple corollary of Malykhin's and Rjutin's result [22, Theorem 1] on estimates of the widths of a product of multi-dimensional octahedra.

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SECOND INFORMATION LETTER

Dear Colleagues!

The L.N. Gumilyov Eurasian National University (Astana, Kazakhstan) and the RUDN University (Moscow, Russia) will hold through January 7-11, 2025, the International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2025), dedicated to the 15th anniversary of the Eurasian Mathematical Journal.

The Eurasian Mathematical Journal (hereinafter EMJ) was founded in 2009 by the L.N. Gumilyov Eurasian National University (hereinafter ENU) with the assistance of the M.V. Lomonosov Moscow State University (hereinafter MSU), the RUDN University (hereinafter RUDN), and the University of Padua (Italy).

Starting from 2018, the journal is published jointly by the ENU and RUDN.

Editors-in-Chief are V.I. Burenkov (RUDN), M. Otelbaev (IITU), V.A. Sadovnichy (MSU), Vice-Editors-in-Chief are K.N. Ospanov (ENU), T.V. Tararykova (RUDN), Executive secretary is A.M. Temirkhanova (ENU).

In the years 2010-2024 about 450 original research and review articles were published in the journal by more than 500 mathematicians from more than 40 countries.

The journal is included in the databases of Web of Sciences (quartile Q3, ESCI, its Web of Science impact factor is 0,6.) and Scopus (percentile 64, quartile Q2 (Mathematics (miscellaneous, SJR 0,54).)), refereed by Mathematical Reviews (USA), Zentralblatt fΓjr Mathematik (Germany) and the Referativnyi Zhurnal "Mathematics" (Russia). The journal is registered in the portals MathSciNet (USA) and Math-Net.Ru (Russia).

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The conference will be held in the following sections.

1. Theory of functions and functional analysis.

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3. Algebra and model theory.

Selected papers by participants will be published in the Eurasian Mathematical Journal and Bulletin of Karaganda State University, Series "Mathematics", which is included in the Web of Science and Scopus databases.

Plenary (30-40 min.), sectional (15-20 min.) and poster presentations are planned. The official languages of the conference are Kazakh, Russian, and English.

E-mail of the Organizing committee: emjconf2025@gmail.com

Those wishing to participate please follow the steps below:

1. Register at the conference web page: https://emj-2025.enu.kz
For this, by December 01, 2024, we ask you to fill out an application with the authors' data on the conference website https://emj-2025.enu.kz.

2. Transfer the registration fee by cashless payment.

Registration fee: 20 000 tenge, or 45 US dollars for conference participants;
10 000 tenge, or 20 US dollars for doctoral, graduate, and undergraduate students;
The registration fee includes the preparation of conference materials and the organization of coffee breaks.

The conference fee can be paid in cash at the registration for the conference (**IF NO RECEIPT IS REQUIRED**).

Attention! If participants need documents for the financial report, we ask them to pay the registration fee according to the following details:

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Submission of abstracts of the report

You can attach the abstracts on the conference website https://emj-2025.enu.kz or send it to the organizing committee's email address: emjconf2025@gmail.com **until December 01, 2024.** Abstracts should be prepared in TeX according to the conference template. The volume should not exceed 2 pages. The abstract template can be downloaded from the conference website (see https://emj-2025.enu.kz).

The title of the thesis file (*.tex and *.pdf) should consist of the surname and initials of the corresponding author in Latin letters (for example, KoshkarovaB.tex and KoshkarovaB.pdf for B. Koshkarova's abstracts). Before publication, all abstracts are reviewed.

Accommodation

The Organizing Committee recommends booking a hotel in advance. The list of hotels where you can stay during the conference is given below:
Hotel "Sultan Beibarys" www.sultanbeibarys.com
Hotel "Belon Life Hotel" https://belon.kz/life/
The hotel complex "Nomad" https://nomadhotel.2gis.biz/
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Important dates:

until 01.12.2024 – registration on the conference web page and presentation of abstracts,
until 10.12.2024 – confirmation of the inclusion of reports in the programme of the conference,
07.01.2025 – day of arrival,
08.01.2025 – registration of participants, opening of the conference,
10.01.2025 – closing of the conference,
11.01.2025 – departure day.

The Organizing Committee invites you to take part in the work of the Conference

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If you have any questions, please, do not hesitate to contact the organizing committee: emjconf2025@gmail.com

Feel free to distribute this information to those who may be interested.

Best regards,
V.I. Burenkov, professor of the RUDN, Editor-in-Chief of the EMJ, Co-Chairman of the International Programme Committee (Russia),
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