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ESTIMATES OF M–TERM APPROXIMATIONS OF FUNCTIONS OF SEVERAL VARIABLES IN THE LORENTZ SPACE BY A CONSTRUCTIVE METHOD

G. Akishev

Communicated by E.D. Nursultanov

Key words: Lorentz space, Nikol'skii–Besov class, best M–term approximation, constructive method.

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Abstract. In the paper, the Lorentz space $L_{q,\tau}(\mathbb{T}^m)$ of periodic functions of several variables, the Nikol'skii–Besov class $S^{\overline{r}}_{q,\tau,\theta}B$ and the associated class $W^{a,b,\overline{r}}_{q,\tau}$ for $1 < q, \tau < \infty$, $1 \leqslant \theta \leqslant \infty$ are considered. Estimates are established for the best M-term trigonometric approximations of functions of the classes $W^{a,b,\overline{r}}_{q,\tau_1}$ and $S^{\overline{r}}_{q,\tau_1,\theta}B$ in the norm of the space $L_{p,\tau_2}(\mathbb{T}^m)$ for different relations between the parameters $q, \tau_1, p, \tau_2, a, \theta$. The proofs of the theorems are based on the constructive method developed by V.N. Temlyakov.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$– be the sets of all natural, integer, real numbers, respectively, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}^m$—m-dimensional Euclidean point space $\overline{x} = (x_1, \ldots, x_m)$ with real coordinates; $\mathbb{T}^m = [0, 2\pi)^m$ and $\mathbb{I}^m = [0, 1)^m$ — m-dimensional cube.

$L_{p,\tau}(\mathbb{T}^m)$ will denote the Lorentz space of all real-valued Lebesgue-measurable functions f that have a 2π-period in each variable and for which the quantity

$$\|f\|_{p,\tau} = \left\{ \frac{\tau}{p} \int\limits_0^1 \big(f^*(t)\big)^{\tau} t^{\frac{\tau}{p}-1} dt \right\}^{\frac{1}{\tau}}, \ 1 < p < \infty, 1 \leqslant \tau < \infty,$$

is finite, where $f^*(t)$ is the non-increasing rearrangement of the function $|f(2\pi\overline{x})|$, $\overline{x} \in \mathbb{I}^m$ (see [34], pp. 213–216).

In the case $\tau = p$, the Lorentz space $L_{p,\tau}(\mathbb{T}^m)$ coincides with the Lebesgue space $L_p(\mathbb{T}^m)$ with the norm (see for example, [26, Chapter 1, Section 1.1, p. 11])

$$\|f\|_p = \left[\int_0^{2\pi} ... \int_0^{2\pi} |f(x_1, ..., x_m)|^p dx_1 ... dx_m \right]^{\frac{1}{p}}, \ 1 \leq p < \infty.$$

We will introduce the notation $a_{\overline{n}}(f)$-Fourier coefficients of the function $f \in L_1(\mathbb{T}^m)$ by system $\{e^{i\langle \overline{n}, \overline{x}\rangle}\}_{\overline{n} \in \mathbb{Z}^m}$ and $\langle \overline{y}, \overline{x}\rangle = \sum\limits_{j=1}^{m} y_j x_j$;

$$\delta_{\overline{s}}(f, \overline{x}) = \sum_{\overline{n} \in \rho(\overline{s})} a_{\overline{n}}(f)\, e^{i\langle \overline{n}, \overline{x}\rangle},$$

where

$$\rho(\overline{s}) = \left\{\overline{k} = (k_1, ..., k_m) \in \mathbb{Z}^m : \quad [2^{s_j - 1}] \leq |k_j| < 2^{s_j}, j = 1, ..., m\right\},$$

$[a]$ is the integer part of a real number a, $\overline{s} = (s_1, ..., s_m)$, $s_j \in \mathbb{Z}_+$.

For a given $p \in [1, \infty)$, a numerical sequence $\{a_{\overline{n}}\}_{\overline{n} \in \mathbb{Z}^m}$ belongs to the space l_p if

$$\left\| \{a_{\overline{n}}\}_{\overline{n} \in \mathbb{Z}^m} \right\|_{l_p} = \left[\sum_{\overline{n} \in \mathbb{Z}^m} |a_{\overline{n}}|^p \right]^{\frac{1}{p}} < \infty.$$

Further, for a vector $\overline{r} = (r_1, ..., r_m)$ and the zero vector $\overline{0} = (0, ..., 0)$, the inequality $\overline{r} > \overline{0}$ means that $r_j > 0$ for all $j = 1, 2, ..., m$. Let $1 \leqslant \theta \leqslant \infty$. We will consider an analogue of the Nikol'skii-Besov class

$$\mathbb{S}^{\overline{r}}_{p,\tau,\theta} B = \left\{ f \in \mathring{L}_{p,\tau}(\mathbb{T}^m) : \left\| \left\{ 2^{\langle \overline{s}, \overline{r} \rangle} \|\delta_{\overline{s}}(f)\|_{p,\tau} \right\}_{\overline{s} \in \mathbb{Z}^m_+} \right\|_{l_\theta} \leqslant 1 \right\}.$$

In the case $\tau = p$, the class $\mathbb{S}^{\overline{r}}_{p,\tau,\theta} B$ coincides with the well-known Nikol'skii-Besov class $S^{\overline{r}}_{p,\theta} B$ in the space $L_p(\mathbb{T}^m)$ (see for example [8], [23]). Currently, there are various generalizations of the Nikol'skii–Besov spaces and their further applications in the theory of approximation of functions, harmonic analysis and in other branches of mathematics (see, for example, [9], [15], [16], [18], [36], [40]).

For a given vector $\overline{r} = (r_1, ..., r_m) > \overline{0} = (0, \ldots, 0)$ put $\overline{\gamma} = \frac{\overline{r}}{r_1}$ and

$$Q_n^{(\overline{\gamma})} = \cup_{\langle \overline{s}, \overline{\gamma} \rangle < n} \rho(\overline{s}),$$

$S^{(\overline{\gamma})}_{Q_{n,\overline{\gamma}}}(f, \overline{x}) = \sum_{\overline{k} \in Q_n^{(\overline{\gamma})}} a_{\overline{k}}(f) e^{i \langle \overline{k}, \overline{x} \rangle}$ will denote a partial sum of the Fourier series of a function f.

Let $\overline{k}^{(j)} \in \mathbb{Z}^m$. The quantity

$$e_M(f)_{p,\tau} = \inf_{\overline{k}^{(j)}, b_j} \left\| f - \sum_{j=1}^{M} b_j e^{\langle i \overline{k}^{(j)}, \overline{x} \rangle} \right\|_{p,\tau}$$

is called the best M–term trigonometric approximation of a function $f \in L_{p,\tau}(\mathbb{T}^m)$, $M \in \mathbb{N}$, $\overline{k}^{(j)} \in \mathbb{Z}^m$. If $F \subset L_{p,\tau}(\mathbb{T}^m)$ is some functional class, then we put

$$e_M(F)_{p,\tau} = \sup_{f \in F} e_M(f)_{p,\tau}.$$

In the case $\tau = p$ instead of $e_M(F)_{p,\tau}$ we will write $e_M(F)_p$.

The best M–term approximation of a function $f \in L_2[0, 1]$ by polynomials via an orthonormal system was first defined by S.B. Stechkin [33] who established a criterion for the absolute convergence of the Fourier series via this system. Further, important results on estimating M-term approximations of functions for various classes of Sobolev, Nikol'skii–Besov, Lizorkin–Triebel were obtained by R.S. Ismagilov [21], Yu. Makovoz [25], V.E. Mayorov [24], E.S. Belinsky [12] – [14], B.S. Kashin [22], R. DeVore [16], V. N. Temlyakov [35] – [39], A.S. Romanyuk [27], [28], Dinh Dung [17], Wang Heping and Sun Yongsheng [41], M. Hansen and W.Sickel [19], [20] , S.A. Stasyuk [30] – [32], A.L. Shidlich [29].

To estimate M–term approximations of functions of the Nikol'skii–Besov class $S^{\overline{r}}_{p,\theta} B$ in the space $L_q(\mathbb{T}^m)$ two methods were used: non-constructive and constructive. The first method is based on Lemma 2.3 [14] (also see [25], [24]) which is proved by probabilistic reasoning. The second method was developed by V.N. Temlyakov [37], [38] and is based on greedy algorithms (see [36], [39]). Further, a constructive method of n–term approximations for the trigonometric system was developed by D.B. Bazarkhanov and V.N. Temlyakov in [10] and in [11]. A survey of the results on this theory can be found in [18]. Estimates for n–term approximations of functions of the Nikol'skii–Besov class in the Lorentz space are investigated in [1] – [3].

For a constructive method for estimating n - term approximations of functions of the Nikol'skii–Besov class $S^{\overline{r}}_{p,\theta}B$ V.N. Temlyakov [37], [38] introduced the class $W^{a,b,\overline{r}}_{q}$. In this article, we will consider an analogue of this class in the Lorentz space.

For a function $f \in L_1(\mathbb{T}^m)$ put

$$f_{l,\overline{r}}(\overline{x}) = \sum_{l \leqslant \langle \overline{s}, \overline{\gamma} \rangle < l+1} \delta_{\overline{s}}(f, \overline{x}), \ l \in \mathbb{Z}_+,$$

where $\overline{\gamma} = (\gamma_1, \ldots, \gamma_m)$, $\gamma_1 = \ldots = \gamma_\nu < \gamma_{\nu+1} \leq \ldots \leq \gamma_m$, $\gamma_j = \frac{r_j}{r_1}$, $r_j > 0$, $j = 1, \ldots, m$.

We will consider the following class defined in [37], [38]

$$W^{a,b,\overline{r}}_{A} = \{f \in L_1(\mathbb{T}^m): \ \|f_{l,\overline{r}}\|_A \leqslant 2^{-la} l_0^{(\nu-1)b}\},$$

where $l_0 = \max\{1, l\}$, $l \in \mathbb{Z}_+$ and

$$\|f_{l,\overline{r}}\|_A = \sum_{l \leqslant \langle \overline{s}, \overline{\gamma} \rangle < l+1} \sum_{\overline{n} \in \rho(\overline{s})} |a_{\overline{n}}(f)|.$$

We also define the class

$$W^{a,b,\overline{r}}_{q,\tau} = \{f \in L_1(\mathbb{T}^m): \ \|f_{l,\overline{r}}\|_{q,\tau} \leqslant 2^{-la} l_0^{(\nu-1)b}\},$$

where $a > 0$, $b \in \mathbb{R}$, $l_0 = \max\{1, l\}$.

We will introduce the following notation

$$\|f\|_{W^{a,b,\overline{r}}_{q,\tau}} = \sup_{l \in \mathbb{Z}_+} \|f_{l,\overline{r}}\|_{q,\tau} 2^{la} l_0^{-(\nu-1)b}, 1 < q, \tau < \infty.$$

In the case $\tau = q$, the class $W^{a,b,\overline{r}}_{q,\tau}$ is defined by V.N. Temlyakov [37], [38] and in this case, instead of $W^{a,b,\overline{r}}_{q,q}$ we will write $W^{a,b,\overline{r}}_{q}$.

For the class $W^{a,b,\overline{r}}_{q,\tau_1}$ we put

$$e_n(W^{a,b,\overline{r}}_{q,\tau_1})_{p,\tau_2} = \sup_{f \in W^{a,b,\overline{r}}_{q,\tau_1}} e_n(f)_{p,\tau_2}, \ 1 < q, p, \tau_1, \tau_2 < \infty.$$

In the case $\tau = q$, the order-sharp estimates for the best n-th trigonometric approximations of functions belonging to the class $W^{a,b,\overline{r}}_{q}$ in the space $L_p(\mathbb{T}^m)$, $1 < q \leqslant p < \infty$ were established by V.N. Temlyakov [37], [38]. In particular, he proved

Theorem 1.1 ([38, Theorem 3.2]). Let $1 < q \leq 2 < p < \infty$ and $(\frac{1}{q} - \frac{1}{p})p' < a < \frac{1}{q}$, $p' = \frac{p}{p-1}$, then

$$e_n(W^{a,b,\overline{r}}_{q})_p \asymp n^{-\frac{p}{2}(a + \frac{1}{p} - \frac{1}{q})} (\log_2 n)^{(\nu-1)(b + a(p-1) - (\frac{1}{q} - \frac{1}{p})p)}.$$

Here and in what follows, the notation $A_n \asymp B_n$ means that there exist positive numbers C_1, C_2 independent of $n \in \mathbb{N}$ such that $C_1 A_n \leqslant B_n \leqslant C_2 A_n$ for $n \in \mathbb{N}$.

In [38], the problem of finding order-sharp estimates for $e_n(W^{a,b,\overline{r}}_{q})_p$ by the constructive method, in the case of $\frac{1}{q} - \frac{1}{p} < a < (\frac{1}{q} - \frac{1}{p})p'$, $1 < q \leqslant 2 < p < \infty$ remains open.

We will consider the problem of estimating the best M–term trigonometric approximations for the Lorentz space. The main results of the article are formulated and proved in the third section (see Theorem 3.1 and Theorem 3.2). In the second section, we formulate some auxiliary assertions required for proving the main results. In the fourth section, as an application of Theorem 3.1, we establish an upper bound for the best M - term approximations of functions of the Nikol'skii-Besov class in the Lorentz space (see Theorem 4.1).

2 Auxiliary statements

Theorem 2.1. (see [5]). *Let $1 < q < \lambda < \infty$, $1 < \tau, \theta < \infty$. If a function $f \in L_{q,\tau}(\mathbb{T}^m)$, then*

$$\|f\|_{q,\tau} \geqslant C\Big(\sum_{\bar{s}\in\mathbb{Z}_+^m}\prod_{l=1}^{m}2^{s_l(1/\lambda-1/q)\tau}\|\delta_{\bar{s}}(f)\|_{\lambda,\theta}^{\tau}\Big)^{1/\tau},$$

where $C > 0$ is independent of f.

Theorem 2.2. (see [5]). *Let $1 < p < q < \infty$, $1 < \tau_1, \tau_2 < \infty$. If the function $f \in L_{p,\tau_1}(\mathbb{T}^m)$ satisfies the condition*

$$\sum_{\bar{s}\in\mathbb{Z}_+^m}\prod_{j=1}^{m}2^{s_j\tau_2(1/p-1/q)}\|\delta_{\bar{s}}(f)\|_{p,\tau_1}^{\tau_2} < \infty,$$

then $f \in L_{q,\tau_2}(\mathbb{T}^m)$ and the following inequality holds

$$\|f\|_{q,\tau_2} \leqslant C\left(\sum_{\bar{s}\in\mathbb{Z}_+^m}\prod_{j=1}^{m}2^{s_j\tau_2(1/p-1/q)}\|\delta_{\bar{s}}(f)\|_{p,\tau_1}^{\tau_2}\right)^{1/\tau_2},$$

where $C > 0$ is independent of f.

Let $A(\mathbb{T}^m)$ be the space $f \in L(\mathbb{T}^m)$ with absolutely converging Fourier series with the norm (see [11], [37], [38])

$$\|f\|_A = \sum_{\bar{k}\in\mathbb{Z}^m}|a_{\bar{k}}(f)|.$$

As a corollary of Theorem 1.1 [38], the following statement is true, which we will often use in the proofs of theorems.

Lemma 2.1. *Let $2 \leqslant p < \infty$ and $1 < \tau < \infty$. There exist constructive approximation methods $G_M(f)$ based on greedy-type algorithms that lead to M–term polynomials with respect the system $\{e^{i\langle\bar{k},\bar{x}\rangle}\}_{\bar{k}\in\mathbb{Z}^m}$ with the following property:*

$$\|f - G_M(f)\|_{p,\tau} \leq CM^{-\frac{1}{2}}p^{\frac{1}{2}}\|f\|_A,$$

for all $f \in A(\mathbb{T}^m)$, where $C > 0$ is independent of $M \in \mathbb{N}$ and of f.

Proof. . We will choose a number $p_0 \in (p, \infty)$. It is known that $L_{p_0}(\mathbb{T}^m) \subset L_{p,\tau}(\mathbb{T}^m)$ and $\|g\|_{p,\tau} \leqslant C\|g\|_{p_0}$ for a function $g \in L_{p_0}(\mathbb{T}^m)$ (see [34, Theorem 3.11]). Now, according to Theorem 1.1 [38] or Theorem 2.6 [37], it is easy to verify that the assertion of Lemma 2.1 is true. □

3 Main results

Theorem 3.1. *Let $0 < r_1 = \ldots = r_\nu < r_{\nu+1} \leq \ldots r_m$, $1 < q < 2 < p < \infty$, $1 < \tau_1, \tau_2 < \infty$, $\frac{1}{q} - \frac{1}{p} < a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, $\tau_2' = \frac{\tau_2}{\tau_2-1}$ and $b \in \mathbb{R}$.*

If $\frac{1}{q} - \frac{1}{p} < a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, $\tau_2' = \frac{\tau_2}{\tau_2-1}$, then

$$e_M(W_{q,\tau_1}^{a,b,\bar{r}})_{p,\tau_2} \asymp M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log_2 M)^{(\nu-1)b},\ M > 1.$$

If $a=(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2^{'}$, *then*

$$e_M(W^{a,b,\bar{r}}_{q,\tau_1})_{p,\tau_2}\leqslant C\begin{cases} M^{-\frac{\tau_2^{'}}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}(\log_2 M)^{(\nu-1)b}(\log_2\log_2 M)^{1/\tau_2}, & \text{if } (\nu-1)b\tau_2+1\neq 0,\\ M^{-\frac{\tau_2^{'}}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}, & \text{if } (\nu-1)b\tau_2+1=0\end{cases}$$

for $M\geqslant 4$, *where* $C>0$ *is independent of* M *and* f.

Proof. For a natural number M, there is a number $n\in\mathbb{N}$ such that $M\asymp 2^n n^{\nu-1}$.

Let $\nu\geqslant 2$ be a natural number. We put

$$n_1=\frac{p}{2}n-p\Big(\frac{1}{2}-\frac{1}{\tau_2}\Big)(\nu-1)\log n,$$

$$n_2=\frac{p}{2}n+\frac{p}{2}(\nu-1)\log n.$$

We will introduce the notation

$$S_l=\Big(2^{la\tau_1}\bar{l}^{-(\nu-1)b\tau_1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}$$

and

$$m_l=\Big[2^{-l\frac{\tau_2^{'}}{p}}S_l^{\tau_1}2^{n\frac{\tau_2^{'}}{2}}n^{(\nu-1)\frac{\tau_2^{'}}{2}}\Big]+1,\ l\in\mathbb{Z}_+,$$

where $\langle\bar{s},\bar{1}\rangle=\sum\limits_{j=1}^{m}s_j$, $p^{'}=\frac{p}{p-1}$ and $[y]$ is an integer part of the number y.

By $G(l)$ we denote the set of indices $\bar{s}$, $l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1$, with the largest $\|\delta_{\bar{s}}(f)\|_2$ and $m_l=|G(l)|$ is the number of elements in the set $G(l)$.

Let us consider the functions

$$F_1(\overline{x})=\sum_{n\leqslant l<n_1}f_l(\overline{x}),$$

$$F_2(\overline{x})=\sum_{n_1\leqslant l<n_2}\sum_{\bar{s}\notin G(l)}\delta_{\bar{s}}(f,\overline{x}),$$

$$F_3(\overline{x})=\sum_{n_1\leqslant l<n_2}\sum_{\bar{s}\in G(l)}\delta_{\bar{s}}(f,\overline{x}).$$

We will estimate $\|F_1\|_A$. Applying Hölder's inequality for the sum and Parseval's equality, we have

$$\|F_1\|_A=\sum_{l=n}^{n_1-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\sum_{\bar{k}\in\rho(\bar{s})}|a_{\bar{k}}(f)|\leqslant$$

$$2^{-\frac{m}{2}}\sum_{l=n}^{n_1-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\bar{s}}(f)\|_2$$

$$=2^{-\frac{m}{2}}\sum_{l=n}^{n_1-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2 2^{\langle\bar{s},\bar{1}\rangle\frac{1}{q}}.\quad(3.1)$$

Now, to the inner sum on the right side of inequality (3.1), applying Hölder's inequality to the inner sum for $\frac{1}{\tau_1}+\frac{1}{\tau_1'}=1$ and $1<\tau_1<\infty$, we get

$$\|F_1\|_A \leqslant 2^{-\frac{m}{2}}\sum_{l=n}^{n_1-1}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle\frac{\tau_1'}{q}}\Big)^{1/\tau_1'}. \tag{3.2}$$

We will choose numbers δ_j such that $\delta_j=\gamma_j$ for $j=1,\ldots,\nu$ and $1<\delta_j<\gamma_j$ for $j=\nu+1,\ldots,m$. Then, by Lemma G [35], we have

$$\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle\frac{\tau_1'}{q}}\Big)^{1/\tau_1'}\leqslant C2^{\frac{l}{q}}l^{\frac{\nu-1}{\tau_1'}}, \tag{3.3}$$

where $C>0$ is independent of l. According to Theorem 2.1, for $1<q<2$ and $\lambda=\theta=2$, we have

$$\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}\leqslant C\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{q,\tau_1}, \tag{3.4}$$

where here and in the rest of the proof C denotes a positive number which depends only on numerical parameters, and may be different on different occurrences.

Now, taking into account that the function $f\in W^{a,b,\bar{r}}_{q,\tau_1}$, $\frac{1}{q}-a>0$, from inequalities (3.2), (3.3) and (3.4), we obtain

$$\|F_1\|_A\leqslant C\sum_{l=n}^{n_1-1}2^{\frac{l}{q}}l^{(\nu-1)/\tau_1'}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{\frac{1}{\tau_1}}$$

$$\leqslant C\sum_{l=n}^{n_1-1}2^{\frac{l}{q}}l^{(\nu-1)/\tau_1'}\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{q,\tau_1}$$

$$\leqslant C\sum_{l=n}^{n_1-1}2^{l(\frac{1}{q}-a)}l^{(\nu-1)(b+\frac{1}{\tau_1'}}\leqslant C2^{n_1(\frac{1}{q}-a)}n_1^{(\nu-1)(b+\frac{1}{\tau_1'})}.$$

Thus,

$$\|F_1\|_A\leqslant C2^{n_1(\frac{1}{q}-a)}n_1^{(\nu-1)(b+\frac{1}{\tau_1'})} \tag{3.5}$$

for a function $f\in W^{a,b,\bar{r}}_{q,\tau_1}$ and $\frac{1}{q}-a>0$, $1<q<2$ and $1<\tau_1<\infty$. By Lemma 2.1 for the function F_1, using a constructive method, one can find an M–term trigonometric polynomial $G_M(F_1,\overline{x})$ such that

$$\|F_1-G_M(F_1)\|_{p,\tau_2}\leqslant CM^{-1/2}\|F_1\|_A,\ 2<p<\infty. \tag{3.6}$$

Now, taking into account the definition of the number n_1 and the condition $a<(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$

from estimates (3.5) and (3.6) we obtain

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leqslant CM^{-1/2}2^{n_1(\frac{1}{q}-a)}n_1^{(\nu-1)(b+\frac{1}{\tau_1'})}$$
$$= CM^{-1/2}2^{n\frac{p}{2}(\frac{1}{q}-a)}n^{-(\nu-1)p(\frac{1}{2}-1)(\frac{1}{\tau_1}-a)}n_1^{(\nu-1)(b+\frac{1}{\tau_1'})}$$
$$\leqslant CM^{-1/2}(2^n n^{\nu-1})^{\frac{p}{2}(\frac{1}{q}-a)}n^{(\nu-1)p(1-\frac{1}{\tau_2})(a-\frac{1}{q})}n^{(\nu-1)(b+\frac{1}{\tau_1'})}$$
$$= CM^{-1/2}(2^n n^{\nu-1})^{\frac{p}{2}(\frac{1}{q}-a)}n^{(\nu-1)(\frac{p}{\tau_2'}(a-\frac{1}{q})+\frac{1}{\tau_1'})}n^{(\nu-1)b}$$
$$= CM^{-1/2}(2^n n^{\nu-1})^{\frac{p}{2}(\frac{1}{q}-a)}n^{(\nu-1)(\frac{p}{\tau_2'}(a-\tau_2'(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2}))}n^{(\nu-1)b}$$
$$\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b}, \quad (3.7)$$

in the case $a \leqslant (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, $1 < q < 2 < p < \infty$, $1 < \tau_1, \tau_2 < \infty$.

Let us estimate $\|F_2\|_{p,\tau_2}$. By Theorem 2.2, for $p = \tau_1 = 2$ and replacing q by p, taking into account that

$$\|\delta_{\bar{s}}(f)\|_2 \leqslant m_l^{-\frac{1}{\tau_1}}2^{-la}l^{(\nu-1)b}2^{-l(\frac{1}{2}-\frac{1}{q})}S_l,$$

for $\bar{s} \notin G(l)$, for $\tau_2 - \tau_1 \geqslant 0$ we have

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1}\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_2}\Big)^{1/\tau_2}$$
$$= C\Big(\sum_{l=n_1}^{n_2-1}\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_2-\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}$$
$$\leqslant C\Big(\sum_{l=n_1}^{n_2-1}\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big(m_l^{-\frac{1}{\tau_1}}2^{-la}l^{(\nu-1)b}2^{-l(\frac{1}{2}-\frac{1}{q})}S_l\Big)^{\tau_2-\tau_1}\Big)^{1/\tau_2}$$
$$= C\Big(\sum_{l=n_1}^{n_2-1}\Big(2^{-la}l^{(\nu-1)b}2^{-l(\frac{1}{2}-\frac{1}{q})}\Big)^{\tau_2-\tau_1}m_l^{-\frac{\tau_2-\tau_1}{\tau_1}}S_l^{\tau_2-\tau_1}$$
$$\times\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}. \quad (3.8)$$

Since $1 < q < 2 < p$, then $(\frac{1}{2} - \frac{1}{p})\tau_2 - (\frac{1}{2} - \frac{1}{q})\tau_1 > 0$. Therefore, taking into account that $1 \leqslant \gamma_j,\ j = 1, ..., m$, it is easy to verify that

$$\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1} \leqslant 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2-(\frac{1}{2}-\frac{1}{q})\tau_1}$$
$$\times\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1} \leqslant 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2-(\frac{1}{2}-\frac{1}{q})\tau_1}\Big(2^{-la}l^{(\nu-1)b}\Big)^{\tau_1}S_l^{\tau_1}.$$

Therefore,

$$S_l^{\tau_2-\tau_1}\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1} \leqslant 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2-(\frac{1}{2}-\frac{1}{q})\tau_1}\Big(2^{-la}l^{(\nu-1)b}\Big)^{\tau_1}S_l^{\tau_2}.$$

Hence, from inequality (3.8) we obtain

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1}\Big(2^{-la}l^{(\nu-1)b}2^{-l(\frac{1}{2}-\frac{1}{q})}\Big)^{\tau_2-\tau_1} m_l^{-\frac{\tau_2-\tau_1}{\tau_1}} 2^{(l+1)((\frac{1}{2}-\frac{1}{p})\tau_2-(\frac{1}{2}-\frac{1}{q})\tau_1)}$$

$$\times\Big(2^{-la}l^{(\nu-1)b}\Big)^{\tau_1} S_l^{\tau_2}\Big)^{1/\tau_2} = C\Big(\sum_{l=n_1}^{n_2-1}\Big(2^{-la}l^{(\nu-1)b}\Big)^{\tau_2} m_l^{-\frac{\tau_2-\tau_1}{\tau_1}} 2^{l(\frac{1}{\tau_1}-\frac{1}{\tau_2})\tau_2} S_l^{\tau_2}\Big)^{1/\tau_2}.$$

Now, substituting the values of the numbers m_l, from here we get

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1} 2^{-l(a+\frac{1}{p}-\frac{1}{q})\tau_2} l^{(\nu-1)b\tau_2}\Big(2^{-l\frac{\tau_2}{p}} S_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{\tau_2-\tau_1}{\tau_1}} S_l^{\tau_2}\Big)^{1/\tau_2}$$

$$= C\Big(2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{\tau_2-\tau_1}{\tau_1\tau_2}} \Big(\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2} S_l^{\tau_1}\Big)^{1/\tau_2}. \quad (3.9)$$

Further, using inequality (3.4) and taking into account that the function $f \in W_{q,\tau_1}^{a,b,\bar{r}}$ and $a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$ we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2} S_l^{\tau_1}$$

$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2}\Big(2^{la}l^{-(\nu-1)b}\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{q,\tau_1}\Big)^{\tau_1}$$

$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2} \leqslant C2^{-n_2(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} n_2^{(\nu-1)b\tau_2}. \quad (3.10)$$

It is easy to verify that if $a = (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, then

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2} \leqslant C\begin{cases} n^{(\nu-1)b\tau_2}\log n, & \text{if } (\nu-1)b\tau_2+1\neq 0,\\ 1, & \text{if } (\nu-1)b\tau_2+1=0.\end{cases} \quad (3.11)$$

If $a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, then from (3.9) and (3.10) we obtain

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{\tau_2-\tau_1}{\tau_1\tau_2}} 2^{-n_2(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')} n_2^{(\nu-1)b}.$$

Now, by the definition of the number n_2 and taking into account that $M \asymp 2^n n^{\nu-1}$, from this formula, we obtain that

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{\tau_2-\tau_1}{\tau_1\tau_2}} (2^n n^{(\nu-1)})^{-\frac{p}{2}(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'))} n^{(\nu-1)b}$$

$$= C(2^n n^{(\nu-1)})^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})} n^{(\nu-1)b} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b} \quad (3.12)$$

for the function $f \in W_{q,\tau_1}^{a,b,\bar{r}}$ in the case of $a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, $1 < q < 2 < p < \infty$, $1 < \tau_1 \leqslant \tau_2 < \infty$.

If $a=(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2^{'}$, then from (3.9) and (3.11) we obtain that

$$\|F_2\|_{p,\tau_2}\leqslant C\Big(2^n n^{(\nu-1)}\Big)^{-\frac{\tau_2^{'}}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}\begin{cases} n^{(\nu-1)b}(\log n)^{\frac{1}{\tau_2}}, & \text{if } (\nu-1)b\tau_2+1\neq 0,\\ 1, & \text{if } (\nu-1)b\tau_2+1=0.\end{cases} \tag{3.13}$$

Next, we estimate $\|F_3\|_A$. By applying Hölder's inequality for the sum and Parseval's equality, we have

$$\|F_3\|_A=\sum_{l=n_1}^{n_2-1}\sum_{l\leqslant\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}\sum_{\bar{k}\in\rho(\bar{s})}|a_{\bar{k}}(f)|$$

$$\leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2. \tag{3.14}$$

Now, to the inner sum on the right side of inequality (3.14) applying Hölder's inequality for $\frac{1}{\tau_1}+\frac{1}{\tau_1^{'}}=1$ and $1<\tau_1<\infty$, we get

$$\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{\tau_1})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}1\Big)^{1/\tau_1^{'}}$$

$$\leqslant\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}m_l^{1/\tau_1^{'}}.$$

Further, substituting the values of the numbers m_l, from this formula, we obtain that

$$\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}\Big(2^{-l\frac{\tau_2^{'}}{p}}S_l^{\tau_1}2^{n\frac{\tau_2^{'}}{2}}n^{(\nu-1)\frac{\tau_2^{'}}{2}}+1\Big)^{1/\tau_1^{'}}$$

$$=C\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q})}l^{(\nu-1)b}S_l\Big(2^{-l\frac{\tau_2^{'}}{p}}S_l^{\tau_1}2^{n\frac{\tau_2^{'}}{2}}n^{(\nu-1)\frac{\tau_2^{'}}{2}}+1\Big)^{1/\tau_1^{'}}$$

$$\leqslant C\Big\{\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q})}l^{(\nu-1)b}S_l\Big(2^{-l\frac{\tau_2^{'}}{p}}S_l^{\tau_1}2^{n\frac{\tau_2^{'}}{2}}n^{(\nu-1)\frac{\tau_2^{'}}{2}}\Big)^{1/\tau_1^{'}}+\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q})}l^{(\nu-1)b}S_l\Big\}$$

$$=C\Big\{2^{n\frac{\tau_2^{'}}{2\tau_1^{'}}}n^{(\nu-1)\frac{\tau_2^{'}}{2\tau_1^{'}}}\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q}+\frac{\tau_2^{'}}{p\tau_1^{'}})}l^{(\nu-1)b}S_lS_l^{\frac{\tau_1}{\tau_1^{'}}}+\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q})}l^{(\nu-1)b}S_l\Big\}. \tag{3.15}$$

Since

$$\frac{\tau_2^{'}}{p\tau_1^{'}}-\frac{1}{q}=\tau_2^{'}(\frac{1}{p\tau_1^{'}}-\frac{1}{q\tau_2^{'}}),\ \ S_lS_l^{\frac{\tau_1}{\tau_1^{'}}}=S_l^{\tau_1},$$

then

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q}+\frac{\tau_2^{'}}{p\tau_1^{'}})} l^{(\nu-1)b} S_l S_l^{\frac{\tau_1}{\tau_1^{'}}} = \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}}))} l^{(\nu-1)b} S_l^{\tau_1}. \tag{3.16}$$

Now, by using inequality (3.4) and taking into account that the function $f \in W_q^{a,b,\bar{r}}$ in the case $a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}})<0$ from equality (3.16), we obtain

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q}+\frac{\tau_2^{'}}{p\tau_1^{'}})} l^{(\nu-1)b} S_l S_l^{\frac{\tau_1}{\tau_1^{'}}}$$

$$\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}}))} l^{(\nu-1)b} \Big(2^{la} l^{-(\nu-1)b} \Big\| \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{q,\tau_1}\Big)^{\tau_1}$$

$$\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}}))} l^{(\nu-1)b} \leqslant C 2^{-n_2(a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}}))} n_2^{(\nu-1)b}. \tag{3.17}$$

and if $a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}})=0$, then according to (3.11)

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q}+\frac{\tau_2^{'}}{p\tau_1^{'}})} l^{(\nu-1)b} S_l S_l^{\frac{\tau_1}{\tau_1^{'}}} \leqslant C \begin{cases} n^{(\nu-1)b}\log n, & \text{if } (\nu-1)b+1\neq 0, \\ 1, & \text{if } (\nu-1)b+1=0. \end{cases} \tag{3.18}$$

Since $a-\frac{1}{q}<0$, then again using Theorem 2. 1 for $\lambda=\theta=2$ and taking into account that the function $f\in W_{q,\tau_1}^{a,b,\bar{r}}$, we get

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q})} l^{(\nu-1)b} S_l \leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q})} l^{(\nu-1)b} \Big(2^{la} l^{-(\nu-1)b} \Big\| \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{q,\tau_1}\Big)$$

$$\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q})} l^{(\nu-1)b} \leqslant C 2^{-n_2(a-\frac{1}{q})} n_2^{(\nu-1)b}. \tag{3.19}$$

Now from inequalities (3.15), (3.17) and (3.19), it follows that

$$\sum_{l=n_1}^{n_2-1} 2^{\frac{l+1}{q}} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\in G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})} \|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant C\Big\{2^{n\frac{\tau_2^{'}}{2\tau_1^{'}}} n^{(\nu-1)\frac{\tau_2^{'}}{2\tau_1^{'}}} 2^{-n_2(a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}}))} n_2^{(\nu-1)b} + 2^{-n_2(a-\frac{1}{q})} n_2^{(\nu-1)b}\Big\}, \tag{3.20}$$

in the case $a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}})<0$. By the definition of the number n_2, we have

$$2^{-n_2(a-\frac{1}{q})} n_2^{(\nu-1)b} = (2^{n\frac{p}{2}} n^{(\nu-1)\frac{p}{2}})^{-(a-\frac{1}{q})} n_2^{(\nu-1)b} \leqslant C(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})} n^{(\nu-1)b}$$

and

$$(2^n n^{\nu-1})^{\frac{\tau_2^{'}}{2\tau_1^{'}}} 2^{-n_2(a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}}))} = (2^n n^{\nu-1})^{\frac{\tau_2^{'}}{2\tau_1^{'}}} (2^n n^{(\nu-1)})^{-\frac{p}{2}(a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}}))}$$

$$= (2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})} (2^n n^{\nu-1})^{-(\frac{p}{2}(\frac{1}{q}-p^{'}(\frac{1}{q}-\frac{1}{p}))-\frac{p^{'}}{2q^{'}})}.$$

Now, taking into account that $\frac{p}{2}(a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'}))-\frac{\tau_2'}{2\tau_1'}=\frac{p}{2}(a-\frac{1}{q})$ according to these relations from formula (3.20), we obtain

$$\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant C(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})}n^{(\nu-1)b},$$

for $a<(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'=\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'})$, $b\in\mathbb{R}$. Therefore, inequality (3.14) implies that

$$\|F_3\|_A\leqslant C(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})}n^{(\nu-1)b} \tag{3.21}$$

for $a<(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$, $b\in\mathbb{R}$.

Since $a-\frac{1}{q}<0$, then from inequalities (3.13), (3.18) and (3.19) it follows that inequality (3.21) is also true in the case $a=(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$.

By Lemma 2.1 for the function F_3 there exists an M-term polynomial $G_M(F_3,\overline{x})$ such that

$$\|F_3-G_M(F_3)\|_{p,\tau_2}\leqslant CM^{-1/2}\|F_3\|_A.$$

Therefore, according to inequality (3.21) from this formula, we obtain that

$$\|F_3-G_M(F_3)\|_{p,\tau_2}\leqslant CM^{-1/2}(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})}n^{(\nu-1)b}\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b} \tag{3.22}$$

in the case $a\leqslant(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$.

We represent the function $f\in W^{a,b,\bar{r}}_{q,\tau_1}$ as a sum

$$f(\overline{x})=S_{Q_{n,\bar{\gamma}}}(f,\overline{x})+F_1(\overline{x})+F_2(\overline{x})+F_3(\overline{x})+\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f,\bar{x}).$$

Therefore, from estimates (3.7), (3.12), (3.22), it follows that

$$\begin{aligned}&\|f-(S_{Q_{n,\bar{\gamma}}}(f)+G_M(F_1)+G_M(F_3))\|_{p,\tau_2}\\&\qquad\leqslant\|F_1-G_M(F_1)\|_{p,\tau_2}+\|F_3-G_M(F_3)\|_{p,\tau_2}+\|F_2\|_{p,\tau_2}\\&\qquad+\Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2}\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b}+\Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2},\end{aligned} \tag{3.23}$$

in the case $a<(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$, $b\in\mathbb{R}$. Since $1<q<p<\infty$, then by Theorem 2.2, inequality (3.4) and the definition of the class $W^{a,b,\bar{r}}_{q,\tau_1}$ and taking into account such that $a+\frac{1}{p}-\frac{1}{q}>0$ and $1<\tau_1\leqslant\tau_2<\infty$, we have

$$\begin{aligned}\Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2}&=\Big\|\sum_{l=n_2}^{\infty}\sum_{l\leqslant\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2}\\&\leqslant C\Big(\sum_{l=n_2}^{\infty}2^{l(\frac{1}{q}-\frac{1}{p})\tau_2}\Big\|\sum_{l\leqslant\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{q,\tau_1}^{\tau_2}\Big)^{\frac{1}{\tau_2}}\leqslant C\Big(\sum_{l=n_2}^{\infty}2^{-l(a+\frac{1}{p}-\frac{1}{q})p}l^{(\nu-1)bp}\Big)^{\frac{1}{p}}\\&\leqslant C2^{-n_2(a+\frac{1}{p}-\frac{1}{q})}n_2^{(\nu-1)b}\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b}.\end{aligned} \tag{3.24}$$

Now from inequalities (3.23) and (3.24), it follows that

$$e_M(f)_{p,\tau_2} \leqslant \|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$ for $\frac{1}{q}-\frac{1}{p} < a < (\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$, $b \in \mathbb{R}$ and $1 < q < 2 < p < \infty$ and $\nu \geqslant 2$.

If $a = (\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$, then

$$\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q}) = \frac{\tau_2'}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2}).$$

Therefore, in the case $a = (\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$ and $(\nu-1)b\tau_2+1 \neq 0$ from inequalities (3.13), (3.22) and (3.7), we obtain

$$\|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2} \leqslant CM^{-\frac{\tau_2'}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}(\log M)^{(\nu-1)b}(\log\log M)^{1/\tau_2}.$$

Hence

$$e_M(f)_{p,\tau_2} \leqslant CM^{-\frac{\tau_2'}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}(\log M)^{(\nu-1)b}(\log\log M)^{1/\tau_2}$$

for the function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, $1 < q < 2 < p < \infty$, $a = (\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$ and $(\nu-1)b\tau_2+1 \neq 0$, $\nu \geqslant 2$.

If $a = (\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$ and $(\nu-1)b\tau_2+1 = 0$, then from inequalities (3.7), (3.13) and (3.21), it follows that

$$e_M(f)_{p,\tau_2} \leqslant CM^{-\frac{\tau_2'}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, $1 < q < 2 < p < \infty$.

Let $\nu = 1$ i.e. $r_1 < r_{\nu+1} \leqslant ... \leqslant r_m$. For $M \asymp 2^n$, there is a natural number n such that $M \asymp 2^n$. In this case, put $n_1 = n\frac{p}{2}$ and consider the function

$$F_1(\overline{x}) = \sum_{l=n}^{n_1-1} f_l(\overline{x}).$$

Now, repeating the arguments in the proof of inequality (3.5) for the function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, we obtain

$$\|F_1\|_A \leqslant C2^{n_1(\frac{1}{q}-a)}, \tag{3.25}$$

in the case $\frac{1}{q}-a > 0$, for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$. Hence, from inequalities (3.25) and (3.6), we obtain

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leqslant CM^{-1/2}\|F_1\|_A \leqslant CM^{-1/2}2^{n_1(\frac{1}{q}-a)} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})} \tag{3.26}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, in the case of $\frac{1}{q}-a > 0$. By the property of the norm and according to (3.26), we have

$$\|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1))\|_{p,\tau_2} \leqslant \|F_1 - G_M(F_1)\|_{p,\tau_2} + \Big\| \sum_{\langle\bar{s},\bar{\gamma}\rangle \geqslant n_1} \delta_{\bar{s}}(f) \Big\|_{p,\tau_2}$$
$$\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})} + \Big\| \sum_{\langle\bar{s},\bar{\gamma}\rangle \geqslant n_1} \delta_{\bar{s}}(f) \Big\|_{p,\tau_2}, \tag{3.27}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, in the case of $a < \frac{1}{q}$. Further, repeating the proof of inequality (3.24) with n_2 replaced by n_1, we have

$$\Big\| \sum_{\langle \bar{s},\bar{\gamma}\rangle \geqslant n_1} \delta_{\bar{s}}(f)\Big\|_{p,\tau_2} \leqslant C2^{-n_1(a+\frac{1}{p}-\frac{1}{q})} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}, \tag{3.28}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, in the case of $\frac{1}{p} - \frac{1}{q} < a$, $1 < q < p < \infty$, $1 < \tau_1 \leqslant \tau_2 < \infty$.

Now from inequalities (3.27) and (3.28), it follows that

$$\|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1))\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})},$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, in the case $\frac{1}{p} - \frac{1}{q} < a < \frac{1}{q}$, $1 < q < 2 < p < \infty$, $1 < \tau_1 \leqslant \tau_2 < \infty$. Hence

$$e_M(W^{a,b,\bar{r}}_{q,\tau_1})_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})},$$

in the case $\nu = 1$ and $\frac{1}{p} - \frac{1}{q} < a < \frac{1}{q}$, $1 < q < 2 < p < \infty$, $1 < \tau_1 \leqslant \tau_2 < \infty$.

Lower bound for $e_M(W^{a,b,\bar{r}}_{q,\tau_1})_{p,\tau_2}$. Let $M \in \mathbb{N}$ and $N = [\frac{p}{2}\log_2 M]$ is an integer part of the number $\frac{p}{2}\log_2 M$.

Let $\bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}^m_+$ such that $\prod_{j=1}^m 2^{s_j} = 2^N$. Consider the function

$$f_0(\bar{x}) = 2^{-N(1-\frac{1}{q})}2^{-Na}N^{(\nu-1)b} \sum_{\bar{k}\in\rho(\bar{s})} e^{\langle \bar{k},\bar{x}\rangle}.$$

Then

$$\|f_{0,l}\|_{q,\tau_1} = \Big\| \sum_{l\leqslant\langle \bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f_0)\Big\|_{q,\tau_1} = 0$$

for $l \neq N$. If $l = N$, then by virtue of the estimate for the norm of the Dirichlet kernel in the Lorentz space (see [5, p. 13]), we have

$$\|f_{0,l}\|_{q,\tau_1} = \|f_0\|_{q,\tau_1} \leqslant C2^{-Na}N^{(\nu-1)b}.$$

Thus, the function $f_0 \in W^{a,b,\bar{r}}_{q,\tau_1}$, $1 < q < \infty$, $1 < \tau_1, \tau_2 < \infty$, $a > 0$, $b \in \mathbb{R}$.

Let K_M be an arbitrary set of M harmonics $\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m_+$ and $\mathbb{T}(K_M)$ is the set of trigonometric polynomials with harmonics from K_M. Consider an additional function

$$h(\bar{x}) = \sum_{\bar{k}\in\rho(\bar{s})\setminus K_M} e^{\langle \bar{k},\bar{x}\rangle}.$$

Then, by the property of the norm, the estimate for the norm of the Dirichlet kernel, and Parseval's equality, we have

$$\|h\|_{p',\tau_2'} \leqslant \|g_0\|_{p',\tau_2'} + \|g_0 - h\|_{p',\tau_2'} \leqslant \|g_0\|_{p',\tau_2'} + C\|g_0 - h\|_2 \leqslant C\{2^{\frac{N}{p}} + \sqrt{M}\} \leqslant C_0\sqrt{M},$$

where $g_0(\bar{x}) = \sum_{\bar{k}\in\rho(\bar{s})} e^{\langle \bar{k},\bar{x}\rangle}$, $2 < p < \infty$, $1 < \tau_2 < \infty$, $\beta' = \frac{\beta}{\beta-1}$. Therefore, for any polynomial $T \in \mathbb{T}(K_M)$, due to Hölder's inequality in the Lorentz space, we have

$$\int_{\mathbb{T}^m} (f_0(\bar{x}) - T(\bar{x}))h(\bar{x})d\bar{x} \leqslant \|f_0 - T\|_{p,\tau_2}\|h\|_{p',\tau_2'} \leqslant C\sqrt{M}\|f_0 - T\|_{p,\tau_2}, \tag{3.29}$$

for $2 < p < \infty$, $1 < \tau_2 < \infty$.

On the other hand, taking into account the orthogonality of the trigonometric system, we have

$$\int\limits_{\mathbb{T}^m}(f_0(\overline{x})-T(\overline{x}))h(\overline{x})d\overline{x}=\int\limits_{\mathbb{T}^m}f_0(\overline{x})h(\overline{x})d\overline{x}=2^{-N(1-\frac{1}{q})}2^{-Na}N^{(\nu-1)b}\sum_{\overline{k}\in\rho(\overline{s})}1$$
$$=2^{-N(1-\frac{1}{q})}2^{-Na}N^{(\nu-1)b}(|\rho(\overline{s})\setminus K_M|-M)\geqslant 2^{-N(a+1-\frac{1}{q})}N^{(\nu-1)b}(2M-M)$$
$$=2^{-N(a+1-\frac{1}{q})}N^{(\nu-1)b}2^N.$$

Therefore, from inequality (3.29), we obtain

$$\|f_0-T\|_{p,\tau_2}\geqslant C2^{-N(a-\frac{1}{q})}N^{(\nu-1)b}2^NM^{-\frac{1}{2}}\geqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b},$$

for any polynomial $T\in\mathbb{T}(K_M)$, $2<p<\infty$, $1<\tau_2<\infty$. Hence

$$e_M(W^{a,b,\overline{r}}_{q,\tau_1})_{p,\tau_2}\geqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b},$$

in the case $\frac{1}{q}-\frac{1}{p}<a<(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$, $1<q<2<p<\infty$, $1<\tau_2<\infty$. □

Theorem 3.2. *Let* $0<r_1=\ldots=r_\nu<r_{\nu+1}\leq\ldots r_m$, $2<p<\infty$, $1<\max\{\tau_1,2\}\leqslant\tau_2<\infty$, $\frac{1}{2}-\frac{1}{p}<a<(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})\tau_2'$, $\tau_2'=\frac{\tau_2}{\tau_2-1}$ *and* $b\in\mathbb{R}$, *then*

$$e_M(W^{a,b,\overline{r}}_{2,\tau_1})_{p,\tau_2}\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log_2 M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}},$$

where $C>0$ *is independent of* $M>1$.

Proof. As in the proof of Theorem 3.1, consider the functions F_j, $j=1,2,3$. By formula(3.1), we have

$$\|F_1\|_A=\sum_{l=n}^{n_1-1}\sum_{l\leq\langle\overline{s},\overline{\gamma}\rangle<l+1}\sum_{\overline{k}\in\rho(\overline{s})}|a_{\overline{k}}(f)|\leqslant 2^{-\frac{m}{2}}\sum_{l=n}^{n_1-1}\sum_{l\leq\langle\overline{s},\overline{\gamma}\rangle<l+1}2^{\langle\overline{s},\overline{1}\rangle\frac{1}{2}}\|\delta_{\overline{s}}(f)\|_2. \quad (3.30)$$

If $2<\tau_1<\infty$, then according to the inequality of different metrics for trigonometric polynomials in the Lorentz space [4] we have

$$\|\delta_{\overline{s}}(f)\|_2\leqslant C\Big(\sum_{j=1}^m(s_j+1)\Big)^{\frac{1}{2}-\frac{1}{\tau_1}}\|\delta_{\overline{s}}(f)\|_{2,\tau_1},$$

where here and in the rest of the proof C denotes a positive number which depends only on numerical parameters, and may be different on different occurrences.

Therefore, from Lemma 1.6 [5] for $p=2$ and $2<\tau_1<\infty$ we obtain

$$\left(\sum_{\overline{s}\in\mathbb{Z}_+}\Big(\sum_{j=1}^m(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\overline{s}}(f)\|_2^{\tau_1}\right)^{\frac{1}{\tau_1}}\leqslant C\left(\sum_{\overline{s}\in\mathbb{Z}_+}\|\delta_{\overline{s}}(f)\|_{2,\tau_1}^{\tau_1}\right)^{\frac{1}{\tau_1}}\leqslant C\|f\|_{2,\tau_1}. \quad (3.31)$$

According to inequality (3.31) and Hölder's inequality, we obtain

$$\sum_{l\leq\langle\overline{s},\overline{\gamma}\rangle<l+1}2^{\langle\overline{s},\overline{1}\rangle\frac{1}{2}}\|\delta_{\overline{s}}(f)\|_2\leqslant\left(\sum_{l\leq\langle\overline{s},\overline{\gamma}\rangle<l+1}\Big(\sum_{j=1}^m(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\overline{s}}(f)\|_2^{\tau_1}\right)^{\frac{1}{\tau_1}}$$

$$\times \left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{1}\rangle\frac{\tau_1'}{2}} \Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac12-\frac{1}{\tau_1})\tau_1'} \right)^{\frac{1}{\tau_1'}}$$

$$\leqslant C\Big\| \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{2,\tau_1} \left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{1}\rangle\frac{\tau_1'}{2}} \Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac12-\frac{1}{\tau_1})\tau_1'} \right)^{\frac{1}{\tau_1'}}, \quad (3.32)$$

where $\tau_1' = \frac{\tau_1}{\tau_1-1}$, $1<\tau_1<\infty$. We will choose numbers δ_j such that $\delta_j=\gamma_j$ for $j=1,\dots,\nu$ and $1<\delta_j<\gamma_j$ for $j=\nu+1,\dots,m$. Then, by Lemma G [35], from inequality (3.32) we have

$$\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{1}\rangle\frac12}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant C\Big\| \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{2,\tau_1} \left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{\delta}\rangle\frac{\tau_1'}{2}} \Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac12-\frac{1}{\tau_1})\tau_1'} \right)^{\frac{1}{\tau_1'}},$$

$$\leqslant C\Big\| \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{2,\tau_1} 2^{\frac{l}{2}} l^{(\nu-1)\frac{1}{\tau_1'}} l^{\frac12-\frac{1}{\tau_1}}, \quad (3.33)$$

in the case $2<\tau_1<\infty$. Therefore, taking into account that the function $f\in W^{a,b,\overline{r}}_{2,\tau_1}$ and $a<\frac12$ from (3.30) and (3.33) we get

$$\|F_1\|_A \leqslant C\sum_{l=n}^{n_1-1} 2^{\frac{l}{2}} l^{(\nu-1)\frac{1}{\tau_1'}} l^{\frac12-\frac{1}{\tau_1}} 2^{-la} l^{(\nu-1)b} \leqslant C2^{-n_1(a-\frac{l}{2})} n_1^{(\nu-1)(b+\frac{1}{\tau_1'})} n_1^{\frac12-\frac{1}{\tau_1}}, \quad (3.34)$$

in the case $q=2<p<\infty$, $2<\tau_1<\infty$, $a<\frac12$. Since $2<p<\infty$, then by Lemma 2.1 for the function F_1 there exists a M-term polynomial $G_M(F_1,\overline{x})$ such that

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leqslant CM^{-\frac12}\|F_1\|_A.$$

Therefore, according to inequality (3.34) and taking into account the definition of the number n_1 and the relation $M\asymp 2^n n^{\nu-1}$ from this formula, we obtain that

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac1p-\frac1q)}(\log M)^{(\nu-1)b}(\log M)^{\frac{l}{2}-\frac{1}{\tau_1}}, \quad (3.35)$$

in the case $q=2<p<\infty$, $2<\tau_1<\infty$, $1<\tau_2<\infty$, $a<\frac12$.

For the estimate $\|F_3\|_A$ by applying Hölder's inequality for the sum and Parseval's equality, we obtain

$$\|F_3\|_A = \sum_{l=n_1}^{n_2-1} \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)} \sum_{\overline{k}\in\rho(\overline{s})} |a_{\overline{k}}(f)| \leqslant 2^{-\frac{m}{2}} \sum_{l=n_1}^{n_2-1} \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)} 2^{\langle\bar{s},\bar{1}\rangle\frac12}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{\frac{l}{2}}(l+1)^{\frac12-\frac{1}{\tau_1}} \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)} \Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac12)}\|\delta_{\bar{s}}(f)\|_2. \quad (3.36)$$

Now, to the inner sum on the right side of inequality (3.36), by applying Hölder's inequality for $\frac{1}{\tau_1}+\frac{1}{\tau_1'}=1$ and $1<\tau_1<\infty$ we will have

$$\|F_3\|_A \leqslant C\sum_{l=n_1}^{n_2-1} 2^{\frac{l}{2}}(l+1)^{\frac12-\frac{1}{\tau_1}} \Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)} \Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac12)\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{\frac{1}{\tau_1}} |G(l)|^{\frac{1}{\tau_1'}}. \quad (3.37)$$

We will put

$$\tilde{S}_l = \Big(2^{la\tau_1} l^{-(\nu-1)b\tau_1} \sum_{l\leq \langle \bar{s},\bar{\gamma}\rangle < l+1} \Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1} \|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}$$

and

$$m_l := |G(l)| := \Big[2^{-l\frac{\tau_2'}{p}} \tilde{S}_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big] + 1.$$

Then from (3.37), it follows that

$$\begin{aligned} \|F_3\|_A &\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l m_l^{\frac{1}{\tau_1'}} \\ &\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l \Big\{2^{-l\frac{\tau_2'}{p}} \tilde{S}_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}} + 1\Big\}^{\frac{1}{\tau_1'}} \\ &\leqslant C\Big\{\Big(2^n n^{\nu-1}\Big)^{\frac{\tau_2'}{2\tau_1'}} \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} + \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l\Big\}. \quad (3.38) \end{aligned}$$

Since $\tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} = \tilde{S}_l^{\tau_1}$ and $-\frac{1}{2} + \frac{\tau_2'}{p\tau_1'} = \tau_2'(-\frac{1}{2}+\frac{1}{p}-\frac{1}{p\tau_1}+\frac{1}{2\tau_2})$, then according to (3.31), we have

$$\begin{aligned} \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} &= \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{\tau_1} \\ &\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \Big(2^{la\tau_1} l^{-(\nu-1)b\tau_1} \Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{2,\tau_1}\Big)^{\tau_1} \\ &\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}. \end{aligned}$$

for a function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$. Since $a - \tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}) < 0$, then taking into account the definition of the number n_2 from this formula, we obtain that

$$\begin{aligned} \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} &\leqslant C 2^{-n_2(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n_2^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \\ &\leqslant C 2^{-n\frac{p}{2}(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n^{-(\nu-1)\frac{p}{2}(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}, \quad (3.39) \end{aligned}$$

for a function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$, $2 < \tau_1 < \infty$.

Further, according to inequality (3.31), taking into account the function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ and $a - \frac{1}{2} < 0$ we have

$$\begin{aligned} \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l &\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \\ &\quad \times \Big(2^{la\tau_1} l^{-(\nu-1)b\tau_1} \Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{2,\tau_1}\Big) \\ &\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \leqslant C 2^{-n_2(a-\frac{1}{2})} n_2^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \\ &\leqslant C 2^{-n\frac{p}{2}(a-\frac{1}{2})} n^{-(\nu-1)\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}. \quad (3.40) \end{aligned}$$

Now from inequalities (3.38), (3.39) and (3.40), it follows that

$$\|F_3\|_A \leqslant C\Big\{\Big(2^n n^{\nu-1}\Big)^{\frac{\tau_2'}{2\tau_1'}} 2^{-n\frac{p}{2}(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n^{-(\nu-1)\frac{p}{2}(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} + (2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}\Big\}$$

for a function $f\in W^{a,b,\bar r}_{q,\tau_1}$, $2<\tau_1<\infty$, $1<\tau_2<\infty$, $a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})<0$.

Since $\frac{p}{2}(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))-\frac{\tau_2'}{2\tau_1'}=\frac{p}{2}(a-\frac{1}{2})$, then it follows that

$$\|F_3\|_A \leqslant C(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}. \tag{3.41}$$

Since $2<p<\infty$, then by Lemma 2.1 for the function F_3 by a constructive method there is a M–term polynomial $G_M(F_3,\overline{x})$ such that

$$\|F_3-G_M(F_3)\|_{p,\tau_2}\leqslant CM^{-\frac{l}{2}}\|F_3\|_A.$$

Therefore, according to (3.41), we have

$$\|F_3-G_M(F_3)\|_{p,\tau_2}\leqslant CM^{-\frac{l}{2}}(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}, \tag{3.42}$$

for a function $f\in W^{a,b,\bar r}_{2,\tau_1}$, $2<p<\infty$, $2<\tau_1<\infty$, $1<\tau_2<\infty$, $a<\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$.

Let us estimate $\|F_2\|_{p,\tau_2}$. In formula (3.8), the inequality is proved

$$\|F_2\|_{p,\tau_2}\leqslant C\Big(\sum_{l=n_1}^{n_2-1}\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1,\bar s\notin G(l)} 2^{\langle\bar s,\bar 1\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar s}(f)\|_2^{\tau_2-\tau_1}\|\delta_{\bar s}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}.$$

Now, taking into account that

$$\|\delta_{\bar s}(f)\|_2\leqslant m_l^{-\frac{1}{\tau_1}}2^{-la}l^{(\nu-1)b}l^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde S_l$$

for $\overline{s}\notin G(l)$ and substituting the values of the numbers m_l, for $\tau_2-\tau_1\geqslant 0$, hence we have

$$\begin{aligned}
\|F_2\|_{p,\tau_2} &\leqslant C\Big(\sum_{l=n_1}^{n_2-1}\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1,\bar s\notin G(l)} 2^{\langle\bar s,\bar 1\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar s}(f)\|_2^{\tau_1}\Big(m_l^{-\frac{1}{\tau_1}}2^{-la}l^{(\nu-1)b}l^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde S_l\Big)^{\tau_2-\tau_1}\Big)^{1/\tau_2}\\
&= C\Big(\sum_{l=n_1}^{n_2-1}\Big(\Big(2^{-l\frac{\tau_2'}{p}}\tilde S_l^{\tau_1}2^{n\frac{\tau_2'}{2}}n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{1}{\tau_1}}2^{-la}l^{(\nu-1)b}\tilde S_l l^{\frac{1}{2}-\frac{1}{\tau_1}}\Big)^{\tau_2-\tau_1}\\
&\qquad\times\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1,\bar s\notin G(l)} 2^{\langle\bar s,\bar 1\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar s}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}\\
&= C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}\Big(\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{\tau_2'}{p\tau_1})(\tau_2-\tau_1)}l^{(\nu-1)b(\tau_2-\tau_1)}l^{(\frac{1}{2}-\frac{1}{\tau_1})(\tau_2-\tau_1)}\\
&\qquad\times\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1,\bar s\notin G(l)} 2^{\langle\bar s,\bar 1\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar s}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}.
\end{aligned}\tag{3.43}$$

Further, taking into account that $1 \leqslant \gamma_j,\ j=1,...,m$ and using inequality (3.31), it is easy to verify that

$$\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}$$
$$\leqslant 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2} l^{-(\frac{1}{2}-\frac{1}{\tau_1})\tau_1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} (\sum_{j=1}^{m}(s_j+1))^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}$$
$$\leqslant C2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2} l^{-(\frac{1}{2}-\frac{1}{\tau_1})\tau_1}\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1}^{\tau_1}$$
$$\leqslant C2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2} l^{-(\frac{1}{2}-\frac{1}{\tau_1})\tau_1}\Big(2^{-la}l^{(\nu-1)b}\Big)^{\tau_1} \quad (3.44)$$

for a function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$, $2<\tau_1\leqslant\tau_2<\infty$.

Now from inequalities (3.43) and (3.44), it follows that

$$\|F_2\|_{p,\tau_2} \leqslant C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}$$
$$\times\Big(\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{\tau_2'}{p\tau_1})(\tau_2-\tau_1)} l^{(\nu-1)b(\tau_2-\tau_1)} l^{(\frac{1}{2}-\frac{1}{\tau_1})(\tau_2-\tau_1)} 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2} l^{-(\frac{1}{2}-\frac{1}{\tau_1})\tau_1}\Big(2^{-la}l^{(\nu-1)b}\Big)^{\tau_1}\Big)^{1/\tau_2}$$
$$= C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}\Big(\sum_{l=n_1}^{n_2-1} 2^{-l\tau_2(a-\frac{\tau_2'}{p\tau_1\tau_2}(\tau_2-\tau_1)-(\frac{1}{2}-\frac{1}{p}))} l^{(\nu-1)b\tau_2} l^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_2}\Big)^{1/\tau_2}.$$

Since

$$a-\frac{\tau_2'}{p\tau_1\tau_2}(\tau_2-\tau_1)-(\frac{1}{2}-\frac{1}{p}) = a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}),$$

then taking into account the definition of the number n_2, from this formula, we get

$$\|F_2\|_{p,\tau_2} \leqslant C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}} 2^{-n_2(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n_2^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}$$
$$\leqslant C2^{-n\frac{p}{2}(a-\frac{1}{p}-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}, \quad (3.45)$$

for a function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ for $2<p<\infty$, $2<\tau_1\leqslant\tau_2<\infty$, $a<\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$.

Now from inequalities (3.35), (3.42) and (3.45), it follows that

$$\|f-(S_{Q_{n,\bar{\gamma}}}(f)+G_M(F_1)+G_M(F_3))\|_{p,\tau_2}$$
$$\leqslant \|F_1-G_M(F_1)\|_{p,\tau_2}+\|F_3-G_M(F_3)\|_{p,\tau_2}+\|F_2\|_{p,\tau_2}$$
$$+\Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f,\bar{x})\Big\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}+\Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f,\bar{x})\Big\|_{p,\tau_2}$$

for a function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ for $2<p<\infty$, $2<\tau_1\leqslant\tau_2<\infty$, $a<\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$.

Further, using inequality (3.24) for $q=2$ and taking into account that $\frac{1}{2}-\frac{1}{\tau_1}\geqslant 0$ from this formula, we obtain

$$e_M(f)_{p,\tau_2} \leqslant \|f-(S_{Q_{n,\bar{\gamma}}}(f)+G_M(F_1)+G_M(F_3))\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}},$$

for a function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ for $2<p<\infty$, $2<\tau_1\leqslant\tau_2<\infty$, $a<\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$, $b\in\mathbb{R}$.

Let $1 < \tau_1 \leqslant 2$. Then, by Lemma 1.5 [5], the following inequality holds

$$\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|_{2,\tau_1}^2\Big)^{1/2}\leqslant C\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1}. \tag{3.46}$$

Since $1 < \tau_1 \leqslant 2$, then (see [34, p. 217, Theorem 3.11])

$$\|\delta_{\bar{s}}(f)\|_2 \leqslant C\|\delta_{\bar{s}}(f)\|_{2,\tau_1}. \tag{3.47}$$

From inequalities (3.30), (3.47) and (3.46), it follows that

$$\|F_1\|_A \leqslant C\sum_{l=n}^{n_1-1}2^{l/2}\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1}.$$

Now taking into account that the function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ and the choice of the number n_1 from this formula, we get that

$$\|F_1\|_A \leqslant CM^{-\frac{p}{2}(a-\frac{1}{2})}(\log M)^{(\nu-1)(b+\frac{p}{\tau_2'}(a-\frac{1}{2})}, \tag{3.48}$$

for $a < 1/2$. Further, arguing as in the proof of inequality (3.35), we obtain

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \ll M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b}(\log M)^{\frac{l}{2}-\frac{1}{\tau_1}} \ll M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b}, \tag{3.49}$$

in the case $q = 2 < p < \infty$, $1 < \tau_1 \leqslant 2$, $1 < \tau_2 < \infty$, $a < \frac{1}{2}$.

In order to estimate $\|F_3\|_A$, we put

$$\tilde{S}_l = \Big(2^{la\tau_1}l^{-(\nu-1)b\tau_1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|_2^2\Big)^{1/2}$$

and

$$\tilde{m}_l := |G(l)| := \Big[2^{-l\frac{\tau_2'}{p}}\tilde{S}_l^2 2^{n\frac{\tau_2'}{2}}n^{(\nu-1)\frac{\tau_2'}{2}}\Big]+1.$$

In inequality (3.36), it was proved that

$$\|F_3\|_A \leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\bar{s}}(f)\|_2$$
$$\leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}2^{(l+1)/2}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}\|\delta_{\bar{s}}(f)\|_2. \tag{3.50}$$

By to the inner sum on the right side of inequality (3.50) applying Hölder's inequality and substituting the value of the number $\tilde{m}_l := |G(l)|$ from (3.50), we obtain

$$\|F_3\|_A \leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}2^{(l+1)/2}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}\|\delta_{\bar{s}}(f)\|_2^2\Big)^{1/2}|G(l)|^{1/2}$$
$$\ll 2^{-\frac{m-1}{2}}\Big\{\sum_{l=n_1}^{n_2-1}2^{l(\frac{1}{2}-a)}l^{(\nu-1)b}2^{-l\frac{\tau_2'}{2p}}\tilde{S}_l^2(2^n n^{(\nu-1)})^{\frac{\tau_2'}{4}}+\sum_{l=n_1}^{n_2-1}2^{l(\frac{1}{2}-a)}l^{(\nu-1)b}\tilde{S}_l\Big\}. \tag{3.51}$$

Now, using inequalities (3.46) and (3.47) and taking into account the value of the numbers $\tilde{S}_l$, we obtain

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2^{'}}{2p})} l^{(\nu-1)b} \tilde{S}_l^2 \leqslant \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2^{'}}{2p})} l^{(\nu-1)b} \Big(2^{la} l^{-(\nu-1)b} \Big\| \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f) \Big\|_{2,\tau_1} \Big). \tag{3.52}$$

Since the function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ and

$$a - \frac{1}{2} + \frac{\tau_2^{'}}{2p} = a - \tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2}) \leqslant a - \tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < 0,$$

then from inequality (3.52) we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2^{'}}{2p})} l^{(\nu-1)b} \tilde{S}_l^2 \leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2^{'}(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))} l^{(\nu-1)b}$$

$$\leqslant C 2^{-n_2(a-\tau_2^{'}(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))} n_2^{(\nu-1)b}. \tag{3.53}$$

Since the function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ and $a - \frac{1}{2} < 0$, then arguing similarly we can prove that

$$\sum_{l=n_1}^{n_2-1} 2^{l(\frac{1}{2}-a)} l^{(\nu-1)b} \tilde{S}_l \leqslant C 2^{n_2(\frac{1}{2}-a)} n_2^{(\nu-1)b}. \tag{3.54}$$

Now from inequalities (3.51), (3.53) and (3.54), it follows that

$$\|F_3\|_A \leqslant C \Big\{ (2^n n^{(\nu-1)})^{\frac{\tau_2^{'}}{4}} 2^{-n_2(a-\tau_2^{'}(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))} n_2^{(\nu-1)b} + 2^{n_2(\frac{1}{2}-a)} n_2^{(\nu-1)b} \Big\}$$

$$\leqslant C (2^n n^{(\nu-1)})^{-\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b}, \tag{3.55}$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ for $2 < p < \infty$, $1 < \tau_1 \leqslant 2$ and $1 < \tau_2 < \infty$, $a < \tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$, $b \in \mathbb{R}$.

Therefore, according to Lemma 2.1 for the function F_3, by a constructive method there is a M–term polynomial $G_M(F_3, \overline{x})$ such that

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \leqslant C M^{-\frac{l}{2}} \|F_3\|_A. \leqslant C M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})} (\log M)^{(\nu-1)b}, \tag{3.56}$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ for $2 < p < \infty$, $1 < \tau_1 \leqslant 2$, $1 < \tau_2 < \infty$, $a < \tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})$, $b \in \mathbb{R}$.

Let us estimate $\|F_2\|_{p,\tau_2}$. To do this, note that if $\bar{s} \notin G(l)$, then

$$\|\delta_{\bar{s}}(f)\|_2 \leqslant \tilde{m}_l^{-\frac{1}{2}} 2^{-la} l^{(\nu-1)b} \tilde{S}_l \tag{3.57}$$

and (see formula (3.8))

$$\|F_2\|_{p,\tau_2} \leqslant C \Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2} \|\delta_{\bar{s}}(f)\|_2^{\tau_2} \Big)^{1/\tau_2}$$

$$= C \Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2} \|\delta_{\bar{s}}(f)\|_2^{\tau_2-2} \|\delta_{\bar{s}}(f)\|_2^2 \Big)^{1/\tau_2}. \tag{3.58}$$

Further, if $\tau_2 - 2 \geqslant 0$, then using inequality (3.57) and repeating the reasoning in the proof (3.45), we obtain

$$\|F_2\|_{p,\tau_2} \leqslant C(2^n n^{\nu-1})^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})} n^{(\nu-1)b} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b}, \tag{3.59}$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ for $q = 2 < p < \infty$, $1 < \tau_1 \leqslant 2 \leqslant \tau_2 < \infty$, $a < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$, $b \in \mathbb{R}$.

Now from inequalities (3.49), (3.56), (3.59), it follows that

$$e_M(f)_{p,\tau_2} \leqslant \|f - (S_{Q_{n,\bar{\gamma}}}(f) + G^p_M(F_1) + G^p_M(F_3))\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}},$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ for $2 < p < \infty$, $1 < \tau_1 \leqslant 2 \leqslant \tau_2 < \infty$, $a < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$, $b \in \mathbb{R}$. □

Remark 1. In the case $\tau_1 = q$ and $\tau_2 = p$ Theorem 3.1 and Theorem 3.2 complement Theorem 3.2 [38].

Remark 2. Estimates for the quantity $e_M(W^{a,b,\bar{r}}_{q,\tau_1})_{p,\tau_2}$ for other values of the parameters q, p, τ_1, τ_2, a are announced in [6].

4 Conclusion

Now, using Theorem 3.1, we can obtain estimates for M–term approximations of a function in the Nikol'skii–Besov class.

Theorem 4.1. *Let* $1 < q < 2 < p < \infty$, $1 < \tau_1 \leqslant 2 \leqslant \tau_2 < \infty$ *and* $\frac{1}{q} - \frac{1}{p} < r_1 = ... = r_{\nu-1} < r_{\nu+1} \leqslant r_m$.

1. If $1 \leqslant \theta \leqslant \tau_1$ *and* $\frac{1}{q} - \frac{1}{p} < r_1 < \tau_2'(\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})$, *then*

$$e_M(\mathbb{S}^{\bar{r}}_{q,\tau_1,\theta}B)_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{q})},$$

where $C > 0$ *is independent of* M.

Proof. Let $f \in S^{\bar{r}}_{q,\tau_1,\theta}B$. Since $1 < \tau_1 \leq 2$ and $1 < q < \infty$, then

$$\|f_l\Big\|_{q,\tau_1} = \Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{q,\tau_1} \leqslant C\Bigg(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \|\delta_{\bar{s}}(f)\|^{\tau_1}_{q,\tau_1}\Bigg)^{1/\tau_1},$$

where $C > 0$ is independent of l and f. If $1 \leqslant \theta \leqslant \tau_1$, then according to Jensen's inequality [26, Lemma 3.3.3] from this formula, we obtain

$$\|f_l\Big\|_{q,\tau_1} \leqslant C\Bigg(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \|\delta_{\bar{s}}(f)\|^{\theta}_{q,\tau_1}\Bigg)^{1/\theta}$$

$$\leqslant C2^{-lr_1}\Bigg(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{r}\rangle\theta}\|\delta_{\bar{s}}(f)\|^{\theta}_{q,\tau_1}\Bigg)^{1/\theta} \leqslant C2^{-lr_1}\Bigg(\sum_{\bar{s}\in\mathbb{Z}_+} 2^{\langle\bar{s},\bar{r}\rangle\theta}\|\delta_{\bar{s}}(f)\|^{\theta}_{q,\tau_1}\Bigg)^{1/\theta}.$$

Hence $\mathbb{S}^{\bar{r}}_{q,\tau_1,\theta}B \subset W^{r_1,0,\bar{r}}_{q,\tau_1}$ in the case $1 \leqslant \theta \leqslant \tau_1 \leqslant 2$ and $1 < q < \infty$. Therefore, according to Theorem 3.1, for $a = r_1$ and $b = 0$, we have the estimate

$$e_M(\mathbb{S}^{\bar{r}}_{q,\tau_1,\theta}B)_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{q})},$$

in the case $\frac{1}{q} - \frac{1}{p} < r_1 < \tau_2'(\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})$, where $C > 0$ is independent of M.

Note that if $1 \leqslant \theta \leqslant \tau_1$, then $\tau_2'(\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2}) \leqslant \frac{1}{q} - \frac{\tau_2'}{p\theta'}$. □

Remark 3. If $1 < \tau_1 < \theta \leqslant \tau_2 < \infty$, then $\frac{1}{q} - \frac{\tau_2'}{p\theta'} < \tau_2'(\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})$. In this case, estimates of the quantity $e_M(\mathbb{S}^{\bar{r}}_{q,\tau_1,\theta}B)_{p,\tau_2}$ are given in [7].

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INTERPOLATION METHODS FOR ANISOTROPIC NET SPACES

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Abstract. In this paper, we study the interpolation properties of anisotropic net spaces $N_{\bar{p},\bar{q}}(M)$, where $\bar{p} = (p_1, ..., p_n)$, $\bar{q} = (q_1, ..., q_n)$. It is shown that, with respect to the multidimensional interpolation method, the following equality holds

$$(N_{\bar{p}_0,\bar{q}_0}(M), N_{\bar{p}_1,\bar{q}_1}(M))_{\bar{\theta},\bar{q}} = N_{\bar{p},\bar{q}}(M), \quad \frac{1}{\bar{p}} = \frac{1-\bar{\theta}}{\bar{p}_0} + \frac{\bar{\theta}}{\bar{p}_1}.$$

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1 Introduction

Let M be the set of all segments from $\mathbb{R}$. For a function $f(x)$, defined and integrable on each segment Q of M, we define the function

$$\bar{f}(t, M) = \sup_{\substack{Q \in M \\ |Q|>t}} \frac{1}{|Q|} \left| \int_Q f(x)dx \right|, \quad t > 0,$$

where the supremum is taken over all segments $Q \in M$, whose length is $|Q| > t$. The function $\bar{f}(t, M)$ is called the averaging of the function f over the net M.

We define the net spaces $N_{p,q}(M)$, $0 < p, q \le \infty$ as the set of all functions f, such that for $q < \infty$

$$\|f\|_{N_{p,q}(M)} = \left(\int_0^\infty \left(t^{\frac{1}{p}} \bar{f}(t, M) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,$$

and for $q = \infty$

$$\|f\|_{N_{p,\infty}(M)} = \sup_{t>0} t^{\frac{1}{p}} \bar{f}(t, M) < \infty.$$

These spaces were introduced in work [18]. Net spaces are an important research tool in the theory of Fourier series, in operator theory and in other areas [1]-[3], [19]-[23], [24], [28], [29].

It was shown in [17] that the scale of spaces $N_{p,q}(M)$ is closed under the real interpolation method, i.e. for $p_0 \ne p_1$ holds

$$(N_{p_0,q_0}(M), N_{p_1,q_1}(M))_{\theta,q} = N_{p,q}(M).$$

If in the definition of the space $N_{p,q}(M)$ instead of $\bar{f}(t, M)$ we consider the function

$$\sup_{\substack{Q \in M \\ |Q|>t}} \frac{1}{|Q|} \int_Q |f(x)|dx,$$

then the corresponding space, as can be seen from [9], coincides with the Morrey space $M^{\alpha}_{p,q}$, where $\alpha = \frac{1}{p} - \frac{1}{q}$, but for the scale of these spaces it is known that it is not closed under the real interpolation method (see [7], [25], [26]).

We consider the following generalization of the space $N_{p,q}(M)$ in the n-dimensional case.

Let $\tau \in \mathbb{Z}$, by G_τ we denote the set of all segments of the form $[0, 2^\tau\} + k^\tau$, $k \in \mathbb{Z}$. Let $G = \bigcup G_\tau$ be the set of all dyadic segments. Let M be a set of all parallelipeds of the form

$$Q = Q_1 \times \cdots \times Q_n$$

where $Q_i \in G$, $i = 1, \ldots, n$. We will call M *dyadic net.*

For the function $f(x) = f(x_1, ..., x_n)$ integrable on every set $Q \in M$ we define

$$\bar{f}(t; M) = \bar{f}(t_1, ..., t_n; M) = \sup_{|Q_i| \geq t_i} \frac{1}{|Q_n|} \left| \int_Q f(x_1, ..., x_n) dx_1 ... dx_n \right|, \quad t_i > 0,$$

where $|Q_i|$ is the length of the segment Q_i.

Let $0 < \bar{p} = (p_1, ..., p_n) < \infty$, $0 < \bar{q} = (q_1, ..., q_n) \leq \infty$. Denote by $N_{\bar{p},\bar{q}}(M)$ the set of all functions $f(x) = f(x_1, ..., x_n)$, for which

$$\|f\|_{N_{\bar{p},\bar{q}}(M)} = \left(\int_0^\infty \cdots \left(\int_0^\infty \left(t_1^{\frac{1}{p_1}} ... t_n^{\frac{1}{p_n}} \bar{f}(t_1, ..., t_n; M) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \cdots \frac{dt_n}{t_n} \right)^{\frac{1}{q_n}} < \infty,$$

here and below, when $q = \infty$, the expression $\left(\int_0^\infty (\varphi(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}$ is understood as $\sup_{t>0} \varphi(t)$.

As can be seen from the definition of the space $N_{\bar{p},\bar{q}}(M)$, this is the space of functions that have different characteristics for each variable. These spaces are called *anisotropic net spaces.*

For spaces with a mixed metric, anisotropic spaces, the real interpolation method does not work. For the interpolation of mixed metric spaces, the interpolation method was introduced by D.L. Fernandez [11]-[13] and modified in [14], [17], [20], [21]. An interpolation theorem regarding this method for Lebesgue spaces $L_{\bar{p}}$ with a mixed metric was obtained in [22]: *let* $0 < \bar{p}_i < \infty$ *and* $p_0^i \neq p_1^i$, $i = 0, 1$, $0 < \bar{q} \leq \infty$, $0 < \bar{\theta} < 1$, *then*

$$(L_{\bar{p}_0}, L_{\bar{p}_1})_{\bar{\theta},\bar{q}} = L_{\bar{p},\bar{q}}, \quad \frac{1}{\bar{p}} = \frac{1-\bar{\theta}}{\bar{p}_0} + \frac{\bar{\theta}}{\bar{p}_1},$$

where $L_{\bar{p},\bar{q}}$ *is the anisotropic Lorentz space.* (see [8])

Other applications of this method can be found in [6], [22].

The purpose of this paper is to obtain an interpolation theorem for anisotropic net spaces.

Given functions F and G, in this paper $F \asymp G$ means that $F \leq CG$ and $G \leq CF$, where C is a positive number, depending only on numerical parameters, that may be different on different occasions.

2 Lemmas

Let $\tau = (\tau_1, ..., \tau_n)$. The system of all sets $G_\tau = G_{\tau_1} \times \cdots \times G_{\tau_n} = \left\{ I_k = I^1_{k_1} \times \cdots \times I^n_{k_n} : \quad I^i_{k_i} \in G_{\tau_i} \right\}$ defines the partition of $\mathbb{R}^n$ into parallelepipeds, i.e. $\mathbb{R}^n = \bigcup\limits_{k \in \mathbb{Z}} I_k$.

Let $E = \{\epsilon = (\epsilon_1, ..., \epsilon_n) : \epsilon_i \in \{0, 1\}\}$ be the vertices of the unit cube in $\mathbb{R}^n$. For a locally integrable function $f(x_1, ..., x_n)$ and a set G_τ we define the functions $f_\epsilon(x)$, $\epsilon \in E$ as follows:

$$f_\epsilon(x) = \frac{1}{\prod\limits_{i=1}^n |I^i_{k_i}|} \int_{I^n_{k_n}} \cdots \int_{I^1_{k_1}} \Delta^\epsilon_x f(x_1', ..., x_n') dx_1' ... dx_n' \quad x \in I^1_{k_1} \times \cdots \times I^n_{k_n}, \tag{2.1}$$

where

$$\Delta_x^{\epsilon} f(x') = \Delta_{x_n}^{\epsilon_n} \dots \Delta_{x_1}^{\epsilon_1} f(x'),$$

$$\Delta_{x_i}^{\epsilon_i} \phi(x_i') = \begin{cases} \phi(x_i'), & \text{for} \quad \epsilon = 0, \\ \phi(x_i) - \phi(x_i'), & \text{for} \quad \epsilon = 1. \end{cases}$$

Note that $f(x) = \sum_{\epsilon \in E} f_\epsilon(x)$. These functions $\{f_\epsilon\}_{\epsilon\in E}$ will be called the expansion of the function $f(x)$, corresponding to the partition G_τ.

Lemma 2.1. *Let $\tau = (\tau_1, ..., \tau_n) \in \mathbb{Z}^n$, G_τ be the partition of $\mathbb{R}^n$ into rectangles, $f(x_1, ..., x_n)$ be locally integrable on $\mathbb{R}^n$. $f = \sum_{\epsilon\in E} f_\epsilon(x)$ be the decomposition corresponding to the partition G_τ. Then for $\epsilon_i = 1$*

$$\frac{1}{|I_k^i|} \int_{I_k^i} f_\epsilon(x_1, ..., x_n) dx_i = \begin{cases} 0, & \epsilon_i = 1 \\ f_\epsilon(x_1, ..., x_n), & \epsilon_i = 0 \end{cases} , \quad k \in \mathbb{Z}.$$

The proof follows from the definitions of the functions f_ϵ.

Lemma 2.2. *Let $\tau = (\tau_1, ..., \tau_n) \in \mathbb{Z}^n$, $\tau_i > 0$, G_τ be a partition of $\mathbb{R}^n$ into rectangles, $f(x_1, ..., x_n)$ be locally integrable on $\mathbb{R}^n$ and $\{f_\epsilon\}_{\epsilon\in E}$ be the decomposition of the function $f(x)$, corresponding to the partition G_τ. Then for an arbitrary $t \in \mathbb{Z}^n$*

$$\bar{f}_\epsilon(2^{t_1}, ..., 2^{t_n}; M)$$

$$\leq \begin{cases} 2^{|\epsilon|} \prod_{i=1}^{n} \min\{2^{\tau_i - t_i}, 1\} \bar{f}(2^{t_1\epsilon_1 + \tau_1(1-\epsilon_1)}, ..., 2^{t_n\epsilon_n + \tau_n(1-\epsilon_n)}; M), & \textit{for} \quad t_i\epsilon_i < \tau_i, \quad i = \overline{0, n}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.2}$$

where $|\epsilon| = \epsilon_1 + ... + \epsilon_n$.

Proof. Let $Q = Q_1 \times \cdots \times Q_n \in M$, $|Q_i| = 2^{s_i}$. Let us prove the following equality

$$\frac{1}{|Q|} \left| \int_Q f_\epsilon(x) dx \right| = \frac{1}{|Q_n|} \left| \int_{Q_n} \Delta_{x_n}^{\epsilon_n} \frac{1}{|Q_{n-1}|} \int_{Q_{n-1}} \Delta_{x_{n-1}}^{\epsilon_{n-1}} \cdots \frac{1}{|Q_1|} \int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n') dx_1' \dots dx_n' \right|. \tag{2.3}$$

Since for $s_i \geq \tau_i$ the segment Q_i splits into segments from G_τ, then if for some index i, $s_i \geq \tau_i$ and $\epsilon_i = 1$ are satisfied, then

$$\frac{1}{|Q|} \left| \int_Q f_\epsilon(x) dx \right| = 0.$$

Therefore, we assume that if $\epsilon_i = 1$, then $s_i < \tau_i$. Further, in the case when $\epsilon_i = 0$ and $s_i < \tau_i$ we have

$$\frac{1}{|Q_i|} \int_{Q_i} \Delta_{x_i}^{\epsilon_i} \frac{1}{|Q_{i-1}|} \int_{Q_{i-1}} \Delta_{x_{i-1}}^{\epsilon_{i-1}} \cdots \frac{1}{|Q_1|} \int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n') dx_1' \dots dx_i'$$

$$= \Delta_{x_i}^{\epsilon_i} \frac{1}{|Q_{i-1}|} \int_{Q_{i-1}} \Delta_{x_{i-1}}^{\epsilon_{i-1}} \cdots \frac{1}{|Q_1|} \int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n') dx_1' \dots dx_{i-1}'.$$

And in the case when $\epsilon = 0$ and $s_i \geq \tau_i$

$$\frac{1}{|Q_i|} \int_{Q_i} \Delta_{x_i}^{\epsilon_i} \frac{1}{|Q_{i-1}|} \int_{Q_{i-1}} \Delta_{x_{i-1}}^{\epsilon_{i-1}} \cdots \frac{1}{|Q_1|} \int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n') dx_1' \dots dx_i'$$

$$= \sum_{I_k^i \subset Q_i} \frac{1}{|Q_i|} \int_{I_k^i} \Delta_{x_i}^{\epsilon_i} \frac{1}{|Q_{i-1}|} \int_{Q_{i-1}} \Delta_{x_{i-1}}^{\epsilon_{i-1}} \dots \frac{1}{|Q_1|} \int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n') dx_1' \dots dx_i'.$$

By the above equalities, we have

$$\frac{1}{|Q|} \left| \int_Q f_\epsilon(x) dx \right| \le 2^{|\epsilon|} \prod_{i=1}^n \min\{2^{\tau_i - s_i}, 1\} \bar{f}(2^{s_1\epsilon_1 + \tau_1(1-\epsilon_1)}, ..., 2^{s_n\epsilon_n + \tau_n(1-\epsilon_n)}; M).$$

Taking into account that $s_i \ge t_i$ we get

$$\frac{1}{|Q|} \left| \int_Q f_\epsilon(x) dx \right| \le 2^{|\epsilon|} \prod_{i=1}^n \min\{2^{\tau_i - t_i}, 1\} \bar{f}(2^{t_1\epsilon_1 + \tau_1(1-\epsilon_1)}, ..., 2^{t_n\epsilon_n + \tau_n(1-\epsilon_n)}; M).$$

□

We will use the classical Hardy inequalities. Let us formulate them as a lemma.

Lemma 2.3 (Hardy's inequality)**.** *Let $1 \le q < \infty$, $\alpha > 0$, then the inequalities hold*

$$\left(\int_0^\infty \left(t^\alpha \int_t^\infty \varphi(s) ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le \alpha^{-1} \left(\int_0^\infty \left(t^{1+\alpha} \varphi(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

$$\left(\int_0^\infty \left(t^{-\alpha} \int_0^t \varphi(s) ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le \alpha^{-1} \left(\int_0^\infty \left(t^{1-\alpha} \varphi(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

3 Main result

Let us consider the interpolation method for anisotropic spaces proposed by Nursultanov E.D. [20]. This method is based on the ideas of G. Sparr [27] , D.L. Fernandez [11]-[13] and others [10], [15], [16]. Some results related to the interpolation of anisotropic net spaces were obtained in papers [4], [5].

Let $\mathbf{A_0} = (A_1^0, ..., A_n^0)$, $\mathbf{A_1} = (A_1^1, ..., A_n^1)$ be two anisotropic spaces, $E = \{\varepsilon = (\varepsilon_1, ..., \varepsilon_n) : \varepsilon_i = 0,$ or $\varepsilon_i = 1, \ \ i = 1, ..., n\}$. For arbitrary $\varepsilon \in E$ we define the space $\mathbf{A}_\varepsilon = (A_1^{\varepsilon_1}, ..., A_n^{\varepsilon_n})$ with the norm

$$\|f\|_{\mathbf{A}_\varepsilon} = \| \dots \|f\|_{A_1^{\varepsilon_1}} \dots \|_{A_n^{\varepsilon_n}}.$$

Let $0 < \bar{\theta} = (\theta_1, ..., \theta_n) < 1$, $0 < \bar{q} = (q_1, ..., q_n) \le \infty$. Via $\mathbf{A}_{\bar{\theta},\bar{q}} = (\mathbf{A_0}, \mathbf{A_1})_{\bar{\theta},\bar{q}}$ denote the linear subset $\sum_{\varepsilon \in E} \mathbf{A}_\varepsilon$, of all elements, for which

$$\|f\|_{\mathbf{A}_{\bar{\theta},\bar{q}}} = \left(\int_0^\infty \dots \left(\int_0^\infty \left(t_1^{-\theta_1} \dots t_n^{-\theta_n} K(t_1, ..., t_n; f) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n} \right)^{\frac{1}{q_n}} < \infty,$$

where

$$K(t, f; \mathbf{A_0}, \mathbf{A_1}) = \inf \left\{ \sum_{\varepsilon \in E} t^\varepsilon \|f_\varepsilon\|_{\mathbf{A}_\varepsilon} \ : \ f = \sum_{\varepsilon \in E} f_\varepsilon, \ f_\varepsilon \in \mathbf{A}_\varepsilon \right\},$$

where $t^\varepsilon = t_1^{\varepsilon_1} ... t_n^{\varepsilon_n}$.

Lemma 3.1. *Let $a_i > 1, i = 1, ..., n$ $0 < \bar{\theta} = (\theta_1, ..., \theta_n) < 1$, $0 < \bar{q} = (q_1, ..., q_n) \le \infty$. Then*

$$\|f\|_{\mathbf{A}_{\bar{\theta},\bar{q}}} \asymp \left(\sum_{k_n \in \mathbb{Z}} \dots \left(\sum_{k_1 \in \mathbb{Z}} \left(a_1^{-\theta_1 k_1} \dots a_n^{-\theta_n k_n} K(a_1^{k_1}, ..., a_n^{k_n}; f) \right)^{q_1} \right)^{\frac{q_2}{q_1}} \dots \right)^{\frac{1}{q_n}} = J_{\bar{\theta},\bar{q}}(f).$$

Proof. From the definition of the space $\mathbf{A}_{\bar{\theta},\bar{q}}$ we have

$$\|f\|_{\mathbf{A}_{\bar{\theta},\bar{q}}} = \left(\int_0^\infty \dots \left(\int_0^\infty \left(t_1^{-\theta_1}\dots t_n^{-\theta_n} K(t_1,...,t_n;f)\right)^{q_1} \frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n}\right)^{\frac{1}{q_n}}$$

$$= \left(\sum_{k_n\in\mathbb{Z}} \int_{a_n^{k_n}}^{a_n^{k_n+1}} \dots \left(\sum_{k_n\in\mathbb{Z}} \int_{a_1^{k_1}}^{a_1^{k_1+1}} \left(t_1^{-\theta_1}\dots t_n^{-\theta_n} K(t_1,...,t_n;f)\right)^{q_1} \frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n}\right)^{\frac{1}{q_n}}.$$

If the function $\Phi(t_i)$ is monotonically non-decreasing in the variable t_i then we get

$$\left(a_i^{-\theta_i(k_i+1)}\Phi(a_i^{-\theta_i k_i})\right)^{q_i} \ln a_i \le \int_{a_i^{k_1}}^{a_i^{k_1+1}} \left(t_i^{-\theta_i}\Phi(t_i)\right)^{q_i} \frac{dt_i}{t_i} \le \left(a_i^{-\theta_1 k_1}\Phi(a_i^{-\theta_1(k_1+1)})\right)^{q_i} \ln a_i.$$

Applying this relation and taking into account that $K(t_1,...,t_n;f)$ is non-decreasing in each variable, we obtain

$$C_1 J_{\bar{\theta},\bar{q}}(f) \le \|f\|_{\mathbf{A}_{\bar{\theta},\bar{q}}} \le C_2 J_{\bar{\theta},\bar{q}}(f),$$

where

$$C_1 = \prod_{i=1}^n a_i^{-\theta_i} (\ln a_i)^{\frac{1}{q_i}},$$

and

$$C_2 = \prod_{i=1}^n a_i^{\theta_i} (\ln a_i)^{\frac{1}{q_i}}.$$

□

Theorem 3.1. *Let M be the dyadic net in $\mathbb{R}^n$, $0 < \bar{p}_1 = (p_1^1,...,p_n^1) < \bar{p}_0 = (p_1^0,...,p_n^0) < \infty$, $0 < \bar{q}_0, \bar{q}, \bar{q}_1 \le \infty$, $0 < \bar{\theta} = (\theta_1,...,\theta_n) < 1$, then*

$$(N_{\bar{p}_0,\bar{q}_0}(M), N_{\bar{p}_1,\bar{q}_1}(M))_{\bar{\theta},\bar{q}} = N_{\bar{p},\bar{q}}(M), \tag{3.1}$$

where $\frac{1}{\bar{p}} = \frac{1-\bar{\theta}}{\bar{p}_0} + \frac{\bar{\theta}}{\bar{p}_1}$.

Proof. Let us prove the continuous embedding

$$N_{\bar{p},\bar{q}}(M) \hookrightarrow (N_{\bar{p}_0,\bar{v}}(M), N_{\bar{p}_1,\bar{v}}(M))_{\bar{\theta},\bar{q}}, \tag{3.2}$$

where $\bar{v} = (v,...,v)$, $v = \min_{1\le i\le n} q_i$.

Let $\tau = (\tau_1,...,\tau_n) \in \mathbb{Z}^n$, G_τ be a partition of $\mathbb{R}^n$, $f \in N_{\bar{p},\bar{q}}(M)$, $f = \sum_{\epsilon\in E} f_\epsilon(x)$ be the decomposition corresponding to the partition G_τ (f_ϵ is defined by the formula (2.1)).

Using Lemma 2.2, we get

$$\|f_\epsilon\|_{N_{\bar{p}_\epsilon,\bar{v}}} \asymp \left(\sum_{t_n\in Z} \dots \sum_{t_1\in Z} \left(2^{\frac{t_1}{p_1^{\epsilon_1}}} \dots 2^{\frac{t_n}{p_n^{\epsilon_n}}} \bar{f}_\epsilon(2^{t_1},...,2^{t_n};M)\right)^v\right)^{\frac{1}{v}}$$

$$\le 2^{|\epsilon|} \left(\sum_{\epsilon_i t_i < \tau_i} \left(\prod_{i=1}^n 2^{\frac{t_i}{p_i^{\epsilon_i}}} \min\{2^{\tau_i - t_i}, 1\} \bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)},...,2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)};M)\right)^v\right)^{\frac{1}{v}}.$$

Hence for $a_i > 1, i = \overline{1,n}$, we have

$$K(a_1^{\tau_1},..,a_n^{\tau_n},f;N_{\bar{p}_\epsilon,\bar{v}},\epsilon\in E)=\sum_{\epsilon\in E}a_1^{\epsilon_1\tau_1}\dots a_n^{\epsilon_n\tau_n}\|f_\epsilon\|_{N_{\bar{p}_\epsilon,\bar{v}}}$$

$$\le 2^n\sum_{\epsilon\in E}a_1^{\epsilon_1\tau_1}\dots a_n^{\epsilon_n\tau_n}\left(\sum_{\epsilon_i t_i<\tau_i}\left(\prod_{i=1}^n 2^{\frac{t_i}{p_i^{\epsilon_i}}}\min\{2^{\tau_i-t_i},1\}\bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)},...,2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)};M)\right)^v\right)^{\frac{1}{v}},$$

and

$$\|f\|_{(N_{\bar{p}_0,\bar{v}}(M),N_{\bar{p}_1,\bar{v}}(M))_{\bar{\theta},\bar{q}}}\asymp\left(\sum_{\tau_n\in Z}\dots\left(\sum_{\tau_1\in Z}\left(a_1^{-\theta_1\tau_1}\dots a_n^{-\theta_n\tau_n}K(a_1^{\tau_1},...,a_n^{\tau_n},f)\right)^{q_1}\right)^{\frac{q_2}{q_1}}\dots\right)^{\frac{1}{q_n}}$$

$$\le C\sum_{\epsilon\in E}\left(\sum_{\tau_n\in Z}\dots\left(\sum_{\tau_1\in Z}\left(a_1^{(\epsilon_1-\theta_1)\tau_1}\dots a_n^{(\epsilon_n-\theta_n)\tau_n}\times\right.\right.\right.$$

$$\left.\left.\left.\times\left(\sum_{\epsilon_i t_i<\tau_i}\left(\prod_{i=1}^n 2^{\frac{t_i}{p_i^{\epsilon_i}}}\min\{2^{\tau_i-t_i},1\}\bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)},...,2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)};M)\right)^v\right)^{\frac{1}{v}}\right)^{q_1}\right)^{\frac{q_2}{q_1}}\dots\right)^{\frac{1}{q_n}}, \quad (3.3)$$

where $C=2^n2^{\sum_{i=1}^n(1-\frac{1}{q_1})_+}$.

Let $\epsilon\in E$, using the definition of v and the generalized Minkowski inequality, we obtain

$$\left(\sum_{\tau_n\in Z}\dots\left(\sum_{\tau_1\in Z}\left(a_1^{(\epsilon_1-\theta_1)\tau_1}\dots a_n^{(\epsilon_n-\theta_n)\tau_n}\times\right.\right.\right.$$

$$\left.\left.\left.\times\left(\sum_{\epsilon_i t_i<\tau_i}\left(\prod_{i=1}^n 2^{\frac{t_i}{p_i^{\epsilon_i}}}\min\{2^{\tau_i-t_i},1\}\bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)},...,2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)};M)\right)^v\right)^{\frac{1}{v}}\right)^{q_1}\right)^{\frac{q_2}{q_1}}\dots\right)^{\frac{1}{q_n}},$$

$$\le\left(\sum_{\tau_n\in Z}\left(a_n^{(\epsilon_n-\theta_n)\tau_n}\left(\sum_{\epsilon_n t_n<\tau_n}\left(2^{\frac{t_n}{p_n^{\epsilon_n}}}\min\{2^{\tau_n-t_n},1\}F_{n-1}(2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)})\right)^v\right)^{\frac{1}{v}}\right)^{q_n}\right)^{\frac{1}{q_n}},$$

where

$$F_{n-1}(y)=\left(\sum_{\tau_{n-1}\in Z}\dots\left(\sum_{\tau_1\in Z}\left(a_1^{(\epsilon_1-\theta_1)\tau_1}\dots a_n^{(\epsilon_n-\theta_n)\tau_n}\times\right.\right.\right.$$

$$\left.\left.\left.\times\left(\sum_{\epsilon_i t_i<\tau_i}\left(\prod_{i=1}^{n-1} 2^{\frac{t_i}{p_i^{\epsilon_i}}}\min\{2^{\tau_i-t_i},1\}\bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)},...,y;M)\right)^v\right)^{\frac{1}{v}}\right)^{q_1}\right)^{\frac{q_2}{q_1}}\dots\right)^{\frac{1}{q_n}}.$$

Let $a_n=2^{\frac{1}{p_n^0}-\frac{1}{p_n^1}}$. If $\epsilon_n=0$, then we have

$$\left(\sum_{\tau_n\in Z}\left(a_n^{(\epsilon_n-\theta_n)\tau_n}\left(\sum_{\epsilon_n t_n<\tau_n}\left(2^{\frac{t_n}{p_n^{\epsilon_n}}}\min\{2^{\tau_n-t_n},1\}F_{n-1}(2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)})\right)^v\right)^{\frac{1}{v}}\right)^{q_n}\right)^{\frac{1}{q_n}}$$

$$= \left(\sum_{\tau_n \in Z} \left(2^{-\theta_n \tau_n (\frac{1}{p_n^0} - \frac{1}{p_n^1})} \left(\sum_{t_n \in Z} \left(2^{\frac{t_n}{p_n^0}} \min\{2^{\tau_n - t_n}, 1\} F_{n-1}(2^{\tau_n}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$= \left(\sum_{\tau_n \in Z} \left(2^{-\theta_n \tau_n (\frac{1}{p_n^0} - \frac{1}{p_n^1})} F_{n-1}(2^{\tau_n}) \left(\sum_{t_n=-\infty}^{\tau_n} \left(2^{\frac{t_n}{p_n^0}} \right)^v + \sum_{t_n=\tau_n+1}^{\infty} \left(2^{\frac{t_n}{p_n^0}} 2^{\tau_n - t_n} \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$\asymp \left(\sum_{\tau_n \in Z} \left(2^{-\theta_n \tau_n (\frac{1}{p_n^0} - \frac{1}{p_n^1})} F_{n-1}(2^{\tau_n}) 2^{\frac{\tau_n}{p_n^0}} \right)^{q_n} \right)^{\frac{1}{q_n}} = \left(\sum_{\tau_n \in Z} \left(2^{\frac{\tau_n}{p_n}} F_{n-1}(2^{\tau_n}) \right)^{q_n} \right)^{\frac{1}{q_n}}.$$

In the last relation, we used the equality $\frac{1}{p_n} = \frac{1-\theta_n}{p_n^0} + \frac{\theta}{p_n^1}$.

If $\epsilon_n = 1$, then we get

$$\left(\sum_{\tau_n \in Z} \left(a_n^{(\epsilon_n - \theta_n)\tau_n} \left(\sum_{\epsilon_n t_n < \tau_n} \left(2^{\frac{t_n}{p_n^{\epsilon_n}}} \min\{2^{\tau_n - t_n}, 1\} F_{n-1}(2^{t_n \epsilon_n + \tau_n(1-\epsilon_n)}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$= \left(\sum_{\tau_n \in Z} \left(2^{(1-\theta)_n \tau_n (\frac{1}{p_n^0} - \frac{1}{p_n^1})} \left(\sum_{t_n=-\infty}^{\tau_n - 1} \left(2^{\frac{t_n}{p_n^1}} F_{n-1}(2^{t_n}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$\le C \left(\sum_{\tau_n \in Z} \left(2^{\frac{\tau_n}{p_n}} F_{n-1}(2^{\tau_n}) \right)^{q_n} \right)^{\frac{1}{q_n}}.$$

where C>0 is independent of f. Here we have used Hardy's inequality and the equality $\frac{1}{p_n} = \frac{1-\theta_n}{p_n^0} + \frac{\theta}{p_n^1}$.

Further, applying to $F_{n-1}(2^{\tau_n})$ the same procedure as above, after $n-1$ steps we obtain the estimate of the form

$$\left(\sum_{\tau_n \in Z} \left(a_n^{(\epsilon_n - \theta_n)\tau_n} \left(\sum_{\epsilon_n t_n < \tau_n} \left(\min\{2^{\tau_n - t_n(1 - \frac{1}{p_n^{\epsilon_n}})}, 2^{\frac{t_n}{p_n^{\epsilon_n}}}\} F_{n-1}(2^{t_n \epsilon_n + \tau_n(1-\epsilon_n)}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$\le C \left(\sum_{\tau_n \in Z} \cdots \left(\sum_{\tau_1 \in Z} \left(2^{\frac{\tau_n}{p_n}} \cdots 2^{\frac{\tau_1}{p_1}} \bar{f}_{n-1}(2^{\tau_1}, ..., 2^{\tau_n}; M) \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_n}} \asymp \|f\|_{N_{\bar{p},\bar{q}}(M)}.$$

where C>0 is independent of f.

Substituting the resulting relation into (3.3) we get (3.2). Thus, taking into account that $v = \min_{1 \le i \le n} q_i$, we get the continuous embedding

$$N_{\bar{p},\bar{q}}(M) \hookrightarrow (N_{\bar{p}_0,\bar{v}}(M), N_{\bar{p}_1,\bar{v}}(M))_{\bar{\theta},\bar{q}} \hookrightarrow (N_{\bar{p}_0,\bar{q}_0}(M), N_{\bar{p}_1,\bar{q}_1}(M))_{\bar{\theta},\bar{q}}.$$

The reverse continuous embedding $(N_{\bar{p}_0,\bar{q}_0}(M), N_{\bar{p}_1,\bar{q}_1}(M))_{\bar{\theta},\bar{q}} \hookrightarrow N_{\bar{p},\bar{q}}(M)$ was proved in [20] (see Theorem 1). □

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ESTIMATE OF THE BEST CONSTANT OF DISCRETE HARDY-TYPE INEQUALITY WITH MATRIX OPERATOR SATISFYING THE OINAROV CONDITION

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Key words: Hardy-type inequality, weight sequence, space of sequences, matrix operator, Oinarov condition.

AMS Mathematics Subject Classification: 26D15.

Abstract. This paper studies the weighted inequality of Hardy-type in discrete form for matrix operators satisfying the Oinarov condition. Necessary and sufficient conditions on the weight sequences under which the Hardy-type inequality holds were found in [13] for the case $1 < p \leq q < \infty$, in [14] for the case $1 < q < p < \infty$, and in [15] for the case $0 < p \leq q < \infty$, $0 < p \leq 1$. In this paper, we extend the result of [13] with a two-sided estimate of the inequality constant.

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1 Introduction

For arbitrary non-negative sequences $f = \{f_i\}_{i=1}^{\infty}$ the modern form of the discrete Hardy-type inequality can be written as follows:

$$\left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j\right)^q\right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} v_i^p f_i^p\right)^{\frac{1}{p}}, \tag{1.1}$$

where $u = \{u_i\}_{i=1}^{\infty}$ and $v = \{v_i\}_{i=1}^{\infty}$ are weight sequences of positive real numbers, and

$$(Af)_i = \sum_{j=1}^{i} a_{i,j} f_j \tag{1.2}$$

is a matrix operator with the kernel $a := \{a_{i,j}\}_{i,j=1}^{\infty}$, $i \geq j$, such that $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and $C > 0$ depends only on p, q, u, v, and a.

In the case $a_{i,j} \equiv 1$, the problem of finding necessary and sufficient conditions on the weight sequences $u = \{u_i\}_{i=1}^{\infty}$ and $v = \{v_i\}_{i=1}^{\infty}$ such that inequality (1.1) holds for any non-negative sequences $f = \{f_i\}_{i=1}^{\infty}$ has been solved for all possible relations between the parameters $0 < p < \infty$ and $0 < q < \infty$ (see [1, 2, 3, 4, 6, 8]).

Suppose that $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and there exists a number $d > 1$ such that

$$\frac{1}{d}(a_{i,k} + a_{k,j}) \leq a_{i,j} \leq d(a_{i,k} + a_{k,j}), \quad \forall\ i \geq k \geq j \geq 1. \tag{1.3}$$

This condition is an analogue of the Oinarov condition for kernels of integral operators introduced in [5] and [12]. Characterizations of the validity of inequality (1.1) for the operators satisfying discrete Oinarov condition (1.3) were found in [13] for the case $1 < p \leq q < \infty$, in [14] for the case $1 < q < p < \infty$ and in [15] for the case $0 < p \leq q < \infty$, $0 < p \leq 1$.

In [12], the integral weighted Hardy-type inequality for the operator satisfying the Oinarov condition was characterized in the case $1 < p \leq q < \infty$. In 2021, in paper [9] this result was extended with a two-sided estimate of the inequality constant. Since estimates of the best constants of Hardy-type inequalities have important applications in the oscillation theory of differential inequalities, paper [9] has got many citations over the past two years. In this paper, motivated by the development in the continuous case, we aim to find a two-sided estimate of the best constant $C > 0$ in inequality (1.1) also in the case $1 < p \leq q < \infty$. The obtained result will be used to establish the oscillatory properties of difference equations.

Let $l_{p,v}$ denote the space of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers whose norm $\|f\|_{p,v} \equiv \|vf\|_p = \left(\sum\limits_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}$ is finite. Then inequality (1.1) can be rewritten in the form: $\|Af\|_{q,u} \leq C\|f\|_{p,v}$. The validity of this inequality is equivalent to the boundedness of matrix operator (1.2) from $l_{p,v}$ into $l_{q,u}$, while for the best constant $C > 0$ we have that $C = \|A\|_{p,v\to q,u}$, where $\|A\|_{p,v\to q,u}$ denotes the norm of operator (1.2) from $l_{p,v}$ to $l_{q,u}$.

Let $p' = \frac{p}{p-1}$. To prove the main result we need the following theorem proved in [4].

Theorem A. *Let $1 < p \leq q < \infty$. Then for any non-negative $f \in l_{p,v}$ the inequality*

$$\left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} f_j\right)^q\right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} v_i^p f_i^p\right)^{\frac{1}{p}}, \tag{1.4}$$

holds if and only if

$$A = \sup_{k\geq 1} \left(\sum_{n=k}^{\infty} u_n^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} v_j^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $A \leq C \leq \widetilde{C}A$, where $\widetilde{C} = \left(1 + \frac{q}{p'}\right)^{\frac{1}{q}} \left(1 + \frac{p'}{q}\right)^{\frac{1}{p'}}$ and C is the best constant in (1.4).

Remark 1. In the case $p = q = 2$, we have that $\widetilde{C} = \left(1 + \frac{2}{2}\right)^{\frac{1}{2}} \left(1 + \frac{2}{2}\right)^{\frac{1}{2}} = 2$.

Note that the Hardy inequality has a long history (see [10]), and its various generalizations and applications have grown into a separate field called the "theory of Hardy-type inequalities", with many papers published every year (see, e.g., most recent publications [7], [11] and [16]).

2 Main result

Theorem 2.1. *Let $1 < p \leq q < \infty$ and a matrix $(a_{i,j})$ satisfy condition* (1.3). *Then for any non-negative $f \in l_{p,v}$ inequality* (1.1) *holds if and only if $B = \max\{B_1, B_2\} < \infty$, where*

$$B_1 = \sup_{k\geq 1} \left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} v_j^{-p'}\right)^{\frac{1}{p'}},$$

$$B_2 = \sup_{k\geq 1} \left(\sum_{n=k}^{\infty} u_n^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} a_{k,j}^{p'} v_j^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover, $B \leq C \leq \overline{C}B$, where $\overline{C} = \left(2(d+1)^q + (d+1)^{2q}(1 + d\,\widetilde{C}^q)\right)^{\frac{1}{q}}$ and C is the best constant in (1.1).

Proof. Necessity. Let inequality (1.1) hold. To estimate C from below, we follow the same steps as in paper [13]. Putting the test sequence $g = \{g_j\}_{j=1}^{\infty}$ such that $g_j = \begin{cases} v_j^{-p'}, & 1 \le j \le k, \\ 0, & j > k, \end{cases}$ for $k \ge 1$, into the right-hand side and then into the left-hand side of inequality (1.1), we get $B_1 \le C$. Putting one more test sequence $h = \{h_j\}_{j=1}^{\infty}$ such that $h_j = \begin{cases} a_{k,j}^{p'-1} v_j^{-p'}, & 1 \le j \le k, \\ 0, & j > k, \end{cases}$ for $k \ge 1$ into the both sides of inequality (1.1), we have $B_2 \le C$. Combining the obtained estimates, we find that

$$B \le C. \tag{2.1}$$

Sufficiency. Let $B < \infty$. For any $i \ge 1$ the set of positive numbers S_i is defined as follows: $S_i = \{k \in \mathbb{Z} : (d+1)^k \le (Af)_i\}$, where d is the number from (1.3). If $k(i) = \max S_i$, then

$$(d+1)^{k(i)} \le (Af)_i \le (d+1)^{k(i)+1}. \tag{2.2}$$

Let $m_1 = 1$ and $M_1 = \{i \in \mathbb{N} : k(i) = k(1) = k(m_1)\}$. We define m_2 as $m_2 = \sup M_1 + 1$. It is obvious that $m_2 > m_1$. Moreover, if the set M_1 is bounded from above, then $m_2 < \infty$ and $m_2 - 1 = \max M_1 = \sup M_1$. Suppose that for $s \ge 1$ the numbers $1 = m_1 < m_2 < ... < m_s < \infty$ are defined. We define the next number m_{s+1} as $m_{s+1} = \sup M_s + 1$, where $M_s = \{i \in \mathbb{N} : k(i) = k(m_s)\}$.

Let $N = \{s \in \mathbb{N} : m_s < \infty\}$. For $s \in N$ the definition of m_s and (2.2) give that

$$(d+1)^{k(m_s)} \le (Af)_i \le (d+1)^{k(m_s)+1}, \quad m_s \le i \le m_{s+1} - 1, \tag{2.3}$$

and $\mathbb{N} = \bigcup_{s \in N} [m_s, m_{s+1})$. Hence,

$$\|Af\|_{q,u}^q = \sum_{s \in N} \sum_{j=m_s}^{m_{s+1}-1} u_j^q (Af)_j^q.$$

We assume that $\sum\limits_{j=m_s}^{m_{s+1}-1} u_j^q (Af)_j^q = 0$ if $m_s = \infty$. Then $\|Af\|_{q,u}^q$ can be presented as follows:

$$\|Af\|_{q,u}^q = \sum_{j=m_1}^{m_2-1} u_j^q (Af)_j^q + \sum_{j=m_2}^{m_3-1} u_j^q (Af)_j^q + \sum_{s \ge 3} \sum_{j=m_s}^{m_{s+1}-1} u_j^q (Af)_j^q. \tag{2.4}$$

Since $m_1 = 1 < \infty$, it belongs to N. Thus, from (2.3) we have

$$\sum_{j=m_1}^{m_2-1} u_j^q (Af)_j^q \le \sum_{j=1}^{m_2-1} u_j^q (d+1)^{(k(m_1)+1)q} \le (d+1)^q (d+1)^{k(m_1)q} \sum_{j=1}^{\infty} u_j^q$$

$$\le (d+1)^q (Af)_1^q \sum_{j=1}^{\infty} u_j^q \le (d+1)^q \left(\sum_{s=1}^{1} a_{1,s}^{p'} v_s^{-p'} \right)^{\frac{q}{p'}} \sum_{j=1}^{\infty} u_j^q \, \|f\|_{p,v}^q \le (d+1)^q B_2^q \, \|f\|_{p,v}^q. \tag{2.5}$$

If $m_2 = \infty$, then $m_s = \infty$ for all $s \ge 2$. Therefore, arguing as above, we get

$$\|Af\|_{q,u}^q \le (d+1)^q B_2^q \|f\|_{p,v}^q.$$

If $m_2 < \infty$, then $s = 2$ belongs to N. Thus, from (2.3) we have

$$\sum_{j=m_2}^{m_3-1} u_j^q (Af)_j^q \le (d+1)^q (d+1)^{k(m_2)q} \sum_{j=m_2}^{\infty} u_j^q \le (d+1)^q (Af)_{m_2}^q \sum_{j=m_2}^{\infty} u_j^q$$

$$= (d+1)^q \left(\sum_{i=1}^{m_2} a_{m_2,i} f_i\right)^q \sum_{j=m_2}^{\infty} u_j^q \le (d+1)^q \left(\sum_{i=1}^{m_2} a_{m_2,i}^{p'} v_i^{-p'}\right)^{\frac{q}{p'}} \sum_{j=m_2}^{\infty} u_j^q \left(\sum_{i=1}^{m_2} v_i^p f_i^p\right)^{\frac{q}{p}}$$

$$\le (d+1)^q \left(\left(\sum_{i=1}^{m_2} a_{m_2,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{j=m_2}^{\infty} u_j^q\right)^{\frac{1}{q}}\right)^q \|f\|_{p,v}^q \le (d+1)^q B_2^q \|f\|_{p,v}^q. \tag{2.6}$$

If $m_3 = \infty$, then from (2.4), (2.5) and (2.6) we get

$$\|Af\|_{q,u}^q \le 2(d+1)^q B_2^q \|f\|_{p,v}^q.$$

Let us consider $s \ge 3$ such that s belongs to N. Since $k(m_{s-2}) < k(m_{s-1}) < k(m_s)$, we have that $k(m_{s-2}) + 1 \le k(m_s) - 1$. Therefore, using (2.3) and (1.3), we obtain

$$(d+1)^{k(m_s)-1} = (d+1)^{k(m_s)} - d(d+1)^{k(m_s)-1} \le (d+1)^{k(m_s)} - d(d+1)^{k(m_{s-2})+1}$$

$$< (Af)_{m_s} - d(Af)_{m_{s-1}-1} = \sum_{i=1}^{m_s} a_{m_s,i} f_i - d \sum_{i=1}^{m_{s-1}-1} a_{m_{s-1}-1,i} f_i$$

$$= \sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + \sum_{i=1}^{m_{s-1}-1} \left[a_{m_s,i} - d a_{m_{s-1}-1,i}\right] f_i$$

$$\le \sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + \sum_{i=1}^{m_{s-1}-1} \left[d(a_{m_s,m_{s-1}-1} + a_{m_{s-1}-1,i}) - d a_{m_{s-1}-1,i}\right] f_i$$

$$= \sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + d \sum_{i=1}^{m_{s-1}-1} a_{m_s,m_{s-1}-1} f_i.$$

The latter, together with (2.3), for $s \ge 3$ gives that

$$\sum_{s\ge3} \sum_{j=m_s}^{m_{s+1}-1} u_j^q (Af)_j^q < \sum_{s\ge3} \sum_{j=m_s}^{m_{s+1}-1} u_j^q (d+1)^{(k(m_s)+1)q} = (d+1)^{2q} \sum_{s\ge3} (d+1)^{(k(m_s)-1)q} \sum_{j=m_s}^{m_{s+1}-1} u_j^q$$

$$\le (d+1)^{2q} \sum_{s\ge3} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + d \sum_{i=1}^{m_{s-1}-1} a_{m_s,m_{s-1}-1} f_i\right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q$$

$$\le (d+1)^{2q} \left[\sum_{s\ge3} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i\right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q\right.$$

$$\left. + d \sum_{s\ge3} \left(\sum_{i=1}^{m_{s-1}-1} a_{m_s,m_{s-1}-1} f_i\right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q\right] = (d+1)^{2q} (I_1 + d\, I_2). \tag{2.7}$$

We estimate I_1 and I_2 separately. Using the Hölder and Jensen inequalities, we obtain

$$I_1 = \sum_{s\ge3} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i\right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q \le \sum_{s\ge3} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s,i}^{p'} v_i^{-p'}\right)^{\frac{q}{p'}} \left(\sum_{i=m_{s-1}}^{m_s} v_i^p f_i^p\right)^{\frac{q}{p}}$$

$$\times \sum_{j=m_s}^{m_{s+1}-1} u_j^q \le \left(\sup_{k\ge1}\left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{j=k}^{\infty} u_j^q\right)^{\frac{1}{q}}\right)^q \sum_{s\ge3}\left(\sum_{i=m_{s-1}}^{m_s} v_i^p f_i^p\right)^{\frac{q}{p}}$$

$$\le B_2^q\left(\sum_{s\ge3}\sum_{i=m_{s-1}}^{m_s} v_i^p f_i^p\right)^{\frac{q}{p}} \le B_2^q\|f\|_{p,v}^q. \tag{2.8}$$

Let us turn to the estimate of I_2. By Theorem A, we have

$$I_2 = \sum_{s\ge3} a^q_{m_s,m_{s-1}-1}\sum_{j=m_s}^{m_{s+1}-1} u_j^q\left(\sum_{i=1}^{m_{s-1}-1} f_i\right)^q$$

$$\le \widetilde{C}^q\left(\sup_{k\ge1}\left(\sum_{m_{s-1}-1\ge k} a^q_{m_s,m_{s-1}-1}\sum_{j=m_s}^{m_{s+1}-1} u_j^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{k} v_j^{-p'}\right)^{\frac{1}{p'}}\right)^q \|f\|_{p,v}^q. \tag{2.9}$$

Since $a_{i,j}$ is non-decreasing in i and non-increasing in j, we deduce that

$$\sum_{m_{s-1}-1\ge k} a^q_{m_s,m_{s-1}-1}\sum_{j=m_s}^{m_{s+1}-1} u_j^q \le \sum_{m_{s-1}-1\ge k}\sum_{j=m_s}^{m_{s+1}-1} a_{j,k}^q u_j^q \le \sum_{j=k}^{\infty} a_{j,k}^q u_j^q.$$

Using the latter, from (2.9) we find

$$I_2 \le \widetilde{C}^q B_1^q\|f\|_{p,v}^q. \tag{2.10}$$

Combining (2.4), (2.5), (2.6), (2.7), (2.8), and (2.10), we get

$$\|Af\|_{q,u}^q \le (d+1)^q B_2^q\,\|f\|_{p,v}^q + (d+1)^q B_2^q\,\|f\|_{p,v}^q + (d+1)^{2q}(B_2^q\,\|f\|_{p,v}^q + d\,\widetilde{C}^q B_1^q\|f\|_{p,v}^q)$$

$$\le \left(2(d+1)^q + (d+1)^{2q}(1+d\,\widetilde{C}^q)\right) B^q\,\|f\|_{p,v}^q. \tag{2.11}$$

Therefore, from (2.11) we obtain

$$C \le \left(2(d+1)^q + (d+1)^{2q}(1+d\,\widetilde{C}^q)\right)^{\frac{1}{q}} B,$$

which, together with (2.1), gives that $B \le C \le \overline{C}B$. □

Remark 2. Taking into account Remark 1, in the case $p = q = 2$ and $d = 1$, we have that $\overline{C} = (2(1+1)^2 + (1+1)^4(1+1\cdot 2^2))^{\frac{1}{2}} = 88^{\frac{1}{2}} \approx 9.38$.

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DYNAMICS OF RELAY SYSTEMS WITH HYSTERESIS AND HARMONIC PERTURBATION

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Key words: multidimensional system of ordinary differential equations; relay hysteresis; harmonic perturbation; decomposition; parametric matrix; subsystems; Jordan block; asymptotically stable periodic solution.

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Abstract. We consider a system of ordinary differential equations with a relay hysteresis and a harmonic perturbation. We propose an approach that allows one to decompose an n-dimensional system into one- and two-dimensional subsystems. The approach is illustrated by a numerical example for the system of dimension 3. As a result of the decomposition, a two-dimensional subsystem with non-trivial Jordan block in right-hand side is studied. For this subsystem we prove a theorem on the existence and uniqueness of an asymptotically stable solution with a period being multiple to period of the perturbation. Moreover, we show how to obtain this solution by tuning the parameters defining the relay. We also provide a supporting example in this regard.

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1 Introduction

It is well known in oscillation theory [1] that many results obtained for linear systems and nonlinear systems with continuous right-hand sides are impossible to use for systems with the nonlinearities being nonlinearized. Such nonlinearities are called "essential" ones (see [23]). In practice, by essentially nonlinear systems including relay systems with hysteresis, one describes numerous automatic control devices [32], in particular, devices with non-ideal relays, which are installed, for instance, on water crafts [7], [30]. Moreover, the mathematical models of these devices are multidimensional. Influence of hysteresis becomes actually important when the devices are utilized in high-precision engineering systems [33]. The readers are referred to monographs [4], [6], [24], [28], and [34] for general information about systems with hysteresis and their applications.

It is clear that, for specific parameters, one can first study these mathematical models by numerical methods, using a powerful computing technique and then perform simulations to support numerical results. At the same time, analytical results provide a basis for examining the multidimensional and essentially nonlinear automatic control systems. Even if the results are obtained for bounded domains in the system parameter space, they might be considered as a scientific background in numerical experiments. It is beyond any doubt that both theoretical and numerical investigations of the models with hysteresis provide more adequate results for applications.

Models with hysteresis were already surveyed in a number of works (see, e.g., [2], [5], [8], [17], [26], [27], [31], and [35]–[37]). From the latest papers, we draw attention to [13], [19], [20], [29], [40], and [41]. Notice that classical methods including methods of fitting, fixed points, and point mappings are still actively employed to investigate piecewise integrable systems, in particular, relay systems [9], [12], [21], [38], and [39].

Definitely, the use of analytical methods in research of n-dimensional nonlinear systems ($n > 2$) yields challenges. As is known, classical methods are usually based on a local approach to the research of system phase space. But even in this case, some numerical calculations are to be carried out. On the other hand, there is an opportunity to study the phase portraits of lower-dimensional systems for $n \leq 2$ in large, non-locally (see, e.g., [18]).

We consider a multidimensional system of ordinary differential equations with a non-ideal relay and a harmonic pertubation. The present study develops the results of the authors obtained, in particular, in [17], [18], [20], [21], and [37]–[39].

The aim of the paper is to propose a new approach for the research of the system under consideration by analytical methods. The approach involves a decomposition of the system into subsystems and the study of these subsystems. The decomposition allows to overcome difficulties related to higher dimensions and investigate multidimensional systems analytically, precisely and so fully as we can do it for systems of lower dimensions using phase planes [18].

The main idea of the approach is the following. For a system of dimension n, we select some domains in its parameter space. Then, in these domains, we reduce the system to subsystems of dimensions 1 and 2, using a nonsingular linear transformation. These subsystems are connected in such a way that one can integrate them consistently, one behind another, if consider some of them as inhomogeneous ones (see [22]), and therefore study them by well-known methods. As a result of the reduction, the system parameter space is decomposed into the direct sum of subspaces, and a number of dynamic behaviour types in these subspaces is put into a dynamic behaviour in the space. Thus, by investigating the subsystems, it is possible to obtain a certain knowledge not only about the dynamics of the system but about the structure of its parameter space, since the points in this space are associated with the various topological phase portraits.

Different approaches with the reduction of matrices to diagonal or Jordan forms are proposed in a number of publications, in particular in [14] for a delayed differential system. For systems with constant matrices in the linear part and relay hysteresis, nonsingular transformations that reduce matrices to the same forms were applied in the papers of authors as well (see, e.g., [38]). The novelty of the present study is the usage of a transformation matrix as the product of two matrices, one of which is parametric. Parameters in the transformation matrix give an opportunity to choose such system parameters to investigate multidimensional relay systems up to the end analytically. In contrast to the earlier works of authors, in this paper the form of the feedback vector depends on the eigenvalues of the system matrix as well as the form of the parametric matrix does.

The decomposition of the system into subsystems is presented in Section 3 and illustrated by Example 1 in Section 6. In Sections 4 and 5, we consider one of subsystems, namely, a system of dimension 2 with non-trivial Jordan block, called *basic system*. Section 5 is concerned with the existence and stability of periodic solutions to the basic system. We prove a theorem on the existence and uniqueness of an asymptotically stable periodic solution (Theorem 5.1). Example 2 in Section 6 shows that periodic oscillations take place in the dynamics of relay systems owing to the periodic perturbation. Interesting results concerning relay systems and perturbations, in particular influence of perturbations, can be found in [9]–[11]. These studies are dedicated to a new type of oscillation, the so-called unpredictable. Their results manifest that the main source for the unpredictable controllable behaviour in the dynamics is the relay perturbations.

2 Statement of the problem

We study a complicated automatic control system the dynamics of which can be governed by the following n-dimensional system of ordinary differential equations

$$\dot{\overline{X}} = A_0\overline{X} + B_0(F(\sigma) + \psi(t)), \quad \sigma(t) = C_0\overline{X}(t). \tag{2.1}$$

Here A_0, B_0, and C_0 are $(n \times n)$, $(n \times m)$, and $(m \times n)$ matrices, respectively $(n \geq m)$; $\dot{\overline{X}}(t)$, $\overline{X}(t)$, and $\sigma(t)$ are $(n \times 1)$, $(n \times 1)$, and $(m \times 1)$ vectors. The nonlinear part of the system is described by the $(m \times 1)$ vector $F(\sigma)$; $\psi(t)$ is a $(m \times 1)$ vector of perturbations.

We also study the two-dimensional system, named *basic system*, of the form

$$\dot{X} = AX + B(F(\sigma) + \psi(t)). \tag{2.2}$$

Here X is the vector of system state such that $X = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2$, where the symbol T means the transposition operation, $A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$, where $\lambda \in \mathbb{R}\backslash\{0\}$; $B = (1, 0)^{\mathrm{T}}$ or $B = (-1, 0)^{\mathrm{T}}$; $F(\sigma)$ is a scalar function and $\sigma = \gamma_1 x_1 + \gamma_2 x_2$, where γ_1, γ_2 are real constants; $\Gamma = (\gamma_1, \gamma_2)^{\mathrm{T}} \in \mathbb{R}^2$ is a nonzero vector; the scalar function $\psi(t)$ stands for a perturbation.

We consider systems (2.1) and (2.2) as models of automatic control systems in which $F(\sigma)$ is a relay-type control, and Γ is a feedback vector. We define $F(\sigma)$ as follows [20]: $F(\sigma) = m_1$ if $\sigma < l_2$, and $F(\sigma) = m_2$ if $\sigma > l_1$, where $m_1, m_2, l_1, l_2 \in \mathbb{R}$, $m_1 < m_2$, $l_1 < l_2$. If $\sigma(0) \leq l_1$ or $\sigma(0) \geq l_2$, then $F(\sigma(t))$ is single-valued. If $\sigma(0) \in (l_1, l_2)$, then $F(\sigma(t))$ is two-valued, therefore we need to specify either $F(\sigma(0)) = m_1$ or $F(\sigma(0)) = m_2$ and follow the positive spin in the plane (σ, F), namely, the value of $F(\sigma(t))$ is kept constant for all $t > 0$ until $\sigma(t)$ crosses the value l_2 from below or the value l_1 from above, respectively. At these instants (when $\sigma(t) = l_i$, $i = 1, 2$) the value of $F(\sigma(t))$ is changed to m_1 or m_2, respectively. Thus, $F(\sigma)$ describes the relay hysteresis with counterclockwise orientation in the plane (σ, F), its figure one can see, for example, in [26].

Notice that the research of systems with relay feedback is quite a task (see [3], [15], and [42]).

The general problem is to investigate system (2.1) completely analytically. The aim of this paper is to show how this problem can be studied by a decomposition of system (2.1) into subsystems of lower dimensions one of which is system (2.2). Also, we pose the problem about sufficient conditions under which system (2.2) has a unique periodic solution. To solve this problem we study system (2.2) in the particular case when $\gamma_2 = 0$, $\psi(t) = F_0 + k\sin(\omega t + \varphi)$, where F_0, $\varphi \in \mathbb{R}$ and k, $\omega \in \mathbb{R}_+$.

On the one hand, if $x_1 = x$, $x_2 = \dot{x}$, then system (2.2) can result from a transformation of the second-order equation in x. On the other hand, system (2.2) is a subsystem of system (2.1) for $n > 2$ after its decomposition. In view of that, we call system (2.2) *basic system*.

3 Decomposition of the n-dimensional system

System (2.1) with arbitrary $F(\sigma)$ and $\psi(t)$ cannot be fully investigated by qualitative methods of the theory of differential equations even for $n = 2$ (see [1]). That is why researchers often use a linear transformation of system (2.1) that reduces the matrix A_0 to diagonal or Jordan form.

It is known [25] that a nonsingular matrix of the transformation is not uniquely determined. Therefore, we look for the transformation $\overline{X}(t) = MX(t)$ with the matrix M being in the form $M = SQ$. Here Q is a nonsingular parametric matrix such that $M^{-1}A_0M = A_j$, where A_j has Jordan form (see [16]). Besides, S and Q are nonsingular matrices such that $S^{-1}A_0S = A_j$, $A_j = Q^{-1}A_jQ$.

Next, we show how to find the matrices Q and Q^{-1} if

$$A_j = \begin{pmatrix} \Lambda_1 & 0 & \dots & 0 \\ 0 & \Lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Lambda_k \end{pmatrix}.$$

Here Λ_i are the r_i-order block diagonal matrix with the Jordan block K_{ij} corresponding to the eigenvalue λ_i $(i = \overline{1,k})$, $\sum\limits_{i=1}^{k} r_i = n$ (r_i is the multiplicity of λ_i). Then

$$Q = \begin{pmatrix} Q_1 & 0 & \dots & 0 \\ 0 & Q_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Q_k \end{pmatrix},$$

where Q_i are also block diagonal matrix with the block Q_{ij} having the same dimension as K_{ij} has. To establish the form of Q_{ij}, we consider K_{ij} of order q corresponding to λ_i, i.e.

$$K_{ij} = \begin{pmatrix} \lambda_i & 0 & \dots & 0 & 0 \\ 1 & \lambda_i & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \lambda_i \end{pmatrix}.$$

Note that K_{ij} can be written down as follows: $K_{ij} = \lambda_i I + H$, where I is the identity matrix, the matrix H is of the form

$$H = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The block Q_{ij} has the form $Q_{ij} = \alpha_0 I + \alpha_1 H + \alpha_2 H^2 + \dots + \alpha_{q-1} H^{q-1}$, where α_0, α_1, $\alpha_2, \dots,$ α_{q-1} are nonzero real numbers. The blocks K_{ij} and Q_{ij} are commutative. Therefore Λ_i and Q_i are commutative and $QA_j = A_j Q$.

To find out Q^{-1}, it suffices to find Q_{ij}^{-1}. Note that

$$Q_{ij}^{-1} = \beta_0 I + \beta_1 H + \beta_2 H^2 + \dots + \beta_{q-1} H^{q-1},$$

where β_0, β_1, β_2, …, β_{q-1} are the solution to the system that follows from the equality $Q_{ij} Q_{ij}^{-1} = I$ taking into account that H^q is a zero matrix.

Thus, after the transformation, system (2.1) acquires the form

$$\dot{X}(t) = A_j X(t) + \overline{B}_M (F(\sigma) + \psi(t)), \quad \sigma(t) = \overline{C}_M X(t), \tag{3.1}$$

where $\overline{B}_M = Q^{-1} S^{-1} B_0$, $\overline{C}_M = C_0 S Q$.

If we consider the elements of B_0 and C_0 as the parameters of system (2.1) for tuning, then Q in the relation $M = SQ$ allows one to simplify and expand the choice of these elements for system (2.1) to be investigated up to the end analytically.

4 Study of the two-dimensional subsystem

Consider system (2.2) with $B = (-1, 0)^{\mathrm{T}}$, namely,

$$\begin{cases} \dot{x}_1 = \lambda x_1 - (F(\sigma) + \psi(t)), \\ \dot{x}_2 = x_1 + \lambda x_2, \end{cases} \tag{4.1}$$

where $\sigma = \gamma_1 x_1$.

Remark 1. If n is even and $n > 3$, then the subsystems in the form (4.1) can also be obtained by the transformation of the initial system with the result that

$$A_j = \begin{pmatrix} \lambda_1 & 0 & & & \\ 1 & \lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_i & 0 \\ & & & 1 & \lambda_i \end{pmatrix},$$

where $i = n/2$.

Next we examine system (4.1) analytically for the purpose of obtaining the conditions on the system parameters such that there exist solutions with periods being other than period of the perturbation and studying the properties of these solutions.

To analyse the solutions to system (4.1) and its phase space, we use Cauchy's form for the solution representation. Thus, for the first equation of (4.1), we have

$$x_1(t) = e^{\lambda t} x_1^0 - \int_0^t e^{-\lambda(\tau - t)} (F(\sigma) + \psi(\tau)) d\tau,$$

where $x_1^0 = x_1(0)$.

Multiplying the latter equation by γ_1 and taking into account $F(\sigma) = m_i$ $(i = 1, 2)$, $\psi(\tau) = F_0 + k \sin(\omega\tau + \varphi)$, we come to the expression

$$\sigma(t) = \gamma_1 x_1(t) = e^{\lambda t} \gamma_1 x_1^0 - \gamma_1 \int_0^t e^{-\lambda(\tau - t)} (m_i + F_0 + k \sin(\omega\tau + \varphi)) d\tau.$$

Note that

$$\int_0^t e^{-\lambda(\tau - t)} (m_i + F_0 + k \sin(\omega\tau + \varphi)) d\tau$$

$$= e^{\lambda t} \left((m_i + F_0) \left(-\frac{e^{-\lambda\tau}}{\lambda} \right) - \frac{\lambda \sin(\omega\tau + \varphi) + \omega \cos(\omega\tau + \varphi)}{\lambda^2 + \omega^2} k e^{-\lambda\tau} \right) \Bigg|_0^t.$$

After integrating, we have

$$\sigma(t) = \left(\sigma_0 - \frac{\gamma_1 \overline{m}_i}{\lambda} - \frac{\gamma_1 k}{\sqrt{\lambda^2 + \omega^2}} \sin(\varphi + \delta) \right) e^{\lambda t} + \frac{\gamma_1 \overline{m}_i}{\lambda} + \frac{\gamma_1 k}{\sqrt{\lambda^2 + \omega^2}} \sin(\omega t + \varphi + \delta).$$

Here $\sigma_0 = \sigma(0)$, $\overline{m}_i = m_i + F_0$ $(i = 1, 2)$, and $\delta = \arctan(\omega/\lambda) + \pi q$ ($q = 0$ if $\lambda > 0$ and $q = 1$ if $\lambda < 0$). Consequently, $x_1(t) = \sigma(t)/\gamma_1$ and $x_1^0 = \sigma_0/\gamma_1$. Integrating the second equation in (4.1) provided that x_1 is known, we have

$$x_2(t) = x_2^0 e^{\lambda t} + \left(\frac{\sigma_0}{\gamma_1} - \frac{\overline{m}_i}{\lambda} - \frac{k}{\sqrt{\lambda^2 + \omega^2}} \sin(\varphi + \delta) \right) t e^{\lambda t}$$

$$-\frac{\overline{m}_i}{\lambda^2}(1-e^{\lambda t})+\frac{ke^{\lambda t}}{\lambda^2+\omega^2}\sin(\varphi+2\delta)+\frac{1}{\lambda^2+\omega^2}\sin(\omega t+\varphi+2\delta), \quad (4.2)$$

where $x_2^0 = x_2(0)$.

Put

$$\Phi(\sigma_0,\overline{m}_i,t)=\left(\frac{\sigma_0}{\gamma_1}-\frac{\overline{m}_i}{\lambda}-\frac{k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)\right)te^{\lambda t}$$

$$-\frac{\overline{m}_i}{\lambda^2}(1-e^{\lambda t})+\frac{ke^{\lambda t}}{\lambda^2+\omega^2}\sin(\varphi+2\delta)+\frac{1}{\lambda^2+\omega^2}\sin(\omega t+\varphi+2\delta). \quad (4.3)$$

Let τ_1 be the time for moving of representative point from the switching line H_2 ($\gamma_1 x_1 = l_2$) to H_1 ($\gamma_1 x_1 = l_1$) and τ_2 transition time from H_1 to H_2. Further we seek for τ_1, τ_2, using the following initial and boundary conditions for $\sigma(t)$:
if $t \in [0,\tau_1]$, then $\sigma_0 = l_2$, $\overline{m}_i = \overline{m}_2$, $\sigma(\tau_1) = l_1$;
if $t \in [0,\tau_2]$, then $\sigma_0 = l_1$, $\overline{m}_i = \overline{m}_1$, $\sigma(\tau_2) = l_2$.

Hence, we come to the transcendental equations with respect to τ_1, τ_2

$$l_1-\frac{\gamma_1\overline{m}_2}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\omega\tau_1+\varphi+\delta)$$

$$=\left(l_2-\frac{\gamma_1\overline{m}_2}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)\right)e^{\lambda\tau_1}, \quad (4.4)$$

$$l_2-\frac{\gamma_1\overline{m}_1}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\omega\tau_2+\varphi+\delta)$$

$$=\left(l_1-\frac{\gamma_1\overline{m}_1}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)\right)e^{\lambda\tau_2}. \quad (4.5)$$

Now we write out the sufficient conditions for the existence of positive roots τ_1, τ_2. Equations (4.4), (4.5) have solutions $\tau_1 > 0$, $\tau_2 > 0$ if the following inequalities hold:

$$\lambda<0,\quad \gamma_1>0,\quad \overline{m}_1<0,\quad \overline{m}_2>0,\quad l_1<0,\quad l_2>0,$$

$$l_2-\frac{\gamma_1\overline{m}_2}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)>0,\quad l_1-\frac{\gamma_1\overline{m}_2}{\lambda}>-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}},$$

$$l_1-\frac{\gamma_1\overline{m}_1}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)<0,\quad l_2-\frac{\gamma_1\overline{m}_1}{\lambda}<\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}. \quad (4.6)$$

Equations (4.4), (4.5) have also solutions $\tau_1 > 0$, $\tau_2 > 0$ under the conditions below

$$\lambda<0,\quad \gamma_1<0,\quad \overline{m}_1<0,\quad \overline{m}_2>0,\quad l_1<0,\quad l_2>0,$$

$$l_2-\frac{\gamma_1\overline{m}_2}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)>0,\quad l_1-\frac{\gamma_1\overline{m}_2}{\lambda}>\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}},$$

$$l_1-\frac{\gamma_1\overline{m}_1}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)<0,\quad l_2-\frac{\gamma_1\overline{m}_1}{\lambda}<-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}. \quad (4.7)$$

5 Existence of asymptotically stable periodic solutions

The existence and uniqueness of a solution are established by the the following theorem.

Theorem 5.1. *Let inequalities* (4.6) *or* (4.7) *be true, equation* (4.4) *have the least solution (or a unique one)* τ_1 *such that* $\tau_1 = 2\pi\nu_1/\omega$, $\nu_1 \in \mathbb{N}$, *and equation* (4.5) *the least solution (or a unique one)* τ_2 *such that* $\tau_2 = 2\pi\nu_2/\omega$, $\nu_2 \in \mathbb{N}$. *Then system* (4.1) *has a unique asymptotically stable* T_f*-periodic solution with* $T_f = \tau_1 + \tau_2$.

Proof. If inequalities (4.6) or (4.7) are satisfied, then equations (4.4), (4.5) have solutions $\tau_1 > 0$ and $\tau_2 > 0$, respectively. But equation (4.4) as well as equation (4.5) can have more than one solution. By τ_i, $i = 1, 2$, we denote transition time from one switching line to the other one. From this, it follows that τ_i is the least solution (or a unique one).

By assumption, $\gamma_2 = 0$. That is why the switching lines H_1 and H_2 are orthogonal to the Ox_1-axis on the plane (x_1Ox_2). By construction of equations (4.4), (4.5), transition time τ_i depends on $x_1(0)$ but does not depend on $x_2(0)$. This implies that the value τ_i is independent of the initial state of representative point on H_i.

Next, we set the initial and boundary conditions for $x_2(t)$ in the case when representative point goes from the point $(l_2/\gamma_1, x_2^0)^{\mathrm{T}} \in H_2$ to the point $(l_1/\gamma_1, x_2^1)^{\mathrm{T}} \in H_1$

$$x_2^0 = x_2(0),\ \overline{m}_i = \overline{m}_2,\ \sigma_0 = l_2,\ x_2^1 = x_2(\tau_1)$$

and from the point $(l_1/\gamma_1, x_2^1)^{\mathrm{T}} \in H_1$ to the point $(l_2/\gamma_1, x_2^2)^{\mathrm{T}} \in H_2$

$$x_2^1 = x_2(0),\ \overline{m}_i = \overline{m}_1,\ \sigma_0 = l_1,\ x_2^2 = x_2(\tau_2).$$

Using (4.2), we obtain

$$x_2(\tau_1) = x_2^0 e^{\lambda\tau_1} + \Phi(l_2, \overline{m}_2, \tau_1),$$
$$x_2(\tau_2) = x_2^1 e^{\lambda\tau_2} + \Phi(l_1, \overline{m}_1, \tau_2),$$

where Φ is defined by (4.3).

Now we write out the point map of the line H_2 into itself in the form

$$x_1^2 = l_2/\gamma_1,\ x_2^2 = x_2^0 e^{\lambda(\tau_1+\tau_2)} + \Theta(\tau_1, \tau_2), \tag{5.1}$$

where $\Theta(\tau_1, \tau_2) = e^{\lambda\tau_2}\Phi(l_2, \overline{m}_2, \tau_1) + \Phi(l_1, \overline{m}_1, \tau_2)$.

By (5.1), we mean the point map such that representative point goes from any initial point $(l_2/\gamma_1, x_2^0)^{\mathrm{T}} \in H_2$ to any point $(l_1/\gamma_1, x_2^1)^{\mathrm{T}} \in H_1$ in τ_1 and from the latter point to the point $(l_2/\gamma_1, x_2^2)^{\mathrm{T}} \in H_2$ in τ_2. Therefore, we consider a set of the points $(l_2/\gamma_1, x_2^0)^{\mathrm{T}} \in H_2$ that are mapped into the line H_2 in T_f, where $T_f = \tau_1 + \tau_2$, by virtue of the solution to system (4.1). Note that (5.1) gives the first return map to the layer between the two hysteretic regimes. Clearly, for obtained τ_1 and τ_2, the value $\Theta(\tau_1, \tau_2)$ is constant.

Consider the second return map and so on. Since $\tau_i = 2\pi\nu_i/\omega$, we have $\sin(\eta) = \sin(\omega(\tau_1+\tau_2)+\eta)$ and $\sin(\omega\tau_i+\eta) = \sin(\omega(\tau_i+\tau_1+\tau_2)+\eta)$, where $\eta = \varphi+\delta$ or $\eta = \varphi+2\delta$. Hence, equations (4.4), (4.5) are kept their forms. It means that τ_i and hence $\Theta(\tau_1, \tau_2)$ are kept constant. Therefore relations (5.1) are met for the following return maps. Under conditions above, there exists a fixed point $(x_1^{\mathrm{fix}}, x_2^{\mathrm{fix}})^{\mathrm{T}}$ of the map defined by (5.1) such that $(x_1^{\mathrm{fix}}, x_2^{\mathrm{fix}})^{\mathrm{T}} = (l_2/\gamma_1, x_2^2)^{\mathrm{T}} = (l_2/\gamma_1, x_2^0)^{\mathrm{T}}$. From (5.1), we obtain $x_2^{\mathrm{fix}} = \Theta(\tau_1, \tau_2)/(1 - e^{\lambda(\tau_1+\tau_2)})$. Moreover, if $\Theta > 0$, then x_2^{fix} is positive, but if $\Theta < 0$, then it is negative.

According to (4.6) or (4.7), we have $\lambda < 0$. Then the fixed point is asymptotically stable. Indeed, we have $x_2(t) = x_2^0 e^{\lambda t} + \Theta(\tau_1, \tau_2)$ for $t \geq 0$. For any $\delta > 0$, take any point $\tilde{x}_2^0$ such that $0 < |x_2^0 - \tilde{x}_2^0| < \delta$. Then $\tilde{x}_2(t) = \tilde{x}_2^0 e^{\lambda t} + \Theta(\tau_1, \tau_2)$ and hence

$$|x_2(t) - \tilde{x}_2(t)| = \left|x_2^0 - \tilde{x}_2^0\right| e^{\lambda t} < \delta e^{\lambda t} \to 0 \ \text{ as } \ t \to \infty.$$

The fixed point corresponds to a closed trajectory on the plane (x_1Ox_2). Thus, the representative point of the solution attains the switching line H_i every time in the same transition time τ_i and returns to the same fixed point in phase space along the same trajectory. This means that system (4.1) has an asymptotically stable T_f-periodic solution with $T_f = \tau_1 + \tau_2$. In view of the compressed map, this solution is unique. □

Theorem 5.1 gives the condition $T_f = nT$, where $n = \nu_1 + \nu_2$, $T = 2\pi/\omega$, under which system (4.1) has a T_f-periodic solution.

Now we show consideration of how to obtain the parameters of $F(\sigma)$ satisfying (4.4), (4.5) such that system (4.1) has a periodic solution with period T_f being multiple to period T of the perturbation.

Let the parameters F_0, k, ω, and φ of the function $\psi(t)$ be given. Let a system of automatic control possess a periodic mode, for example, with $\tau_i = 2\pi\nu_i/\omega$ for some $\nu_i \in \mathbb{N}$. Then from (4.5) we obtain

$$l_2 - l_1 e^{2\pi\nu_2\lambda/\omega} = \gamma_1 \left(\frac{m_1 + F_0}{\lambda} + \frac{k \sin(\varphi + \delta)}{\sqrt{\lambda^2 + \omega^2}} \right) \left(1 - e^{2\pi\nu_2\lambda/\omega}\right). \tag{5.2}$$

Relation (5.2) associates the parameters l_1, l_2, m_1 and can be used for their tuning. From (4.4) we express

$$m_2 = \frac{\lambda \left(l_1 - l_2 e^{2\pi\nu_1\lambda/\omega}\right)}{\gamma_1 \left(1 - e^{2\pi\nu_1\lambda/\omega}\right)} - \frac{\lambda k \sin(\varphi + \delta)}{\sqrt{\lambda^2 + \omega^2}} - F_0. \tag{5.3}$$

The parameter m_2 is defined by (5.3) under the proper choice of the parameters l_1, m_1, and l_2 according to (5.2) for some ν_i determining τ_i.

Corollary. *Let the characteristic equation of system* (4.1) *have the root* $\lambda < 0$ *corresponding to Jordan block,* F_0*,* k*,* ω*, and* φ *be given,* $\tau_i = 2\pi\nu_i/\omega$*,* $i = 1, 2$*,* $\nu_i \in \mathbb{N}$*. Also, let* (5.2), (5.3) *defining the parameters* γ_1*,* l_i*,* m_i*,* $i = 1, 2$*, of the function* $F(\sigma)$ *be fulfilled for some* ν_1*,* ν_2 *such that* $\tau_1 + \tau_2 = nT$*,* $n \in \mathbb{N}$*,* $T = 2\pi/\omega$*. Besides, let these parameters satisfy* (4.6) *or* (4.7)*. Then system* (4.1) *has a unique asymptotically stable* T_f*-periodic solution with* $T_f = nT$*,* $n \geq 2$*, provided that* τ_1 *is the least solution (or a unique one) of equation* (4.4) *and* τ_2 *is the least solution (or a unique one) of equation* (4.5) *for chosen parameters.*

Stability or unstability of periodic solutions depends on the type of the iterative process that is defined by formula (5.1) containing the multiplier $e^{\lambda(\tau_1+\tau_2)}$ with $\lambda < 0$ or $\lambda > 0$.

Remark 2. If $\lambda = 0$ in case of the considered Jordan block, then it is necessary to explore the system

$$\begin{cases} \dot{x}_1 = F(\sigma) + \psi(t), \\ \dot{x}_2 = x_1 + F(\sigma) + \psi(t), \end{cases}$$

which can be successfully integrated when $\gamma_1 \neq 0$, $\gamma_2 = 0$.

6 Examples

To illustrate the decomposition approach given in Section 3, we provide the following example for system (2.1) with B_0 being a (3×1) matrix, C_0 being a (1×3) matrix, and $F(\sigma)$, $\psi(t)$ being scalar.

Example 1. Consider the system with the parameters

$$\begin{cases} \dot{\overline{x}}_1 = -6\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 + b_{11}(F(\sigma) + \psi(t)), \\ \dot{\overline{x}}_2 = -2\overline{x}_1 - 2\overline{x}_2 + b_{21}(F(\sigma) + \psi(t)), \\ \dot{\overline{x}}_3 = -2\overline{x}_3 + b_{31}(F(\sigma) + \psi(t)), \end{cases}$$

where $\sigma = c_{11}\overline{x}_1 + c_{12}\overline{x}_2 + c_{13}\overline{x}_3$. Here $A_0 = \begin{pmatrix} -6 & 2 & 2 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. The eigenvalues of A_0 are $\lambda_{1,2} = -4$, $\lambda_3 = -2$. The excess of the matrix $(-4I - A_0)$ is equal to 1, i.e. the (2×2) Jordan block corresponds to the eigenvalues $\lambda_{1,2} = -4$. The Jordan block A_j has the form $A_j = \begin{pmatrix} -4 & 0 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. There exists a nonsingular matrix S such that $SA_j = A_0S$. Next we point out one of the possible matrices S meeting this condition. Let $S = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, then $S^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -0.5 & -0.5 \\ 0 & 0 & 1 \end{pmatrix}$. Now we need to obtain the matrices Q and Q^{-1}. The matrix A_j has the block form $A_j = \begin{pmatrix} A_1 & 0 \\ 0 & -2 \end{pmatrix}$, where A_1 is a (2×2) matrix. The matrix Q has the same form as A_j has, i.e. $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, where Q_1 is a (2×2) matrix, Q_2 is a number. At this, $Q_1 = \alpha_0 I + \alpha_1 H$, $Q_2 = \alpha_2$, where α_0, α_1, α_2 are the real parameters such that $\det Q \neq 0$, $H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus, $Q = \begin{pmatrix} \alpha_0 & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}$ and Q is commutative with A_j. Since $\det Q \neq 0$, we set $\alpha_0 \neq 0$, $\alpha_2 \neq 0$. In addition, we put $\alpha_1 \neq 0$. Then $Q^{-1} = \begin{pmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{pmatrix}$, where $Q_1^{-1} = \beta_0 I + \beta_1 H = \begin{pmatrix} \beta_0 & 0 \\ \beta_1 & \beta_0 \end{pmatrix}$, $Q_2^{-1} = \alpha_2^{-1}$. From the relation $Q_1Q_1^{-1} = I$, we obtain the parameters β_0, β_1 such that $\beta_0 = \alpha_0^{-1}$, $\beta_1 = -\alpha_1/\alpha_0^2$. Thus, the matrices Q and Q^{-1} are found.

Further we use the transformation with the matrix in the form $M = SQ$. Put $B_M = S^{-1}B_0$, where $B_0 = (b_{11} \quad b_{21} \quad b_{31})^{\mathrm{T}}$. Then $B_M = \begin{pmatrix} -b_{11} + b_{21} + b_{31} \\ b_{11} - 0.5b_{21} - 0.5b_{31} \\ b_{31} \end{pmatrix}$ and

$$\overline{B}_M = Q^{-1}S^{-1}B_0 = Q^{-1}B_M = \begin{pmatrix} \frac{1}{\alpha_0}(-b_{11} + b_{21} + b_{31}) \\ \frac{\alpha_1+\alpha_0}{\alpha_0^2}b_{11} - \frac{2\alpha_1+\alpha_0}{2\alpha_0^2}b_{21} - \frac{2\alpha_1+\alpha_0}{2\alpha_0^2}b_{31} \\ \frac{b_{31}}{\alpha_2} \end{pmatrix} = \begin{pmatrix} b_{11}^M \\ b_{21}^M \\ b_{31}^M \end{pmatrix}.$$

Put $\overline{C}_M = C_0SQ = C_MQ$, where $C_0 = (c_{11} \quad c_{12} \quad c_{13})$. Subsequently,

$$C_M = (c_{11} + 2c_{12} \quad 2c_{11} + 2c_{12} \quad -c_{12} + c_{13}),$$

$$\overline{C}_M = ((\alpha_0 + 2\alpha_1)c_{11} + 2(\alpha_0 + \alpha_1)c_{12} \quad \alpha_0(2c_{11} + 2c_{12}) \quad \alpha_2(-c_{12} + c_{13})) = (c_{11}^M \quad c_{12}^M \quad c_{13}^M).$$

Next, we point out two sets of the parameters under which the considered system acquires the canonical form. Suppose $b_{21}^M = b_{31}^M = 0$, $c_{11}^M \neq 0$, and $c_{12}^M = c_{13}^M = 0$. In addition, first let $b_{11}^M = 1$. Then

$$\begin{cases} b_{11}^M = \frac{1}{\alpha_0}(-b_{11} + b_{21} + b_{31}) = 1, \\ b_{21}^M = \frac{\alpha_1+\alpha_0}{\alpha_0^2}b_{11} - \frac{2\alpha_1+\alpha_0}{2\alpha_0^2}b_{21} - \frac{2\alpha_1+\alpha_0}{2\alpha_0^2}b_{31} = 0, \\ b_{31}^M = \frac{b_{31}}{\alpha_2} = 0. \end{cases} \tag{6.1}$$

From the last equation of system (6.1), we have $b_{31} = 0$. From the first two equations, it follows that $b_{11} = \alpha_0 + 2\alpha_1$, $b_{21} = 2(\alpha_0 + \alpha_1)$. If b_{11} and b_{21} are other than zero in the initial system, then, except for the conditions $\alpha_0 \neq 0$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, we obtain the two additional conditions $\alpha_0 \neq -2\alpha_1$ and $\alpha_0 \neq -\alpha_1$.

Now, in addition, let $b_{11}^M = -1$. Then $b_{11} = -(\alpha_0 + 2\alpha_1)$, $b_{21} = -2(\alpha_0 + \alpha_1)$. Put $\alpha_0 \neq -2\alpha_1$, $\alpha_0 \neq -\alpha_1$. For $\overline{C}_M$, we have

$$\begin{cases} c_{11}^M = (\alpha_0 + 2\alpha_1)c_{11} + 2(\alpha_0 + \alpha_1)c_{12} \neq 0, \\ c_{12}^M = 2\alpha_0(c_{11} + c_{12}) = 0, \\ c_{13}^M = \alpha_2(-c_{12} + c_{13}) = 0. \end{cases} \tag{6.2}$$

Since $\alpha_0 \neq 0$ and $\alpha_2 \neq 0$, we have $c_{11} = -c_{12}$ and $c_{12} = c_{13}$. From the first equation of system (6.2), we obtain $-\alpha_0 c_{11} \neq 0$. It follows from here that we may choose any number other than zero as c_{11}. Here $c_{12} = c_{13} = -c_{11}$.

Thus, in the considered system, the feedback vector can consist of all nonzero elements. It is obvious that for other matrices A_0 and S, the numerical coefficients in (6.1) and (6.2) are others, but the main sense does not change. Introducing the parameters α_0 and α_1 expands the number of options for choosing the elements in B_0 and C_0 such that, for $b_{11}^M = -1$, the initial system is reduced to the two-dimensional system (basic system)

$$\begin{cases} \dot{x}_1 = -4x_1 - (F(\sigma) + \psi(t)), \\ \dot{x}_2 = x_1 - 4x_2, \end{cases} \tag{6.3}$$

and the one-dimensional system

$$\dot{x}_3 = -2x_3,$$

where $\sigma = c_{11}^M x_1$. We have the vector Γ such that $\Gamma = (\gamma_1, 0)^{\mathrm{T}} = (c_{11}^M, 0)^{\mathrm{T}}$. The systems can be successfully integrated and investigated analytically.

Consider system (6.3). Setting the parameters of $F(\sigma)$, $\psi(t)$ and using Theorem 5.1, we can establish whether there exists a periodic solution with period T_f. However, setting the parameters of $\psi(t)$ and using Corollary, we can determine the parameters of $F(\sigma)$ under which system (6.3) has a solution with T_f given. Next we solve the latter task.

Example 2. In system (6.3) we have $\lambda = -4 < 0$. Put $F_0 = 1$, $k = 0.02$, $\omega = 2$, and $\varphi = -1$. Then $T = 2\pi/\omega = \pi$. Also, put $\nu_1 = \nu_2 = 1$. Then $\tau_1 = \tau_2 = \pi$ and $T_f = 2\pi$ are given.

According to (4.6), we take $\gamma_1 = 1 > 0$, $l_1 = -100 < 0$, and $m_1 = -100$ ($\overline{m}_1 = m_1 + F_0 = -99 < 0$). Using (5.2), we obtain $l_2 \approx 24.754011 > 0$. From (5.3) we calculate $m_2 \approx 399.019526$ and hence $\overline{m}_2 = m_2 + F_0 \approx 400.019526 > 0$.

Now we need to verify the inequalities in (4.6). So, we have

$$l_2 - \gamma_1 \left(\frac{\overline{m}_2}{\lambda} + \frac{k \sin(\varphi + \delta)}{\sqrt{\lambda^2 + \omega^2}} \right) \approx 124.754446 > 0,$$

$$l_1 - \gamma_1 \left(\frac{\overline{m}_2}{\lambda} - \frac{k}{\sqrt{\lambda^2 + \omega^2}} \right) \approx 0.009354 > 0,$$

$$l_1 - \gamma_1 \left(\frac{\overline{m}_1}{\lambda} + \frac{k \sin(\varphi + \delta)}{\sqrt{\lambda^2 + \omega^2}} \right) \approx -124.754446 < 0,$$

$$l_2 - \gamma_1 \left(\frac{\overline{m}_1}{\lambda} + \frac{k}{\sqrt{\lambda^2 + \omega^2}} \right) \approx -0.0004607 < 0.$$

Inequalities (4.6) are true.

Under the values of parameters stated above, consider equation (4.4) for τ_1 and equation (4.5) for τ_2. Feasible solution analysis makes it possible to assert that $\tau_1 = \pi$ is the least solution of equation (4.4) and $\tau_2 = \pi$ is a unique solution of equation (4.5) on the interval $(0, 2\pi)$. Therefore all the conditions of Corollary hold.

We thus come to the conclusion that there exists a unique asymptotically stable $2T$-periodic solution to the system with $\gamma_1 = 1$, $l_1 = -100$, $m_1 = -100$, $l_2 \approx 24.754011$, and $m_2 \approx 399.019526$. This task is solved.

To compare the dynamics of the system in both nonautonomous and autonomous cases, consider the system with $\psi(t) \equiv 0$ in the vector form $\dot{X} = AX + Bm_i$ $(i = 1, 2)$. On the plane (x_1Ox_2), the autonomous system has the stability centerpoints $X_i = -A^{-1}Bm_i$ with coordinates $(25, 6.25)$ and $(-99.754881, -24.938720)$. As is known, if both centerpoints of stability lay out of the ambiguity zone, then the autonomous system would have at least one periodic solution with two switch points. As we see, the point $(-99.754881, -24.938720)$ lies in the interior of the ambiguity zone $-100 \leq x_1 \leq 24.754011$ $(l_1/\gamma_1 \leq x_1 \leq l_2/\gamma_1)$. This means that the autonomous system $(\psi(t) \equiv 0)$ has no periodic solutions. Hence the perturbation $\psi(t) = 1 + 0.02\sin(2t - 1)$ has a strong effect on the system. Thus, the oscillatory process exists in the system only due to the periodic perturbation.

7 Conclusion

We have proposed the approach for the study of the multidimensional system with a non-ideal relay and a harmonic perturbation by analytical methods. It consists in the decomposition of the system into subsystems of dimensions 1 and 2 and the research these subsystems. The decomposition is based on the transformation with the parametric matrix that enables one to choose system parameters.

The main object for the research by analytical methods is a two-dimensional subsystem with the linear part being reduced to non-trivial Jordan block. We have obtained the sufficient conditions on the parameters under which the subsystem possesses a unique asymptotically stable periodic solution (see Theorem 5.1). These conditions are expressed as inequalities, which has allowed us to set the relay parameters such that there exists the solution with given period (see Corollary). Moreover, we have pointed out the parameters such that a periodic solution takes place in the dynamics of the subsystem with a harmonic perturbation but it does not in the dynamics of the same subsystem with no perturbations (see Example 2).

In the future, the results of Theorem 5.1 can be used to perform numerical simulations of the n-dimensional system dynamics, since theoretical investigations together with simulations provide more reliable results for applications. Also, the present study is worthwhile to develop for the system with the matrix A having complex eigenvalues and the truncated Fourier series (the sum of a constant and sine functions with different but commensurable periods) as the function of perturbations. Moreover, the proposed approach can be applied to a class of systems broader than the one considered in this study, namely, to systems with the nonlinearity being a monotone function of a non-ideal relay.

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CURVILINEAR PARALLELOGRAM IDENTITY AND MEAN-VALUE PROPERTY FOR A SEMILINEAR HYPERBOLIC EQUATION OF THE SECOND ORDER

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Abstract. In this paper, we discuss some of important qualitative properties of solutions of second-order hyperbolic equations, whose coefficients of the terms involving the second-order derivatives are independent of the desired function and its derivatives. Solutions of these equations have a special property called curvilinear parallelogram identity (or mean-value property), which can be used to solve some initial-boundary value problems.

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1 Introduction

The terms "mean value theorem", "mean value property", "mean formula", and "mean value" are quite common in mathematics (e.g. real analysis, complex analysis, probability theory, partial differential equations) and physics. But they may pertain to diverse phenomena.

In the theory of partial differential equations mean value theorems for harmonic functions and solutions of various elliptic equations are best known. They include the classical mean value property for harmonic functions [12] and the results obtained in works [9, 7, 8, 27] for more general elliptic equations and elliptic operators. Similar theorems are formulated for (hypoelliptic) parabolic equations [16, 17, 18].

Such facts can be established not only for elliptic and parabolic equations but also for hyperbolic ones. Foremost, it should be noted the classical Asgeirson's mean value theorem [3, 6] for the ultrahyperbolic differential equation and the mean value theorem of Bitsadze and Nakhushev for the wave equation [2]. Spherical means can be used to solve initial-value problems as it is done in work [10] for the wave equation and the Darboux equation. Using a symbolic approach [28] several results [24, 22, 23, 30, 25, 31, 29, 26, 33, 32] associated with mean values of solutions of various differential equations were obtained in works of Polovinkin and Meshkov et al. It should also be said that in these works the parallelogram identity (parallelogram rule) for the wave equation (which the authors call 'difference mean-value formula') was generalized to the following cases: a (nonstrictly) hyperbolic equation with constant coefficients of the third-order [24], fourth-order [22], higher-order [32], an equation with constant coefficients and with the operator represented by the product of the first order hyperbolic operators and the second-order elliptic operators [29]. These results can be used to obtain analytical and numerical solutions to differential equations as it was done in [12, 14, 11, 20, 21]. However, these results are mainly given for equations with constant coefficients because of the methods used (Fourier transform, search for accompanying distribution with compact support).

Moreover, the characteristic parallelogram of differential equations has some applications in hydrodynamics [19].

In this paper, we derive the identity of a curvilinear characteristic parallelogram for a general semilinear second-order hyperbolic equation using the method of characteristics [12]. This identity can be considered as the mean value theorem in some sense.

2 Semilinear hyperbolic equation

In the domain $\Omega \subseteq \mathbb{R}^2$ of two independent variables $\mathbf{x} = (x_1, x_2) \in \Omega$ we consider the following semilinear hyperbolic equation of the second-order

$$Au(x_1, x_2) = f(x_1, x_2, u(x_1, x_2), \partial_{x_1} u(x_1, x_2), \partial_{x_2} u(x_1, x_2)), \tag{2.1}$$

where the operator A is defined as

$$Au(x_1, x_2) := a(x_1, x_2)\partial^2_{x_1} u(x_1, x_2) + 2b(x_1, x_2)\partial_{x_1}\partial_{x_2} u(x_1, x_2) + c(x_1, x_2)\partial^2_{x_2} u(x_1, x_2),$$

and is hyperbolic (this means $b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x}) > 0$ for any $x \in \Omega$).

Equation (2.1) has two families of characteristics: $\gamma_1(x_1, x_2)$ and $\gamma_2(x_1, x_2)$, which are the first integrals of the ordinary differential equation [12]

$$a(\mathbf{x})(\mathrm{d}x_2)^2 - 2b(\mathbf{x})\mathrm{d}x_1\mathrm{d}x_2 + c(\mathbf{x})(\mathrm{d}x_1)^2 = 0, \tag{2.2}$$

and solutions of the equation of characteristics [12]

$$a\left(\frac{\partial \gamma_i}{\partial x_1}\right)^2 + 2b\frac{\partial \gamma_i}{\partial x_1}\frac{\partial \gamma_i}{\partial x_2} + c\left(\frac{\partial \gamma_i}{\partial x_2}\right)^2 = 0, \quad i = 1, 2. \tag{2.3}$$

It is known [12] that equation (2.2), generally speaking, can be decomposed into two equations

$$a(\mathbf{x})\mathrm{d}x_2 - (b(\mathbf{x}) \pm \sqrt{b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x})})\mathrm{d}x_1 = 0, \text{ if } a(\mathbf{x}) \neq 0,$$

or

$$c(\mathbf{x})\mathrm{d}x_1 - (b(\mathbf{x}) \pm \sqrt{b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x})})\mathrm{d}x_2 = 0, \text{ if } c(\mathbf{x}) \neq 0,$$

or

$$\mathrm{d}x_1\mathrm{d}x_2 = 0, \text{ if } a(\mathbf{x}) = c(\mathbf{x}) = 0.$$

Therefore, we can assume that γ_1 and γ_2 are the first integrals of different differential equations and they are functionally independent since the Jacobian $\left|\dfrac{\partial(\gamma_1, \gamma_2)}{\partial(x_1, x_2)}\right|$ is nonzero [12].

If the curves γ_i, $i = 1, 2$, have a parametric representation $(x_1^{(i)}(t), x_2^{(i)}(t))$, where $x_j^{(i)}$, $j = 1, 2$, are some twice continuously differentiable functions, then the following equality holds [4]

$$a\left(Dx_2^{(i)}\right)^2 - 2bDx_1^{(i)}Dx_2^{(i)} + c\left(Dx_1^{(i)}\right)^2 = 0, \quad i = 1, 2,$$

where D is the ordinary differential operator.

3 Curvilinear characteristic parallelogram

Definition 1. Curvilinear characteristic parallelogram of hyperbolic differential equation (2.1) is the set $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$, where l_1, l_2, r_1, r_2 are some real numbers and γ_i, $i = 1, 2$ are two different functionally independent characteristics.

Remark 1. Definition 1 is well defined. It is known [1] that any other first integral of (2.2) has the form $q \circ \gamma_1$, where q is some continuously differentiable function. If $\gamma_1(\mathbf{x}) \in [l_1, l_2]$, then, due to the continuity of q, $q(\gamma_1(\mathbf{x})) \in q([l_1, l_2]) = [\widetilde{l}_1, \widetilde{l}_2]$. So the curvilinear characteristic parallelogram does not depend on considered characteristics.

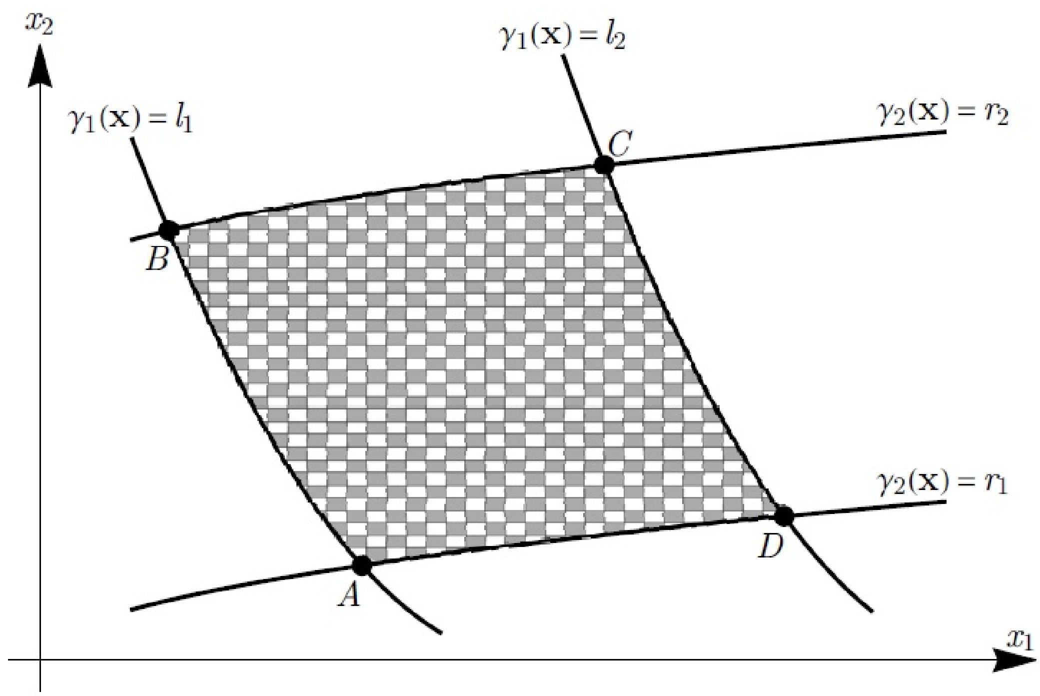

Fig. 1. Curvilinear characteristic parallelogram

Definition 2. Vertices of the curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$ are points $\mathbf{x}$ such that $\gamma_1(x) = l_i \wedge \gamma_2(x) = r_j$, $(i, j) \in \{1, 2\} \times \{1, 2\}$.

Remark 2. Definition 2 is well defined. We should show that $q \circ \gamma_1$, where q is some continuously differentiable function, maps $[l_1, l_2]$ into $[\widetilde{l}_1, \widetilde{l}_2]$ and $\partial([l_1, l_2])$ into $\partial([\widetilde{l}_1, \widetilde{l}_2])$. Obviously, if the function q is increasing or decreasing these properties must be true. But if the the function q does not satisfy these conditions, then there exists at least one point $l_0 \in (l_1, l_2)$ such that $q'(l_0) = 0$. Due to the continuity of q, there exists a point $\mathbf{x} \in \Pi$ such that $\gamma_1(\mathbf{x}) = l_0 \in (l_1, l_2)$ This implies

$$\left|\frac{\partial(q \circ \gamma_1, \gamma_2)}{\partial(x_1, x_2)}\right|(\mathbf{x}) = \begin{vmatrix} q'(\gamma_1(\mathbf{x}))\partial_{x_1}\gamma_1(\mathbf{x}) & q'(\gamma_1(\mathbf{x}))\partial_{x_2}\gamma_1(\mathbf{x}) \\ \partial_{x_1}\gamma_2(\mathbf{x}) & \partial_{x_2}\gamma_2(\mathbf{x}) \end{vmatrix} = 0 \text{ when } \gamma_1(\mathbf{x}) = l_0.$$

But we consider only characteristics with nonzero Jacobian. The statement is proved.

Definition 3. Opposite vertices of the curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$ are its vertices $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\gamma_1(\mathbf{x}_1) \neq \gamma_1(\mathbf{x}_2)$ and $\gamma_2(\mathbf{x}_1) \neq \gamma_2(\mathbf{x}_2)$.

Point transformation of variables of the form $y_1 = \gamma_1(x_1, x_2)$, $y_1 = \gamma_2(x_1, x_2)$ is invertible [34], i.e. there is the inverse change of variables $x_1 = \gamma_1^{-1}(y_1, y_2)$, $x_2 = \gamma_2^{-1}(y_1, y_2)$.

Lemma 3.1. *Let* $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$ *be a curvilinear characteristic parallelogram and the conditions* $a \in C^2(\Pi)$, $b \in C^2(\Pi)$, $c \in C^2(\Pi)$, *and* $f \in C^1(\Pi \times \mathbb{R}^3)$ *be satisfied. The function* u *belongs to the class* $C^2(\Pi)$ *and satisfies equation* (2.1) *if and only if it can be represented as*

$$
\begin{aligned}
u(\mathbf{x}) &= g_1(\gamma_1(\mathbf{x})) + g_2(\gamma_2(\mathbf{x})) \\
&+ \int\limits_{l^{(0)}}^{\gamma_1(\mathbf{x})} dz_1 \int\limits_{r^{(0)}}^{\gamma_2(\mathbf{x})} \frac{1}{2\left(a\partial_{x_1}\gamma_1\partial_{x_1}\gamma_2 + b\left(\partial_{x_2}\gamma_2\partial_{x_1}\gamma_1 + \partial_{x_2}\gamma_1\partial_{x_1}\gamma_2\right) + c\partial_{x_2}\gamma_1\partial_{x_2}\gamma_2\right)\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)} \\
&\times \Big[f\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}), u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right),\right. \\
&\left.\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\right) \\
&- A\gamma_1\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\left(\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\partial_{y_1}\gamma_1^{-1}(\mathbf{z})\right. \\
&\left.+ \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\partial_{y_1}\gamma_2^{-1}(\mathbf{z})\right) \\
&- A\gamma_2\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\left(\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\partial_{y_2}\gamma_1^{-1}(\mathbf{z})\right. \\
&\left.+ \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\partial_{y_2}\gamma_2^{-1}(\mathbf{z})\right)\Big]\, dz_2,
\end{aligned}
\tag{3.1}
$$

where $l^{(0)} \in [l_1, l_2]$, $r^{(0)} \in [r_1, r_2]$, *and the functions* g_1, g_2 *belong to the classes* $C^2(\mathfrak{D}(g_1))$, $C^2(\mathfrak{D}(g_2))$ *respectively.*

Proof. Let a function $u \in C^2(\Pi)$ satisfy equation (2.1). Making the nonlinear nondegenerate change of independent variables $y_1 = \gamma_1(x_1, x_2)$, $y_1 = \gamma_2(x_1, x_2)$ and denoting $u(x_1, x_2) = v(y_1, y_2)$ we obtain a new differential equation

$$
\begin{aligned}
&2\left(a\partial_{x_1}\gamma_1\partial_{x_1}\gamma_2 + b\left(\partial_{x_2}\gamma_2\partial_{x_1}\gamma_1 + \partial_{x_2}\gamma_1\partial_{x_1}\gamma_2\right) + c\partial_{x_2}\gamma_1\partial_{x_2}\gamma_2\right)\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right) \\
&\times \partial_{y_1}\partial_{y_2}v(\mathbf{y}) + A\gamma_1\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\partial_{y_1}v(\mathbf{y}) + A\gamma_2\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\partial_{y_2}v(\mathbf{y}) \\
&= f\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y}), u\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right), \partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right),\right. \\
&\left.\partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\right) = f\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y}), v(\mathbf{y}), \partial_{y_1}v(\mathbf{y})\partial_{x_1}\gamma_1\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\right. \\
&+ \partial_{y_2}v(\mathbf{y})\partial_{x_1}\gamma_2\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right), \partial_{y_1}v(\mathbf{y})\partial_{x_2}\gamma_1\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right) \\
&\left.+ \partial_{y_2}v(\mathbf{y})\partial_{x_2}\gamma_2\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\right)
\end{aligned}
$$

Let us integrate it twice to obtain the equality

$$
\begin{aligned}
v(\mathbf{y}) &= g_1(\mathbf{y}) + g_2(\mathbf{y}) \\
&+ \int\limits_{l^{(0)}}^{y_1} dz_1 \int\limits_{r^{(0)}}^{y_2} \frac{1}{2\left(a\partial_{x_1}\gamma_1\partial_{x_1}\gamma_2 + b\left(\partial_{x_2}\gamma_2\partial_{x_1}\gamma_1 + \partial_{x_2}\gamma_1\partial_{x_1}\gamma_2\right) + c\partial_{x_2}\gamma_1\partial_{x_2}\gamma_2\right)\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)} \\
&\times \Big[f\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}), u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right),\right. \\
&\left.\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\right) \\
&- A\gamma_1(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}))\partial_{y_1}v(\mathbf{z}) - A\gamma_2(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}))\partial_{y_2}v(\mathbf{z})\Big]\, dz_2,
\end{aligned}
$$

Returning to the variables x_1 and x_2 we obtain equation (3.1). This also implies that the functions g_j belong to the class $C^2(\mathfrak{D}(g_1))$, $j = 1, 2$.

Substituting representations (3.1) into equation (2.1), we verify that the function u satisfies this equation in Π. □

Remark 3. Under some additional conditions on the functions f, a, b, c, g_1, g_2, we can show the solvability of integro-differential equation (3.1) using the methods proposed in the works [5, 13, 35].

For the convenience of further presentation, we introduce the notation

$$
\begin{aligned}
&\beta = 2\left(a\partial_{x_1}\gamma_1\partial_{x_1}\gamma_2 + b\left(\partial_{x_2}\gamma_2\partial_{x_1}\gamma_1 + \partial_{x_2}\gamma_1\partial_{x_1}\gamma_2\right) + c\partial_{x_2}\gamma_1\partial_{x_2}\gamma_2\right),\\
&K(\mathbf{z},p,q,r) = f(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}),p,q,r)\\
&- A\gamma_1(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}))(q\partial_{y_1}\gamma_1^{-1}(\mathbf{z}) + r\partial_{y_1}\gamma_2^{-1}(\mathbf{z}))\\
&- A\gamma_2(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}))(q\partial_{y_2}\gamma_1^{-1}(\mathbf{z}) + r\partial_{y_2}\gamma_2^{-1}(\mathbf{z})),\\
&\widetilde{K}(\mathbf{z},p,q,r) = (\beta(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})))^{-1}K(\mathbf{z},p,q,r)
\end{aligned}
$$

4 Curvilinear parallelogram identity

Theorem 4.1. *Let a function u belong to the class $C^2(\Omega)$ and be a solution to hyperbolic equation (2.1), where $a \in C^2(\Omega)$, $b \in C^2(\Omega)$, $c \in C^2(\Omega)$, and $f \in C^1(\Omega \times \mathbb{R}^3)$. Then for any curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1,l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1,r_2]\} \subseteq \Omega$ with vertices $A(\gamma_1^{-1}(l_1,r_1),\gamma_2^{-1}(l_1,r_1))$, $B(\gamma_1^{-1}(l_1,r_2),\gamma_2^{-1}(l_1,r_2))$, $C(\gamma_1^{-1}(l_2,r_2),\gamma_2^{-1}(l_2,r_2))$, $(\gamma_1^{-1}(l_2,r_1),\gamma_2^{-1}(l_2,r_1))$, the following equality holds*

$$
\begin{aligned}
&u(A) - u(B) + u(C) - u(D)\\
&\quad= \int_{l_1}^{l_2} dz_1 \int_{r_1}^{r_2} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\\
&\qquad \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\Big)dz_2.
\end{aligned}
\tag{4.1}
$$

Proof. According to Lemma 3.1, the function u is representable in the form

$$
\begin{aligned}
&u(\mathbf{x}) = g_1(\gamma_1(\mathbf{x})) + g_2(\gamma_2(\mathbf{x}))\\
&\quad+ \int_{l_1}^{\gamma_1(\mathbf{x})} dz_1 \int_{r_1}^{\gamma_2(\mathbf{x})} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\\
&\qquad \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\Big)dz_2.
\end{aligned}
\tag{4.2}
$$

where $g_i \in C^2(\mathfrak{D}(g_i))$, $i = 1,2$. Using expression (4.2) we calculate

$$
\begin{aligned}
&u(A) = g_1(l_1) + g_2(r_1), u(B) = g_1(l_1) + g_2(r_2), u(D) = g_1(l_2) + g_2(r_1),\\
&u(C) = g_1(l_2) + g_2(r_2)\\
&+ \int_{l_1}^{l_2} dz_1 \int_{r_1}^{r_2} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\\
&\quad \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\Big)dz_2.
\end{aligned}
\tag{4.3}
$$

Substituting representations (4.3) into (4.1) we obtain the correct equality. □

Theorem 4.2. *Let functions $u \in C^2(\Omega)$, $a \in C^2(\Omega)$, $b \in C^2(\Omega)$, $c \in C^2(\Omega)$, $f \in C^1(\Omega \times \mathbb{R}^3)$, and the condition $b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x}) > 0$ be satisfied, where $\Omega \subseteq \mathbb{R}^2$. If for any curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1,l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1,r_2]\} \subseteq \Omega$ with vertices $A(\gamma_1^{-1}(l_1,r_1),\gamma_2^{-1}(l_1,r_1))$, $B(\gamma_1^{-1}(l_1,r_2),\gamma_2^{-1}(l_1,r_2))$, $C(\gamma_1^{-1}(l_2,r_2),\gamma_2^{-1}(l_2,r_2))$, $(\gamma_1^{-1}(l_2,r_1),\gamma_2^{-1}(l_2,r_1))$, where γ_i, $i = 1,2$ are solutions of equations (2.2) and γ_i^{-1} are defined as before, equality (4.1) is satisfied, then the function u is a solution to equation (2.1).*

Proof. Let $l_2 = l + l_1$, $r_2 = r + r_1$. So, we can write the coordinates of points A, B, C and D in the form

$$A(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1)),\ B(\gamma_1^{-1}(l_1, r + r_1), \gamma_2^{-1}(l_1, r + r_1)),$$
$$C(\gamma_1^{-1}(l + l_1, r + r_1), \gamma_2^{-1}(l + l_1, r + r_1)),\ D(\gamma_1^{-1}(l + l_1, r_1), \gamma_2^{-1}(l + l_1, r_1)).$$

Let us consider the expression

$$\frac{u(A) - u(B)}{r} = \frac{u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1)) - u(\gamma_1^{-1}(l_1, r + r_1), \gamma_2^{-1}(l_1, r + r_1))}{r} \xrightarrow[r\to 0]{} \xrightarrow[r\to 0]{} -\partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1)).$$

In the same way

$$\frac{u(C) - u(D)}{r} \xrightarrow[r\to 0]{} \partial_r u(\gamma_1^{-1}(l_1 + l, r_1), \gamma_2^{-1}(l_1 + l, r_1)).$$

Now since

$$\frac{\partial_r u(\gamma_1^{-1}(l_1 + l, r_1), \gamma_2^{-1}(l_1 + l, r_1)) - \partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))}{l} \xrightarrow[l\to 0]{} \xrightarrow[l\to 0]{} \partial_l \partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1)),$$

we obtain $\lim\limits_{(r,l)\to(0,0)} (lr)^{-1}(u(A) - u(B) + u(C) - u(D)) = \partial_l\partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))$. Similarly, we get

$$\lim_{(r,l)\to(0,0)} \frac{1}{lr} \int\limits_{l_1}^{l+l_1} dz_1 \int\limits_{r_1}^{r+r_1} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right),$$
$$\partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\Big) dz_2 =$$
$$= \widetilde{K}\Big(\mathbf{z} = (l_1, r_1), u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\Big).$$

Thus

$$\lim_{(r,l)\to(0,0)} \frac{1}{lr} \Bigg(u(A) - u(B) + u(C) - u(D)$$
$$- \int\limits_{l_1}^{l+l_1} dz_1 \int\limits_{r_1}^{r+r_1} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\Big) dz_2 \Bigg)$$
$$= \lim_{(r,l)\to(0,0)} \frac{u(A) - u(B) + u(C) - u(D)}{lr}$$
$$- \lim_{(r,l)\to(0,0)} \frac{1}{lr} \int\limits_{l_1}^{l+l_1} dz_1 \int\limits_{r_1}^{r+r_1} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right),$$
$$\partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\Big) dz_2 = \partial_l \partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))$$
$$- \frac{K\left(\mathbf{z} = (l_1, r_1), u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\right)}{\beta(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))}.$$

This means that the function u satisfies at the point

$$(\gamma_1^{-1}(\mathbf{z} = (y_1 = l_1, y_2 = r_1)), \gamma_2^{-1}(\mathbf{z}))$$

the differential equation

$$
\begin{aligned}
&\beta(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}))\partial_{y_1}\partial_{y_2}u(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}))\\
&= f\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}),u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\right.\\
&\left.\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\right)\\
&- A\gamma_1\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\left(\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\partial_{y_1}\gamma_1^{-1}(\mathbf{z})\right.\\
&\left.+\partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\partial_{y_1}\gamma_2^{-1}(\mathbf{z})\right)\\
&- A\gamma_2\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\left(\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\partial_{y_2}\gamma_1^{-1}(\mathbf{z})\right.\\
&\left.+\partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\partial_{y_2}\gamma_2^{-1}(\mathbf{z})\right),
\end{aligned}
\tag{4.4}
$$

where $x_1=\gamma_1^{-1}(y_1,y_2)$, $x_2=\gamma_2^{-1}(y_1,y_2)$. By virtue of the arbitrariness of $\Pi\subseteq\Omega$, equality (4.4) is true for any point $(x_1=\gamma_1^{-1}(\mathbf{z}=(l_1,r_1)),x_2=\gamma_2^{-1}(\mathbf{z}=(l_1,r_1)))\in\Omega$.

Making the change of variables $x_1=\gamma_1^{-1}(y_1,y_2)$, $x_2=\gamma_2^{-1}(y_1,y_2)$ in equation (4.4), we obtain equation (2.1). □

Note that formula (4.1) can be considered as a kind of a mean value theorem.

5 Applications

5.1 Wave equation

Let us consider $Au(x_1,x_2)=\partial_{x_1}^2u(x_1,x_2)-a^2\partial_{x_2}^2u(x_1,x_2)$, where $a>0$ (for definiteness). Then we have $\gamma_1(x_1,x_2)=x_2-ax_1$, $\gamma_2(x_1,x_2)=x_2+ax_1$, $\gamma_1^{-1}(y_1,y_2)=(y_2-y_1)/(2a)$, $\gamma_2^{-1}(y_1,y_2)=(y_1+y_2)/2$, $A\gamma_1\equiv0$, $A\gamma_2\equiv0$.

5.1.1 Parallelogram identity

Let $f\equiv0$. In this case, formula (4.1) transforms to

$$
\begin{aligned}
&u\left(\frac{r_1-l_1}{2a},\frac{l_1+r_1}{2}\right)-u\left(\frac{r_2-l_1}{2a},\frac{l_1+r_2}{2}\right)\\
&+u\left(\frac{r_2-l_2}{2a},\frac{l_2+r_2}{2}\right)-u\left(\frac{r_1-l_2}{2a},\frac{l_2+r_1}{2}\right)=0,
\end{aligned}
\tag{5.1}
$$

where l_1, l_2, r_1 and r_2 are some real numbers. Equality (5.1) is the well-known parallelogram identity for the wave equation.

5.1.2 Goursat problem

Let us consider the Goursat problem [15]

$$
\begin{cases}
(\partial_{x_1}^2-a^2\partial_{x_2}^2)u(\mathbf{x})=f(\mathbf{x}), & 0<x_1,-ax_1<x_2<ax_1,\\
u(x_1,x_2=ax_1)=\phi^{(1)}(x_1), & u(x_1,x_2=-ax_1)=\phi^{(1)}(x_2),\quad x_1>0,
\end{cases}
\tag{5.2}
$$

where $f\in C^1(\{\mathbf{x}\mid 0\leqslant x_1,-ax_1\leqslant x_2\leqslant ax_1\})$, $\phi^{(1)}\in C^2([0,\infty))$, $\phi^{(2)}\in C^2([0,\infty))$ and $\phi^{(1)}(0)=\phi^{(2)}(0)$. We can write the classical solution of (5.2) using formula (4.1). If we select $C(x_1,x_2)$, $B\left(\frac{ax_1+x_2}{2a},\frac{ax_1+x_2}{2}\right)$, $D\left(\frac{ax_1-x_2}{2a},\frac{x_2-ax_1}{2}\right)$, $A(0,0)$ and apply (4.1), then we obtain

$$
u(x_1,x_2)=u(C)=\phi^{(1)}\left(\frac{ax_1+x_2}{2a}\right)+\phi^{(2)}\left(\frac{ax_1-x_2}{2a}\right)-\phi^{(1)}(0)
$$

$$-\frac{1}{4a^2}\int\limits_0^{x_2-ax_1} dy_1 \int\limits_0^{x_2+ax_1} f\left(\frac{y_2-y_1}{2a},\frac{y_1+y_2}{2}\right) dy_2, \quad 0<x_1, -ax_1<x_2<ax_1.$$

5.1.3 Mixed problem

Let us consider the first mixed problem [12]

$$\begin{cases} (\partial_{x_1}^2 - a^2\partial_{x_2}^2)u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x}\in(0,\infty)\times(0,\infty), \\ u(0,x_2)=\phi(x_1), \quad \partial_{x_1}u(0,x_2)=\psi(x_2), \quad x_1>0, \\ u(x_1,0)=\mu(x_1), \quad x_2>0, \end{cases} \tag{5.3}$$

where $f\in C^1([0,\infty)\times[0,\infty))$, $\phi\in C^2([0,\infty))$, $\psi\in C^1([0,\infty))$, $\mu\in C^2([0,\infty))$.

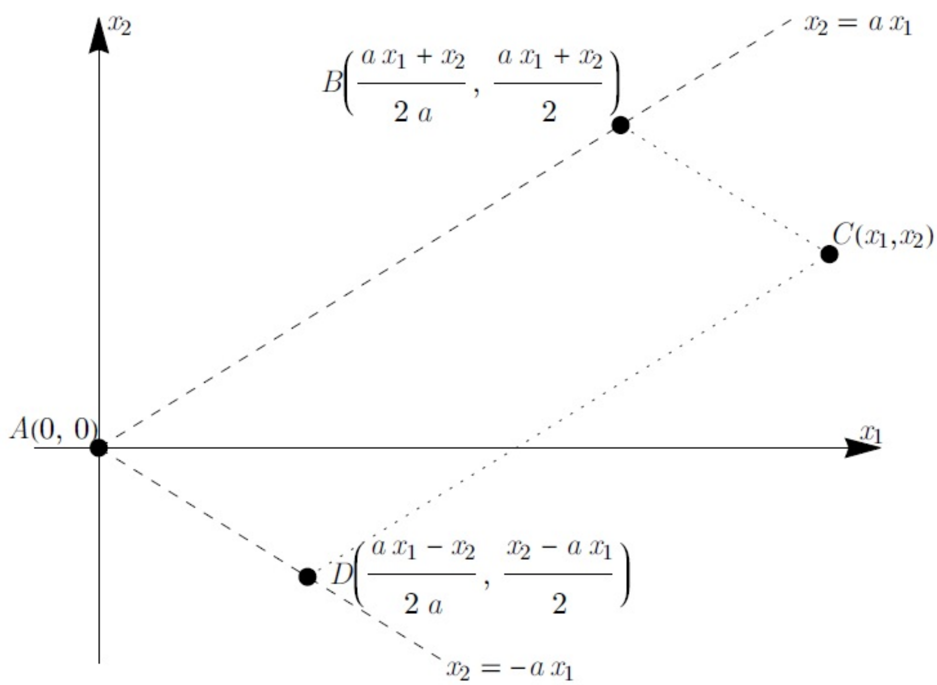

Fig. 2. To the Goursat problem (5.2).

If $x_2-ax_1>0$, then the solution of (5.3) at the point (x_1,x_2) can be defined by d'Alembert formula

$$u(x_1,x_2)=\frac{\phi(x_2-ax_1)+\phi(x_2+ax_1)}{2}+\frac{1}{2a}\int\limits_{x_2-ax_1}^{x_2+ax_1}\psi(\xi)\,d\xi + \\ +\frac{1}{2a}\int\limits_0^{x_1}d\tau\int\limits_{x_2-a(x_1-\tau)}^{x_2+a(x_1-\tau)} f(\tau,\xi)\,d\xi, \quad x_2-ax_1>0, x_1>0, x_2>0. \tag{5.4}$$

If $x_2-ax_1<0$, then we can use parallelogram identity (4.1) to derive the solution of (5.3) at the point (x_1,x_2). We can select $C(x_1,x_2)$, $B\left(x_1-\dfrac{x_2}{a},0\right)$, $D\left(\dfrac{x_2}{a},ax_1\right)$, $A(0,ax_1-x_2)$, apply (4.1) and obtain

$$u(x_1,x_2) = \mu\Big(x_1 - \frac{x_2}{a}\Big) + \frac{\phi(ax_1+x_2) - \phi(ax_1-x_2)}{2} + \frac{1}{2a}\int\limits_{ax_1-x_2}^{ax_1+x_2} \psi(\xi)\,d\xi$$
$$+ \frac{1}{2a}\int\limits_0^{\frac{x_2}{a}} d\tau \int\limits_{ax_1-x_2+a\tau}^{ax_1+x_2-a\tau} f(\tau,\xi)\,d\xi - \frac{1}{4a^2}\int\limits_{ax_1-x_2}^{x_2-ax_1} dy_1 \int\limits_{ax_1-x_2}^{ax_1+x_2} f\Big(\frac{y_2-y_1}{2a}, \frac{y_2+y_1}{2}\Big)\,dy_2, \tag{5.5}$$
$$x_2 - ax_1 < 0, x_1 > 0, x_2 > 0.$$

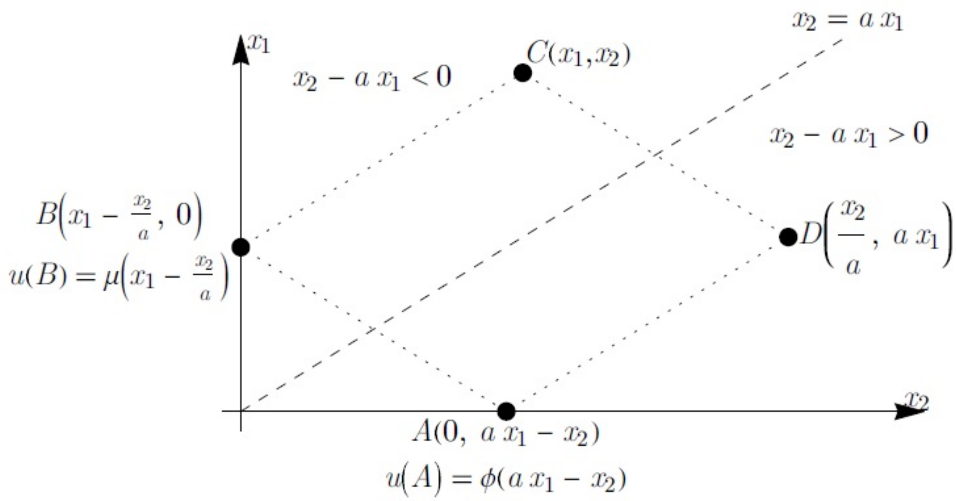

Fig. 3. To the first mixed problem (5.3).

Using representations (5.4) and (5.5), we can easily derive necessary and sufficient matching conditions $\mu(0) = \phi(0)$, $\mu'(0) = \psi(0)$ and $\mu''(0) = a^2\phi''(0) + f(0,0)$ under which the solution u of the first mixed problem (5.3) will be classical.

5.2 Nonlinear wave equation

For convenience, further in this chapter we will present equations in divergence form. Let us consider $Au(x_1,x_2) = \partial_{x_1}\partial_{x_2}u(x_1,x_2)$. Then we have $\gamma_1(x_1,x_2) = x_1$, $\gamma_2(x_1,x_2) = x_2$, $\gamma_1^{-1}(y_1,y_2) = y_1$, $\gamma_2^{-1}(y_1,y_2) = y_2$, $A\gamma_1 \equiv 0$, $A\gamma_2 \equiv 0$.

5.2.1 Darboux problem

Let us consider the second Darboux problem for a nonlinear wave equation in divergence form [11]

$$\begin{cases} \partial_{x_1}\partial_{x_2}u(\mathbf{x}) + \lambda g(\mathbf{x}, u(\mathbf{x})) = f(\mathbf{x}), & 0 < x_1, \alpha x_1 < x_2 < \beta x_1, \\ u(x_1, x_2 = \alpha x_1) = u(x_1, x_2 = \beta x_1) = 0, & x_1 > 0, \end{cases} \tag{5.6}$$

where $\lambda \in \mathbb{R}$, $0 < \alpha < 1 < \beta < \infty$, $f \in C^1(\{\mathbf{x} \mid 0 \leqslant x_1 \wedge \alpha x_1 \leqslant x_2 \leqslant \beta x_1\})$, $g \in C^1(\{\mathbf{x} \mid 0 \leqslant x_1 \wedge \alpha x_1 \leqslant x_2 \leqslant \beta x_1\} \times \mathbb{R})$, $|g(x_1,x_2,z)| \leqslant L_1 + L_2|z|$, $L_1 \geqslant 0$, $L_2 \geqslant 0$.

We want to obtain an expression for the classical solution u of problem (5.6) at the point $P_0(x_1,x_2)$. Let us denote by $P_1M_0P_0N_0$ the characteristic parallelogram, whose vertices N_0 and M_0 lie, respectively, on the segments $x_2 = \alpha x_1$ and $x_2 = \beta x_1$, that is: $N_0 := (x_1, \alpha x_1)$, $M_0 := (\beta^{-1}x_2, x_2)$,

$P_1 := (\beta^{-1}x_2, \alpha x_1)$. Since $P_1 \in \{\mathbf{x} \mid 0 < x_1 \wedge \alpha x_1 < x_2 < \beta x_1\}$, we construct analogously the characteristic parallelogram $P_2M_1P_1N_1$ whose vertices N_1 and M_1 lie, respectively, on the segments $x_2 = \alpha x_1$ and $x_2 = \beta x_1$. Continuing this process, we obtain the characteristic parallelogram $P_{i+1}M_iP_iN_i$ for which $N_i \in \{\mathbf{x} \mid x_2 = \alpha x_1\}$, $M_i \in \{\mathbf{x} \mid x_2 = \beta x_1\}$, and $N_i := \left(x_1^{(i)}, \alpha x_1^{(i)}\right)$, $M_i := \left(\beta^{-1}x_2^{(i)}, x_2\right)$, $P_{i+1} := \left(\beta^{-1}x_2^{(i)}, \alpha x_1^{(i)}\right)$ if $P_i := \left(x_1^{(i)}, x_2^{(i)}\right)$.

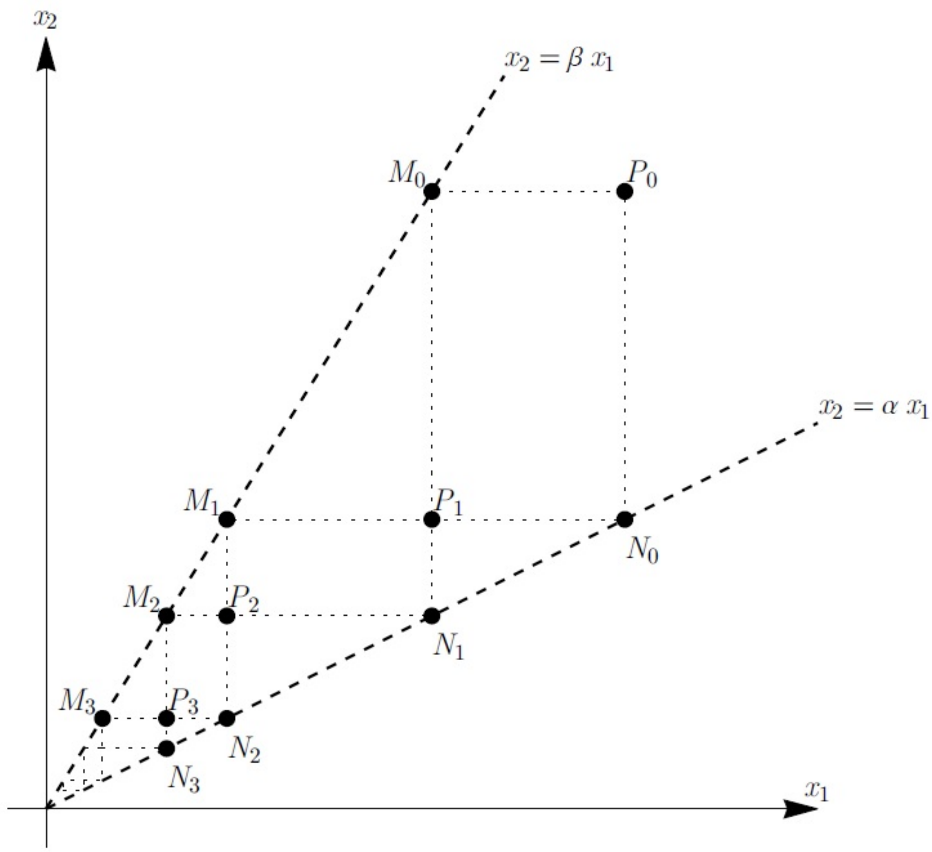

Fig. 4. To the second Darboux problem (5.6).

By virtue of (4.1) and (5.6) we have

$$u(P_i) = u(M_i) + u(N_i) - u(P_{i+1}) + \iint\limits_{P_{i+1}M_iP_iN_i} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z}$$

$$= -u(P_{i+1}) + \iint\limits_{P_{i+1}M_iP_iN_i} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z}, \quad i \in \mathbb{N} \cup \{0\}.$$

Thus it follows that

$$u(x_1, x_2) = u(P_0) = \iint\limits_{P_1M_0P_0N_0} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z} - u(P_1)$$

$$= u(P_2) + \iint\limits_{P_1M_0P_0N_0} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z} - \iint\limits_{P_2M_1P_1N_1} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z}$$

$$= (-1)^n u(P_n) + \sum_{i=0}^{n-1} (-1)^i \iint\limits_{P_{i+1}M_iP_iN_i} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z}.$$

Clearly that $\lim_{n\to\infty} u(P_n) = u\left(\lim_{n\to\infty} P_n\right) = u(0,0) = 0$. Hence, passing to the limit, as $n \to \infty$, we obtain the following integral representation

$$u(x_1, x_2) = \sum_{i=0}^{\infty} (-1)^i \iint\limits_{P_{i+1}M_iP_iN_i} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z}. \tag{5.7}$$

The further solution of problem (5.6) is connected with the study of the solvability of equation (5.7), and it is given in the work [11]. And it turns out that under the conditions specified in the formulation of problem (5.6), it has a unique classical solution. But we still notice that in the linear case (i.e., when $\lambda = 0$), formula (5.7) transforms into

$$u(x_1, x_2) = \sum_{i=0}^{\infty} (-1)^i \iint\limits_{P_{i+1}M_iP_iN_i} f(\mathbf{z}) d\mathbf{z}, \tag{5.8}$$

The series in the right-hand side of equality (5.8) is uniformly and absolutely convergent [11]. So, in the linear case, there is a solution u of (5.6) written in the explicit analytic form (5.8).

5.3 Linear second-order hyperbolic equation

As in the prevoius subsection, we consider $Au(x_1, x_2) = \partial_{x_1}\partial_{x_2} u(x_1, x_2)$. Then we have $\gamma_1(x_1, x_2) = x_1$, $\gamma_2(x_1, x_2) = x_2$, $\gamma_1^{-1}(y_1, y_2) = y_1$, $\gamma_2^{-1}(y_1, y_2) = y_2$, $A\gamma_1 \equiv 0$, $A\gamma_2 \equiv 0$.

5.3.1 Goursat problem

Let us consider the Goursat problem for a linear second-order hyperbolic equation [12]

$$\begin{cases} \partial_{x_1}\partial_{x_2} u(\mathbf{x}) + a(\mathbf{x})\partial_{x_1} u(\mathbf{x}) + b(\mathbf{x})\partial_{x_2} u(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), & x_1^{(0)} < x_1, x_2^{(0)} < x_2, \\ u(x_1 = x_1^{(0)}, x_2) = \phi(x_2), & x_2 > x_2^{(0)}, \\ u(x_1, x_2 = x_2^{(0)}) = \psi(x_1), & x_1 > x_1^{(0)}, \end{cases} \tag{5.9}$$

where $f \in C(\{\mathbf{x} \mid x_1^{(0)} \leqslant x_1 \wedge x_2^{(0)} \leqslant x_2\})$, $\phi \in C^2([x_2^{(0)}, \infty))$, $\psi \in C^1([x_1^{(0)}, \infty))$ and $\phi(x_2^{(0)}) = \psi(x_1^{(0)})$. We can write the classical solution of (5.9) using formula (4.1). If we select $C(x_1, x_2)$, $B\left(x_1^{(0)}, x_2\right)$, $D\left(x_1, x_2^{(0)}\right)$, $A\left(x_1^{(0)}, x_2^{(0)}\right)$ and apply (4.1), then we obtain

$$\begin{aligned} u(\mathbf{x}) = u(C) &= \phi(x_2) + \psi(x_1) - \psi(x_2^{(0)}) \\ &+ \int\limits_{x_1^{(0)}}^{x_1} dy_1 \int\limits_{x_2^{(0)}}^{x_2} [f(\mathbf{y}) - a(\mathbf{y})\partial_{x_1} u(\mathbf{y}) - b(\mathbf{y})\partial_{x_2} u(\mathbf{y}) - c(\mathbf{y})u(\mathbf{y})] dy_2. \end{aligned} \tag{5.10}$$

A representation of the solution in the form of integro-differential equation (5.10) is obtained. Under the conditions specified in the formulation of problem (5.9), equation (5.10) will be solvable [12] and the function u will have the required smoothness. This proves the solvability of problem (5.9).

6 Conclusion

In the paper, the property of the characteristic parallelogram for the wave equation is generalized to the case of a semilinear hyperbolic equation of the second order. This identity connects not only the values of points at the vertices of the parallelogram but also the continuum of function values on the parallelogram, in contrast to the linear cases with constant coefficients considered earlier. It is shown how the obtained results, combined with other methods, can be used to solve various mixed problems.

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AN EXISTENCE RESULT FOR A $(p(x), q(x))$-KIRCHHOFF TYPE SYSTEM WITH DIRICHLET BOUNDARY CONDITIONS VIA TOPOLOGICAL DEGREE METHOD

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Key words: weak solutions, $(p(x), q(x))$-Kirchhoff type systeme, variable-exponent Sobolev spaces, topological degree methods.

AMS Mathematics Subject Classification: 35J35, 47H11, 47H30

Abstract. This paper focuses on the existence of at least one weak solution for a nonlocal elliptic system of $(p(x), q(x))$-Kirchhoff type with Dirichlet boundary conditions. The results are obtained by applying the topological degree method of Berkovits applied to an abstract Hammerstein equation associated to our system and also by the theory of the generalized Sobolev spaces.

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1 Introduction

The study of nonlinear boundary value problems involving variable exponents has received considerable attention in the last decades. This is motivated by the developments in elastic mechanics, electrorheological fluids, and image restoration [4, 7, 12, 13, 21, 32, 33].

In this work, we aim to prove the existence of a weak solution for the following nonlocal elliptic system of $(p(x), q(x))$-Kirchhoff type with the Dirichlet boundary conditions:

$$\begin{cases} \mathcal{T}_1(u) = \lambda h(x, u, \nabla u) + \mathcal{Q}(x)|u|^{r_1(x)-2}u & \text{in } \Omega, \\ \mathcal{T}_2(v) = \kappa g(x, v, \nabla v) + \mathcal{O}(x)|v|^{r_2(x)-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{T}_1(u) = -\mathcal{N}_1\Big(\int_\Omega (A_1(x, \nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})dx\Big) \times \operatorname{div}\Big(a_1(x, \nabla u) + |\nabla u|^{p(x)-2}\nabla u\Big),$$

and

$$\mathcal{T}_2(v) = -\mathcal{N}_2\Big(\int_\Omega (A_2(x, \nabla v) + \frac{1}{q(x)}|\nabla v|^{q(x)})dx\Big) \times \operatorname{div}\Big(a_2(x, \nabla v) + |\nabla v|^{q(x)-2}\nabla v\Big).$$

Here and in the sequel, Ω designates a bounded open set in $\mathbb{R}^N (N \geq 2)$, with a Lipschitz boundary denoted by $\partial\Omega$. $p, q, r \in C_+(\overline{\Omega})$, λ and κ are two real parameters, $-\text{div } a_i(x, \nabla u)$ $(i = 1, 2)$ are Leray-Lions operators, $h, g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are two Carathéodory's functions that satisfy the assumption of growth, $\mathcal{Q}, \mathcal{O} \in L^\infty(\Omega)$ and $\mathcal{N}_i : \mathbb{R}^+ \to \mathbb{R}^+$ are functions that satisfy some conditions which will be stated later. As is well known, problem (1.1) is related to the stationary problem of a model presented by Kirchhoff in 1883 [16]. More precisely, Kirchhoff introduced a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

which extends the classical d'Alembert's wave equation that takes into account the effects of length changes of the string produced by transverse vibrations, the parameters in (1.2) have the following meanings: h is the cross-section area, E is the Young modulus, ρ is the mass density, L is the length of the string, and ρ_0 is the initial tension.

The Kirchhoff type equations involving variable exponent growth conditions have been a very interesting topic in recent years, it has been studied in many papers; we refer to [10, 11, 19, 23, 29] in which variational methods have been used to get the existence and multiplicity of solutions, on the other hand, many authors used the topological degree methods to prove the existence of solutions see for example (see, for example,[9, 24, 25, 27, 28]).

The purpose of this work is to study the existence of solutions to the problem (1.1) in the Sobolev spaces with variable exponents by using another approach based on the topological degree of Berkovits based on the Leray-Schauder principle, presented in [5, 6] for a class of demicontinuous operators of generalized (S_+) type, and the theory of the variable-exponent Sobolev spaces.

This article is arranged as follows. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces and we introduce, some classes of operators of generalized (S_+) type and the topological degree, while Section 3 is devoted to the existence of at least one weak solution for problem (1.1).

2 Preliminary results

2.1 The generalized Lebesgue-Sobolev spaces:

First, we introduce some definitions and basic properties of the Lebesgue-Sobolev spaces with variable exponents $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. In this context, we refer to [14, 18, 31] for more details.

Let us set $C_+(\overline{\Omega})=\Big\{p: p\in C(\overline{\Omega}) \text{ and is such that } p(x)>1 \text{ for all } x\in\overline{\Omega}\Big\}$.

For each $p\in C_+(\overline{\Omega})$, we define $p^+ := \max\big\{p(x),\ x\in\overline{\Omega}\big\}$ and $p^- := \min\big\{p(x),\ x\in\overline{\Omega}\big\}$.

For every $p\in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega)=\Big\{v:\Omega\to\mathbb{R} \text{ is measurable and such that } \int_\Omega |v(x)|^{p(x)}dx<+\infty\Big\},$$

equipped with the Luxemburg norm given by

$$|v|_{p(x)}=\inf\Big\{\varepsilon>0,\ \int_\Omega\Big|\frac{v(x)}{\varepsilon}\Big|^{p(x)}dx\le 1\Big\},$$

$\big(L^{p(\cdot)}(\Omega),|\cdot|_{p(\cdot)}\big)$, we call it the generalized Lebesgue space, is a separable, and reflexive Banach space (see, [18]).

Proposition 2.1 ([14]). *Set*

$$\varrho_{p(x)}(v)=\int_\Omega |v(x)|^{p(x)}dx,\ \forall\ v\in L^{p(x)}(\Omega),$$

then,

$$|v|_{p(x)} < 1(respectively = 1; > 1) \;\Leftrightarrow\; \varrho_{p(x)}(v) < 1(respectively = 1; > 1), \tag{2.1}$$

$$|v|_{p(x)} > 1 \Rightarrow |v|^{p^-}_{p(x)} \le \varrho_{p(x)}(v) \le |v|^{p^+}_{p(x)}, \tag{2.2}$$

$$|v|_{p(x)} < 1 \;\Rightarrow\; |v|^{p^+}_{p(x)} \le \varrho_{p(x)}(v) \le |v|^{p^-}_{p(x)}, \tag{2.3}$$

$$\lim_{n\to\infty} |v_k - v|_{p(x)} = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \varrho_{p(x)}(v_k - v) = 0. \tag{2.4}$$

Remark 1. From (2.2) and (2.3), we can deduce the follwingin inequalities:

$$|v|_{p(x)} \le \varrho_{p(x)}(v) + 1, \tag{2.5}$$

$$\varrho_{p(x)}(v) \le |v|^{p^-}_{p(x)} + |v|^{p^+}_{p(x)}. \tag{2.6}$$

Proposition 2.2 ([18])**.** *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, $\forall x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Hölder-type inequality*

$$\Big| \int_\Omega uv \; dx \Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.7}$$

Remark 2. If $k_1,\ k_2 \in C_+(\overline{\Omega})$ with $k_1(x) \le k_2(x)$ for any $x \in \overline{\Omega}$ then, the embedding $L^{k_2(x)}(\Omega) \hookrightarrow L^{k_1(x)}(\Omega)$ is continuous.

$L^{p(x),q(x)}(\Omega)$ refers to the generalized Lebesgue space $L^{p(x)}(\Omega) \times L^{q(x)}(\Omega)$ equipped with the norm $\|\cdot\|_{p(x),q(x)}$ given by

$$\|(u, v)\|_{p(x),q(x)} = |u|_{p(x)} + |v|_{q(x)}, \quad \forall (u, v) \in L^{p(x),q(x)}(\Omega).$$

Now, we define the generalized Sobolev space $W^{1,p(x)}(\Omega)$, for all $p \in C_+(\overline{\Omega})$:

$$W^{1,p(x)}(\Omega) = \Big\{ v \in L^{p(x)}(\Omega) \text{ such that } |\nabla v| \in L^{p(x)}(\Omega) \Big\},$$

equipped with the norm

$$|v|_{1,p(x)} = |v|_{p(x)} + |\nabla v|_{p(x)}.$$

We define $W^{1,p(\cdot)}_0(\Omega)$ as the subspace of $W^{1,p(\cdot)}(\Omega)$, which is the closure of $C^\infty_0(\Omega)$ with respect to the norm $|\cdot|_{1,p(x)}$.

Proposition 2.3 ([15, 22])**.** *If the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $\alpha > 0$ such that for every $x,\ y \in \Omega,\ x \neq y$ with $|x - y| \le \dfrac{1}{2}$ one has*

$$|p(x) - p(y)| \le \frac{\alpha}{-\log|x - y|}, \tag{2.8}$$

then we have the Poincaré inequality, i.e. the exists a constant $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \le C|\nabla u|_{p(x)},\ \forall\ u \in W^{1,p(\cdot)}_0(\Omega). \tag{2.9}$$

In particular, the space $W^{1,p(x)}_0(\Omega)$ has the norm $\|v\|_{1,p(x)}$ which is equivalent to $|v|_{1,p(x)}$, defined by

$$\|v\|_{1,p(x)} = |\nabla v|_{p(x)}.$$

Proposition 2.4 ([14, 18])**.** *The spaces* $\left(W^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\right)$ *and* $\left(W_0^{1,p(x)}(\Omega), \|\cdot\|_{1,p(x)}\right)$ *are separable and reflexive Banach spaces.*

Furthermore, we have the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *(see [18]).*

Remark 3. The dual space of $W_0^{1,p(x)}(\Omega)$ denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$\|u\|_{-1,p'(x)} = \inf\left\{|u_0|_{p'(x)} + \sum_{i=1}^{N} |u_i|_{p'(x)}\right\},\ \forall u \in W^{-1,p'(x)}(\Omega)$$

where the infinimum is taken on all possible decompositions $u = u_0 - \mathrm{div} F$ with $u_0 \in L^{p'(x)}(\Omega)$ and $F = (u_1, \ldots, u_N) \in (L^{p'(x)}(\Omega))^N$

In the sequel, the notation $\mathcal{X}^{1,p(x),q(x)}(\Omega)$ refers to the Orlicz-Sobolev space $W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)$, equipped with th norm $\|(u,v)\| = \|(u,v)\|_{1,p(x),q(x)}$ given by

$$\|(u,v)\| = \|(u,v)\|_{1,p(x),q(x)} = \|u\|_{1,p(x)} + \|v\|_{1,q(x)},\ \forall (u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$$

$\left(\mathcal{X}^{1,p(x),q(x)}(\Omega)\right)^* = \mathcal{X}^{-1,p'(x),q'(x)}(\Omega)$ is the dual space of $\mathcal{X}^{1,p(x),q(x)}(\Omega)$, corresponding to the Orlicz-Sobolev space $W^{-1,p'(x)}(\Omega) \times W^{-1,q'(x)}(\Omega)$ equipped with the norme

$$\|(\varphi,\phi)\|_{-1,p'(x),q'(x)} = \|\varphi\|_{-1,p'(x)} + \|\varphi\|_{-1,q'(x)},\ \forall (\varphi,\phi) \in \mathcal{X}^{-1,p'(x),q'(x)}(\Omega).$$

The continuous pairing between $\mathcal{X}^{1,p(x),q(x)}(\Omega)$ and $\mathcal{X}^{-1,p'(x),q'(x)}(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle_{1,p(x),q(x)}$ satisfying

$$\langle (u,v),(\varphi,\phi)\rangle_{1,p(x),q(x)} = \langle u,\varphi\rangle_{1,p(x)} + \langle v,\phi\rangle_{1,q(x)},$$

for all $(\varphi,\phi) \in \mathcal{X}^{-1,p'(x),q'(x)}(\Omega)$ and $(u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$.

2.2 Topological degree theory

Let $\mathcal{X}$ be a real separable and reflexive Banach space, $\mathcal{X}^*$ its dual space with dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\mathcal{D}$ be a nonempty subset of $\mathcal{X}$. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$), and let $\mathcal{O}$ be the collection of all bounded open sets in $\mathcal{X}$. The readers can find more information about the history of this theory in [1, 8, 25, 27, 17].

Definition 1. Let $\mathcal{Y}$ be a real Banach space. An operator $F : \mathcal{D} \subset \mathcal{X} \to \mathcal{Y}$ is said to be

1) bounded, if it takes any bounded set into a bounded set.

2) demicontinuous, if for any $(u_n) \subset \mathcal{D}$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.

3) compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 2. A mapping $F : \mathcal{D} \subset \mathcal{X} \to \mathcal{X}^*$ is said to be

1) of type (S_+), if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty} \langle Fu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$.

2) quasimonotone, if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$, it follows that $\limsup\limits_{n\to\infty} \langle Fu_n,\ u_n - u\rangle \geq 0$.

For any bounded operator $T : \mathcal{D}_1 \subset \mathcal{X} \to \mathcal{X}^*$ such that $\mathcal{D} \subset \mathcal{D}_1$ and for any operator $F : \mathcal{D} \subset \mathcal{X} \to \mathcal{X}$, we say that F of type $(S_+)_T$, if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup_{n\to\infty} \langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$.

Remark 4 (see [30]). 1) If a mapping is compact in a set, then it is quasi-monotone in that set.

2) If the mapping is demicontinuous and satisfies the condition (S_+) in a set, then it is quasi-monotone in that set.

In the sequel, we consider the following classes of operators :

$$\mathcal{F}_1(\mathcal{D}) := \Big\{F : \mathcal{D} \to \mathcal{X}^* \mid F \text{ is bounded, demicontinuous and of type } (S_+)\Big\},$$
$$\mathcal{F}_T(\mathcal{D}) := \Big\{F : \mathcal{D} \to \mathcal{X} \mid F \text{ is demicontinuous and of type } (S_+)_T\Big\},$$
$$\mathcal{F}_{T,B}(\mathcal{D}) := \{F : \mathcal{D} \to \mathcal{X} \mid F \text{ is bounded, demicontinuous and of class } (S_+)_T\}.$$

An operator $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 2.1 ([17]). *Let $T \in \mathcal{F}_1(\overline{G})$ be continuous and $S : D_S \subset \mathcal{X}^* \to \mathcal{X}$ be demicontinuous such that $T(\overline{G}) \subset D_s$, where $G \in \mathcal{O}$. Then the following statements are true:*

1) if S *is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{G})$, where I denotes the identity operator,*

2) *if S is of type (S_+), then $S \circ T \in \mathcal{F}_T(\overline{G})$.*

Definition 3. Let $G \in \mathcal{O}$, $T \in \mathcal{F}_1(\overline{G})$ be continuous and consider the mappings $F, S : \overline{G} \subset \mathcal{X} \to \mathcal{X}^*$. The affine homotopy $\mathcal{H} : [0, 1] \times \overline{G} \to \mathcal{X}$, defined by

$$\mathcal{H}(t, u) := (1 - t)Fu + tSu \quad \text{for all } (t, u) \in [0, 1] \times \overline{G},$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Lemma 2.2 ([17]). *If the mappings $F, S \in \mathcal{F}_T(\overline{G})$, then the affine homotopy $\mathcal{H} : [0, 1] \times \overline{G} \to \mathcal{X}$ defined in Definition 3 of type $(S_+)_T$.*

Now we give the Berkovits topological degree for a class of demicontinuous operators satisfying condition $(S_+)_T$ for more details, see [17].

Theorem 2.1. *There exists a unique degree function*

$$d : \mathcal{M} = \Big\{(F, G, h) \mid G \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{G}),\ F \in \mathcal{F}_T(\overline{G}),\ h \notin F(\partial G)\Big\} \longrightarrow \mathbb{Z}$$

which satisfies the following properties.

1) (Existence) *If $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in G.*

2) (Normalization) *For any $h \in F(G)$, we have $d(I, E, h) = 1$.*

3) (Additivity) *Let $F \in \mathcal{F}_{T,B}(\overline{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\overline{G}\backslash(G_1 \cup G_2))$ then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

4) (Homotopy invariance) *If $\mathcal{H}$: $[0, 1] \times \overline{G} \to \mathcal{X}$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0, 1] \to \mathcal{X}$ is a continuous path in $\mathcal{X}$ such that $h(t) \notin \mathcal{H}(t, \partial G)$ $\forall t \in [0, 1]$, then*

$$d(\mathcal{H}(t, \cdot), G, h(t)) = \text{constant for all } t \in [0, 1].$$

3 Assumptions and main results

In this section, we will discuss the existence of a weak solution to problem (1.1).

Let $\xi,\xi'\in\mathbb{R}^N$ with $\xi\neq\xi'$. For almost every x in Ω and $i=1,2$, we assume the following hypothesis: $a_i(x,\xi):\Omega\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ is a Carathéodory function, is the gradient with respect to ξ of the mapping $A_i(x,\xi):\Omega\times\mathbb{R}^N\longrightarrow\mathbb{R}$, that is $a_i(x,\xi)=\nabla_\xi A_i(x,\xi)$, and is such that

(M_1) $\quad A_i(x,0)=0,$

(M_2) $\quad \sigma|\xi|^{p(x)}\leq a_1(x,\xi)\cdot\xi\leq p(x)A_1(x,\xi)$ and $\iota|\xi|^{q(x)}\leq a_2(x,\xi)\cdot\xi\leq q(x)A_2(x,\xi),$

(M_3) $\quad \big|a_1(x,\xi)\big|\leq\eta\big(\rho(x)+|\xi|^{p(x)-1}\big)$ and $\big|a_2(x,\xi)\big|\leq\beta\big(\theta(x)+|\xi|^{q(x)-1}\big),$

(M_4) $\quad \big[a_i(x,\xi)-a_i(x,\xi')\big]\cdot(\xi-\xi')>0,$

where $\sigma,\eta,\iota,\theta,\beta$ are some positive constants, $\rho(x)$ is a positive function belonging to $L^{p'(x)}(\Omega)$ and $\theta(x)$ is a positive function belonging to $L^{q'(x)}(\Omega)$, ($p'(x)$ is the conjugate exponent of $p(x)$).

(H_1) $h:\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ is a Carathéodory function satisfying the following growth condition:

$$|h(x,\xi,\xi')|\leq\mu(\gamma(x)+|\xi|^{r_1(x)-1}+|\xi'|^{r_1(x)-1}),$$

where $\mu>0$, $\gamma\in L^{p'(x)}(\Omega)$ and $1\leq r_1^-\leq r_1(x)\leq r_1^+<p^-$.

(H_2) $g:\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ is a Carathéodory function satisfying the following growth condition:

$$|g(x,\xi,\xi')|\leq\alpha(e(x)+|\xi|^{r_2(x)-1}+|\xi'|^{r_2(x)-1}).$$

where $\alpha>0$ and $e\in L^{q'(x)}(\Omega)$ and $1\leq r_2^-\leq r_2(x)\leq r_2^+<q^-$.

(M_5) $\mathcal{N}_i$: $\mathbb{R}^+\to\mathbb{R}^+$ ($i=1,2$) are continuous and nondecreasing function, for which there exist two functions l,j such that,

$$k_0t^{l(x)-1}\leq\mathcal{N}_1(t)\leq k_1t^{l(x)-1},$$
$$m_0t^{j(x)-1}\leq\mathcal{N}_2(t)\leq m_1t^{j(x)-1},$$

where m_i,k_i ($i=0,1$) are positive constants $l,j\in C_+(\overline{\Omega})$ $1\leq l^-\leq l(x)\leq l^+<p^-$, and $1\leq j^-\leq j(x)\leq j^+<q^-$.

Finally, we recall that the $\mathcal{Q},\mathcal{O}\in L^\infty(\Omega)$ and $\mathcal{Q}(x),\mathcal{O}(x)>0$ for almost every x in Ω.

The definition of a weak solution for problem(1.1) can be stated as follows:

Definition 4. A couple $(u,v)\in\mathcal{X}^{1,p(x),q(x)}(\Omega)$ is called a weak solution of (1.1) if

$$\begin{aligned}\langle f_pu,\varphi\rangle+\langle f_qv,\psi\rangle&+\int_\Omega\mathcal{Q}(x)|u|^{r_1(x)-2}u\varphi(x)dx+\int_\Omega\mathcal{O}(x)|v|^{r_2(x)-2}v\psi(x)dx\\&=\int_\Omega\lambda h(x,u,\nabla u)\varphi(x)dx+\int_\Omega\kappa g(x,v,\nabla v)\psi(x)dx,\end{aligned}$$

where

$$\langle f_pu,\varphi\rangle=\mathcal{N}_1\Big(\int_\Omega(A_1(x,\nabla u)+\frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big)\Big[\int_\Omega a_1(x,\nabla u)\nabla\varphi+\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla\varphi\Big],$$

and

$$\langle f_qv,\psi\rangle=\mathcal{N}_2\Big(\int_\Omega(A_2(x,\nabla v)+\frac{1}{q(x)}|\nabla v|^{q(x)})\,dx\Big)\Big[\int_\Omega a_2(x,\nabla v)\nabla\psi+\int_\Omega|\nabla v|^{q(x)-2}\nabla v\nabla\psi\Big],$$

for every $(\varphi,\psi)\in\mathcal{X}^{1,p(x),q(x)}(\Omega)$.

Lemma 3.1 ([2]). *Let* $g \in L^{r(x)}(\Omega)$ *and* $(g_n) \subset L^{r(x)}(\Omega)$ *such that* $\sup_{n\in\mathbb{N}} \|g_n\|_{r(x)} < \infty$, *If* $g_n(x) \to g(x)$ *for almost every* $x \in \Omega$, *then* $g_n \rightharpoonup g$ *weakly in* $L^{r(x)}(\Omega)$.

Lemma 3.2 ([2]). *Assume that* (M_2)-(M_4) *hold. Let* $(u_m)_m$ *be a sequence in* $W_0^{1,n(x)}(\Omega)$ *such that* $u_m \rightharpoonup u$ *weakly in* $W_0^{1,n(x)}(\Omega)$ *and*

$$\int_\Omega \big[a(x,\nabla u_m) - a(x,\nabla u)\big]\nabla(u_m - u)dx \longrightarrow 0, \tag{3.1}$$

then $u_m \longrightarrow u$ *strongly in* $W_0^{1,n(x)}(\Omega)$.

Before giving our main result, we first give two important lemmas that will be used later. Let us consider the following functionals:

$$\begin{aligned}\mathcal{L}(u,v) &:= \widehat{\mathcal{N}_1}\Big(\mathcal{J}_1(u)\Big) + \widehat{\mathcal{N}_2}\Big(\mathcal{J}_2(v)\Big)\\ &:= \widehat{\mathcal{N}_1}\Big(\int_\Omega (A_1(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big) + \widehat{\mathcal{N}_2}\Big(\int_\Omega (A_2(x,\nabla v) + \frac{1}{q(x)}|\nabla v|^{q(x)})dx\Big),\end{aligned}$$

for all $(u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, where the functionals $\mathcal{J}_1 : W_0^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$ and $\mathcal{J}_2 : W_0^{1,q(x)}(\Omega) \longrightarrow \mathbb{R}$, are defined by
$\mathcal{J}_1(u) = \displaystyle\int_\Omega (A_1(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx$ and $\mathcal{J}_2(v) = \displaystyle\int_\Omega (A_2(x,\nabla v) + \frac{1}{q(x)}|\nabla v|^{q(x)})\,dx$,
then $\mathcal{J}_1 \in C^1(W_0^{1,p(x)}(\Omega),\mathbb{R})$, and $\mathcal{J}_2 \in C^1(W_0^{1,q(x)}(\Omega),\mathbb{R})$, $\widehat{\mathcal{N}_i} : [0,+\infty[\longrightarrow [0,+\infty[$ be the primitive of the functions $\mathcal{N}_i$ $(i = 1,2)$, defned by

$$\widehat{\mathcal{N}_i}(t) = \int_0^t \mathcal{N}_i(\xi)\,d\xi.$$

On the other hand, we consider the functional $\mathbf{J} : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to \mathbb{R}$ defined by:

$$\begin{aligned}\mathbf{J}(u,v) &= \mathcal{J}_1(u) + \mathcal{J}_2(v)\\ &= \int_\Omega (A_1(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx + \int_\Omega (A_2(x,\nabla v) + \frac{1}{q(x)}|\nabla v|^{q(x)})\,dx,\end{aligned}$$

for all $(u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, then $\mathbf{J} \in C^1\big(\mathcal{X}^{1,p(x),q(x)}(\Omega),\mathbb{R}\big)$ and,

$$\begin{aligned}\langle \mathbf{J}'(u,v)\,,(\varphi,\psi)\rangle &= \langle \mathcal{J}_1'(u,\varphi)\rangle + \langle \mathcal{J}_2'(v,\psi)\rangle\\ &= \int_\Omega a_1(x,\nabla u)\nabla\varphi\,dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx + \int_\Omega a_2(x,\nabla v)\nabla\psi\,dx\\ &\quad + \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi\,dx.\end{aligned}$$

It is obvious that the functional $\mathcal{L}$ is defined and continuously Gâteaux differentiable and whose Gâteaux derivative at the point $(u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ is the functional $\mathcal{F} := \mathcal{L}'(u,v) \in (\mathcal{X}^{1,p(x),q(x)}(\Omega))^*$ given by

$$\langle \mathcal{L}'(u,v),(\varphi,\psi)\rangle = \langle \mathcal{F}(u,v),(\varphi,\psi)\rangle = \langle f_p u,\varphi\rangle + \langle f_q v,\psi\rangle.$$

Lemma 3.3. *Suppose that hypotheses* (M_1)-(M_5) *hold, then*

i) $\mathcal{F}$ *is continuous, bounded, strictly monotone operator.*

ii) $\mathcal{F}$ *is a mapping of type* (S_+).

Proof. i) It is obvious that $\mathcal{F}$ is continuous because $\mathcal{F}$ is the Fréchet derivative of $\mathcal{L}$. Now, we verify that $\mathcal{F}$ is bounded. For all (u,v) and $(\varphi,\psi)\in\mathcal{X}^{1,p(x),q(x)}(\Omega)$ we have,

$$
\begin{aligned}
|\langle \mathcal{F}(u,v),(\varphi,\psi)\rangle| \leq &\Big|\mathcal{N}_1\Big(\mathcal{J}_1(u)\Big)\Big[\int_\Omega a_1(x,\nabla u)\nabla\varphi\,dx+\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx\Big]\Big|\\
&+\Big|\mathcal{N}_2\Big(\mathcal{J}_2(v)\Big)\Big[\int_\Omega a_2(x,\nabla v)\nabla\psi\,dx+\int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi dx\Big]\Big|.
\end{aligned}
$$

Applying (M_5) and Hölder's inequality, from the last inequality, it follows that

$$
\begin{aligned}
|\langle \mathcal{F}(u,v),(\varphi,\psi)\rangle| \leq\, & k_1\big(\mathcal{J}_1(u)\big)^{l(x)-1}\Big[\int_\Omega |a_1(x,\nabla u)\nabla\varphi|dx+\int_\Omega |\nabla u|^{p(x)-1}|\nabla\varphi|dx\Big]\\
&+m_1\big(\mathcal{J}_2(v)\big)^{j(x)-1}\Big[\int_\Omega |a_2(x,\nabla v)\nabla\psi|dx+\int_\Omega |\nabla v|^{q(x)-1}|\nabla\psi|dx\Big]\\
\leq\, & C_1\Big(\Big(\int_\Omega A_1(x,\nabla u)dx\Big)^{l(x)-1}+\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^{l(x)-1}\Big)\\
&\times\Big[|a_1(x,\nabla u)|_{p'(x)}|\nabla\varphi|_{p(x)}+|\nabla u^{p(x)-1}|_{p'(x)}|\nabla\varphi|_{p(x)}\Big]\\
&+C_2\Big(\Big(\int_\Omega A_2(x,\nabla v)dx\Big)^{j(x)-1}+\Big(\int_\Omega |\nabla v|^{q(x)}dx\Big)^{j(x)-1}\Big)\\
&\times\Big[|a_2(x,\nabla v)|_{q'(x)}|\nabla\psi|_{q(x)}+|\nabla v^{q(x)-1}|_{q'(x)}|\nabla\psi|_{q(x)}\Big].
\end{aligned}
$$

Bearing (2.5) and (2.6) in mind, we obtain

$$
\begin{aligned}
|\langle \mathcal{F}(u,v),(\varphi,\psi)\rangle| \leq\, & \mathrm{C}_3\Big(\Big(\int_\Omega A_1(x,\nabla u)dx\Big)^{l(x)-1}+\|u\|_{1,p(x)}^{p^-(l(x)-1)}+\|u\|_{1,p(x)}^{p^+(l(x)-1)}\Big)\\
&\times\Big[|a_1(x,\nabla u)|_{p'(x)}+\varrho_{p'(x)}(\nabla u^{p(x)-1})+1\Big]\|\varphi\|_{1,p(x)}\\
&+\mathrm{C}_4\Big(\Big(\int_\Omega A_2(x,\nabla v)dx\Big)^{j(x)-1}+\|v\|_{1,q(x)}^{q^-(j(x)-1)}+\|v\|_{1,q(x)}^{q^+(j(x)-1)}\Big)\\
&\times\Big[|a_2(x,\nabla v)|_{q'(x)}+\varrho_{q'(x)}(\nabla v^{q(x)-1})+1\Big]\|\psi\|_{1,q(x)}\\
\leq\, & C_5\Big(\Big(\int_\Omega A_1(x,\nabla u)dx\Big)^{l(x)-1}+\|u\|_{1,p(x)}^{p^-(l(x)-1)}+\|u\|_{1,p(x)}^{p^+(l(x)-1)}\Big)\\
&\times\Big[|a_1(x,\nabla u)|_{p'(x)}+\|u\|_{1,p(x)}^{p^-}+\|u\|_{1,p(x)}^{q^+}+1\Big]\|\varphi\|_{1,p(x)}\\
&+C_6\Big(\Big(\int_\Omega A_2(x,\nabla v)dx\Big)^{j(x)-1}+\|v\|_{1,q(x)}^{q^-(j(x)-1)}+\|v\|_{1,q(x)}^{q^+(j(x)-1)}\Big)\\
&\times\Big[|a_2(x,\nabla v)|_{q'(x)}+\|v\|_{1,q(x)}^{q^-}+\|v\|_{1,q(x)}^{q^+}+1\Big]\|\psi\|_{1,q(x)},
\end{aligned}
$$

where $C_1,...,C_6>0$ are independent of u and v.

By (M_1), we have for any $x\in\Omega$, $\xi\in\mathbb{R}^n$ and $(i=1,2)$,

$$
A_i(x,\xi)=\int_0^1\frac{d}{ds}A_i(x,s\xi)ds=\int_0^1 a_i(x,s\xi)\xi ds,
$$

by combining (M_3), Fubini's theorem and Young's inequality, we have

$$
\begin{aligned}
\int_\Omega A_1(x,\nabla u)dx&=\int_\Omega\int_0^1 a_1(x,s\nabla u)\nabla u ds\,dx=\int_0^1\Big[\int_\Omega a_1(x,s\nabla u)\nabla u dx\Big]ds\\
&\leq\int_0^1\Big[c_0\int_\Omega \big|a_1(x,s\nabla u)\big|^{p'(x)}dx+c_1\int_\Omega|\nabla u|^{p(x)}dx\Big]ds
\end{aligned}
$$

$$\leq \int_0^1 \Big[c_2 \int_\Omega |\rho(x)|^{p'(x)} + |s\,\nabla u|^{p(x)}\,dx + c_1 \int_\Omega |\nabla u|^{p(x)} dx\Big] ds$$
$$\leq c_3 + c_4 \varrho_{p(x)}(\nabla u)$$
$$\leq c_3 + c_4 \big(\|u\|_{1,p(x)}^{p-} + \|u\|_{1,p(x)}^{p+}\big)$$
$$\leq c_5 \Big(\|u\|_{1,p(x)}^{p-} + \|u\|_{1,p(x)}^{p+} + 1\Big), \tag{3.2}$$

where $c_0, ..., c_5 > 0$ are independent of u and v.

The same reasoning is used to prove that,

$$\int_\Omega A_2(x, \nabla v) dx \leq c_6 \Big(\|v\|_{1,q(x)}^{q-} + \|v\|_{1,q(x)}^{q+} + 1\Big).$$

From (M_3), we can easily show that $|a_1(x, \nabla u)|_{p'(x)}$ and $|a_2(x, \nabla v)|_{q'(x)}$ are bounded for all (u, v) in $\mathcal{X}^{1,p(x),q(x)}(\Omega)$. Therefore,

$$|\langle \mathcal{F}(u, v), (\varphi, \psi)\rangle| \leq C_7 \Big(\|\varphi\|_{1,p(x)} + \|\psi\|_{1,q(x)}\Big),$$

where $C_7 > 0$ is independent of ϕ and ψ. Hence, the operator $\mathcal{F}$ is bounded.

Next, we prove that the operator $\mathcal{F}$ is coercive. For each $(u, v) \in \mathcal{X}^{1,p(x,q(x))}(\Omega)$, we have

$$\frac{\langle \mathcal{F}(u, v), (u, v)\rangle}{\|(u, v)\|} = \frac{\mathcal{N}_1\big(\mathcal{J}_1(u)\big) \Big[\int_\Omega a_1(x, \nabla u)\nabla u + \int_\Omega |\nabla u|^{p(x)} dx\Big]}{\|(u, v)\|}$$
$$+ \frac{\mathcal{N}_2\big(\mathcal{J}_2(v)\big) \Big[\int_\Omega a_2(x, \nabla v)\nabla v + \int_\Omega |\nabla v|^{q(x)} dx\Big]}{\|(u, v)\|}.$$

From (M_2) and (M_5), we obtain

$$\frac{\langle \mathcal{F}(u, v), (u, v)\rangle}{\|(u, v)\|} \geq k_0 \frac{\Big(\int_\Omega (A_1(x, \nabla u) + \frac{1}{p^+}|\nabla u|^{p(x)})\,dx\Big)^{l(x)-1} \Big[\sigma \int_\Omega |\nabla u|^{p(x)} + \int_\Omega |\nabla u|^{p(x)} dx\Big]}{\|(u, v)\|}$$
$$+ m_0 \frac{\Big(\int_\Omega (A_2(x, \nabla v) + \frac{1}{q^+}|\nabla v|^{q(x)})\,dx\Big)^{j(x)-1} \Big[\iota \int_\Omega |\nabla v|^{q(x)} + \int_\Omega |\nabla v|^{q(x)} dx\Big]}{\|(u, v)\|}$$
$$\geq k_0 \frac{\Big(\frac{\sigma}{p^+} \int_\Omega |\nabla u|^{p(x)} + \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)})\,dx\Big)^{l(x)-1} \Big[\sigma \int_\Omega |\nabla u|^{p(x)} + \int_\Omega |\nabla u|^{p(x)} dx\Big]}{\|(u, v)\|}$$
$$+ m_0 \frac{\Big(\frac{\iota}{q^+} \int_\Omega |\nabla v|^{q(x)} + \frac{1}{q^+} \int_\Omega |\nabla v|^{q(x)})\,dx\Big)^{j(x)-1} \times \Big[(1+\iota) \int_\Omega |\nabla v|^{q(x)}\,dx\Big]}{\|(u, v)\|}$$
$$\geq k_0 \frac{\Big(\frac{\sigma}{p^+} \int_\Omega |\nabla u|^{p(x)} + \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)})\,dx\Big)^{l(x)-1} \Big[\sigma \int_\Omega |\nabla u|^{p(x)} + \int_\Omega |\nabla u|^{p(x)} dx\Big]}{\|(u, v)\|}$$
$$+ \geq m_0 \frac{\Big(\frac{\iota}{q^+} \int_\Omega |\nabla v|^{q(x)} + \frac{1}{q^+} \int_\Omega |\nabla v|^{q(x)})\,dx\Big)^{j(x)-1} \times \Big[(1+\iota) \int_\Omega |\nabla v|^{q(x)}\,dx\Big]}{\|(u, v)\|}$$
$$\geq C_1 \frac{\|u\|_{1,p(x)}^{\gamma l(x)} + \|v\|_{1,q(x)}^{\beta j(x)}}{\|(u, v)\|}$$
$$\geq C_1 \frac{\|u\|_{1,p(x)}^{\gamma l^-} + \|v\|_{1,q(x)}^{\beta j^-}}{\|u\|_{1,p(x)} + \|v\|_{1,q(x)}},$$

where $C_1 > 0$ is independent of u and v, $\gamma = \begin{cases} p^- & \text{if} \quad \|u\| \leq 1 \\ p^+ & \text{if} \quad \|u\| \geq 1. \end{cases}$ and $\beta = \begin{cases} q^- & \text{if} \quad \|v\|_{1,a(x)} \leq 1 \\ q^+ & \text{if} \quad \|v\|_{1,q(x)} \geq 1. \end{cases}$

Since $\lim\limits_{x+y \longrightarrow \infty} \frac{x^s+y^t}{x+y} = +\infty$ for $s, t > 1$, then $\lim\limits_{\|(u,v)\| \to \infty} \dfrac{\langle \mathcal{F}(u,v) \rangle}{\|(u,v)\|} = \infty.$

Next, we prove that $\mathcal{F}$ is a strictly monotone operator, we show first the monotonicity of $\mathcal{J}'_i$ $(i = 1, 2)$. Using (M_4) and taking into account the following inequality (see [20]), for all $x, y \in \mathbb{R}^N$,

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \cdot (|x|^p + |y|^p)^{\frac{2-p}{p}} \geq (p-1)|x-y|^p \quad \text{if} \quad 1 < p < 2,$$
$$(|x|^{p-2}x - |y|^{p-2}y) \cdot (x-y) \geq (\frac{1}{2})^p |x-y|^p \quad \text{if} \qquad p \geq 2,$$

we obtain, for all $(u_1, v_1), (u_2, v_2) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ with $(u_1, v_1) \neq (u_2, v_2)$, that

$$\langle \mathcal{J}_1'(u_1) - \mathcal{J}_1'(u_2), u_1 - u_2 \rangle > 0 \text{ and } \langle \mathcal{J}_2'(v_1) - \mathcal{J}_2'(v_2), v_1 - v_2 \rangle > 0,$$

which implies that $\mathcal{J}_1'$, $\mathcal{J}_2'$ are strictly monotone.

Thus, by [30, Proposition 25.10], $\mathcal{J}_i$ are strictly convex. Furthermore, as $\mathcal{N}_i \quad (i = 1, 2)$ are nondecreasing, then $\widehat{\mathcal{N}_i}$ are convex in $\mathbb{R}^+$. So, for each $(u_1, v_1), (u_2, v_2) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ with $(u_1, v_1) \neq (u_2, v_2)$, and every $s, t \in (0, 1)$ with $s + t = 1$, we have

$$\widehat{\mathcal{N}_1}(\mathcal{J}_1(su_1 + tu_2)) < \widehat{\mathcal{N}_1}(s\mathcal{J}_1(u_1) + t\mathcal{J}_1(u_2)) \leq s\widehat{\mathcal{N}_1}(\mathcal{J}_1(u_1)) + t\widehat{\mathcal{N}_1}(\mathcal{J}_1(u_2)),$$

and

$$\widehat{\mathcal{N}_2}(\mathcal{J}_2(sv_1 + tv_2)) < \widehat{\mathcal{N}_2}(s\mathcal{J}_2(v_1) + t\mathcal{J}_2(v_2)) \leq s\widehat{\mathcal{N}_2}(\mathcal{J}_2(v_1)) + t\widehat{\mathcal{N}_2}(\mathcal{J}_2(v_2)).$$

This proves that $\mathcal{L} = \widehat{\mathcal{N}_1}(\mathcal{J}_1) + \widehat{\mathcal{N}_2}(\mathcal{J}_2)$ is strictly convex. Since $\mathcal{L}'(u, v) = \mathcal{F}(u, v)$ for all $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, finally, we infer that $\mathcal{F}$ is strictly monotone on $\big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*$.

ii) Now, we verify that the operator $\mathcal{F}$ is of type (S_+). Assume that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u, v) & \text{in } \mathcal{X}^{1,p(x),q(x)}(\Omega) \\ \limsup\limits_{n \to \infty} \langle \mathcal{F}(u_n, v_n), (u_n - u, v_n - v) \rangle \leq 0. \end{cases} \tag{3.3}$$

We will show that $(u_n, v_n) \to (u, v)$ in $\mathcal{X}^{1,p(x),q(x)}(\Omega)$. By the strict monotonicity of $\mathcal{F}$ we get,

$$\limsup_{n \to \infty} \langle \mathcal{F}(u_n, v_n) - \mathcal{F}(u, v),\ (u_n - u, v_n - v) \rangle = \lim_{n \to \infty} \langle \mathcal{F}(u_n, v_n) - \mathcal{F}(u, v),\ (u_n - u, v_n - v) \rangle = 0.$$

Then,

$$\lim_{n \to \infty} \langle \mathcal{F}(u_n, v_n),\ (u_n - u, v_n - v) \rangle = 0.$$

Therefore,

$$\lim_{n \to \infty} \langle f_p(u_n), u_n - u \rangle + \langle f_q(v_n), v_n - v \rangle = 0.$$

Since f_p and f_q are monotone,

$$\lim_{n \to \infty} \langle f_p(u_n), u_n - u \rangle = 0 \text{ and } \lim_{n \to \infty} \langle f_q(v_n), v_n - v \rangle = 0. \tag{3.4}$$

which means that

$$\lim_{n \to \infty} \mathcal{N}_1\Big(\mathcal{J}_1(u)\Big) \Big[\int_\Omega a_1(x, \nabla u_n) \nabla(u_n - u) + \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla(u_n - u) dx \Big] = 0, \tag{3.5}$$
$$\lim_{n \to \infty} \mathcal{N}_2\Big(\mathcal{J}_2(v)\Big) \Big[\int_\Omega a_2(x, \nabla v_n) \nabla(v_n - v) + \int_\Omega |\nabla u_n|^{q(x)-2} \nabla v_n \nabla(v_n - v) dx \Big] = 0.$$

By (3.2), we infer that $\mathcal{J}_1(u_n)$ and $\mathcal{J}_2(v_n)$ are bounded.

As $\mathcal{N}_1$ is continuous, up to a subsequence there is $y, z \geq 0$ such that

$$\begin{aligned} \mathcal{N}_1(\mathcal{J}_1(u_n)) &\longrightarrow \mathcal{N}_1(y) \geq k_0 y^{l(x)-1} \qquad \text{as} \quad n \to \infty, \\ \mathcal{N}_2(\mathcal{J}_2(v_n)) &\longrightarrow \mathcal{N}_2(z) \geq m_0 z^{j(x)-1} \qquad \text{as} \quad n \to \infty. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we get

$$\lim_{n\to\infty} \int_\Omega a_1(x, \nabla u_n) \nabla(u_n - u) dx + \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx = 0.$$

Using the continuous embedding $W_0^{1,r(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$, we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx = 0 \text{ and } \lim_{n\to\infty} \int_\Omega |\nabla v_n|^{q(x)-2} \nabla v_n (\nabla v_n - \nabla v) dx = 0.$$

Then,

$$\lim_{n\to\infty} \int_\Omega a_1(x, \nabla u_n) \nabla(u_n - u) dx = 0. \text{ and } \lim_{n\to\infty} \int_\Omega a_2(x, \nabla v_n) \nabla(v_n - v) dx = 0.$$

In the light of Lemma 3.2, we obtain

$$(u_n, v_n) \longrightarrow (u, v) \quad \text{strongly in } \mathcal{X}^{1,p(x),q(x)}(\Omega),$$

which implies that $\mathcal{F}$ is of type (S_+).

□

Lemma 3.4. *Assume that assumptions (H_1) and (H_2) hold, then the operator*

$$\mathcal{S} : \mathcal{X}^{1,p(x),q(x)}(\Omega) \longrightarrow \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*, \text{ defined for all } (\varphi, \psi) \in \mathcal{X}^{1,p(x),q(x)}(\Omega) \text{ by}$$

$$\begin{aligned} \langle \mathcal{S}(u, v), (\varphi, \psi) \rangle = &-\lambda \int_\Omega h(x, u, \nabla u) \varphi dx - \kappa \int_\Omega g(x, v, \nabla v) \psi dx \\ &+ \int_\Omega \mathcal{Q}(x) |u|^{r_1(x)-2} u \varphi(x) dx + \int_\Omega \mathcal{O}(x) |v|^{r_2(x)-2} v \psi(x) dx, \end{aligned}$$

where $\lambda, \kappa \in \mathbb{R}$, is compact.

Proof. In order to prove this lemma, we proceed in three steps.
Step 1. Let us define the operator $\Psi : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to L^{p'(x),q'(x)}(\Omega)$ by

$$\Psi(u, v) := \big(\mathcal{Q}(x)|u|^{r_1(x)-2} u\,,\, \mathcal{O}(x)|v|^{r_2(x)-2} v\big),$$

that is for all $(\varphi, \psi) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ by

$$\langle \Psi(u, v), (\varphi, \psi) \rangle = \int_\Omega \mathcal{Q}(x)|u|^{r_1(x)-2} u \varphi dx + \int_\Omega \mathcal{O}(x)|v|^{r_2(x)-2} v \psi dx.$$

We will show that Ψ is bounded and continuous.

It is clear that Ψ is continuous. Next, we prove that Ψ is bounded. Let $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$. Since $r_1^+ < p^- < p(x)$ and $r_2^+ < q^- < q(x)$, then

$$\begin{aligned} |\Psi(u, v)|_{p'(x),q'(x)} &= |\mathcal{Q}(x)|u|^{r_1(x)-2} u|_{p'(x)} + |\mathcal{O}(x)|v|^{r_2(x)-2} v|_{q'(x)} \\ &\leq \varrho_{p'(x)}(\mathcal{Q}(x)|u|^{p(x)-2} u) + \varrho_{q'(x)}(\mathcal{O}(x)|v|^{q(x)-2} v) + 2 \\ &= \int_\Omega |\mathcal{Q}(x)|u|^{p(x)-2} u|^{p'(x)} dx + \int_\Omega |\mathcal{O}(x)|v|^{q(x)-2} v|^{q'(x)} dx + 2 \\ &\leq \int_\Omega |\mathcal{Q}(x)|^{p'(x)} |u|^{p(x)} dx + \int_\Omega |\mathcal{O}(x)|^{q'(x)} |v|^{q(x)} dx + 2 \\ &\leq \|\mathcal{Q}^{p'+}\|_\infty \varrho_{p(x)}(u) + \|\mathcal{O}^{q'+}\|_\infty \varrho_{q(x)}(v) + 2 \\ &\leq C_1 \Big(|u|_{p(x)}^{p+} + |u|_{p(x)}^{p-} + |v|_{q(x)}^{q+} + |v|_{q(x)}^{q-} \Big) \\ &\leq C_2 \Big(\|u\|_{1,p(x)}^{p+} + \|u\|_{1,p(x)}^{p-} + \|v\|_{1,q(x)}^{q+} + |v\|_{1,q(x)}^{q-} \Big), \end{aligned}$$

where $C_1, C_2 > 0$ are independent of u, v. Consequently, Ψ is bounded on $\mathcal{X}^{1,p(x),q(x)}(\Omega)$.
Step 2. Let us define the operator $\varsigma : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to L^{p'(x),q'(x)}(\Omega)$ by

$$\varsigma(u,v) := \big(-\lambda h(x,u,\nabla u), -\kappa g(x,v,\nabla v)\big),$$

that is for $(\varphi,\psi) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, by

$$\langle \varsigma(u,v),(\varphi,\psi)\rangle = -\lambda \int_\Omega h(x,u,\nabla u)\varphi dx - \kappa \int_\Omega g(x,v,\nabla v)\psi dx.$$

We will show that ς is bounded. Let $(u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, then

$$\begin{aligned}
|\varsigma(u,v)|_{p'(x),q'(x)} &\leq |\lambda h(x,u,\nabla u)|_{p'(x)} + |\kappa g(x,v,\nabla v)|_{q'(x)}\\
&= \int_\Omega |\lambda h(x,u,\nabla u)|^{p'(x)}dx + \int_\Omega |\kappa g(x,v,\nabla v)|^{q'(x)}dx + 2\\
&\leq \big(|\lambda|^{p+} + |\lambda|^{p-}\big) \int_\Omega \Big|\mu\big(\gamma(x) + |u|^{r_1(x)-1} + |\nabla u|^{r_1(x)-1}\big)\Big|^{p'(x)} dx\\
&+ \big(|\kappa|^{q+} + |\kappa|^{q-}\big) \int_\Omega \Big|\alpha\big(e(x) + |v|^{r_2(x)-1} + |\nabla v|^{r_2(x)-1}\big)\Big|^{q'(x)} dx\\
&\leq C_1 \int_\Omega (|u|^{p(x)} + |\nabla u|^{p(x)})dx + C_2 \int_\Omega (|v|^{q(x)} + |\nabla v|^{q(x)})dx\\
&\leq C_3\Big(\|u\|^{p+}_{1,p(x)} + \|u\|^{p-}_{1,p(x)}\Big) + C_4\Big(\|v\|^{q+}_{1,q(x)} + \|v\|^{q-}_{1,q(x)}\Big)\\
&\leq C_5\Big(\|u\|^{p+}_{1,p(x)} + \|u\|^{p-}_{1,p(x)} + \|v\|^{q+}_{1,q(x)} + \|v\|^{q-}_{1,q(x)}\Big),
\end{aligned}$$

where $C_1, ..., C_5 > 0$ are independent of u and v. Therefore, ς is bounded.

Next, we show that ς is continuous. Let $(u_n, v_n) \to (u,v)$ in $\mathcal{X}^{1,p(x),q(x)}(\Omega)$ then,$(u_n,v_n) \to (u,v)$ in $L^{p(x),q(x)}(\Omega)$ and $(\nabla u_n, \nabla v_n) \to (\nabla u, \nabla v)$ in $(L^{p(x),q(x)}(\Omega))^N$. Then

$$\begin{aligned}
\|\varsigma(u_n,v_n) - \varsigma(u,v)\|_{p'(x),q'(x)} &= \|\lambda\big(f(x,u_n,\nabla u_n) - f(x,u,\nabla u)\big)\|_{p'(x)}\\
&+ \|\kappa\big(h(x,v_n,\nabla v_n) - h(x,v,\nabla v)\big)\|_{q'(x)}.
\end{aligned}$$

First, we prove that

$$\lim_{n\to+\infty} \|\lambda\big(h(x,u_n,\nabla u_n) - h(x,u,\nabla u)\big)\|_{p'(x)} = 0.$$

By Proposition 2.4, it is equivalent to prove that

$$\lim_{n\to+\infty} \varrho_{p'(x)}\Big(\lambda\big(h(x,u_n,\nabla u_n) - h(x,u,\nabla u)\big)\Big) = 0.$$

Since $u_n \to u$ in $L^{p(x),q(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in$(L^{p(x),q(x)}(\Omega))^N$. Then, there exist a subsequence still denoted by (u_n) and δ in $L^{p(x)}$ and Υ in $(L^{p(x)}(\Omega))^N$ such that

$$u_n(x) \to u(x) \quad \text{and} \quad \nabla u_n(x) \to \nabla u(x), \tag{3.7}$$

$$|u_n(x)| \leq \delta(x) \quad \text{and} \quad |\nabla u_n(x)| \leq \Upsilon(x), \tag{3.8}$$

for almost every $x \in \Omega$ and all $n \in \mathbb{N}$. Thus, from assumption (H_1) and (3.7), we have

$$h(x,u_n,\nabla u_n) \to h(x,u,\nabla u)\,\text{as}\,n\to\infty, \;\text{ for almost every } x \in \Omega,$$

by (3.8) and (H_1), we can deduce

$$|h(x,u_n(x),\nabla u_n(x))| \leq \mu(\gamma(x) + |\delta(x)|^{p(x)-1} + |\Upsilon(x)|^{p(x)-1}),$$

for almost every $x \in \Omega$ and for all $n \in \mathbb{N}$. Taking into account that

$$\gamma(x) + |\delta(x)|^{p(x)-1} + |\Upsilon(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

by applying Lebesgue's theorem, we have

$$\lim_{n\to+\infty} \varrho_{p'(x)}\Big(\lambda h(x, u_n, \nabla u_n) - \lambda h(x, u, \nabla u)\Big) = 0.$$

The same reasoning is used to prove that

$$\lim_{n\to+\infty} \varrho_{q'(x)}\Big(\kappa g(x, v_n, \nabla v_n) - \kappa g(x, v, \nabla v)\Big) = 0.$$

We conclude that ζ is continuous.

Step 3. Since the embedding $i : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to L^{p(x),q(x)}(\Omega)$ is compact, then the adjoint operator $i^* : L^{p'(x),q'(x)}(\Omega) \to \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*$ is also compact. Hence, the compositions $i^* \circ \Psi : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*$ and $i^* \circ \zeta : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*$ are compact, that means $\mathcal{S} = i^* \circ \Psi + i^* \circ \zeta$ is compact. With this last step the proof of Lemma 3.4 is completed. □

Our main result is the following existence theorem.

Theorem 3.1. *Assume that assumptions* (M_1)-(M_5) *and* (H_1),(H_2) *are satisfied. Then problem* (1.1), *admits at least one weak solution* (u, v) *in* $\mathcal{X}^{1,p(x),q(x)}(\Omega)$.

Proof. The couple $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ is a weak solution of (1.1) if and only if

$$\mathcal{F}(u, v) = -\mathcal{S}(u, v), \tag{3.9}$$

where $\mathcal{F}, \mathcal{S}$ are defined as in Lemmas 3.3 and 3.4, respectively by

$$\begin{aligned} &\mathcal{F} : \mathcal{X}^{1,p(x),q(x)}(\Omega) \longrightarrow \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^* \\ &\langle \mathcal{F}(u, v), (\varphi, \psi)\rangle = \langle f_p u, \varphi\rangle + \langle f_q v, \psi\rangle, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{S} : \mathcal{X}^{1,p(x),q(x)}(\Omega) \longrightarrow \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^* \\ &\langle \mathcal{S}(u, v), (\varphi, \psi)\rangle = -\lambda \int_\Omega h(x, u, \nabla u)\varphi dx - \kappa \int_\Omega g(x, v, \nabla v)\psi dx \\ &\qquad + \int_\Omega \mathcal{Q}(x)|u|^{r_1(x)-2} u\varphi(x) dx + \int_\Omega \mathcal{O}(x)|v|^{r_2(x)-2} v\psi(x) dx. \end{aligned}$$

By Lemma 3.3, the operator $\mathcal{F}$ is continuous, bounded, strictly monotone and of class (S_+), therefore, by the Minty-Browder Theorem (see [30]), the inverse operator

$$\mathcal{T} := \mathcal{F}^{-1} : (\mathcal{X}^{1,p(x),q(x)}(\Omega))^* \to \mathcal{X}^{1,p(x),q(x)}(\Omega),$$

$$\mathcal{T}(\phi, \psi) = (T_p\phi, T_q\psi),$$

is also bounded, continuous, strictly monotone, and of class (S_+). The operator $\mathcal{T}$ is such that

$$\mathcal{T}(\phi, \psi) = (u, v) \text{ if and only if } \quad (\phi, \psi) = \mathcal{F}(u, v).$$

Consequently, following Zeidler's terminology [30], equation (3.9) is equivalent to the following abstract Hammerstein equation

$$(u, v) = \mathcal{T}(\phi, \psi) \text{ and } (\phi, \psi) + \mathcal{S} \circ \mathcal{T}(\phi, \psi) = 0, \tag{3.10}$$

for all $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ and $(\phi, \psi) \in (\mathcal{X}^{1,p(x),q(x)}(\Omega))^*$. To say that a couple $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ is a solution to (3.9) is equivalent to say that (ϕ, ψ) is a dual solution of (3.10). Then to solve (3.9) it suffices to solve (3.10), and we will apply the Berkovits topological degree introduced in Section 2.2. To do this, we, first, claim that the set

$$B := \Big\{(\phi,\psi) \in (\mathcal{X}^{1,p(x),q(x)}(\Omega))^* : \exists\, t \in [0,1] \ \text{ such that } \ (\phi,\psi) + t\mathcal{S}o\mathcal{T}(\phi,\psi) = 0\Big\}.$$

is bounded. To verify this, we show that the set $\Big\{\mathcal{T}(\phi,\psi) \mid\, (\phi,\psi) \in B\Big\}$ is bounded. Indeed, taking into account that

$$\|\mathcal{T}(\phi,\psi)\|_{1,p(x),q(x)} = \|(u,v)\|_{1,p(x),q(x)} = \|\nabla u\|_{p(x)} + \|\nabla v\|_{q(x)}.$$

We denote $\mathtt{D} = \mathcal{X}^{1,p(x),q(x)}(\Omega) \cap \mathcal{T}(B)$ and define

$$\begin{aligned}
\mathtt{D}_1 &= \Big\{(u,v) \in \mathtt{D}\Big|\, 1 \geq \|\nabla u\|_{p(x)},\ \|\nabla v\|_{q(x)}\Big\},\\
\mathtt{D}_2 &= \Big\{(u,v) \in \mathtt{D}\Big|\, 1 < \|\nabla u\|_{p(x)},\ \|\nabla v\|_{q(x)}\Big\},\\
\mathtt{D}_3 &= \Big\{(u,v) \in \mathtt{D}\Big|\, 1 < \|\nabla u\|_{p(x)} \text{ and } \|\nabla v\|_{q(x)} < 1\Big\},\\
\mathtt{D}_4 &= \Big\{(u,v) \in \mathtt{D}\Big|\, 1 > \|\nabla u\|_{p(x)} \text{ and } \|\nabla v\|_{q(x)} > 1\Big\}.
\end{aligned}$$

Then we have the following cases:
First case. If $(u,v) \in \mathtt{D}_1$, then $\|\mathcal{T}(\phi,\Psi)\|_{1,p(x),q(x)}$ is bounded by definition.
Second case. If $(u,v) \in \mathtt{D}_2$, we deduce from (2.2), (M_2)-(M_3) that

$$\begin{aligned}
\|\mathcal{T}(\phi,\psi)\|_{1,p(x),q(x)} &\leq \|\nabla u\|_{p(x)}^{p-} + \|\nabla v\|_{q(x)}^{q-} \leq \varrho_{p(x)}(\nabla u) + \varrho_{q(x)}(\nabla v)\\
&\leq \frac{1}{\sigma}\int_\Omega |\nabla u|^{p(x)}dx + \frac{1}{\iota}\int_\Omega |\nabla v|^{q(x)}dx\\
&\leq \int_\Omega a_1(x,\nabla u)\nabla u\, dx + \int_\Omega a_2(x,\nabla v)\nabla v\, dx\\
&\leq max\{\frac{1}{\sigma},\frac{1}{\iota}\}\ \langle \mathcal{F}(u,v),\ (u,v)\rangle_{1,p(x),q(x)}\\
&= -t\ max\{\frac{1}{\sigma},\frac{1}{\iota}\}\ \langle \mathcal{S}o\mathcal{T}(\phi,\psi),\ \mathcal{T}(\phi,\psi)\rangle_{1,p(x),q(x)}.
\end{aligned}$$

Moreover, by assumptions (H_1)-(H_2), Young's inequality and bearing (2.7), (2.6) in mind, we obtain

$$\begin{aligned}
\|\mathcal{T}(\phi,\psi)\|_{1,p(x),q(x)} &\leq C_1\Big(\int_\Omega \lambda f(x,u,\nabla u)u dx + \int_\Omega \kappa h(x,v,\nabla v)v dx + \int_\Omega \lambda \mathcal{Q}(x)|u|^{r_1(x)}dx\\
&\quad + \int_\Omega \kappa\, \mathcal{O}(x)|v|^{r_2(x)}dx\Big)\\
&\leq C_2\Big[\varrho_{p(x)}(u) + \varrho_{q(x)}(v) + \int_\Omega \mu(\gamma(x) + |u|^{r_1(x)-1} + |\nabla u|^{r_1(x)-1})u\, dx\\
&\quad + \int_\Omega \alpha(e(x) + |v|^{r_2(x)-1} + |\nabla v|^{r_2(x)-1})v\, dx\Big]\\
&\leq C_3\Big[\rho_{p(x)}(u) + \rho_{q(x)}(v) + \int_\Omega \gamma(x)u dx + \int_\Omega |u|^{p(x)-1}u\, dx + \int_\Omega |\nabla u|^{p(x)-1}u\, dx\\
&\quad + \int_\Omega e(x)v\, dx + \int_\Omega |\nabla v|^{q(x)-1}v\, dx + \int_\Omega |v|^{q(x)-1}v\, dx\Big]\\
&\leq C_4\Big[|u|_{p(x)}^{p-} + |u|_{p(x)}^{p+} + |v|_{q(x)}^{q-} + |v|_{q(x)}^{q+} + |\gamma|_{p'(x)}|u|_{p(x)} + |e|_{q'(x)}|v|_{q(x)}\\
&\quad + C_5\, \varrho_{p(x)}(\nabla u) + C_6\, \varrho_{p(x)}(u) + C_7\, \varrho_{q(x)}(\nabla v) + C_8\, \varrho_{q(x)}(v)\Big]\\
&\leq C_9\big[\|u\|_{p(x)}^{p-} + \|u\|_{p(x)}^{p+} + \|v\|_{q(x)}^{q-} + \|v\|_{q(x)}^{q+}\big],
\end{aligned}$$

where $C_1, ..., C_9 > 0$ are independent of u, v. Hence, $\|\mathcal{T}(\phi, \psi)\|_{1,p(x),q(x)}$ is bounded.
Third case. If $(u, v) \in \mathtt{D}_3$, then

$$\begin{aligned}\|\mathcal{T}(\phi, \psi)\|_{1,p(x),q(x)} &= \|\nabla u\|_{p(x)} + \|\nabla v\|_{q(x)} \\ &\leq \|\nabla u\|_{p(x)}^{p-} + 1 \leq 1 + \|\nabla u\|_{p(x)}^{p-} + \|\nabla v\|_{q(x)}^{q+} \\ &\leq \varrho_{p(x)}(\nabla u) + \varrho_{q(x)}(\nabla v) + 1.\end{aligned}$$

From here, we proceed in the same manner as in the prior case to arrive at the conclusion that $\|\mathcal{T}(\phi, \psi)\|_{1,p(x),q(x)}$ is bounded.
Fourth case. Similarly to the previous case, if $(u, v) \in \mathtt{D}_4$ inversing the positions of u and v, we get that $\Big\{\mathcal{T}(\phi, \psi) : (\phi, \psi) \in B\Big\}$ is bounded.

On the other hand, we have that the operator $\mathcal{S}$ is bounded. Thus, thanks to (3.10), we have that the set B is bounded in $(\mathcal{X}^{1,p(x),q(x)}(\Omega))^*$. Consequently, there exists $R > 0$ such that

$$\|(\phi, \psi)\|_{1,p'(x),q'(x)} < R \quad \text{for all} \quad (\phi, \psi) \in B.$$

Hence, it follows that

$$(\phi, \psi) + t\mathcal{S} \circ \mathcal{T}(\phi, \psi) \neq 0 \quad \text{for all} \quad (\phi, \psi) \in \partial B_R(0) \text{ and } t \in [0, 1].$$

Moreover, $\mathcal{S}$ is compact, then it is known that $\mathcal{S}$ is continuous, quasimonotne and by Lemma 2.1, we conclude that

$$I + \mathcal{S} \circ \mathcal{T} \in \mathcal{F}_{\mathcal{T}}(\overline{B_R(0)}) \quad \text{and} \quad I = \mathcal{F} \circ \mathcal{T} \in \mathcal{F}_{\mathcal{T}}(\overline{B_R(0)}).$$

Since $I, \mathcal{S}$ and $\mathcal{T}$ are bounded, then

$$I + \mathcal{S} \circ \mathcal{T} \in \mathcal{F}_{\mathcal{T},B}(\overline{B_R(0)}) \quad \text{and} \quad I = \mathcal{F} o \mathcal{T} \in \mathcal{F}_{\mathcal{T},B}(\overline{B_R(0)}).$$

Consequently, the homotopy

$$\begin{aligned}\mathcal{H} : [0, 1] \times \overline{B_R(0)} &\to (\mathcal{X}^{1,p(x),q(x)}(\Omega))^* \\ (t, \phi, \psi) &\mapsto \mathcal{H}(t, \phi, \psi) := (\phi, \psi) + t\mathcal{S} \circ \mathcal{T}(\phi, \psi)\end{aligned}$$

is such that $\mathcal{H} \in \mathcal{F}_{\mathcal{T},B}(\overline{B_R(0)})$, and thanks to the homotopy invariance and normalization property of the degree d, seen in Theorem 2.1, we obtain

$$d(I + \mathcal{S} \circ \mathcal{T}, B_R(0), 0) = d(I, B_R(0),\ 0) = 1 \neq 0,$$

which implies that there exists $(\phi, \psi) \in B_R(0)$ satisfying the equality

$$(\phi, \psi) + \mathcal{S} \circ \mathcal{T}(\phi, \psi) = 0.$$

Finally, we conclude that $(u, v) = \mathcal{T}(\phi, \psi)$ is a weak solutions of (1.1). □

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INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS IN PERIODIC MORREY SPACES

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Abstract. A detailed exposition of Bernstein's inequality, inequalities of different metrics and of different dimensions for trigonometic polynomials in Lebesgue spaces is given in the book of S.M. Nikol'skii [4]. In this paper, we state analogues of these inequalities in peridoic Morrey spaces.

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1 Introduction

Definition 1. Let $n \in \mathbb{N},\ \mu \in \mathbb{N}_0$. Let $\mathfrak{M}^*_\mu(\mathbb{R}^n)$ denote the set all real valued trigonometric polynomials of order less than or equal to μ:

$$T_\mu(x) = T_\mu(x_1, \dots, x_n) = \sum_{\substack{-\mu \le k_j \le \mu \\ j=1,\dots,n}} c_{k_j} e^{ik\cdot x} \tag{1.1}$$

$$= \sum_{-\mu \le k_1 \le \mu} \cdots \sum_{-\mu \le k_n \le \mu} c_{k_1,\dots,k_n} e^{i(k_1 x_1 + \dots k_n x_n)}.$$

where $x_1, \dots, x_n \in \mathbb{R},\ c_{k_1,\dots,k_n} \in \mathbb{C}$ are constant coefficients such that $c_{-k} = \bar{c}_k$ (hence $T_\mu(x) \in \mathbb{R}$ for any $x \in \mathbb{R}^n$).

Let hereafter $0 < p \le \infty$. A function $f \in L^*_p$ if it is 2π-periodic Lebesgue measurable and

$$\|f\|^*_{L_p} = \|f\|_{L_p(Q(0,\pi))} < \infty, \tag{1.2}$$

where $Q(x,r) = \{y \in \mathbb{R}^n : |x_j - y_j| < r, j = 1, \dots, n\}$.

In book [9] the following inequalities are proven for trigonometric polynomials $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$, where the space $\mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$ is $\mathfrak{M}^*_\mu(\mathbb{R}^n)$ equiped with the quasinorm $\|\cdot\|^*_{L_p}$.

1. (Bernstein's inequality) Let $1 \le p \le \infty$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$

$$\left\|\frac{\partial T_\mu}{\partial x_j}\right\|^*_{L_p} \le \mu \|T_\mu\|^*_{L_p}, \quad j = 1, \dots, n. \tag{1.3}$$

2. (Inequality of different metrics) Let $1 \le p < q \le \infty$, then for any trigonometric polynomials $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$

$$\|T_\mu\|^*_{L_q} \le 3^n \mu^{n(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{L_p}. \tag{1.4}$$

3. (Inequality of different dimensions) Let $1 \le p \le \infty$, $1 \le m < n$, $x = (u, v)$, $u = (x_1, \dots, x_m) \in \mathbb{R}^m$, $v = (x_{m+1}, \dots, x_n) \in \mathbb{R}^{n-m}$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$

$$\Big\| \|T_\mu(u,v)\|_{L_{\infty,v}(\mathbb{R}^{n-m})} \Big\|^*_{L_{p,u}} \le 3^{n-m} \mu^{\frac{n-m}{p}} \|T_\mu\|^*_{L_p}, \tag{1.5}$$

in particular,

$$\|T_\mu(u,0)\|^*_{L_p} \le 3^{n-m} \mu^{\frac{n-m}{p}} \|T_\mu\|^*_{L_p}. \tag{1.6}$$

The purpose of this work is to present similar inequalities in which the space L^*_p is replaced by the periodic Morrey space $(M_p^\lambda)^*$.

Note also that Bernstein's inequality, inequalities of different metrics and different dimensions for entire functions of exponential type for the spaces $L_p(\mathbb{R}^n)$ were proved by S.M. Nikolsky [9], and for the Morrey spaces in the works [2], [3]

2 Morrey spaces

The spaces $M_p^\lambda(\mathbb{R}^n)$, now called Morrey spaces, were first considered by Charles Morrey [8] in connection with the study of the regularity of solutions of partial differential equations.

Definition 2. Let $0 < p \le \infty$ and $0 \le \lambda \le \frac{n}{p}$, then $f \in M_p^\lambda(\mathbb{R}^n)$, if $f \in L_p^{loc}(\mathbb{R}^n)$ and

$$\|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty, \tag{2.1}$$

where $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$.

Periodic analogues $(M_p^\lambda)^*(\mathbb{R}^n)$ of the Morrey space were considered in [10]

Definition 3. Let $0 < p \le \infty$ and $0 \le \lambda \le \frac{n}{p}$, then $f \in (M_p^\lambda)^*(\mathbb{R}^n)$, if it has period 2π, is Lebesgue measurable on $\mathbb{R}^n$ and

$$\|f\|^*_{M_p^\lambda} = \sup_{x\in Q(0,\pi)} \sup_{0<r\le\pi} r^{-\lambda} \|f\|_{L_p(Q(x,r))} < \infty. \tag{2.2}$$

We note some properties of these spaces.

1. It is immediately clear from the definition that for $\lambda = 0$

$$\|f\|^*_{M_p^0} = \|f\|^*_{L_p}.$$

2. For $\lambda = \frac{n}{p}$

$$\|f\|^*_{M_p^{\frac{n}{p}}} = \|f\|^*_{L_\infty},$$

3. If $\lambda < 0$ or $\lambda > \frac{n}{p}$, then the spaces $(M_p^\lambda)^*(\mathbb{R}^n)$ consist only of functions equivalent to 0 on $Q(0, \pi)$.
4. Note that the space $(M_p^\lambda)^*(\mathbb{R}^n)$ has the property of monotonicity with respect to the parameter λ:

$$(M_p^\lambda)^* \subset (M_p^\mu)^*,\ 0 \le \mu < \lambda \le \frac{n}{p},\ 0 < p < \infty \tag{2.3}$$

and

$$\|f\|^*_{M_p^\mu} \le \pi^{\lambda-\mu}\|f\|^*_{M_p^\lambda}. \tag{2.4}$$

In particular, for $\mu = 0$

$$(M_p^\lambda)^*(\mathbb{R}^n) \subset (L_p)^*(\mathbb{R}^n)$$

and

$$\|f\|^*_{L_p} \le \pi^{\lambda}\|f\|^*_{M_p^\lambda}. \tag{2.5}$$

5. In [4] it is proven that for any $f \in (M_p^\lambda)^*$

$$\|f\|^*_{M_p^\lambda} = \|f\|^{**}_{M_p^\lambda} \equiv \sup_{x\in\mathbb{R}^n} \sup_{0<r\le\pi} r^{-\lambda}\|f\|^*_{L_p(Q(x,r))}. \tag{2.6}$$

6. (Shift invariance) For any $f \in (M_p^\lambda)^*$

$$\|f(y+h)\|^*_{M_p^\lambda} = \|f(y)\|^*_{M_p^\lambda} \quad \forall h \in \mathbb{R}^n. \tag{2.7}$$

3 Inequalities for trigonometric polynomials in periodic Morrey spaces

3.1 Bernstein's inequality

In the one-dimensional case, the interpolation formula for an arbitrary trigonometric polynomial T_μ of order $\mu > 0$ has the form (see [9]):

$$T'_\mu(x) = \frac{1}{4\mu}\sum_{k=1}^{2\mu}(-1)^{k+1}\frac{1}{\sin^2\frac{x_k}{2}}T_\mu(x+x_k), \tag{3.1}$$

where x_k are the zeros of the polynomial $\cos(nx)$.

If $T_\mu(x) = \sin(\mu x)$ and $x = 0$, then we get

$$\mu = \frac{1}{4\mu}\sum_{k=1}^{2\mu}\frac{1}{\sin^2\frac{x_k}{2}}. \tag{3.2}$$

Theorem 3.1. *Let Z^* be a normed space of 2π-periodic functions in each variable, and let $\|\cdot\|^*_Z$ be a shift invariant norm, i.e. for any function $f \in Z^*$*

$$\|f(x+h)\|^*_Z = \|f\|^*_Z \quad \forall h \in \mathbb{R}^n. \tag{3.3}$$

Then for any trigonometric polynomials $T_\mu \in Z^*(\mathbb{R}^n)$

$$\left\|\frac{\partial T_\mu}{\partial x_j}\right\|^*_Z \le \mu\|T_\mu\|^*_Z,\ j = 1,\dots,n. \tag{3.4}$$

The proof is based on representation (3.1).

Corollary 3.1. *Let $1 \le p \le \infty$, $0 \le \lambda \le \frac{n}{p}$, then for any trigonometric polynomial $T_\mu \in (M_p^\lambda)^*$*

$$\left\|\frac{\partial T_\mu}{\partial x_j}\right\|^*_{M_p^\lambda} \le \mu\|T_\mu\|^*_{M_p^\lambda}, \quad j = 1,\dots,n. \tag{3.5}$$

3.2 Inequality of different metrics

Definition 4. Let $1 \le p \le \infty$, $0 \le \lambda \le \frac{n}{p}$, $r > 0$, $\mu, N \in \mathbb{N}$, $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$ and

$$((T_\mu))^*_{M^\lambda_{p,N}} = \sup_{x\in Q(0,\pi)} \sup_{0<r\le\pi} r^{-\lambda}\Bigg(\Big(\frac{r}{N}\Big)^n \sum_{k_1=-N}^{N-1} \cdots \sum_{k_n=-N}^{N-1}$$

$$\left|T_\mu\Big(x_1 + \frac{r}{N}k_1, \ldots, x_n + \frac{r}{N}k_n\Big)\right|^p\Bigg)^{1/p}.$$

Lemma 3.1. *Let $1 \le p \le \infty$, $n, \mu, N \in \mathbb{N}$, $0 \le \lambda \le \frac{n}{p}$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$*

$$\|T_\mu\|^*_{M^\lambda_p} \le ((T_\mu))^*_{M^\lambda_{p,N}} \le (1 + \frac{\pi}{N}\mu)^n \|T_\mu\|^*_{M^\lambda_p}. \tag{3.6}$$

Lemma 3.2. *Let $1 \le p \le q \le \infty$ $n, \mu, N \in \mathbb{N}$, $0 \le \lambda \le \frac{n}{q}$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$*

$$((T_\mu))^*_{M^{\lambda-n(\frac{1}{p}-\frac{1}{q})}_{q,N}} \le N^{n(\frac{1}{p}-\frac{1}{q})}((T_\mu))^*_{M^\lambda_{p,N}}. \tag{3.7}$$

Theorem 3.2. *Let $1 \le p \le q \le \infty$, $n\big(\frac{1}{p}-\frac{1}{q}\big) \le \lambda \le \frac{n}{p}$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$*

$$\|T_\mu\|^*_{M_q^{\lambda-n(\frac{1}{p}-\frac{1}{q})}} \le (1+\pi)^n \mu^{n(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{M^\lambda_p}. \tag{3.8}$$

Consider the convolution of functions $\varphi, g \in L_1(Q(0,\pi))$ $2\pi-$ periodic in each variable

$$(\varphi * g)(x) = \int_{Q(0,\pi)} \varphi(x-y)g(y)dy,\ x \in \mathbb{R}^n. \tag{3.9}$$

Recall that $\forall k \in \mathbb{Z}^n$

$$c_k(\varphi * g) = (2\pi)^n c_k(\varphi)c_k(g). \tag{3.10}$$

If $c_k(\varphi) = (2\pi)^{-n}$ then

$$c_k(g) = c_k(\varphi * g). \tag{3.11}$$

Lemma 3.3. *Let $n \in \mathbb{N}$, $\mu \in \mathbb{N}, \varphi \in L_1(Q(0,\pi))$ be a 2π-periodic trigonometric polynomial in each variable. In order for any trigonometric polynomial T_μ of order μ to satisfy the equality*

$$T_\mu = \varphi * T_\mu, \tag{3.12}$$

it is necessary and sufficient condition that

$$c_k(\varphi) = (2\pi)^{-n}\ \forall k \in \mathbb{Z}^n : |k_j| \le \mu,\ j = 1, \ldots, n. \tag{3.13}$$

Definition 5. (Dirichlet kernel) Let

$$D_\mu(x) = \frac{1}{2}\sum_{k=-\mu}^{\mu} e^{ikx} = \frac{1}{2} + \sum_{k=1}^{\mu}\cos(kx) = \frac{\sin(\mu+\frac{1}{2})x}{2\sin\frac{x}{2}} \tag{3.14}$$

and

$$\widetilde{D}_\mu(x) = \frac{1}{\pi}D_\mu(x). \tag{3.15}$$

Note that

$$\|\widetilde{D}_\mu\|^*_{L_2} = \sqrt{\frac{2\mu+1}{2\pi}} \tag{3.16}$$

and

$$\|\widetilde{D}_\mu\|^*_{L_\infty} = \frac{2\mu+1}{2\pi}. \tag{3.17}$$

From equalities (3.16) and (3.17) it follows that for any $2 < p < \infty$

$$\|\widetilde{D}_\mu\|^*_{L_p} \le \left(\frac{2\mu+1}{2\pi}\right)^{1-\frac{1}{p}}. \tag{3.18}$$

A special case of equality (3.12) is the well-known equality

$$T_\mu(x) = \widetilde{D}_\mu(x) * T_\mu(x).$$

Remark 1. If φ is a trigonometric polynomial of order μ in each variable, then equality (3.12) holds for any trigonometric polynomials T_μ of order μ in each variable if and only if

$$\varphi(x) = \frac{1}{(2\pi)^n} \sum_{\substack{|k_j|\le\mu \\ j=1,\dots,n}} e^{ik\cdot x} = \frac{1}{(2\pi)^n} \prod_{j=1}^n \sum_{|k_j|\le\mu} e^{ik_jx_j} = \frac{1}{\pi^n} \prod_{j=1}^n D_\mu(x_j) = \prod_{j=1}^n \widetilde{D}_\mu(x_j).$$

Remark 2. Let $\alpha, n \in \mathbb{N}$

$$\Delta_\alpha(j) = \{k \in \mathbb{Z}^n, |k_j| \le \alpha\}$$

and

$$\Delta_\alpha = \Delta_\alpha(1) \times \cdots \times \Delta_\alpha(n).$$

If φ is a trigonometric polynomial of order $\nu > \mu$ in each variable, then equality (3.12) holds for any trigonometric polynomials T_μ of order μ in each variable if and only if

$$\varphi(x) = \sum_{k\in\Delta_\nu} c_k e^{ik\cdot x} = \prod_{j=1}^n \widetilde{D}_\mu(x_j) + \sum_{k\in\Delta_\nu\setminus\Delta_\mu} c_k e^{ik\cdot x}. \tag{3.19}$$

$\Big($In particular, for $n = 1$ $\varphi(x) = \widetilde{D}_\mu(x) + \big(\sum_{k=-\nu}^{-\mu-1} + \sum_{k=\mu+1}^{\nu}\big) c_k e^{ik\cdot x}.\Big)$

Definition 6. Let, for $\mu \in \mathbb{N}$, J^*_μ denote the set of all 2π-periodic functions $\varphi \in L_1(Q(0,\pi))$, satisfying condition (3.13) (hence, having form (3.19) for some $\nu \in \mathbb{N}, \nu \ge \mu$).

According to Lemma 3.3 for such functions φ equality (3.12) holds.

Definition 7. (see [10]) Let $\mu, \nu \in \mathbb{N}$ and $\nu > \mu$. The Vallee Poussin kernels are defined as follows:

$$\mathfrak{V}_{\mu,\nu}(x) = (\nu-\mu)^{-1} \sum_{l=\mu}^{\nu-1} D_l(x), \ x \in \mathbb{R}, \tag{3.20}$$

in particular,

$$\mathfrak{V}_\mu(x) = \mathfrak{V}_{\mu,2\mu}(x), \quad \mu \ge 1, \quad \mathfrak{V}_0(x) = 1, \ x \in \mathbb{R}. \tag{3.21}$$

Remark 3. For $\nu > \mu$ we represent the Dirichlet kernel as

$$D_\nu(x) = \frac{1}{2} + \cos x + \cdots + \cos\mu x + (\cos(\mu+1)x + \cdots + \cos\nu x) \tag{3.22}$$

$$= D_\mu(x) + D_{\mu,\nu}(x), \tag{3.23}$$

where

$$D_{\mu,\nu}(x) = \sum_{l=\mu+1}^{\nu} \cos lx. \tag{3.24}$$

Then for $\nu > \mu + 1$

$$\mathfrak{V}_{\mu,\nu}(x) = D_\mu(x) + \frac{1}{\nu-\mu}\sum_{l=\mu+1}^{\nu-1} D_{\mu,l}(x). \tag{3.25}$$

Let us put

$$\widetilde{\mathfrak{V}}_{\mu,\nu}(x) = \frac{1}{\pi}\mathfrak{V}_{\mu,\nu}(x), \quad \widetilde{D}_{\mu,\nu}(x) = \frac{1}{\pi}D_{\mu,\nu}(x), \tag{3.26}$$

then

$$\widetilde{\mathfrak{V}}_{\mu,\nu}(x) = \widetilde{D}_\mu(x) + \frac{1}{\nu-\mu}\sum_{l=\mu+1}^{\nu-1} \widetilde{D}_{\mu,l}(x). \tag{3.27}$$

in particular,

$$\widetilde{\mathfrak{V}}_{\mu}(x) = \widetilde{D}_\mu(x) + \frac{1}{\mu}\sum_{l=\mu+1}^{2\mu-1} \widetilde{D}_{\mu,l}(x). \tag{3.28}$$

A special case of equality (3.12) is the equality

$$T_\mu(x) = \widetilde{\mathfrak{V}}_{\mu,\nu}(x) * T_\mu(x), \tag{3.29}$$

in particular,

$$T_\mu(x) = \widetilde{\mathfrak{V}}_{\mu}(x) * T_\mu(x).$$

Remark 4. Note that

$$\widetilde{D}_\mu(x),\ \widetilde{\mathfrak{V}}_{\mu,\nu},\ \nu > \mu,\ \widetilde{\mathfrak{V}}_\mu \in J_\mu^* \tag{3.30}$$

Theorem 3.3 (see, for example, [10]). *Let $\mu \in \mathbb{N}$, $1 \le p \le \infty$, then*

$$\|\widetilde{\mathfrak{V}}_\mu\|_{L_p}^* \le 3^n \mu^{n(1-1/p)}. \tag{3.31}$$

Theorem 3.4. *(Corollary of the Young-type inequality for periodic Morrey spaces, see* [4] *)*

Let

$$0 \le \lambda < \frac{n}{p}, 1 \le r, p < q \le \infty, \quad 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p},$$

$f_1 \in L_r(\mathbb{R}^n)$ *and* $f_2 \in (M_p^\lambda)^*$. *Then*

$$\|f_1 * f_2\|^*_{M_q^{\frac{p\lambda}{q}}} \le \|f_1\|^*_{L_r}(\|f_2\|^*_{M_p^\lambda})^{\frac{p}{q}}(\|f_2\|^*_{L_p})^{1-\frac{p}{q}}. \tag{3.32}$$

Theorem 3.5. *Let $1 \le r, p < q \le \infty$, $n, \mu \in \mathbb{N}$ $0 \le \lambda \le \frac{n}{p}, 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$. Then*

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le c(\|T_\mu\|^*_{M_p^\lambda})^{\frac{p}{q}}(\|T_\mu\|^*_{L_p})^{1-\frac{p}{q}} \tag{3.33}$$

for any $T_\mu \in (M_p^\lambda)^*$, where

$$c = c(\mu, r) = \inf_{\varphi \in J_\mu^*} \|\varphi\|^*_{L_r}. \tag{3.34}$$

Corollary 3.2. *Let $1 \le p \le q \le \infty$, $n, \mu \in \mathbb{N}$ $0 \le \lambda \le \frac{n}{p}$, then for any $T_\mu \in (M_p^\lambda)^*$*

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le 3^n \mu^{n(\frac{1}{p}-\frac{1}{q})}(\|T_\mu\|^*_{M_p^\lambda})^{\frac{p}{q}}(\|T_\mu\|^*_{L_p})^{1-\frac{p}{q}}. \tag{3.35}$$

Inequality (3.35) follows from inequalities (3.31) and (3.33) since $\widetilde{\mathfrak{V}}_\mu \in J_\mu^*$ and in (3.33) $c \le \|\widetilde{\mathfrak{V}}_\mu\|^*_{L_r}$.

Corollary 3.3. *If $1 \le p \le 2$, $q \ge \frac{2p}{2-p}$, then for any $T_\mu \in L_p^*$*

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le \left(\frac{2\mu+1}{2\pi}\right)^{n(\frac{1}{p}-\frac{1}{q})} (\|T_\mu\|^*_{M_p^\lambda})^{\frac{p}{q}} (\|T_\mu\|^*_{L_p})^{1-\frac{p}{q}}, \tag{3.36}$$

in particular, for $0 \le \lambda \le \frac{n}{2}$

$$\|T_\mu\|^*_{LM_2^{\frac{\lambda}{2}}} \le \left(\frac{2\mu+1}{2\pi}\right)^{\frac{n}{2}} (\|T_\mu\|^*_{LM_1^\lambda} \|T_\mu\|^*_{L_1})^{\frac{1}{2}}, \tag{3.37}$$

and

$$\|T_\mu\|^*_{L_\infty} \le \left(\frac{2\mu+1}{2\pi}\right)^{\frac{n}{2}} \|T_\mu\|^*_{L_2}. \tag{3.38}$$

Inequality (3.38) *follows from inequalities* (3.31), (3.33) *and* (3.16) *since $\widetilde{D}_\mu \in J_\mu^*$ and in* (3.33) *$c \le \|\widetilde{D}_\mu\|^*_{L_2}$. In the last inequality the constant is sharp, the equality is attained for $T_\mu(x) = \prod_{l=1}^n \widetilde{D}_\mu(x_l)$. Regarding generalizations, see* [7].

Corollary 3.4. *By inequality* (2.5) *inequalities* (3.33)-(3.36) *imply that*

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le c\pi^{\lambda(1-\frac{p}{q})} \|T_\mu\|^*_{M_p^\lambda}, \tag{3.39}$$

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le 3^n \pi^{\lambda(1-\frac{p}{q})} \mu^{n(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{M_p^\lambda}, \tag{3.40}$$

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le \left(\frac{2\mu+1}{2\pi}\right)^{n(\frac{1}{p}-\frac{1}{q})} \pi^{\lambda(1-\frac{p}{q})} \|T_\mu\|^*_{M_p^\lambda}. \tag{3.41}$$

Remark 5. Inequality (3.35) is a periodic analogue of the inequality of different metrics for entire functions of exponential type (see [2],[3]).

Remark 6. Note that inequailies (2.4) and (3.40) imply that

$$\|T_\mu\|^*_{M_q^{\lambda-n(\frac{1}{p}-\frac{1}{q})}} \le \pi^{(\lambda p-n)(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{M_q^{\frac{\lambda p}{q}}} \le 3^n (\pi\mu)^{n(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{M_p^\lambda}. \tag{3.42}$$

So, inequality (3.40) has a better exponent $\frac{p\lambda}{q}$ compared with the exponent $\lambda - n(\frac{1}{p} - \frac{1}{q})$ in (3.8). However, for some values of λ, p, q the constant $(1+\pi)^n$ in (3.8) is better than the constant $3^n \pi^{\lambda(1-\frac{p}{q})}$ in (3.40).

3.3 Inequality of different dimensions

Definition 8. Let

$$0 < p_1, p_2 \le \infty, \quad m_1, m_2 \in \mathbb{N}$$
$$0 \le \lambda_1 \le \frac{m_1}{p_1}, \quad 0 \le \lambda_2 \le \frac{m_2}{p_2}.$$

Let us define the space

$$(M_{p_1}^{\lambda_1})^*(\mathbb{R})^{m_1} \times (M_{p_2}^{\lambda_2})^*(\mathbb{R}^{m_2}) \tag{3.43}$$

with a mixed quasinorm as the set of all measurable functions f on $\mathbb{R}^{m_1+m_2}$ for which

$$\|T_\mu\|^*_{M_{p_1}^{\lambda_1}(\mathbb{R}^{m_1}) \times M_{p_2}^{\lambda_2}(\mathbb{R}^{m_2})} = \left\| \|T_\mu(u_1,u_2)\|^*_{M_{p_1,u_1}^{\lambda_1}(\mathbb{R}^{m_1})} \right\|^*_{M_{p_2,u_2}^{\lambda_2}(\mathbb{R}^{m_2})}$$
$$= \sup_{y \in Q_{m_2}(0,\pi)} \sup_{0<\rho\le\pi} \rho^{-\lambda_2} \| \sup_{x \in Q_{m_1}(0,\pi)} \sup_{0<r\le\pi} r^{-\lambda_1} \|T_\mu(u_1,u_2)\|_{L_{p_1,u_1}(Q(x,r))} \|_{L_{p_2,u_2}(Q(x,r))}, \tag{3.44}$$

where $Q_{m_1}(0,\pi) = \{u_1 \in \mathbb{R}^{m_1} : |u_{1j}| < \pi, j = 1, \ldots, m_1\}$ and $Q_{m_2}(0,\pi)$ is defined similarly.

Let us note some properties of these spaces.

Lemma 3.4. *Let* $0 < p \le \infty,\ m_1, m_2 \in \mathbb{N}, 0 < \lambda_1 \le \frac{m_1}{p}, 0 < \lambda_2 \le \frac{m_2}{p},\ f_1 \in (M_p^{\lambda_1})^*(\mathbb{R}^{m_1})\ f_2 \in (M_p^{\lambda_2})^*(\mathbb{R}^{m_2})$ $f_1 \sim 0$ *on* $\mathbb{R}^{m_2}$ $f_2 \sim 0$ *on* $\mathbb{R}^{m_1}$, *then*

$$\|f_1 f_2\|^*_{M_p^{\lambda_1}(\mathbb{R}^{m_1})\times M_p^{\lambda_2}(\mathbb{R}^{m_2})} = \|f_1\|^*_{M_p^{\lambda_1}(\mathbb{R}^{m_1})}\|f_2\|^*_{M_p^{\lambda_2}(\mathbb{R}^{m_2})} \tag{3.45}$$

Lemma 3.5. *Let* $0 < p \le \infty,\ m_1, m_2 \in \mathbb{N},\ 0 \le \lambda_1 \le \frac{m_1}{p},\ 0 \le \lambda_2 \le \frac{m_2}{p}$. *Then*

$$(M_p^{\lambda_1})^*(\mathbb{R}^{m_1}) \times (M_p^{\lambda_2})^*(\mathbb{R}^{m_2}) \subset (M_p^{\lambda_1+\lambda_2})^*(\mathbb{R}^{m_1+m_2}), \tag{3.46}$$

and

$$\|f\|^*_{M_p^{\lambda_1+\lambda_2}(\mathbb{R}^{m_1+m_2})} \le \|f\|^*_{M_p^{\lambda_1}(\mathbb{R}^{m_1})\times M_p^{\lambda_2}(\mathbb{R}^{m_2})} \tag{3.47}$$

for any $f \in (M_p^{\lambda_1})^*(\mathbb{R}^{m_1}) \times (M_p^{\lambda_2})^*(\mathbb{R}^{m_2})$.

If $0 < \lambda_1 + \lambda_2 < \frac{m_1+m_2}{p}$, *then inclusion* (3.46) *is strict.*

Theorem 3.6. *Let* $1 \le p < \infty,\ m, n \in \mathbb{N},\ m < n,\ 0 \le \lambda \le \frac{n}{p}$, *then*

$$\|T_\mu\|^*_{L_\infty(\mathbb{R}^{n-m})\times M_p^{\lambda}(\mathbb{R}^m)} \le 3^{n-m}\mu^{\frac{n-m}{p}}\|T_\mu\|^*_{L_{p,v}(\mathbb{R}^{n-m})\times M_p^{\lambda}(\mathbb{R}^m)}, \tag{3.48}$$

in particular, if $x = (u, v),\ u = (x_1 \ldots x_m),\ v = (x_{m+1}, \ldots, x_n)$, *then*

$$\|T_\mu(u,0)\|^*_{M_p^{\lambda}(\mathbb{R}^m)} \le 3^{n-m}\mu^{\frac{n-m}{p}}\|T_\mu\|^*_{L_p(\mathbb{R}^{n-m})\times M_p^{\lambda}(\mathbb{R}^m)}. \tag{3.49}$$

Remark 7. If $\lambda = 0$, then it is obvious that

$$L_p^*(\mathbb{R}^{n-m}) \times (M_p^0)^*(\mathbb{R}^m) = L_p^*(\mathbb{R}^{n-m}) \times L_p^*(\mathbb{R}^m) = L_p^*(\mathbb{R}^n) \tag{3.50}$$

however, for $0 < \lambda \le \frac{m}{p}$ according to Lemma 3.5

$$L_p^*(\mathbb{R}^{n-m}) \times (M_p^{\lambda})^*(\mathbb{R}^m) \subset (M_p^{\lambda})^*(\mathbb{R}^n), \tag{3.51}$$

but

$$L_p^*(\mathbb{R}^{n-m}) \times (M_p^{\lambda})^*(\mathbb{R}^m) \ne (M_p^{\lambda})^*(\mathbb{R}^n). \tag{3.52}$$

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Events

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INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF ANALYSIS, DIFFERENTIAL EQUATIONS AND ALGEBRA" DEDICATED TO THE 15TH ANNIVERSARY OF THE EURASIAN MATHEMATICAL JOURNAL

(JANUARY 7-11, 2025, ASTANA, KAZAKHSTAN)

FIRST INFORMATION LETTER

Dear Colleagues!

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The Eurasian Mathematical Journal (hereinafter EMJ) was founded in 2009 by the L.N. Gumilyov Eurasian National University (hereinafter ENU) with the assistance of the M.V. Lomonosov Moscow State University (hereinafter MSU), the RUDN University (hereinafter RUDN) and the University of Padua (Italy).

Starting from 2018, the journal is published jointly by ENU and RUDN.

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The work of the Conference will consist of plenary and sectional talks, and poster presentations. Selected papers by participants will be published in the EMJ.

The official languages of the conference are Kazakh, Russian and English. It is planned to publish electronic versions of abstracts of talks before the beginning of the Conference.

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11/01/2025 - the day of departure.

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Best regards,

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