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KHARIN STANISLAV NIKOLAYEVICH

(to the 85th birthday)

On December 4, 2023 Doctor of Physical and Mathematical Sciences, Academician of the National Academy of Sciences of the Republic of Kazakhstan, member of the editorial board of the Eurasian Mathematical Journal Stanislav Nikolaevich Kharin turned 85 years old.

Stanislav Nikolayevich Kharin was born in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis "Heat phenomena in electrical contacts and associated singular integral equations", and in 1990 his doctoral thesis "Mathematical models of thermo-physical processes in electrical contacts" in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. For these outstanding achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research as evidenced by his scientific publications in high-ranking journals with his students in recent years.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Stanislav Nikolayevich on the occasion of his 85th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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CORRECT AND COERCIVE SOLVABILITY CONDITIONS FOR A DEGENERATE HIGH ORDER DIFFERENTIAL EQUATION

R.D. Akhmetkaliyeva, T.D. Mukasheva, K.N. Ospanov

Communicated by Ya.T. Sultanaev

Key words: degenerate fifth-order differential equation, unbounded coefficient, generalized solution, correct solvability, coercive estimate.

AMS Mathematics Subject Classification: 35J70.

Abstract. In the work, we consider a fifth-order singular differential equation with variable coefficients. The singularity means, firstly, that the equation is given on the real axis $\mathbb{R} = (-\infty, \infty)$, and secondly, its coefficients are unbounded functions. We study a new degenerate case, when the intermediate coefficients of the equation grow faster than the lowest coefficient (potential), and also the potential is not sign-definite. We obtain sufficient conditions for the existence and uniqueness of the generalized solution of the equation. We also prove a coercive estimate for the solution. The coefficients of the equation are assumed to be smooth, but we do not impose any restrictions on their derivatives to prove the results. Note that the well-known stationary Kawahara equation can be reduced to the considered equation after linearization.

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1 Introduction

The Kawahara equation or the generalized Korteweg-de Vries equation describes the propagation of one-dimensional nonlinear waves in a dispersive medium. It was presented for the first time in the paper [7] and is written as follows

$$-\beta y_x^{(5)}(x, t) + \alpha y_x^{(3)}(x, t) + \frac{3}{2} y(x, t) y_x'(x, t) + y_t'(x, t) = 0, \tag{1.1}$$

where α, β are real numbers. Boundary value problems for equation (1.1) were studied in many works (see [4,5,16]).

In [11], a Kawahara-type equation with variable coefficients defined in a non-compact domain was studied. Conditions for the solvability of the Cauchy problem were obtained. The authors of [11] assumed that the coefficients of the equation are bounded. Related references can also be found in [11]. A natural continuation of these studies is the study of Kawahara-type equations with unbounded coefficients. The present work is devoted to this case.

Let us consider the following stationary Kawahara-type equation

$$-y^{(5)} + r_0(x)\, y^{(3)} + q_0(x)\, y y' = f_0(x), \tag{1.2}$$

where $x \in \mathbb{R}$. As a result of the linearization of (1.2), we will get one of the following two differential equations

$$-y^{(5)} + r_1(x)\, y^{(3)} + q_1(x)\, y = f_1(x) \tag{1.3}$$

and

$$-y^{(5)} + r_2\left(x\right) y^{(3)} + q_2\left(x\right) y' = f_2\left(x\right). \tag{1.4}$$

Equation (1.3) with a positive potential $q_1\left(x\right)$ was considered in [9] (see also references therein). It should be noted that the problem of the correctness of (1.3) with sign-variable $q_1(x)$, as well as of equation (1.4), remained open.

In the present work, we will study the following linear equation

$$L_0 y \equiv -y^{(5)} + r\left(x\right) y^{(3)} + q\left(x\right) y' + p\left(x\right) y = f\left(x\right), \tag{1.5}$$

where $x \in \mathbb{R}$, $f(x) \in L_2(\mathbb{R})$, r is a three times continuously differentiable function, q is a continuously differentiable function, and $p(x)$ is a continuous but not a sign-constant function. Equation (1.5) generalizes both (1.3) and (1.4). Using some modifications of methods of [9, 10, 13], we prove sufficient conditions for the correct solvability of equation (1.5) and the fulfillment of the following the so-called maximal regularity estimate

$$\left\|y^{(5)}\right\|_2 + \left\|ry^{(3)}\right\|_2 + \left\|qy'\right\|_2 + \|py\|_2 \le c\|f\|_2 \tag{1.6}$$

for the solution y, where $c > 0$ depends only on r, q, p. Here $\|\cdot\|_2$ is the norm in $L_2(\mathbb{R})$.

Note that in [1, 8-15, 17], the stationary singular differential equations of the second and high orders with intermediate coefficients were studied.

Let

$$L_0 y = -y^{(5)} + r\left(x\right) y^{(3)} + q\left(x\right) y' + p\left(x\right) y$$

be a differential operator defined on the set $C_0^{(5)}\left(\mathbb{R}\right)$ of all five times continuously differentiable functions with compact support. Due to the conditions imposed on the functions $r(x)$, $q(x)$ and $p(x)$, the operator L_0 can be closed by the norm of $L_2(\mathbb{R})$. We denote its closure by L.

Definition 1. A function $y \in D\left(L\right)$ satisfying the equality $Ly = f$ is called a solution to the differential equation (1.5).

Let g and $h \neq 0$ be real-valued continuous functions. We introduce the following notations:

$$\alpha_{g,h,j}\left(x\right) = \left(\int_0^x g^2\left(t\right) dt\right)^{\frac{1}{2}} \left(\int_x^{+\infty} t^{2j} h^{-2}\left(t\right) dt\right)^{\frac{1}{2}} \ (x > 0),$$

$$\beta_{g,h,j}\left(\tau\right) = \left(\int_\tau^0 g^2\left(t\right) dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\tau} t^{2j} h^{-2}\left(t\right) dt\right)^{\frac{1}{2}} \ (\tau < 0),$$

$$\gamma_{g,h,j} = \max\left(\sup_{x>0}\alpha_{g,h,j}\left(x\right), \sup_{\tau<0}\beta_{g,h,j}\left(\tau\right)\right) \ (j = 1,\ 2).$$

Lemma 1.1. [3]. *If the functions g, h satisfy the condition $\gamma_{g,h,j} < \infty$ $(j = 1,\ 2)$, then for each $y \in C_0^{(j+1)}\left(\mathbb{R}\right)$ the following inequality*

$$\int_{-\infty}^{+\infty} \left|g\left(x\right) y\left(x\right)\right|^2 dx \le \frac{2}{j}\gamma_{g,h,j} \int_{-\infty}^{+\infty} \left|h\left(x\right) y^{(j+1)}\left(x\right)\right|^2 dx \tag{1.7}$$

holds.

2 Auxiliary statements

We consider the differential operator $l_0y = -y^{(5)} + r(x)\,y^{(3)}$ defined on the set $C_0^{(5)}(\mathbb{R})$. By l we denote the closure of l_0 in $L_2(\mathbb{R})$. Consider the following equation

$$ly = -y^{(5)} + r(x)\,y^{(3)} = h(x), \tag{2.1}$$

where $h \in L_2(\mathbb{R})$.

Definition 2. A function $y \in D(l)$ satisfying the equality $ly = f$ is called a solution to differential equation (2.1).

Lemma 2.1. *If the function $r(x)$ is three times continuously differentiable and*

$$r \geq 1, \gamma_{1,\sqrt{r},2} < \infty,$$

then for each $f \in L_2(\mathbb{R})$, there exists a unique solution y of equation (2.1). *Moreover, for the solution y the following estimate holds*

$$\left\|\sqrt{r}y^{(3)}\right\|_2 + \|y\|_2 \leq (\gamma_{1,\sqrt{r},2} + 1)\,\|ly\|_2\,. \tag{2.2}$$

Proof. Let $y \in C_0^{(5)}(\mathbb{R})$. Integrating by parts we get

$$\left(l_0y, y^{(3)}\right) = -\int_{\mathbb{R}} y^{(5)}\bar{y}^{(3)}dx + \int_{\mathbb{R}} r\left|y^{(3)}\right|^2 dx = \left\|y^{(4)}\right\|_2^2 + \left\|\sqrt{r}y^{(3)}\right\|_2^2 \geq \left\|\sqrt{r}y^{(3)}\right\|_2^2.$$

By condition $r \geq 1$ and Hölder's inequality, we have

$$\left|\left(l_0y,\, y^{(3)}\right)\right| \leq \left(\int_{\mathbb{R}}\left|\frac{1}{\sqrt{r}}ly\right|^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left|\sqrt{r}y^{(3)}\right|^2 dx\right)^{\frac{1}{2}}.$$

It follows from the last two inequalities that

$$\left\|\sqrt{r}y^{(3)}\right\|_2 \leq \left\|\frac{1}{\sqrt{r}}l_0y\right\|_2. \tag{2.3}$$

By inequality (1.7) in Lemma 1.1, (2.3) implies that

$$\|y\|_2 \leq \gamma_{1,\sqrt{r},2}\left\|\sqrt{r}y^{(3)}\right\|_2 \leq \gamma_{1,\sqrt{r},2}\|l_0y\|_2. \tag{2.4}$$

From inequalities (2.3) and (2.4) we obtain that

$$\|y\|_2 + \left\|\sqrt{r}y^{(3)}\right\|_2 \leq (\gamma_{1,\sqrt{r},2} + 1)\,\|l_0y\|_2\,, y \in C_0^{(5)}(R)\,. \tag{2.5}$$

Now we show that estimate (2.5) holds for any $y \in D(l)$. For $y \in D(l)$ there exists a sequence $\{y_n\}_{n=1}^{\infty} \subseteq C_0^{(5)}(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - ly\|_2 \to 0$ $(n \to \infty)$. By (2.5),

$$\|y_n\|_2 + \left\|\sqrt{r}y_n^{(3)}\right\|_2 \leq (\gamma_{1,\sqrt{r},2} + 1)\,\|l_0y_n\|_2 \;\; (n = 1,\, 2,\, ...). \tag{2.6}$$

Therefore, the inequality

$$\|y_n - y_m\|_2 + \left\|\sqrt{r}\left(y_n^{(3)} - y_m^{(3)}\right)\right\|_2 \leq \left[\gamma_{1,\sqrt{r},2} + 1\right]\|l_0y_n - l_0y_m\|_2 \to 0$$

holds for any natural numbers n and m. Let $\dot{W}^3_{2,r}(\mathbb{R})$ be the closure of $C_0^{(3)}(\mathbb{R})$ with respect to the norm $\|y\|_W = \left\|\sqrt{r}y^{(3)}\right\|_2 + \|y\|_2$. The sequence $\{y_n\}_{n=1}^{\infty}$ converges to $y \in \dot{W}^3_{2,r}(\mathbb{R})$. Passing to the limit in (2.6), we obtain that estimate (2.2) holds for $y \in D(l)$.

Inequality (2.2) shows that there exists the inverse operator l^{-1} to the operator l. By (2.2) and Definition 2, the solution of equation (2.1) is unique (if exists).

Now we will show the solvability of equation (2.1). If we denote $y^{(3)} = z$ and $\mathfrak{I}z = -z'' + r(x)z$, then equation (2.1) takes the following form

$$\mathfrak{I}z = -z'' + r(x)z = h(x).$$

It follows from estimate (2.2) that $D(\mathfrak{I}) \subseteq L_2(\mathbb{R})$. By Definition 2, it suffices to show that $R(\Im) = L_2(\mathbb{R})$. Assume the contrary: $R(\Im) \neq L_2(\mathbb{R})$. Due to inequality (2.2), the set $R(\Im)$ is closed, so there exists a nonzero element $v \in L_2(\mathbb{R})$ and $v \perp R(\Im)$ [18]. It is easy to check that

$$\Im^* v = -v'' + r(x)v = 0,$$

where $\Im^*$ is the adjoint operator to $\Im$. By condition $r \geq 1$ and known properties of the Sturm-Liouville equation, $v \notin L_2(\mathbb{R})$. We obtain a contradiction that shows $R(\Im) = L_2(\mathbb{R})$. □

Remark 1. The condition $r \geq 1$ in Lemma 2.1 can be replaced by the inequality $r \geq \delta > 0$. To check of this fact, it is enough to make the substitution $x = \frac{1}{\sqrt{\delta}}t$ in (2.1), where the condition $r \geq \delta > 0$ is fulfilled.

Lemma 2.2. *Let a function r satisfy the conditions of Lemma 2.1 and*

$$C^{-1} \leq \frac{r(x)}{r(\eta)} \leq C, \forall x, \eta \in \mathbb{R} : |x - \eta| \leq 1, \tag{2.7}$$

where $C > 1$. Then the following estimate

$$\left\|y^{(5)}\right\|_2 + \left\|ry^{(3)}\right\|_2 \leq C_1 \|ly\|_2 \tag{2.8}$$

holds for the solution y of equation (2.1). Here C_1 depends only on r.

Proof. Let y be the solution of equation (2.1). By estimate (2.2), we have $y^{(3)} \in L_2(\mathbb{R})$. If we denote $y^{(3)} = z$, then equation (2.1) takes the following form:

$$\mathfrak{I}z = -z'' + r(x)z = h(x).$$

It is known that the solution z of last equation satisfies the following inequality

$$\left\|z^{(2)}\right\|_2 + \|rz\|_2 \leq C_1 \|f\|_2,$$

if condition $r \geq 1$ in Lemma 2.1 and (2.7) are fulfilled [8]. Since $y^{(3)} = z$, we obtain estimate (2.8) for the solution y of equation (2.1). □

3 Main result

Theorem 3.1. *Let a function $r(x)$ satisfy the conditions of Lemma 2.2, and functions $q(x)$ and $p(x)$ be such that*

$$\gamma_{q,r,1} < \infty, \gamma_{p,r,2} < \infty. \tag{3.1}$$

Then for any $f \in L_2(\mathbb{R})$ there exists a solution y of equation (1.5) and it is unique. Furthermore, the following estimate holds for the solution y:

$$\left\|y^{(5)}\right\|_2 + \left\|ry^{(3)}\right\|_2 + \|qy'\|_2 + \|py\|_2 \leq C_2 \|f\|_2, \tag{3.2}$$

where $C_2 > 0$ depends only on r, q, p.

Proof. Let us make the substitution $t = \frac{x}{a}$ in equation (1.5), where a is a positive number. If we denote $y\,(at) = \tilde{y}\,(t)$, $r\,(at) = \tilde{r}\,(t)$, $p\,(at) = \tilde{p}\,(t)$, $q\,(at) = \tilde{q}\,(t)$, $a^5 f\,(at) = \tilde{f}\,(at)$, then (1.5) takes the following form

$$\tilde{L}_a \tilde{y} = -\tilde{y}^{(5)} + a^2 \tilde{r} \tilde{y}^{(3)} + a^4 \tilde{q} \tilde{y}' + a^5 \tilde{p} \tilde{y} = \tilde{f}. \tag{3.3}$$

Let l_a be the closure in $L_2(\mathbb{R})$ of the differential operator

$$l_{0a} \tilde{y} = -\tilde{y}^{(5)} + a^2 \tilde{r} \tilde{y}^{(3)}, \tag{3.4}$$

defined in $C_0^{(5)}(\mathbb{R})$. Since $a^2 \tilde{r}\,(t) \geq \delta_0$ for some $\delta_0 > 0$, by Lemmas 2.1 and 2.2 and Remark 1, the operator l_a is continuously invertible and for any $\tilde{y} \in D\,(l_a)$ the following estimate holds

$$\left\| \tilde{y}^{(5)} \right\|_2 + \left\| a^2 \tilde{r} \tilde{y}^{(3)} \right\|_2 \leq C_{l_a} \left\| l_a \tilde{y} \right\|_2, \tag{3.5}$$

where C_{l_a} depends only on $\tilde{r}$ and a. It is easy to check that $\gamma_{\tilde{q},\tilde{r},1} = a^{-2} \gamma_{q,r,1}$ and $\gamma_{\tilde{p},\tilde{r},2} = a^{-3} \gamma_{p,r,2}$. By (3.5), condition (3.1) and Lemma 1.1, we have

$$\left\| a^4 \tilde{q} \tilde{y}' \right\|_2 \leq 2a^4 \gamma_{q,r,1} C_{l_a} \left\| l_a \tilde{y} \right\|_2 \tag{3.6}$$

and

$$\left\| a^5 \tilde{p} \tilde{y} \right\|_2 \leq a^4 \gamma_{p,r,2} C_{l_a} \left\| l_a \tilde{y} \right\|_2. \tag{3.7}$$

If we choose a number a such that $a \leq \frac{1}{\sqrt[4]{2(2\gamma_{q,r,1} + \gamma_{p,r,2}) C_{l_a}}}$, then by (3.6) and (3.7), we obtain

$$\left\| a^4 \tilde{q} \tilde{y}' \right\|_2 + \left\| a^5 \tilde{p} \tilde{y} \right\|_2 \leq \frac{1}{2} \| l_a \tilde{y} \|_2. \tag{3.8}$$

Then, according to the well-known theorem on small perturbation of a linear operator (see, for example, [6]), there exists the inverse $(\tilde{L}_a)^{-1}$ to the operator $\tilde{L}_a = l_a + a^4 \tilde{q} E + a^5 \tilde{p} E$ (E is the unit operator) and $(\tilde{L}_a)^{-1}$ is defined on all $L_2\,(\mathbb{R})$. By Definition 1, for each $f \in L_2(\mathbb{R})$ there exists a solution $\tilde{y}$ of equation (3.3) and it is unique.

From (3.5) and (3.8) we have

$$\left\| \tilde{y}^{(5)} \right\|_2 + \left\| a^2 \tilde{r} \tilde{y}^{(3)} \right\|_2 + \left\| a^4 \tilde{q} \tilde{y}' \right\|_2 + \left\| a^5 \tilde{p} \tilde{y} \right\|_2 \leq \left(C_{l_a} + \frac{1}{2} \right) \| l_a \tilde{y} \|_2. \tag{3.9}$$

By (3.8),

$$\| l_a \tilde{y} \|_2 \leq 2 \left\| \tilde{L}_a \tilde{y} \right\|_2.$$

Then (3.9) implies that

$$\left\| \tilde{y}^{(5)} \right\|_2 + \left\| a^2 \tilde{r} \tilde{y}^{(3)} \right\|_2 + \left\| a^4 \tilde{q} \tilde{y}' \right\|_2 + \left\| a^5 \tilde{p} \tilde{y} \right\|_2 \leq (2C_{l_a} + 1) \| \tilde{f} \|_2.$$

Taking into account that $x = at$ in the last inequality and passing to the variable x, we obtain that estimate (3.2) holds for the solution y of equation (1.5). □

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EQUIVALENT SEMI-NORMS FOR NIKOL'SKII-BESOV SPACES

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Abstract. The aim of this paper is to establish the equivalence of various semi-norms involving differences for Nikol'skii-Besov spaces on an interval.

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1 Introduction

We start with recalling the definitions of Niko'skii-Besov spaces $B^l_{p,\theta}(a,b)$ and semi-normed Nikol'skii-Besov spaces $b^l_{p,\theta}(a,b)$.

Definition 1. Let $l > 0$, $k \in \mathbb{N}$, $k > l$, $1 \le p$, $\theta \le \infty$, $\alpha_1 \ge 0$, $\alpha_2 \ge k$, and $-\infty \le a < b \le +\infty$. Then $f \in b^l_{p,\theta}(a,b)$ if f is measurable on (a,b) and the following semi-norm is finite:

$$\|f\|_{b^l_{p,\theta}(a,b)} = \left(\int_0^{\frac{b-a}{k}} \left(\frac{\|\Delta_h^k f\|_{L_p(a,b-kh)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}} \tag{1.1}$$

if $1 \le \theta < \infty$ and

$$\|f\|_{b^l_{p,\theta}(a,b)} = \sup_{h \in (0,\frac{b-a}{k})} \frac{\|\Delta_h^k f\|_{L_p(a,b-kh)}}{h^l}, \tag{1.2}$$

if $\theta = \infty$.

Moreover, $B^l_{p,\theta}(a,b) = b^l_{p,\theta}(a,b) \cap L_p(a,b)$ with the norm

$$\|f\|_{B^l_{p,\theta}(a,b)} = \|f\|_{L_p(a,b)} + \|f\|_{b^l_{p,\theta}(a,b)}.$$

Here $\Delta_h^k f$ is the difference of order k of f with step h:

$$(\Delta_h^k f)(x) = \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(x+mh).$$

Let for $\alpha_1 \ge 0$, $\alpha_2 \ge k$,

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} = \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k f\|_{L_p(a+\alpha_1 h,b-\alpha_2 h)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}} \tag{1.3}$$

if $1 \le \theta < \infty$ and

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} = \sup_{h\in(0,\frac{b-a}{\alpha_1+\alpha_2})} \|\Delta^k_h f\|_{L_p(a+\alpha_1 h, b-\alpha_2 h)} \tag{1.4}$$

if $\theta = \infty$.

Respectively,

$$\|f\|^{(1)}_{B^l_{p,\theta}(a,b)} = \|f\|_{L_p(a,b)} + \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}.$$

We shall prove the equivalence of (1.1) and (1.3), (1.2) and (1.4) respectively, for an arbitrary interval (a,b). We note the following results related to this statement.

The following theorem was proved in [5].

Theorem 1.1. *Let $l>0$, $k\in\mathbb{N}$, $k>l$, $1\le p, \theta\le\infty$, $0<\delta\le\infty$, $s\ge 2$, $a\in\mathbb{R}$, $0<\alpha<\infty$.*

Then there exists $c_1>0$, depending only on α, s, k and l, such that

$$\left(\int_0^\delta \left(\frac{\|\Delta^k_h f\|_{L_p(a,a+\alpha h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}$$

$$\le c_1 \sup_{\substack{m\in\mathbb{N}_0,\\ m\le k(s-1)-1}} \left(\int_0^{\frac{\delta}{s}} \left(\frac{\|\Delta^k_h f\|_{L_p(a+(\alpha+m)h, a+(s\alpha+m+1)h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \tag{1.5}$$

for all read-valued functions If f measurable on $(a, a+(k+\alpha)\delta)$ for which the left-hand side of this inequality is finite, in partcular, for all $f\in C^\infty([a, a+(k+\alpha)\delta])$.

Corollary 1.1. *If $s=2$ inequality (1.5) takes the form*

$$\left(\int_0^\delta \left(\frac{\|\Delta^k_h f\|_{L_p(a,a+\alpha h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}$$

$$\le c_2 \left(\int_0^{\frac{\delta}{2}} \left(\frac{\|\Delta^k_h f\|_{L_p(a+\alpha h, a+(2\alpha+k)h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}, \tag{1.6}$$

where $c_2>0$ depends only on α, k and l.

For differences of order one the following statement was proved in [10].

Theorem 1.2. *Let $1\le p,\theta\le\infty$, $l>0$, $a\in\mathbb{R}$, $0<\delta\le\infty$, $t>0$, $0\le b<c$, $T>0$, and $0\le B<C$.*

Then there exists $c_3>0$, depending only on t, b, c, T, B, C, and l, such that

$$\left(\int_0^\delta \left(\frac{\|\Delta_{th} f\|_{L_p(a+bh, a+ch)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}$$

$$\le c_3 \left(\int_0^{\frac{c+t}{B+T}\delta} \left(\frac{\|\Delta_{Th} f\|_{L_p(a+Bh, a+Ch)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \tag{1.7}$$

for all measurable functions $f:[a,\infty)\to\mathbb{R}$.

Corollary 1.2. *If $b = 0, c = \alpha, B = \alpha, C = \beta, 0 \le \alpha < \beta, t = T = 1$, inequality (1.7) takes the form*

$$\left(\int_0^\delta \left(\frac{\|\Delta_h f\|_{L_p(a,a+\alpha h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}$$

$$\le c_4 \left(\int_0^\delta \left(\frac{\|\Delta_h^k f\|_{L_p(a+\alpha h,a+\beta h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}. \tag{1.8}$$

where $c_4 > 0$ depends only on α, β and l.

The proof of the equivalences of (1.1) and (1.3), (1.2) and (1.4) will be based on Corollary 1.1, a general statement (Lemma 2.1) for semi-normed space, connected with application of the Banach theorem on the boundedness of an inverse operator, and the inclusion $b^l_{p,\theta}(a,b) \subset L_p(a,b)$, proved in [6].

2 Equivalent semi-norms

Theorem 2.1. *Let $l > 0$, $k \in \mathbb{N}$, $k > l$, $1 \le p$, $\theta \le \infty$, $\alpha_1 \ge 0$, $\alpha_2 \ge k$.*

Then for an arbitrary interval (a,b) the semi-norms $\|f\|_{b^l_{p,\theta}(a,b)}$ and $\|f\|^{(1)}_{b^l_{p,\theta}(a,b)}$ are equivalent. Moreover, there exists $c_5 > 0$ is depending only on l, k, p, θ, α_1 and α_2 such that

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} \le \|f\|_{b^l_{p,\theta}(a,b)} \le c_5 \|f\|^{(1)}_{b^l_{p,\theta}(a,b)} \tag{2.1}$$

for all $f \in b^l_{p,\theta}(a,b)$.

Lemma 2.1. *Let E_1, E_2 be semi-normed spaces with the corresponding semi-norms $\|.\|_{E_1}$, $\|.\|_{E_2}$, $E_1 \subset E_2$ and*

$$\theta_1 = \{g \in E_1 : \|g\|_{E_1} = 0\} = \theta_2 = \{g \in E_2 : \|g\|_{E_2} = 0\}.$$

Furthermore, let the space E_1 be complete with respect to the semi-norms $\|.\|_{E_1}$ and $\|.\|_{E_1} + \|.\|_{E_2}$.

Then there exists $c_6 > 0$ such that

$$\|f\|_{E_2} \le c_6 \|f\|_{E_1} \tag{2.2}$$

for all $f \in E_1$.

Proof. We consider the factor spaces

$$\tilde{E}_1 = E_1/\theta_1, \tilde{E}_2 = E_2/\theta_1 \text{ and } \tilde{E}_{12} = E_{12}/\tilde{\theta}_1,$$

where E_{12} is the space $E_1 \cap E_2 = E_1$, equipped with the semi-norms $\|.\|_{E_{12}} = \|.\|_{E_1} + \|.\|_{E_2}$.

By the definitions of a factors-space and corresponding semi-norm, $\tilde{E}_1$ is the set of all non-intersecting classes $\tilde{f}$ generated by elements $f \in E_1$: $\tilde{f} = \{f + g : g \in \theta_1\}$, and $\|\tilde{f}\|_{\tilde{E}_1} = \inf_{h \in \tilde{f}} \|h\|_{E_1}$.

Since θ_1 is the null-set of E_1, it follows that $\|\tilde{f}\|_{\tilde{E}_1} = \|f\|_{E_1} \ \forall \ f \in \tilde{f}$.

Since $\theta_2 = \theta_1$, $\tilde{E}_2 = E_2/\theta_2$ and similarly $\|\tilde{f}\|_{\tilde{E}_2} = \|f\|_{E_2} \ \forall \ f \in \tilde{f}$. Finally, for each $\tilde{f} \in \tilde{E}_{12}$

$$\|\tilde{f}\|_{\tilde{E}_{12}} = \|\tilde{f}\|_{\tilde{E}_1} + \|\tilde{f}\|_{\tilde{E}_2}. \tag{2.3}$$

We note that $\tilde{E}_1$ and $\tilde{E}_{12}$ are Banach spaces. Next we consider the identity operator

$$I : \tilde{E}_{12} \to \tilde{E}_1.$$

This operator is linear, continuous, and such that $\|\tilde{f}\|_{\tilde{E}_1} \leq \|\tilde{f}\|_{\tilde{E}_{12}}$. Moreover it bijectively maps $\tilde{E}_1$ onto $\tilde{E}_{12}$. By the theorem on boundedness of an inverse operator (corollary of the Banach theorem on an open map), the operator $I^{-1} : \tilde{E}_1 \to \tilde{E}_{12}$ is also continuous. Hence it is bounded, therefore there exists $M > 0$, such that

$$\|\tilde{f}\|_{\tilde{E}_{12}} \leq M\|\tilde{f}\|_{\tilde{E}_1}$$

which implies inequality (2.2). □

Remark 1. For the case of normed spaces E_1 and E_2 this statement is proved in book [7] (Theorem 2, pp. 268-269).

Proof of Theorem 2.1. Step 1. The inequality

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} \leq \|f\|_{b^l_{p,\theta}(a,b)}$$

being trivial, it is required to prove the right-hand side inequality of (2.1).

Assume that $-\infty < a < b < \infty$.

By applying Minkowski's inequality, we obtain for any $1 \leq p,\ \theta \leq \infty$(if $\theta = \infty$,then integrals should be replaced by appropriate supremums)

$$\begin{aligned}
\|f\|_{b^l_{p,\theta}(a,b)} \leq\ & \left(\int_{\frac{b-a}{\alpha_1+\alpha_2}}^{\frac{b-a}{k}} \left(\frac{\|\Delta^k_h f\|_{L_p(a,b-kh))}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& + \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta^k_h f\|_{L_p(a,b-kh)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& \leq \left(\int_{\frac{b-a}{\alpha_1+\alpha_2}}^{\frac{b-a}{k}} \left(\frac{\|\Delta^k_h f\|_{L_p(a,b-kh)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& + \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta^k_h f\|_{L_p(a,a+\alpha_1 h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& + \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta^k_h f\|_{L_p(a+\alpha_1 h,b-\alpha_2 h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& = \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta^k_h f\|_{L_p(b-\alpha_2 h,b-kh)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& \equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Since

$$\|\Delta^k_h f\|_{L_p(a,b-kh)} \leq 2^k\|f\|_{L_p(a,b)}$$

we have that for some $c_7 > 0$ independent of f

$$I_1 \leq c_7\|f\|_{L_p(a,b)}.$$

Let in Corollary 1.1, $\delta = \frac{b-a}{(\alpha_1+\alpha_2)}$. Then, since

$$(a + \alpha_1 h, a + (2\alpha_1 + k)h) \subset (a + \alpha_1 h, b - \alpha_2 h)$$

for $0 \le h \le \frac{b-a}{2(\alpha_1+\alpha_2)}$, we obtain

$$\begin{aligned} I_2 &\le c_1 \left(\int_0^{\frac{b-a}{2(\alpha_1+\alpha_2)}} \left(\frac{\|\Delta_h^k f\|_{L_p(a+\alpha_1 h, b-\alpha_2 h)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}} \\ &\le c_8 \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}, \end{aligned}$$

where $c_8 > 0$ depends only on α_1, α_2, k, l..

Next, $I_3 = \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}$.

To estimate I_4 we first note that

$$\|\Delta_h^k f\|_{L_p(b-\alpha_2 h, b-kh)} = \left\| \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(x+mh) \right\|_{L_p(b-\alpha_2 h, b-kh)}$$

$$= \left\| \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(b-kh+a-y+mh) \right\|_{L_p(a, a+(\alpha_2-k)h)}$$

$$= \left\| \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(a+b-(y+(k-m)h)) \right\|_{L_{p(a,a+(\alpha_2-k)h)}}$$

$$= \left\| \sum_{s=0}^{k} (-1)^{s} \binom{k}{k-s} f(a+b-(y+sh)) \right\|_{L_p(a, a+(\alpha_2-k)h)}$$

$$= \left\| \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} f(a+b-(y+(k-s)h)) \right\|_{L_p(a, a+(\alpha_2-k)h)}$$

$$= \left\| \Delta_h^k g \right\|_{L_p(a, a+(\alpha_2-k)h)},$$

where $g(x) = f(a+b-x)$. (We changed the variable $x = b-kh+a-y$ and the summation index $m = k-s$). Consequently

$$I_4 = \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k g\|_{L_p(a, a+(\alpha_2-k)h)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}}.$$

By Corollary 1.1, with $\delta = \frac{b-a}{\alpha_1+\alpha_2}$, we obtain

$$I_4 \le c_9 \left(\int_0^{\frac{b-a}{2(\alpha_1+\alpha_2)}} \left(\frac{\|\Delta_h^k g\|_{L_p(a+(\alpha_2-k)h, a+(2\alpha_2-k)h)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}},$$

where $c_9 > 0$ depends only on α_1, α_2, k, l..

By changing the variable $y = b-kh+a-x$ similarly to the above we get

$$\|\Delta_h^k g\|_{L_p(a+(\alpha_2-k)h, a+(2\alpha_2-k)h)}$$

$$= \|\Delta_h^k f(a+b-y)\|_{L_p(a+(\alpha_2-k)h, a+(2\alpha_2-k)h)}$$

$$= \|\Delta_h^k f\|_{L_p(b-2\alpha_2 h, b-\alpha_2 h)}.$$

Therefore,

$$I_4 \;\le\; c_9 \left(\int_0^{\frac{b-a}{2(\alpha_1+\alpha_2)}} \left(\frac{\|\Delta_h^k f\|_{L_p(b-2\alpha_2 h, b-\alpha_2 h)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}}.$$

Since

$$(b-2\alpha_2 h, b-\alpha_2 h) \subset (a+\alpha_1 h, b-\alpha_2 h)$$

for $0 \le h \le \frac{b-a}{2(\alpha_1+\alpha_2)}$, we have

$$\begin{aligned} I_4 \;&\le\; c_9 \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k f\|_{L_p(a+\alpha_1 h, b-\alpha_2 h)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}} \\ &= \; c_7 \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}. \end{aligned}$$

Finally, we get

$$\|f\|_{b^l_{p,\theta}(a,b)} \le c_{10}(\|f\|_{L_p(a,b)} + \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}) \tag{2.4}$$

where $c_{10} = \max(c_7, 1, c_8, c_9)$.

Inequality (2.4) immediately implies that $\|f\|_{B^l_{p,\theta}(a,b)}$ is equivalent to $\|f\|^{(1)}_{B^l_{p,\theta}(a,b)}$.

Step 2. Let $E_2 = b^l_{p,\theta}(a,b)$ and E_1 be the set of all function f measurable on (a,b) for which

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} < \infty.$$

If $f \in E_1$ then by the result in [3] it follows that $f \in L_p(a,b)$. Hence, by inequality (2.4) $f \in E_2$, so $E_1 \subset E_2$.

By Lemma 2.1 it follows that

$$\|f\|_{b^l_{p,\theta}(a,b)} \le c_{11} \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}, \tag{2.5}$$

where $c_{11} = c_{11}(a,b,p,\theta,l,k,\alpha_1,\alpha_2) > 0$ is independent of the function f.

Step 3. In fact, it follows that one can assume that in the inequality c_9 is also independent of a and b. Namely,

$$c_{11}(a,b,p,\theta,l,k,\alpha_1,\alpha_2) = c_{11}(0,1,p,\theta,l,k,\alpha_1,\alpha_2). \tag{2.6}$$

To prove this we note that, if $-\infty < a < b < +\infty$, then

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} = (b-a)^{\frac{1}{p}-l} \|g\|^{(1)}_{b^l_{p,\theta}(0,1)} \tag{2.7}$$

where $g(y) = f(a + y(b-a))$, $y \in (0,1)$. In particular, if $\alpha_1 = 0$ and $\alpha_2 = k$,

$$\|f\|_{b^l_{p,\theta}(a,b)} = (b-a)^{\frac{1}{p}-l} \|g\|_{b^l_{p,\theta}(0,1)}. \tag{2.8}$$

Indeed by substituting $y = \frac{x-a}{b-a}$ we get

$$\|\Delta_h^k f\|_{L_p(a+\alpha_1 h, b-\alpha_2 h)} = \left(\int\limits_{a+\alpha_1 h}^{b-\alpha_2 h} \left| \sum_{k=0}^{m} (-1)^{k-m} \binom{k}{m} f(x+mh) \right|^p dx \right)^{\frac{1}{p}}$$

$$= \left(\int\limits_{\frac{\alpha_1 h}{b-a}}^{1-\frac{\alpha_2 h}{b-a}} \left| \sum_{k=0}^{m} (-1)^{k-m} \binom{k}{m} f(a + y(b-a) + mh) \right|^p (b-a) dy \right)^{\frac{1}{p}}$$

$$= \left(\int\limits_{\frac{\alpha_1 h}{b-a}}^{1-\frac{\alpha_2 h}{b-a}} \left| \sum_{k=0}^{m} (-1)^{k-m} \binom{k}{m} g\left(y + \frac{mh}{b-a} \right) \right|^p dy \right)^{\frac{1}{p}} (b-a)^{\frac{1}{p}}$$

$$\left\| \Delta^k_{\frac{h}{b-a}} g \right\|_{L_p(\frac{\alpha_1 h}{b-a}, 1-\frac{\alpha_2 h}{b-a})} (b-a)^{\frac{1}{p}}.$$

Hence, by substituting $t = \frac{h}{b-a}$, we get

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} = \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta^k_{\frac{h}{b-a}} g\|_{L_p(\frac{\alpha_1 h}{b-a}, 1-\frac{\alpha_2 h}{b-a})}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}} (b-a)^{\frac{1}{p}}$$

$$\left(\int_0^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta^k_t g\|_{L_p(\alpha_1 t, 1-\alpha_2 t)}}{(t(b-a))^l} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}} (b-a)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}-l} \|g\|^{(1)}_{b^l_{p,\theta}(0,1)}.$$

By (2.8), (2.5) with $a = 0, b = 1$ and (2.7), we obtain

$$\begin{aligned} \|f\|_{b^l_{p,\theta}(a,b)} &= (b-a)^{\frac{1}{p}-l} \|g\|_{b^l_{p,\theta}(0,1)} \\ &\le c_{11}(0,1,p,\theta,l,k,\alpha_1,\alpha_2)(b-a)^{\frac{1}{p}-l} \|g\|^{(1)}_{b^l_{p,\theta}(0,1)} \\ &= c_{11}(0,1,p,\theta,l,k,\alpha_1,\alpha_2) \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}, \end{aligned}$$

which is inequality (2.5) with c_9 defined by (2.6) for the case of a finite interval.

Inequality (2.5) also holds with the same c_{11} for infinite interval which follows by passing in (2.5) to the limit.

□

Remark 2. It may be of interest to consider a similar problem for the Nikol'skii-Besov-Morrey spaces in the definition of which L_p-norms are replaced by the norms in Morrey spaces M_p^λ. The required information on Morrey spaces and Nikol'skii-Besov-Morrey spaces can be found in [2], [3], [4], [8] and [9].

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COMPARISON OF POWERS OF DIFFERENTIAL POLYNOMIALS

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Abstract. Necessary and sufficient conditions are obtained for a polynomial P to be more powerful then a polynomial Q. These conditions are formulated in terms of the orders of generalized-homogeneous sub-polynomials, corresponding to these polynomials, and the multiplicity of their zeros. Applying these results, conditions are obtained, under which a monomial ξ^ν for a certain set of multi-indices $\nu \in \Re^*$ can be estimated via terms of a given degenerate polynomial P.

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1 Introduction

Let $\mathbb{E}^n$ and $\mathbb{R}^n$ be n-dimensional Euclidean spaces of points (vectors) $x = (x_1, \cdots, x_n)$ and $\xi = (\xi_1, \cdots, \xi_n)$ respectively, $\mathbb{R}^{n,+} := \{\xi \in \mathbb{R}^n, \xi_j \geq 0,\ j = 1, \cdots, n\}$, $\mathbb{R}^{n,0} := \{\xi \in \mathbb{R}^n, \xi_1 \cdots \xi_n \neq 0\}$. Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}_0^n = \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ be the set of all $n-$dimensional multi-indices, i.e. the set of all points with non-negative integer coordinates $\{\alpha = (\alpha_1, ..., \alpha_n) : \ \alpha_i \in \mathbb{N}_0 \ (i = 1, ..., n)\}$.

For $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n : \ \lambda_j > 0 \ (j = 1, ..., n)$ and $\nu \in \mathbb{R}^{n,+}$ we denote $|\xi| := \sqrt{\xi_1^2 + \cdots + \xi_n^2}$, $|\xi, \lambda| := \sqrt{|\xi_1|^{2/\lambda_1} + \cdots + |\xi_n|^{2/\lambda_n}}$, $|\nu| := \nu_1 + ... + \nu_n$, $\xi^\nu := \xi_1^{\nu_1} \cdots \xi_n^{\nu_n}$, $|\xi^\nu| := |\xi_1|^{\nu_1} \cdots |\xi_n|^{\nu_n}$.

For $\alpha \in \mathbb{N}_0^n$, we denote $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_j = \frac{1}{i}\, \partial/\partial x_j$ or $D_j = \partial/\partial \xi_j$ $(j = 1, ..., n)$.

Let $\mathcal{A} = \{\nu^j = (\nu_1^j, \cdots, \nu_n^j)\}_{j=1}^M$ be a finite set of points $\nu^j \in \mathbb{R}^{n,+}$. By the Newton polyhedron (further, when it does not cause misunderstanding, we will briefly write N.P.) of the set $\mathcal{A}$ we mean the least convex hull (which is a polyhedron) $\Re = \Re(\mathcal{A})$ in $\mathbb{R}^n$, containing all points of $\mathcal{A}$ (see [23] or [33]).

A polyhedron $\Re$ with vertices in $\mathbb{R}^{n,+}$ is said to be complete if $\Re$ has a vertex at the origin of coordinates and further vertices on each coordinate axis of $\mathbb{R}^{n,+}$.

The $k-$dimensional faces of a polyhedron $\Re$ are denoted by $\Re_i^k$ $(i = 1, ..., M_k', k = 0, 1, ..., n-1)$. The faces of the N.P. (by definition) are closed sets.

The unit outward normal to a supporting hyper-plane of a polyhedron $\Re$, containing some face $\Re_i^k$ and not containing any other face of dimension greater than k, will be simply called the outward normal (or $\Re-$normal) of the face $\Re_i^k$. Thus, a given unit vector λ can serve as an outward normal to one (and only one) face of $\Re$. We denote by Λ_i^k the set of all outward normals of the face $\Re_i^k$ $(i = 1, ..., M_k', k = 0, 1, ..., n-1)$. Note that either the set Λ_i^k consists of one vector (when $k = n-1$), or it is an open set (when $0 \leq k < n-1$).

For any $\lambda \in \Lambda_i^k$ $(1 \leq i \leq M_k',\ k = 0 \leq k \leq n-1)$ there exists a number $d = d_{i,k} = d_{i,k}(\lambda) \geq 0$ such that $(\lambda, \alpha) = d$ for all $\alpha \in \Re_i^k$, and $(\lambda, \alpha) < d$ for any $\alpha \in \Re \setminus \Re_i^k$. Moreover, the $\Re-$normal of the $(n-1)-$dimensional (and only $(n-1)-$dimensional) face $\Re_i^{n-1}$ of the polyhedron $\Re$ and the number $d_{i,n-1}(\lambda)$ $(1 \le i \le M_{n-1})$ are determined uniquely.

Definition 1. A face $\Re_i^k$ of a polyhedron $\Re$ is said to be principal, if one of the following (obviously equivalent) conditions is satisfied: 1) $\Re_i^k$ does not go through the origin, 2) among the $\Re-$normals of this face there is one with at least one positive component. We say that a point $\alpha \in \Re$ is principal if α lies on some (recall, closed) principal face.

Obviously, all sub-faces of a principal face are principal. The number of $k-$dimensional principal faces of the polyhedron $\Re$ is denoted by M_k, obviously $M_k \le M_k'$.

Let $P(D) = P(D_1, ..., D_n) = \sum_\beta \gamma_\beta D^\beta$ be a linear differential operator with constant coefficients and $P(\xi) = \sum_\beta \gamma_\beta \xi^\beta$ be its complete symbol (the characteristic polynomial). Here the sum goes over a finite set of multi-indices $(P) := \{\beta \in \mathbb{N}_0^n; \gamma_\beta \ne 0\}$.

The Newton polyhedron of the set $(P) \cup \{0\}$ is called the Newton polyhedron of the operator $P(D)$ (polynomial $P(\xi)$) and is denoted by $\Re(P)$. Thus, the Newton polyhedron of any operator $P(D)$ (polynomial $P(\xi)$) is actually constructed as the Newton polyhedron of the operator $I+P(D)$ (polynomial $1 + P(\xi)$), where I is the identity operator. Note that a polyhedron $\Re(P)$ may have dimensionality less than n. However, in our considerations, we will assume that for both general and generalized-homogeneous polynomials P the polyhedrons $\Re(P)$ are $n-$dimensional (for complete polyhedrons this is obvious).

Let $\Re(P)$ be the N.P. of a polynomial $P(\xi)$ and $\Re_i^k$ $(i = 1, ..., M_k'; k = 0, 1, ..., n-1)$ be its faces. The polynomial $P^{i,k}(\xi) := \sum_{\alpha \in \Re_i^k} \gamma_\alpha \xi^\alpha$ $(1 \le i \le M_k; 0 \le k \le n))$ will be called the sub-polynomial of polynomial $P(\xi)$, corresponding to the face $\Re_i^k$.

Definition 2. Let $\mu \in \mathbb{R}^n$ be a vector with rational components. A polynomial $R(\xi) = R(\xi_1, ..., \xi_n)$ is called $\mu-$homogeneous (generalized-homogeneous) of $\mu-$order $d = d(\mu)$ (which is also a rational number), if $R(t^\mu \xi) := R(t^{\mu_1} \xi_1, ..., t^{\mu_n} \xi_n) = t^d R(\xi)$ for all $t > 0$, $\xi \in \mathbb{R}^n$. When $\lambda_1 = \lambda_2 = ... = \lambda_n (= 1)$, it is an ordinary homogeneous polynomial, wherein $|\xi, \lambda| = |\xi|$.

We will often use the following proposition, proved by V.P. Mikhailov

Lemma 1.1. ([33]) *Let $\Re = \Re(P)$ be the N.P. of a polynomial $P(\xi)$ and λ be any $\Re-$normal to the face $\Re_i^k$ $(\lambda \in \Lambda_i^k,\ 1 \le i \le M_k'; 0 \le k \le n-1)$ of the polyhedron $\Re$. Then the sub-polynomial $P^{i,k}$ is $\lambda-$homogeneous.*

Remark 1. It is obvious that if λ is a unit vector and $P(\xi) = \sum_{\alpha \in (P)} \gamma_\alpha \xi^\alpha$ is a polynomial, then there exist (uniquely defined) numbers $d_j(\lambda)$ and $\lambda-$homogeneous polynomials $P_j := P_{d_j(\lambda)}$ $(j = 0, 1, ..., M(\lambda))$: $d_0(\lambda) > d_1(\lambda) > ... > d_{M(\lambda)}(\lambda)$, such that the polynomial $P(\xi)$ can be represented in the form

$$P(\xi) = \sum_{j=0}^{M(\lambda)} P_j(\xi) = \sum_{j=0}^{M(\lambda)} P_{d_j(\lambda)}(\xi) = \sum_{j=0}^{M(\lambda)} \sum_{(\lambda,\alpha)=d_j(\lambda)} \gamma_\alpha \xi^\alpha, \tag{1.1}$$

where the set of numbers $\{d_j = d_j(\lambda)\}$ coincides with the finite set of values $\{(\lambda, \alpha)\}$ for all $\alpha \in \Re(P)$.

Note that

1) if $\Re_i^k$ $(i = 1, ..., M_k; k = 0, 1, ..., n-1)$ is some principal face of $\Re(P)$ and $\lambda \in \Lambda(\Re_i^k)$, then $(\lambda, \alpha) = d_0(\lambda)$ is the equation of the $(n-1)-$dimensional supporting hyperplane to $\Re(P)$ with the outward (with respect to $\Re(P)$) normal λ, containing the face $\Re_i^k$, where $P_{d_0(\lambda)}(\xi) \equiv P^{i,k}(\xi)$.

2) it follows from Lemma 1.1 that a sub-polynomial $P^{i,k}$ $(1 \le M_k', 0 \le k \le n-1)$ of the polynomial P is $\lambda-$homogeneous for any $\lambda \in \Lambda_i^k(\Re(P))$, i.e. there exists a number $d_{i,k} = d_{i,k}(\lambda) \ge 0$ such that $P^{i,k}$ can be represented in the form $P^{i,k}(\xi) = \sum_{(\lambda,\beta)=d_{i,k}} \gamma_\beta \xi^\alpha$.

A face $\Re_i^k$ $(1 \le i \le M_k',\ 0 \le k \le n-1)$ of the polyhedron $\Re(R)$ of a polynomial $R(\xi)$ is said to be non-degenerate ([33]) if $R^{i,k}(\xi) \neq 0$ for $\xi \in \mathbb{R}^{n,0}$. If there exists a point $\eta \in \mathbb{R}^{n,0}$, such that $P^{i,k}(\eta) = 0$, then the face $\Re_i^k$ is said to be degenerate. A polynomial $P(\xi)$ with P° N.P. $\Re(P)$ is said to be non-degenerate, if all its principal faces are non-degenerate.

Definition 3. An operator $P(D)$ (a polynomial $P(\xi)$) is called hypoelliptic ([12], Definition 11.1.2 and Theorem 11.1.1) if the following equivalent conditions are satisfied:

1) all the solutions $u \in D' = D'(\mathbb{E}^n)$ of the equation $P(D)u = f$ are continuously differentiable (belong to C^∞) for any $f \in C^\infty$,

2) $P^{(\alpha)}(\xi)/P(\xi) := D^\alpha P(\xi)/P(\xi) \to 0$ if $|\xi| \to \infty$, and $0 \neq \alpha \in \mathbb{N}_0^n$.

Definition 4. 1) ([36] or [16]) We say that a polynomial P is more powerful than a polynomial Q (a polynomial Q is less powerful than a polynomial P) and write $P > Q$ ($Q < P$), if there exists a constant $c > 0$ such that

$$|Q(\xi)| \leq c[|P(\xi)| + 1] \ \forall \xi \in \mathbb{R}^n, \tag{1.2}$$

2) ([12], Definition 10.3.4) We say that a polynomial P is stronger (by L.Hörmander) than a polynomial Q (Q is weaker than P) and write $P \succ Q$ ($Q \prec P$), if there exists a constant $c > 0$ such that

$$\tilde{Q}(\xi) \leq c\,\tilde{P}(\xi) \quad \forall \xi \in \mathbb{R}^n, \tag{1.3}$$

where for a polynomial R the function $\tilde{R}$ is defined by the formula

$$\tilde{R}(\xi) = [\sum_{|\alpha| \geq 0} |D^\alpha R(\xi)|^2]^{1/2}, \quad \xi \in \mathbb{R}^n.$$

Denote by $\mathbb{I}_n$ the set of all polynomials in n variables, such that $|P(\xi)| \to \infty$ for $|\xi| \to \infty$.

Many properties of the solutions of a general linear differential equation $P(D)u = 0$ are determined by the behavior at infinity of the symbol $P(\xi)$ of corresponding operator $P(D)$ as the modulus of the argument tends to infinity. For example, the symbol of a hypoelliptic operator tends to infinity (i.e. $P \in \mathbb{I}_n$).

In this case, it is important (and sometimes determining) not only that the symbol of a given operator tends to infinity, but also that this happens at a certain rate. For example, the symbol of an elliptic (and only elliptic) operator tend to infinity at an "optimal" rate, i.e. if $P(D)$ is an elliptic operator of order m, then there exits a number $c > 0$ such that

$$c^{-1}\,[1 + |\xi|^m] \leq 1 + |P(\xi)| \leq c\,[1 + |\xi|^m] \ \forall \xi \in \mathbb{R}^n.$$

In accordance with this, all continuous solutions of the elliptic equation $R(D)u = 0$ are real-analytic functions.

Solutions to a hypoelliptic equation (the symbols $P(\xi)$ of which belongs to $\mathbb{I}_n$) are infinitely differentiable functions. But they can also have better smoothness properties, for example, they can belong to certain Gevrey classes ([9], [37] or [38]). As is known, the Gevrey class $G^{(\sigma)}$ $(0 < \sigma < 1)$ is intermediate between the class of all infinitely continuously differentiable functions and the class of all real-analytic functions. Moreover, if for a differential operator $P(D)$ there are positive constants c and k such that

$$1 + |P(\xi)| \geq c\,[1 + |\xi|^k] \forall \xi \in \mathbb{R}^n,$$

then the value of σ directly depends on the value of k ([9], [37], [3], [29], [30]). Therefore, the need naturally arises to describe the set of multi-indices $\mathbb{B} = \mathbb{B}(P) := \{\beta\}$ for which the estimate

$$1 + |P(\xi)| \geq c \sum_{\beta \in \mathbb{B}} |\xi^\beta| \ \ \forall \xi \in \mathbb{R}^n \tag{1.4}$$

is valid with some constant $c > 0$.

V.P. Mikhailov in [33] described the class of all non-degenerate polynomials P with a complete Newton polyhedron, for which the set $\mathbb{B}$ coincides with the set $\Re(P)$, which is (in a certain sense) an "optimal" result. Similar result for an incomplete polyhedron was obtained by S. G. Gindikin in [10]. The classes of polynomials considered by these authors are certainly different from the class of elliptic ones, but they are close in character to an elliptic operator in the sense that they are non-degenerate.

The case in which the polynomial P is degenerate was first considered in the work [17]. The following proposition was proved there.

Theorem 1.1. ([17]) *Let $\Re = \Re(P)$ be the complete N.P. of a polynomial P. Suppose that all principal faces $\Re_i^k$ $(i = 1, 2, ..., M_k < M_k^{'}, k = 0, 1, ..., n-1)$ of the polyhedron $\Re$ except one $(n-1)-$dimensional principal face $\Gamma := \Re_{i_0}^{n-1}$ are non-degenerate, and the face Γ with the outward normal μ (which in this case is determined uniquely) is degenerate. Let the polynomial P be represented as the sum of $\mu-$ homogeneous polynomials* (*see representation* (1.1))

$$P(\xi) = \sum_{j=0}^{M} P_j(\xi) = \sum_{j=0}^{M} P_{d_j(\mu)}(\xi) = \sum_{j=0}^{M} \sum_{(\mu,\alpha)=d_j(\mu)} \gamma_\alpha \, \xi^\alpha, \tag{1.5}$$

where $P_0(\xi) = P^{i_0, n-1}(\xi)$, $M = M(P) = M(P, \mu)$.

Suppose that $P_1(\eta) \neq 0$ for all $\eta \in \Sigma(P_0) := \{\eta \in \mathbb{R}^{n,0}, |\eta, \lambda| = 1, P_0(\eta) = 0\}$ and denote by $\Re^*$ the N.P. of the set $\{\beta \in \Re, (\mu, \beta) \leq d_1\}$.

Then

1) *in order to have the estimate*

$$|\xi^\nu| \leq c \, [|P(\xi)| + 1] \ \ \forall \xi \in \mathbb{R}^n \tag{1.6}$$

for all points $\nu \in \Re^*$ with some constant $c = c(\nu, P) > 0$, it is necessary and sufficient, that for each point $\eta \in \Sigma(P_0)$ there exists a neighbourhood $U(\eta)$ such that $P_1(\eta) \neq 0$ and $P_0(\xi) \, P_1(\xi) \geq 0$ for all $\xi \in U(\eta)$.

2) *if $\nu \notin \Re^*$, then inequality (1.4) cannot hold for any constant c.*

As for the fact that only one face is degenerate, moreover it is a $(n-1)-$dimensional face, it is obvious that in case of the presence of several $(n-1)-$dimensional degenerate faces, the set $\nu \in \Re^*$ narrows and is obtained as the intersection of the sets. In [19], the case, in which $k-$dimensional faces for $k < n-1$ were present was also studied. Namely, the following proposition was proved in [19] (see also [17], Lemma 1.1), which in terms of the set $\mathbb{I}_n$ can be rephrased as follows (below $\Re^* := \{\nu \in \Re, (\lambda, \nu) \leq d_1(\lambda) \ \forall \lambda \in \Lambda(\Gamma)\}$)

Theorem 1.2. *Let $\Re$ be the complete Newton polyhedron of a polynomial $P \in \mathbb{I}_n$. Let all the principal faces $\Re_i^k$ $(i = 1, ..., M_k, \ k = 0, 1, ..., n-1)$ of the polyhedron $\Re$, except for (possibly) one k_0-dimensional face $\Gamma := \Re_{i_0}^{k_0}$ $(1 \leq i \leq M_k : 1 \leq k_0 < n-1)$ are non-degenerate, and the face Γ is degenerate. Let the polynomial P be represented with respect to any vector $\lambda \in \Lambda(\Gamma)$ in form* (1.1)(*for the definition of the set $\Lambda(\Gamma)$ see* [17]).

Then inequality (2.1) *holds for $\nu \in \Re^*$ if and only if $P_{d_1(\lambda)}(\eta) \neq 0$ for all $\eta \in \Sigma(\Gamma)$ and for all $\lambda \in \Lambda(\Gamma)$.*

The main limitation in these theorems is that at the points of the set $\Sigma(P_0)$, on which the polynomial P_0 vanishes, the next (or, which is the same, the first after P_0) polynomial P_1 must be nonzero. The author (and not only him) has not yet been able to overcome this limitation.

Our goal in this work is to overcome this limitation. Namely, we consider the casein which $P_1(\eta) = P_2(\eta) = ... = P_{l-1}(\eta) = 0$, $P_l(\eta) \neq 0$, $l \geq 2$ for some point $\eta \in \Sigma(P_0)$.

First, let us make the following remarks important for the sequel.

Remark 2. 1) When we compare a monomial ξ^ν and P° polynomial P, (or two polynomials Q and P), we can assume that the coefficients of these polynomials are real. Otherwise, we can compare the polynomials $|Q(\xi)|^2$ and $|P(\xi)|^2$. This is possible thanks to a simple lemma proved in [21], which says that if $\Re = \Re(R)$ is the N.P. of a polynomial R and $\mathfrak{M} = \mathfrak{M}(|R|^2)$ is the N.P. of the polynomial $|R|^2$, then $\Re$ is similar to $\mathfrak{M}$ with a similarity coefficient is equal to 2 and the similarity center at the origin. Moreover, if the similar faces are denoted by the same indices (i,k), then $[|P|^2]^{i,k}(\xi) = |P^{i,k}(\xi)|^2$. In particular, this means that if the face $\Re_i^k$ of the polyhedron $\Re$ is principal (degenerate, non-degenerate), then the face $\mathfrak{M}_i^k$ of the polyhedron $\mathfrak{M}$ is also principal (degenerate, non - degenerate) and vice versa.

2) If a polynomial P satisfies the conditions of Theorem 1.1 and the polyhedron $\Re^*$ is complete, then $P \in \mathbb{I}_n$.

3) If $P \in \mathbb{I}_n$, then outside of some ball the polynomial P does not change its sign. Therefore, if necessary, multiplying by (-1) and adding a positive constant (which does not affect their power), we can assume that the polynomials $P \in \mathbb{I}_n$ are everywhere positive. [21]

4) For polynomials $P \in \mathbb{I}_n$, the following simple proposition holds.

Lemma 1.2. *Let $\Re = \Re(P)$ be the N.P. of a polynomial $P \in I_n$ and $\Re_i^k$ $(i = 1, 2, ..., M_k,\ k = 0, 1, ..., n-1)$ be the principal faces of $\Re$. Then*

a) *the polyhedron $\Re = \Re(P)$ is complete,*

b) *$P^{i,k}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ $(i = 1, 2, ..., M_k,\ k = 0, 1, ..., n-1)$,*

c) *let a pair of indices (i,k) $(1 \leq i \leq M_k,\ 0 \leq k \leq n-1)$, a vector $\lambda \in \Lambda(\Re_i^k)$ and a point $\eta \in \Sigma(P^{i,k})$ be fixed; moreover (see representation (1.5)) $P_j(\eta) = 0$ $(j = 0, 1, ..., l-1)$, $P_l(\eta) \neq 0$ $(1 \leq l \leq M)$, then $P_l(\eta) > 0$.*

Proof. Property a) is obvious. In both cases b) and c), assuming the converse, that $P^{i,k}(\eta) < 0$ (respectively, $P_l(\eta) < 0$) for some point $\eta \in \Sigma(P^{i,k})$, we get that on the sequence $\{\xi^s := s^\lambda\,\eta\}_{s=1}^\infty$ $P(\xi^s) \to -\infty$ for $s \to \infty$, which contradicts our assumption $P(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$. □

With all this in mind, Theorem 1.1 can be rephrased as follows.

Theorem 1.1′ *Let $\Re = \Re(P)$ be the Newton polyhedron of a polynomial $P \in \mathbb{I}_n$. Let all the principal faces $\Re_i^k$ $(i = 1, ..., M_k,\ k = 0, 1, ..., n-1)$ of the polyhedron $\Re$, except (possibly) one $(n-1)-$dimensional face $\Gamma := \Re_{i_0}^{n-1}$ $(1 \leq i \leq M_k : 1 \leq k_0 \leq n-1)$ (with the outward normal μ), are non-degenerate. Then, if Γ is also non-degenerate, for any $\nu \in \Re$ estimate (1.6) holds. If Γ is degenerate, then with respect to the vector μ, we represent the polynomial P by formula (1.5). Suppose $P_1(\eta) \neq 0$ for all $\eta \in \Sigma(P_0)$ and let $\Re^*$ denote the Newton polyhedron of the set $\{\beta \in \Re, (\mu, \beta) \leq d_1\}$.*

Then estimate (1.6) holds if and only if $\nu \in \Re^*$.

Corollary 1.1. *Obviously, under the assumptions of Theorem 1.1′ $P_0 < P$ and $P_1 < P$.*

Remark 3. It goes without saying that in Theorems 1.1 and 1.2, in essence, only the cases in which the polyhedron $\Re^*$ is complete are interesting. Moreover in this case, obviously, Theorems 1.1 and 1.1′ are equivalent (see also Lemma 1.2).

We are now in a position to move on to our main task. Namely, let a degenerate polynomial $P \in \mathbb{I}_n$ be represented in form (1.5), with $P_l(\eta) \neq 0$ $(1 < l \leq M)$ for all $\eta \in \Sigma(P_0)$, and each of the polynomials P_j $(1 \leq j \leq l-1)$ vanishes at least at one point $\eta \in \Sigma(P_0)$. Let $\Re^{**}$ denote the Newton polyhedron of the set $\{\beta \in \Re, (\mu, \beta) \leq d_l\}$. Under what conditions on the polynomials P_j $(1 \leq j \leq l-1)$ inequality (1.6) is valid for all $\nu \in \Re^{**}$?

Let us paraphrase the problem in terms convenient for us. Let a polynomial P be represented in form (1.5) and satisfy the above conditions. Denote $\mathcal{P}(\xi) := P_0(\xi) + P_l(\xi) + P_{l+1}(\xi) + ... + P_M(\xi)$,

$\mathcal{P}_1(\xi) := P_1(\xi)+...+P_{l-1}(\xi)$. Then $P(\xi) = \mathcal{P}(\xi) + \mathcal{P}_1(\xi)$. If $l = 1$, then $\mathcal{P}(\xi) \equiv P(\xi)$, $\Re^{**} = \Re^*$ and from Theorem 1.1′ it follows that $\xi^\nu < \mathcal{P}$ for all $\nu \in \Re^{**}$. Let $l \geq 2$, what should be the polynomials P_j $(j = 1, ..., l-1)$ in order for the polynomial P to satisfy the conditions $\xi^\nu < P = \mathcal{P} + \mathcal{P}_1$ for all $\nu \in \Re^{**}$?

Since the polynomial $\mathcal{P}$ satisfies the assumptions of Theorem 1.1′ (with P_1 replaced by P_l) then $\xi^\nu < \mathcal{P}$ for all $\nu \in \Re^{**}$, it is clear that the polynomials P_j $(j = 1, ..., l-1)$ must be such, that the relation $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1$ holds.

The question posed is a special case of the following more general question (which, in addition to having numerous applications in the general theory of linear differential equations, is of independent interest): what polynomials $\{r(\xi)\}$ can be added to a polynomial $R(\xi)$, so that

a) $\Re(R + r) = \Re(R)$,

b) $r < R$,

c) the polynomials R and $R + r$ have the same power, i.e. $R < R + r < R$

We will call such polynomials r the lower-order terms of R.

Except that (as we will see below) the method of adding lower-order terms to a given differential operator (polynomial) that preserve (do not change) the power of the original operator (polynomial) will be directly applied to solving the problem posed by us, we present a number of other uses to illustrate the importance of this capability.

1) ([12], Theorem 11.1.9) If the operators $P(D)$ and $Q(D)$ have the same strength (by L. Hörmander) and $P(D)$ is hypoelliptic, then $Q(D)$ is also hypoelliptic.

2) ([31], Theorem 2) Let P and Q be polynomials with real coefficients with degrees m_P and m_Q respectively ($m_P > m_Q$). If for any real number a the polynomial $P + a\,Q$ is hypoelliptic, then the polynomial Q is also hypoelliptic.

3) ([20],Theorem 1) Let a hypoelliptic polynomial P be represented by a vector $\lambda \in \mathbb{E}^n$ in form (1.5), where $M = 1$. Let R be a $\lambda-$homogeneous polynomial of $\lambda-$degree $d(R) : d_1 < d(R) < d_0$ and $R < P_0$. Then $P + R$ is also hypoelliptic

4) ([20], Theorem 2) If a polynomial P (with generally speaking complex coefficients) is hypoelliptic and $Q < P$, then there exists a number $\varepsilon > 0$, such that for any complex number $a : |a| < \varepsilon$ the polynomial $P + a\,Q$ is hypoelliptic.

5) ([12, Section 12.4], [11], [3]) Let P_m be a homogeneous polynomial, hyperbolic with respect to the vector $N \in \mathbb{R}^n$ and Q be a polynomial such that $ord\,Q < m$. Then the polynomial $P_m + Q$ is hyperbolic (with respect to the N) if and only if $Q \prec P_m$ (see Definition 4)

These and other examples show the importance of finding the widest possible classes of lower-order terms for a given (in particular, generalized-homogeneous) polynomial.

Thus, our problem is reduced to finding conditions under which the polynomials $P_1, \cdots, P_{l-1}$ are the lower- order terms of the polynomial $\mathcal{P}$, i.e. for which a) $P_j < \mathcal{P}\,(j = 1, 2, ..., l-1)$, b) $P < \mathcal{P} < P$.

We will deal with this issue in the next section.

2 Comparison of powers of polynomials

Note that everywhere below, when comparing polynomials (or monomials and polynomials), we will consider only the case of the presence of an $(n-1)-$dimensional degenerate face. The case of the presence of a degenerate face of dimension $k < n-1$ is staded by comparing the method of proving Theorem 2.2 of this paper and the method of proving Theorem 1.2 formulated in present paper and proved in [19].

Let $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{E}^n$ be a vector with positive rational coordinates and $R(\xi) = \sum_{(\lambda,\alpha)=d_R} \gamma_\alpha^R\, \xi^\alpha$ be a $\lambda-$homogeneous polynomial. As usual, we denote by (R) the set of multi

- indices $\{\alpha\}$ for which $\gamma_\alpha^R \neq 0$ and by $\Re(R)$ we denote the Newton polyhedron of the set $(R) \cup 0$. Further, we will assume that the polyhedron $\Re(R)$ has a dimension n. Also put $\Sigma(R) := \{\xi; \xi \in \mathbb{R}^{n,0}, |\xi, \lambda| = 1, R(\xi) = 0\}$ and for the points $\eta \in \Sigma(R)$ denote

$$\mathfrak{A}(\eta, R) := \{\nu; \nu \in \mathbb{N}_0^n, D^\nu R(\eta) \neq 0\},\ \Delta(\eta, R) := \min_{\nu \in \mathfrak{A}(\eta,R)} (\lambda, \nu). \tag{2.1}$$

It is natural to start the comparison with the simplest case, namely with the comparison of generalized- homogeneous polynomials.

2.1 Comparison of powers of generalized-homogeneous polynomials

First, let us make the following remark

Remark 4. It is geometrically obvious that a sub - polynomial corresponding to the face $\Re_i^k$ of the polyhedron $\Re(R) = \Re(R \cup 0)$ has the form $R^{i,k}(\xi)$ or $R^{i,k}(\xi) + 1$. Therefore, only the faces $\Re_i^k$ that are formed without taking into account the point zero (that is the faces $\Re_i^k$ to which the polynomials $R^{i,k}(\xi)$ correspond without the participation of unity), can be degenerate, because the remaining faces correspond polynomials of the form $R^{i,k}(\xi) + 1$, where $R^{i,k}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ (see Remark 2).

The next proposition was proved in [16]

Theorem 2.1. *Let R be a $\lambda-$homogeneous polynomial of $\lambda-$order d_R. Let all the principal faces $\Re(R)$ of the polynomial R be non - degenerate, except possibly the $(n-1)-$dimensional face Γ containing the set (R). Then*

I) If the face Γ is non - degenerate, then $r < P$ for any $\lambda-$homogeneous polynomial r of $\lambda-$order $d_r \leq d_R$ such that $\Re(r) \subset \Re(R)$;

II) If the face Γ is degenerate, then $r < R$ if and only if the following conditions are simultaneously satisfied

1) $d_r \leq d_R$,

2) $\Sigma(r) \supset \Sigma(R)$,

3) $\Re(r) \subset \Re(R)$,

4)(*see notation* (2.1))

$$\frac{d_r}{d_R} \leq \frac{\Delta(\eta, r)}{\Delta(\eta, R)} \quad \forall \eta \in \Sigma(R), \tag{2.2}$$

5) *for each point $\eta \in \Sigma(R)$ there exists a neighborhood $U(\eta)$ and a constant $c = c(\eta) > 0$ such that*

$$|r(\xi)|^{1/\Delta(\eta,r)} \leq c\,|R(\xi)|^{1/\Delta(\eta,R)} \quad \forall \xi \in U(\eta). \tag{2.3}$$

In general, for generalized polynomials Q and P the relation $Q < P$ does not guarantee, that the polynomial Q is a lower-order term of the polynomial P, that is, $P < P + Q < P$ However, it turns out that for generalized - homogeneous polynomials r and R from $r < R$ it follows that $R < R + r < R$.

Let us prove the last statement. Firstly note, that since the numbers $\lambda_1, ..., \lambda_n$ are positive and rational, and any $\lambda-$homogeneous polynomial R is also $(k\,\lambda)-$homogeneous, then choosing a natural number k in an appropriate way (which does not affect the d_Q/d_P ratios), we can assume, that the numbers d_Q and d_P are natural, hence the functions P^{d_Q} and Q^{d_P} are also polynomials. Therefore, we can compare their power. Moreover, the following proposition holds

Lemma 2.1. *Let P and Q be $\lambda-$homogeneous polynomials of $\lambda-$orders d_P and d_Q respectively, where $d_P \geq d_Q$. Then*

a) $Q < P$ *if and only if* $Q^{d_P} < P^{d_Q}$, *(what is the same* $Q < P^{d_Q/d_P}$) *i.e. there is a number* $c > 0$ *such that*

$$|Q(\xi)|^{d_P} \le c\,[1+|P(\xi)|^{d_Q}] \;\; \forall \xi \in \mathbb{R}^n, \tag{2.4}$$

b) *if* $P > Q$ *and* $d_Q < d_P$, *then*

b.1) $|Q(\xi)|/|P(\xi)| \to 0$ *and* $|Q(\xi)|/|P(\xi)+Q(\xi)| \to 0$ *for* $|Q(\xi)| \to \infty$ *(hence* $|P(\xi)| \to \infty$),

b.2) $P < P+Q < P$.

Proof. Let us prove item a). Since $d_P \ge d_Q$, it is obviously followed from $Q^{d_P} < P^{d_Q}$ that $Q < P$. Consequently, the sufficiency of estimation (2.4) for the ratio $Q < P$ is obvious. We need to prove that estimation (2.4) follows from $Q < P$. On the other hand, to prove the estimate (2.4), it suffices to prove it for sequences $\{\xi^s\}$ such that $|Q(\xi^s)| \to \infty$ for $|\xi^s| \to \infty$.

So, let $\{\xi^s\}$ be such a sequence. From the condition $P > Q$ it also follows that $|P(\xi^s)| \to \infty$ for $s \to \infty$.

Denote $t_s := |P(\xi^s)|$ and $\tau_i^s := t_s^{-\lambda_i/d_P}\,\xi_i^s$ $(i = 1,...,n)$ i.e. $\xi^s = t_s^{\lambda/d_P}\,\tau^s$, $P(\tau^s) = 1$ $(i = 1,2,...,n;\ s = 1,2,...\cdot)$ Consider the individual parts of the inequality (2.4) on this sequence.

Due to the $\lambda-$homogeneity of the polynomials Q and P and bearing in mind that $P(\tau^s) = 1$ $(s = 1,2,...)$, we will have

$$|Q(\xi^s)| = t_s^{d_Q/d_P}\,|Q(\tau^s)|, \quad |P(\xi^s)|^{d_Q/d_P} = t_s^{d_Q}\,|P(\tau^s)|^{d_Q/d_P} = t_s^{d_Q}.$$

Since $Q < P$, from these representations and from $P(\tau^s) = 1$ $(s = 1,2,...)$ we have

$$|Q(\xi)^s|/[1+|P(\xi^s)|^{d_Q/d_P}] = t_s^{d_Q/d_P}\,|Q(\tau^s)|/[1+t_s^{d_Q}]$$

$$\le c\,t_s^{d_Q/d_P}\,[1+P(\tau^s)]/[1+t_s^{d_Q}] = 2\,c\,t_s^{d_Q/d_P-d_Q} = 2\;c\,t_s^{d_Q(1/d_P-1)}.$$

Since $d_P \ge 1$ and $t_s \to \infty$ for $|\xi^s| \to \infty$ we obtain item a) of the lemma.

Item b.1) directly follows from item a), if both sides of the (already proved) inequality (2.4) are divided by $|P(\xi)|^{d_P}$ and $|P(\xi)|$ tends to infinity. Similarly, it turns out that b.1) implies b.2). □

Corollary 2.1. *From part* b.2) *of Lemma* 2.1 *it follows that if* P *and* Q *are generalized - homogeneous polynomials satisfying the conditions* $P > Q$ *and* $d_Q < d_P$, *then polynomial* Q *is the lower - order term of the polynomial* P, *that is* $P < P+Q < P$. *As mentioned above, below we will make sure that, generally speaking, this is not the case for general polynomials (see examples* 2.3 *and* 3.1 *below).*

2.2 Comparison of powers of general polynomials

In this section, we set ourselves the task of comparing the powers of two general polynomials. Exactly: let P be a given polynomial with the complete Newton polyhedron $\Re(P)$ and Q be some polynomial. Find the conditions under which $Q < P$. If polynomial P is non - degenerate and $\Re(Q) \subset \Re(P)$, then by Theorem 2.1 $Q < P$. Therefore, we only need to consider the case when polynomial P is degenerate.

Before proceeding to the comparison of general polynomials, we prove one simple proposition, which, comparing a general polynomial with a generalized homogeneous polynomial reduces to comparing two generalized homogeneous polynomials and which, in our opinion, is also independent interest.

Lemma 2.2. *Let* R *be a* $\lambda-$*homogeneous polynomial of* $\lambda-$*order* d_R *and* Q *be a general polynomial represented in form* (1.1) *of as the sum of* $\lambda-$*homogeneous polynomials, i.e.*

$$Q(\xi) = \sum_{j=1}^{N(Q)} Q_j(\xi) = \sum_{j=1}^{N(Q)} \sum_{(\lambda,\alpha)=\delta_j} \gamma_\alpha^Q \, \xi^\alpha, \;\; \delta_1 > \delta_2 > ... > \delta_{N(Q)} \geq 0.$$

Then relation $R > Q$ *holds if and only if* $Q_j < R \;\; (j = 1, ..., N = N(Q))$

Proof. The proof of sufficiency is obvious. Let us prove the **necessity**. Let $R > Q$. We must proof, that $Q_k < R \;\; (k = 1, .., N)$.

Since $R > Q$, for any $t > 0 \;\; Q(t^\lambda \xi) < R(t^\lambda \xi) = t^{d_R} R(\xi)$. Consequently $Q(t^\lambda \xi) < R(\xi)$ for any $t > 0$.

Choose (and fix) N positive numbers $t_1, ..., t_N$ sPs that the matrix $(t_j^{\delta_k})$ is non - degenerate. We obtain from representation of Q that $Q(t_j^\lambda \xi) = \sum_{k=1}^N t_j^{\delta_k} Q_k(\xi)$. Therefore each of polynomials $Q_k(\xi) \;\; (k = 1, .., N)$ is a linear combination of polynomials $Q(t_j^\lambda \xi)$. This means that there exist numbers $\{a_i^j = a_i^j(t_1, ..., t_N)\}_{i,j=1}^N$ and $\{b_i^j = b_i^j(t_1, ..., t_N)\}_{i,j=1}^N$ such that for all j = 1,...,N

$$Q_j(\xi) = a_1^j \, Q(t_1^\lambda \xi) + ... + a_N^j \, Q(t_N^\lambda \xi) \;\; \leq b_1^j \, [1 + |R(t_1^\lambda \xi)|\,]$$
$$+ ... + b_N^j(t)[1 + |R(t_N^\lambda \xi)|\,] = b_1^j \, [1 + t_1^{d_R} \, |R(\xi)|\,] + ... + b_1^j \, [1 + t_N^{d_R} \, |R(\xi)|.$$

Since the vector $t = (t_1, ..., t_N)$ is fixed, denoting $B_j := max \; \{b_i^j; i = 1, ..., N\} \;\; (j = 1, ..., N)$ and $T := max\{t_i^{d_R}, i = 1, ..., N\}$, we obtain for some constant $c_j = c_j(B_j, T, R, Q) > 0$

$$|Q_j(\xi)| \leq c_j[1 + |R(\xi)|] \;\; \forall \xi \in \mathbb{R}^n, \;\; j = 1, ..., N.$$

□

To describe the set of all polynomials which are estimated via a given non-homogeneous, degenerate polynomial P, as above, we first consider the simplest case in which $P \in \mathbb{I}_n$, only one principal $(n-1)-$dimensional face of the polyhedron $\Re(P)$ of the polynomial P is degenerate, and $P_1(\eta) \neq 0$ for all $\eta \in \Sigma(P_0)$.

So, let us compare a degenerate polynomial P, represented as the sum of $\mu-$homogeneous polynomials in form (1.5) and a polynomial Q represented in the form (below $\delta_j = \delta_j(\mu) = \delta_j(Q, \mu)$ $(j = 0, 1, ..., M(Q); \; \delta_0 > \delta_1 > ... > \delta_{M(Q)})$

$$Q(\xi) = \sum_{j=0}^{M(Q)} Q_j(\xi) = \sum_{j=0}^{M(Q)} Q_{\delta_j(\mu)}(\xi) = \sum_{j=0}^{M(Q)} \sum_{(\mu,\alpha)=\delta_j} \gamma_\alpha^Q \, \xi^\alpha. \tag{2.5}$$

We want to find under which conditions $Q < P$.

First, note the following

1)if $\Re(Q) \subset \Re(P)$, and $\delta_{j_0} \leq d_1$, for some number $j_0 : 1 \leq j_0 \leq M(Q)$, then by Theorem 1.1′ $Q_j < P$ for all $j = j_0, j_0 + 1, ..., M(Q)$. Therefore, it remains to consider the polynomials Q_j for $j = 0, 1, ..., j_0 - 1$.

2) If $Q_j < P_0$ for all $j = 0, 1, ..., j_0-1$, then by Corollary 1.1 $Q_j < P_0 < P$ for all $j = 0, 1, ..., j_0-1$. As a result, we get that $Q < P$.

So, it suffices to consider the case $Q_{j_1} \not< P_0$ for some number $j_1 : 0 \leq j_1 \leq j_0 - 1$, wherein $d_1 < \delta_{j_1} \leq d_0$.

Let us prove two numerical inequalities which will be used in the proof of Theorem 2.2.

Lemma 2.3. *In order the inequality*

$$x^a \, y^b \leq 1 + x^c \, y^d$$

hold for all $x \geq 1, y \in [0, 1]$, *it is necessary and sufficient that the positive numbers* a, b, c, d *satisfy the inequalities:*

1) $a \leq c$

2) $d/b \le c/a$.

Proof. The necessity of condition 1) is obvious. Let us prove **the necessity** of condition 2).

Let condition 2) be violated, i.e. $d/b > c/a$. Let us prove that the required inequality cannot hold. Put $y = x^{-c/d}$. Then

$$x^a y^b = x^{a-b\,(c/d)} = x^{b[(a/b)-(c/d)]}; \;\; x^c y^d = 1,$$

Since, according to our assumption $(a/b)-(c/d) > 0$, the obtained relations show that for sufficiently large values of x the required inequality does not hold.

Sufficiency. If $b \ge d$, then the required inequality is obvious. Let $b < d$. Denoting $x^a =: u$, $y^b =: v$, we arrive at the equivalent inequality

$$u\,v \le 1 + u^{\frac{c}{a}}\,v^{\frac{d}{b}} \quad \forall u \ge 1, v \in [0,1].$$

When $u\,v \le 1$ this inequality is obvious. If $u\,v > 1$, then by the conditions of the lemma and the assumption $b < d$ we have

$$u\,v \le (u\,v)^{\frac{d}{b}} = u^{\frac{d}{b}}\,v^{\frac{d}{b}} \le u^{\frac{c}{a}}\,v^{\frac{d}{b}},$$

which proves the required inequality. □

Lemma 2.4. *In order the inequality*

$$x^a\,y^b \le 1 + C[\sigma_1 x^c\,y^d + \sigma_2\,x^{c-d}]$$

to hold for all $x \ge 1, y \in [0,1]$ *and a pair of positive numbers* σ_1, σ_2, *with some constant* $C = C(\sigma_1, \sigma_2) > 0$, *it is necessary and sufficient that the positive numbers* a, b, c, d *satisfy the inequalities:*

1) $a \le c$,

2) $a - b \le c - d$.

Proof. The necessity of condition 1) is obvious. We prove the necessity of condition 2). Let condition 2) be violated, i.e. $a - b > c - d$. and let $y = x^{-1}$, then for $x \to \infty$ we have

$$x^a\,y^b/\{1 + [\sigma_1 x^c\,y^d + \sigma_2\,x^{c-d}]\} = x^{a-b}/[1 + C\,(\sigma_1 + \sigma_2)\,x^{c-d}] \to \infty,$$

which proves the necessity of condition 2).

Sufficiency. If $b \ge d$ or $d/b \le c/a$, then the required inequality is a corollary of inequalities in Lemma 2.3. If, however, $d/b > c/a \ge 1$, then the substitution $y = t/x$ yields the equivalent inequality

$$x^{a-b}\,t^b \le C\,[1 + \sigma_1 x^{c-d}\,t^d + \sigma_2 x^{c-d}],$$

which can be easily proved (with any constant $C \ge max\{1, |\sigma_1|, |\sigma_2|\,\}$) if we consider separately the cases $t \ge 1$ and $t < 1$. □

Now, let us turn to the comparison of generalized polynomials P and Q represented forms (1.5) and (2.5), respectively. Moreover, it is obvious that to prove the relation $Q < P$, it is suffices to prove the relations $Q_j < P$ for each $j = 0, 1, ..., M(Q)$. It means, that it is suffices for us to compare the generalized-homogeneous polynomial Q with the generalized polynomial P. In a certain sense the following theorem allows to solve the problem in this case.

Theorem 2.2. I) *Let* $P \in \mathbb{I}_n$ *be a degenerate polynomial with a complete Newton polyhedron* $\Re$, *all principal faces of which are non - degenerate, except one* $(n-1)-$*dimensional face* $\Gamma = \Re^{n-1}_{i_0}$, *(with the outward normal* μ*), which is degenerate. Assume that the polynomial* P *is represented by formula* (1.5) *and that* $P_1(\eta) \ne 0$ *for all* $\eta \in \Sigma(P_0)$. *Let* Q *be a* $\mu-$ *homogeneous polynomial of* $\mu-$*order* $\delta_Q : d_1 < \delta_Q < d_0$ *and* $\Re(Q) \subset \Re(P)$. *Then* $Q < P$ *if and only if*

1) $\Sigma(P_0) \subset \Sigma(Q)$,

2) $(d_0 - d_1)/(\delta_Q - d_1) \geq \Delta(\eta, P_0)/\Delta(\eta, Q), \ \forall \eta \in \Sigma(P_0)$,

3) *if* $n > 2$, *then for every point* $\eta \in \Sigma(P_0)$ *there exists a constant* $c = c(\eta) > 0$ *and a neighborhood* $U(\eta)$ *such that*

$$|Q(\xi)| \leq c|P_0(\xi)|^{(\delta_Q - d_1)/(d_0 - d_1)} \ \forall \xi \in U(\eta).$$

II) *Moreover, if* $Q < P$, *the points of the set* (Q) *are interior points of the polyhedron* $\Re(P)$ *and for each point* $\eta \in \Sigma(P_0)$ *there exists a neighborhood* $U(\eta)$ *such that* $Q(\xi) \geq 0$ *for all* $\xi \in U(\eta)$, *then* $P < P + Q < P$.

Proof. The necessity of condition I.1) is obvious.

The necessity of condition I.2). Assume the converse, i.e. the condition $Q < P$ is satisfied, but there exists a point $\eta \in \Sigma(P_0)$ such that

$$(d_0 - d_1)/(\delta_Q - d_1) < (\Delta(\eta, P_0)/(\Delta(\eta, Q)). \tag{2.6}$$

For $t > 0, \theta = (\theta_1, ..., \theta_n) \in \mathbb{R}^n, \ \kappa > 0$ set $\xi_i = \xi_i(t) = \xi_i(t, \theta, \kappa) = t^{\mu_i}(\eta_i + \theta_i t^{-\kappa \mu_i}), \quad i = 1, ..., n.$

Since $D^\alpha P(\eta) = 0$ for all $\alpha \in \mathbb{N}_0^n$ and the condition $(\mu, \alpha) < \Delta(\eta, P_0)$, is satisfied, then according to Taylor's formula, for sufficiently large values of t we have

$$Q(\xi(t)) = t^{\delta_Q} Q(\eta + \theta t^{-\kappa \mu}) = t^{\delta_Q} \sum_\alpha t^{-\kappa (\mu, \alpha)} [D^\alpha Q(\eta)/(\alpha!)] \, \theta^\alpha$$

$$= t^{\delta_Q - \kappa \Delta(\eta, Q)} \sum_{(\mu, \alpha) = \Delta(\eta, Q)} [D^\alpha Q(\eta)/(\alpha!)] \, \theta^\alpha + o(t^{\delta_Q - \kappa \Delta(\eta, Q)}).$$

Choose θ in such a way that

$$c = c(\theta) := \sum_{(\mu, \alpha) = \Delta(\eta, Q)} [D^\alpha Q(\eta)/(\alpha!)] \, \theta^\alpha \neq 0.$$

The existence of such a vector θ obviously follows from the definition of the number $\Delta(\eta, Q)$. In fact, otherwise, it turns out that all the coefficients of the polynomial $c(\theta)$ are equal to zero, which contradicts the definition of $\Delta(\eta, Q)$. Then (for a fixed such θ), we have

$$|Q(\xi(t))| \geq c \, t^{\delta_Q - \kappa \Delta(\eta, Q)}. \tag{2.7}$$

For the polynomials P_0 and P_1 we obviously have for a constant $c_1 > 0$ such that for sufficiently large t

$$|P_0(\xi(t))| \leq c_1 \, t^{d_0 - \kappa \Delta(\eta, P_0)}, \ \ |P_1(\xi(t))| = t^{d_1} \, P_1(\eta) \, (1 + o(1)). \tag{2.8}$$

Obvious geometric arguments show that as $t \to +\infty$

$$r((\xi(t)) := P((\xi(t)) - [P_0((\xi(t)) + P_1((\xi(t))] = o(t^{d_1}). \tag{2.9}$$

We put $\kappa = (d_0 - d_1)/\Delta(\eta, P_0)$, then $d_0 - \kappa \, \Delta(\eta, P_0) = d_1$, and from (2.8) - (2.9), for a constant $c_2 > 0$ we have

$$|P(\xi(t))| \leq c_2 t^{d_1}. \tag{2.10}$$

It is easy to calculate, that from assumption (2.6) it follows that $d_1 < \delta_Q - \kappa \, \Delta(\eta, Q)$. From estimates (2.7), (2.10) it follows, that $|Q(\xi(t))|/[1 + P(\xi(t))] \to \infty$ for $t \to \infty$, which contradicts the condition $Q < P$ and proves the necessity of condition I.2).

The necessity of condition I.3). Assume that for some point $\eta \in \Sigma(P_0)$ there ib a sequence $\{\eta^s\}$ such that $P_0(\eta^s) \neq 0$ $(s = 1, 2, ...)$, $\eta^s \to \eta$ for $s \to \infty$ and

$$R(\eta^s) := |Q(\eta^s)|/[|P_0(\eta^s)|^{(\delta_Q - d_1)/(d_0 - d_1)}] \to \infty. \tag{2.11}$$

Set $t_s = |P_0(\eta^s)|^{-1/(d_0-d_1)}$, $\xi^s = t_s^\mu \eta^s$, $s = 1, 2, ...\cdot$ Since $\eta^s \to \eta \in \Sigma(P_0)$ we have $t_s \to \infty$ as $s \to \infty$. Then, as a corollary of the $\mu-$homogeneity of $P_0(\xi)$, $P_1(\xi)$ and $Q(\xi)$, for sufficiently large s we have

$$|P_1(\xi^s)| = t_s^{d_1} |P_1(\eta^s)| = t_s^{d_1} |P_1(\eta)| (1 + o(1)), \tag{2.12}$$

$$|P_0(\xi^s)| = t_s^{d_0} |P_0(\eta^s)| = t_s^{d_1}, \;\; r(\xi) = o(t_s^{d_1}) \tag{2.13}$$

Representations (2.12), (2.13) show that a constant $c_3 > 0$ exists such that for sufficiently large s

$$|P(\xi^s)| + 1 \leq c_3 t_s^{d_1}. \tag{2.14}$$

For $Q(\xi)$ we obtain analogously (see also (2.11))

$$|Q(\xi^s)| = t_s^{\delta_Q} |Q(\eta^s)| = t_s^{\delta_Q} R(\eta^s) |P_0(\eta^s)|^{(\delta_Q - d_1)/(d_0 - d_1)} = R(\eta^s) t_s^{d_1}. \tag{2.15}$$

Estimates (2.14) and (2.15), together with assumption (2.11), show that as $s \to \infty$ we have $|Q(\xi^s)|/[|P(\xi^s)| + 1] \geq [1/c_3] R(\eta^s) \to \infty$. This proves the necessity of condition I.3) for $Q < P$.
Sufficiency. When proving sufficiency, we will use the method, proposed by Mikhailov in the study of non - degenerate polynomials (see [33]) and the method, modified by us, which was used in the study of degenerate polynomials (see, for example, [16] or [19]).

Assume that $Q \not< P$ under the hypotheses of Theorem 2.2, i.e. there exists a sequence $\{\xi^s\}$ such that $\xi^s \to \infty$ as $s \to \infty$ and

$$|Q(\xi^s)|/[|P(\xi^s)| + 1] \to \infty. \tag{2.16}$$

Without loss of generality, it can assumed, that all coordinates of the vectors ξ^s are positive. Let

$$\rho_s := exp\sqrt{\sum_{k=1}^{n} (\ln \xi_k^s)^2}, \;\; \lambda_i^s := \frac{\ln \xi_i^s}{\ln \rho_s} \;\; (i = 1, ..., n, \; s = 1, 2, ...). \tag{2.17}$$

Then $\lambda^s = (\lambda_1^s, ..., \lambda_n^s)$ is a unit vector and

$$\xi^s = \rho_s^{\lambda^s} \;\; (\xi_i^s = \rho_s^{\lambda_i^s}, \; i = 1, ..., n), \tag{2.17'}$$

It is clear, that $\rho_s \to \infty$ if $|\xi_i^s| \to \infty$ or $|\xi_i^s| \to +0$ for some $i = 1, 2, ..., n$.

Since the vectors λ^s are placed on the unit sphere, the sequence $\{\lambda^s\}$ has a limit point λ^∞. It can be assumed, that $\lambda^s \to \lambda^\infty$, $|\lambda^\infty| = 1$. From the convexity of the polyhedron $\Re(P)$ it follows, that λ^∞ is an outward normal to one and only one face of $\Re(P)$.

Denote λ^∞ by $e^{1,1}$, and choose $n-$ dimensional vectors $(e^{1,1}, e^{1,2}, ..., e^{1,n})$ so that this system forms an orthonormal basis in $\mathbb{R}^n$. Then $\lambda^s = \sum_{i=1}^n \lambda_{1,i}^s e^{1,i}$ $(s = 1, 2, ...)$. Since $\lambda^s \to \lambda^\infty = e^{1,1}$ for $s \to \infty$, then $\lambda_{1,1}^s \to 1$, $\lambda_{1,i}^s = o(\lambda_{1,1}^s)$ for $i = 2, 3, ..., n$.

If it is possible to choose a sub-sequence in a such way, that $\sum_{j=2}^n \lambda_{1,i}^s e^{1,j} = 0$ for all sufficiently large s,, then the basis $(e^{1,2}, ..., e^{1,n})$ we shall denote by $e^1, ..., e^n$. Otherwise, by appropriate choice of a sub-sequence we may assume that $\sum_{j=2}^n \lambda_{1,i}^s e^{1,j} \neq 0$ for all $s = 1, 2, ...$ and for $s \to \infty$

$$[\sum_{i=2}^{n} \lambda_{1,i}^s e^{1,i}]/|\sum_{i=2}^{n} \lambda_{1,i}^s e^{1,i}| \to e^{2,2}.$$

In the subspace spanned by $(e^{1,2}, e^{1,3}, ..., e^{1,n})$ we pass to a new orthonormal basis $(e^{2,2}, e^{2,3}, ..., e^{2,n})$ with the vector $e^{2,2}$ defined above. Then, if $n \geq 3$

$$\lambda^s = \lambda^s_{1,1}\, e^{1,1} + \lambda^s_{2,2}\, e^{2,2} + \sum_{i=3}^{n} \lambda^s_{2,i}\, e^{2,i}, \ (s = 1, 2, ...),$$

hence $\lambda^s_{1,1} \to 1$, $\lambda^s_{2,2} = o(\lambda^s_{1,1})$, $\lambda^s_{2,i} = o(\lambda^s_{2,2})$, $i = 3, ..., n$ for $s \to \infty$.

Reasoning analogously, as in the subspace with the basis $(e^{2,3}, ..., e^{2,n})$ etc., we finally obtain (after modifying the notation) that $\lambda^s = \sum_{i=1}^{n} \lambda^s_i\, e^i$, where $(e^1, ..., e^n)$ is an orthonormal basis, and $\lambda^s_1 \to 1, \lambda^s_{i+1} = o(\lambda^s_i)$, $i = 1, ..., n-1$ for $s \to \infty$.

Moreover, there exist numbers s_0 and $m : 1 \leq m \leq n$ such that for all $s \geq s_0$ we have $\lambda^s_i > 0$ for $(i = 1, ..., m)$ and $\lambda^s_i = 0$ $(i = m+1, ..., n)$. By choosing a sub-sequence, we may assume, that $s_0 = 1$, $\lambda^s_i > 0$ for all $(i = 1, ..., m)$ and $s \in \mathbb{N}$.

Now we associate the constructed basis with the polyhedron $\Re$. We select the faces $\Re^{k_1}_{i_1}, \Re^{k_2}_{i_2}, ..., \Re^{k_m}_{i_m}$ as follows: denote by $\Re^{k_1}_{i_1}$ the faces of $\Re(P)$ which lie in the supporting hyperplane of $\Re(P)$ with the outward normal e^1, and each face $\Re^{k_j}_{i_j}$ $(j = 2, ..., n)$ either coincides with the previous one, or is its sub-face, which lies in the supporting hyperplane with the normal e^j. If there are several sub-faces $\Re^{k_j}_{i_j}$ with the normal e^{j+1}, then as $\Re^{k_{j+11}}_{i_{j+1}}$ we agree to take the one for which points α the expression (e^{j+1}, α) is maximal.

From the construction of the faces $\Re^{k_1}_{i_1}, \Re^{k_2}_{i_2}, ..., \Re^{k_m}_{i_m}$ it is obvious, that their dimensions are subject to the relation: $k_1 \geq k_2 \geq ..., \geq k_m$ and (see (2.17) - (2.17′))

$$\xi^s = \rho_s^{\sum_{i=1}^{n} \lambda^s_i e^i} \quad (s = 1, 2, ...),$$

wherein, it can be assumed that $\rho_s \to \infty$ for $s \to \infty$ and some r $(1 \leq r \leq m)$

$$\rho_s^{\lambda^s_j} \to \infty \ (j = 1, ..., r), \ \ \rho_s^{\lambda^s_{r+1}} \to b \geq 1, (s = 1, 2, ..).$$

When $r = m = n$, then we shall assume, that $\lambda^s_{n+1} = 0$ $(s = 1, 2, ..)$, and e^{n+1} is an arbitrary unit vector.

Let, as above, $P^{i_j,k_j}(\xi)$ be the sub-polynomial of $P(\xi)$, corresponding to the face $\Re^{k_j}_{i_j}$, i.e $P^{i_j,k_j}(\xi) := \sum_{\beta \in \Re^k_i} \gamma_\beta\, \xi^\beta$, and α be an arbitrary multi-index belonging to all $\Re^{k_j}_{i_j}$ $(j = 1, ..., m)$, i.e $\alpha \in \Re^{k_m}_{i_m}$. We will study the behaviour of polynomials $P(\xi)$ and $Q(\xi)$ for $\rho_s \to \infty$ and $\xi^s = \rho_s^{\lambda^s_1 e^1 + \lambda^s_2 e^2 + ... + \lambda^s_n e^n}$.

Further, for brevity, when this does not cause misunderstanding, we omit the index s in the notation.

Then, from e^j-homogeneity of polynomials $\{P^{i_j,k_j}(\xi)\}$ and convexity of $\Re(P)$ and its faces, for certain positive numbers $\sigma_1, ..., \sigma_r$ and multi-index $\alpha \in \Re^{k_r}_{i_r}(P)$ we get

$$\begin{aligned} P(\xi) &= \rho^{(\alpha, \lambda_1\, e^1)}[P^{i_1,k_1}(\rho^{\sum_{j=2}^{n+1} \lambda_j e^j}) + o(\rho^{-\sigma_1\, \lambda_1})] \\ &= \rho^{(\alpha, \lambda_1\, e^1 + \lambda_2\, e^2)}[P^{i_2,k_2}(\rho^{\sum_{j=3}^{n+1} \lambda_j e^j}) + o(\rho^{-\sigma_2\, \lambda_2})] = ... \\ &= \rho^{(\alpha, \sum_{j=1}^{r} \lambda_j\, e^j)}[P^{i_r,k_r}(\rho^{\sum_{j=r+1}^{n+1} \lambda_j e^j}) + o(\rho^{-\sigma_r\, \lambda_r})]. \end{aligned} \tag{2.18}$$

Similarly, for the polynomial Q, for a number σ'_r and a multi-index $\beta \in \Re^{k_r}_{i_r}(Q)$ we have

$$Q(\xi) = \rho^{(\beta, \sum_{j=1}^{r} \lambda_j\, e^j)}[Q^{i_r,k_r}(\rho^{\sum_{j=r+1}^{n+1} \lambda_j e^j}) + o(\rho^{-\sigma'_r\, \lambda_r})]. \tag{2.18′}$$

Since $\rho_s^{\lambda_{r+1}^s} \to b \geq 1$, it follows, that $\rho_s^{(\alpha, \sum_{r+1}^{n+1} \lambda_j^s e^j)} \to b^{e^{r+1}} := \eta$ for $s \to \infty$. It is clear, that $0 < \eta_i < \infty$ for all $i = 1, ..., n$ (in accordance with the definition of η_i).

Let us consider two cases: a) $(e^1, \alpha) > 0$ and b) $(e^1, \alpha) = 0$. The case $(e^1, \alpha) < 0$ is impossible because of the fact, that equation for supporting hyperplane with the outward normal λ of $\Re$ can be written in the form $(\lambda, \alpha) = d$, where $d \geq 0$ is the distance from the origin to the given hyperplane and α is a point of the hyperplane (see, for example, [1]).

Case a.1) Firstly suppose, that $P^{i_r, k_r}(\eta) \neq 0$. Since $(e^1, \alpha) > 0$, $\lambda_1^s \to 1$ and $\lambda_i = o(\lambda_1)$ for $i = 2, ..., n,$, for sufficiently large s eventually we have, that $(\alpha, \sum_1^r \lambda_j e^j) > 0$. Therefore, (2.18) implies that

$$P(\xi) = \rho^{(\alpha, \sum_{j=1}^{r} \lambda_j e^j)} [P^{i_r, k_r}(\eta) + o(1)]. \tag{2.19}$$

Similarly, for the polynomial $Q(\xi)$

$$Q(\xi) = \rho^{(\beta, \sum_{j=1}^{r} \lambda_j e^j)} [Q^{i_r, k_r}(\eta) + o(1)]. \tag{2.20}$$

We show, that

$$(\beta, \sum_{j=1}^{r} \lambda_j e^j) \leq (\alpha, \sum_{j=1}^{r} \lambda_j e^j). \tag{2.21}$$

Since $\beta \in \Re_{i_r}^{k_r}(\Re(Q))$, $\alpha \in \Re_{i_r}^{k_r}(\Re(P))$, $\Re(Q) \subset \Re(P)$ and e^1 is the normal of the face $\Re_{i_1}^{k_1}(\Re(P))$, hence $(\beta, e^1) \leq (\alpha, e^1)$. If $(\beta, e^1) < (\alpha, e^1)$, then inequality (2.21) follows from the fact that $\lambda_1 \to 1$ and $\lambda_{j+1} = o(\lambda_j)$ for $j = 1, 2, ..., r-1$. If $(\beta, e^1) = (\alpha, e^1)$, then this means that the points β and α belong to the same face $\Re_{i_1}^{k_1}$. Since $\beta \in \Re_{i_r}^{k_r}(\Re(Q))$ and $\Re(Q) \subset \Re(P)$, hence $(\beta, e^2) \leq (\alpha, e^2)$. If $(\beta, e^2) < (\alpha, e^2)$, then inequality (2.21) follows from the same fact, regarding the numbers λ_j. If $(\beta, e^2) = (\alpha, e^2)$, this means that the points β and α belong to the same face $\Re_{i_2}^{k_2}$, $(\beta, e^3) \leq (\alpha, e^3)$ and so on.

Continuing this process, after a finite number of steps, we either arrive at the equality $(\beta, e^j) = (\alpha, e^j)$ $j = 1, 2, ..., q-1$, $(\beta, e^j) < (\alpha, e^j)$ for some $q < r$, or for the relation $(\beta, e^j) = (\alpha, e^j)$ for all $j = 1, ..., r$. In both cases, the inequality (2.21) is obvious. Thus, inequality (2.21) is proved.

So, relations (2.19) - (2.21) together contradict our assumption (2.16) and complete the consideration of sub - case a.1) of case a).

Consider the case a.2): $P^{i_r, k_r}(\eta) = 0$. In this case, the face $\Re_{i_r}^{k_r}$ coincides with the $(n-1)-$ dimensional degenerate face $\Gamma := \Re_{i_0}^{n-1}$ (with the outward normal μ) and $r = m = 1$, $k_r = k_1 = n-1$, $e^1 = \mu, \eta \in \Sigma(\Gamma)$.

With respect to the vector $e^1 = \mu$, we represent the polynomial $P(\xi)$ in form (1.5)

$$P(\xi) = \sum_{j=0}^{M} P_j(\xi) := \sum_{j=0}^{M} \sum_{(e^1, \alpha) = d_j} \gamma_\alpha \xi^\alpha \tag{2.22}$$

and denote $q(\xi) := P(\xi) - [P_0(\xi) + P_1(\xi)]$. Then,

$$P(\xi) = P_0(\xi) + P_1(\xi) + q(\xi). \qquad (2.23).$$

Substituting

$$\xi (= \xi^s) = \rho^{\sum_{j=1}^{n+1} \lambda_j e^j} = \rho_s^{\sum_{j=1}^{n+1} \lambda_j^s e^j}$$

in (2.23) and using e^1-homogeneity of the polynomials $P_0(\xi)$, $P_1(\xi)$ and $Q(\xi)$ we get (below $h^s := \sum_{j=2}^{n+1} \lambda_j^s e^j$)

$$P(\xi^s) = \rho_s^{\lambda_1^s(\mu,\alpha)} P_0(\rho_s^{h^s}) + \rho_s^{\lambda_1^s(\mu,\beta)} P_1(\rho^{h^s}) + q(\xi), \tag{2.24}$$

ord

$$Q(\xi^s) = \rho_s^{\lambda_1^s(\mu,\gamma)} Q(\rho^{h^s}) \tag{2.25}$$

Let $s \to \infty$, i.e. $\rho_s \to \infty$, then $\rho_s^{h^s} \to \eta$, $\lambda_1^s(\mu,\alpha) \to d_0$, $\lambda_1^s(\mu,\beta) \to d_1$, $\lambda_1^s(\mu,\gamma) \to \delta_Q$ and by Lemma 1.2 $P_0(\eta) \geq 0$, $P_1(\eta) > 0$.

On the other hand, since $ord\, q \leq d_2 < d_1$, so with some constant $c_4 > 0$ we have $|q(\xi^s)| \leq c_4\, \rho_s^{d_2}$, for all $s = 1, 2, ...,$ i.e. $|q(\xi^s)| = o(\rho_s^{d_1})$ for $s \to \infty$.

Thus, from (2.24) - (2.25) (for sufficiently large value of s) we have

$$P(\xi^s) = \rho_s^{\lambda_1^s(\mu,\alpha)}\, P_0(\rho_s^{h^s}) + \rho_s^{\lambda_1^s(\mu,\beta)}\, P_1(\rho_s^{h^s}) + o(\rho_s^{\lambda_1^s(\mu,\beta)}), \tag{2.24$'$}$$

and representation (2.25) for polynomial $Q(\xi)$, where $\rho_s^{h^s} \to \eta$, $\lambda_1^s(\mu,\alpha) \to d_0$, $\lambda_1^s(\mu,\beta) \to d_1$, $\lambda_1^s(\mu,\gamma) \to \delta_Q$, as $s \to \infty$.

Since $\rho_s^{h^s} \to \eta \in \Sigma(P_0)$, then for sufficiently large s (i.e. for sufficiently large ρ_s) condition I.3) of our theorem is satisfied. Then, from (2.25) and condition I.3) for sufficiently large s and for a constant $c_5 > 0$ we obtain

$$|Q(\xi^s)| = \rho_s^{\lambda_1^s(\mu,\gamma)}\, |Q(\rho_s^{h^s})| \leq c_5\, \rho_s^{\lambda_1^s(\mu,\gamma)} |P_0(\rho_s^{h^s})|^{(\delta_Q - d_1)/(d_0-d_1)}. \tag{2.26}$$

According to the conditions of our theorem $P \in \mathbb{I}_n$, therefore, for indicated s $P_0(\rho_s^{h^s}) \geq 0$ and $P_0(\eta) = 0$, $P_1(\eta) > 0$. So, we can assume, that $0 \leq P_0(\rho_s^{h^s}) \leq 1$, $P_1(\rho_s^{h^s}) \geq \frac{1}{2} P_1(\eta) > 0$ for sufficiently large s and $|q(\xi^s)|/|P(\xi^s)| \to 0$ for $s \to \infty$. This and (2.24$'$) in turn show that

$$|P(\xi^s)| \geq \sigma_1 \rho_s^{d_0}\, P_0(\rho_s^{h^s}) + \sigma_2 \rho_s^{d_1}. \tag{2.27}$$

for sufficiently large s and for positive constants σ_1 and σ_2.

From estimates (2.26) - (2.27) it follows that, in order to obtain a contradiction with (2.16), it suffices to prove the existence of a constant $C = C(\sigma_1, \sigma_2) > 0$ such that for sufficiently large s

$$\rho_s^{\delta_Q} P_0(\rho_s^{h^s})^{(\delta_Q - d_1)/(d_0-d_1)} \leq C[1 + \sigma_1\, \rho_s^{d_0}\, P_0(\rho_s^{h^s}) + \sigma_2 \rho_s^{d_1}]. \tag{2.28}$$

To prove the estimates (2.28), let us apply Lemma 2.4 with the following notations

$$a := \delta_Q, b := \delta_Q - d_1, c = d_0, d := d_0 - d_1, x := \rho_s, y := [P_0(\rho_s^{h^s})]^{[1/(d_0-d_1)]}.$$

After introducing these notations, the inequality (2.28) takes the following form

$$x^a\, y^b \leq 1 + C\, |\sigma_1\, x^c\, y^d + \sigma_2\, x^{c-d}|. \tag{2.28$'$}$$

Since $x \geq 1$, $y \in [0,1]$, $a \leq c$, $a - b = c - d = d_1$, then all conditions of Lemma 2.4 are satisfied. According to this lemma, inequality (2.28$'$) holds, therefore, inequality (2.28) holds. Resulting inequality (2.28) contradicts our assumption (2.16) and completes the consideration of sub-case a.2) and, therefore, completes the consideration of case a).

Let us move to case b) $(e^1, \alpha) = 0$.

Firstly, note, that if the Newton polyhedron $\Re$ of the polynomial $P(\xi) = P(\xi_1, ..., \xi_n)$ is complete, then the Newton polyhedron of polynomial $P(\xi)|_{\xi_j=0}$ for $j \in [1, n]$ is also complete in the appropriate $(n-1)-$dimensional subspace. Secondly, in the case b) under consideration, the face, whose outward normal is e^1, clearly passes through the origin and hence is not a principal face of $\Re$; consequently, $e_i^1 \leq 0$ $(i = 1, ..., n)$. In this connection, if non principal face with outward normal e^1 has dimension

$l \leq n-1$, then l if the numbers e_i^1 $(1 \leq i \leq n)$ are equal to zero with the remaining numbers being negative. Without loss of generality it can clearly be assumed, that $e_1^1 = ... = e_l^1 = 0$, $e_{l+1}^1 < 0, ..., e_n^1 < 0$.

Since

$$e_j^1 = \lim_{\xi\to\infty}[ln\xi_j/(\sum_{k=1}^{n}(ln\xi_k)^2)^{1/2}] \; < 0 \; (j = l+1, ..., n),$$

beginning with some number s_0 (we assume that $s_0 = 1$) we have, that $\xi_j^s < 1$ $(j = l+1, ..., n)$ $(s = 1, 2, ...)$. On the other hand, since $|\xi^s| \to \infty$ for $s \to \infty$, we have $\xi_i^s \to \infty$ for certain $i \in [1, l]$. But since $e_i^1 = 0$ for such i, hence (at least for some subsequence of the sequence ξ^s) $\xi_j^s \to 0$ for $s \to \infty$ and at least one $j \in (l, n]$.

Suppose, that (after a possible renumbering) $\xi_l^s \to \infty, ..., \xi_{l_0}^s \to \infty$ $(l_0 \geq l)$ for $s \to \infty$ and $\xi_{l_0+1}^s \to 0, ..., \xi_{l_0+l_1}^s \to 0$ $(l_0 + l_1 \leq n)$.

Let $\psi(\xi) := max_{1\leq j\leq l_0}\xi_j$, then it is obvious, that as $s \to \infty$

$$ln\psi(\xi^s)/[\sum_{k=1}^{n}(ln\xi_k^s)^2]^{1/2} \to 0. \tag{2.29}$$

On the other hand, there clearly exist positive constants c_6, c_7 such that

$$c_6 \leq \sum_{k=1}^{l_0}(ln\xi_k^s)^2\,(ln\psi(\xi^s))^2 \leq c_7 \quad (s = 1, 2, ...). \tag{2.30}$$

From (2.29)-(2.30) it follows that

$$\sum_{k=l_0+1}^{n}(ln\xi_k^s)^2\,(ln\psi(\xi^s))^2 \to \infty \quad as \quad s \to \infty. \tag{2.31}$$

From this result, going over a sub-sequence, if necessary, we can get, that for some $j \in [l_0+1, n]$

$$|ln\xi_j^s|/ln\psi(\xi^s) \to \infty \quad as \quad s \to \infty, \tag{2.32}$$

i.e. $|ln\xi_j^s| \to \infty$ "faster" than $ln\psi(\xi^s) \to \infty$. Hence $\xi_j^s = o([\psi(\xi^s)]^{-\sigma})$ for some $\sigma > 0$ or, equivalently,

$$(\xi_j^s)^{\alpha_1}.[\psi(\xi^s)]^{\alpha_2} \to 0 \quad as \quad |\xi| \to \infty \tag{2.33}$$

for $\alpha_1 > 0$ and $\alpha_2 \geq 0$.

Let $\breve{\xi} = (\breve{\xi}_1, ..., \breve{\xi}_n)$, where $\breve{\xi}_j = 0$ if j satisfies the condition (2.32) and $\breve{\xi} = \xi_j$ otherwise.

In view of (2.33) from (2.16) it follows that

$$|Q(\breve{\xi}^s)|/[1 + |P(\breve{\xi}^s)|] \to \infty \quad as \quad s \to \infty \tag{2.34}$$

(under our limit process, i.e. with the possibility of repeatedly going over the sub-sequences of the sequence $\{\xi^s\}$ of (2.16)).

As a result, the polynomial $P(\xi) = P(\xi_1, ..., \xi_n)$ can be transformed into the polynomial $\breve{P}(\xi) := P(\breve{\xi})$ on less than n variables. Consequently, dimension of the polyhedron $\breve{\Re}(P) := \Re(\breve{P})$ is less than the dimension of the polyhedron $\Re(P)$, while the non - degenerate faces of $\Re$ correspond to the non - degenerate faces of $\breve{\Re}$ and vice versa.

Thus, in the process of proving Theorem, relation (2.16) leads either to a contradiction or to relation (2.34), which is analogous to (2.16) but corresponds to a space of dimension less than or equal to $n-1$.

Repeating the arguments, presented above within the proof of this theorem, now with respect to the polynomial $\breve{P}$, and so on, we clearly arrive after a finite number of steps at either a contradiction or relation (2.34) for polynomials of one variable.

But for polynomials of one variable, the polyhedrons $\Re(P)$ and $\Re(Q)$ have the shape of segment, and a contradiction with (2.16) is due to the fact, that $\Re(Q) \subset \Re(P)$.

Thus, the first part of Theorem 2.2 is proved.

Let us prove the second part of Theorem. Repeating reasoning, carried out in the sufficiency proof of the first part of Theorem, i.e. assuming the converse, that there exists a sequence $\{\xi^s\}$ such that $\xi^s \to \infty$ and

$$|P(\xi^s)|/[1 + |P(\xi^s) + Q(\xi^s)] \to \infty \ \ as \ s \to \infty, \tag{2.35}$$

In the case a.1) we obtain representation (2.19) for polynomial P and following representation for polynomial $P+Q$

$$P(\xi^s) + Q(\xi^s) = \rho^{(\alpha, \sum\limits_{j=1}^{r} \lambda_j^s e^j)} P^{i_r,k_r}(\eta) + \rho^{(\beta, \sum\limits_{j=1}^{r} \lambda_j^s e^j)} Q^{i_r,k_r}(\eta) + o(1). \tag{2.36}$$

Since, based on the condition II) of Theorem, the points of the set (Q) are interior points of the set $\Re(P)$, that is $(\beta, e^1) < (\alpha, e^1)$ and $\lambda_1^s \to 1$, $\lambda_j^s = o(\lambda_1^s)$ for $s \to \infty$ $(j = 2, ..., r)$, then $(\beta, \sum\limits_{j=1}^{r} \lambda_j^s e^j) < (\alpha, \sum\limits_{j=1}^{r} \lambda_j^s e^j)$ for sufficiently large s. Then

$$\rho^{(\beta, \sum\limits_{j=1}^{r} \lambda_j^s e^j)} / \rho^{(\alpha, \sum\limits_{j=1}^{r} \lambda_j^s e^j)} \to 0 \ \ as \ s \to \infty$$

and representations (2.19), (2.36) together contradict (2.35).

In the case a.2): $P^{i_r,k_r}(\eta) = Q^{i_r,k_r}(\eta) = 0$, the face $\Re_{i_r}^{k_r}$ coincides with $(n-1)-$dimensional degenerate face $\Gamma = \Re_{i_0}^{n-1}$ (with the outward normal μ) and $r = m = 1$, $k_r = k_1 = n-1, e^1 = \mu$, $\eta \in \Sigma(\Gamma)$.

In this case, we obtain the representations (2.24') and (2.25) for the polynomials P and Q, respectively, and following representation for the polynomial $P+Q$

$$P(\xi^s) + Q(\xi^s) = \rho^{\lambda_1^s(\mu,\alpha))} P_0(\rho_s^{h^s}) + \rho^{\lambda_1^s(\mu,\gamma))} Q(\rho_s^{h^s})$$

$$+\rho^{\lambda_1^s(\mu,\alpha))} P_1(\rho_s^{h^s}) + o(1). \tag{2.37}$$

Since, according to the conditions (first and second parts) of Theorem $P_0(\rho_s^{h^s}) \geq 0$, $Q(\rho_s^{h^s}) \geq 0$, $P_1(\rho_s^{h^s}) > 0$ for sufficiently large s, it follows from (2.24'), (2.37) that $|P(\xi^s) + Q(\xi^s)| \geq |P(\xi^s)|$ for sufficiently large s. This contradicts our assumption (2.35). □

Let us give examples, illustrating this theorem.

Example 1. Let us compare the polynomial $Q(\xi) = (\xi_1 - \xi_2)^2(\xi_1^6 + \xi_2^6)$ with the following two polynomials $P^1(\xi) := P_0^1(\xi) + P_1^1(\xi) = (\xi_1 - \xi_2)^4(\xi_1^6 + \xi_2^6) + (\xi_1^6 + \xi_2^6)$ and $P^2(\xi) := P_0^2(\xi) + P_2^1(\xi) = (\xi_1 - \xi_2)^4(\xi_1^6 + \xi_2^6) + (\xi_1^4 + \xi_2^4)$.

Here $d_0^1 = d_0^2 =: d_0$, $d_1^1 = 6$, $d_1^2 = 4$, $\Delta(\eta, P_0^1) = \Delta(\eta, P_0^2) := 4$, $\eta = \pm(1/\sqrt{2}, 1/\sqrt{2})$, Simple calculations show, that the pair (P^1, Q) satisfies all conditions of Theorem 2.2, while the pair (P^2, Q) does not satisfy condition 2) of this theorem. Indeed, $(d_0 - d_1^2)/(\delta_Q - d_1^2) = 3/2 < 2 = \Delta(\eta, P_0^2)/\Delta(\eta, Q)$. Therefore, $Q < P^1$, but $Q \not< P^2$.

Remark 5. Note, that conditions of Theorem 2.2 do not guarantee the $Q < P_0$, which can be seen from the following example.

Example 2. Let $n = 2$, $P(\xi) := P_0(\xi) + P_1(\xi) = (\xi_1 - \xi_2)^8 + (\xi_1^2 + \xi_2^2)^2$, $Q(\xi) = (\xi_1 - \xi_2)^4 (\xi_1^2 + \xi_2^2)$. Here $d_0 = 8, d_1 = 4$, $\delta_Q = 6$, $\eta = \pm(1/\sqrt{2}, 1/\sqrt{2})$, $\Delta(\eta, P_0) = 8$, $\Delta(\eta, Q) = 4$.

It is easy to verify, that all conditions of Theorem 2.2 are satisfied, hence $Q < P$. Moreover, applying the arithmetic inequality $a\,b \leq (1/2)(a^2 + b^2))$, we obtain, that $P < P + Q$. However, in this case the (necessary) condition II.4) of Theorem 2.1 is violated, and, therefore, $Q \not< P_0$. This can also be verified directly (without resorting to the help of Theorem 2.1) by taking, for example, $\xi_1^s = s+1,\ \xi_2^s = s\ \ s = 1, 2, ...$.

Remark 6. Note, that as we saw above (see the Corollary 2.1), for a pair of generalized - homogeneous polynomials P and Q the relations $Q < P$ and $P < P + Q < P$ are equivalent, however, in general, this does not apply to generalized polynomials. Here are some examples conforming this.

Example 3. Let $n = 2$. Compare the polynomials $P(\xi) = (\xi_1 - \xi_2)^8 + (\xi_1^2 + \xi_2^2)^2$, and $Q(\xi) = (-2,5)(\xi_1 - \xi_2)^6(\xi_1 + \xi_2)$. Here $P_0(\xi) = (\xi_1 - \xi_2)^8$, $P_1(\xi) = (\xi_1^2 + \xi_2^2)^2$, $\Sigma(P_0) = \{\pm\eta = \pm(1/\sqrt{2}, 1/\sqrt{2})\}$, $d_0 = 8, d_1 = 4$, $\Delta(\eta, P_0) = 8$, $\delta_Q = 7, \Delta(\eta, Q) = 6$, $(\delta_Q - d_1)/(d_0 - d_1) = \Delta(\eta, Q)/\Delta(\eta, P_0) = 3/4$, $\eta = \pm(1/\sqrt{2}, 1/\sqrt{2})$.

Conditions 1) - 2) of Theorem 2.2 are obvious, because $\Sigma(P_0) \subset \Sigma(Q)$ and $(d_0 - d_1)/(\delta_Q - d_1) = \Delta(\eta, P_0)/\Delta(\eta, Q)$, $\forall \eta \in \Sigma(P_0)$.

To prove condition 3) of Theorem 2.2 for the couple (P, Q), as a neighborhood of $U(\eta)$ for both η and $-\eta$ one can take, for example, a circle, centered at the point η (or $-\eta$) with unit radius. Then, the condition 3) reduces to the existence of a constant $c > 0$ such that the inequality $|(\xi_1 - \xi_2)^6\,(\xi_1 + \xi_2) \le c\,|\xi_1 - \xi_2|^6$ holds for all $\xi \in U(\eta)$. In this case, this inequality is obvious, since $|\xi - \eta|| \le 1$ for the points $\xi \in U(\eta)$. Thus, by Theorem 2.2 $Q < P$

Let us show, that $P \not< P+Q$, i.e., that Q is not of lower order term for the polynomial P. Indeed, simple calculations show, that on the sequence $\{\xi^s = (s + \sqrt{s}, s)\}$ for $s \to \infty$ $|P(\xi^s)| = O(s^4)$ and $|P(\xi^s) + Q(\xi)| = O(s^{3,5})$, i.e. $|P(\xi^s)|/|P(\xi^s) + Q(\xi)| \to \infty$ for $s \to \infty$. It is also easy to see, that $Q \not< P + Q$.

Thus, in general case, Theorem 2.2 does not answer the question: when (under what conditions on the polynomials $P_j\,(j = 1, 2, ..., l-1)$) $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1$? We will do this in the next section. But, before moving to the next section, we note the following

Remark 7. 1)from Theorem 1.1′ it follows, that if the polynomial P, with the complete Newton polyhedron $\Re(P)$, is non - degenerate, then $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1 < \mathcal{P}$

2) when $\Gamma := \Re_{i_0}^{n-1}$ is a (unique) degenerate principal face of the polyhedron $\Re(P)$, the conditions (necessary and sufficient) for the fulfillment of the right - hand side of this estimation ($P = \mathcal{P} + \mathcal{P}_1 < \mathcal{P}$) are given by Theorem 2.2 (first part): it means, that each pair of polynomials $(P_j, \mathcal{P})\,(j = 1, ..., l-1)$ must satisfy the conditions of Theorem 2.2,

3) in a particular case (sufficient), the validity conditions for relation $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1$ are given in the second part of Theorem 2.2.

Remark 8. From the course of the proof of Theorem 2.2, it became obvious that under the conditions of this theorem a) $P_0 < P_0 + P_1 < P$ (however, this is clear from the proof of Theorem 1.1′ also), b) the polynomials $P_2, ..., P_M$ do not affect the behavior at infinity of the polynomial P (although they can participate in the construction of the Newton polyhedron $\Re(P)$).

3 Adding lower-order terms and main result

Recall, that in Theorem 1.1′ we considered only the case, when in the studied degenerate polynomial $P = P_0 + P_1 + P_2 + ...$ at all points $\eta \in \Sigma(P_0) := \{\eta \in \mathbb{R}^{n,0} : P_0(\eta) = 0\}$ it was the first of polynomials $\{P_j\}$ that did not vanish: $P_1(\eta) \ne 0\ \forall \eta \in \Sigma(P_0)$. Now we want to free ourselves from this restriction.

Namely, let, like to Theorem 1.1′, $\Gamma := \Re_{i_0}^{n-1}$ be the only degenerate principal face (with the outward normal μ) of the complete Newton polyhedron $\Re(P)$ of polynomial $P \in \mathbb{I}_n$ and with respect to the vector μ the polynomial P is represented as a sum of $\mu-$homogeneous polynomials

$$P(\xi)=\sum_{j=0}^{M} P_j(\xi)=\sum_{j=0}^{M}\sum_{(\mu,\alpha)=d_j}\gamma_\alpha\,\xi^\alpha, \tag{3.1}$$

where $d_0>d_1>...>d_l>...>d_M\geq 0$.

Suppose, that $P_l(\eta)\neq 0$ $(1\leq l\leq M)$ for all $\eta\in\Sigma(P_0)$ and each polynomial P_j $(j=1,2...,l-1)$ vanishes at least at one point $\eta\in\Sigma(P_0)$ and put $\Re^*:=\{\beta\in\Re,(\mu,\beta)\leq d_l\}$, $\mathcal{P}(\xi):=P_0(\xi)+P_l(\xi)+P_{l+1}(\xi)+...+P_M(\xi)$, $\mathcal{P}_1(\xi):=P_1(\xi)+...+P_{l-1}(\xi)$. If $l=1$, then $\mathcal{P}(\xi)\equiv P(\xi)$ and it follows from Theorem 1.1$'$, that $\xi^\nu<\mathcal{P}$ for all $\nu\in\Re^*$.

A question naturally arises: suppose $l\geq 2$, and polynomial $\mathcal{P}$ satisfies the conditions of Theorem 1.1$'$. Therefore, $\xi^\nu<\mathcal{P}$ for all $\nu\in\Re^*$. Which conditions must the polynomials P_j $(j=1,...,l-1)$ satisfy, so that for newly introduced set $\Re^*$ the relation $\xi^\nu<P=\mathcal{P}+\mathcal{P}_1$ also holds for all $\nu\in\Re^*$?

To do this, we need to answer the following question (which, besides of numerous applications in differential equations, of course, is also of independent interest): which lower - order terms Q can be added to the polynomial $P=P_0+P_1+...$, so that a) $\Re(P+Q)=\Re(P)$, b) the polynomials P and $R:=P+Q$ have the same power, i. e. $P<R<P$? In this case, we will call the polynomial Q of lower - order term with respect to the polynomial P.

It is clear, that in this case our question sounds like this: what should be polynomials $P_1,P_2,...,P_{l-1}$ so that the polynomials P and $\mathcal{P}$ had the same power, i.e., that the relation $\mathcal{P}<P=\mathcal{P}+\mathcal{P}_1<\mathcal{P}$ held ?

The next proposition in a sense solves the question posed in the class of polynomials that we considered above.

Theorem 3.1. *Suppose that a degenerate polynomial P and a $\mu-$homogeneous polynomial Q of $\mu-$order $\delta_Q\in(d_1,d_o)$ satisfy the conditions of the first part of Theorem 2.2 (consequently $Q<P$). Let for each point $\eta\in\Sigma(P_0)$ and for any sequence $\{\eta^s\}:\ \eta^s\to\eta$ for $s\to\infty$ the following relation is true*

$$\psi(\eta^s):=|Q(\eta^s)|/|P_0(\eta^s)|^{(\delta_Q-d_1)/(d_0-d_1)}\to 0. \tag{3.2}$$

Then

1) $|Q(\xi)|/[|P(\xi)|+1]\to 0$ *as* $|\xi|\to\infty$,

2) $Q<P+Q,\ \ P<P+Q<P.$

Proof of statement 1). Suppose, to the contrary, that conditions of Theorem are satisfied, but there exist a sequence $\{\xi^s\}$ and a number $c_1>0$ such that $\xi^s\to\infty$ for $s\to\infty$ and

$$|Q(\xi^s)|/[|P(\xi^s)|+1]\geq c_1\ ,(s=1,2,...). \tag{3.3}$$

Reasoning as in the proof of Theorem 2.2 we obtain (for sufficiently large s) the following estimations for the polynomial P (see the representation (2.24))

$$P(\xi^s)=\rho_s^{\lambda_1^s(\mu,\alpha)}\,P_0(\rho_s^{h^s})+\rho_s^{\lambda_1^s(\mu,\beta)}\,P_1(\rho_s^{h^s})+o(\rho_s^{\lambda_1^s(\mu,\beta)}).$$

Since $\lambda_1^s\to 1$ as $s\to\infty$, from this, for a number $c_2>0$ and sufficiently large s we have

$$|P(\xi^s)|+1\geq c_2\,[1+|\rho_s^{d_0}\,P_0(\rho_s^{h^s})+\rho_s^{d_1}\,P_1(\rho_s^{h^s})|]. \tag{3.4}$$

Taking into account condition (3.2), for polynomial Q and the same s we have

$$|Q(\xi^s)|=\rho_s^{\delta_Q}\,|Q(\rho_s^{h^s})|=\rho_s^{\delta_Q}\,|P_0(\rho_s^{h^s})|^{(\delta_Q-d_1)/(d_0-d_1)}\ \ \psi(\rho_s^{h^s}). \tag{3.5}$$

Then, from (3.4) - (3.5), with some constant $c_3>0$ we have

$$[|Q(\xi^s)|/|P(\xi^s)|+1]\leq c_3\,M(\rho_s^{h^s})\ \ \psi(\rho_s^{h^s}), \tag{3.6}$$

where
$$M(\rho_s^{h^s}) := \rho_s^{\delta_Q} |P_0(\rho_s^{h^s})|^{(\delta_Q-d_1)/(d_0-d_1)}]/[|\rho_s^{d_0} P_0(\rho_s^{h^s}) + \rho_s^{d_1} P_1(\eta) | + 1|].$$

Let us prove the existence of some constant $c_4 > 0$ for which the following inequality holds

$$M(\rho_s^{h^s}) \le c_4 \ (s = 1, 2, ...). \tag{3.7}$$

We introduce the notation $x = x_s = \rho_s^{\delta_Q}$, $y = y_s = |P_0(\rho_s^{h^s})|^{1/(d_0-d_1)}$, $a = \delta_Q$, $b = \delta_Q - d_1$, $c = d_0$, $d = d_0 - d_1$. Then inequality (3.7) takes the form

$$x^{\delta_Q} y^{\delta_Q - d_1} \le c_4[1 + |x^{d_0} y^{d_0-d_1} + P_1(\eta) x^{c-d}|], \tag{3.8}$$

where $P_1(\eta) > 0$, $x \ge 1$, $y \in [0, 1]$ for sufficiently large s.

To prove the inequality ((3.8) we apply the Lemma 2.4. The conditions of this lemma are satisfied, because $a = \delta_Q < d_0 = c$, $c - a = d - b = d_0 - \delta_Q$, $c - d = d_1$, $\sigma_1 = 1$, $\sigma_2 = P_1(\eta) > 0$.

Thus, inequality (3.7) is proved. Since $\psi(\rho_s^{h^s}) \to 0$ for $s \to \infty$, the inequalities (3.6), (3.7) together contradict the assumption (3.3) and prove the first part of Theorem.

The second part of Theorem is an immediate consequence of the first part. It is only necessary to reverse the fact, that now the behavior of polynomial Q does not affect the behavior of $P + Q$ when $|\xi| \to \infty$, (i.e. $P(\xi) \to \infty$). □

Let us give an example of a pair of polynomials (P, Q) satisfying the conditions of Theorem 3.1.

Example 4. Let $n = 2$, $P(\xi) = (\xi_1 - \xi_2)^8 + (\xi_1^2 + \xi_2^2)^2$, $Q(\xi) = (\xi_1 - \xi_2)^5(\xi_1 + \xi_2)$.

Here $P_0(\xi) = (\xi_1 - \xi_2)^8$, $P_1(\xi) = (\xi_1^2 + \xi_2^2)^2$, $\Sigma(P_0) = \{\pm\eta = \pm(1/\sqrt{2}, 1/\sqrt{2}),\}$, $d_0 = 8, d_1 = 4$, $\Delta(\eta, P_0) = 8$, $\delta_Q = 6$, $\Delta(\eta, Q) = 5$, $(\delta_Q - d_1)/(d_0 - d_1) = 1/2$.

Conditions I.1) - I.3) of Theorem 2.2 can be easily verified, and condition (3.2) of Theorem 3.1 is satisfied, since for any sequence $\{\eta^s\}$: $\eta^s \to \eta$ for $s \to \infty$ we have

$$\psi(\eta^s) := |Q(\eta^s)|/[|P_0(\eta^s)]^{1/2} = ((\eta_1^s)^2 - (\eta_2^s)^2) \to \eta_1^2 - \eta_2^2 = 0.$$

At the same time, it is obvious, that $Q \not< P_0$.

As for the pair of polynomials from the Example 2.2 for any sequence $\{\eta^s\}$: $\eta^s \to (1/\sqrt{2}, 1/\sqrt{2})$ as $s \to \infty$, $\psi(\eta^s) = |Q(\eta^s)/|P_0(\eta^s)|^{1/2} = (\eta_1^s)^2 + (\eta_2^s)^2 \to 1$, i.e. condition (3.2) is violated. Despite this, as we saw above, $P < P + Q$ and $Q < P + Q$ because the pair (P, Q) satisfies the condition of the second part of Theorem 2.2.

Now we are already in a position to turn into the question, posed at the beginning of this section. Namely, let with respect to a vector $\mu \in \mathbb{R}^n$ a (generalized) polynomial P be represented as a sum of $\mu-$homogeneous polynomials in the form (3.1). We need to describe those multi - indites $\nu \in \mathbb{N}_0^n$ for which $\xi^\nu < P$, i.e. a constant $c = c(\nu, P) > 0$ exists, such that

$$|\xi^\nu| \le c\,[|P(\xi)| + 1] \ \forall \xi \in \mathbb{R}^n. \tag{3.9}$$

Theorem 3.2 (main result). *Let $\Re = \Re(P)$ be the complete Newton polyhedron of a polynomial $P \in I_n$. Let all of the principal faces of $\Re$ with exception of a $(n-1)-$dimensional face $\Gamma := \Re_{i_0}^{n-1}$ (with the outward normal μ) be non-degenerate and the face Γ be degenerate.*

Let the polynomial P be represented as a sum of $\mu-$homogeneous polynomials in form (3.1) and

$$P = P_0 + P_1 + ...P_l + ... + P_M,$$

where $P_0(\xi) =: P^{i_0,n-1}(\xi)$, P_j is a $\mu-$ homogeneous polynomial of $\mu-$order d_j $j = 0, 1, ..., l, ..., M$, $d_0 > d_1 > ... > d_l > ... > d_M \ge 0$.

Suppose, that $P_l(\eta) \ne 0$ for all $\eta \in \Sigma(P_0) := \{\xi \in \mathbb{R}^{n,0} \ |\xi, \mu| = 1\}$ and each polynomial $P_j \in \mathfrak{M} := \{P_1, P_2, ..., P_{l-1}\}$ vanishes at least at one point $\eta \in \Sigma(P_0)$.

Let $\mathcal{P}(\xi) := P_0(\xi) + P_l(\xi)$, $\mathcal{P}_1(\xi) := P_1(\xi) + ... + P_{l-1}(\xi)$, $p(\xi) := P_{l+1}(\xi) + ... + P_M(\xi)$, $\Re^* := \{\beta \in \Re, (\mu, \beta) \le d_l\}$ *and suppose, that* $\Re(\mathcal{P}) = \Re(P)$.

Then

a) *if* $\nu \notin \Re^*$, *inequality* (3.9) *cannot hold,*

b) *inequality* (3.9) *holds for any multi-index* $\nu \in \Re^*$ *if each of polynomials* $P_j \in \mathfrak{M}$ *satisfies one of the following conditions*

b.1) *for the pair of* ($\mu-$*homogeneous) polynomials* (P_j, P_0) $(1 \le j \le l-1)$ *the assumptions of Theorem* 2.1 *are satisfied,*

b.2) *for the pair of polynomials* $(P_j, \mathcal{P})$ $(1 \le j \le l-1)$ *the assumptions* $I) - II)$ *of Theorem* 2.2 *are satisfied,*

b.3) *for the pair of polynomials* $(P_j, \mathcal{P})$ $(1 \le j \le l-1)$ *the assumptions of Theorem* 3.1 *are satisfied.*

Remark 9 Before proceeding to the proof of the theorem, we note that

1) conditions b.2) and b.3) should be set not for a pair of polynomials $(P_j, \mathcal{P})$ but for a pair $(P_j, \mathcal{P} + p)$. On one hand, the notation $(P_j, \mathcal{P})$ simplifies writing and reasoning, on the other hand, it is legitimate, since by Remark 8, the polynomial $p(\xi)$ does not affect the behaviour of polynomials P and $\mathcal{P}$ at infinity,

2) from condition II.2) of Theorem 2.1, condition I.3) of Theorem 2.2 and (3.2) of Theorem 3.1 it follows that in all b.1) - b.3) cases of this theorem the polynomials $P_j \in \mathfrak{M}$ $(j = 1, ..., l-1)$ must vanish at all points $\eta \in \Sigma(P_0)$.

Proof of Theorem 3.2. Bearing in mind that for the polynomial $\mathcal{P}$ estimate (3.9) is valid for all $\nu \in \Re^*$, it is sufficient for us to prove that $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1$.

Firstly, let us add to the polynomial $\mathcal{P}$ those polynomials from $\mathfrak{M}$ that (together with the polynomial P_0) satisfy condition b.1) of Theorem (i.e.conditions of Theorem 2.1). Let these be polynomials $\mathfrak{M}_1 = \{P_{i_1}, P_{i_2}, ..., P_{i_{k_1}}\} \subset \mathfrak{M}$ $(1 \le i_j \le l-1$, $j = 1, ..., k_1)$, $k_1 \le l-1$ i.e $P_{i_j} < P_0$ $j = 1, ..., k_1)$.

Since $d_{i_j} < d_0$ $(j = 1, ..., k_1)$, by Lemma 2.1 $P_{i_j}(\xi) = o(|P_0(\xi)|)$ for $|P_0(\xi)| \to \infty$ i.e. $|P_{i_1}(\xi)| + |P_{i_2}(\xi)| + ..., |P_{i_k}(\xi)| = o(|P_0(\xi)|)$ for $|P_0(\xi)| \to \infty$.

Remark 8. implies that $P_0 < \mathcal{P} = P_0 + P_l$, hence $|P_{i_1}(\xi)| + |P_{i_2}(\xi)| + ..., |P_{i_{k_1}}(\xi)| = o(|\mathcal{P}(\xi)|)$ for $|P_0(\xi)| \to \infty$. Thus, there exists a constant $c > 0$ such that, for sufficiently large $|P_0(\xi)|$, the inequality

$$|\mathcal{P}(\xi)| \le c\,[1 + |\mathcal{P}(\xi) + P_{i_1}(\xi) + P_{i_2}(\xi) + ..., P_{i_{k_1}}(\xi)|] \tag{3.10}$$

holds. If $|P_0(\xi)|$ is bounded for $|\xi| \to \infty$, then the polynomials $\{P_{i_j}\}$ are also bounded on this sequence (recall that $P_{i_j} < P_0$ $(j = 1, ..., k_1)$). On the other hand, since $\mathcal{P} \in \mathbb{I}_n$ hence $\mathcal{P}(\xi) \to \infty$, and inequality (3.10) (perhaps with a different constant) is obvious. As a result, we get that $\mathcal{P} < \mathcal{P} + P_{i_1} + P_{i_2} + ... + P_{i_{k_1}} < \mathcal{P}$. It means, that further, when comparing the polynomials $\mathcal{P}$ and P, it suffices to compare the polynomials $\mathcal{P}^1 := \mathcal{P} + P_{i_1} + P_{i_2} + ... + P_{i_{k_1}}$ and P.

If $k_1 = l-1$, i.e $\mathcal{P}^1(\xi) := \mathcal{P}(\xi) + \mathcal{P}_1(\xi) = P(\xi)$ $\forall \xi \in \mathbb{R}^n$, then this proves Theorem.

Consider the case when $k_1 < l-1$, i.e. $\mathfrak{M}_1 \ne \mathfrak{M}$.

Let us first consider those polynomials $P_j \in \mathfrak{M} \setminus \mathfrak{M}_1$ that satisfy condition b.3). Let these be polynomials $\mathfrak{M}_3 := \{P_{k_1+i_1}, P_{k_1+i_2}, ..., P_{k_1+k_2}$ $(1 \le i_j \le l-1$, $j = 1, ..., k_2)$, $k_1 + k_2 \le l-1\}$ i.e $|P_{i_j}(\xi)|/[\,P(\xi)| + 1] \to 0$ as $|\xi| \to \infty$ and $P < P + P_{i_j} < P$ for all $j = k_1 + 1, ..., k_1 + k_2$.

Arguing as in the previous case, we find that $\mathcal{P} < \mathcal{P}^2 := \mathcal{P}^1 + P_{k_1+i_1} + P_{k_1+i_2} + ... + P_{i_{k_1}+k_2} < \mathcal{P}$, i.e. further, when comparing the polynomials $\mathcal{P}$ and P, it suffices to compare the polynomials $\mathcal{P}^2$ and P.

Finally, to the polynomial $\mathcal{P}^2$ we add the remaining polynomials from $\mathfrak{M}$ that satisfy condition b.2) of Theorem (i.e. conditions of Theorem 2.2). Let these be polynomials $\mathfrak{M}_2 :=$

$\{P_{k_1+k_2+i_1}, P_{k_1+k_2+i_2}, ..., P_{k_1+k_2+k_3}\}$, $k_1+k_2+k_3 = l-1$. Then $\mathcal{P}^2(\xi)+ P_{k_1+k_2+i_1}(\xi) + P_{k_1+k_2+i_2}(\xi) + ... + P_{k_1+k_2+k_3}(\xi) = P(\xi)$ for all $\xi \in \mathcal{R}^n$.

As a result of the previous two cases we have that $\mathcal{P} < \mathcal{P}^2 < \mathcal{P}$. From Theorem 2.2 it follows that $\mathcal{P} < \mathcal{P}+ P_{k_1+k_2+i_1} + P_{k_1+k_2+i_2} + ... + P_{k_1+k_2+k_3}$. Hence $\mathcal{P}^2 < \mathcal{P}^2 + P_{k_1+k_2+i_1} + P_{k_1+k_2+i_2} + ... + P_{k_1+k_2+k_3} = P$. So we obtain that $\mathcal{P} < P < \mathcal{P}$. □

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TWO–DIMENSIONAL BILINEAR INEQUALITY FOR RECTANGULAR HARDY OPERATOR AND NON–FACTORIZABLE WEIGHTS

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Abstract. Necessary conditions and sufficient conditions are given for the validity of two–dimensional bilinear norm inequalities with rectangular Hardy operators in weighted Lebesgue spaces. The results are applicable for non–factorizable weights.

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1 Introduction

Let $\mathfrak{M}$ be the set of all Lebesgue measurable functions on $\mathbb{R}^2_+ := (0,\infty)^2$, and let $\mathfrak{M}^+ \subset \mathfrak{M}$ be the subset of all non-negative functions.

For fixed parameters $1 < p_1, p_2, q < \infty$ and weight functions $u, v_1, v_2 \in \mathfrak{M}^+$, we consider the problem of characterizing of the bilinear Hardy inequality

$$\left(\int_0^\infty\int_0^\infty (I_2f)^q\,(I_2g)^q\,u\right)^{\frac{1}{q}} \le C\left(\int_0^\infty\int_0^\infty f^{p_1}v_1\right)^{\frac{1}{p_1}}\left(\int_0^\infty\int_0^\infty g^{p_2}v_2\right)^{\frac{1}{p_2}} \tag{1.1}$$

for all $f, g \in \mathfrak{M}^+$, where

$$I_2f(x_1,x_2) := \int_0^{x_1}\int_0^{x_2} f(t_1,t_2)\,dt_1\,dt_2$$

is the two–dimensional Hardy operator. Here, $C > 0$ is supposed to be the best (least possible) constant that does not depend on f and g.

Integral transforms, which map a product of function spaces into another function space (multi–linear integral operators), have applications, in particular, to smoothness properties and approximation of function classes (see e.g. [13] and references therein). In the one–dimensional case a multi–linear analogue of (1.1) was considered in [3, 4] as an illustration of results about multi–linear inequalities. Other types of one–dimensional linear and bilinear integral operators in Lebesgue spaces and subclasses were studied in [1, 2, 5, 6, 7, 8, 9, 10, 11, 14, 15, 17, 21]. For product type weight functions (or factorizable weights) inequality (1.1) was completely studied in [16].

The goal of our work is to solve the same problem without such restrictions on weight functions.

The paper is organized as follows. In Section 2, we review auxiliary results which pertain to estimates of the best constant C in weighted two-dimensional linear Hardy inequality (2.1). In Section 3, the results are given on characterization of the best constant C in bilinear Hardy inequality (1.1). We consider thirteen cases depending on relations between the norm parameters p_1, p_2 and q.

The subsections correspond those relations between the numerical parameters for which the proofs of estimates are similar.

The dual operator to I_2 is defined as

$$I_2^* f(x_1,x_2) := \int_{x_1}^{\infty}\int_{x_2}^{\infty} f(t_1,t_2)dt_1dt_2.$$

The results of this paper for the operator I_2 can be proved in a similar way for the operator I_2^*.

Throughout the paper, products of the form $0\cdot\infty$ are taken to be equal to 0. By $A \lesssim B$ we mean that there exists $k>0$, which depends only on some insignificant numerical parameters, such that $A \le kB$. If $A\lesssim B$ and $B \lesssim A$ then we write $A \approx B$. If $p>1$ then $p' = p/(p-1)$.

2 Auxiliary results

Let us first recall Sawyer's Theorem (see [12, Theorem A] or [18, Theorem 1]).

Theorem 2.1. *Let $1<p\le q<\infty$ and $w,v \in \mathfrak{M}^+$ be weights. Then the inequality*

$$\left(\int_0^\infty\int_0^\infty (I_2 f)^q\, w\right)^{\frac1q} \le C\left(\int_0^\infty\int_0^\infty f^p v\right)^{\frac1p} \tag{2.1}$$

holds for some $C>0$ and for all $f\in\mathfrak{M}^+$ if and only if

$$D_1 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \big(I_2^* w(t_1,t_2)\big)^{\frac1q}\big(I_2\sigma(t_1,t_2)\big)^{\frac{1}{p'}} < \infty,$$

$$D_2 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\int_0^{t_2} (I_2\sigma)^q w\right)^{\frac1q} \big(I_2\sigma(t_1,t_2)\big)^{-\frac1p} < \infty,$$

$$D_3 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} (I_2^* w)^{p'}\sigma\right)^{\frac{1}{p'}} \big(I_2^* w(t_1,t_2)\big)^{-\frac{1}{q'}} < \infty,$$

where $\sigma := v^{1-p'}$. Moreover, if C is the best constant in (2.1), then

$$C \approx D_1 + D_2 + D_3. \tag{2.2}$$

Now, if $p<q$ there is an alternative estimate ([18], Theorem 2).

Theorem 2.2. *Let $1<p<q<\infty$ and $w,v\in\mathfrak{M}^+$ be weights. Then inequality (2.1) holds for some $C>0$ and for all $f\in\mathfrak{M}^+$ if and only if $D_1<\infty$. Moreover, if C is the best constant in (2.1), then*

$$C \approx D_1. \tag{2.3}$$

For the case $q<p$ the following results are known [20], Theorem 3).

Theorem 2.3. *Let $1<q<p<\infty$, $1/r := 1/q-1/p$ and let $w,v\in\mathfrak{M}^+$ be weights. Then inequality (2.1) holds for some $C>0$ and for all $f\in\mathfrak{M}^+$ if*

$$B_v := \left(\int_0^\infty\int_0^\infty \sigma(u,z)\left(\int_u^\infty\int_z^\infty (I_2\sigma)^{q-1} w\right)^{\frac rq} du\, dz\right)^{\frac1r} < \infty.$$

Reversely, if inequality (2.1) *true then* $B < \infty$, *where*

$$B := \left(\int_0^\infty \int_0^\infty d_y \big(I_2\sigma(x,y)\big)^{\frac{r}{p'}} d_x \Big(- \big(I_2^* w(x,y)\big)^{\frac{r}{q}} \Big) \right)^{\frac{1}{r}}$$
$$= \left(\int_0^\infty \int_0^\infty \big(I_2\sigma(x,y)\big)^{\frac{r}{p'}} d_x\, d_y \big(I_2^* w(x,y)\big)^{\frac{r}{q}} \right)^{\frac{1}{r}}$$
$$= \left(\int_0^\infty \int_0^\infty \big(I_2^* w(x,y)\big)^{\frac{r}{q}} d_x\, d_y \big(I_2\sigma(x,y)\big)^{\frac{r}{p'}} \right)^{\frac{1}{r}}.$$

Moreover, if C *is the best constant in* (2.1), *then*

$$B \lesssim C \lesssim B_v.$$

Theorem 2.4. *Let* $1 < q < p < \infty$ *and* $w, v \in \mathfrak{M}^+$ *be weights. Assume that the following is satisfied:*

(1) *there exists* $\gamma \in [\frac{q}{p}, 1)$ *such that* $\dfrac{\partial^2 \Big(\big[I_2\sigma(x,y)\big]^{\gamma} \Big)}{\partial x \partial y} \geq 0$ *for almost all* $(x,y) \in \mathbb{R}^2_+$;

(2) *there exists* $\gamma^* \in [\frac{p'}{q'}, 1)$ *such that* $\dfrac{\partial^2 \Big(\big[I_2 w(x,y)\big]^{\gamma^*} \Big)}{\partial x \partial y} \geq 0$ *for almost all* $(x,y) \in \mathbb{R}^2_+$.

Then inequality (2.1) *holds for some* $C > 0$ *and for all* $f \in \mathfrak{M}^+$ *if and only if* $B < \infty$. *Moreover, if* C *is the best constant in* (2.1), *then*

$$C \approx B. \tag{2.4}$$

3 Main results

We denote $\sigma_i := v_i^{1-p_i'}$ and $V_i := I_2\sigma_i$, where $i = 1, 2$.

There are thirteen ways to arrange three numbers p_1, p_2, q considering that some of them may be equal. We will break the thirteen cases into subcases based upon similarity of the proof.

3.1 Case $\max(p_1, p_2) \leq q$

The following cases arise when $\max(p_1, p_2) \leq q$:

1) $p_1 < p_2 = q$,

2) $p_2 < p_1 = q$,

3) $\max\{p_1, p_2\} < q$,

4) $p_1 = p_2 = q$.

Theorem 3.1. *Let* $p_1 \neq p_2$ *and* $\max(p_1, p_2) = q$ *or* $\max\{p_1, p_2\} < q$. *Assume* $w, v \in \mathfrak{M}^+$ *are weights. Then the best constant* C *in inequality* (1.1) *can be estimated as:*

1) $C \approx \mathcal{A}_1$, *if* $p_1 < p_2 = q$, *where*

$$\mathcal{A}_1 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \Big(\widetilde{D}_1(x,y) + \widetilde{D}_2(x,y) + \widetilde{D}_3(x,y) \Big)$$

and $\widetilde{D}_i(x,y)$, $i = 1, 2, 3$ *are defined by equations* (3.3), (3.4), (3.5), *respectively;*

2) $C \approx \mathcal{A}_2$, *if* $p_2 < p_1 = q$, *where*

$$\mathcal{A}_2 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_2(x,y)\big)^{\frac{1}{p_2'}} \Big(\widehat{D}_1(x,y) + \widehat{D}_2(x,y) + \widehat{D}_3(x,y)\Big)$$

and $\widehat{D}_i(x,y)$, $i = 1,2,3$ *are defined by equations* (3.6), (3.7), (3.8), *respectively;*

3) $C \approx \min\big\{\mathcal{B}_1, \mathcal{B}_2\big\}$, *if* $p_1 < p_2 < q$ *or* $p_2 < p_1 < q$ *or* $p_1 = p_2 < q$, *where*

$$\mathcal{B}_1 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \Big(\widetilde{D}_1(x,y)\Big),$$

$$\mathcal{B}_2 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_2(x,y)\big)^{\frac{1}{p_2'}} \Big(\widehat{D}_1(x,y)\Big).$$

Proof. For a given weight $v \in \mathfrak{M}^+$ and a fixed parameter $p > 1$ denote by

$$\|h\|_{p,v} := \left(\int_0^\infty\!\!\int_0^\infty |h|^p v\right)^{\frac{1}{p}}$$

the weighted Lebesgue norm of h. Then we have the following equality for the best constant C in (1.1):

$$C = \sup_{g\neq 0}\sup_{f\neq 0} \frac{\left(\int_0^\infty\!\!\int_0^\infty (I_2 f)^q\,(I_2 g)^q\,u\right)^{\frac{1}{q}}}{\|f\|_{p_1,v_1}\|g\|_{p_2,v_2}}. \tag{3.1}$$

1) Consider the case $p_1 < p_2 = q$. By virtue of (2.3) and (2.2),

$$\begin{aligned} C &\overset{(2.3)}{\approx} \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2} \sup_{(x,y)\in\mathbb{R}^2_+} \left(\int_0^\infty\!\!\int_0^\infty (I_2 g)^q \chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{1}{q}} \left(I_2 v_1^{1-p_1'}(x,y)\right)^{\frac{1}{p_1'}} \qquad (3.2)\\ &= \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \sup_{g\neq 0} \left(\int_0^\infty\!\!\int_0^\infty (I_2 g)^q \chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{1}{q}} \|g\|^{-1}_{p_2,v_2}\\ &\overset{(2.2)}{\approx} \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \Big(\widetilde{D}_1(x,y) + \widetilde{D}_2(x,y) + \widetilde{D}_3(x,y)\Big), \end{aligned}$$

where

$$\widetilde{D}_1(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{\frac{1}{p_2'}}, \tag{3.3}$$

$$\widetilde{D}_2(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\!\!\int_0^{t_2} (V_2)^q \chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}}, \tag{3.4}$$

$$\widetilde{D}_3(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{\left(\int_{t_1}^\infty\!\!\int_{t_2}^\infty \Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)\Big)^{p_2'} \sigma_2\right)^{\frac{1}{p_2'}}}{\Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)(t_1,t_2)\Big)^{\frac{1}{q'}}}. \tag{3.5}$$

2) The proof for the case $p_2 < p_1 = q$ is analogous to case 1). In this case

$$C \approx \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_2(x,y)\big)^{\frac{1}{p_2'}} \Big(\widehat{D}_1(x,y) + \widehat{D}_2(x,y) + \widehat{D}_3(x,y)\Big),$$

where

$$\widehat{D}_1(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{\frac{1}{p_1'}}, \tag{3.6}$$

$$\widehat{D}_2(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\int_0^{t_2} (V_1)^q \chi_{(x,\infty)\times(y,\infty)}u\right)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{-\frac{1}{p_1}}, \tag{3.7}$$

$$\widehat{D}_3(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} \Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)\Big)^{p_1'}\sigma_1\right)^{\frac{1}{p_1'}}}{\Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)(t_1,t_2)\Big)^{\frac{1}{q'}}}. \tag{3.8}$$

3) Let $p_1 < p_2 < q$ or $p_2 < p_1 < q$ or $p_1 = p_2 < q$. Analogously to (3.2),

$$\begin{aligned} C &\overset{(2.3)}{\approx} \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \sup_{g\neq 0} \left(\int_0^{\infty}\int_0^{\infty} (I_2 g)^q \chi_{(x,\infty)\times(y,\infty)}u\right)^{\frac{1}{q}} \|g\|^{-1}_{p_2,v_2} \\ &\overset{(2.3)}{\approx} \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \Big(\widetilde{D}_1(x,y)\Big) := \mathcal{B}_1. \end{aligned}$$

Similarly, we can obtain an alternative estimate:

$$C \approx \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_2(x,y)\big)^{\frac{1}{p_2'}} \Big(\widehat{D}_1(x,y)\Big) := \mathcal{B}_2.$$

Therefore, $C \approx \min\big\{\mathcal{B}_1, \mathcal{B}_2\big\}$. □

It remains to consider the case 4) $p_1 = p_2 = q$. By (2.2), we have

$$C \approx C_1 + C_2 + C_3,$$

where

$$C_1 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \sup_{g\neq 0} \left(\int_0^{\infty}\int_0^{\infty} (I_2 g)^q \chi_{(x,\infty)\times(y,\infty)}u\right)^{\frac{1}{q}} \|g\|^{-1}_{p_2,v_2},$$

$$C_2 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{-\frac{1}{p_1}} \sup_{g\neq 0} \left(\int_0^{\infty}\int_0^{\infty} (V_1)^q (I_2 g)^q \chi_{(0,x)\times(0,y)}u\right)^{\frac{1}{q}} \|g\|^{-1}_{p_2,v_2},$$

$$C_3 := \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \sup_{(x,y)\in\mathbb{R}^2_+} \frac{\left(\int_x^{\infty}\int_y^{\infty} \left(\int_0^{\infty}\int_0^{\infty} (I_2 g)^q \chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{p_1'} \sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}}}{\left(\int_0^{\infty}\int_0^{\infty} (I_2 g)^q \chi_{(x,\infty)\times(y,\infty)}u\right)^{\frac{1}{q'}}}. \tag{3.9}$$

From (2.2) it follows that

$$C_1 \approx \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p'_1}} \Big(S_1(x,y)+S_2(x,y)+S_3(x,y)\Big) =: Q_1,$$

where

$$S_1(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^\infty\int_0^\infty u\chi_{(\max(t_1,x),\infty)\times(\max(t_2,y),\infty)}\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{\frac{1}{p'_2}},$$

$$S_2(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_x^{t_1}\int_y^{t_2} (V_2)^q u\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}},$$

$$S_3(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{\left(\int_{t_1}^\infty\int_{t_2}^\infty \left(\int_0^\infty\int_0^\infty u\chi_{(\max(x,\rho),\infty)\times(\max(y,\tau),\infty)}\right)^{p'_2} \sigma_2(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p'_2}}}{\left(\int_0^\infty\int_0^\infty u\chi_{(\max(t_1,x),\infty)\times(\max(t_2,y),\infty)}\right)^{\frac{1}{q'}}},$$

and

$$C_2 \approx \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{-\frac{1}{p_1}} \Big(T_1(x,y)+T_2(x,y)+T_3(x,y)\Big) =: Q_2,$$

where

$$T_1(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^x\int_{t_2}^y (V_1)^q\, u\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{\frac{1}{p'_2}},$$

$$T_2(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^\infty\int_0^\infty (V_1V_2)^q u\chi_{(0,\min(t_1,x)\times(0,\min(t_2,y))}\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}},$$

$$T_3(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^\infty\int_{t_2}^\infty \left(\int_\rho^x\int_\tau^y (V_1)^q u\right)^{p'_2} \sigma_2(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p'_2}} \left(\int_{t_1}^x\int_{t_2}^y (V_1)^q u\right)^{-\frac{1}{q'}}.$$

Since $\chi_{(\rho,\infty)\times(\tau,\infty)} \le \chi_{(x,\infty)\times(y,\infty)}$ in (3.9) and $p'=q'$ then

$$\begin{aligned} C_3 &\lesssim \sup_{g\neq 0} \frac{\sup\limits_{(x,y)\in\mathbb{R}^2_+} \left(\int_x^\infty\int_y^\infty \left(\int_0^\infty\int_0^\infty (I_2g)^q\,\chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{p'_1-1} \sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p'_1}}}{\|g\|_{p_2,v_2}} \\ &= \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left(\left(\int_0^\infty\int_0^\infty \left(\int_0^\infty\int_0^\infty (I_2g)^q\,\chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{p'_1-1} \sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p'_1-1}}\right)^{\frac{1}{p_1}}. \end{aligned} \tag{3.10}$$

Let $p_1'-1\geq 1$. Applying Minkowskii's integral inequality with $p_1'-1$, we obtain

$$C_3 \lesssim \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty \big(I_2g(t,z)\big)^q u(t,z)\left(\int_0^t\int_0^z \sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'-1}} dt\,dz\right)^{\frac{1}{p_1}}$$

$$= \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty (I_2g)^q u (V_1)^{p_1-1}\right)^{\frac{1}{p_1}} \overset{(2.2)}{\lesssim} R_1+R_2+R_3.$$

Here,

$$R_1 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_{t_1}^\infty\int_{t_2}^\infty u(V_1)^{p_1-1}\right)^{\frac{1}{q}}\big(V_2(t_1,t_2)\big)^{\frac{1}{p_1'}},$$

$$R_2 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_0^{t_1}\int_0^{t_2} (V_2)^q u(V_1)^{p_1-1}\right)^{\frac{1}{q}}\big(V_2(t_1,t_2)\big)^{-\frac{1}{p_1}},$$

$$R_3 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_{t_1}^\infty\int_{t_2}^\infty\left(\int_t^\infty\int_z^\infty u(V_1)^{p_1-1}\right)^{p_1'}\sigma_2(t,z)\,dt\,dz\right)^{\frac{1}{p'}}\left(\int_{t_1}^\infty\int_{t_2}^\infty u(V_1)^{p_1-1}\right)^{-\frac{1}{q'}}.$$

If $p_1'-1<1$ then from (3.10) it follows that

$$C_3 \lesssim \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty\left(\int_0^\infty\int_0^\infty (I_2g)^q\,\chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{p_1'-1}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}}$$

$$\leq \left(\int_0^\infty\int_0^\infty\left(\sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty (I_2g)^q\,\chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{\frac{1}{p_1}}\right)^{p_1'}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}}$$

$$\overset{(2.2)}{\lesssim} \left(\int_0^\infty\int_0^\infty\Big(J_1(\rho,\tau)+J_2(\rho,\tau)+J_3(\rho,\tau)\Big)^{p_1'}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}} =: Q_3, \tag{3.11}$$

where

$$J_1(\rho,\tau) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_0^\infty\int_0^\infty u\chi_{(\max(t_1,\rho),\infty)\times(\max(t_2,\tau),\infty)}\right)^{\frac{1}{q}}\big(V_2(t_1,t_2)\big)^{\frac{1}{p_2'}},$$

$$J_2(\rho,\tau) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_\rho^{t_1}\int_\tau^{t_2}(V_2)^q u\right)^{\frac{1}{q}}\big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}}, \tag{3.12}$$

$$J_3(\rho,\tau) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\frac{\left(\int_{t_1}^\infty\int_{t_2}^\infty\left(\int_0^\infty\int_0^\infty u\chi_{(\max(\rho,z_1),\infty)\times(\max(\tau,z_2),\infty)}\right)^{p_2'}\sigma_2(z_1,z_2)\,dz_1\,dz_2\right)^{\frac{1}{p_2'}}}{\left(\int_0^\infty\int_0^\infty u\chi_{(\max(t_1,\rho),\infty)\times(\max(t_2,\tau),\infty)}\right)^{\frac{1}{q'}}}.$$

Note that estimates (3.11)–(3.12) hold for the case $p'-1\geq 1$ as well.

For a lower bound for C_3 we obtain from (3.9) by setting $g=\sigma_2\chi_{(0,t_1)\times(0,t_2)}$:

$$C_3 \geq \sup_{\substack{(x,y)\in\mathbb{R}^2_+\\(t_1,t_2)\in\mathbb{R}^2_+}} \frac{\left(\int\limits_x^\infty\int\limits_y^\infty\left(\int\limits_\rho^\infty\int\limits_\tau^\infty\big(V_2(\min(t_1,t),\min(t_2,z))\big)^q u(t,z)\,dt\,dz\right)^{p_1'}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}}}{\left(\int\limits_x^\infty\int\limits_y^\infty\big(V_2(\min(t_1,t),\min(t_2,z))\big)^q u(t,z)\,dt\,dz\right)^{\frac{1}{q'}}\big(V_2(t_1,t_2)\big)^{\frac{1}{p_2}}}.$$

Summarizing the above, we can state the following theorem.

Theorem 3.2. *Let $p_1=p_2=q$ and $w,v\in\mathfrak{M}^+$ be weights. Then the best constant C in inequality* (1.1) *can be estimated from above as follows.*

1. *If $p_1'-1<1$ then*
$$C\lesssim Q_1+Q_2+Q_3.$$
2. *If $p_1'-1\geq 1$ then*
$$C\lesssim Q_1+Q_2+\min\big\{R_1+R_2+R_3,Q_3\big\}.$$

A lower bound for C, independently of relations between $p_1'-1$ and 1, is

$$C\gtrsim Q_1+Q_2+ \\ +\sup_{\substack{(x,y)\in\mathbb{R}^2_+\\(t_1,t_2)\in\mathbb{R}^2_+}} \frac{\left(\int\limits_x^\infty\int\limits_y^\infty\left(\int\limits_\rho^\infty\int\limits_\tau^\infty\big(V_2(\min(t_1,t),\min(t_2,z))\big)^q u(t,z)\,dt\,dz\right)^{p_1'}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}}}{\left(\int\limits_x^\infty\int\limits_y^\infty\big(V_2(\min(t_1,t),\min(t_2,z))\big)^q u(t,z)\,dt\,dz\right)^{\frac{1}{q'}}\big(V_2(t_1,t_2)\big)^{\frac{1}{p_2}}}.$$

3.2 Case $q<\max\{p_1,p_2\}$

There following cases arise when $q<\max\{p_1,p_2\}$:

1) $p_1<q<p_2$ or $p_1<q<p_2$,

2) $q<\min\{p_1,p_2\}$,

3) $q=p_1<p_2$ or $q=p_2<p_1$.

Theorem 3.3. *Let $\min\{p_1,p_2\}<q<\max\{p_1,p_2\}$ and $w,v\in\mathfrak{M}^+$ be weights. Then the best constant C in inequality* (1.1) *can be estimated as*

$$\sup_{(x,y)\in\mathbb{R}^2_+}\big(V_1(x,y)\big)^{\frac{1}{p_1'}}\widetilde{B}_v(x,y)\lesssim C\lesssim \sup_{(x,y)\in\mathbb{R}^2_+}\big(V_1(x,y)\big)^{\frac{1}{p_1'}}\widetilde{B}(x,y)$$

in the case $p_1<q<p_2$ with $\widetilde{B}_v(x,y)$ and $\widetilde{B}(x,y)$ given by equalities (3.14) *and* (3.15)*. If $p_2<q<p_1$ then we have* (3.16).

Proof. Let $p_1 < q < p_2$. Then

$$C \overset{(2.3)}{\approx} \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \sup_{(x,y)\in\mathbb{R}^2_+} \left(\int\limits_0^\infty\int\limits_0^\infty (I_2 g)^q \,\chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{1}{q}} \left(I_2 v_1^{1-p_1'}(x,y)\right)^{\frac{1}{p_1'}}$$

$$= \sup_{(x,y)\in\mathbb{R}^2_+} \left(V_1(x,y)\right)^{\frac{1}{p_1'}} \sup_{g\neq 0} \left(\int\limits_0^\infty\int\limits_0^\infty (I_2 g)^q \,\chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{1}{q}} \|g\|^{-1}_{p_2,v_2}. \tag{3.13}$$

Therefore, from Theorem 2.3 it follows that

$$\sup_{(x,y)\in\mathbb{R}^2_+} \left(V_1(x,y)\right)^{\frac{1}{p_1'}} \widetilde{B}(x,y) \lesssim C \lesssim \sup_{(x,y)\in\mathbb{R}^2_+} \left(V_1(x,y)\right)^{\frac{1}{p_1'}} \widetilde{B}_v(x,y),$$

where with $1/r_2 := 1/q - 1/p_2$

$$\widetilde{B}_v(x,y) := \left(\int\limits_0^\infty\int\limits_0^\infty \sigma_2(t,z)\left(\int\limits_t^\infty\int\limits_z^\infty (V_2)^{q-1}\chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{r_2}{q}} dt\,dz\right)^{\frac{1}{r_2}}, \tag{3.14}$$

$$\widetilde{B}(x,y) := \left(\int\limits_0^\infty\int\limits_0^\infty d_z\left(V_2(t,z)\right)^{\frac{r_2}{p_2'}} d_t\left(-\left(I_2^*(\chi_{(x,\infty)\times(y,\infty)} u)(t,z)\right)^{\frac{r_2}{q}}\right)\right)^{\frac{1}{r_2}}. \tag{3.15}$$

The proof of the case $p_2 < q < p_1$ is analogous. Here,

$$\sup_{(x,y)\in\mathbb{R}^2_+} \left(V_2(x,y)\right)^{\frac{1}{p_2'}} \widehat{B}(x,y) \lesssim C \lesssim \sup_{(x,y)\in\mathbb{R}^2_+} \left(V_2(x,y)\right)^{\frac{1}{p_2'}} \widehat{B}_v(x,y) \tag{3.16}$$

and with $1/r_1 := 1/q - 1/p_1$

$$\widehat{B}_v(x,y) := \left(\int\limits_0^\infty\int\limits_0^\infty \sigma_1(t,z)\left(\int\limits_t^\infty\int\limits_z^\infty (V_1)^{q-1}\chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{r_1}{q}} dt\,dz\right)^{\frac{1}{r_1}},$$

$$\widehat{B}(x,y) := \left(\int\limits_0^\infty\int\limits_0^\infty d_z\left(V_1(t,z)\right)^{\frac{r_1}{p_1'}} d_t\left(-\left(I_2^*(\chi_{(x,\infty)\times(y,\infty)} u)(t,z)\right)^{\frac{r_1}{q}}\right)\right)^{\frac{1}{r_1}}.$$

□

Remark 1. *If the weights* u *and* σ_2 *in* (3.13) *satisfy properties* (1) *and* (2)*, respectively, from Theorem* 2.4*, then by virtue of* (2.4) *we obtain*

$$C \approx \sup_{(x,y)\in\mathbb{R}^2_+} \left(V_1(x,y)\right)^{\frac{1}{p_1'}} \left(\int\limits_0^\infty\int\limits_0^\infty \left(I_2^*[u\chi_{(x,\infty)\times(y,\infty)}](\rho,\tau)\right)^{\frac{r_2}{q}} d_\rho\, d_\tau\left(V_2(\rho,\tau)\right)^{\frac{r_2}{p_2'}}\right)^{\frac{1}{r_2}}.$$

Analogically, in the case $p_2 < q < p_1$*, if the weights* u *and* σ_1 *satisfy properties* (1) *and* (2) *from Theorem* 2.4*, then we obtain*

$$C \approx \sup_{(x,y)\in\mathbb{R}^2_+} \left(V_2(x,y)\right)^{\frac{1}{p_2'}} \left(\int\limits_0^\infty\int\limits_0^\infty \left(I_2^*(u\chi_{(x,\infty)\times(y,\infty)})(\rho,\tau)\right)^{\frac{r_1}{q}} d_\rho\, d_\tau\left(V_1(\rho,\tau)\right)^{\frac{r_1}{p_1'}}\right)^{\frac{1}{r_1}}.$$

Theorem 3.4. *Let $q < p_1 = p_2$ or $q < p_1 < p_2$ or $q < p_2 < p_1$ and $w, v \in \mathfrak{M}^+$ be weights. Then the best constant C in inequality* (1.1) *can be estimated from above as*

$$C \lesssim \min\left\{ \left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z)\big(E(t,z)\big)^{r_1} dt\, dz \right)^{\frac{1}{r_1}}, F \right\},$$

where E is defined by equation (3.17) *and F is defined by equation* (3.19).

If, in addition, the weights $w_1 := u(V_1)^{q-1}$ and $w_2 := u(V_1)^{\frac{q}{p_1'}}$ satisfy condition (1) *of Theorem* 2.4 *and the weight σ_2 is of type* (2), *then*

$$C \lesssim \min\{\tilde{J}, \hat{J}\},$$

where $\tilde{J}$ and $\hat{J}$ are defined by equations (3.18) *and* (3.20), *respectively.*

A lower bound for the best constant C in (1.1) *is given in* (3.22).

Proof. From (3.1) and Theorem 2.3 we have

$$\begin{aligned} C &\lesssim \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left(\int_t^\infty\!\!\int_z^\infty (I_2 g)^q\, (V_1)^{q-1} u \right)^{\frac{r_1}{q}} dt\, dz \right)^{\frac{1}{r_1}} \\ &= \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left(\int_t^\infty\!\!\int_z^\infty (I_2 g)^q\, w_1 \right)^{\frac{r_1}{q}} dt\, dz \right)^{\frac{1}{r_1}} \\ &\lesssim \left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left(\sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left(\int_0^\infty\!\!\int_0^\infty (I_2 g)^q\, w_1 \chi_{(t,\infty)\times(z,\infty)} \right)^{\frac{1}{q}} \right)^{r_1} dt\, dz \right)^{\frac{1}{r_1}} \\ &\lesssim \left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \big(E(t,z)\big)^{r_1} dt\, dz \right)^{\frac{1}{r_1}}. \end{aligned}$$

Here,

$$E(t,z) = \left(\int_0^\infty\!\!\int_0^\infty \sigma_2(\rho,\tau) \left(\int_0^\infty\!\!\int_0^\infty (V_1 V_2)^{q-1} u \chi_{(\max(t,\rho),\infty)\times(\max(z,\tau),\infty)} \right)^{\frac{r_2}{q}} d\rho\, d\tau \right)^{\frac{1}{r_2}}. \tag{3.17}$$

If the weights w_1 and σ_2 satisfy properties (1), (2) of Theorem 2.4, then

$$\begin{aligned} C &\lesssim \left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left(\sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left(\int_0^\infty\!\!\int_0^\infty (I_2 g)^q\, w_1 \chi_{(t,\infty)\times(z,\infty)} \right)^{\frac{1}{q}} \right)^{r_1} dt\, dz \right)^{\frac{1}{r_1}} \\ &\lesssim \left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \big(J(t,z)\big)^{r_1} dt\, dz \right)^{\frac{1}{r_1}} =: \tilde{J}, \end{aligned} \tag{3.18}$$

where

$$J(t,z) = \left(\int_0^\infty\!\!\int_0^\infty \Big(\big(I_2^* w_1 \chi_{(t,\infty)\times(z,\infty)}\big)(x,y) \Big)^{\frac{r_2}{q}} d_x\, d_y \big(V_2(x,y)\big)^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}}.$$

Alternatively, we can write

$$C \lesssim \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left(\int_t^\infty\!\!\int_z^\infty (I_2 g)^q \,(V_1)^{q-1} u \right)^{\frac{r_1}{q}} dt\,dz \right)^{\frac{1}{r_1}}$$

$$= \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \left(\left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left(\int_t^\infty\!\!\int_z^\infty (I_2 g)^q \,(V_1)^{q-1} u \right)^{\frac{r_1}{q}} dt\,dz \right)^{\frac{q}{r_1}} \right)^{\frac{1}{q}}.$$

Application of Minkowski's integral inequality with the exponent r_1/q yields

$$C \lesssim \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \left(\int_0^\infty\!\!\int_0^\infty \big((I_2 g)(x,y)\big)^q \big(V_1(x,y)\big)^{\frac{q}{p_1'}} u(x,y)\,dx\,dy \right)^{\frac{1}{q}}$$

$$\lesssim \left(\int_0^\infty\!\!\int_0^\infty \sigma_2(\tilde t,\tilde z) \left(\int_{\tilde t}^\infty\!\!\int_{\tilde z}^\infty (V_1)^{\frac{q}{p_1'}} (V_2)^{q-1} u \right)^{\frac{r_2}{q}} d\tilde t\,d\tilde z \right)^{\frac{1}{r_2}} := F. \tag{3.19}$$

Therefore,

$$C \lesssim \min \left\{ \left(\int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \big(E(t,z)\big)^{r_1} dt\,dz \right)^{\frac{1}{r_1}}, F \right\}.$$

If the weights w_2 and σ_2 are of types (1) and (2) from Theorem 2.4, then

$$C \lesssim \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \left(\int_0^\infty\!\!\int_0^\infty (I_2 g)^q\,(x,y) w_2(x,y)\,dx\,dy \right)^{\frac{1}{q}}$$

$$\lesssim \left(\int_0^\infty\!\!\int_0^\infty \Big(\big(I_2^* w_2\big)(x,y) \Big)^{\frac{r_2}{q}} d_x\,d_y \big(V_2(x,y)\big)^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}} =: \hat J. \tag{3.20}$$

For the lower bound for C we use Theorem 2.3, first, to obtain

$$C \gtrsim \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \left(\int_0^\infty\!\!\int_0^\infty \Big(I_2^* \big((I_2 g)^q w\big)(x,y) \Big)^{\frac{r_1}{q}} d_x\,d_y \big(I_2\sigma_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}$$

$$= \sup_{g\neq 0} \left(\int_0^\infty\!\!\int_0^\infty \left(\|g\|_{p_2,v_2}^{-1} \left(\int_0^\infty\!\!\int_0^\infty \chi_{(x,\infty)\times(y,\infty)} (I_2 g)^q w \right)^{\frac{1}{q}} \right)^{r_1} d_x\,d_y \big(V_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}. \tag{3.21}$$

After this, substituting the test function

$$g_0(s,\tau,x,y) := \sigma_2(s,\tau) \left(\int_s^\infty \big(V_2(\rho,\tau)\big)^{\frac{r_1}{q'}} \Big(\big(I_2^* w_0(x,y)\big)(\rho,\tau) \Big)^{\frac{r_1}{p_1}} \left(\int_\tau^\infty w_0(\rho,z)\,dz \right) d\rho \right)^{\frac{1}{p_1}}$$

with $w_0 := \chi_{(x,\infty)\times(y,\infty)} u$ into (3.21) (see [18, pages 627–631] for details) implies

$$C \gtrsim \left(\int_0^\infty\!\!\int_0^\infty \left(\int_0^\infty\!\!\int_0^\infty \Big(\big(I_2^* w_0(x,y)\big)(s,t) \Big)^{\frac{r_2}{q}} d_s\,d_t \big(V_2(s,t)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{r_1}{r_2}} d_x\,d_y \big(V_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}. \tag{3.22}$$

□

Theorem 3.5. *Let $w, v \in \mathfrak{M}^+$ be weights.*

1. *If $q = p_2 < p_1 < \infty$ then the best constant C in inequality (1.1) can be estimated as*
$$C \lesssim \min\{G, K\}.$$
where functionals G, K are defined in (3.23) and (3.24) respectively. A lower bound for C is as given in (3.27).

2. *If $q = p_1 < p_2 < \infty$ then the best constant C in inequality (1.1) can be estimated as*
$$C \lesssim \min\{\tilde{G}, \tilde{K}\},$$
where functionals $\tilde{G}$ and $\tilde{K}$ are defined in (3.25) and (3.26) respectively. A lower estimate for C is as given in (3.28).

Proof. 1. Let $q = p_2 < p_1 < \infty$. By Theorem 2.3 and (2.2), we have

$$C \lesssim \left(\int_0^\infty\int_0^\infty \sigma_1(t,z)\left(\sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty (I_2 g)^q\,(V_1)^{q-1}u\chi_{(t,\infty)\times(z,\infty)}\right)^{\frac{1}{q}}\right)^{r_1} dt\,dz\right)^{\frac{1}{r_1}}$$
$$\approx \left(\int_0^\infty\int_0^\infty \sigma_1(t,z)\Big(G_1(t,z) + G_2(t,z) + G_3(t,z)\Big)^{r_1} dt\,dz\right)^{\frac{1}{r_1}} := G, \tag{3.23}$$

where

$$G_1(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big(\chi_{(t,\infty)\times(z,\infty)}(V_1)^{q-1}u\big)(t_1,t_2)\Big)^{\frac{1}{q}}\big(V_2(t_1,t_2)\big)^{\frac{1}{p_2'}},$$
$$G_2(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\int_0^{t_2}(V_2)^q(V_1)^{q-1}\chi_{(t,\infty)\times(z,\infty)}u\right)^{\frac{1}{q}}\big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}},$$
$$G_3(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{\left(\int_{t_1}^\infty\int_{t_2}^\infty\Big(I_2^*\big(\chi_{(t,\infty)\times(z,\infty)}(V_1)^{q-1}u\big)\Big)^{p_2'}\sigma_2\right)^{\frac{1}{p_2'}}}{\Big(I_2^*\big(\chi_{(t,\infty)\times(z,\infty)}(V_1)^{q-1}u\big)(t_1,t_2)\Big)^{\frac{1}{q'}}}.$$

Alternatively, analogously to the proof of Theorem 3.5 we obtain

$$C \lesssim \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\left(\int_0^\infty\int_0^\infty\sigma_1(t,z)\left(\int_t^\infty\int_z^\infty (I_2 g)^q(V_1)^{q-1}u\right)^{\frac{r_1}{q}}dt\,dz\right)^{\frac{q}{r_1}}\right)^{\frac{1}{q}}$$
$$\Big[\text{by Minkowski's integral inequality with the exponent } r_1/q\Big]$$
$$\lesssim \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty\big((I_2 g)(x,y)\big)^q(V_1)^{\frac{q}{p_1'}}(x,y)u(x,y)\,dx\,dy\right)^{\frac{1}{q}}$$
$$\lesssim K_1 + K_2 + K_3 := K. \tag{3.24}$$

Here,

$$K_1 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big((V_1)^{\frac{q}{p_1'}}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{\frac{1}{p_2'}},$$

$$K_2 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\int_0^{t_2} (V_2)^q (V_1)^{\frac{q}{p_1'}}u\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}},$$

$$K_3 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} \Big(I_2^*\big((V_1)^{\frac{q}{p_1'}}u\big)\Big)^{p_2'}\sigma_2\right)^{\frac{1}{p_2'}} \Big(I_2^*\big((V_1)^{\frac{q}{p_1'}}u\big)(t_1,t_2)\Big)^{-\frac{1}{q'}}.$$

Therefore,

$$C \lesssim \min\big\{G, K\big\}.$$

2. If $q = p_1 < p_2 < \infty$ then, analogously to Case 1,

$$C \lesssim \left(\int_0^{\infty}\int_0^{\infty} \sigma_2(t,z)\left(\widetilde{G}_1(t,z) + \widetilde{G}_2(t,z) + \widetilde{G}_3(t,z)\right)^{r_1} dt\,dz\right)^{\frac{1}{r_1}} := \tilde{G}, \tag{3.25}$$

where

$$\widetilde{G}_1(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big(\chi_{(t,\infty)\times(z,\infty)}(V_2)^{q-1}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{\frac{1}{p_1'}},$$

$$\widetilde{G}_2(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\int_0^{t_2} (V_1)^q (V_2)^{q-1}\chi_{(t,\infty)\times(z,\infty)}u\right)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{-\frac{1}{p_1}},$$

$$\widetilde{G}_3(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} \Big(I_2^*\big(\chi_{(t,\infty)\times(z,\infty)}(V_2)^{q-1}u\big)^{p_1'}\sigma_1\right)^{\frac{1}{p_1'}}}{\Big(I_2^*\big(\chi_{(t,\infty)\times(z,\infty)}(V_2)^{q-1}u\big)(t_1,t_2)\Big)^{\frac{1}{q'}}}.$$

Alternatively,

$$C \lesssim \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3 := \tilde{K}. \tag{3.26}$$

Here,

$$\tilde{K}_1 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big((V_2)^{\frac{q}{p_2'}}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{\frac{1}{p_1'}},$$

$$\tilde{K}_2 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\int_0^{t_2} (V_1)^q (V_2)^{\frac{q}{p_2'}}u\right)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{-\frac{1}{p_1}},$$

$$\tilde{K}_3 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} \Big(I_2^*\big((V_2)^{\frac{q}{p_2'}}u\big)\Big)^{p_1'}\sigma_1\right)^{\frac{1}{p_1'}} \Big(I_2^*\big((V_2)^{\frac{q}{p_2'}}u\big)(t_1,t_2)\Big)^{-\frac{1}{q'}}.$$

Therefore,

$$C \lesssim \min\{\tilde{G}, \tilde{K}\}.$$

To derive the lower estimate for C in Case 1, we start from (3.21)

$$C \gtrsim \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \left(\int_0^\infty\int_0^\infty \left(\int_0^\infty\int_0^\infty \chi_{(x,\infty)\times(y,\infty)}(I_2 g)^q w \right)^{\frac{r_1}{q}} d_x\, d_y \big(V_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}$$

and obtain, by setting $g = \sigma_2 \chi_{(0,s)\times(0,t)}$, that

$$C \gtrsim \sup_{(s,t)\in\mathbb{R}_+^2} \frac{\left(\int_0^\infty\int_0^\infty \left(\int_x^\infty\int_y^\infty (I_2 \sigma_2 \chi_{(0,s)\times(0,t)})^q w \right)^{\frac{r_1}{q}} d_x\, d_y \big(V_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}}{\big(V_2(s,t)\big)^{\frac{1}{p_2}}}. \qquad (3.27)$$

Analogously, in Case 2:

$$C \gtrsim \sup_{(s,t)\in\mathbb{R}_+^2} \frac{\left(\int_0^\infty\int_0^\infty \left(\int_x^\infty\int_y^\infty (I_2 \sigma_1 \chi_{(0,s)\times(0,t)})^q w \right)^{\frac{r_2}{q}} d_x\, d_y \big(V_2(x,y)\big)^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}}}{\big(V_1(s,t)\big)^{\frac{1}{p_1}}}. \qquad (3.28)$$

□

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BOUNDEDNESS OF THE GENERALIZED RIEMANN-LIOUVILLE OPERATOR IN LOCAL MORREY-TYPE SPACES

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Key words: Morrey-type spaces, generalized Riemann-Liouville operator, boundedness.

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Abstract. The objective of this paper is to establish the boundedness of the generalized multidimensional Riemann-Liouville integral operator from one local Morrey-type space to another one under some conditions on numerical parameters p and q.

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1 Introduction

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $0 < p \leq \infty, 0 \leq \lambda \leq \dfrac{n}{p}$. We denote by $LM_p^\lambda(\Omega)$, local Morrey-type spaces, the spaces of all functions $f \in L_p^{loc}\Omega$ with finite quasi-norms

$$\|f\|_{LM_p^\lambda(\Omega)} = \sup_{r>0} r^{-\lambda} \|f\|_{L_p(\Omega \bigcap B(0,r)} < \infty.$$

For properties of Morrey-type spaces, introduced in [7], see, for example, [1]-[5].

Definition 2. The left multidimensional fractional Riemann-Liouville integral operator $I_{a+}^{\overline{\alpha}} f$ of order $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$, $0 < \alpha_i < 1$, $i = 1, ..., n$, $a = (a_1, ..., a_n) \in \mathbb{R}^n$, is defined as follows

$$\left(I_{a+}^{\overline{\alpha}} f\right)(x) = \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_{a_n}^{x_n} ... \int_{a_1}^{x_1} \left(\prod_{i=1}^n (x_i - t_i)^{\alpha_i - 1}\right) f(t_1, ..., t_n) dt_1 ... dt_n \tag{1.1}$$

for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$ such that $x_i > a_i$, $i = 1, .., n$, where Γ is the Euler Gamma-function.

Let $\tau_i = t_i - a_i$, then

$$\left(I_{a+}^{\overline{\alpha}} f\right)(x)$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^{x_n - a_n} ... \int_0^{x_1 - a_1} \left(\prod_{i=1}^n (x_i - a_i - \tau_i)^{\alpha_i - 1}\right) f(\tau_1 + a_1, ..., \tau_n + a_n) d\tau_1 ... d\tau_n$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^{x_n - a_n} ... \int_0^{x_1 - a_1} \left(\prod_{i=1}^n (x_i - a_i - \tau_i)^{\alpha_i - 1}\right) g(\tau_1, ..., \tau_n) d\tau_1 ... d\tau_n$$

$$= (I_{0+}^{\overline{\alpha}} g)(x_1 - a_1, ..., x_n - a_n), \tag{1.2}$$

where $g(\tau_1, ..., \tau_n) = f(\tau_1 + a_1, ..., \tau_n + a_n)$.

The right multidimensional fractional Riemann-Liouville integral operator of order $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$, $0 < \alpha_i < 1$, $b = (b_1, ..., b_n) \in \mathbb{R}^n$ is defined similarly:

$$\left(I_{b_-}^{\overline{\alpha}} f\right)(x) = \frac{1}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \int_{x_n}^{b_n} ... \int_{x_1}^{b_1} \left(\prod_{i=1}^{n} (t_i - x_i)^{\alpha_i - 1} \right) f(t_1 ... t_n) dt_1 ... dt_n$$

for all $x \in \mathbb{R}^n$ such that $x_i < b_i$, $i = 1, ..., n$.

Definition 3. Let $f \in L_p(\Omega)$, where $0 < p \leq \infty$, $\overline{k} = (k_1, ..., k_n)$, $k_i \geq 0$, $i = 1, ..., n$. The generalized Riemann-Liouville fractional integral operator $I^{\overline{\alpha}, \overline{k}} f$ of order $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$, $0 < \alpha_i < 1$, $i = 1, ..., n$, $n \in \mathbb{N}$, is defined by

$$\left(I_{a_+}^{\overline{\alpha}, \overline{k}} f\right)(x)$$
$$= \prod_{i=1}^{n} \frac{(k_i + 1)^{1 - \alpha_i}}{\Gamma(\alpha_i)} \int_{a_n}^{x_n} ... \int_{a_1}^{x_1} \left(\prod_{i=1}^{n} \left[(x_i^{k_i + 1} - t_i^{k_i + 1})^{\alpha_i - 1} t_i^{k_i} \right] \right) f(t_1, ..., t_n) dt_1 ... dt_n \qquad (1.3)$$

Remark 1. If $k_i = 0$, $\forall i = 1, ..., n$, $\overline{k} = 0$ in Definition 3; we get the usual Riemann-Liouville integral operator defined by (1.1).

For the operator I_{0+}^{α}, the following theorem was proved in [8].

Let a, $b \in \mathbb{R}^n$, $0 < a_i < b_i < \infty$, $i = 1, ..., n$, and

$$Q(a, b) = \{x \in \mathbb{R}^n, a_i < x_i < b_i, i = 1, ..., n\}.$$

Theorem 1.1. *Let $1 < p \leq \infty$, $0 < q \leq \infty$, $0 \leq \lambda \leq \frac{n}{p}$, $0 \leq \mu \leq \frac{n}{q}$, $\frac{1}{p} < \alpha_i < 1$, $i = 1, ..., n$. Then there exists $C_1 > 0$ such that*

$$\| I_{a_+}^{\alpha} f \|_{M_q^{\mu}(Q(a,b))} \leq C_1 \mid b - a \mid^{\nu} \| f \|_{M_p^{\lambda}(Q(a,b))}, \qquad (1.4)$$

where

$$\nu = \lambda + \alpha_1 + ... + \alpha_n - \frac{n}{p} + \frac{n}{q} - \mu, \qquad (1.5)$$

for all finite parallelepipeds $Q(a, b)$ and for all $f \in M_p^{\lambda}(Q(a, b))$.

The exponent ν cannot be replaced by any other one.

2 Main results

Lemma 2.1. ([8]) *Let $0 < p \leq \infty$, $0 \leq \lambda \leq \frac{n}{p}$. Then*

$$\| f \|_{L_p(Q(0,y))} \leq |y|^{\lambda} \| f \|_{LM_p^{\lambda}(Q(0,b))} \qquad (2.1)$$

for any parallelepipeds $Q(0, b)$ and for any $y \in Q(0, b)$.

Theorem 2.1. *Let $1 < p < \infty$, $0 < q \leq \infty$, $0 \leq \lambda \leq \frac{n}{p}$, $0 \leq \mu \leq \frac{n}{q}$, $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$, $\frac{1}{p} < \alpha_i < 1$, $\overline{k} = (k_1, ..., k_n)$, $k_i \geq 0$, $i = 1, ..., n$.*

Then there exists $C_2 > 0$ such that

$$\| I_{0_+}^{\overline{\alpha} \overline{k}} f \|_{LM_q^{\mu}(Q(a,b))} \leq C_2 \mid b \mid^{\nu} \| f \|_{LM_p^{\lambda}(Q(a,b))}, \qquad (2.2)$$

where

$$\nu = \lambda + \frac{n}{q} - \frac{n}{p} + \sum_{i=1}^{n} (k_i + 1) \alpha_i - \mu, \qquad (2.3)$$

for all finite parallelepipeds $Q(a, b)$ and for all $f \in LM_p^{\lambda}(Q(a, b))$.

The exponent ν cannot be replaced by any other one.

Proof. Part 1. By (1.1) with $a = 0$ and Hölder's inequality it follows that

$$\left| \left(I_{0+}^{\overline{\alpha},\overline{k}} f \right)(x) \right|$$

$$\leq C_3 \left\| \prod_{i=1}^{n} (x_i^{k_i+1} - t_i^{k_i+1})^{(\alpha_i-1)} t_i^{k_i} \right\|_{L_{p'}(Q(0,x))} \|f\|_{L_p(Q(0,x))} (t_i = x_i \tau_i)$$

$$\leq C_3 \prod_{i=1}^{n} x_i^{\alpha_i(k_i+1)-\frac{1}{p}} \prod_{i=1}^{n} \left(\int_0^1 (1-\tau_i^{k_i+1})^{(\alpha_i-1)p'} \tau_i^{k_i p'} d\tau_i \right)^{\frac{1}{p'}} \|f\|_{L_p(Q(0,x))},$$

where

$$C_3 = \prod_{j=1}^{n} \frac{(k_j+1)^{1-\alpha_j}}{\Gamma(\alpha_j)}.$$

By changing variable $\tau_i^{k_i+1} = z_i$ and taking into account that

$$(\alpha_i - 1)p' = \frac{\alpha_i p - 1}{p-1} - 1$$

and $\frac{(p'-1)k_i}{k_i+1} = \frac{p'k_i+1}{k_i+1} - 1$, we obtain

$$\int_0^1 (1-\tau_i^{k_i+1})^{(\alpha_1-1)p'} \tau_i^{k_i p'} d\tau_i = \frac{1}{k_i+1} B\left(\frac{\alpha_i p - 1}{p-1}, \frac{p'k_i+1}{k_i+1} \right),$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta-function.

So

$$\left| \left(I_{0+}^{\overline{\alpha},\overline{k}} f \right)(x) \right| \leq C_4 \left(\prod_{i=1}^{n} x_i^{(k_i+1)\alpha_i - \frac{1}{p}} \right) \|f\|_{L_p(Q(0,x))},$$

where

$$C_4 = C_3 \prod_{i=1}^{n} \frac{1}{(k_i+1)^{1-\frac{1}{p}}} \left(B\left(\frac{\alpha_i p - 1}{p-1}, \frac{p'k_i+1}{k_i+1} \right) \right)^{1-\frac{1}{p}}.$$

By Lemma 2.1, since $(k_i+1)\alpha_i - \frac{1}{p} > 0$, we get

$$\left| \left(I_{0+}^{\overline{\alpha},\overline{k}} f \right)(x) \right| \leq C_4 \mid x \mid^{\lambda - \frac{n}{p} + \sum_{i=1}^{n}(k_i+1)\alpha_i} \| f \|_{LM_p^{\lambda}(Q(0,b))}$$

and

$$\left\| | I_{0+}^{\overline{\alpha},\overline{k}} f \right\|_{L_q(Q(0,b) \bigcap B(0,r))}$$

$$\leq C_4 \mid b \mid^{\lambda - \frac{n}{p} + \sum_{i=1}^{n}(k_i+1)\alpha_i} \| 1 \|_{L_q(Q(0,b)) \bigcap B(0,r))} \| f \|_{LM_p^{\lambda}(Q(0,b))}.$$

We consider two cases:

1) If $r < |b|$, since $0 < \mu \leq \frac{n}{q}$, then

$$r^{-\mu} \left\| I_{0+}^{\overline{\alpha},\overline{k}} f \right\|_{L_q(Q(0,b) \bigcap B(0,r))}$$

$$\leq C_5 \mid b \mid^{\lambda - \frac{n}{p} + \frac{n}{q} - \mu + \sum_{i=1}^{n}(k_i+1)\alpha_i} \| f \|_{LM_p^{\lambda}(Q(0,b))},$$

where

$$C_5 = C_4 \nu_n^{\frac{n}{q}}.$$

2) If $r \geq |b|$, then

$$r^{-\mu} \left\| I_{0_+}^{\overline{\alpha},\overline{k}} f \right\|_{L_q(Q(0,b) \bigcap B(0,r))}$$

$$\leq C_4 \mid b \mid^{\lambda - \frac{n}{p} + \frac{n}{q} - \mu + \sum_{i=1}^n (k_i+1)\alpha_i} \| f \|_{LM_p^\lambda (Q(0,b))} .$$

Hence (2.2) follows.

Part 2. Suppose that $I_{0_+}^{\overline{\alpha},\overline{k}} f$ is bounded from $LM_p^\lambda(Q(0,b))$ to $LM_q^\mu(Q(0,b))$, that is: for some $C_5(b) > 0$ depending on b, p, q, λ, μ and k.

$$\left\| I_{0_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^\mu(Q(0,b))} \leq C_6(b) \| f \|_{LM_p^\lambda(Q(0,b))} . \tag{2.4}$$

Assume that $b_1 = ... = b_n = \beta$, then $\beta = \frac{|b|}{\sqrt{n}}$

Let

$$f(x) = \begin{cases} 1 & x \in Q(\frac{b}{2}, b) \\ 0 & x \in Q(0,b) \backslash Q(\frac{b}{2}, b). \end{cases}$$

$$\|f\|_{L_p(Q(0,b) \bigcap B(0,r))} = \| f \|_{L_p(Q(0,b) \backslash Q(0,\frac{b}{2}) \bigcap B(0,r))}$$

$$= \| 1 \|_{L_p(Q(\frac{b}{2},b) \bigcap B(0,r))} = \left| Q\left(\frac{b}{2}, b\right) \bigcap B(0,r) \right|^{\frac{1}{p}} \leq \left| Q\left(\frac{b}{2}, b\right) \right|^{\frac{1}{p}} = \left(\frac{|b|}{2}\right)^{\frac{n}{p}} \tag{2.5}$$

and

$$\|f\|_{LM_p^\lambda(Q(\frac{b}{2},b) \bigcap B(0,r))} \leq \sup_{r \geq \frac{|b|}{2}} r^{-\lambda} \left| Q\left(\frac{b}{2}, b\right) \right|^{\frac{1}{p}} = C_7 |b|^{\frac{n}{p} - \lambda},$$

where

$$C_7 = 2^{\lambda - \frac{n}{p}}.$$

Let estimate $\left\| I_{a_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^\mu(Q(\frac{b}{2},b))}$, so

$$\left\| I_{a_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^\mu(Q(\frac{b}{2},b))}$$

$$\geq C_3 \left\| \int_0^{x_n} ... \int_0^{x_1} \prod_{i=1}^n (x_i^{k_i+1} - t_i^{k_i+1})^{\alpha_i - 1} t_i^{k_i} dt_1 ... dt_n \right\|_{LM_q^\mu(Q(\frac{b}{2},b))} \quad (t_i = x_i \tau_i)$$

$$\geq C_8 r^{-\mu} \left\| \prod_{i=1}^n x_i^{\alpha_i(k_i+1)} \right\|_{L_q(Q(\frac{b}{2},b) \bigcap B(0,r))} \Bigg|_{r=|b|} = C_8 |b|^{-\mu} \left\| \prod_{i=1}^n x_i^{\alpha_i(k_i+1)} \right\|_{L_q(Q(\frac{b}{2},b))},$$

where

$$C_8 = C_3 \beta^{\frac{1}{q}} (k_i + 1, \alpha_i).$$

$$\left\| \prod_{i=1}^n x_i^{\alpha_i(k_i+1)} \right\|_{L_q(Q(\frac{b}{2},b))}$$

$$\geq \left(\frac{|b|}{2}\right)^{\sum_{i=1}^n (k_i+1)\alpha_i} \left| Q\left(\frac{b}{2}, b\right) \right|^{\frac{1}{q}} = 2^{-(\frac{n}{q} + \sum_{i=1}^n (k_i+1)\alpha_i)} |b|^{\frac{n}{q} + \sum_{i=1}^n (k_i+1)\alpha_i}$$

$$\geq C_9 |b|^{\frac{n}{q} + \sum_{i=1}^n (k_i+1)\alpha_i},$$

where

$$C_9 = 2^{-(\frac{n}{q}+\sum_{i=1}^{n}(k_i+1)\alpha_i)}.$$

Consequently

$$\left\| I_{0_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^{\mu}(Q(0,b))} \geq C_{10}|b|^{\frac{n}{q}-\mu+\sum_{i=1}^{n}(k_i+1)\alpha_i}, \tag{2.6}$$

where $C_{10} = C_8 C_9$.

Finally, by (2.4), (2.5) and (2.6), we get

$$C_{10}|b|^{\frac{n}{q}-\mu+\sum_{i=1}^{n}(k_i+1)\alpha_i} \leq \left\| I_{0_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^{\mu}(Q(0,b))} \leq C_6 C_7 |b|^{\frac{n}{p}-\lambda},$$

where $C_{10} = C_8 C_9$.

Hence

$$C_6(b) \geq \frac{C_1 0}{C_7}|b|^{\nu},$$

where ν is defined by (2.3). □

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ON THE HADAMARD AND MARCHAUD-HADAMARD-TYPES MIXED FRACTIONAL INTEGRO-DIFFERENTIATION

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Abstract. The paper is devoted to the integral representations of the Marchaud-Hadamard and Marchaud-Hadamard-type truncated mixed fractional derivatives in weighted mixed Lebesgue spaces. Inversion theorems and characterization of tne Hadamard and Hadamard-type mixed fractional integrals of functions in weighted mixed Lebesgue spaces are proven.

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1 Introduction

It is known that the Riemann-Liouville fractional integro-differentiation is formally a fractional power of $\left(\frac{d}{dx}\right)^{\alpha}$ and is invariant relative to translation [19, 20]. J. Hadamard [10] suggested a construction of fractional integro-differentiation, which is a fractional power of the type $\left(x\frac{d}{dx}\right)^{\alpha}$. This construction is well suited to the case of the half-axis and is invariant relative to dilation.

We consider the Hadamard and Hadamard-type fractional integro-differentiation of functions of several variables in mixed Lebesgue spaces. Lebesgue spaces with a mixed norm were introduced and studied in [2]. The boundedness of operators on mixed norm spaces was studied in [1, 3, 17, 23]. A number of properties of mixed Lebesgue spaces can be found in [5]. Since the function spaces with mixed norm have finer structures than the corresponding classical function spaces, they naturally arise in studies of solutions of partial differential equations used to model physical processes involving spatial and time variables, such as thermal or wave equations [9, 11, 16].

The one-dimensional Hadamard and Hadamard-type fractional integro-differentiation has been studied by many researchers [6-8], [12-15], [21-22], [25]. A number of properties of the Hadamard fractional integration can be found in [20, 19]. In this paper, we extended the operation of the Hadamard and Hadamard-type fractional integro-differentiation to the case of multivariable functions, when these operators, applied to each variable or to some of them, give the so-called partial and mixed fractional integrals and derivatives in the framework of spaces $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ with a mixed norm.

Partial and mixed Marchaud fractional derivatives in the case of two variables were considered in [20]. In [13], [14], the conditions were obtained for the existence of unique solutions to problems of Cauchy type for nonlinear differential equations with fractional Hadamard and Marchaud-Hadamard-type derivatives in spaces of summable functions and for the solutions in a closed form of Cauchy type problems for linear differential equations of fractional order.

In [4], the properties of some integro-differential operators that generalize the fractional differentiation operators in the Hadamard and Hadamard-Marchaud sense in the class of harmonic functions were considered. As an application of the obtained properties, the solvability of nonlocal problems for the Laplace equation in a ball was studied.

In this paper, we obtain integral representations for the Marchaud-Hadamard and Marchaud-Hadamard-type of truncated fractional derivatives. In addition, the inversion theorems and characterization of ordinary Hadamard-type fractional integrals of functions from $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ are proven.

The consideration is conducted in the framework of spaces with a mixed norm

$$\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\left(\mathbb{R}^n_+,\ \frac{dx}{x}\right) =$$

$$= \left\{ f : \left\| f;\ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| = \left\{ \int\limits_0^\infty [\ldots (\int\limits_0^\infty |f(x)|^{p_1} x_1^{-\gamma_1} \frac{dx_1}{x_1})^{\frac{p_2}{p_1}} \ldots]^{\frac{p_n}{p_{n-1}}} x_n^{-\gamma_n} \frac{dx_n}{x_n} \right\}^{\frac{1}{p_n}} < \infty \right\},$$

$$C_{\overline{\gamma}}\left(\mathbb{R}^n_+\right) = \left\{ f : \|f;\ C_{\overline{\gamma}}\| = \sup_{x\in\mathbb{R}^n_+} \left| x^{-\overline{\gamma}} f(x) \right| < \infty,\ \lim_{|x|\to 0} x^{-\overline{\gamma}} f(x) = \lim_{|x|\to\infty} x^{-\overline{\gamma}} f(x) \right\},$$

where $\gamma_i \geq 0$, $i = \overline{1,n}$. Norm in $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ is determined by the formula

$$\|f\|_{\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}} = \left\| f;\ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| = \left\| x^{-\overline{\gamma}*} f;\ \mathfrak{L}^{\overline{p}} \right\|, 1 \leq \overline{p} \leq \infty, \tag{1.1}$$

where $x^{-\overline{\gamma}^*} = x_1^{-\gamma_1^*} \cdot \ldots \cdot x_n^{-\gamma_n^*}$,

$$\gamma_i^* = \begin{cases} \frac{\gamma_i}{p_i}, & 1 \leq p_i < \infty, \\ \gamma_i,\ p_i = \infty, & i = \overline{1,n}. \end{cases} \tag{1.2}$$

The paper has the following structure. In Sections 2, 3, 4, we give definitions and various auxiliary features of multiple integro-differentiation of Hadamard and Hadamard-type for multivariable functions (in terms of tensor products), and the auxiliary lemmas for spaces $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ are given in Section 5. Sections 6, 7, 8, 9 contain the proofs of basic results: the boundedness of the fractional integration of Hadamard and Hadamard type in spaces with mixed norms is proven in Section 6; in Section 7 we describe the integral representations of truncated mixed fractional derivatives of Marchaud-Hadamard and Marchaud-Hadamard-type in weighted mixed Lebesgue spaces. Sections 8 and 9 contain the inversion theorem and characterization the Hadamard and Hadamard-type mixed fractional integrals on functions from $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$.

Notations. $\mathbb{N}, \mathbb{R} = \mathbb{R}^1, \mathbb{C}$ are the sets of all positive integers, real numbers and complex numbers respectively; $\mathbb{R}^1_+ = (0; +\infty)$ is the semi-axis; $\mathbb{R}^n$ is the n-dimensional Euclidean space of points $x = (x_1, x_2, \ldots, x_n)$; $\dot{\mathbb{R}}^n-$ compactification of $\mathbb{R}^n$ by one infinitely remote point. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0,\ x_2 > 0, \ldots, x_n > 0\}$. Everywhere below: E is the identity operator; $(\Pi_\delta f)(x) = f(x \circ \delta)$, $x, \delta \in \mathbb{R}^n_+$ is the dilation operator. Introduce mixed finite difference of function f of vector order $l = (l_1, l_2, \ldots, l_n)$, $l_k \in \mathbb{N}$ with a "multiplicative" vector step of $t \in \mathbb{R}^n_+$:

$$(\tilde{\Delta}^l_t f)(x) = \tilde{\Delta}^{l_1}_{\xi_1} [\tilde{\Delta}^{l_2}_{\xi_2} \ldots (\tilde{\Delta}^{l_n}_{\xi_n} f)](x) = \sum_{0\leq|k|\leq l} (-1)^{|k|} \tbinom{l}{k} f\left(x \circ t^k\right), \tag{1.3}$$

here $x \circ t^k = \left(x_i \cdot t_1^{k_1}, \ldots, x_n \cdot t_n^{k_n}\right)$ and $\binom{l}{k} = \prod\limits_{i=1}^n \binom{l_i}{k_i}$, $\binom{l_i}{k_i}$ are the binomial coefficients, k is a multi-index. Let us agree that the record $1 \leq \overline{p} < \infty$ and $\overline{p} = \overline{\infty}$, where $\overline{p} = (p_1, , \ldots, p_n)$, $\overline{\infty} = (\infty, \ldots, \infty)$ means that, $1 \leq p_i < \infty, p_i = \infty$, $i = \overline{1,n}$. $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\left(\mathbb{R}^n_+,\ \frac{dx}{x}\right)$, $1 \leq \overline{p} < \infty$; $C\left(\dot{\mathbb{R}}^n_+\right) = \{\ f\ :\ f \in C\left(\dot{\mathbb{R}}^n_+\right), f(0) = f(\infty)\ \}, \overline{p} = \overline{\infty}$. Let $\omega = (\omega_1, \ldots, \omega_n)$, then $\rho^\omega = (\rho_1^{\omega_1}, \ldots, \rho_n^{\omega_n})$,

$x \circ \rho^{\omega} = (x_1 \cdot \rho_1^{\omega_1}, \ldots, x_n \cdot \rho_n^{\omega_n})$, $(x : \rho^{\omega}) = (x \cdot \rho^{-\omega}) = \left(\frac{x_1}{\rho_1^{\omega_1}}, \ldots, \frac{x_n}{\rho_n^{\omega_n}}\right)$. If $u = (u_1, u_2, \ldots, u_n)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then $u_+^{\alpha} = \prod\limits_{i=1}^{n} (u_i)_+^{\alpha_i}$, $(u_i)_+^{\alpha_i} = \begin{cases} u_i^{\alpha_i}, & u_i > 0, \\ 0, & u_i < 0. \end{cases}$ We use $\aleph(\alpha, l) = \prod\limits_{i=0}^{n} \aleph(\alpha_i, l_i)$, $\aleph(\alpha_i, l_i) = \int\limits_0^{\infty} t^{-1-\alpha_i} (1 - e^{-t})^{l_i} \, dt$ as the normalization constant, known in the theory of fractional differentiation; $C_0^{\infty}(\mathbb{R}_+^n)$ is the class of all infinitely continuously differentiable functions with compact support in $\mathbb{R}_+^n$.

2 Partial and mixed Hadamard and Hadamard-type fractional integrals and derivatives

We start with defining the partial and mixed Hadamard and Hadamard-type fractional integrals and derivatives.

Definition 1. Let $x \in \mathbb{R}_+^n$. The left and the right partial Hadamard-type fractional integrals of order $\alpha_k \in \mathbb{R}$ $(\alpha_k > 0)$ of a function φ with respect to the variable x_k are defined by

$$\left(J_{+,\mu_k}^{\alpha_k}\varphi\right)(x) := \frac{1}{\Gamma(\alpha_k)} \int\limits_0^{x_k} \left(\frac{t}{x_k}\right)^{\mu_k} \left(\ln \frac{x_k}{t}\right)^{\alpha_k - 1} \varphi(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n) \frac{dt}{t}$$
$$= \frac{1}{\Gamma(\alpha_k)} \int_0^1 u^{\mu_k} \left(\ln \frac{1}{u}\right)^{\alpha_k - 1} \varphi(x \circ u\mathbf{e}_k) \frac{du}{u},$$

and

$$\left(J_{-,\mu_k}^{\alpha_k}\varphi\right)(x) := \frac{1}{\Gamma(\alpha_k)} \int\limits_{x_k}^{\infty} \left(\frac{x_k}{t}\right)^{\mu_k} \left(\ln \frac{t}{x_k}\right)^{\alpha_k - 1} \varphi(x_1, \ldots, x_{k-1}, t, x_{k+1}, ..., x_n) \frac{dt}{t}$$
$$= \frac{1}{\Gamma(\alpha_k)} \int\limits_1^{\infty} u^{-\mu_k} (\ln u)^{\alpha_k - 1} \varphi(x \circ u\mathbf{e}_k) \frac{du}{u},$$

respectively, where $\mathbf{e}_k = (\underbrace{0, \ldots, 0,}_{k-1} 1, 0, \ldots, 0)$, $x \circ u\mathbf{e}_k = (x_1, \ldots, x_{k-1}, x_k \cdot u, x_{k+1}, \ldots, x_n)$.

Definition 2. Let $x \in \mathbb{R}_+^n$. The left and the right partial Hadamard-type fractional derivatives of order α_k $(0 < \alpha_k < 1)$ of a function φ with respect to the variable x_k are defined by

$$\left(\mathfrak{D}_{+,\mu_k}^{\alpha_k}\varphi\right)(x) = \frac{x_k^{1-\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int\limits_0^{x_k} \left(\frac{t}{x_k}\right)^{\mu_k} \left(\ln \frac{x_k}{t}\right)^{-\alpha_k} \varphi(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n) \frac{dt}{t}$$
$$= \frac{x_k^{1-\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int\limits_0^1 u^{\mu_k} \left(\ln \frac{1}{u}\right)^{-\alpha_k} \varphi(x \circ u\mathbf{e}_k) \frac{du}{u} = x_k^{1-\mu_k} \frac{\partial}{\partial x_k} \left(J_{+,\mu_k}^{1-\alpha_k}\varphi\right)(x), \tag{2.1}$$

$$\left(\mathfrak{D}_{-,\mu_k}^{\alpha_k}\varphi\right) = \frac{-x_k^{1+\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int\limits_{x_k}^{\infty} \left(\frac{x_k}{t}\right)^{\mu_k} \left(\ln \frac{t}{x_k}\right)^{-\alpha_k} \varphi(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n) \frac{dt}{t}$$
$$= \frac{-x_k^{1+\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int\limits_1^{\infty} u^{-\mu_k} (\ln u)^{-\alpha_k} \varphi(x \circ u\mathbf{e}_k) \frac{du}{u} = -x_k^{1+\mu_k} \frac{\partial}{\partial x_k} \left(J_{-,\mu_k}^{1-\alpha_k}\varphi\right)(x) \tag{2.2}$$

respectively.

Definition 3. For a function $\varphi(x)$, defined on $\mathbb{R}^n_+$, the following integrals

$$\left(J^{\alpha}_{+\cdots+}\varphi\right)(x) = \int\limits_0^{x_1}\dots\int\limits_0^{x_n}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\ln\frac{x_i}{t_i}\right)^{\alpha_i-1}\frac{dt_1}{t_1}\dots\frac{dt_1}{t_1}, \tag{2.3}$$

$$\left(J^{\alpha}_{-\cdots-}\varphi\right)(x) = \int\limits_{x_1}^{\infty}\dots\int\limits_{x_n}^{\infty}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\ln\frac{t_i}{x_i}\right)^{\alpha_i-1}\frac{dt_1}{t_1}\dots\frac{dt_1}{t_1} \tag{2.4}$$

are called the integrals of fractional order α ($\alpha_i > 0,\ i = \overline{1,n}$) in the sense of Hadamard (left and right, respectively).

Definition 4. For a function $\varphi(x)$, defined on $\mathbb{R}^n_+$, the integrals

$$\left(J^{\alpha}_{+\cdots+,\mu}\varphi\right)(x) = \int\limits_0^{x_1}\dots\int\limits_0^{x_n}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\frac{t_i}{x_i}\right)^{\mu_i}\left(\ln\frac{x_i}{t_i}\right)^{\alpha-1}\frac{dt_1}{t_1}\dots\frac{dt_n}{t_n}, \tag{2.5}$$

$$\left(J^{\alpha}_{-\cdots-,\mu}\varphi\right)(x) = \int\limits_{x_1}^{\infty}\dots\int\limits_{x_n}^{\infty}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\frac{x_i}{t_i}\right)^{\mu_i}\left(\ln\frac{t_i}{x_i}\right)^{\alpha_i-1}\frac{dt_1}{t_1}\dots\frac{dt_n}{t_n}, \tag{2.6}$$

$$\left(\Im^{\alpha}_{+\cdots+,\mu}\varphi\right)(x) = \int\limits_0^{x_1}\dots\int\limits_0^{x_n}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\frac{t_i}{x_i}\right)^{\mu_i}\left(\ln\frac{x_i}{t_i}\right)^{\alpha-1}\frac{dt_1}{x_1}\dots\frac{dt_n}{x_n},$$

$$\left(\Im^{\alpha}_{-\cdots-,\mu}\varphi\right)(x) = \int\limits_{x_1}^{\infty}\dots\int\limits_{x_n}^{\infty}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\frac{x_i}{t_i}\right)^{\mu_i}\left(\ln\frac{t_i}{x_i}\right)^{\alpha_i-1}\frac{dt_1}{x_1}\dots\frac{dt_n}{x_n}$$

are called the mixed integrals of fractional order α ($\alpha_i > 0,\ i = \overline{1,n}$) of the Hadamard type (left and right, respectively).

Operators (2.3)-(2.6) commute with the dilation operator $\Pi_\rho J^{\alpha}_{\pm\cdots\pm} = J^{\alpha}_{\pm\cdots\pm}\Pi_\rho$, $\Pi_\rho J^{\alpha}_{\pm\cdots\pm,\mu} = J^{\alpha}_{\pm\cdots\pm,\mu}\Pi_\rho$, and are related to the Riemann-Liouville operator $I^{\alpha}_{\pm\cdots\pm}$ by the following equalities

$$J^{\alpha}_{\pm\cdots\pm}\varphi = Q^{-1}I^{\alpha}_{\pm\cdots\pm}Q\varphi,\ \left(J^{\alpha}_{\pm\cdots\pm,\mu}\varphi\right)(x) = \left(M_{\mp\mu}Q^{-1}I^{\alpha}_{\pm\cdots\pm}QM_{\pm\mu}\varphi\right)(x),$$

where $(Q\varphi)(x) = \varphi(e^x) = \varphi(e^{x_1},\dots,e^{x_n})$, $(Q^{-1}\varphi)(x) = \varphi(\ln x) = \varphi(\ln x_1,\dots,\ln x_n)$, $(M_{\pm\mu}\varphi)(x) = x_1^{\pm\mu_1}\dots x_n^{\pm\mu_n}\varphi(x_1,\dots,x_n)$ (see [20], p. 251 and [8], p. 11).

The operators $J^{\alpha}_{\pm\cdots\pm}$ and $J^{\alpha}_{\pm\cdots\pm,\mu}$ have semi-group properties:

$$J^{\alpha}_{\pm\cdots\pm}J^{\beta}_{\pm\cdots\pm}\varphi = J^{\alpha+\beta}_{\pm\cdots\pm}\varphi\,(\alpha\ge 0,\ \beta\ge 0),$$

$$J^{\alpha}_{\pm\cdots\pm,\mu}J^{\beta}_{\pm\cdots\pm,\mu}\varphi = J^{\alpha+\beta}_{\pm\cdots\pm,\mu}\varphi\,(\alpha\ge 0,\ \beta\ge 0).$$

The expressions

$$\left(\mathfrak{D}^{\alpha}_{+\dots+,\mu}f\right)(x)$$

$$= \prod_{k=1}^{n}\frac{x_k^{1-\mu_k}}{\Gamma(1-\alpha_k)}\frac{\partial^n}{\partial x_1\dots\partial x_n}\int\limits_0^{x_1}\dots\int\limits_0^{x_n}\prod_{k=1}^{n}t_k^{\mu_k}\left(\ln\frac{x_k}{t_k}\right)^{-\alpha_k}f(t)\frac{dt_1}{t_1}\dots\frac{dt_n}{t_n},$$

$$\left(\mathfrak{D}^{\alpha}_{-\dots-,\mu}f\right)(x)$$

$$= \prod_{k=1}^{n} \frac{(-1)^n x_k^{1+\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial^n}{\partial x_1 \dots \partial x_n} \int\limits_{x_1}^{\infty} \dots \int\limits_{x_n}^{\infty} \prod_{k=1}^{n} t_k^{-\mu_k} \left(\ln \frac{t_k}{x_k}\right)^{-\alpha_k} f(t) \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n}$$

are called the mixed fractional derivatives of the Hadamard-type of order $\alpha = (\alpha_1, \dots, \alpha_n)$ $(0 < \alpha_k < 1,\ k = \overline{1,n})$.

For $\alpha_k \geq 1,\ k = \overline{1,n}$, the mixed fractional derivatives of the Hadamard-type are introduced in the following way

$$(\mathfrak{D}^{\alpha}_{+\cdots+,\mu} f)(x) = \prod_{k=1}^{n} \frac{x_k^{[\alpha_k]+1-\mu_k}}{\Gamma([\alpha_k]+1-\alpha_k)} \times$$

$$\times \frac{\partial^{[\alpha_1]+\cdots+[\alpha_n]+n}}{\partial x_1^{[\alpha_1]+1} \dots \partial x_n^{[\alpha_n]+1}} \int\limits_{0}^{x_1} \dots \int\limits_{0}^{x_n} \prod_{k=1}^{n} t_k^{\mu_k} \left(\ln \frac{x_k}{t_k}\right)^{[\alpha_k]-\alpha_k} f(t) \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n}, \tag{2.7}$$

$$(\mathfrak{D}^{\alpha}_{-\cdots-,\mu} f)(x) = \prod_{k=1}^{n} \frac{(-1)^{[\alpha_1]+\cdots+[\alpha_n]+n}\, x_k^{[\alpha_k]+1+\mu_k}}{\Gamma([\alpha_k]+1-\alpha_k)} \times$$

$$\times \frac{\partial^{[\alpha_1]+\cdots+[\alpha_n]+n}}{\partial x_1^{[\alpha_1]+1} \dots \partial x_n^{[\alpha_n]+1}} \int\limits_{x_1}^{\infty} \dots \int\limits_{x_n}^{\infty} \prod_{k=1}^{n} t_k^{-\mu_k} \left(\ln \frac{t_k}{x_k}\right)^{[\alpha_k]-\alpha_k} f(t) \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n}, \tag{2.8}$$

where $\alpha_k > 0,\ k = \overline{1,n}$ and $[\alpha_k],\ k = \overline{1,n}$ are the integral parts of $\alpha_k,\ k = \overline{1,n}$. Substituting $t_i = x_i \cdot y_i,\ t_i = x_i \cdot y_i^{-1}, i = \overline{1,n}$, integrals (2.5), (2.6) can be written in the following way:

$$(J^{\alpha}_{+\cdots+,\mu}\varphi)(x) = \int\limits_{0}^{\infty} \dots \int\limits_{0}^{\infty} \varphi(x \circ y) \prod_{i=1}^{n} k^{+}_{\mu_i,\alpha_i}(y_i) \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n},$$

$$(J^{\alpha}_{-\cdots-,\mu}\varphi)(x) = \int\limits_{0}^{\infty} \dots \int\limits_{0}^{\infty} \varphi\left(x \circ y^{-1}\right) \prod_{i=1}^{n} k^{+}_{\mu_i,\alpha_i}(y_i) \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n},$$

where $x \circ y = (x_1 \cdot y_1, \dots, x_n \cdot y_n)$, $x \circ y^{-1} = \left(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n}\right)$,

$$k^{+}_{\mu_i,\alpha_i}(y_i) = \begin{cases} \frac{1}{\Gamma(\alpha_i)} y_i^{\mu} \left(\ln \frac{1}{y_i}\right)^{\alpha_i - 1}, & 0 < y_i < 1, \\ 0, & y_i > 1, \end{cases} \quad , i = \overline{1,n}.$$

Next we introduce a modification of mixed fractional integrals with a kernel "improved" at infinity:

$$(I^{\alpha,l}_{+\cdots+,\mu;\tau}\varphi)(x) = \int\limits_{0}^{\infty} \dots \int\limits_{0}^{\infty} \left(\tilde{\Delta}^{l}_{\tau^{-1}} k^{+}_{\mu,\alpha}\right)(y)\, \varphi(x \circ y) \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n}, \tag{2.9}$$

$$(I^{\alpha,l}_{-\cdots-,\mu;\tau}\varphi)(x) = \int\limits_{0}^{\infty} \dots \int\limits_{0}^{\infty} \left(\tilde{\Delta}^{l}_{\tau^{-1}} k^{+}_{\mu,\alpha}\right)(y)\, \varphi\left(x \circ y^{-1}\right) \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n}, \tag{2.10}$$

where $\tau \in \mathbb{R}^n_+$, $l_i > \alpha_i > 0$, $i = \overline{1,n}$,

$$\left(\tilde{\Delta}^{l}_{\tau^{-1}} k^{+}_{\mu,\alpha}\right)(y) = \tilde{\Delta}^{l_1}_{\tau_1^{-1}} \tilde{\Delta}^{l_2}_{\tau_2^{-1}} \dots (\tilde{\Delta}^{l_n}_{\tau_n^{-1}} k^{+}_{\mu,\alpha})(y), \quad k^{+}_{\mu,\alpha}(y) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} y_i^{\mu_i} \left(\ln \frac{1}{y_i}\right)_{+}^{\alpha_i - 1}.$$

It is obvious that $I^{\alpha,l}_{\pm\cdots\pm,\mu;\tau}\varphi=\tilde{\Delta}^{l}_{\tau}I^{\alpha}_{\pm\cdots\pm,\mu}\varphi$ on sufficiently good functions $\varphi\left(x\right)$, i.e. operators (2.9)-(2.10) are obtained by applying the definition in (1.3) of the difference operators $\tilde{\Delta}^{(l_1,\ldots,l_n)}_{(\tau_1,\ldots,\tau_n)}$ with a "multiplicative" step to the operators $J^{\alpha}_{\pm\cdots\pm,\mu}\varphi$. They have the advantage over $J^{\alpha}_{\pm\cdots\pm,\mu}\varphi$, at $l_i>\alpha_i>0,\ i=\overline{1,n}$, as they are limited in the space $L_{\bar{p},\bar{\gamma}}\left(\mathbb{R}^n_+,\frac{dx}{x}\right)$ for all $1\le p_i<\infty,\ \gamma_i>0,\ i=\overline{1,n}$ (i.e., including the case of $\gamma_i=0,\ i=\overline{1,n}$).

At $\mu=0$ the partial and mixed Hadamard fractional integrals and derivatives are obtained.

3 Mixed fractional integro-differentiation in terms of tensor products

It is convenient to use the concept of the tensor product of operators, introduced by the following definition.

Definition 5. Let $A_1u_1,\,A_2u_2,\ldots,A_nu_n$ be the linear operators defined on functions $u_1\left(x\right),\,u_2\left(x\right),\ldots,u_n\left(x\right)$ of one variable. The tensor product of operators $A_1,\,A_2,\ldots,A_n$ is an operator$A_1\otimes\,A_2\otimes\ldots\otimes A_n$ which is defined on functions of the form

$$\varphi\left(x_1,x_2,\ldots,x_n\right)=\sum_i u^i_1\left(x_1\right)\cdot\ldots\cdot u^i_n\left(x_n\right) \tag{3.1}$$

by the relation

$$(A_1\otimes\,A_2\otimes\ldots\otimes A_n)\varphi\left(x_1,x_2,\ldots,x_n\right)=\sum_i A_1\,u^i_1\left(x_1\right)\cdot\ldots\cdot A_nu^i_n\left(x_n\right).$$

From Definition 5, it follows that the operators of mixed fractional integro-differentiation $J^{\alpha}_{\pm\cdots\pm}\varphi,\ J^{\alpha}_{\pm\cdots\pm,\mu}\varphi,\ \mathfrak{D}^{\alpha}_{\pm\cdots\pm}f,\ \mathfrak{D}^{\alpha}_{\pm\cdots\pm,\mu}f,\ \alpha=(\alpha_1,\ldots,\alpha_n)$ are the tensor products of the corresponding one-dimensional operators

$$J^{\alpha}_{\pm\cdots\pm}\varphi=J^{\alpha_1}_{\pm}\otimes\cdots\otimes J^{\alpha_n}_{\pm}\varphi, \tag{3.2}$$

$$J^{\alpha}_{\pm\cdots\pm,\mu}\varphi=J^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes J^{\alpha_n}_{\pm,\mu_n}\varphi, \tag{3.3}$$

$$\mathfrak{D}^{\alpha}_{\pm\cdots\pm}f=\mathfrak{D}^{\alpha_1}_{\pm}\otimes\cdots\otimes\mathfrak{D}^{\alpha_n}_{\pm}f, \tag{3.4}$$

$$\mathfrak{D}^{\alpha}_{\pm\cdots\pm,\mu}f=\mathfrak{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\mathfrak{D}^{\alpha_n}_{\pm,\mu_n}f. \tag{3.5}$$

The following operators are also considered

$$J^{\alpha}_{\pm\cdots\mp\cdots\pm}\varphi=J^{\alpha_1}_{\pm}\otimes\cdots\otimes J^{\alpha_i}_{\mp}\otimes\cdots\otimes J^{\alpha_n}_{\pm}\varphi,$$

$$J^{\alpha}_{\pm\cdots\mp\cdots\pm,\mu}\varphi=J^{\alpha_1}_{\pm,\mu_1}\otimes\ldots\otimes J^{\alpha_i}_{\mp,\mu_i}\otimes\cdots\otimes J^{\alpha_n}_{\pm,\mu_n}\varphi,$$

$$\mathfrak{D}^{\alpha}_{\pm\cdots\mp\cdots\pm}f=\mathfrak{D}^{\alpha_1}_{\pm}\otimes\cdots\otimes\mathfrak{D}^{\alpha_i}_{\mp}\otimes\cdots\otimes\mathfrak{D}^{\alpha_n}_{\pm}f,$$

$$\mathfrak{D}^{\alpha}_{\pm\cdots\mp\cdots\pm,\mu}f=D^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\mathfrak{D}^{\alpha_i}_{\mp,\mu_i}\otimes\cdots\otimes\mathfrak{D}^{\alpha_n}_{\pm,\mu_n}f,$$

with the appropriate choice of signs. The case $\alpha_i=0$ for some i means the absence of integro-differentiation in (3.2) - (3.5) in the i-th variable

$$J^{(\alpha_1,\ldots,0,\ldots,\alpha_n)}_{\pm\cdots\pm\cdots\pm}\varphi=J^{\alpha_1}_{\pm}\otimes\ldots\otimes J^{\alpha_{i-1}}_{\pm}\otimes E\otimes J^{\alpha_{i+1}}_{\pm}\otimes\cdots\otimes J^{\alpha_n}_{\pm}\varphi,$$

$$J^{(\alpha_1,\ldots,0,\ldots,\alpha_n)}_{\pm\cdots\pm\cdots\pm,\mu}\varphi=J^{\alpha_1}_{\pm,\mu_1}\otimes\ldots\otimes J^{\alpha_{i-1}}_{\pm,\mu_{i-1}}\otimes E\otimes J^{\alpha_{i+1}}_{\pm,\mu_{i+1}}\otimes\ldots\otimes J^{\alpha_n}_{\pm,\mu_n}\varphi,$$

$$\mathfrak{D}^{(\alpha_1,\ldots,0,\ldots,\alpha_n)}_{\pm\cdots\pm\cdots\pm}f=\mathfrak{D}^{\alpha_1}_{\pm}\otimes\cdots\otimes\mathfrak{D}^{\alpha_{i-1}}_{\pm}\otimes E\otimes D^{\alpha_{i+1}}_{\pm}\otimes\cdots\otimes\mathfrak{D}^{\alpha_n}_{\pm}f,$$

$$\mathfrak{D}^{(\alpha_1,\ldots,0,\ldots,\alpha_n)}_{\pm\cdots\pm\cdots\pm,\mu}f=\mathfrak{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\mathfrak{D}^{\alpha_{i-1}}_{\pm,\mu_{i-1}}\otimes E\otimes\mathfrak{D}^{\alpha_{i+1}}_{\pm,\mu_{i+1}}\otimes\cdots\otimes\mathfrak{D}^{\alpha_n}_{\pm,\mu_n}f.$$

4 Mixed Marchaud-Hadamard and Marchaud-Hadamard-type fractional differentiation

Derivatives (2.7), (2.8) can be easily reduced on sufficiently good functions $f(x)$ to a form similar to the fractional Marchaud derivative.

Definition 6. For a function $f(x)$ defined on $\mathbb{R}^n_+$, the expression

$$\left(D^{\alpha}_{\pm\cdots\pm}f\right)(x)=\frac{1}{\prod\limits_{k=1}^{n}\aleph(\alpha_k,l_k)}\int\limits_0^1\cdots\int\limits_0^1\prod_{k=1}^{n}\left(\ln\frac{1}{t_k}\right)^{-1-\alpha_k}\left(\tilde{\Delta}^{l}_{t^{\pm1}}f\right)(x)\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n},$$

is called the mixed fractional Marchaud-Hadamard derivative of order $\alpha=(\alpha_1,\ldots,\alpha_n),\alpha_i>0,\ i=\overline{1,n}$.

The mixed fractional Marchaud-Hadamard derivatives $D^{\alpha}_{\pm\cdots\pm}f$ are related to the fractional Marchaud derivatives $\mathbb{D}^{\alpha}_{\pm\cdots\pm}f$ by the equalities

$$D^{\alpha}_{\pm\cdots\pm}f=Q^{-1}\mathbb{D}^{\alpha}_{\pm\cdots\pm}Qf,$$

where $(Qf)(x)=f(e^{x_1},\ldots,e^{x_n})$, $(Q^{-1}f)(x)=f(\ln x_1,\ldots,\ln x_n)$.

The partial fractional derivatives of the Hadamard-type (2.1)-(2.2) can be written (on sufficiently good functions) in the Marchaud form

$$\left(D^{\alpha_k}_{\pm,\mu_k}f\right)=\frac{\alpha_k}{\Gamma(1-\alpha_k)}\int\limits_0^1 t^{\mu_k}\left(\ln\frac{1}{t}\right)^{-\alpha_k-1}\left[f(x)-f\left(x\circ t^{\pm1}\mathbf{e}_k\right)\right]\frac{dt}{t}+\mu_k^{\alpha_k}f(x)$$

$$=\frac{\alpha_k}{\Gamma(1-\alpha_k)}\int\limits_0^\infty e^{-\mu_k t}\frac{f(x)-f\left(x\circ e^{\pm t\mathbf{e}_k}\right)}{t^{\alpha_k+1}}dt+\mu_k^{\alpha_k}f(x), \tag{4.1}$$

where $\mathbf{e}_k=(\underbrace{0,\ldots,0}_{k-1},1,0,\ldots,0)$, $x\circ t^{\pm1}\mathbf{e}_k=(x_1,\ldots,x_{k-1},x_k\cdot t^{\pm1},x_{k+1},\ldots,x_n)$. Hence it is easy to see that for the mixed fractional derivatives of the Marchaud-Hadamard-type, instead of (4.1) we obtain

$$D^{\alpha}_{\pm\cdots\pm,\mu}f=\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}+\mu_1^{\alpha_1}E\right)\otimes\left(\tilde{D}^{\alpha_2}_{\pm,\mu_2}+\mu_2^{\alpha_2}E\right)\otimes\ldots\otimes\left(\tilde{D}^{\alpha_n}_{\pm,\mu_n}+\mu_n^{\alpha_n}E\right)f$$

$$=\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}f+\sum_{i=1}^{n}\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}\right)_{\mu_i^{\alpha_i}E}f$$

$$+\sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n}\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}\right)_{\mu_{ij}^{\alpha_{ij}}E}f+\cdots+\sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n}\left(\mu_1^{\alpha_1}E\otimes\ldots\otimes\mu_n^{\alpha_n}E\right)_{\tilde{D}^{\alpha_{ij}}_{\pm,\mu_{ij}}}f$$

$$+\sum_{i=1}^{n}\left(\mu_1^{\alpha_1}E\otimes\cdots\otimes\mu_n^{\alpha_n}E\right)_{\tilde{D}^{\alpha_i}_{\pm,\mu_i}}f+\mu_1^{\alpha_1}E\otimes\cdots\otimes\mu_n^{\alpha_n}Ef,$$

where $0<\alpha_k<1,\ k=\overline{1,n}$,

$$\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}\right)_{\mu_i^{\alpha_i}E}=\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_{i-1}}_{\pm,\mu_{i-1}}\otimes\mu_i^{\alpha_i}E\otimes\tilde{D}^{\alpha_{i+1}}_{\pm,\mu_{i+1}}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n},$$

$$\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}\right)_{\mu_{ij}^{\alpha_{ij}}E}$$

$$=\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_{i-1}}_{\pm,\mu_{i-1}}\otimes\mu_i^{\alpha_i}E\otimes\tilde{D}^{\alpha_{i+1}}_{\pm,\mu_{i+1}}\otimes\cdots\otimes\tilde{D}^{\alpha_{j-1}}_{\pm,\mu_{j-1}}\otimes\mu_j^{\alpha_j}E\otimes\tilde{D}^{\alpha_{j+1}}_{\pm,\mu_{j+1}}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n},$$

$$\left(\tilde{D}^{\alpha_i}_{\pm,\mu_i}+\mu_i^{\alpha_i}E\right)g(x)=\frac{\alpha_i}{\Gamma(1-\alpha_i)}\int_0^1 u_i^{\mu_i}\frac{(\tilde{\Delta}^1_{u_i^{\pm1}}g)(x)}{\left(\ln\frac{1}{u_i}\right)^{\alpha_i+1}}\frac{du_i}{u_i}+\mu_i^{\alpha_i}g(x).$$

In particular, at $n=2$

$$D^{\alpha}_{\pm\ldots\pm,\mu}f=\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}+\mu_1^{\alpha_1}E\right)\otimes\left(\tilde{D}^{\alpha_2}_{\pm,\mu_2}+\mu_2^{\alpha_2}E\right)f$$

$$=\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\tilde{D}^{\alpha_2}_{\pm,\mu_2}\right)f+\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\mu_2^{\alpha_2}E\right)f+\left(\mu_1^{\alpha_1}E\otimes\tilde{D}^{\alpha_2}_{\pm,\mu_2}\right)f+\left(\mu_1^{\alpha_1}E\otimes\mu_2^{\alpha_2}E\right)f$$

$$=\frac{\alpha_1\alpha_2}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}\int_0^1\int_0^1 u_1^{\mu_1}u_2^{\mu_2}\frac{[\tilde{\Delta}^1_{u_2^{\pm1}}(\tilde{\Delta}^1_{u_1^{\pm1}}f)](x)}{\left(\ln\frac{1}{u_1}\right)^{\alpha_1+1}\left(\ln\frac{1}{u_2}\right)^{\alpha_2+1}}\frac{du_1}{u_1}\frac{du_2}{u_2}$$

$$+\mu_2^{\alpha_2}\frac{\alpha_1}{\Gamma(1-\alpha_1)}\int_0^1 u_1^{\mu_1}\frac{(\tilde{\Delta}^1_{u_1^{\pm1}}f)(x)}{\left(\ln\frac{1}{u_1}\right)^{\alpha_1+1}}\frac{du_1}{u_1}$$

$$+\mu_1^{\alpha_1}\frac{\alpha_2}{\Gamma(1-\alpha_2)}\int_0^1 u_2^{\mu_2}\frac{(\tilde{\Delta}^1_{u_2^{\pm1}}f)(x)}{\left(\ln\frac{1}{u_2}\right)^{\alpha_2+1}}\frac{du_2}{u_2}+\mu_1^{\alpha_1}\mu_2^{\alpha_2}f(x_1,x_2),$$

where $0<\alpha_k<1,\ k=1,2.$

Definition 7. The expression

$$\left(D^{\alpha}_{\pm\cdots\pm;\rho}f\right)(x)=\frac{1}{\prod\limits_{k=1}^{n}\aleph(\alpha_k,l_k)}\int_0^{\rho_1}\ldots\int_0^{\rho_n}\prod_{k=1}^{n}\left(\ln\frac{1}{t_k}\right)^{-1-\alpha_k}\left(\tilde{\Delta}^l_{t^{\pm1}}f\right)(x)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n},$$

$0<\rho_i<1$, $i=\overline{1,n}$, is called the "truncated" mixed fractional Marchaud-Hadamard derivative of order $\alpha=(\alpha_1,\ldots,\alpha_n),\alpha_i>0,\ i=\overline{1,n}$.

In the sequel, we assume by definition that

$$D^{\alpha}_{\pm\cdots\pm}f=\lim_{\rho\to1}D^{\alpha}_{\pm\cdots\pm,\rho}f(\alpha_i>0,\ i=\overline{1,n}),$$

$$D^{\alpha}_{\pm\cdots\pm,\mu}f=\lim_{\rho\to1}D^{\alpha}_{\pm\cdots\pm,\mu;\rho}f,(0<\alpha_i<1,\ i=\overline{1,n}),$$

where the limit is taken in the space $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$.

5 Auxiliary lemmas for spaces $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$

Lemma 5.1. *The space* $C_0^\infty\left(\mathbb{R}_+^n\right)$ *is dense in* $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\left(\mathbb{R}_+^n, \frac{dx}{x}\right)$, $1 \le \overline{p} < \infty$, *and in*

$$C_{\overline{\gamma},0}\left(\dot{\mathbb{R}}_+^n\right) = \left\{f : f(x) = x^{\overline{\gamma}} g(x)\,,\ g(x) \in C\left(\dot{\mathbb{R}}_+^n\right),\ \lim_{|x|\to 0} g(x) = \lim_{|x|\to\infty} g(x) = 0\right\},$$

for any $-\infty < \gamma_i < \infty$, $i = \overline{1,n}$.

This lemma is proven by standard means.

Lemma 5.2. *Let* $\varphi \in \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, $1 \le \overline{p} \le \infty$, $\gamma_i \in \mathbb{R}$, $i = \overline{1,n}$, *then the following inequality is true:*

$$\left\|\Pi_\rho \varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| = C\left(\rho^{\gamma^*}\right) \cdot \left\|\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|, \tag{5.1}$$

where

$$C\left(\rho^{\gamma^*}\right) = \prod_{i=1}^n C\left(\rho_i^{\gamma_i^*}\right),\ C\left(\rho_i^{\gamma_i^*}\right) = \begin{cases} \rho_i^{\frac{\gamma_i}{p_i}}, & 1 \le p_i < \infty, \\ \rho_i^{\gamma_i}, & \rho = \infty, i = \overline{1,n}. \end{cases} \tag{5.2}$$

In addition, the dilation operator approximates the unit operator in the space $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$:

$$\lim_{\rho\to 1-0} \left\|\Pi_\rho \varphi - \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| = 0. \tag{5.3}$$

Proof. Equality (5.1) is proved by obvious changes of variables. Let us prove the statement (5.3). We have

$$\left\|\Pi_\rho\varphi - \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le \left\|[1 - \left(C\left(\rho^{\gamma^*}\right)\right)^{-1}]\Pi_\rho\varphi + \left(C\left(\rho^{\gamma^*}\right)\right)^{-1}\Pi_\rho\varphi - \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|,$$

where $C\left(\rho^{\gamma^*}\right)$ is the function given in (5.2). Hence, on the basis of the generalized Minkowski inequality (see [5], p. 22), we obtain

$$\left\|\Pi_\rho\varphi - \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le \left\|[1 - \left(C\left(\rho^{\gamma^*}\right)\right)^{-1}]\Pi_\rho\varphi\ \ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| + \left\|\left(C\left(\rho^{\gamma^*}\right)\right)^{-1}\Pi_\rho\varphi - \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|.$$

By (5.1) and (1.1), we have

$$\left\|\Pi_\rho\varphi - \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le \left|1 - C\left(\rho^{\gamma^*}\right)\right| \left\|\varphi\ \ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| + \left\|\Pi_\rho g - g\ ; \mathfrak{L}^{\overline{p}}\right\|, \tag{5.4}$$

where $g(x) := x^{-\overline{\gamma}\cdot\overline{p}}\varphi(x)\,,\ g(x) \in \mathfrak{L}^{\overline{p}}\left(\mathbb{R}_+^n, \frac{dx}{x}\right)$ at $1 \le \overline{p} < \infty$, $g(x) := x^{-\overline{\gamma}}\varphi(x)\,,\ g(x) \in C\left(\dot{\mathbb{R}}_+^n\right)$ at $\overline{p} = \overline{\infty}$. Statement (5.3) follows from inequality (5.4). □

The following lemmas relate to convolution-type operators that are invariant with respect to dilation and to their approximation of the unities in the spaces $\mathfrak{L}_{\overline{p},\overline{\gamma}}$. Consider the operators of the form:

$$(A_\rho\varphi)(x) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} K(y_1, \dots, y_n)\, \varphi\left(x_1 \cdot \rho_1^{y_1}, \dots, x_n \cdot \rho_n^{y_n}\right) dy_1 \dots dy_n$$

and

$$(B_\omega\varphi)(x) = \int_0^\infty \dots \int_0^\infty B(\xi_1, \dots, \xi_n)\, \varphi\left(x_1 \cdot \xi_1^{\omega_1}, \dots, x_n \cdot \xi_n^{\omega_n}\right) d\xi_1 \dots d\xi_n,$$

where $\rho_i > 0$, $\omega_i > 0$, $i = \overline{1,n}$.

Lemma 5.3. *Let* $1 \le p_i \le \infty,\ \gamma_i \in \mathbb{R}, \rho_i > 0,\ \omega_i > 0,\ i = \overline{1,n}.$

1) *If* $K\left(\rho^{\gamma^*}\right) := \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} |K(y_1, \dots, y_n)| \prod\limits_{i=1}^{n} \rho_i^{\gamma_i^* \cdot y_i} dy_1 \dots dy_n < \infty$, *where* $\gamma_i^*,\ i = \overline{1,n}$- *are the constants from* (1.2), *then the operator* A_ρ *is bounded in the space* $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, *and*

$$\left\|A_\rho \varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le K\left(\rho^{\gamma^*}\right) \left\|\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|. \tag{5.5}$$

2) *If* $d(\gamma^*, \omega) := \int\limits_{0}^{\infty} \cdots \int\limits_{0}^{\infty} |B(\xi_1, \dots, \xi_n)| \prod\limits_{i=1}^{n} \xi_i^{\gamma_i^* \cdot \omega_i} d\xi_1 \dots d\xi_n < \infty$, *where* $\gamma_i^*,\ i = \overline{1,n}$- *are the constants from* (1.2), *then the operator* B_ω *is bounded in the space* $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ *and*

$$\left\|B_\omega \varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le d(\gamma^*, \omega) \left\|\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|$$

Proof. Representing $A_\rho \varphi$ as

$$(A_\rho \varphi)(x) = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} K(y_1, \dots, y_n) (\Pi_{\rho^y} \varphi)(x)\, dy_1 \dots dy_n$$

and using the generalized Minkowski inequality, we have

$$\left\|A_\rho \varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} |K(y_1, \dots, y_n)| \left\|(\Pi_{\rho^y} \varphi)(x)\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| dy_1 \dots dy_n.$$

Taking into account equality (5.1) we obtain (5.5). The operator $B_\omega \varphi$ is considered similarly. □

Lemma 5.4. *Let* $K(y) = k_1(y_1) \dots k_n(y_n), k_i(y_i) \in L_1(\mathbb{R}^1),\ k_i(y_i) = 0$ *at* $y_i < 0,\ i = \overline{1,n}$. *Then*

$$\left\|A_\rho \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le \left\|k_1\ ;\ L_1\left(\mathbb{R}^1\right)\right\| \dots \left\|k_n\ ;\ L_1\left(\mathbb{R}^1\right)\right\| \cdot \left\|\varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|$$

at $0 < \rho_i \le 1,\ i = \overline{1,n}$.

The proof of Lemma 5.4 follows from Lemma 5.3.

Lemma 5.5. *Let* $K(y) = k_1(y_1) \cdots k_n(y_n), k_i(y_i) \in L_1(\mathbb{R}^1),\ k_i(y_i) = 0$ *at* $y_i < 0,\ i = \overline{1,n}$ *and* $\int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} K(y_1, \dots, y_n)\, dy_1 \dots dy_n = 1$. *Then*

$$\lim_{\rho \to 1-0} \left\|A_\rho \varphi - \varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| = 0 \tag{5.6}$$

for all $1 \le p_i \le \infty,\ \gamma_i \ge 0,\ 0 < \rho_i \le 1, i = \overline{1,n}$.

Proof. First, note that $A_\rho \varphi \in\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ for $\varphi \in\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ at $0 < \rho_i < 1,\ i = \overline{1,n}$, according to Lemma 5.4. To prove equality (5.6) note that since $\int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} K(y_1, \dots, y_n)\, dy_1 \dots dy_n = 1$, then

$$(A_\rho \varphi)(x) - \varphi(x) = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} K(y_1, \dots, y_n) ((\Pi_{\rho^y} \varphi)(x) - \varphi(x)) dy_1 \dots dy_n.$$

Using the generalized Minkowski inequality, we obtain

$$\left\|A_\rho \varphi - \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|$$

$$\leq \int_0^\infty \dots \int_0^\infty |k_1(y_1)| \dots |k_n(y_n)| \cdot \left\| (\Pi_{\rho^y}\varphi)(x) - \varphi(x) \ ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| dy_1 \dots dy_n. \tag{5.7}$$

Since $0 < \rho_i \leq 1$, $i = \overline{1,n}$, then in (5.7) the passage to the limit under the sign of the integral is possible on the basis of the majorant Lebesgue theorem. The application of the latter is substantiated by statements (5.1), (5.3) of Lemma 5.2. □

6 On the boundedness of mixed fractional Hadamard and Hadamard type integration in space $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$

Theorem 6.1. *Let $\gamma_i \in \mathbb{R}^1$, $1 \leq p_i \leq \infty$, $\alpha_i > 0$ and $\mu_i \in \mathbb{C}$, $i = \overline{1,n}$. If $Re\mu_i > -\gamma_i^*$, , $i = \overline{1,n}$, where γ_i^*, $i = \overline{1,n}$ are the constants from (1.2), then the operator $J^{\alpha}_{+\dots+,\mu}$ is bounded in the $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, and*

$$\left\| J^{\alpha}_{+\dots+,\mu}\varphi \ ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| \leq \prod_{i=1}^{n} (\mu_i + \gamma_i^*)^{-\alpha_i} \left\| \varphi \ ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\|. \tag{6.1}$$

Proof. First consider the case $1 \leq \overline{p} < \overline{\infty}$. By the generalized Minkowski inequality, we have

$$\left\| J^{\alpha}_{+\dots+,\mu}\varphi ; \ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| \leq \int_0^\infty \dots \int_0^\infty \left\| \varphi(x \circ y) \, ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| \prod_{i=1}^{n} \left| k^{+}_{\mu_i,\alpha_i}(y_i) \right| \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n}.$$

After substitution $\tau_i = x_i \cdot y_i$, $i = \overline{1,n}$, we obtain

$$\left\| J^{\alpha}_{+\dots+,\mu}\varphi ; \ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| \leq \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{n} \left| k^{+}_{\mu_i,\alpha_i}(y_i) \right| y_i^{\frac{\gamma_i}{p_i}} \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n} \left\| \varphi ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\|.$$

So,

$$\begin{aligned} \left\| J^{\alpha}_{+\dots+,\mu}\varphi ; \ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| &\leq \int_0^1 \dots \int_0^1 \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} y_i^{\mu_i + \frac{\gamma_i}{p_i}} \left(\ln \frac{1}{y_i} \right)^{\alpha_i - 1} \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n} \left\| \varphi ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| \\ &\leq \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} e^{-(\mu_i + \frac{\gamma_i}{p_i})\xi_i} (\xi_i)^{\alpha_i - 1} d\xi_1 \dots d\xi_n \left\| \varphi ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| \\ &\leq \prod_{i=1}^{n} \left(\frac{p_i}{\mu_i \, p_i + \gamma_i} \right)^{\alpha_i} \left\| \varphi ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\|. \end{aligned} \tag{6.2}$$

At $\overline{p} = \overline{\infty}$ in (6.2) substitute p_i, $i = \overline{1,n}$ for 1. Then we get (6.1). □

Theorem 6.2. 1) *Let $\gamma_i \in \mathbb{R}^1$, $1 \leq p_i \leq \infty$, $\alpha_i > 0, i = \overline{1,n}$. If $\gamma_i > 0$, $i = \overline{1,n}$, then operator $J^{\alpha}_{+\dots+}$ is bounded in the $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, and*

$$\left\| J^{\alpha}_{+\dots+}\varphi \ ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| \leq \prod_{i=1}^{n} (\gamma_i^*)^{-\alpha_i} \left\| \varphi \ ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\|.$$

2) *Let $1 \leq p_i \leq \infty$, $1 \leq q_i \leq \infty$, $0 < \alpha_i < 1$, $i = \overline{1,n}$. Operators of fractional integration $J^{\alpha}_{+\dots+}\varphi$ and $J^{\alpha}_{-\dots-}\varphi$ are bounded from $\mathfrak{L}^{\overline{p}}\left(\mathbb{R}^n_+, \frac{dx}{x}\right)$ into $\mathfrak{L}^{\overline{q}}\left(\mathbb{R}^n_+, \frac{dx}{x}\right)$ if and only if $1 < p_i < \frac{1}{\alpha_i}$, $q_i = \frac{p_i}{1-\alpha_i p_i}$, $i = \overline{1,n}$.*

Proof. The first statement follows from Lemma 5.3. Then, the operators $J^{\alpha}_{+\cdots+}\varphi$ and $J^{\alpha}_{-\cdots-}\varphi$ are related to the Riemann-Liouville operators $I^{\alpha}_{\pm\cdots\pm}\varphi$ by the equalities

$$J^{\alpha}_{+\cdots+}\varphi = Q^{-1}I^{\alpha}_{+\cdots+}Q\varphi, J^{\alpha}_{-\cdots-}\varphi = Q^{-1}I^{\alpha}_{-\cdots-}Q\varphi, \tag{6.3}$$

where $(Q\varphi)(x) = \varphi(e^x) = \varphi(e^{x_1}, \dots, e^{x_n})$. By virtue of (6.3), the second statement of the theorem follows from the well-known Hardy-Littlewood theorem for ordinary fractional integration over $\mathbb{R}^n$ (see [20], p. 494). □

Theorem 6.3. *Operators $J^{\alpha,l}_{+\cdots+,\mu;\tau}$, $J^{\alpha,l}_{+\cdots+,\tau}$ is bounded in the space $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ for all $1 \le p_i \le \infty$, $\gamma_i \ge 0$, $i = \overline{1,n}$,*

$$\left\| J^{\alpha,l}_{\pm\cdots\pm,\tau}\varphi\ ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| \le \prod_{i=1}^{n} c_i(\tau_i, \mu_i) \left\|\varphi\ ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\|,$$

where $0 < c_i(\tau_i, \mu_i) < 1$ at $Re\mu_i + \gamma_i^* \ge 0,\ 0 < \tau_i \le 1,\ l_i > \alpha_i > 0,\ i = \overline{1,n}$,

$$\left\| J^{\alpha,l}_{\pm\cdots\pm,\tau}\varphi\ ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}} \right\| \le \prod_{i=1}^{n} c_i(\tau_i) \left\|\varphi\ ; \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\|,$$

where $0 < c_i(\tau_i) < 1$ at $0 < \tau_i \le 1,\ l_i > \alpha_i > 0,\ i = \overline{1,n}$.

The proof of this theorem follows from Lemma 5.3.

7 Integral representation of the truncated mixed Marchaud-Hadamard and Marchaud-Hadamard-type fractional derivatives

Lemma 7.1. *Let $f(x) = \left(J^{\alpha}_{+\cdots+,\mu}\varphi\right)(x)$, $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, where $1 \le p_i < \infty$, $\gamma_i > 0$, $\mu_i \ge 0$, $\mu_i > -\frac{\gamma_i}{p_i}$, $0 < \alpha_i < 1$, $i = \overline{1,n}$, and $0 < \rho_i < 1$, $i = \overline{1,n}$, the truncated mixed fractional derivative $D^{\alpha}_{+\cdots+,\mu;\,\rho}f$ has the following integral representation*

$$D^{\alpha}_{+\cdots+,\mu;\,\rho}f = \int\limits_{R^n} K^{+}_{\alpha,\,\mu}(t,\,\rho)\,\varphi\left(x\circ\rho^t\right)dt, \tag{7.1}$$

where

$$K^{+}_{\alpha,\,\mu}(t,\,\rho) = K^{+}_{\alpha_1,\,\mu_1}(t_1,\,\rho_1)\dots K^{+}_{\alpha_n,\,\mu_n}(t_n,\,\rho_n)\,, K^{+}_{\alpha_i,\,\mu_i}(t_i,\,\rho_i) =$$

$$= \frac{\sin\alpha_i\pi}{\pi}\frac{\rho_i^{\mu_i\,t_i}}{t_i}[(\alpha_i\,\Gamma\left(-\alpha_i,\,\mu_i\,\ln\frac{1}{\rho_i}\right) + \Gamma(1-\alpha_i))\left(\mu_i\,\ln\frac{1}{\rho_i}\right)^{\alpha_i}(t_i)^{\alpha}_{+} - (t_i-1)^{\alpha_i}_{+}],$$

$\Gamma\left(-\alpha_i,\,\mu_i\ln\frac{1}{\rho_i}\right)$, $i = \overline{1,n}$, the upper incomplete gamma function. In this case, the kernel $K^{+}_{\alpha_i,\,\mu_i}(t_i,\,\rho_i) \in L_1\left(\mathbb{R}^1_+\right)$ is an averaging one:

$$\int\limits_0^{\infty} K^{+}_{\alpha_i,\,\mu_i}(t_i,\,\rho_i)\,dt_i = 1, K^{+}_{\alpha_i,\,\mu_i}(t_i,\,\rho_i) > 0 \tag{7.2}$$

at $0 < t_i < 1$.

Proof. The proof is easily reduced to known facts for the one-dimensional case ([25]). Namely, we have

$$J^{\alpha}_{+\cdots+,\mu}\varphi = J^{\alpha_1}_{+,\mu_1} \otimes \cdots \otimes J^{\alpha_n}_{+,\mu_n}\varphi,$$
$$D^{\alpha}_{+\cdots+,\mu;\,\rho}f = D^{\alpha_1}_{+,\mu_1;\,\rho_1} \otimes \cdots \otimes D^{\alpha_n}_{+,\mu_n;\,\rho_n}f\ .$$

Since $f(x) = \left(J^{\alpha}_{+\cdots+,\mu}\varphi\right)(x)$, then

$$D^{\alpha}_{+\cdots+,\mu;\,\rho}f = D^{\alpha_1}_{+,\mu_1;\,\rho_1}J^{\alpha_1}_{+,\mu_1} \otimes D^{\alpha_2}_{+,\mu_2;\,\rho_2}J^{\alpha_2}_{+,\mu_2} \otimes \cdots \otimes D^{\alpha_n}_{+,\mu_n;\,\rho_n}J^{\alpha_n}_{+,\mu_n}\varphi\ .$$

It is known, that (see [15], [25])

$$D^{\alpha_i}_{+,\mu_i;\,\rho_i}J^{\alpha_i}_{+,\mu_i}g = \mathrm{K}^{+}_{\alpha_i,\,\mu_i}(\tau,\,\rho_i)\,g,\, i = \overline{1,n},\, g = g(t) \in \mathfrak{L}^p\left(\mathbb{R}_+,\, t^{-\gamma}\frac{dx}{x}\right)$$

for the function of one variable and

$$D^{\alpha_i}_{+,\mu_i;\,\rho_i}J^{\alpha_i}_{+,\mu_i}g = \int\limits_0^{\infty} K^{+}_{\alpha_i,\,\mu_i}(\tau,\,\rho_i)\,g\left(t\cdot\rho_i^{\tau}\right)d\tau.$$

Then

$$D^{\alpha}_{+\cdots+,\mu;\,\rho}f = K^{+,\alpha_1}_{\mu_1;\,\rho_1} \otimes K^{+,\alpha_2}_{\mu_2;\,\rho_2} \otimes \cdots \otimes K^{+,\alpha_n}_{\mu_n;\,\rho_n}\varphi,$$

for $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, $1 \le p_i < \infty$, $\gamma_i > 0$, $i = \overline{1,n}$, taking into account the density of functions of form (3.1). This implies representation (7.1). Operator (7.1) on the right-hand side is also bounded by Lemma 5.4. Therefore, by virtue of Lemma 5.1, identity (7.1) applies with $C_0^{\infty}\left(\mathbb{R}^n_+\right)$ to all functions $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, $1 \le p_i < \infty$, $\gamma_i > 0$, $i = \overline{1,n}$. □

Lemma 7.2. *Let* $f(x) = \left(J^{\alpha}_{+\cdots+}\varphi\right)(x)$, $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ *, where* $\alpha_i > 0$, $1 \le p_i < \infty$, $\gamma_i > 0$, $i = \overline{1,n}$, *or* $0 < \alpha_i < 1$, $1 < p_i < \frac{1}{\alpha}$, $\gamma_i = 0$, $i = \overline{1,n}$ *and* $0 < \rho_i < 1$, $i = \overline{1,n}$. *Then the truncated mixed fractional derivative* $D^{\alpha}_{+\cdots+,\,\rho}f$ *has the following integral representation*

$$\left(D^{\alpha}_{+\cdots+,\rho}f\right)\ (x) = \int\limits_0^{\infty}\ldots\int\limits_0^{\infty}\prod_{i=1}^{n}K^{+}_{l_i,\alpha_i}(y_i)\varphi(x\circ\rho^{y})dy_1\ldots dy_n, \tag{7.3}$$

where the kernel

$$K^{+}_{l_i,\,\alpha_i}(y_i) = \frac{\sum\limits_{k=0}^{l_i}(-1)^k\begin{pmatrix} l_i \\ k\end{pmatrix}(y_i-k)^{\alpha_i}_{+}}{\vartheta\left(\alpha_i,l_i\right)\Gamma(1+\alpha_i)y_i} \in L_1\left(\mathbb{R}^1_+\right) \tag{7.4}$$

at $l > \alpha > 0$,

$$\int\limits_0^{\infty}K^{+}_{l_i,\,\alpha_i}(y_i)dy_i = 1,\ l_i > \alpha_i > 0. \tag{7.5}$$

The proof of Lemma 7.2 is similar to the proof of Lemma 7.1

Lemma 7.3. *Let* $f \in \mathfrak{L}^{\overline{r}}_{\overline{\lambda}}$, $1 \le r_i \le \infty, \lambda_i \ge 0$, $i = \overline{1,n}$ *be such that its difference* $(\tilde{\Delta}^l_t f)(x)$ *of order* l *is represented by a modified mixed Hadamard fractional integral* (2.7) *of a function from* $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$*:*

$$(\tilde{\Delta}^l_{\tau}f)(x) = (J^{\alpha,l}_{+\cdots+,\,\tau}\varphi)(x) = \int\limits_0^{\infty}\ldots\int\limits_0^{\infty}\left(\tilde{\Delta}^l_{\tau^{-1}}k^{+}_{\alpha}\right)(y)\,\varphi\left(x\circ y\right)\frac{dy_1}{y_1}\ldots\frac{dy_n}{y_n}, \tag{7.6}$$

where $l_i > \alpha_i > 0,\ 0 < \tau_i < 1,\ \varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}},\ 1 \le p_i \le \infty, \quad \gamma_i \ge 0,\ i = \overline{1,n}$ *and* $0 < \ h_i < 1,\ i = \overline{1,n}$. *Then the truncated mixed fractional derivative* $D^{\alpha}_{+\cdots+,\rho} f$ *allows integral representation* (7.3) *for all* $1 \le p_i < \infty,\ \gamma_i \ge 0,\ i = \overline{1,n}$ *and integral representation*

$$\left(D^{\alpha}_{+\cdots+,\rho} f\right)\ (x) = K_1 \left(\Pi^1_{\rho_1^{t_1}} - \Pi^1_0\right) \otimes \cdots \otimes K_n \left(\Pi^n_{\rho_n^{t_n}} - \Pi^n_0\right) \varphi\,(x) \tag{7.7}$$

for all $p_i = \infty,\ \gamma_i = 0,\ i = \overline{1,n}$*, where the operator* Π^i_0 *is:*

$$\left(\Pi^i_0 \varphi\right)(x) = \varphi\left(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n\right)$$

In particular, for $n = 2$

$$\left(D^{\alpha}_{++,\rho} f\right)(x) = K_1 \left(\Pi^1_{\rho_1^{t_1}} - \Pi^1_0\right) \otimes K_2 \left(\Pi^2_{\rho_2^{t_2}} - \Pi^2_0\right) \varphi\,(x)$$

$$= \int\limits_0^\infty \int\limits_0^\infty K^+_{l_1,\alpha_1}(t_1)\, K^+_{l_2,\alpha_2}(t_2)\, \varphi\left(x_1 \cdot \rho_1^{t_1}, x_2 \cdot \rho_2^{t_2}\right) dt_1 dt_2$$

$$- \int\limits_0^\infty K^+_{l_1,\alpha_1}(t_1)\, \varphi\left(x_1 \cdot \rho_1^{t_1}, 0\right) dt_1 - \int\limits_0^\infty K^+_{l_2,\alpha_2}(t_2)\, \varphi\left(0, x_2 \cdot \rho_2^{t_2}\right) dt_2 + \varphi\,(0,0)\,,$$

for all $p_i = \infty,\ \gamma_i = 0,\ i = 1,2$*, where* $K^+_{l_i,\alpha_i}(t_i)$ *is kernel* (7.4).

Proof. Lemma 7.3 is proven in the same way as Lemma 6.3 from [25]. Since at $\gamma_i > 0,\ i = \overline{1,n}$ the procedure is substantiated in the proof of Lemma 7.1, it suffices to consider the case at $\gamma_i = 0,\ i = \overline{1,n}$ for any $1 \le p_i \le \infty$, $\alpha_i > 0,\ i = \overline{1,n}$. Similarly, as in the one-dimensional case, it is necessary to substantiate the following equality

$$\int\limits_{\ln\frac{1}{\rho_1}}^{\infty} \cdots \int\limits_{\ln\frac{1}{\rho_n}}^{\infty} \prod_{i=1}^{n} \frac{d\xi_i}{\xi_i^2} \int\limits_0^\infty \cdots \int\limits_0^\infty (\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi}) \varphi(x \circ e^{-\tau}) d\tau$$

$$= \int\limits_0^\infty \cdots \int\limits_0^\infty \varphi(x \circ e^{-\tau}) d\tau \int\limits_{\ln\frac{1}{\rho_1}}^{\infty} \cdots \int\limits_{\ln\frac{1}{\rho_n}}^{\infty} (\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi}) \prod_{i=1}^{n} \frac{d\xi_i}{\xi_i^2},$$

where $(\Delta_1^l k_\alpha^+)(y) = \Delta_1^{l_1}[\Delta_1^{l_2} \ldots (\Delta_1^{l_n} k_\alpha^+)](y),\ \ (k_\alpha^+)(y) = \prod\limits_{i=1}^{n} \frac{(y_i)_+^{\alpha_i - 1}}{\Gamma(\alpha_i)}$. Here the change of order of integration is substantiated by Fubini's theorem at $1 \le p_i < \infty,\ \gamma_i = 0,\ i = \overline{1,n}$. Next, we prove that at $1 \le p_i < \infty,\ i = \overline{1,n}$, the iterated integral converges (for almost all x, $x \in \mathbb{R}^n$)

$$\int\limits_0^\infty \cdots \int\limits_0^\infty \left|\varphi(x \circ e^{-\tau})\right| d\tau_1 \ldots d\tau_n \int\limits_{\ln\frac{1}{\rho_1}}^{\infty} \cdots \int\limits_{\ln\frac{1}{\rho_n}}^{\infty} \left|(\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi})\right| \frac{d\xi_1}{\xi_1^2} \cdots \frac{d\xi_n}{\xi_n^2},$$

for all $\varphi \in \mathfrak{L}^{\overline{p}}\left(\mathbb{R}^n_+, \frac{dx}{x}\right)$. Changing of the variables $\frac{\tau_i}{\xi_i} = s_i$ and $\tau_i = t_i \ln\frac{1}{h_i},\ i = \overline{1,n}$, leads to the necessity to prove the convergence of the integral

$$A := \int\limits_0^\infty \cdots \int\limits_0^\infty \left|\varphi(x \circ h^t)\right| K^*(t) dt_1 \ldots dt_n, \tag{7.8}$$

where $K^*(t) = \frac{1}{t_1 \ldots t_n} \int_0^{t_1} \ldots \int_0^{t_n} \left|(\Delta_1^l k_\alpha^+)(s)\right| ds$. Since $(\Delta_1^l k_\alpha^+)(s) \in L_1(\mathbb{R}^n)$ (see Theorem 6.3), then $K^*(t) \le \frac{c}{t_1 \ldots t_n}$ at $t \to \infty$. Then it is evident that $K^*(t) \le ct^{\alpha-1}$ at $t \to 0$ and, $K^*(t)$ is continuous at $t \in \mathbb{R}_+^n$. We have

$$\overline{K^*(t)} = \sum_{1 \le |j| < n} k_j(t), \quad k_j(t) = t^{a_j(t)} = t_1^{a_{j_1}(t_1)} \ldots t_n^{a_{j_n}(t_n)},$$

where $a_{j_i}(t_i) = \begin{cases} \alpha_i - 1, & 0 < t_i < 1, \\ -1, & t_i \ge 1. \end{cases}$ Then from (7.8) we obtain

$$A \le \int_0^\infty \ldots \int_0^\infty \left|\varphi(x \circ \rho^t)\right| \overline{K^*(t)} dt_1 \ldots dt_n$$

and it remains to refer to Young's theorem for spaces with mixed norm ([5], p. 25).

Substantiate the case $p_i = \infty$, $i = \overline{1,n}$, for $\varphi \in C\left(\mathbb{R}_+^n\right)$. Consider the "two-sided" mixed truncated Marchaud-Hadamard fractional derivative, i.e.

$$\left(D_{+\cdots+,\rho,\delta}^\alpha f\right)(x) = \frac{1}{\aleph(\alpha,l)} \int_{\delta_1}^{\rho_1} \ldots \int_{\delta_n}^{\rho_n} \left(\ln\frac{1}{t}\right)^{-1-\alpha} \left(\tilde{\Delta}_t^l f\right)(x) \frac{dt_1}{t_1} \ldots \frac{dt_n}{t_n}, \tag{7.9}$$

at $l_i > \alpha_i > 0$, where $0 < \delta_i < \rho_i < 1$, $i = \overline{1,n}$, then refer to the limit $\delta \to 0$. From (7.6) we have

$$\left(\tilde{\Delta}_t^l f\right)(x) = \left(\ln\frac{1}{t}\right)^\alpha \int_0^\infty \ldots \int_0^\infty (\Delta_1^l k_\alpha^+)(y) \varphi\left(x \circ t^y\right) dy_1 \ldots dy_n, \tag{7.10}$$

where $0 < t_i < 1$, $i = \overline{1,n}$. Substituting (7.10) into (7.9), we obtain

$$\left(D_{+\cdots+,\rho,\delta}^\alpha f\right)(x)$$

$$= \frac{1}{\aleph(\alpha,l)} \int_{\delta_1}^{\rho_1} \ldots \int_{\delta_n}^{\rho_n} \prod_{i=1}^n \left(\ln\frac{1}{t_i}\right)^{-1} \frac{dt_i}{t_i} \int_0^\infty \ldots \int_0^\infty (\Delta_1^l k_\alpha^+)(y) \varphi\left(x \circ t^y\right) dy_1 \ldots dy_n.$$

The changes of variables $\ln\frac{1}{t_i} = \xi_i$ and $y_i \xi_i = \tau_i$, $i = \overline{1,n}$, give:

$$\left(D_{+\cdots+,\rho,\delta}^\alpha f\right)(x)$$

$$= \frac{1}{\aleph(\alpha,l)} \int_{\ln\frac{1}{\rho_1}}^{\ln\frac{1}{\delta_1}} \ldots \int_{\ln\frac{1}{\rho_n}}^{\ln\frac{1}{\delta_n}} \frac{d\xi_1}{\xi_1^2} \ldots \frac{d\xi_n}{\xi_n^2}{}_i \int_0^\infty \ldots \int_0^\infty (\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi}) \varphi\left(x \circ e^{-\tau}\right) d\tau_1 \ldots d\tau_n$$

and the change of the order of integration leads to the equality

$$\left(D_{+\cdots+,\rho,\delta}^\alpha f\right)(x) =$$

$$= \frac{1}{\aleph(\alpha,l)} \int_0^\infty \ldots \int_0^\infty \varphi\left(x \circ e^{-\tau}\right) d\tau \int_{\ln\frac{1}{\rho_1}}^{\ln\frac{1}{\delta_1}} \ldots \int_{\ln\frac{1}{\rho_n}}^{\ln\frac{1}{\delta_n}} (\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi}) \prod_{i=1}^n \xi_i^{-2} d\xi_i. \tag{7.11}$$

Here the change of order of integration is easily substantiated by introducing $\delta = (\delta_1, \ldots, \delta_n)$, $0 < \delta_i < 1$, $i = \overline{1,n}$ (considering that $|\varphi| \le c$ and $\int\limits_{\ln \frac{1}{\rho_1}}^{\ln \frac{1}{\delta_1}} \ldots \int\limits_{\ln \frac{1}{\rho_n}}^{\ln \frac{1}{\delta_n}} \frac{d\xi_1}{\xi_1^2} \ldots \frac{d\xi_n}{\xi_n^2} \int\limits_0^\infty \ldots \int\limits_0^\infty \left|(\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi})\right| d\tau_1 \ldots d\tau_n < \infty$). Equality (7.11) means that

$$\left(D^\alpha_{+\cdots+,\rho,\delta} f\right)(x) = \frac{1}{\aleph(\alpha, l)} \int\limits_0^\infty \ldots \int\limits_0^\infty \varphi\left(x \circ e^{-\tau}\right) \times$$

$$\times \prod_{i=1}^n [\frac{1}{\ln \frac{1}{\rho_i}} K^+_{l_i,\alpha_i}\left(\frac{\tau_i}{\ln \frac{1}{\rho_i}}\right) - \frac{1}{\ln \frac{1}{\delta_i}} K^+_{l_i,\alpha_i}\left(\frac{\tau_i}{\ln \frac{1}{\delta_i}}\right)] d\tau_1 \cdots d\tau_n,$$

where $K^+_{l_i,\alpha_i}(t_i)$, $i = \overline{1,n}$, is kernel (7.4). Here, the integral representation can be written in terms of tensor products, i.e.

$$\left(D^\alpha_{+\cdots+,1-\rho,\delta} f\right)(x) = K_1\left(\Pi^1_{\rho_1^{t_1}} - \Pi^1_{\delta_1^{t_1}}\right) \otimes \cdots \otimes K_n\left(\Pi^n_{\rho_n^{t_n}} - \Pi^n_{\delta_n^{t_n}}\right)\varphi(x), \tag{7.12}$$

where

$$K_i\left(\Pi^i_{\rho_i^{t_i}} - \Pi^i_{\delta_i^{t_i}}\right) g(x_i) = \int\limits_0^\infty K^+_{l_i,\alpha_i}(t_i)\left[g\left(x_i \rho_i^{t_i}\right) - g\left(x_i \delta_i^{t_i}\right)\right] dt_i,$$

$\left(\Pi^i_{\rho_i^{t_i}} \varphi\right)(x) = \varphi\left(x_1, \ldots, x_{i-1}, x_i \rho_i^{t_i}, x_{i+1}, \ldots, x_n\right)$ is the dilation operator. Since $\varphi \in C\left(\dot{\mathbb{R}}^n_+\right)$ and $K^+_{l_i,\alpha_i}(t_i) \in L_1(\mathbb{R}^1)$, $i = \overline{1,n}$, a passage to the limit is possible at $\delta \to 0$ under the sign of the integral. By (7.5) from (7.12) we obtain (7.7). □

8 Inversion of mixed fractional integrals of functions belonging to $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$

Theorem 8.1. . *Let* $f = J^\alpha_{+\cdots+,\mu}\varphi$, $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, *where* $\gamma_i > 0$, $0 < \alpha_i < 1$, $1 \le p_i \le \infty$, $\mu_i \ge 0$, $\mu_i > -\gamma_i^*$, $i = 1, \ldots, n$. *Then*

$$\left(D^\alpha_{+\cdots+,\mu} f\right)(x) = \lim_{\substack{\rho \to 1 \\ (\mathfrak{L}^{\overline{p}}_{\overline{\gamma}})}} \left(D^\alpha_{+\cdots+,\mu;\rho} f\right)(x) = \varphi(x).$$

Proof. Convergence in norm follows from Lemmas 7.1 and 5.5. □

Theorem 8.2. *Let* $f = J^\alpha_{+\cdots+}\varphi$, $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, *where either* $\gamma_i > 0$, $\alpha_i > 0, 1 \le p_i \le \infty$, $i = 1, \ldots, n$, *or* $\gamma_i = 0$, $0 < \alpha_i < 1, 1 \le p_i < \frac{1}{\alpha_i}, i = 1, \ldots, n$. *Then*

$$\left(D^\alpha_{+\cdots+} f\right)(x) = \lim_{\substack{\rho \to 1 \\ (\mathfrak{L}^{\overline{p}}_{\overline{\gamma}})}} \left(D^\alpha_{+\cdots+,\rho} f\right)(x) = \varphi(x),$$

where the limit is taken both in $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, *and almost everywhere.*

Proof. Convergence in norm follows from Lemmas 7.2 and 5.5. The proof of convergence almost everywhere is obtained by using Theorem 2 ([24], p. 77-78), applying it for each variable. In this case, equality (7.4) and the property of the kernel $\left|K^+_{l_i,\alpha_i}(y_i)\right| \le \frac{c}{(1+y_i)^{l_i+1-\alpha_i}}$ at $l_i > \alpha_i$, $y_i > 1$, $i = 1, \ldots, n$ (see [20], p. 379) are taken into account, so, the kernel $K^+_{l_i,\alpha_i}(y_i)$ has a monotone summable majorant. □

Theorem 8.3. *Let* $\left(\tilde{\Delta}_{\tau}^{l} f\right)(x) = J_{+\cdots+,\tau}^{\alpha,l}\varphi,\ \varphi \in \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, *where* $\gamma_i \geq 0,\ \ l > \alpha_i > 0,\ 1 \leq p_i \leq \infty,\ 0 < \tau_i < 1,\ \ i = 1, \ldots, n.$ *Then*

$$\left(D_{+\cdots+}^{\alpha} f\right)(x) = \lim_{\substack{\rho \to 1 \\ \left(\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right)}} \left(D_{+\cdots+,\,\rho}^{\alpha} f\right)(x) = \varphi\left(x\right),$$

where the limit is taken both in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, *and almost everywhere.*

The proof of the convergence in norm follows from Lemmas 7.3 and 5.5. The convergence almost everywhere is proven as in Theorem 8.2.

Remark 1. One can admit the case in which $\alpha_i = 0$ for some i. In particular, if $f = J_{+\cdots+}^{\alpha}\varphi,\ \varphi \in \mathfrak{L}_{\overline{p}}\left(\mathbb{R}_{+}^{n}, \frac{dx}{x}\right)$, where $\alpha_i > 0$ at $i = 1, \ldots, k-1, k+1, \ldots, n,\ \alpha_k = 0,\ \ 1 \leq p_i < \frac{1}{\alpha_i}$ at $i = 1, \ldots, k-1, k+1, \ldots, n$ and $1 \leq p_k \leq \infty$. Then

$$\left(D_{+\cdots+}^{\alpha} f\right)(x) = \lim_{\rho \to 1} (D_{+\cdots+;\rho}^{(\alpha_1,\ldots,\alpha_{k-1},0,\alpha_{k+1},\ldots,\alpha_n)} f)\left(x\right) = \varphi\left(x\right),$$

where the limit is taken both in $\mathfrak{L}_{\overline{p}}$, and almost everywhere.

9 Characterization of mixed fractional integrals of functions from $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$

Denote by $J_{\pm\cdots\pm,\mu}^{\alpha}\left(\mathfrak{L}^{\overline{p}}\right)$ the operator image of mixed fractional integration

$$J_{\pm\cdots\pm,\mu}^{\alpha}\left(\mathfrak{L}^{\overline{p}}\right) = \left\{\ f : f = J_{\pm\cdots\pm,\mu}^{\alpha}\varphi,\ \ \varphi \in \mathfrak{L}^{\overline{p}}\left(\mathbb{R}_{+}^{n}, \frac{dx}{x}\right)\ \right\}$$

defined for $0 < \alpha_i < 1,\ \ 1 \leq p_i < \frac{1}{\alpha_i}, i = 1, \ldots, n.$ Actually, at $1 < p_i < \frac{1}{\alpha_i}, i = 1, \ldots, n,$ they coincide, that is, do not depend on the sign choice, so, denote them by

$$J^{\alpha} := J_{+\cdots+,\mu}^{\alpha}\left(\mathfrak{L}^{\overline{p}}\right) = J_{++--\cdots+,\mu}^{\alpha}\left(\mathfrak{L}^{\overline{p}}\right) = \cdots = J_{-\cdots-,\mu}^{\alpha}\left(\mathfrak{L}^{\overline{p}}\right)$$

Denote similar modified operator of mixed fractional integration by $J_{\pm\cdots\pm,\mu}^{\alpha,l}\left(\mathfrak{L}^{\overline{p}}\right)$:

$$J_{\pm\cdots\pm,\mu}^{\alpha,l}\left(\mathfrak{L}^{\overline{p}}\right) = \left\{g : g = J_{\pm\cdots\pm,\mu;\tau}^{\alpha,l}\varphi,\ \ \varphi \in \mathfrak{L}^{\overline{p}}\ \right\}.$$

This space is defined for $l_i > \alpha_i > 0,\ 1 \leq p_i \leq \infty,\ \mu_i > 0,\ 0 < \tau_i < 1,\ \ i = \overline{1, n}.$

Introduce into consideration the space

$$\mathfrak{L}_{\overline{\gamma},\overline{\lambda}}^{\overline{p},\overline{r},\alpha}\left(\mathbb{R}_{+}^{n}\right) = \left\{f : f \in \mathfrak{L}_{\overline{\lambda}}^{\overline{r}},\ \lim_{\delta \to 0} D_{+\cdots+,\mu;\delta}^{\alpha,l} f = \varphi,\ \ \varphi \in \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\ \right\},$$

where $\gamma_i \geq 0,\ \ \lambda_i \geq 0, 1 \leq p_i, r_i \leq \infty,\ \alpha_i > 0,\ i = 1, ..., n.$

Lemma 9.1. *The operator*

$$\left(B_{h}^{\alpha}\varphi\right)(x) = \int_{0}^{1} \cdots \int_{0}^{1} \left(\Delta_{\ln\frac{1}{h}}^{\alpha} k_{\alpha}^{+}\right)\left(\ln\frac{1}{t}\right)\varphi\left(x \circ t\right)\frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}, \tag{9.1}$$

where $\left(\Delta_{\xi}^{\alpha} k_{\alpha}^{+}\right)(y) = \Delta_{\xi_1}^{\alpha_1}[\Delta_{\xi_2}^{\alpha_2} \ldots (\Delta_{\xi_n}^{\alpha_n} k_{\alpha}^{+})](y), k_{\alpha}^{+}\left(y\right) = \prod_{i=1}^{n} \frac{(y_i)_{+}^{\alpha_i - 1}}{\Gamma(\alpha_i)}$, *is bounded in the space* $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ *for every* $1 \leq p_i \leq \infty,\ \alpha_i > 0, \gamma_i \geq 0,\ \ 0 < h_i < 1,\ i = 1, \ldots, n,$ *and*

$$\left\|B_{h}^{\alpha}\varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\ \right\| \leq C\prod_{i=1}^{n}\left(\ln\frac{1}{h_i}\right)^{\alpha_i}\left\|\varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\ \right\|,$$

where C *does not depend on* h_i, $i = 1, \ldots, n.$

Proof. From (9.1) by the generalized Minkowski inequality, we have

$$\left\|B_h^\alpha \varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\ \right\| \leq \int\limits_0^1 \ldots \int\limits_0^1 \left(\Delta^\alpha_{\ln\frac{1}{h}} k_\alpha^+\right)\left(\ln\frac{1}{t}\right)\left\|\varphi\left(x\circ t\right);\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}.$$

By (5.1) we obtain

$$\left\|B_h^\alpha \varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\ \right\| \leq \int\limits_0^1 \ldots \int\limits_0^1 \left(\Delta^\alpha_{\ln\frac{1}{h}} k_\alpha^+\right)\left(\ln\frac{1}{t}\right) \prod_{i=1}^n t_i^{\gamma_i^*} \frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}\left\|\varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|,$$

where $\gamma_i^*,\ \ i=1,\ldots,n$, are constants from (1.2). The substitution $\ln\frac{1}{t_i}=\xi_i\ln\frac{1}{h_i}$, $i=1,\ldots,n$ gives

$$\left\|B_h^\alpha \varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\ \right\| \leq \prod_{i=1}^n\left(\ln\frac{1}{h_i}\right)^{\alpha_i}\int\limits_0^\infty \ldots \int\limits_0^\infty \prod_{i=1}^n\left|P_{\alpha_i}\left(z_i\right)\right| dz_1\ldots dz_n\left\|\varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|, \tag{9.2}$$

where $P_{\alpha_i}\left(z_i\right)=\frac{1}{\Gamma(\alpha_i)}\sum\limits_{j_i=1}^{\infty}\left(-1\right)^{j_i}\binom{\alpha_i}{j_i}\left(z_i-j_i\right)_+^{\alpha_i-1}\in L_1\left(\mathbb{R}_+^1\right)$ (see 20, p. 282). So, inequality (9.1) follows from (9.2). □

Theorem 9.1. *Let* $f\in\mathfrak{L}_{\overline{\gamma},\overline{\lambda}}^{\overline{p},\overline{r},\alpha}\left(\mathbb{R}_+^n\right)$, $\gamma_i\geq 0,\ \ \lambda_i\geq 0,\ 1\leq p_i,r_i<\infty,\ \ \alpha_i>0,\ \ i=1,\ldots,n$. *Then for the mixed fractional difference* $\left(\tilde{\Delta}_h^\alpha f\right)(x)$, *at fixed* $h=\left(h_1,\ldots,h_n\right),\ 0<h_i<1,\ i=1,\ldots,n$, *the following integral representation is true*

$$\left(\tilde{\Delta}_h^\alpha f\right)(x)=\int\limits_0^\infty \ldots \int\limits_0^\infty \left(\Delta^\alpha_{\ln\frac{1}{h}} k_\alpha^+\right)\left(\ln\frac{x}{t}\right)\left(D^\alpha_{+\cdots+}f\right)(t)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}, \tag{9.3}$$

where $\left(\Delta_\xi^\alpha k_\alpha^+\right)(y)=\Delta_{\xi_1}^{\alpha_1}[\Delta_{\xi_2}^{\alpha_2}\ldots(\Delta_{\xi_n}^{\alpha_n}k_\alpha^+)](y)$, $k_\alpha^+\left(y\right)=\prod\limits_{i=1}^n\frac{(y_i)_+^{\alpha_i-1}}{\Gamma(\alpha_i)}$.

Proof. Consider the operator

$$\left(B_h^\alpha\varphi\right)(x)=\int\limits_0^\infty \ldots \int\limits_0^\infty \left(\Delta^\alpha_{\ln\frac{1}{h}} k_\alpha^+\right)\left(\ln\frac{x}{t}\right)\varphi\left(t\right)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}.$$

Since $\left(\Delta_\xi^\alpha k_\alpha^+\right)(y)\in L_1\left(\mathbb{R}^n\right)$, the operator $B_h^\alpha\varphi$ is bounded in the space $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ in virtue of Lemma 9.1. Denote $\varphi_\delta=D^\alpha_{+\cdots+,\mu;\delta}f$ and

$$\left(B_h^\alpha\varphi_\delta\right)(x)=\int\limits_0^\infty \ldots \int\limits_0^\infty \left(\Delta^\alpha_{\ln\frac{1}{h}} k_\alpha^+\right)\left(\ln\frac{x}{t}\right)\left(D^\alpha_{+\cdots+,\delta}f\right)(t)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}.$$

Note, that B_h^α is a convolution with the summable kernel $\left(\Delta_\xi^\alpha k_\alpha^+\right)(y)\in L_1\left(\mathbb{R}^n\right)$ and so the composition $B_h^\alpha D^\alpha_{+\cdots+,\delta}f$ is (at fixed $\delta=\left(\delta_1,\ldots,\delta_n\right),\ \delta_i>0,\ i=\overline{1,n}$) a bounded operator in $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ at all $\lambda_i\geq 0,\ 1\leq r_i<\infty,\ \ i=1,\ldots,n$. Prove presentation (9.3) first for $f\in C_0^\infty\left(\mathbb{R}_+^n\right)$. We have

$$\left(B_h^\alpha\varphi_\delta\right)(x)=\frac{1}{\aleph\left(\alpha,l\right)}\int\limits_0^\infty \ldots \int\limits_0^\infty \left(\Delta^\alpha_{\ln\frac{1}{h}} k_\alpha^+\right)\left(\ln\frac{x}{t}\right)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}\times$$

$$\times \int\limits_0^{1-\delta_1} \cdots \int\limits_0^{1-\delta_n} \prod_{i=1}^{n} \left(\ln \frac{1}{t_i} \right)^{-\alpha_i - 1} \left(\tilde{\Delta}_y^l f \right)(t) \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n}. \tag{9.4}$$

Since $\left(\Delta_\xi^\alpha k_\alpha^+ \right)(y) \in L_1(\mathbb{R}^n)$, then in (9.4) the change of the order of integration is justified by the Fubini's theorem. So

$$\left(B_h^\alpha \varphi_\delta \right)(x) = \frac{1}{\aleph(\alpha, l)} \int\limits_0^{1-\delta_1} ... \int\limits_0^{1-\delta_n} \prod_{i=1}^{n} \left(\ln \frac{1}{t_i} \right)^{-\alpha_i - 1} \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n} \times$$

$$\times \sum_{0 \le |k| \le l} (-1)^{|k|} \binom{l}{k} \int\limits_0^\infty \cdots \int\limits_0^\infty \left(\Delta_{\ln \frac{1}{h}}^\alpha k_\alpha^+ \right) \left(\ln \frac{x}{t} \right) f\left(y^k \circ t \right) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}.$$

The substitution $t_i = x_i \cdot \xi_i \cdot y_i^{-k_i} \cdot h_i^{j_i}, i = 1, \ldots, n$ gives

$$\left(B_h^\alpha \varphi_\delta \right)(x) = \frac{1}{\aleph(\alpha, l)} \int\limits_0^{1-\delta_1} ... \int\limits_0^{1-\delta_n} \prod_{i=1}^{n} \left(\ln \frac{1}{t_i} \right)^{-\alpha_i - 1} \frac{dy_1}{y_1} ... \frac{dy_n}{y_n} \sum_{0 \le |k| \le l} (-1)^{|k|} \binom{l}{k} \times$$

$$\times \int\limits_0^{y_1^{k_1}} \cdots \int\limits_0^{y_n^{k_n}} \left(\ln \frac{y^k}{\xi} \right)^{\alpha - 1} \sum_{0 \le |j| \le l} (-1)^{|j|} \binom{\alpha}{j} f\left(x \circ h^j \circ t \right) \frac{d\xi_1}{\xi_1} \cdots \frac{d\xi_n}{\xi_n}.$$

Hence,

$$\left(B_h^\alpha \varphi_\delta \right)(x) = \frac{1}{\aleph(\alpha, l)} \int\limits_0^1 \cdots \int\limits_0^1 \left(\tilde{\Delta}_h^\alpha f \right) f(x \circ \xi) \frac{d\xi_1}{\xi_1} \cdots \frac{d\xi_n}{\xi_n} \times$$

$$\times \int\limits_0^{1-\delta_1} \cdots \int\limits_0^{1-\delta_n} \prod_{i=1}^{n} \left(\ln \frac{1}{t_i} \right)^{-\alpha_i - 1} \left(\Delta_{\ln \frac{1}{y}}^l k_\alpha^+ \right) \left(\ln \frac{1}{\xi} \right) \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n}.$$

Here, the change of the order of integration is possible on the basis of Fubini's theorem, since $\left(\Delta_\xi^\alpha k_\alpha^+ \right)(y) \in L_1\left(\mathbb{R}^n, \frac{dx}{x} \right)$. Substituting $\ln \frac{1}{y_i} = \frac{1}{s_i} \cdot \ln \frac{1}{\xi_i}$, $\ln \frac{1}{\xi_i} = u_i \cdot \ln \frac{1}{1-\delta_i}, i = 1, \ldots, n$, we obtain

$$\left(B_h^\alpha \varphi_\delta \right)(x) = \frac{1}{\aleph(\alpha, l)} \int\limits_0^\infty ... \int\limits_0^\infty \left(\tilde{\Delta}_h^\alpha f \right) f\left(x \circ (1-\delta)^u \right) \frac{du_1}{u_1} \cdots \frac{du_n}{u_n} \times$$

$$\times \int\limits_0^{u_1} \cdots \int\limits_0^{u_n} \left(\Delta_1^l k_\alpha^+ \right)(s) \, ds_1 \cdots ds_n. \tag{9.5}$$

The equality $\left(\Delta_1^l k_\alpha^+ \right)(s) = \left(\Delta_1^{l_1} k_{\alpha_1}^+ \right)(s_1) \cdots \left(\Delta_1^{l_n} k_{\alpha_n}^+ \right)(s_n)$, from (9.5), we have

$$\left(B_h^\alpha \varphi_\delta \right)(x) = \int\limits_0^\infty \cdots \int\limits_0^\infty \left(K_{l_1,\alpha_1}^{+1} \right)(u_1) \ldots \left(K_{l_n,\alpha_n}^{+} \right)(u_1) \left(\tilde{\Delta}_h^\alpha f \right) \left(x \circ (1-\delta)^u \right) du_1 \cdots du_n, \tag{9.6}$$

where $K_{l_i,\alpha_i}^{+,\mu_i}$ is kernel (7.4). In (7.4) the right-hand side in (9.6) is an operator bounded in $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ by Lemma 5.4. Since $B_h^\alpha D_{+\cdots+,\delta}^\alpha f$ is also an operator bounded in $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$, then (9.6) follows for functions

f belonging to $C_0^\infty\left(\mathbb{R}_+^n\right)$. Therefore, from (9.6), by passing to the limit at $\delta \to 0$, identity (9.3) is obtained.

Since $\varphi = \lim\limits_{\delta\to 0} \varphi_\delta$ in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, the left-hand side of (9.6) converges in norm $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ due to the boundedness of operator B_h^α in the $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$. On the other hand, the right-hand side in (9.6) converges at $\delta \to 0$ to $\left(\tilde{\Delta}_h^\alpha f\right)(x)$ in norm $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ by virtue of Lemma 5.5.

Due to the identical coincidence of the left-hand and right-hand sides in (9.6), their limits at $\delta \to 0$, although in different norms $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$, must coincide almost everywhere. This leads to (9.3). □

Theorem 9.2. *In order $f(x)$ to be representable by a mixed fractional Hadamard integral $f(x) = \left(J_{+\cdots+}^\alpha \varphi\right)(x)\,,\ \varphi \in \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, where either*

1) $\gamma_i > 0,\ \alpha_i > 0,\ 1 \le p_i < \infty,\ i = 1, \ldots, n,$

or

2) $\gamma_i = 0,\ 0 < \alpha_i < 1,\ \ 1 < p_i < \frac{1}{\alpha_i},\ i = 1, \ldots, n,$

it is necessary and sufficient that $f \in \mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$, where $\lambda_i > 0,\ 1 \le r_i < \infty,\ i = 1, \ldots, n,$ in case 1) or $\lambda_i = 0,\ r_i = \frac{p_i}{1-\alpha_i p_i}$ in case 2) and the limit exists $\varphi = \lim\limits_{\delta\to 0} D_{+\cdots+,\delta}^\alpha f$ in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$.

Proof. The necessity of this theorem follows from Theorem 6.2 and Theorem 8.2.The sufficiency is obtained by the scheme of the proof of Theorem 9.1. □

Theorem 9.3. *In order $\left(\tilde{\Delta}_\tau^l f\right)(x)$ to be representable by a mixed fractional Hadamard integral $\left(\tilde{\Delta}_\tau^l f\right)(x) = J_{+\cdots+,\tau}^{\alpha,l}\varphi,\ \varphi \in \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, where $\gamma_i \ge 0,\ \ l > \alpha_i > 0,\ 1 \le p_i \le \infty,\ \ 0 < \tau_i < 1,\ \ i = 1, \ldots, n$ it is necessary and sufficient that, $\left(\tilde{\Delta}_\tau^l f\right)(x) \in \mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$, where $\lambda_i \ge 0,\ 1 \le r_i < \infty,\ \ 0 < \tau_i < 1,\ \ i = 1, \ldots, n$ and the limit exists*

$$\varphi = \lim_{\delta\to 0} D_{+\cdots+,\delta}^\alpha f, \tag{9.7}$$

where the limit is in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$.

Proof. The necessity of this theorem follows from Theorem 6.3 and Theorem 8.3.

Sufficiency. Let $\left(\tilde{\Delta}_\tau^l f\right)(x) \in \mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ and condition (9.7) be satisfied. It is required to prove that

$$\left(\tilde{\Delta}_\tau^l f\right)(x) = J_{+\cdots+,\tau}^{\alpha,l}\varphi. \tag{9.8}$$

From (9.8) we have

$$\tilde{\Delta}_h^\alpha \left(\tilde{\Delta}_\tau^l f\right)(x) = \tilde{\Delta}_h^\alpha J_{+\cdots+,\tau}^{\alpha,l}\varphi. \tag{9.9}$$

At $h = (h_1, \ldots, h_n)\,,\ 0 < h_i < 1,\ i = 1, \ldots, n.$ Introduce the notation

$$\left(B_h^{\alpha,l}\varphi\right)(x) = \left(\ln\frac{1}{h}\right)^\alpha \int_0^\infty \ldots \int_0^\infty P_\alpha(z) \left(\tilde{\Delta}_\tau^l \varphi\right)(x \circ h^z)\, dz_1 \ldots dz_n,$$

where $P_\alpha(z) = \frac{1}{\Gamma(\alpha)} \sum\limits_{0\le|j|<\infty} (-1)^{|j|} \begin{pmatrix} \alpha \\ j \end{pmatrix} (z-j)^{\alpha-1} \in L_1(\mathbb{R}^n)\,.$

Consider the expression $B_h^{\alpha,l}\varphi_\delta,\ \varphi_\delta = D_{+\cdots+,\delta}^\alpha f$. For functions $f(x)$, belonging to $C_0^\infty\left(\mathbb{R}_+^n\right)$, we have

$$B_h^{\alpha,l}\varphi_\delta = B_h^{\alpha,l} D_{+\cdots+,\delta}^\alpha f = \tilde{\Delta}_h^\alpha \tilde{\Delta}_\tau^l J_{+\cdots+,\delta}^\alpha D_{+\cdots+,\delta}^\alpha f. \tag{9.10}$$

With well-known integral representation of finite differences (see [17], p. 101-102), we obtain

$$(\tilde{\Delta}_\tau^l J^\alpha_{+\cdots+,\delta} D^\alpha_{+\cdots+,\delta} f)\,(x) =$$

$$= \int_0^\infty \ldots \int_0^\infty \left(K^+_{l_1,\alpha_1}\right)(y_1)\ldots\left(K^+_{l_n,\alpha_n}\right)(y_n)\left(\tilde{\Delta}_\tau^l f\right)(x\circ(1-\delta)^y)\,dy_1\ldots dy_n,$$

where $\left(K^+_{l_i,\alpha_i}\right)(y_i)$ is kernel (7.4). Then from (9.10) we have

$$B_h^{\alpha,l}\varphi_\delta = \int_0^\infty \ldots \int_0^\infty \left(K^{+1}_{l_1,\alpha_1}\right)(y_1)\ldots\left(K^+_{l_n,\alpha_n}\right)(y_n)\left(\tilde{\Delta}_h^\alpha\tilde{\Delta}_\tau^l f\right)(x\circ(1-\delta)^y)\,dy_1\ldots dy_n. \tag{9.11}$$

With (7.4) in mind, the right-hand side in (9.11) is an operator bounded in $\mathfrak{L}^{\overline{r}}_{\overline{\lambda}}$. According to Lemma 5.4, since the composition $B_h^{\alpha,l}D^\alpha_{+\cdots+,\delta}f$ is (at fixed $\delta=(\delta_1,\ldots,\delta_n)\,,\ 0<\delta_i<1,\ i=1,\ldots,n$) an operator bounded in $\mathfrak{L}^{\overline{r}}_{\overline{\lambda}}$ for $\lambda_i\geq 0,\ 1\leq r_i<\infty,\ \ i=1,\ldots,n$, then (9.11) follows for functions f belonging to $C_0^\infty\left(\mathbb{R}^n_+\right)$. By (7.5) the right-hand side in (9.11) converges in the norm of the space $\mathfrak{L}^{\overline{r}}_{\overline{\lambda}}$ to $\left(\tilde{\Delta}_h^\alpha\tilde{\Delta}_\tau^l f\right)(x)$.

So, there exists a limit of the left-hand side

$$\lim_{\delta\to 0} B_h^{\alpha,l}D^\alpha_{+\cdots+,\delta}f = \left(\tilde{\Delta}_h^\alpha\tilde{\Delta}_\tau^l f\right)(x)\,.$$

Since $\varphi=\lim\limits_{\delta\to 0}\varphi_\delta$ in $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, the left-hand side of (9.11) converges in $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ due to the boundedness of the operator $B_h^{\alpha,l}$ in $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$. Then there exists a limit

$$\lim_{\delta\to 0} B_h^{\alpha,l}D^\alpha_{+\cdots+,\delta}f = B_h^{\alpha,l}\lim_{\delta\to 0}(D^\alpha_{+\cdots+,\delta}f) = B_h^{\alpha,l}\varphi, \tag{9.12}$$

where $\varphi=D^\alpha_{+\cdots+}f$. Since $B_h^{\alpha,l}D^\alpha_{+\cdots+,\delta}f$ converges both in the norm $\mathfrak{L}^{\overline{r}}_{\overline{\lambda}}$ and norm $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, the limiting functions must coincide almost everywhere. Then from (9.12), we obtain

$$B_h^{\alpha,l}\varphi = \left(\tilde{\Delta}_h^\alpha\tilde{\Delta}_\tau^l f\right)(x)\,,$$

which coincides with (9.9). It should be noted that functions $\tilde{\Delta}_\tau^l f$and $J^{\alpha,l}_{+\cdots+,\tau}D^\alpha_{+\cdots+}f$ have identically coinciding mixed finite differences. Therefore, they can differ only by a polynomial (see [19], p. 103)

$$\tilde{\Delta}_\tau^l f = J^{\alpha,l}_{+\cdots+,\tau}D^\alpha_{+\cdots+}f + P\,(x)\,,$$

where $P\,(x)$ is a polynomial. Then from (9.9) follows (9.8) taking into account that $\tilde{\Delta}_\tau^l f,\ J^{\alpha,l}_{+\cdots+,\tau}\varphi\in\mathfrak{L}^{\overline{r}}_{\overline{\lambda}}$. □

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MAPS BETWEEN FRÉCHET ALGEBRAS WHICH STRONGLY PRESERVES DISTANCE ONE

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Abstract. We prove that if $T : X \longrightarrow Y$ is a 2-isometry between real linear 2-normed spaces, then T is affine whenever Y is strictly convex. Also under some conditions we show that every surjective mapping $T : A \longrightarrow B$ between real Fréchet algebras, which strongly preserves distance one, is affine.

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1 Introduction and preliminaries

An algebra A over the complex field $\mathbb{C}$, is called a *Fréchet algebra* if it is a complete metrizable topological linear space. The topology of a Fréchet algebra A can be generated by a sequence (p_k) of separating submultiplicative seminorms, i.e.,

$$p_k(xy) \leq p_k(x)p_k(y),$$

for all $k \in \mathbb{N}$ and $x, y \in A$, such that $p_k(x) \leq p_{k+1}(x)$, whenever $k \in \mathbb{N}$ and $x \in A$, [6, 7]. A Fréchet algebra A with the above generating seminorms (p_k) is denoted by (A, p_k).

A map $f : X \longrightarrow Y$ between real normed spaces is an *isometry* if $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in X$, and f is *affine* if

$$f(ta + (1-t)b) = tf(a) + (1-t)f(b),$$

for all $a, b \in X$ and $0 \leq t \leq 1$.

An isometry need not be affine [12]. There are two important results describing cases in which every isometry is affine. The first basic result is due to Baker.

Theorem 1.1. [1] *Let X and Y be two real normed linear spaces and suppose that Y is strictly convex. If $T : X \longrightarrow Y$ is an isometry, then T is affine.*

The second result is the Mazur-Ulam theorem.

Theorem 1.2. [8] *Every bijective isometry $T : X \longrightarrow Y$ between normed spaces is affine.*

This result was proved by S. Mazur and S. Ulam [8] in 1932, and their proof is also given in the book [2, p. 166]. See also [9] and [11] for different proofs. Theorem 1.2 was improved by relaxing the subjectivity condition in [5], and is generalized for Fréchet algebras by the author in [12].

Recently, Chu in [3] proved that Theorem 1.2 holds when X and Y are linear 2-normed spaces, i.e., he proved that every 2-isometry between two linear 2-normed spaces is affine.

A mapping $T : X \longrightarrow Y$ between real normed spaces X and Y is said to strongly preserve distance n if for all $x, y \in X$ with $\|x - y\| = n$ it follows that $\|T(x) - T(y)\| = n$ and conversely. In particular, T strongly preserves distance one if $n = 1$.

A different kind of generalization of the Mazur-Ulam theorem was given by Rassias and Semrl in [10]. They proved under a special hypotheses that every surjective mapping $T : X \longrightarrow Y$ between real normed linear spaces X and Y which strongly preserves distance one, is affine [10, Theorem 5].

Also it is shown that the Rassias's theorem holds when X is a linear 2-normed space under some conditions [4].

In this paper, we prove that Theorem 1.1 holds when X and Y are linear 2-normed spaces. We also give an extension of the Rassias's theorem for Fréchet algebras.

2 The Baker result for linear 2-normed spaces

Definition 1. Let X be a real linear space with $dimX \geq 2$ and let $p(\cdot,\cdot) : X^2 \longrightarrow \mathbb{R}^+$ be function. Then $(X, p(\cdot,\cdot))$ is called a linear 2-seminormed space if

(a) $p(x, y) = 0 \Leftrightarrow x, y$ are linearly dependent,

(b) $p(x, y) = p(y, x)$,

(c) $p(\lambda x, y) = |\lambda| p(x, y)$,

(d) $p(x, y + z) \leq p(x, y) + p(x, z)$,

(e) $p(x, z) \leq p(x, y) + p(y, z)$,

for all $x, y, z \in X$, $\lambda \in \mathbb{R}$.

In particular, if we define $p(x, y) = \|x, y\|$, then we obtain a real linear 2-normed space in the sense of [3, Definition 2.1]. For example, define $p : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $p\big((a, b), (x, y)\big) = |ay - bx|$. Then $(\mathbb{R}^2, p)$ is a linear 2-normed space.

Let $(X, p(\cdot,\cdot))$ be a linear 2-seminormed space. Then we say that X is strictly convex if the equality

$$p(a, x) + p(a, y) = p(a, x + y), \quad a, x, y \in X,$$

implies that $x = \lambda y$ for some $\lambda > 0$.

Let $(X, p(\cdot,\cdot))$ and $(Y, q(\cdot,\cdot))$ be linear 2-seminormed spaces. A map $T : X \longrightarrow Y$ is called 2-*isometry* if

$$p(x - z, y - z) = q(T(x) - T(z), T(y) - T(z)),$$

for all $x, y, z \in X$. Moreover, T is called affine if

$$T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1 - \lambda)T(y),$$

for all $\lambda \in [0, 1]$ and every $x, y \in X$.

Lemma 2.1. *Let $(X, p(\cdot,\cdot))$ be linear 2-seminormed space. Then for all $x, y \in X$ and $\lambda \in \mathbb{R}$, we have $p(x, y) = p(x, y + \lambda x)$.*

Proof. Since $p(x,\lambda x)=0$, we have $p(x,y+\lambda x)\le p(x,y)+p(x,\lambda x)=p(x,y)$. On the other hand, let $a=y+\lambda x$. Then $y=a-\lambda x$ and so

$$p(x,y)=p(x,a-\lambda x)\le p(x,a)+p(x,\lambda x)=p(x,y+\lambda x),$$

for all $x,y\in X$. □

Lemma 2.2. *Suppose that X is a strictly convex 2-seminormed space and $a,b\in X$. Then $u=\frac{1}{2}(a+b)$ is the unique element of X satisfying*

$$2p(a-c,a-u)=2p(b-u,b-c)=p(a-c,b-c), \tag{2.1}$$

for some $c\in X$ with $p(a-c,b-c)\neq 0$.

Proof. The result is clear if $a=b$. By Lemma 2.1, we have that for all $c\in X$,

$$\begin{aligned}2p(a-c,a-u)&=2p(a-c,a-\frac{1}{2}(a+b))\\&=p(a-c,a-b)\\&=p(a-c,a-b+(-1)(a-c))\\&=p(a-c,b-c).\end{aligned}$$

Similarly, $2p(b-u,b-c)=p(a-c,b-c)$. Therefore it suffices to prove the uniqueness of u. Suppose that $w\in X$ is such that

$$2p(a-c,a-w)=2p(b-w,b-c)=p(a-c,b-c), \tag{2.2}$$

for some $c\in X$ with $p(a-c,b-c)\neq 0$. By (2.1) and (2.2), we get

$$p(a-c,a-\frac{1}{2}(u+w))\le\frac{1}{2}p(a-c,a-u)+\frac{1}{2}p(a-c,a-w)=\frac{1}{2}p(a-c,b-c). \tag{2.3}$$

Similarly,

$$p(b-\frac{1}{2}(u+w),b-c)\le\frac{1}{2}p(b-u,b-c)+\frac{1}{2}p(b-w,b-c)=\frac{1}{2}p(a-c,b-c). \tag{2.4}$$

If either of these inequalities were strict, then by using Lemma 2.1, we obtain

$$\begin{aligned}p(a-c,b-c)&\le p(a-c,c-\frac{1}{2}(u+w))+p(c-\frac{1}{2}(u+w),b-c)\\&=p(a-c,a-\frac{1}{2}(u+w))+p(b-\frac{1}{2}(u+w),b-c)\\&<p(a-c,b-c),\end{aligned}$$

which is a contradiction. Thus, the equality holds in (2.3) and (2.4). Therefore

$$p(a-c,a-\frac{1}{2}(u+w))=\frac{1}{2}p(a-c,a-u)+\frac{1}{2}p(a-c,a-w).$$

Since X is strictly convex, we conclude that $a-u=\lambda(a-w)$ for some $\lambda>0$. On the other hand, by (2.1) and (2.2), $p(a-c,a-u)=p(a-c,a-w)$. Hence $\lambda=1$ and $u=w$. □

Now we prove our main theorem. The idea of the proof can be found in [1].

Theorem 2.1. *Let $(X, p(\cdot,\cdot))$ and $(Y, q(\cdot,\cdot))$ be two real 2-seminormed spaces, where Y is strictly convex. If $T : X \longrightarrow Y$ is a 2-isometry, then T is affine.*

Proof. We may assume without loss of generality that $T(0) = 0$. Indeed, if $T(0) \neq 0$, then $\phi(x) = T(x) - T(0)$ is an isometry and $\phi(0) = 0$. Since T is 2-isometry, from Lemma 2.1 we have

$$\begin{aligned}2q(T(a)-T(c), T(a)-T(\frac{a+b}{2})) &= 2p(a-c, a-\frac{a+b}{2})\\ &= p(a-c, b-c)\\ &= q(T(a)-T(c), T(b)-T(c)).\end{aligned}$$

Similarly,

$$\begin{aligned}2q(T(b)-T(\frac{a+b}{2}), T(b)-T(c)) &= 2p(b-\frac{a+b}{2}, b-c)\\ &= p(a-c, b-c)\\ &= q(T(a)-T(c), T(b)-T(c)),\end{aligned}$$

for every $a, b, c \in X$. As Y is strictly convex, replacing in (2.1), a by $T(a)$, b by $T(b)$, c by $T(c)$ and u by $T(\frac{a+b}{2})$ and using the uniqueness of u, we get that

$$T(\frac{a+b}{2}) = \frac{1}{2}(T(a) + T(b)).$$

Thus, T preserves the midpoints of line segments, and hence T is affine by Lemma 2.2 of [12]. □

Corollary 2.1. *Let X and Y be two real linear 2-normed spaces, and suppose that Y is strictly convex. If $T : X \longrightarrow Y$ is a 2-isometry, then T is affine.*

3 Maps that preserve distance one

Let A and B be two linear spaces equipped the seminorms p and q, respectively. A map $T : A \longrightarrow B$ is called *isometry* if

$$p(x-y) = q(T(x) - T(y)),$$

for all $x, y \in A$.

Theorem 3.1. *Let A and B be two linear spaces equipped the seminorms p and q, respectively. Suppose that $dimB \geq 2$ and $T : A \longrightarrow B$ is a surjective mapping that strongly preserves distance one. Then T strongly preserves distance n.*

Proof. We first prove that $dimA \geq 2$ and T is one to one. Since $dimB \geq 2$, there exist elements $x, y, z \in B$ such that

$$q(x-y) = q(x-z) = q(y-z) = 1.$$

The mapping T is given to be surjective and strongly preserving distance one, so there exist elements $a, b, c \in A$ such that $T(a) = x$, $T(b) = y$, $T(c) = z$ and

$$p(a-b) = p(a-c) = p(b-c) = 1.$$

Therefore $dimA \geq 2$. Now let $a, b \in A$ such that $a \neq b$ and $T(a) = T(b)$. We can fined $c \in A$ such that $p(a-c) = 1$ and $p(b-c) \neq 1$. Hence

$$q(T(b) - T(c)) = q(T(a) - T(c)) = 1.$$

This implies that $p(b-c)=1$, which is not possible. Thus, T is one to one. Hence T is bijective and both T and T^{-1} preserve distance one.

Next we prove by induction that T strongly preserves distance n for all $n\in\mathbb{N}$. Suppose that T strongly preserves distance n and $a,b\in A$ with $p(a-b)=n+1$. Similar to the proof of [10, Theorem 1], we have that

$$|q\big(T(a)-T(b)\big)-p(a-b)|<1,$$

for all $a,b\in A$. Therefore $q(T(a)-T(b))\leq n+1$. Define

$$u=T(a)+\frac{1}{k}(T(b)-T(a)),$$

where $k=q(T(b)-T(a))$. There exists $v\in A$ such that $u=T(v)$. From $q(u-T(a))=1$ we deduce $p(v-a)=1$. If $q(u-T(b))<n$, then we have $p(v-b)<n$. So

$$p(a-b)\leq p(a-v)+p(v-b)<n+1,$$

which is a contradiction. Therefore, $q(u-T(b))\geq n$. This implies that

$$n\leq q(u-T(b))=q\big((T(a)-T(b))(1-\frac{1}{k})\big)=|1-\frac{1}{k}|\ k=|k-1|.$$

Thus, $q(T(a)-T(b))=k=n+1$. Similarly, T^{-1} strongly preserves distance $n+1$.

Conversely, let $a,b\in A$ such that $q(T(a)-T(b))=n+1$. Assume that $x=T(a)$ and $y=T(b)$. Then $q(x-y)=n+1$. Since T^{-1} strongly preserves distance $n+1$, hence

$$n+1=q(T^{-1}(x)-T^{-1}(y))=p(a-b).$$

□

Let A and B be two linear spaces equipped with seminorms p and q, respectively. We call $T:A\longrightarrow B$ a *Lipschitz mapping* if there is a $L\geq 0$ such that

$$q\big(T(a)-T(b)\big)\leq L\ p(a-b),$$

for all $a,b\in A$. In this case, the constant L is called the Lipschitz constant.

Theorem 3.2. *Let (A,p) and (B,q) be two real linear spaces with $dimB\geq 2$. Suppose that $T:A\longrightarrow B$ is a Lipschitz mapping with $L=1$ and let T be a surjective strongly preserves distance one. Then T is an isometry.*

Proof. By Theorem 3.1, T is one to one and strongly preserves distance n for all $n\in\mathbb{N}$. Let $a,b\in A$ and N be a positive integer satisfying $p(a-b)<N$. Let

$$q(T(a)-T(b))<p(a-b). \tag{3.1}$$

Take

$$c=a+\frac{N}{p(a-b)}(b-a).$$

Then $p(c-a)=N$ and since $p(a-b)<N$, we get

$$p(c-b)=p\big((a-b)(1-\frac{N}{p(b-a)})\big)=(\frac{N}{p(b-a)}-1)p(b-a)=N-p(b-a).$$

Therefore, we obtain

$$\begin{aligned} N = q(T(c) - T(a)) \leq & q(T(c) - T(b)) + q(T(b) - T(a)) \\ < & q(T(c) - T(b)) + p(a - b) \\ < & N - p(b - a) + p(a - b) \\ = & N, \end{aligned}$$

which is not possible. Hence the equality holds in (3.1), and T is an isometry. □

Combining Theorem 3.2 and Theorem 2.3 of [12], we get the following result.

Corollary 3.1. *Let (A, p) and (B, q) be two real linear spaces with $dimB \geq 2$. Suppose that $T : A \longrightarrow B$ is a Lipschitz mapping with $L = 1$ and let T be a surjective strongly preserves distance one. Then T is affine.*

Corollary 3.2. *Let (A, p) and (B, q) be two real linear spaces, where B is strictly convex and $dimB \geq 2$. Suppose that $T : A \longrightarrow B$ is a surjective mapping strongly preserving distance one. Then T is affine.*

Proof. By Theorem 3.1, T is one to one and strongly preserves distance n for all $n \in \mathbb{N}$. Let $a, b \in A$ and $p(a - b) = \frac{1}{n}$. We can find $c \in A$ such that $p(a - c) = p(b - c) = 1$. Let $u = c + n(b - c)$. Then

$$p(u - c) = n, \quad \textit{and} \quad p(b - u) = n - 1.$$

Therefore,

$$q(T(b) - T(c)) = 1, \qquad q(T(u) - T(c)) = n, \qquad q(T(b) - T(u)) = n - 1.$$

Since B is strictly convex, $T(u) - T(b) = (n - 1)\big(T(b) - T(c)\big)$ and so we have

$$T(b) = \frac{1}{n} T(u) + \frac{n - 1}{n} T(c).$$

Similarly, if we take $v = c + n(a - c)$, then we obtain

$$T(a) = \frac{1}{n} T(v) + \frac{n - 1}{n} T(c).$$

Using $p(u - v) = 1$, we get

$$q(T(a) - T(b)) = \frac{1}{n} q(T(u) - T(v)) = \frac{1}{n}.$$

Thus, T preserves distance $\frac{1}{n}$ for all $n \in \mathbb{N}$.

Now let $p(a - b) \leq \frac{m}{n}$, where $m \in \mathbb{N}$ with $m \geq 2$. As $dimA \geq 2$, we can find a finite sequence $x_i \in A$ for $i = 0, 1, ..., m$ with $x_0 = a$ and $x_m = b$ such that $p(x_i - x_{i+1}) = \frac{1}{n}$. Hence

$$q(T(a) - T(b)) \leq \sum_{i=0}^{m-1} q(T(x_i) - T(x_{i+1})) = \frac{m}{n}.$$

Thus, for all $a, b \in A$,

$$q(T(a) - T(b)) \leq p(a - b).$$

Consequently, T is a Lipschitz mapping with $L = 1$ and by Theorem 3.2, it is affine. □

Next we give an example of an isometry between Fréchet algebras which is not affine. This example shows that the strict convexity of Fréchet algebra B in Theorem 3.2 of [12] is essential.

Example 1. Let $I_k = [-k, k]$ for $k \in \mathbb{N}$, and consider the Fréchet algebra $B = C(\mathbb{R})$, the algebra of continuous functions on $\mathbb{R}$ with a compact open topology, and the sequence

$$p_k(f) = sup\{|f(x)| : \quad x \in I_k\},$$

of submultiplicative seminorms. It is easy to check that B is not strictly convex. Choose $f, g \in B$ such that $p_k(f) = p_k(g) = 1$, $p_k(f + g) = p_k(f) + p_k(g)$ and f, g is linearly independent. For all $\lambda \in \mathbb{R}$, define $T : \mathbb{R} \longrightarrow B$ by

$$T(\lambda) = \begin{cases} \lambda f & \lambda \leq 1 \\ f + (\lambda - 1)g & \lambda > 1. \end{cases}$$

Then by the same method as in [1] we conclude that T is an isometry, but is not affine.

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Events

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INTERNATIONAL SCIENTIFIC AND PRACTICAL CONFERENCE “ANALYSIS, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS” DEDICATED TO THE 100TH ANNIVERSARY OF THE BIRTHDAY OF CORRESPONDING MEMBER OF ACADEMY OF SCIENCES OF KAZSSR TOLEUBAY IDRISOVICH AMANOV

The L.N. Gumilyov Eurasian National University (ENU) and Kazakhstan branch of M.V. Lomonosov Moscow State University held the international scientific and practical conference “Analysis, differential equations and their applications”, dedicated to the 100th anniversary of Professor Toleubay Idrisovich Amanov, Doctor of Sciences in physics and mathematics, corresponding member of the National Academy of Sciences of KazSSR, one of the founders of the theory of functions and functional analysis in Kazakhstan through June 22 – 23, 2023 (ENU, 13 Kazhymukan St, Astana).

The purpose of the conference was to discuss scientific achievements in the field of analysis, the theory of differential equations, partial differential equations and related areas.

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Programme of the conference

June 22, 2023

PLENARY REPORTS

10:00 – 10:20, Opening ceremony (E.B. Sydykov, B.T. Zhumagulov)
10:20 – 11:00, E.S. Smailov, *Biography and scientific heritage of Corresponding Member of the Academy of Sciences of the Kazakh SSR, Doctor of Physical and Mathematical Sciences, Professor T.I. Amanov*
11:00 – 11:30, V.I. Burenkov, *Interpolation theorems for nonlinear operators and general Morrey-type spaces*
12:00 – 12:30, N.K. Bliev, *Multidimensional singular integrals and integral equations in fractional spaces*
12:30 – 13:00, G.A. Chechkin, *Boyarsky-Meyers estimates for Zaremba's problems. Example*
14:00 – 18:00, Sectional reports

June 30, 2023

PLENARY REPORTS

10:00 –10:30, M.A. Sadybekov, B. Derbisali, *Direct and inverse initial boundary value problems for heat equation with non-classical boundary condition*
10:30 – 11:00, V.V. Chepyzhov, *Strong zero-viscosity limit for attractors of 2D Navier-Stokes systems with Ekman friction on a two-dimensional torus*

11:00 – 11:30, B.Kh. Turmetov, F. Dadabaeva *Inverse problems for a fractional parabolic equation with a nonlocal biharmonic operator*
12:00 – 12:30, B.M. Sabitbek, *Global existence and nonexistence of solutions to semilinear wave equation*
12:30 – 13:00, T. Nurlybeluly, *Atomic decomposition for symmetric space of noncommutative martingales*
14:00 – 16:00, Sectional reports

V.I. Burenkov, E.S. Smailov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev.

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