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ALGEBRAIC PROOFS OF CHARACTERIZING REVERSE ORDER LAW FOR CLOSED RANGE OPERATORS IN HILBERT SPACES

S.K. Athira, K. Kamaraj, P.S. Johnson

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Key words: Moore-Penrose inverse, reverse order law, closed range operator.

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Abstract. We present more than 60 results, including some range inclusion results to characterize the reverse order law for the Moore-Penrose inverse of closed range Hilbert space operators. We use the basic properties of the Moore-Penrose inverse to prove the results. Some examples are also provided to illustrate failure cases of the reverse order law in an infinite-dimensional setting.

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1 Introduction

One of the fundamental research problems in the theory of generalized inverses of matrices is to establish reverse order laws for generalized inverses of matrix products. It was Erik Ivar Fredholm who seemed to have first mentioned the concept of generalized inverse in 1903. He formulated a pseudoinverse for a linear integral operator, which is not invertible in the ordinary sense. Hilbert, Schmidt, Bounitzky, Hurwitz and other mathematicians had studied the generalized inverses of integral operators and differential operators before Moore introduced the generalized inverse of matrices by algebraic methods in 1920 [17]. Bjerhammar rediscovered Moore's inverse and also noted the relationship of generalized inverses to solutions of linear systems in 1951 [5]. In 1955, Penrose [21] extended Bjerhammar's results and showed that Moore's inverse for a given matrix A is the unique matrix X satisfying the four equations:

$$AXA = A; XAX = X; (AX)^* = AX; (XA)^* = XA.$$

In honour of Moore and Penrose, this unique inverse is now commonly called the Moore-Penrose inverse and is denoted by $A^\dagger$. Meanwhile, generalized inverses were defined for operators by Tseng [24], Murray and von Neumann [19], Nashed [20] and others. Beutler discussed generalized inverses for both bounded and unbounded operators with closed and arbitrary ranges [3, 4]. Throughout the years, the Moore-Penrose inverse was extensively studied. One of the primary reasons for considering the Moore-Penrose inverse is solving systems of linear equations, which constitutes an important application in various fields.

It is well known that the reverse order law $(AB)^{-1} = B^{-1}A^{-1}$ is not true in general for various generalized inverses such as the Moore-Penrose inverse, Drazin inverse etc. Cline attempted to find a reasonable representation for the Moore-Penrose inverse of the product of matrices [9] and Greville found some necessary and sufficient conditions for the reverse order law to hold in matrix setting [13]. The reverse order law problem for bounded linear operators on Hilbert spaces was analyzed by Bouldin [6, 7] and Izumino [16]. The theory of generalized inverses on infinite-dimensional Hilbert spaces can be found in [2, 15, 25].

In this paper, we present algebraic proofs of some characterizations of reverse order law for the Moore-Penrose inverses of closed range Hilbert space operators. In the second section, we collect some definitions and lemmas which will be used in the sequel. We start the main section with some examples to show that the reverse order law does not hold for Hilbert space operators in general. In total, we present 61 results, including some range inclusion results to characterize the reverse order law in this setting. We extend the results of Arghiriade [1] and Tian [22, 23] to infinite-dimensional Hilbert spaces.

2 Preliminaries

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all linear bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. We abbreviate $\mathcal{B}(\mathcal{H}_1) = \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$. For $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we denote by A^*, $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the adjoint, the null-space and the range of A. An operator $A \in \mathcal{B}(\mathcal{H}_1)$ is said to be self-adjoint (Hermitian) if $A = A^*$. An operator $A \in \mathcal{B}(\mathcal{H}_1)$ is said to be a projection if $A^2 = A$. A projection is said to be orthogonal if $A^2 = A = A^*$. The Moore-Penrose inverse of $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is the operator $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ which satisfies the Penrose equations

$$AXA = A \tag{2.1}$$
$$XAX = X \tag{2.2}$$
$$(AX)^* = AX \tag{2.3}$$
$$(XA)^* = XA. \tag{2.4}$$

A matrix X is called a $\{i, \ldots, j\}$-generalized inverse of A, denoted by $A^{(i,\ldots,j)}$ if it satisfies the $i^{th}, \ldots, j^{th}$ conditions of the Penrose equations. The collection of all $\{i, \ldots, j\}$-generalized inverses of A is denoted by $A\{i, \ldots, j\}$. If the Moore-Penrose inverse of A exists, then it is unique and it is denoted by $A^\dagger$. It should be noted that $A^\dagger$ is bounded if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}_2$.

For the sake of clarity as well as for easier reference, we mention the following properties of the Moore-Penrose inverse without proof [25].

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a closed range operator. The following statements hold:*

(i) $(A^\dagger)^\dagger = A$.

(ii) $(A^\dagger)^* = (A^*)^\dagger$.

(iii) $A = AA^*(A^*)^\dagger = (A^*)^\dagger A^*A$.

(iv) $A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^*$.

(v) $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$, $(A^*A)^\dagger = A^\dagger (A^*)^\dagger$.

(vi) $A^* = A^*AA^\dagger = A^\dagger AA^*$.

(vii) $\mathcal{R}(A) = \mathcal{R}(AA^*) = \mathcal{R}(AA^\dagger)$.

(viii) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^*A)$.

(ix) $AA^\dagger = P_{\mathcal{R}(A)}$ *and* $A^\dagger A = P_{\mathcal{R}(A^*)} = P_{\mathcal{R}(A^\dagger)}$.

(x) *If* $\mathcal{H}_1 = \mathcal{H}_2$, *then* $(A^n)^\dagger = (A^\dagger)^n$ *for* $n \geq 1$.

Here, $P_{\mathcal{R}(A)}$ *and* $P_{\mathcal{R}(A^*)}$ *denote the projections onto* $\mathcal{R}(A)$ *and* $\mathcal{R}(A^*)$*, respectively. We use* $A^{\dagger *}$ *instead of* $(A^\dagger)^*$ *throughout the paper.*

Lemma 2.2 ([8], Theorem 1)**.** *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a closed range operator such that $\mathcal{R}(A) = \mathcal{R}(A^*)$. Then $AA^\dagger = A^\dagger A$ and $A^n A^\dagger = A^{n-1}, n \geq 2$.*

Lemma 2.3. *Let $\mathcal{H}$ be a Hilbert space and $P \in \mathcal{B}(\mathcal{H})$ be a projection. Then P is Hermitian if and only if $P = PP^*P$.*

Proof. Suppose $P = PP^*P$ and P is a projection. Let $B = P - P^*$. Then it is easy to verify that $B^3 = 0$ and $\mathcal{R}(B) = \mathcal{R}(B^*)$. By Lemma 2.2, $B^3(B^\dagger)^2 = 0$ gives $B = 0$. Thus, P is Hermitian. Converse follows directly. □

Remark 1. If P is an orthogonal projection, then P satisfies all the Penrose equations and hence $P^\dagger = P$.

Lemma 2.4 ([26], Lemma 1.3)**.** *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Then*

(i) $B \in A\{1, 3\} \Leftrightarrow A^*AB = A^*$,

(ii) $B \in A\{1, 4\} \Leftrightarrow BAA^* = A^*$.

Theorem 2.1 ([12], Theorem 1)**.** *Let A and B be bounded operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;

(ii) *there exists a bounded operator C on $\mathcal{H}$ so that $A = BC$.*

Theorem 2.2 ([14], Theorem 7.20)**.** *Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then there exist a measure space (X, Σ, μ), a bounded measurable real-valued function f on X and a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$ such that*

$$A = U^*TU,$$

where T is the multiplication operator given by $T\psi = f\psi,\ \forall \psi \in L^2(X, \mu)$.

Definition 1. Let $(\mathcal{H}, \langle ., . \rangle)$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. The operator A is called a *positive semi-definite operator* if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$.

Lemma 2.5. *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ be a positive semi-definite operator such that $A^m = A^n$ for some natural numbers $m \neq n$. Then $A^2 = A$.*

Proof. We know that a positive semi-definite operator is self adjoint. By Theorem 2.2, we can write

$$A = U^*TU,$$

where T is the multiplication operator given by $T\psi = f\psi,\ \forall \psi \in L^2(X, \mu)$. Using the positive semi-definiteness of the operator, we get $f(x) \geq 0\ \forall x \in X$.

It is given that $A^m = A^n$ which implies

$$f^m\psi = f^n\psi,\ \forall \psi \in L^2(X, \mu). \tag{2.5}$$

Let $x_0 \in X$ and E be a subset of X such that $x_0 \in E$ and $\mu(E) \neq 0$. Since equation (2.5) holds for the characteristic function on E, we get $f^m(x_0)(1 - f^{n-m}(x_0)) = 0$, from which we can conclude $f(x_0) = 0$ or $f(x_0) = 1$ as $f(x) \geq 0\ \forall x \in X$. As x_0 is arbitrary $f(x) = 0$ or $f(x) = 1$ for all $x \in X$.

Now, $T^2\psi(x) = T(f(x)\psi(x)) = f(x)^2\psi(x) = f(x)\psi(x) = T\psi(x)$ for all $\psi(x) \in L^2(X, \mu)$. Also, $U^*T^2U = U^*TU \Rightarrow A^2 = A$. □

Lemma 2.6. *Let A and B be orthogonal projections on a Hilbert space $\mathcal{H}$ and $m > n \geq 1$. If $(ABA)^m = (ABA)^n$, then $AB = BA$.*

Proof. $ABA = ABBA = ABB^*A^* = AB(AB)^*$. Thus ABA is Hermitian and positive semi-definite as $AB(AB)^*$ is so. Then by Lemma 2.5, $(ABA)^m = (ABA)^n$ implies $(ABA)^2 = ABA$. Consider $(ABA-AB)(ABA-AB)^* = (ABA-AB)(ABA-BA) = (ABA)^2 - ABABA - ABABA + ABA = 0$. Thus $ABA = AB$. Similarly, we can verify $(ABA-BA)(ABA-BA)^* = 0$, which gives $ABA = BA$. Thus, we get $AB = BA$. □

Lemma 2.7. *Let A and B be orthogonal projections on a Hilbert space $\mathcal{H}$ and $m > n \geq 1$. If $(AB)^m = (AB)^n$, then $AB = BA$.*

Proof. Since $(AB)^2A = ABABA = ABAABA = (ABA)^2$, thus $(AB)^mA = (ABA)^m$ for all $m \geq 1$. Now it is clear that$(AB)^m = (AB)^n$ gives $(ABA)^m = (ABA)^n$. Then by Lemma 2.6, we get $AB = BA$. □

3 Main results

We start the section with some examples to show that the reverse order law does not hold good for closed range Hilbert space operators in general.

Example 1. Let $\mathcal{H} = \ell_2$ be the space of all square summable sequences. For $x = (x_1, x_2, x_3, \dots) \in \mathcal{H}$, define $Ax = (x_1 + x_2, x_2, x_3, x_4, \dots)$ and $Bx = (x_1, 0, x_3, 0, x_5, \dots)$. Then

$$AB(x) = A(x_1, 0, x_3, 0, x_5, \dots) = (x_1, 0, x_3, 0, x_5, \dots) = Bx.$$

It can be verified easily that A, B and AB are bounded and have closed ranges. We see that

$$A^*(x) = (x_1, x_1 + x_2, x_3, x_4, \dots) \text{ and } B^*(x) = (x_1, 0, x_3, 0, x_5, \dots) = Bx.$$

Using the Euler-Knopp method for finding the Moore-Penrose inverses of operators ([25], p.327) we get

$$A^\dagger(x) = (x_1 - x_2, x_2, x_3, x_4, \dots).$$

By Remark 1, we get $B^\dagger = B$ *and* $(AB)^\dagger = B^\dagger$. Hence, $B^\dagger A^\dagger(x) = B^\dagger(x_1 - x_2, x_2, x_3, x_4, \dots) = (x_1 - x_2, 0, x_3, 0, x_5, \dots) \neq (AB)^\dagger(x)$, thus $(AB)^\dagger \neq B^\dagger A^\dagger$.

Example 2. Let $\mathcal{H} = \ell_2$. For $x = (x_1, x_2, x_3, \dots) \in \mathcal{H}$, define $Ax = (0, x_2, 0, x_4, 0, \dots)$ and $Bx = (x_1 + x_2, 2x_1 + 2x_2, x_3, x_4, \dots)$. Then $AB(x) = (0, 2x_1 + 2x_2, 0, x_4, \dots)$. It is easy to verify that A, B and AB are bounded and have closed ranges. Since $A^*(x) = (0, x_2, 0, x_4, 0, x_6, \dots) = Ax$, by Remark 1 we get $A^\dagger x = Ax$. Also, $B^*(x) = (x_1 + 2x_2, x_1 + 2x_2, x_3, x_4, \dots)$ and $B^\dagger(x) = (\frac{1}{10}(x_1 + 2x_2), \frac{1}{10}(x_1 + 2x_2), x_3, x_4, \dots)$ by the Euler-Knopp method. Thus, we get

$$B^\dagger A^\dagger(x) = B^\dagger(0, x_2, 0, x_4, 0, x_6, \dots) = (\frac{x_2}{5}, \frac{x_2}{5}, 0, x_4, 0, \dots)$$

and

$$(AB)^\dagger x = (\frac{x_2}{4}, \frac{x_2}{4}, 0, x_4, 0, \dots).$$

Hence $(AB)^\dagger \neq B^\dagger A^\dagger$. One can also check that $B^\dagger A^\dagger$ satisfies the third and fourth but not the first and second Penrose equations.

Lemma 3.1 ([16], Proposition 2.1)**.** *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be Hilbert spaces, and let $A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B have closed ranges. Then AB has a closed range if and only if $A^\dagger ABB^\dagger$ has a closed range.*

The results mentioned below in Theorems 3.1 to 3.4 are proved in C^*-algebra setting [18] and for the sake of completeness, we give the proof of those in Hilbert space setting. However, our proofs are much simpler than those available for the reverse order law for closed range Hilbert space operators.

In the following result, the existence of $(A^\dagger ABB^\dagger)^\dagger$ is guaranteed by Lemma 3.1.

Theorem 3.1. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be Hilbert spaces, and let $A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(i) $ABB^\dagger A^\dagger AB = AB$;

(ii) $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$;

(iii) $BB^\dagger A^\dagger A$ *is a projection* ;

(iv) $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$;

(v) $A^\dagger ABB^\dagger$ *is a projection* ;

(vi) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

(vii) $B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger$.

Proof. (i)⇒ (ii): If $ABB^\dagger A^\dagger AB = AB$, then

$$\begin{aligned} B^\dagger A^\dagger &= (B^*B)^\dagger B^*A^*(AA^*)^\dagger \quad \text{(by Lemma 2.1 (iv))} \\ &= (B^*B)^\dagger (AB)^*(AA^*)^\dagger \\ &= (B^*B)^\dagger (ABB^\dagger A^\dagger AB)^*(AA^*)^\dagger \quad \text{(by the assumption)} \\ &= (B^*B)^\dagger B^*A^\dagger ABB^\dagger A^*(AA^*)^\dagger \\ &= B^\dagger A^\dagger ABB^\dagger A^\dagger \quad \text{(by Lemma 2.1 (iv))}. \end{aligned}$$

(ii)⇒(iii): Using (ii) we see that $(BB^\dagger A^\dagger A)^2 = BB^\dagger A^\dagger ABB^\dagger A^\dagger A = BB^\dagger A^\dagger A$. Hence, it shows that $BB^\dagger A^\dagger A$ is a projection.

(iii)⇒(iv): We have

$$\begin{aligned} BB^\dagger A^\dagger A(BB^\dagger A^\dagger A)^*BB^\dagger A^\dagger A &= BB^\dagger A^\dagger A(A^\dagger A)^*(BB^\dagger)^*BB^\dagger A^\dagger A \\ &= BB^\dagger A^\dagger A(A^\dagger A)(BB^\dagger)BB^\dagger A^\dagger A \\ &= BB^\dagger A^\dagger ABB^\dagger A^\dagger A = BB^\dagger A^\dagger A. \end{aligned}$$

Then by Lemma 2.3, we get $(BB^\dagger A^\dagger A)^* = BB^\dagger A^\dagger A$, since $BB^\dagger A^\dagger A$ is a projection. Thus $BB^\dagger A^\dagger A = A^\dagger ABB^\dagger$.

(iv)⇒(v): It is given that $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$. We have

$$\begin{aligned} (A^\dagger ABB^\dagger)^2 &= A^\dagger ABB^\dagger A^\dagger ABB^\dagger = BB^\dagger A^\dagger AA^\dagger ABB^\dagger \\ &= BB^\dagger A^\dagger ABB^\dagger = A^\dagger ABB^\dagger BB^\dagger = A^\dagger ABB^\dagger. \end{aligned}$$

(v)⇒(vi): Using the fact that A is a projection if and only if A^* is a projection, it is easy to verify all Penrose equations.

(vi)⇒(vii): Pre- and post-multiplying by $B^\dagger$ and $A^\dagger$ respectively in (vi), we get the desired result.

(vii)⇒(i): It is given that $B^\dagger A^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger$. We have

$$\begin{aligned} ABB^\dagger A^\dagger AB &= ABB^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger AB \\ &= AA^\dagger ABB^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger ABB^\dagger B \\ &= AA^\dagger ABB^\dagger B = AB, \end{aligned}$$

where all the equalities follow using the first Penrose equation. □

Next, we give ten equivalent conditions for $B^\dagger A^\dagger$ to be a $\{1,2,3\}$-generalized inverse of AB in Hilbert space setting. The existence of $(ABB^\dagger)^\dagger$ follows as the ranges of $ABB^\dagger$ and AB are equal.

Theorem 3.2. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(i) $AB(AB)^\dagger = ABB^\dagger A^\dagger$;

(ii) $B^\dagger A^\dagger \in AB\{1,2,3\}$;

(iii) $BB^\dagger A^* AB = A^* AB$;

(iv) $(AB)(AB)^\dagger A = ABB^\dagger$;

(v) $A^* ABB^\dagger = BB^\dagger A^* A$;

(vi) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$;

(vii) $B^\dagger (ABB^\dagger)^\dagger = B^\dagger A^\dagger$;

(viii) $B\{1,3\}A\{1,3\} \subseteq AB\{1,3\}$;

(ix) $B^\dagger A^\dagger \in AB\{1,3\}$;

(x) $(BB^*)^\dagger A^\dagger \in ABB^*\{1,2,3\}$.

Proof. We prove the equivalence of all the statements in the following order of implications:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(vi)} \Rightarrow \text{(vii)} \Rightarrow \text{(ix)} \Leftrightarrow \text{(viii)}, \text{(ix)} \Rightarrow \text{(i)}.$$

(i)⇒(ii): Since $ABB^\dagger A^\dagger = AB(AB)^\dagger$, post-multiplying by AB we get,

$$ABB^\dagger A^\dagger AB = AB(AB)^\dagger AB = AB.$$

Hence, $B^\dagger A^\dagger \in AB\{1\}$. By Theorem 3.1, $B^\dagger A^\dagger \in AB\{2\}$. Now using the assumption we get, $(ABB^\dagger A^\dagger)^* = (AB(AB)^\dagger)^* = AB(AB)^\dagger = ABB^\dagger A^\dagger$. Thus $B^\dagger A^\dagger \in AB\{1,2,3\}$.

(ii)⇒(iii): Suppose $B^\dagger A^\dagger \in AB\{1,2,3\}$. Then by Theorem 3.1, $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$. Thus, we get

$$\begin{aligned} A^* AB &= A^* AA^\dagger ABB^\dagger B = A^* ABB^\dagger A^\dagger AB \\ &= A^* (ABB^\dagger A^\dagger)^* AB \quad (\text{since } B^\dagger A^\dagger \in AB\{3\} \\ &= A^* A^{\dagger *} BB^\dagger A^* AB = (A^\dagger A)^* BB^\dagger A^* AB \\ &= A^\dagger ABB^\dagger A^* AB = BB^\dagger A^\dagger AA^* AB \\ &= BB^\dagger A^* AB \quad (\text{by Lemma 2.1 (vi)}). \end{aligned}$$

(iii)⇒(iv): We have

$$\begin{aligned} (AB)(AB)^\dagger A &= ((AB)(AB)^\dagger)^* A^{**} = (A^* AB(AB)^\dagger)^* \\ &= (BB^\dagger A^* AB(AB)^\dagger)^* \quad (\text{by the assumption}) \\ &= ((AB)(AB)^\dagger)^* (BB^\dagger A^*)^* = AB(AB)^\dagger ABB^\dagger \\ &= ABB^\dagger. \end{aligned}$$

(iv)⇒(v): Pre-multiplying the given condition by A^*, we get $A^* ABB^\dagger = A^* AB(AB)^\dagger A$. As the RHS of the previous equality is Hermitian, $A^* ABB^\dagger$ is also Hermitian and

$$A^* ABB^\dagger = (A^* ABB^\dagger)^* = BB^\dagger A^* A.$$

(v)⇒(vi): We show this by verifying all Penrose equations. Given that $A^*ABB^\dagger = BB^\dagger A^*A$. Pre-multiplying by $A^{\dagger *}$, we get $ABB^\dagger = AA^\dagger ABB^\dagger = (AA^\dagger)^*ABB^\dagger = A^{\dagger *}A^*ABB^\dagger = A^{\dagger *}BB^\dagger A^*A$. Hence

$$\begin{aligned}(ABB^\dagger)(BB^\dagger A^\dagger)(ABB^\dagger) &= ABB^\dagger A^\dagger ABB^\dagger = A^{\dagger *}BB^\dagger A^*AA^\dagger ABB^\dagger \\ &= A^{\dagger *}A^*ABB^\dagger BB^\dagger = (AA^\dagger)^*ABB^\dagger = ABB^\dagger.\end{aligned}$$

This shows that $BB^\dagger A^\dagger \in ABB^\dagger\{1\}$. Now,

$$\begin{aligned}BB^\dagger A^\dagger &= BB^\dagger A^*(AA^*)^\dagger \quad \text{(by Lemma 2.1 (iv))} \\ &= (ABB^\dagger)^*(AA^*)^\dagger = (ABB^\dagger BB^\dagger A^\dagger ABB^\dagger)^*(AA^*)^\dagger \\ &= (ABB^\dagger A^\dagger ABB^\dagger)^*(AA^*)^\dagger = BB^\dagger A^\dagger ABB^\dagger A^*(AA^*)^\dagger \\ &= BB^\dagger A^\dagger ABB^\dagger BB^\dagger A^\dagger \quad \text{(by Lemma 2.1 (iv))}.\end{aligned}$$

Thus, $BB^\dagger A^\dagger \in ABB^\dagger\{1,2\}$. Also,

$$(ABB^\dagger)(BB^\dagger A^\dagger) = (A^{\dagger *}A^*ABB^\dagger)(BB^\dagger A^\dagger) = A^{\dagger *}BB^\dagger A^*ABB^\dagger A^\dagger.$$

As the RHS of the last equality is Hermitian, $(ABB^\dagger)(BB^\dagger A^\dagger)$ is so. Similarly, we can prove $(BB^\dagger A^\dagger)(ABB^\dagger)$ is Hermitian. It ensures that $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$.
(vi)⇒(vii): Pre-multiplying the given condition by $B^\dagger$, we get (vii).
(vii)⇒(ix): It is clear that $ABB^\dagger(ABB^\dagger)^\dagger ABB^\dagger = ABB^\dagger$. Then by (vii), we have $ABB^\dagger A^\dagger ABB^\dagger = ABB^\dagger$. Post-multiplying by B we get $ABB^\dagger A^\dagger AB = AB$. Thus $B^\dagger A^\dagger \in AB\{1\}$. Also, $ABB^\dagger A^\dagger$ is Hermitian since $ABB^\dagger A^\dagger = ABB^\dagger(ABB^\dagger)^\dagger$. Thus $B^\dagger A^\dagger \in AB\{1,3\}$.
(ix)⇒(viii): Let $CD \in B\{1,3\}A\{1,3\}$ where $C \in B\{1,3\}$ and $D \in A\{1,3\}$. By Lemma 2.4, C and D satisfy $B^*BC = B^*$ and $A^*AD = A^*$. Also, we note that $B^\dagger BC = (B^*B)^\dagger B^*BC = (B^*B)^\dagger B^* = B^\dagger$ and, similarly we can prove $A^\dagger AD = A^\dagger$. By using $B^\dagger A^\dagger \in AB\{1,3\}$, we get

$$\begin{aligned}(AB)^*(AB)CD &= (ABB^\dagger A^\dagger AB)^*ABCD = (AB)^*ABB^\dagger A^\dagger ABCD \\ &= (AB)^*AA^\dagger ABB^\dagger BCD \quad \text{(by Theorem 3.1)} \\ &= (AB)^*AA^\dagger ABB^\dagger D = (AB)^*ABB^\dagger A^\dagger AD \\ &= (AB)^*ABB^\dagger A^\dagger = (ABB^\dagger A^\dagger AB)^* = (AB)^*.\end{aligned}$$

(viii)⇒(ix): Obvious.
(ix)⇒(i): By the assumption, we have $ABB^\dagger A^\dagger AB = AB$ and $(ABB^\dagger A^\dagger)^* = ABB^\dagger A^\dagger$. Post-multiplying by $(AB)^\dagger$ in the first equation and taking adjoint on both sides, we get $AB(AB)^\dagger = ABB^\dagger A^\dagger$.
(ix)⇒(x): The first and third Penrose conditions follow easily from (ix). The second Penrose condition can be verified with the help of Theorem 3.1 (iv).
(x)⇒(ix): Since $(BB^*)^\dagger A^\dagger \in ABB^*\{1\}$, we get $ABB^*(BB^*)^\dagger A^\dagger ABB^* = ABB^*$ i.e., $ABB^\dagger A^\dagger ABB^* = ABB^*$ by Lemma 2.1 (iv) and Theorem 3.1. Post-multiplying by $B^{\dagger *}$ and using Lemma 2.1 (iii), we get

$$ABB^\dagger A^\dagger AB = AB.$$

Also, $ABB^*(BB^*)^\dagger A^\dagger = ABB^\dagger A^\dagger$ is Hermitian. It shows that $B^\dagger A^\dagger \in AB\{1,3\}$. □

The following result is similar to Theorem 3.2. It gives ten equivalent conditions for $B^\dagger A^\dagger$ to be a $\{1,2,4\}$-generalized inverse of AB in Hilbert space setting. Here, the existence of $(A^\dagger AB)^\dagger$ is guaranteed as the ranges of $(A^\dagger AB)^*$ and $(AB)^*$ are the same.

Theorem 3.3. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(i) $(AB)^\dagger AB = B^\dagger A^\dagger AB$;

(ii) $B^\dagger A^\dagger \in AB\{1,2,4\}$;

(iii) $ABB^* = ABB^* A^\dagger A$;

(vi) $B(AB)^\dagger AB = A^\dagger AB$;

(v) $A^\dagger ABB^* = BB^* A^\dagger A$;

(vi) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(vii) $(A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger$;

(viii) $B\{1,4\}A\{1,4\} \subseteq AB\{1,4\}$;

(ix) $B^\dagger A^\dagger \in AB\{1,4\}$;

(x) $B^\dagger (A^* A)^\dagger \in A^* AB\{1,2,4\}$.

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 3.4. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(i) $(AB)^\dagger = B^\dagger A^\dagger$;

(ii) $(AB)(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$;

(iii) $A^* AB = BB^\dagger A^* AB$ *and* $ABB^* = ABB^* A^\dagger A$;

(iv) $AB(AB)^\dagger A = ABB^\dagger$ *and* $B(AB)^\dagger AB = A^\dagger AB$;

(v) $A^* ABB^\dagger = BB^\dagger A^* A$ *and* $BB^* A^\dagger A = A^\dagger ABB^*$;

(vi) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$*and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(vii) $B^\dagger (ABB^\dagger)^\dagger = B^\dagger A^\dagger$*and* $(A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger$;

(viii) $B\{1,3\}A\{1,3\} \subseteq AB\{1,3\}$ *and* $B\{1,4\}A\{1,4\} \subseteq AB\{1,4\}$;

(ix) $B^\dagger A^\dagger \in AB\{1,3,4\}$;

(x) $(BB^*)^\dagger A^\dagger \in ABB^*\{1,2,3\}$ *and* $B^\dagger (A^* A)^\dagger \in A^* AB\{1,2,4\}$.

Proof. Follows from Theorems 3.2 and 3.3. □

Remark 2. Consider the operators A and B on $\mathcal{H}$ defined in Example 1. Then for all $x \in \mathcal{H}$, $A^* ABx = (x_1, x_1, x_3, 0, x_5, \ldots)$ and $BB^\dagger A^* ABx = (x_1, 0, x_3, 0, x_5, \ldots)$. Hence

$$A^* AB \neq BB^\dagger A^* AB \text{ and } ABB^* x = (x_1, 0, x_3, 0, x_5, \ldots) = ABB^* A^\dagger Ax.$$

Note that the conditions in (iii) of Theorem 3.4 are not satisfied and $(AB)^\dagger \neq B^\dagger A^\dagger$ which was shown in Example 1.

Theorem 3.5. *Let the conditions of Theorem* 3.1 *hold. Then the following statements hold:*

(i) $B^\dagger = (AB)^\dagger A \Leftrightarrow \mathcal{R}(B) = \mathcal{R}(A^*AB)$.

(ii) $A^\dagger = B(AB)^\dagger \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(BB^*A^*)$.

Proof. (i) Suppose that $B^\dagger = (AB)^\dagger A$. Pre-multiplying by AB, we get $ABB^\dagger = (AB)(AB)^\dagger A$, which is equivalent to $BB^\dagger A^*AB = A^*AB$ by Theorem 3.2. This implies $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ as $BB^\dagger$ is the projection onto $\mathcal{R}(B)$. Now by Lemma 2.1 (iv),

$$B^\dagger = (AB)^\dagger A = [(AB)^*(AB)]^\dagger (AB)^* A$$

implies that $B^{\dagger^*} = A^*AB[(AB)^*(AB)]^\dagger$, as $AB)^*(AB)$ is Hermitian. Thus

$$\mathcal{R}(B) = \mathcal{R}(B^{\dagger^*}) = \mathcal{R}(A^*AB[(AB)^*(AB)]^\dagger) \subseteq \mathcal{R}(A^*AB).$$

Conversely, $\mathcal{R}(B) = \mathcal{R}(A^*AB) \implies BB^\dagger A^*AB = A^*AB$. By Theorem 3.2, $B^\dagger A^\dagger \in AB\{1,2,3\}$, $\mathcal{R}(B^{\dagger^*}) = \mathcal{R}(B) = \mathcal{R}(A^*AB) \subseteq \mathcal{R}(A^*)$ and $A^\dagger A$ is the projection onto $\mathcal{R}(A^*)$ gives $A^\dagger AB = B$ and $A^\dagger AB^{\dagger^*} = B^{\dagger^*} \implies B^\dagger A^\dagger A = B^\dagger$. It shows that $(A^\dagger AB)^\dagger = B^\dagger = B^\dagger A^\dagger A$. By Theorem 3.3, $B^\dagger A^\dagger \in AB\{1,2,4\}$ and hence $(AB)^\dagger = B^\dagger A^\dagger$. Therefore $B^\dagger = B^\dagger A^\dagger A = (AB)^\dagger A$.
(ii) Proof is similar to (i). □

Theorem 3.6. *Let the conditions of Theorem* 3.1 *hold. Then the following statements hold:*

(i) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathcal{R}(AA^*AB) = \mathcal{R}(AB)$.

(ii) $(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger \Leftrightarrow \mathcal{R}(B^*B(AB)^*) = \mathcal{R}((AB)^*)$.

Proof. (i) If we replace A by $A^\dagger$ and B by AB in Theorem 3.5 (i), we get $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathcal{R}(AB) = \mathcal{R}(A^{\dagger^*}A^\dagger AB) = \mathcal{R}(A^{\dagger^*}B)$. Now by Theorem 2.1, there exists a bounded operator C such that $AB = A^{\dagger^*}BC$. Pre-multiplying by AA^* we get $AA^*AB = AA^*A^{\dagger^*}BC = A(A^\dagger A)^*BC = ABC$. Thus we get $\mathcal{R}(AA^*AB) \subseteq \mathcal{R}(AB)$. Similarly, we can prove $\mathcal{R}(AB) \subseteq \mathcal{R}(AA^*AB)$.
(ii) Replace A by AB and B by $B^\dagger$ in Theorem 3.5 (ii) and use a similar argument as above. □

Theorem 3.7. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(i) $(AB)^\dagger = B^\dagger A^\dagger$;

(ii) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$;

(iii) $\mathcal{R}(AA^*AB) = \mathcal{R}(AB)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$;

(iv) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}[(ABB^*B)^*] = \mathcal{R}[(AB)^*]$;

(v) $\mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A)$.

Proof. First we note that the condition $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ is equivalent to Theorem 3.2 (iii) and the condition $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ is equivalent to Theorem 3.3 (iii).
(i)⇔(ii): Follows from Theorem 3.4 (iii).
(i)⇒(iii): By Theorem 3.4 (vii), $(AB)^\dagger = B^\dagger A^\dagger = (A^\dagger AB)^\dagger A^\dagger$. Then (iii) follows from Theorem 3.6 (i).
(iii)⇒(v): By Theorem 2.1, there exists an operator T such that $AA^*AB = ABT$. Pre-multiplying by $A^\dagger$ we get $A^*AB = A^\dagger ABT = A^\dagger ABB^*B^{\dagger^*}T = BB^*A^\dagger AB^{\dagger^*}T$ by Theorem 3.3 (v). Also, $A^*ABB^* = BB^*A^\dagger AB^{\dagger^*}TB^*$. Thus $\mathcal{R}(A^*ABB^*) \subseteq \mathcal{R}(BB^*A^*A)$. Similarly, $AB = AA^*ABS$, for some operator S. Pre- and post-multiplying by $A^\dagger$ and B^*A^*A respectively, we get $A^\dagger ABB^*A^*A = A^*ABSB^*A^*A$.

Then by Theorem 3.3 (v), $BB^*A^\dagger AA^*A = A^*ABSB^*A^*A$. Thus, $BB^*A^*A = A^*ABSB^*A^*A$. It shows that

$$\mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A).$$

(v)⇒(ii): By Theorem 2.1, there exists an operator T such that $A^*ABB^* = BB^*A^*AT$. Pre-multiplying by $BB^\dagger$ and post-multiplying by $B^{\dagger^*}$ we get,

$$BB^\dagger A^*ABB^*B^{\dagger^*} = BB^*A^*ATB^{\dagger^*}.$$

Hence, $BB^\dagger A^*AB = BB^*A^*ATB^{\dagger^*} = A^*ABB^*B^{\dagger^*} = A^*AB$, thus $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$. Similarly, we can prove $A^\dagger ABB^*A^* = BB^*A^*$ and hence $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.
We can give the proof of (i)⇒(iv) and (iv)⇒(v) in a similar fashion. □

Remark 3. Let A and B be as defined in Example 1. Then we get $A^*ABx = (x_1, x_1, x_3, 0, x_5, \dots)$. Thus $\mathcal{R}(A^*AB) \nsubseteq \mathcal{R}(B)$, $A^*(x) = (x_1, x_1 + x_2, x_3, x_4, \dots)$ and $BB^*A^*x = (x_1, 0, x_3, 0, x_5, \dots)$ implies $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$. This shows that $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ is indispensable for the reverse order law to hold.

Lemma 3.2. *Let the conditions of Theorem 3.1 hold. If $P = (AA^*)^m A$ and $Q = B(B^*B)^n$, then P, Q and PQ have closed ranges.*

Proof. By Lemma 3.1, AB has a closed range if and only if $A^\dagger ABB^\dagger$ has a closed range. Take A as $(AA^*)^m$ and B as A to apply Lemma 3.1. Then, we have $(AA^*)^{m\dagger}(AA^*)^m AA^\dagger = ((AA^*)^\dagger(AA^*))^m AA^\dagger = (AA^\dagger)^m AA^\dagger = AA^\dagger AA^\dagger = AA^\dagger$. Now, $\mathcal{R}(AA^\dagger) = \mathcal{R}(A)$ is closed implies $\mathcal{R}(P)$ is closed. Similar argument works for Q also.

Now again by Lemma 3.1, PQ has a closed range if and only if $P^\dagger PQQ^\dagger$ has a closed range. For,

$$\begin{aligned} P^\dagger PQQ^\dagger &= [(AA^*)^m A]^\dagger (AA^*)^m AB(B^*B)^n [B(B^*B)^n]^\dagger \\ &= A^\dagger [(AA^*)^m]^\dagger (AA^*)^m AB(B^*B)^n [(B^*B)^n]^\dagger B^\dagger \\ &= A^\dagger AA^\dagger ABB^\dagger BB^\dagger = A^\dagger ABB^\dagger. \end{aligned}$$

Here, the reverse order law is applied for $[(AA^*)^m A]^\dagger$ and $[B(B^*B)^n]^\dagger$ as they satisfy condition (ii) in Theorem 3.7. □

The next result is an extension of Theorem 11.1 of [23] to infinite e-dimensional setting. Djordjević and Dinčić [10, 11] have extended the results of Tian [22, 23] using the operator matrix method to different settings. By Lemma 3.1, $\mathcal{R}(AB)$ is closed if and only if $\mathcal{R}(A^\dagger ABB^\dagger) = \mathcal{R}(A^*A^{\dagger*}BB^\dagger) = \mathcal{R}(A^\dagger AB^{\dagger*}B^*)$ is closed. This happens if and only if $\mathcal{R}(A^{\dagger*}B)$ and $\mathcal{R}(AB^{\dagger*})$ are closed. Thus $(A^{\dagger*}B)^\dagger$ and $(AB^{\dagger*})^\dagger$ exist. Also, $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A^*)$ is closed implies $\mathcal{R}(BB^\dagger A^\dagger A)$ is closed and hence $\mathcal{R}(B^\dagger A^\dagger)$ is closed. For natural numbers m and n, the existence of the Moore-Penrose inverse of $(AA^*)^m$ and $(B^*B)^n$ is guaranteed as they are powers of Hermitian operators with closed ranges, according to the spectral mapping theorem. The existence of the Moore-Penrose inverse of all other operators discussed below can be guaranteed with the closedness of the ranges of $AB, A^{\dagger*}B, AB^{\dagger*}$ and $B^\dagger A^\dagger$.

Theorem 3.8. *Let the conditions of Theorem 3.1 hold. Then the following statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(2) $B(AB)^\dagger A = BB^\dagger A^\dagger A$;

(3) $AA^*(B^*A^*)^\dagger B^*B = AB$;

(4) $(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger$;

(5) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(6) $(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger$ *and* $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$;

(7) $(AB)^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger$ *and* $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

(8) $B^\dagger A^\dagger \in AB\{1,3,4\}$;

(9) $(AB)(AB)^\dagger = ABB^\dagger A^\dagger = A^{\dagger *} BB^\dagger A^*$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB = B^* A^\dagger A (B^\dagger)^*$;

(10) $(A^{\dagger *} B)^\dagger = B^\dagger A^*$;

(11) $A^\dagger (B^* A^\dagger)^\dagger B^* = A^\dagger ABB^\dagger$;

(12) $AA^\dagger (B^* A^\dagger)^\dagger B^* B = AB$;

(13) $(A^{\dagger *} B)^\dagger = B^\dagger A^\dagger ABB^\dagger A^*$;

(14) $(A^{\dagger *} B)^\dagger = (A^\dagger AB)^\dagger A^*$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(15) $(A^{\dagger *} B)^\dagger = B^\dagger (A^{\dagger *} BB^\dagger)^\dagger$ *and* $(A^{\dagger *} BB^\dagger)^\dagger = BB^\dagger A^*$;

(16) $(A^{\dagger *} B)^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^*$ *and* $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

(17) $B^\dagger A^* \in A^{\dagger *} B\{1,3,4\}$;

(18) $(B^* A^\dagger)^\dagger B^* A^\dagger = ABB^\dagger A^\dagger = A^{\dagger *} BB^\dagger A^*$ *and* $B^* A^\dagger (B^* A^\dagger)^\dagger = B^\dagger A^\dagger AB = B^* A^\dagger AB^{\dagger *}$;

(19) $(AB^{\dagger *})^\dagger = B^* A^\dagger$;

(20) $B^{\dagger *} (AB^{\dagger *})^\dagger A = BB^\dagger A^\dagger A$;

(21) $AA^* (B^\dagger A^*)^\dagger B^\dagger B = AB$;

(22) $(AB^{\dagger *})^\dagger = B^* A^\dagger ABB^\dagger A^\dagger$;

(23) $(AB^{\dagger *})^\dagger = (A^\dagger AB^{\dagger *})^\dagger A^\dagger$ *and* $(A^\dagger AB^{\dagger *})^\dagger = B^* A^\dagger A$;

(24) $(AB^{\dagger *})^\dagger = B^* (ABB^\dagger)^\dagger$ *and* $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$;

(25) $(AB^{\dagger *})^\dagger = B^* (A^\dagger ABB^\dagger)^\dagger A^\dagger$ *and* $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

(26) $B^* A^\dagger \in AB^{\dagger *}\{1,3,4\}$;

(27) $(B^\dagger A^*)^\dagger B^\dagger A^* = ABB^\dagger A^\dagger = A^{\dagger *} BB^\dagger A^*$ *and* $B^\dagger A^* (B^\dagger A^*)^\dagger = B^\dagger A^\dagger AB = B^* A^\dagger AB^{\dagger *}$;

(28) $(B^\dagger A^\dagger)^\dagger = AB$;

(29) $A^\dagger (B^\dagger A^\dagger)^\dagger B^\dagger = A^\dagger ABB^\dagger$;

(30) $(AA^*)^\dagger (B^\dagger A^\dagger)^\dagger (B^* B)^\dagger = A^{\dagger *} A^{\dagger *}$;

(31) $(B^\dagger A^\dagger)^\dagger = ABB^\dagger A^\dagger AB$;

(32) $(B^\dagger A^\dagger)^\dagger = A(B^\dagger A^\dagger A)^\dagger$ *and* $(B^\dagger A^\dagger A)^\dagger = A^\dagger AB$;

(33) $(B^\dagger A^\dagger)^\dagger = (BB^\dagger A^\dagger)^\dagger B$ *and* $(BB^\dagger A^\dagger)^\dagger = ABB^\dagger$;

(34) $(B^\dagger A^\dagger)^\dagger = A(BB^\dagger A^\dagger A)^\dagger B$ *and* $(BB^\dagger A^\dagger A)^\dagger = A^\dagger ABB^\dagger$;

(35) $AB \in B^\dagger A^\dagger\{1,3,4\}$;

(36) $B^\dagger A^\dagger (B^\dagger A^\dagger)^\dagger = B^\dagger A^\dagger AB = B^* A^\dagger AB^{\dagger *}$ *and* $(B^\dagger A^\dagger)^\dagger B^\dagger A^\dagger = ABB^\dagger A^\dagger = A^{\dagger *} BB^\dagger A^*$;

(37) $(AB)^\dagger = (A^*AB)^\dagger A^*$ *and* $(A^*AB)^\dagger = B^\dagger (A^*A)^\dagger$;

(38) $(AB)^\dagger = B^*(ABB^*)^\dagger$ *and* $(ABB^*)^\dagger = (BB^*)^\dagger A^\dagger$;

(39) $(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*$ *and* $(A^*ABB^*)^\dagger = (BB^*)^\dagger (A^*A)^\dagger$;

(40) $(AB)^\dagger = (B^*B)^n((AA^*)^m AB(B^*B)^n)^\dagger (AA^*)^m$ *and*

$$((AA^*)^m AB(B^*B)^n)^\dagger = (B(B^*B)^n)^\dagger ((AA^*)^m A)^\dagger;$$

(41) $(AB)^\dagger = B^*(BB^*)^n((A^*A)^{m+1}(BB^*)^{n+1})^\dagger (A^*A)^m A^*$ *and*

$$((A^*A)^{m+1}(BB^*)^{n+1})^\dagger = ((BB^*)^\dagger)^{n+1}((A^*A)^\dagger)^{m+1}.$$

Proof. (1) $\Rightarrow$ (2): Straightforward.
(2) $\Rightarrow$ (3): Pre- and post-multiplying the given condition by B^* and A^*, respectively, we get $B^*B(AB)^\dagger AA^* = B^*A^*$, equivalently $AA^*(B^*A^*)^\dagger B^*B = AB$.
(3) $\Rightarrow$ (1): We have $B^*B(AB)^\dagger AA^* = B^*A^*$. Pre- and post-multiplying by $(B^*B)^\dagger$ and $(AA^*)^\dagger$ respectively, we get $B^\dagger B(AB)^\dagger AA^\dagger = B^\dagger A^\dagger$. It is clear that $\mathcal{R}((AB)^\dagger) = \mathcal{R}((AB)^*) \subseteq R(B^*)$ and $R((AB)^{\dagger *}) = R(AB) \subseteq R(A)$. Thus $B^\dagger B(AB)^\dagger = (AB)^\dagger$ and $(AB)^\dagger AA^\dagger = (AB)^\dagger$. Hence, $B^\dagger B(AB)^\dagger AA^\dagger = (AB)^\dagger = B^\dagger A^\dagger$.
(1) $\Rightarrow$ (4): It is easy to see from the assumption that

$$(AB)^\dagger = (AB)^\dagger AB(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger.$$

(4) $\Rightarrow$ (5): Pre-multiplying the given condition by B and post-multiplying by A, we get $B(AB)^\dagger A = BB^\dagger A^\dagger ABB^\dagger A^\dagger A = (BB^\dagger A^\dagger A)^2$. $(BB^\dagger A^\dagger A)^4 = (BB^\dagger A^\dagger A)^2(BB^\dagger A^\dagger A)^2 = B(AB)^\dagger AB(AB)^\dagger A = B(AB)^\dagger A = (BB^\dagger A^\dagger A)^2$. Since $BB^\dagger$ and $A^\dagger A$ are orthogonal projections by Lemma 2.7, $BB^\dagger A^\dagger A = A^\dagger ABB^\dagger$. The statements $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger$ and $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$ can be proved by verifying all Penrose equations using $BB^\dagger A^\dagger A = A^\dagger ABB^\dagger$.
(5) $\Rightarrow$ (6): Note that by substituting the second condition in the first condition of (5), we get $(AB)^\dagger = B^\dagger A^\dagger$. Thus, we have $AB = AB(AB)^\dagger AB = ABB^\dagger A^\dagger AB$, $(AB)^\dagger = (AB)^\dagger AB(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger$, $ABB^\dagger A^\dagger$ is a projection and Hermitian. Now $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ is easily verifiable, using Theorem 3.1 (iii). Moreover, $(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger (AB)(AB)^\dagger = B^\dagger BB^\dagger A^\dagger AB(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger (AB)(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger$.
(6) $\Rightarrow$ (7): Suppose $(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger$. Then

$$(AB)(AB)^\dagger = ABB^\dagger (ABB^\dagger)^\dagger = ABB^\dagger BB^\dagger A^\dagger = ABB^\dagger A^\dagger.$$

Thus, $AB = ABB^\dagger A^\dagger AB$. Now by Theorem 3.1 (vi), we get $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$. Since $BB^\dagger A^\dagger = (ABB^\dagger)^\dagger$, $(A^\dagger ABB^\dagger)^\dagger = (ABB^\dagger)^\dagger A$. Therefore $B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger (ABB^\dagger)^\dagger AA^\dagger = (AB)^\dagger AA^\dagger =$

$(AB)^\dagger$.
$(7) \Rightarrow (8)$: We have

$$AB = AB(AB)^\dagger AB = ABB^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger AB$$
$$= ABB^\dagger BB^\dagger A^\dagger AA^\dagger AB = ABB^\dagger A^\dagger AB$$

and $(AB)(AB)^\dagger = ABB^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger = ABB^\dagger BB^\dagger A^\dagger AA^\dagger = ABB^\dagger A^\dagger$. Similarly, $(AB)^\dagger AB = B^\dagger A^\dagger AB$. Thus $B^\dagger A^\dagger \in AB\{1,3,4\}$.
$(8) \Rightarrow (9)$: Since $B^\dagger A^\dagger \in AB\{1,3,4\}$, $AB = ABB^\dagger A^\dagger AB$ and $ABB^\dagger A^\dagger = (ABB^\dagger A^\dagger)^*$. Now,

$$\begin{aligned}(AB)(AB)^\dagger = ABB^\dagger A^\dagger AB(AB)^\dagger &= (ABB^\dagger A^\dagger)^* AB(AB)^\dagger\\ &= (B^\dagger A^\dagger)^*(AB)^* AB(AB)^\dagger\\ &= (B^\dagger A^\dagger)^*(AB)^*(\text{ by Lemma 2.1 (vi)})\\ &= (ABB^\dagger A^\dagger)^* = ABB^\dagger A^\dagger\\ &= A^{\dagger *}BB^\dagger A^*.\end{aligned}$$

Similarly, we can prove the other relation.
$(9) \Rightarrow (10)$: Since $AB(AB)^\dagger = ABB^\dagger A^\dagger$, $AB(AB)^\dagger AB = ABB^\dagger A^\dagger AB$. Then by Theorem 3.1 (iv), $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$. It is clear from the assumption that $B^\dagger A^* \in A^{\dagger *}B\{3,4\}$. Also, it is easy to verify that $B^\dagger A^* \in A^{\dagger *}B\{1,2\}$.
$(10) \Rightarrow (1)$: Applying Theorem 3.1 for $A^{\dagger *}$ and B, we get $A^* A^{\dagger *} BB^\dagger = BB^\dagger A^* A^{\dagger *}$ i.e., $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$. Using the third and fourth Penrose conditions for (10), we get

$$\begin{aligned}ABB^\dagger A^\dagger AB &= AA^\dagger ABB^\dagger B = AB,\\ B^\dagger A^\dagger ABB^\dagger A^\dagger &= B^\dagger BB^\dagger A^\dagger AA^\dagger = B^\dagger A^\dagger,\\ (ABB^\dagger A^\dagger)^* &= A^{\dagger *}BB^\dagger A^* = ABB^\dagger A^\dagger,\\ (B^\dagger A^\dagger AB)^* &= B^* A^\dagger AB^{\dagger *} = B^\dagger A^\dagger AB.\end{aligned}$$

The equivalences of (10)-(18) can be established by replacing A by $A^{\dagger *}$ in (1)-(9). Similarly, the equivalences of (19)-(27) can be established by replacing B by $B^{\dagger *}$ in (1)-(9) and the equivalences of (28)-(36) can be established by replacing A by $B^\dagger$ and B by $A^\dagger$ in (1)-(9). The equivalence of (1) and (19) is similar to that of (1) and (10). The equivalence of (1) and (28) follows by applying the Moore-Penrose inverse on both sides of (1) and (28).
$(1) \Rightarrow (37)$: We use Theorem 3.1 to prove $(A^*AB)^\dagger = B^\dagger(A^*A)^\dagger$. We get the first Penrose equation verified as below. By using Lemma 2.1 (iv) and (vi), we get

$$A^*ABB^\dagger(A^*A)^\dagger A^*AB = A^*ABB^\dagger A^\dagger AB = A^*AA^\dagger ABB^\dagger B = A^*AB.$$

By Lemma 2.1, Theorem 3.1 and Theorem 3.2 (v), we get

$$[A^*ABB^\dagger(A^*A)^\dagger]^* = [BB^\dagger A^*A(A^*A)^\dagger]^* = (BB^\dagger A^\dagger A)^* = A^\dagger ABB^\dagger.$$

The right-hand side is Hermitian, so is the left-hand side. Similarly, we can prove the second and fourth Penrose equations. Also, we get

$$(A^*AB)^\dagger A^* = B^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger = (AB)^\dagger.$$

(1) ⇒ (38): Similar to (1) ⇒ (37).
(1) ⇒ (39): By using Lemma 2.1 and Theorem 3.1, we get

$$\begin{aligned} A^*ABB^*(BB^*)^\dagger(A^*A)^\dagger A^*ABB^* &= A^*ABB^\dagger A^\dagger ABB^* \\ &= A^*AA^\dagger ABB^\dagger BB^* \\ &= A^*ABB^*. \end{aligned}$$

Now, using Lemma 2.1 and Theorem 3.2 (v), we have

$$\begin{aligned} [A^*ABB^*(BB^*)^\dagger(A^*A)^\dagger]^* &= [A^*ABB^\dagger(A^*A)^\dagger]^* = [BB^\dagger A^*A(A^*A)^\dagger]^* \\ &= (BB^\dagger A^\dagger A)^*, \end{aligned}$$

which is Hermitian by Theorem 3.1. Hence the first and third conditions of the Penrose equations are satisfied. The second and fourth conditions follow similarly.
(1) ⇒ (40): Let $P = (AA^*)^m A$ and $Q = B(B^*B)^n$. Then $\mathcal{R}(P)$, $\mathcal{R}(Q)$ and $\mathcal{R}(PQ)$ have closed ranges by Lemma 3.2. We prove $[(AA^*)^m AB(B^*B)^n]^\dagger = [B(B^*B)^n]^\dagger[(AA^*)^m A]^\dagger$ i.e., $(PQ)^\dagger = Q^\dagger P^\dagger$ by verifying the Penrose equations. By Lemma 2.1 and Theorem 3.1, we get

$$\begin{aligned} PQQ^\dagger P^\dagger PQ &= (AA^*)^m AB(B^*B)^n[(B^*B)^n]^\dagger B^\dagger A^\dagger[(AA^*)^m]^\dagger(AA^*)^m AB(B^*B)^n \\ &= (AA^*)^m ABB^\dagger BB^\dagger A^\dagger AA^\dagger AB(B^*B)^n \\ &= (AA^*)^m ABB^\dagger A^\dagger AB(B^*B)^n = (AA^*)^m AA^\dagger ABB^\dagger B(B^*B)^n \\ &= (AA^*)^m AB(B^*B)^n = PQ. \end{aligned}$$

Similarly, we can prove the second Penrose equation. For proving the third one we use the following facts that for all $m \geq 1$, $A^\dagger(AA^*)^m = A^*(AA^*)^{m-1}$, $[(AA^*)^m]^\dagger A = [(AA^*)^{m-1}]^\dagger A^{\dagger *} = A^{\dagger *}$ and $(ABB^\dagger A^\dagger)^* = ABB^\dagger A^\dagger$. We have

$$\begin{aligned} (PQQ^\dagger P^\dagger)^* &= ((AA^*)^m ABB^\dagger A^\dagger[(AA^*)^m]^\dagger)^* \\ &= [(AA^*)^m]^\dagger ABB^\dagger A^\dagger(AA^*)^m \\ &= A^{\dagger *}BB^\dagger A^* = (ABB^\dagger A^\dagger)^*. \end{aligned}$$

The right-hand side is Hermitian so is the left-hand side. Similarly, the fourth Penrose equation can be proved. Also, we have

$$\begin{aligned} (B^*B)^n[(AA^*)^m AB(B^*B)^n]^\dagger(AA^*)^m &= (B^*B)^n[B(B^*B)^n]^\dagger[(AA^*)^m A]^\dagger(AA^*)^m \\ &= (B^*B)^n[(B^*B)^n]^\dagger B^\dagger A^\dagger[(AA^*)^m]^\dagger(AA^*)^m \\ &= B^\dagger BB^\dagger A^\dagger AA^\dagger = B^\dagger A^\dagger = (AB)^\dagger. \end{aligned}$$

(1) ⇒ (41): Let $P = (AA^*)^{m+1}$ and $Q = (B^*B)^{n+1}$. We can prove the existence of $(PQ)^\dagger$ by a similar argument in Lemma 3.2.

$$\begin{aligned} PQQ^\dagger P^\dagger PQ &= (AA^*)^{m+1}B^\dagger BAA^\dagger(B^*B)^{n+1} \\ &= (AA^*)^{m+1}AA^\dagger B^\dagger B(B^*B)^{n+1} \\ &= (AA^*)^{m+1}(B^*B)^{n+1} = PQ. \end{aligned}$$

Using the fact $AA^*B^\dagger B = B^\dagger BAA^*$, we have

$$\begin{aligned} (PQQ^\dagger P^\dagger)^* &= (AA^*)^{m+1}B^\dagger B[(AA^*)^{m+1}]^\dagger \\ &= B^\dagger B(AA^*)^{m+1}[(AA^*)^{m+1}]^\dagger \\ &= B^\dagger BAA^\dagger. \end{aligned}$$

The right-hand side is Hermitian so is the left-hand side. Similarly, the second and fourth Penrose equations can be proved. Now we have

$$\begin{aligned} B^*(BB^*)^n((A^*A)^{m+1}(BB^*)^{n+1})^\dagger(A^*A)^m A^* &= B^*(BB^*)^n[(BB^*)^{n+1}]^\dagger \\ &\quad [(A^*A)^{m+1}]^\dagger(A^*A)^m A^* \\ &= B^*BB^\dagger(BB^*)^\dagger(A^*A)^\dagger A^\dagger AA^* \\ &= B^*(BB^*)^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger = (AB)^\dagger. \end{aligned}$$

(37) − (41) ⇒ (1): Using the given conditions we get

$$(AB)^\dagger = (A^*AB)^\dagger A^* = B^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger.$$

Similarly, other implications also follow by substituting the second set of equations into the first ones and using the properties of the Moore-Penrose inverse given in Lemma 2.1. □

Proposition 3.1. *Let the conditions of Theorem 3.1 hold. Then the following statements are true.*

(i) $A^*ABB^\dagger A^\dagger A$ *is Hermitian if and only if* $B^\dagger A^\dagger \in AB\{3\}$.

(ii) $BB^\dagger A^\dagger ABB^*$ *is Hermitian if and only if* $B^\dagger A^\dagger \in AB\{4\}$.

Proof. Since $ABB^\dagger A^\dagger = A^{\dagger *}A^*ABB^\dagger A^\dagger AA^\dagger$ we get $B^\dagger A^\dagger \in AB\{3\}$ if and only if $A^*ABB^\dagger A^\dagger A$ is Hermitian. Similarly, we can prove (ii). □

The next result is a continuation of Theorem 3.8.

Theorem 3.9. *Let the conditions of Theorem 3.1 hold. Then the following statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(42) $(ABB^*)^\dagger = (BB^*)^\dagger A^\dagger$ *and* $BB^\dagger A^\dagger ABB^*$ *is Hermitian;*

(43) $(A^*AB)^\dagger = B^\dagger(A^*A)^\dagger$ *and* $A^*ABB^\dagger A^\dagger A$ *is Hermitian;*

(44) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ *and* $BB^\dagger A^\dagger ABB^*$ *is Hermitian;*

(45) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$ *and* $A^*ABB^\dagger A^\dagger A$ *is Hermitian;*

(46) $(A^*ABB^*)^\dagger = (BB^*)^\dagger(A^*A)^\dagger$; $A^*ABB^\dagger A^\dagger A$ *and* $BB^\dagger A^\dagger ABB^*$ *are Hermitian;*

(47) $(A^\dagger ABB^*)^\dagger = (BB^*)^\dagger A^\dagger A$; $A^*ABB^\dagger A^\dagger A$ *and* $BB^\dagger A^\dagger ABB^*$ *are Hermitian;*

(48) $(A^*ABB^\dagger)^\dagger = BB^\dagger(A^*A)^\dagger$; $A^*ABB^\dagger A^\dagger A$ *and* $BB^\dagger A^\dagger ABB^*$ *are Hermitian;*

(49) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$; $A^*ABB^\dagger A^\dagger A$ *and* $BB^\dagger A^\dagger ABB^*$ *are Hermitian.*

Proof. In all the implications of the proof, Proposition 3.1 is also used.
(1) ⇒ (42): Follows from Theorem 3.8 (38).
(42) ⇒ (1): Follows from Theorem 3.2 (x).
(1) ⇒ (43): Follows from Theorem 3.8 (37).
(43) ⇒ (1): Follows from Theorem 3.3 (x).
(1) ⇒ (44): Follows from Theorem 3.8 (6).
(44) ⇒ (1): Follows from Theorem 3.2 (vi).

(1) $\Rightarrow$ (45): Follows from Theorem 3.8 (5).
(45) $\Rightarrow$ (1): Follows from Theorem 3.3 (vi).
(1) $\Rightarrow$ (46): Follows from Theorem 3.8 (39).
(46) $\Rightarrow$ (1): It is easy to verify that $B^\dagger A^\dagger \in AB\{1, 2\}$.
(46) $\Leftrightarrow$ (47): Replacing B by BB^* in the equivalence (43) $\Leftrightarrow$ (45).
(46) $\Leftrightarrow$ (48): Replacing A by A^*A in the equivalence (42) $\Leftrightarrow$ (44).
(47) $\Leftrightarrow$ (49): Replacing A by $A^\dagger A$ in the equivalence (42) $\Leftrightarrow$ (44). □

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COUNTABLY GENERATED EXTENSIONS OF *QTAG*-MODULES

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Abstract. Let the $QTAG$-module M be the set-theoretic union of a countable collection of isotype submodules S_k of countable length. For $0 \leq k < \omega$ we prove that M is totally projective if S_k is totally projective. Certain related assertions in this direction are also presented.

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1 Introduction and terminology

Modules are the natural generalizations of abelian groups. Among many generalizations of torsion abelian groups the notion of TAG-modules and its related properties have attracted considerable attention since 1976 (see, for example, [1, 19]). Following [17], a module M_R is called a TAG-module if it satisfies the following two conditions.

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

A module M_R satisfying only condition (I) is called a $QTAG$-module (see [18]). This is a very fascinating structure that has been the subject of research of many authors. They studied different notions and structures of $QTAG$-modules and developed the theory of these modules by introducing several notions, investigated some interesting properties and characterized different submodules of $QTAG$-modules. Not surprisingly, many of the developments parallel the earlier development of the structure of torsion abelian groups. The present work translates a few of the ideas of the abelian p-groups over to the area of $QTAG$-modules and certainly contributes to the overall knowledge of the structure of $QTAG$-modules.

Throughout our discussion all the rings R here are associative with unity $(1 \neq 0)$ and modules M are unital $QTAG$-modules. A module M over a ring R is called uniserial if it has a unique decomposition series of finite length. A module M is called uniform if intersection of any two of its non-zero submodules is non-zero. An element x in M is called uniform if xR is a non-zero uniform (hence uniserial) module. For any module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For any uniform element x of M, its exponent $e(x)$ is defined to be equal to the decomposition length $d(xR)$. For any $0 \neq x \in M$, $H_M(x)$ (the height of x in M) is defined by $H_M(x) = \sup\{d(yR/xR) : y \in M,\ x \in yR \text{ and } y \text{ uniform}\}$. For $k \geq 0$,

$H_k(M) = \{x \in M | \ H_M(x) \geq k\}$ denotes the submodule of M generated by the elements of height at least k and for some submodule N of M, $H^k(M) = \{x \in M | \ d(xR/(xR \cap N)) \leq k\}$ is the submodule of M generated by the elements of exponents at most k.

Let us denote by M^1, the submodule of M, containing uniform elements of infinite height. The module M is h-divisible if $M = M^1 = \cap_{k=0}^{\infty} H_k(M)$. The module M is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded [17], if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$. A submodule N of M is h-pure [10] in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$.

For a $QTAG$-module M and an ordinal α, $H_\alpha(M)$ is defined as $H_\alpha(M) = \cap_{\beta<\alpha} H_\beta(M)$. For an ordinal α, a submodule N of M is said to be α-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \alpha$ and a submodule N of M is said to be isotype in M, if it is α-pure for every ordinal α [15]. For an ordinal α, a submodule $N \subseteq M$ is an α-high submodule [14] of M if N is maximal among the submodules of M that intersect $H_\alpha(M)$ trivially.

A submodule $N \subset M$ is nice [12] in M, if $H_\alpha(M/N) = (H_\alpha(M) + N)/N$ for all ordinals α, i.e. every coset of M modulo N may be represented by an element of the same height. The sum of all simple submodules of M is called the socle of M and is denoted by $Soc(M)$. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-Ulm invariant of M and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$.

The major aim here is to extend Theorem 1 from [8] to two important classes of $QTAG$-modules the first one the class of summable modules, whereas the second one the class of α-modules, where α is a limit ordinal. The work is organized thus: in the first section, i.e. here, we have studied the basic notation as well as the terminology necessary for applicable purposes. In the second section, we proceed by proving the preliminary results, and in the third one we obtain a new simplified but more convenient for us major result, when a countable number of h-pure submodules can be a countable number of isotype submodules that seem to be interesting. In the fourth section, several applications of Theorem 3.1 in terms of total projectivity are provided which are of some importance.

It is interesting to note that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules [15]. Many results, stated in the present paper, are clearly generalizations from the papers [7, 8, 9]. For the better understanding of the mentioned topic here one must go through the papers [2, 3]. Most of our notations and terminology will be standard being in agreement with [4] and [5].

2 Preliminary results

We begin by defining a μ-module.

Definition 1. Let μ be a cardinal. We say that a $QTAG$-module M is a μ-module if M has cardinality μ and each submodule of M having cardinality less than μ is a direct sum of uniserial modules.

The question whether all μ-modules are direct sums of uniserial modules, has a significance in the theory of $QTAG$-modules. For every infinite cardinal μ there exists a μ-module that is a direct sum of uniserial modules with $\mu \geq \aleph_k$ and $k \geq 0$. We conjecture that the problem has a negative answer in general, but nevertheless we shall inspect in the sequel its validity for a finite cardinal μ. This follows immediately from the well-known structure of finitely generated $QTAG$-modules. However, we now have the following result.

Theorem 2.1. *Suppose that M is a QTAG-module and ω is a first limit ordinal. If M is an $\aleph_\omega$-module, then M is a direct sum of uniserial modules.*

Proof. Let M be a $QTAG$-module of cardinality $\aleph_\omega$ such that each submodule of M having cardinality less than $\aleph_\omega$ is a direct sum of uniserial modules. Since any infinite submodule can be imbedded in an h-pure submodule of the same cardinality, it easily follows that M is the union of an ascending chain of h-pure submodules S_k of M such that $g(S) = \aleph_k$ for $0 \leq k < \omega$. For each $k < \omega$, consider $S_k = \Sigma_{i\in I_k} x_i R$ and let α denote the smallest ordinal having cardinality $\aleph_\omega$. Then there exist submodules N_β of M, for $\beta < \alpha$, such that
(1) $N_0 = 0$.
(2) N_β is h-pure in M for each $\beta < \alpha$.
(3) $N_\beta + S_k$ is h-pure in M for each $\beta < \alpha$ and $k < \omega$.
(4) $N_{\beta+1} \supseteq N_\beta$ for each β such that $\beta + 1 < \alpha$.
(5) $N_{\beta+1}/N_\beta$ is countably generated for each β such that $\beta + 1 < \alpha$.
(6) $N_\beta \cap S_k = \Sigma_{i\in I_{k,\beta}} x_i R$ for $\beta < \alpha$ and $k < \omega$, where $I_{k,\beta}$ is a subset of I_k.
(7) $N_\gamma = \cup_{\beta<\gamma} N_\beta$ if γ is a limit ordinal less than α.
(8) $M = \cup_{\beta<\alpha} N_\beta$.

Let $\lambda < \alpha$, and suppose that a submodule N_β of M with $\beta < \lambda$ such that conditions $(1) - (7)$ hold when α is replaced by λ. After this, let us assume that a submodule N_λ of M satisfying these conditions also. Then we have two cases to consider:

Case (i). λ is a limit ordinal. In this case, let us consider $N_\lambda = \cup_{\beta<\lambda} N_\beta$. Since N_β is h-pure for each $\beta < \lambda$, so that N_λ is h-pure in M. This, in tern, implies that $N_\lambda + S_k$ is h-pure in M. Thus, we see that condition (6) from definition of N_λ is satisfied. Now, we set $I_{k,\lambda} = \cup_{\beta<\lambda} I_{k,\beta}$ for each k, then it is easy to verify that $N_\lambda \cap S_k = \Sigma_{i\in I_{k,\lambda}} x_i R$. Henceforth, all the conditions $(1) - (7)$ are satisfied for $\beta < \lambda$.

Case (ii). $\lambda - 1$ exists. Consider the submodule N_λ of M such that N_λ is a countably generated extension of $N_{\lambda-1}$ and
(2^+) N_λ is h-pure in M.
(3^+) $N_\lambda + S_k$ is h-pure in M for each $k < \omega$.
(6^+) $N_\lambda \cap S_k = \Sigma_{i\in I_{k,\lambda}} x_i R$ for each k, where $I_{k,\lambda}$ is a subset of I_k.

Let P be any submodule of M containing $N_{\lambda-1}$. If $P/N_{\lambda-1}$ is countably generated, there exists a submodule Q of M containing P with $g(Q/N_{\lambda-1}) \leq \aleph_0$ such that $Q/N_{\lambda-1}$ is h-pure in $M/N_{\lambda-1}$ and $[(Q/N_{\lambda-1}) + (S_k + N_{\lambda-1})/N_{\lambda-1}]/[(S_k + N_{\lambda-1})/N_{\lambda-1}]$ is h-pure in $(M/N_{\lambda-1})/[(S_k + N_{\lambda-1})/N_{\lambda-1}]$ for each $k < \omega$. From the h-purity of $N_{\lambda-1}$ and $S_k + N_{\lambda-1}$, we get that $Q + S_k = Q + S_k + N_{\lambda-1}$ is an h-pure submodule of M. Next, let J_k be a countably generated extension of the subset $I_{k,\lambda-1}$ such that $Q \cap S_k = \Sigma_{i\in J_k} x_i R$. It follows that there is an ascending chain

$$Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_t \subseteq \ldots$$

of h-pure submodules of M such that Q_t is countably generated and $Q_t + S_k$ is h-pure in M for all $t, k < \omega$. Letting $Q_t \cap S_k \subseteq \Sigma_{i\in J_{k,t}} x_i R$, where $J_{k,t}$ is a countably generated extension of the subset $I_{k,\lambda-1}$ of I_k such that $Q_{t+1} \supseteq \Sigma_{i\in J_{k,t}} x_i R$ for all k. Define $N_\lambda = \cup_{t<\omega} Q_t$ and we set $I_{k,\lambda} = \cup_{t<\omega} J_{k,t}$, then $N_\lambda \cap S_k = \Sigma_{i\in I_{k,\lambda}} x_i R$. Thus, all conditions $(1) - (7)$ are satisfied for $\beta < \lambda$.

In addition, if the index set I_k is chosen to be the set of ordinals less than $\aleph_k$, then we can easily continue along condition (8) that $\beta \in I_{k,\beta}$ for all $k < \omega$ provided that $\beta \in I_k$.

In order to show that M is a direct sum of uniserial modules, it remains only to show that N_β is a direct sum of $N_{\beta+1}$ for each $\beta < \alpha$. Since N_β is h-pure and $N_{\beta+1}/N_\beta$ is countably generated, it is enough to show that $H_\omega(N_{\beta+1}/N_\beta) = 0$. Suppose that $y + N_\beta \in H_\omega(N_{\beta+1}/N_\beta) \subseteq H_\omega(M/N_\beta)$. Since $N_\beta + S_k$ is h-pure in M, then $y + N_\beta \in H_\omega((N_\beta + S_k)/N_\beta)$ where $y \in S_k$ for some k. In this connection, observe that $H_\omega((N_\beta + S_k)/N_\beta) = 0$. This completes the argument showing that

$(N_\beta + S_k)/N_\beta \cong S_k/(N_\beta \cap S_k)$ is a direct sum of uniserial modules. Setting $N_{\beta+1} = N_\beta + Q_\beta$, we obtain that $M = \Sigma_{\beta<\alpha} Q_\beta$, and the theorem is proved. □

The same idea is applicable even to $QTAG$-modules having cardinality $\aleph_\beta$, where β is cofinal with ω. So, we state without proof the following direct corollary.

Theorem 2.2. *If a $QTAG$-module M has cardinality $\aleph_\beta$ where β is cofinal with ω, then M is a direct sum of uniserial modules provided each submodule of M having cardinality less than $\aleph_\beta$ is a direct sum of uniserial modules.*

For freely use in the sequel, we obtain the following

Theorem 2.3. *Suppose that a $QTAG$-module M is a set-theoretic union of a countable number of h-pure submodules S_k for each k. If S_k is a direct sum of uniserial modules, then so is M.*

Proof. By appealing to the same reasoning as in Theorem 2.1, one may infer that the assertion follows. □

Now, we proceed by proving

Corollary 2.1. *Let S be a submodule of a $QTAG$-module M such that M is a direct sum of uniserial modules. Then S is a direct sum of uniserial modules.*

Proof. Suppose that M is a union of an ascending chain of h-pure submodules S_k such that S_k is bounded. Choose $P_k = S \cap S_k$ for each k. Let $Q_k \supseteq P_k$ be maximal in S with respect to $Q_k \cap H_k(S) = 0$. It is easy to see that Q_k is h-pure in S. Therefore, since Q_k is bounded, we get that Q_k is a direct sum of uniserial modules. Since $Q_k \supseteq P_k$, it follows immediately that S is a set-theoretic union of its submodules Q_k. Henceforth, according to Theorem 2.3, S is a direct sum of uniserial modules, as required. □

3 Main results

In Section 2, we have shown that if a $QTAG$-module M is the set-theoretic union of a countable number of h-pure submodules S_k, then M is a direct sum of uniserial modules if S_k is a direct sum of uniserial modules for each k. In the present section, we generalize this result by proving that if the submodules S_k are isotype then M must be totally projective provided that S_k is totally projective of countable length for each k. In particular, an ascending chain of isotype and totally projective submodules of countable length leads to a totally projective module.

Recall from [11] that an h-reduced $QTAG$-module M is said to be totally projective if it possesses a collection $\mathcal{N}$ consisting of nice submodules of M such that (i) $0 \in \mathcal{N}$ (ii) if $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\Sigma_{i\in I} N_i \in \mathcal{N}$ (iii) given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated. Call a collection $\mathcal{N}$ of nice submodules of M which satisfies conditions (i), (ii) and (iii) a nice system for M. It is well-known that any countably generated h-reduced $QTAG$-module is totally projective by induction on the length of M. Thus the direct sum of any number of countably generated h-reduced $QTAG$-modules is totally projective.

Before presenting our main attainment, we prove the following working lemma.

Lemma 3.1. *Let α be an arbitrary ordinal and M a $QTAG$-module of countable length. Suppose that*

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_\beta \subseteq \ldots,\ \beta < \alpha$$

is a chain of nice submodules of M satisfying the following conditions:
(a) $N_{\beta+1}/N_\beta$ *is countably generated.*
(b) $N_\gamma = \cup_{\beta<\gamma} N_\beta$ *where* γ *is a limit.*
(c) $M = \cup_{\beta<\alpha} N_\beta$.
Then M is totally projective.

Proof. Let μ be the first uncountable ordinal such that $\alpha < \mu$. Therefore, M satisfies a nice system of countability. In fact, for an arbitrary α, it is not evident that conditions $(a)-(c)$ imply a nice system of countability.

By hypothesis, M embeds as an isotype submodule of a totally projective module, for a height-preserving monomorphism from N_β to $N_{\beta+1}$. Since the length of M is countable, M itself is totally projective and hence the result follows. □

The main result is now the following.

Theorem 3.1. *Let a QTAG-module M be a set-theoretic union of a countable number of isotype submodules S_k. If S_k is totally projective of countable length for each k, then M is totally projective.*

Proof. First we note that M has countable length. Let us assume that length of $M = \eta$ and let the submodules S_k be indexed by the nonnegative integers. For $k < \omega$, let $S_k = \Sigma_{i\in I_k} T_i$ where T_i is countably generated for each i. Suppose that

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_\beta \subseteq \ldots, \ \beta < \lambda$$

is a chain of submodules of M satisfying the following conditions:
(a) $N_{\beta+1}/N_\beta$ is countably generated.
(b) $N_\gamma = \cup_{\beta<\gamma} N_\beta$ where γ is a limit.
(c) $N_\beta \cap S_k = \Sigma_{i\in I_{k,\beta}} T_i$ for each k and β, where $I_{k,\beta}$ is a subset of I_k.
(d) $\langle H_\lambda(M), N_\beta\rangle \cap \langle S_k, N_\beta\rangle = \langle H_\lambda(S_k), N_\beta\rangle$ for each k and β and for each $\lambda \leq \eta$.

We consider two possibilities. Firstly, if λ is a limit ordinal, we define $N_\lambda = \cup_{\beta<\lambda} N_\beta$ and see that conditions $(a)-(d)$ are satisfied for the chain of submodules N_β. Secondly, if $\lambda - 1$ exists. For an arbitrary countably generated extension T of $N_{\lambda-1}$ in M, there exists a countably generated extension P of T such that $P \cap S_k = \Sigma_{i\in J_k} T_i$, for each k, where J_k is a countable generated extension of $I_{k,\lambda-1}$.

Next, with this in hand, we ascertain the same argument that there exists a countably generated extension Q of $P \supseteq T \supseteq N_{\lambda-1}$ in M such that

$$\langle H_\lambda(M), Q\rangle \cap \langle S_k, Q\rangle = \langle H_\lambda(S_k), Q\rangle,$$

for all $\lambda \leq \eta$ and all $k < \omega$. Let $\{x_i\}_{i<\omega}$ be a set of representatives for the cosets of $P/N_{\lambda-1}$. For each triple (i, k, λ) with $i, k < \omega$ and $\lambda \leq \eta$ such that $y + z_k \in x_i + N_{\lambda-1}$ where $y \in H_\lambda(M)$ and $z_k \in S_k$. If we choose a representative $y = y_{i,k,\lambda}$ for the triple (i, k, λ), then clearly there are only a countable number of such representatives. Setting $Q_1 = \langle P, y_{i,k,\lambda}\rangle$, one may see that

$$\langle H_\lambda(M), P\rangle \cap \langle S_k, P\rangle \subseteq \langle H_\lambda(S_k), Q_1\rangle.$$

If we replace Q_1 by P, then Q_{j+1} is replaced by Q_j such that $Q_{j+1} = \langle Q_j, y_{i,k,\lambda}\rangle$. Hence, the desired properties follows if $Q = \cup_{j<\omega} Q_j$.

Furthermore, suppose that the conditions (a) and (d) holds. Then there exists a countable generated extension N_λ of $N_{\lambda-1}$ containing T that satisfies both conditions (c) and (d). Hence, a chain of submodules satisfying conditions $(a)-(d)$ is applicable to deduce that M is totally projective.

To complete the proof of the theorem, it remains only to show that N_β is nice in M for each β. It suffices to show that

$$H_\lambda(M/N_\beta) = \langle H_\lambda(M), N_\beta\rangle/N_\beta \tag{3.1}$$

for all $\lambda \leq \eta$. The proof is by induction on λ in conjunction with

$$H_\lambda(M/N_\beta) \cap \langle S_k, N_\beta\rangle/N_\beta = \langle H_\lambda(S_k), N_\beta\rangle/N_\beta = H_\lambda(\langle S_k, N_\beta\rangle/N_\beta) \tag{3.2}$$

Clearly, for a given λ the second equality in condition (3.2) is a consequence of the first equality. However, the second equality is valid, because of condition (c). We claim that condition (3.2) hold good for $\lambda = \sigma$, where σ is a limit. Then it is suffices to show that condition (3.2) holds for all $\lambda < \sigma$. By the choice of σ, condition (3.1) holds for all $\lambda < \sigma$. Hence, if $\lambda < \sigma$, we observe that

$$H_\lambda(M/N_\beta) \cap \langle S_k, N_\beta\rangle/N_\beta = (\langle H_\lambda(M), N_\beta\rangle \cap \langle S_k, N_\beta\rangle)/N_\beta.$$

Thus, by condition (d), we write

$$H_\lambda(M/N_\beta) \cap \langle S_k, N_\beta\rangle/N_\beta = \langle H_\lambda(S_k), N_\beta\rangle/N_\beta,$$

and so condition (3.2) holds for $\lambda = \sigma$. This gives that

$$H_\sigma(M/N_\beta) \subseteq \cup_{k<\omega}\langle H_\sigma(S_k), N_\beta\rangle/N_\beta \subseteq \langle H_\sigma(M), N_\beta\rangle/N_\omega,$$

which allows us to infer that N_β is nice in M for each β. □

4 Applications

The purpose of the present section is to explore some structural corollaries of Theorem 3.1. Several such applications are now presented.

4.1 Summability

Singh [17] proved that a $QTAG$-module M is a direct sum of uniserial modules if and only if M is the union of an ascending chain of bounded submodules. Apparently, M is a direct sum of uniserial modules if and only if $Soc(M) = \oplus_{k\in\omega} S_k$ and $H_M(x) = k$ for every $x \in S_k$. This led to the notion of summable modules, see, [16]. Let us recall the definition: an h-reduced $QTAG$-module M is summable if $Soc(M) = \oplus_{\beta<\alpha} N_\beta$, where N_β is the set of all elements of $H_\beta(M)$ which are not in $H_{\beta+1}(M)$, where α is the length of M. It is self-evident that a $QTAG$-module of length ω is a direct sum of uniserial modules if and only if the $QTAG$-module is summable. However, for the sake of completeness, the following corollaries are immediate.
(i) Countably generated h-reduced $QTAG$-modules are summable.
(ii) Direct sums of countably generated h-reduced $QTAG$-modules are summable.
(iii) Isotype submodules of summable modules of countable length are summable.

We start here with the following easy observation.

Theorem 4.1. *Let M be a summable $QTAG$-module of countable length α. If $M/H_\beta(M)$ is totally projective for each limit ordinal $\beta < \alpha$, then M is totally projective.*

Proof. The proof is by induction on α. If there is a limit ordinal β such that both $H_\beta(M)$ and $M/H_\beta(M)$ are totally projective, then M is itself totally projective. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \dots$ be an increasing sequence of ordinals whose limit is α. We choose $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq \dots$ be an ascending chain of submodules, so that S_k is α_k-high in M. Observe that $Soc(S) = Soc(M)$, where $S = \cup_{k<\omega} S_k$. Therefore, since S is h-pure in M, we get that $S = M$. Thus in view of Theorem 3.1, it suffices to show that S_k is totally projective for each k. However, we know that S_k is isomorphic to an isotype submodule of $M/H_{\alpha_k}(M)$ under the natural map. Henceforth, a simple technical argument applies to get that S_k is totally projective which gives the desired total projectivity of M. □

The following statement generalizes Theorem 2.2.

Theorem 4.2. *Let M be a QTAG-module of cardinality $\aleph_\beta$ where β is cofinal with ω. If each submodule of M having cardinality less than $\aleph_\beta$ is contained in a totally projective submodule of M having countable length, then M is totally projective.*

Proof. Assume that S_k is a submodule of M having cardinality $\aleph_{\beta_k}$, where $\beta_1 < \beta_2 < \cdots < \beta_k < \dots$ be an increasing sequence of ordinals whose limit is β. Note that if S_k is contained in an isotype submodule P_k of M having the same cardinality $\aleph_{\beta_k}$ as that of S_k, then P_k is contained in a totally projective submodule Q_k of M having countable length. Since P_k is isotype in Q_k, P_k is totally projective. It follows that M is the union of a countable ascending chain $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_k \subseteq \dots$ of isotype and totally projective submodules P_k. One seeing readily in view of Theorem 3.1 that M is totally projective, as wanted. □

This brings us to another technical observation.

Theorem 4.3. *Let M_1 and M_2 be QTAG-modules of countable type λ and suppose that M_1 is totally projective. If, for each ordinal $\beta \leq \lambda$, there exists a height-preserving isomorphism between $Soc(M_1/H_{\beta\omega}(M_1))$ and $Soc(M_2/H_{\beta\omega}(M_2))$, then M_2 is totally projective and $M_1 \cong M_2$.*

Proof. By hypothesis, there exists a height-preserving isomorphism between $Soc(M_1)$ and $Soc(M_2)$. It is easy to see that M_1 and M_2 have the same *Ulm* invariants (and are therefore isomorphic) if M_2 is totally projective.

We induct on λ to show that M_2 is totally projective. Since M_1 is summable, M_2 is also summable. If $\lambda = 1$, then M_2 is a direct sum of uniserial modules. Thus, assuming that $\lambda > 1$ and that $\lambda - 1$ exists. Observe that

$$M_1/H_{\omega(\lambda-1)}(M_1)/H_{\omega\beta}(M_1/H_{\omega(\lambda-1)}(M_1)) \cong M_1/H_{\omega\beta}(M_1),$$

for all $\beta \leq \lambda - 1$, and similarly for M_2. Hence in virtue of inductive hypothesis, $M_2/H_{\omega(\lambda-1)}(M_2)$ is totally projective. Since both $Soc(H_{\omega(\lambda-1)}(M_1))$ and $Soc(H_{\omega(\lambda-1)}(M_2))$ have a height-preserving isomorphism, we deduce that $H_{\omega(\lambda-1)}(M_2)$ is a direct sum of uniserial modules. This guarantees that M_2 is totally projective.

In the remaining case when λ is a limit ordinal, we assume that $Soc(M_1) = \Sigma U_\beta$ and $Soc(M_2) = \Sigma V_\beta$ be decompositions of $Soc(M_1)$ and $Soc(M_2)$, respectively, such that $H_{U_\beta}(x) = H_{V_\beta}(y) = \beta$, for some $x \in U$, $y \in V$. Let $1 \leq \lambda_1 \leq \lambda_2 < \cdots < \lambda_k < \dots$ be an increasing sequence of ordinals whose limit is λ, we choose $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k \subseteq \dots$ and $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_k \subseteq \dots$ be the ascending chain of submodules such that $Soc(N_k) = \Sigma_{\beta<\omega\lambda_k} U_\beta$ and $Soc(L_k) = \Sigma_{\beta<\omega\lambda_k} V_\beta$. Note that $M_1 = \cup_{k<\omega} N_k$ and $M_2 = \cup_{k<\omega} L_k$ since $\cup_{k<\omega} N_k$ and $\cup_{k<\omega} L_k$ are h-pure submodules of M_1 and M_2, respectively, containing $Soc(M_1)$ and $Soc(M_2)$.

What remains to show is that L_k is totally projective. Our future aim, which we pursue, is to check the existence of a height-preserving isomorphism between $Soc(N_k/H_{\omega\beta}(N_k))$ and $Soc(L_k/H_{\omega\beta}(L_k))$

for each $\beta \leq \lambda_k$. To that goal, we have two cases to consider. First, if $\beta = \lambda_k$, then there is a height-preserving isomorphism between $Soc(N_k/H_{\omega\beta}(N_k)) = Soc(N_k)$ and $Soc(L_k/H_{\omega\beta}(L_k)) = Soc(L_k)$ by the choice of N_k and L_k. For the second case where $\beta < \lambda_k$, it is easily observed that $M_1 = \langle N_k, H_{\omega\beta}(M_1)\rangle$ since N_k is $\omega\lambda_k$-high in M_1. Similarly, $M_2 = \langle L_k, H_{\omega\beta}(M_2)\rangle$. Therefore, $M_1/H_{\omega\beta}(M_1) \cong N_k/H_{\omega\beta}(N_k)$ and $M_2/H_{\omega\beta}(M_2) \cong L_k/H_{\omega\beta}(L_k)$. Then there exists a height-preserving isomorphism between $Soc(N_k/H_{\omega\beta}(N_k))$ and $Soc(L_k/H_{\omega\beta}(L_k))$ for each $\beta \leq \lambda_k$. It follows by induction hypothesis that L_k is totally projective.

In addition, since L_k is isotype in M_2 and $M_2 = \cup_{k<\omega} L_k$, so referring to Theorem 3.1, we can conclude that M_2 is totally projective, as promised. □

4.2 α-modules

For the definition of an α-module, the reader can see [13] or [6] where it is given in all details. However, for a convenience of the reader, we shall include it in the text. A $QTAG$-module M is an α-module, where α is a limit ordinal, if $M/H_\beta(M)$ is totally projective for every ordinal $\beta < \alpha$.

It is well-known that every totally projective module is an α-module. Besides, it is simple to checked that an α-module of length α is a direct sum of countably generated modules if and only if it is summable.

Now, we are ready to formulate the following

Theorem 4.4. *Let M be a $QTAG$-module of length α such that M is an α-module. If M is a set-theoretic union of countable number of submodules S_k where the heights of the nonzero uniform elements of S_k in M are bounded by some ordinal $\alpha_k < \alpha$, then M is totally projective.*

Proof. Suppose that $M = \cup_{k<\omega} S_k$ where S_k is isotype of length $\alpha_k < \alpha$ for each k. Since M has countable length, then S_k is totally projective. Because $S_k \cong \langle S_k, H_{\alpha_k}(M)\rangle/H_{\alpha_k}(M)$ is isomorphic to an isotype submodule of a totally projective module $M/H_{\alpha_k}(M)$ having countable length α_k, it easily follows that M is totally projective by the usage of Theorem 3.1 if α is countable.

Similarly, if the interval (γ, α) of ordinals is countable for some γ less than α, then $H_\gamma(M)$ is totally projective. Indeed, since $H_\gamma(M)$ has countable length and $H_\gamma(M) = \cup_{k<\omega} H_\gamma(S_k)$ such that $H_\gamma(S_k)$ is isotype in $H_\gamma(M)$. Therefore, $H_\gamma(M)$ is totally projective if (γ, α) is countable for some $\gamma < \alpha$. Moreover, since M is an α-module, then $M/H_\gamma(M)$ is totally projective. This means that M is totally projective, and the result follows for (γ, α) is countable with $\gamma < \alpha$.

We next assume that (γ, α) is uncountable for every ordinal γ less than the length α of M. In particular, if $\gamma < \alpha$, then $\gamma + \omega < \alpha$. Without loss of generality, we may assume that S_k is maximal with $S_k \cap H_{\alpha_k}(M) = 0$. Then S_k is an α_k-high submodule of M.

Finally, consider the case $\alpha_k = \sigma_k + \omega$ for some ordinal σ_k. Suppose now the ordinal α_k has the form $\alpha_k + \omega$ for some $\alpha_k < \alpha$. Since $\alpha_k = \sigma_k + \omega$ and S_k is an α_k-high in M, we have $M = \langle S_k, H_{\sigma_k}(M)\rangle$ and $M/H_{\sigma_k}(M) \cong S_k/H_{\sigma_k}(S_k)$ for each k. Consequently, $S_k/H_{\sigma_k}(S_k)$ is totally projective, and we get $H_{\sigma_k}(S_k)$ is totally projective since it is isomorphic to an isotype submodule of the totally projective module $H_{\sigma_k}(M)/H_{\alpha_k}(M)$. Therefore, we conclude that S_k is totally projective, and again the application of Theorem 3.1 leads to M being totally projective, as expected. □

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ON SOME SYSTEMS OF NONLINEAR INTEGRAL EQUATIONS ON THE WHOLE AXIS WITH MONOTONOUS HAMMERSTEIN-VOLTERRA TYPE OPERATORS

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Abstract. The work is devoted to studying and solving some systems of nonlinear integral equations with monotonous Hammerstein-Volterra integral operators. In specific cases of matrix kernels and nonlinearities the specified systems have applications in various fields of mathematical physics and mathematical biology. Firstly, a quasilinear system of integral equations on the whole axis with monotonous nonlinearity will be investigated, and a constructive theorem of existence of a one-parameter family of componentwise nonnegative (nontrivial) bounded solutions will be proved. Then, the asymptotic behaviour of the constructed solutions will be studied at $-\infty$. Then, using the obtained results, a system of integral equations with two nonlinearities with different characteristics will be investigated. Under certain limitations on the first nonlinearity we will prove the existence of componentwise nonnegative and bounded solution for such systems. In addition, the limit of the constructed solution at $-\infty$ will be calculated, and the asymptotics of the difference between the limit and the solution will be established. At the end of this paper specific examples of matrix kernels and nonlinearities will be given for the illustration of the obtained results.

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1 Introduction

In the present paper we study the following quasilinear and essentially nonlinear integral equations with monotonous Hammerstein-Volterra operator on the whole axis $\mathbb{R} := (-\infty, +\infty)$:

$$f_i(x) = \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{f_j(t) + \omega_{ij}(t, f_j(t))\}dt, \ i = 1, 2, ..., n, \ x \in \mathbb{R}, \tag{1.1}$$

$$\varphi_i(x) = \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{G_j(\varphi_j(t)) + \omega_{ij}(t, \varphi_j(t))\}dt, \ i = 1, 2, ..., n, \ x \in \mathbb{R}, \tag{1.2}$$

with respect to the unknown measurable on $\mathbb{R}$ vector-functions $f(x) = (f_1(x), ..., f_n(x))^T$ and $\varphi(x) = (\varphi_1(x), ..., \varphi_n(x))^T$ respectively (T is the sign of transposition). In systems (1.1) and (1.2) the matrix kernel $K(x,t) = (K_{ij}(x,t))_{i,j=1}^{n\times n}$ satisfies the following conditions:

a) $K_{ij}(x,t) > 0$, $(x,t) \in \mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, $K_{ij} \in L_\infty(\mathbb{R}^2)$, $i, j = 1, 2, ..., n$, where $L_\infty(\mathbb{R}^2)$ is the space of all essentially bounded functions on the set $\mathbb{R}^2$,

$b)$ there exists a symmetric matrix $A = (a_{ij})_{i,j=1}^{n\times n}$ with positive elements a_{ij} and with a unit spectral radius such that

$b_1)$
$$\gamma_{ij}(x) := a_{ij} - \int_{-\infty}^{x} K_{ij}(x,t)dt \geq 0,\ \ \gamma_{ij}(x) \not\equiv 0,\ \ x \in \mathbb{R},$$
$$\lim_{x\to-\infty} \gamma_{ij}(x) = 0,\ \ i,j = 1,2,...,n,$$

$b_2)$
$$\int_{t}^{\infty} K_{ij}(x,t)dx \leq a_{ij},\ \ t \in \mathbb{R},\ \ i,j = 1,2,...,n,$$

$b_3)$
$$\int_{-\infty}^{0} (-x)\gamma_{ij}(x)dx < +\infty,\ \ i,j = 1,2,...,n,$$

$c)$ there exists a number $\delta_0 > 0$ such that

$$\varepsilon_{ij} := \inf_{x\in(-\infty,0]} \int_{\delta_0}^{\infty} K_{ij}(x+y,x)dy > 0,\ \ i,j = 1,2,...,n.$$

From the properties of the matrix A, by Perron's theorem (see [12]), follows the existence of a vector $\eta = (\eta_1, ..., \eta_n)^T$ with positive coordinates $\eta_i,\ i = 1,2,...,n,$ such that

$$A\eta = \eta. \tag{1.3}$$

The nonlinearities $\{G_j(u)\}_{j=1}^{n}$ and $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ satisfy the following conditions:

I) $G_j \in C[0,+\infty)$, $G_j(u)$ is a concave function on the set $[0,+\infty)$, $G_j(0) = 0$, $j = 1,2,...,n,$

II) $G_j(u)$ are increasing with respect to u on the set $[0,+\infty)$, $j = 1,2,...,n,$

III) there exists a number $\alpha > 0$, such that $G_j(\eta_j^*) = \eta_j^*$, $G_j(u) \geq u$, $u \in [0,\eta_j^*]$, where $\eta_j^* = \alpha\eta_j$, $j = 1,2,...,n,$

A) $\omega_{ij}(t,0) \equiv 0$, $t \in \mathbb{R}$, $i,j = 1,2,...,n,$

B) for every fixed $t \in \mathbb{R}$ the functions $\omega_{ij}(t,u)$, $i,j = 1,2,...,n$ monotonically increase with respect to u on the set $[0,+\infty)$,

C) there exist functions
$$\beta_{ij}(t) := \sup_{u\in[0,+\infty)} (\omega_{ij}(t,u)),\ \ i,j = 1,2,...,n,$$
such that $\beta_{ij}(t)$, $i,j = 1,2,...,n$ are monotone nondecreasing with respect to t on the set $\mathbb{R}$ and satisfy the following inequality

$$\sum_{j=1}^{n} \beta_{ij}(x)\left(a_{ij} - \gamma_{ij}(x)\right) \leq \sum_{j=1}^{n} \eta_j\gamma_{ij}(x),\ \ x \in \mathbb{R},\ \ i = 1,2,...,n, \tag{1.4}$$

D) $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ satisfy the Caratheodory condition with respect to the argument u on the set $\mathbb{R}\times[0,+\infty)$, i.e. for every fixed $u\in[0,+\infty)$ the functions $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ are measurable with respect to t on $\mathbb{R}$ and for almost every $t\in\mathbb{R}$ these functions are continuous with respect to u on the set $[0,+\infty)$.

The study of systems of nonlinear integral equations (1.1) and (1.2), besides purely mathematical interest, has also an important interest in different applied problems of mathematical physics and mathematical biology. In particular, for specific representations of matrix kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$ and nonlinearities $\{G_j(u)\}_{j=1}^{n}$ and $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ such systems of nonlinear integral equations can be found in the kinetic theory of gases, radiative transfer theory, Markovian processes and in the mathematical theory of space-time epidemic spread (see [1]-[5], [10], [13], [14]).

In the case, when the kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$ depend on the difference of their arguments and satisfy the supercritical condition (the spectral radius of the matrix A is greater than one) with particular restrictions on the functions $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ system (1.1) on $(-\infty,0]$ (and the corresponding system of nonlinear integral equations on $[0,+\infty)$, whose right-hand-side integrals have limits from $x\geq 0$ to $+\infty$) is studied in sufficient detail in the work [9]. In the present paper a one-parameter family of positive summable and bounded on $(-\infty,0]$ (on $[0,+\infty)$) solutions is constructed and the set of the corresponding parameters is described.

It should also be noted that in the case when $K_{ij}(x,t)=K_{ij}(x-t)$, $(x,t)\in\mathbb{R}^2$, $i,j=1,2,...,n$ the corresponding systems of convolution type nonlinear integral equations (NIE) (i.e. when the integral in the right-hand sides of (1.1) and (1.2) has the limits from $-\infty$ to $+\infty$) were studied in the works [6]-[8].

In the present paper under conditions $a)-c)$, I) - III) and A) - D) we will deal with the problems of existence of nonnegative (nontrivial) and bounded solutions of systems of nonlinear integral equations (1.1) and (1.2) and also will study the asymptotic behaviour of the constructed solutions on $-\infty$. Firstly, a constructive theorem of existence of a one-parameter family of componentwise nonnegative (nontrivial) and bounded solutions, which have finite limit values in $-\infty$ will be proved. Then, we will prove the integrability on the set $(-\infty,0]$ of the difference between the limit (at $-\infty$) and the constructed solution for every value of the corresponding parameter on the set $(0,+\infty)$ (see Theorem 2.1). Owner furthermore, by using these results, we will construct componentwise nonnegative and bounded on $\mathbb{R}$ solution $\varphi(x)=(\varphi_1(x),...,\varphi_n(x))^T$ of system of nonlinear integral equations (1.2). Additionally, we will prove the existence of

$$\lim_{x\to-\infty}\varphi_j(x)=\eta_j^*$$

and that $\eta_j^*-\varphi_j\in L_1(-\infty,0)$, $j=1,2,...,n$ (see Theorem 3.1). At the end of the work we will provide specific examples of matrix kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$ and nonlinearities $\{G_j(u)\}_{j=1}^{n}$, $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ that satisfy all the conditions of the proved theorem. Note that a part of those examples have applied character (they arise in specific problems of mathematical physics and biology).

2 One parameter family of solutions for system (1.1)

In the current section we will prove the following result for system of NIE (1.1):

Theorem 2.1. *Under conditions a) - c) and A) - D) system of NIE* (1.1) *has a one-parameter family of componentwise nonnegative (nontrivial) and bounded solutions* $f^\gamma(x)=(f_1^\gamma(x),...,f_n^\gamma(x))^T$, $\gamma\in(0,+\infty)$, *such that*

$$\lim_{x\to-\infty}f_j^\gamma(x)=\eta_j\gamma$$

and $\eta_j\gamma-f_j^\gamma\in L_1(-\infty,0)$, $j=1,2,...,n$, *where* η *is defined by* (1.3).

Proof. Firstly, let us consider the first auxiliary system of linear nonhomogeneous Volterra type integral equations:

$$\psi_i(x) = g_i(x) + \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\psi_j(t)dt, \;\; i = 1,2,...,n, \;\; x \in \mathbb{R} \tag{2.1}$$

with respect to an unknown summable on $\mathbb{R}$ vector-function $\psi(x) = (\psi_1(x), ..., \psi_n(x))^T$, where the vector-function $g(x) = (g_1(x), ..., g_n(x))^T$ has the following structure:

$$g_i(x) = \sum_{j=1}^{n} \beta_{ij}(x)\left(a_{ij} - \gamma_{ij}(x)\right), \;\; i = 1,2,...,n, \;\; x \in \mathbb{R}. \tag{2.2}$$

We introduce the following iterations for system (2.1):

$$\begin{gathered} \psi_i^{(m+1)}(x) = g_i(x) + \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\psi_j^{(m)}(t)dt, \\ \psi_i^{(0)}(x) = g_i(x), \;\; x \in \mathbb{R}, \;\; i = 1,2,...,n, \;\; m = 0,1,... \, . \end{gathered} \tag{2.3}$$

By mathematical induction it is not hard to verify that

1) $\psi_i^{(m)}(x)$ are measurable on $\mathbb{R}$, $i = 1,2,...,n$, $m = 0,1,2,...$,

2) $\psi_i^{(m)}(x) \uparrow$ with respect to m, $i = 1,2,...,n$, $x \in \mathbb{R}$.

We will prove that

3) $\psi_i^{(m)}(x) \le \eta_i$, $m = 0,1,2,...$, $i = 1,2,...,n$, $x \in \mathbb{R}$.

Indeed, estimate 3) for $m = 0$ directly follows from b_1), (1.3) and (1.4):

$$\psi_i^{(0)}(x) = g_i(x) \le \sum_{j=1}^{n} \eta_j\gamma_{ij}(x) \le \sum_{j=1}^{n} a_{ij}\eta_j = \eta_i, \;\; x \in \mathbb{R}, \;\; i = 1,2,...,n.$$

Assume that 3) holds for some $m \in \mathbb{N}$. Then, with consideration of b_1), (1.3), (1.4), a) and (2.2) from (2.3) we get

$$\psi_i^{(m+1)}(x) \le \sum_{j=1}^{n} \beta_{ij}(x)\left(a_{ij} - \gamma_{ij}(x)\right) + \sum_{j=1}^{n} \eta_j \int_{-\infty}^{x} K_{ij}(x,t)dt \le$$

$$\le \sum_{j=1}^{n} \eta_j\gamma_{ij}(x) + \sum_{j=1}^{n} \eta_j\left(a_{ij} - \gamma_{ij}(x)\right) = \sum_{j=1}^{n} a_{ij}\eta_j = \eta_i, \;\; i = 1,2,...,n, \;\; x \in \mathbb{R}.$$

Now we will prove that

4) $\psi_i^{(m)} \in L_1(-\infty, 0)$, $i = 1,2,...,n$, $m = 0,1,2,...$.

Indeed, in the case when $m = 0$ inclusion 4) follows from the definition of $g_i(x)$, $i = 1,2,...,n$, with consideration of (1.4), conditions b_1) and b_3). Let $\psi_i^{(m)} \in L_1(-\infty, 0)$, $i = 1,2,...,n$ for some natural

m. Then, considering (1.4), a), b), c) and (2.2), from (2.3) for every $\delta < 0$ by Fubini's theorem (see [11]) we have

$$0 \leq \int\limits_{\delta}^{0} \psi_i^{(m+1)}(x)dx \leq \sum_{j=1}^{n} \int\limits_{\delta}^{0} \eta_j \gamma_{ij}(x)dx + \sum_{j=1}^{n} \int\limits_{\delta}^{0} \int\limits_{-\infty}^{x} K_{ij}(x,t)\psi_j^{(m)}(t)dtdx =$$

$$= \sum_{j=1}^{n} \eta_j \int\limits_{\delta}^{0} \gamma_{ij}(x)dx + \sum_{j=1}^{n} \int\limits_{\delta}^{0} \int\limits_{-\infty}^{\delta} K_{ij}(x,t)\psi_j^{(m)}(t)dtdx + \sum_{j=1}^{n} \int\limits_{\delta}^{0} \int\limits_{\delta}^{x} K_{ij}(x,t)\psi_j^{(m)}(t)dtdx \leq$$

$$\leq \sum_{j=1}^{n} \eta_j \int\limits_{-\infty}^{0} \gamma_{ij}(x)dx + \sum_{j=1}^{n} \int\limits_{-\infty}^{\delta} \psi_j^{(m)}(t) \int\limits_{\delta}^{0} K_{ij}(x,t)dxdt + \sum_{j=1}^{n} \int\limits_{\delta}^{0} \psi_j^{(m)}(t) \int\limits_{t}^{0} K_{ij}(x,t)dxdt \leq$$

$$\leq \sum_{j=1}^{n} \eta_j \int\limits_{-\infty}^{0} \gamma_{ij}(x)dx + \sum_{j=1}^{n} a_{ij} \int\limits_{-\infty}^{0} \psi_j^{(m)}(t)dt < +\infty.$$

By passing to the limit as $\delta \to -\infty$, we conclude that $\psi_i^{(m+1)} \in L_1(-\infty, 0)$. Now let $t \leq 0$ be an arbitrary number. We multiply both sides of (2.3) by η_i, $i = 1, 2, ..., n$ and taking into account conditions a), b), c), (1.4) and also the proven inclusions 1)-4), we integrate both sides of the obtained equality by $x \in (-\infty, t]$, then we add the equations for $i = 1, 2, ..., n$. As a result we obtain

$$\sum_{i=1}^{n} \eta_i \int\limits_{-\infty}^{t} \psi_i^{(m+1)}(x)dx \leq$$

$$\leq \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \eta_j \int\limits_{-\infty}^{t} \gamma_{ij}(x)dx + \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \int\limits_{-\infty}^{t} \int\limits_{-\infty}^{x} K_{ij}(x,y)\psi_j^{(m+1)}(y)dydx =$$

$$= \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \eta_j \int\limits_{-\infty}^{t} \gamma_{ij}(x)dx + \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \int\limits_{-\infty}^{t} \int\limits_{-\infty}^{0} K_{ij}(x,x+\tau)\psi_j^{(m+1)}(x+\tau)d\tau dx =$$

$$= \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \eta_j \int\limits_{-\infty}^{t} \gamma_{ij}(x)dx + \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \int\limits_{-\infty}^{0} \int\limits_{-\infty}^{t} K_{ij}(x,x+\tau)\psi_j^{(m+1)}(x+\tau)dxd\tau =$$

$$= \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \eta_j \int\limits_{-\infty}^{t} \gamma_{ij}(x)dx + \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \int\limits_{-\infty}^{-\delta_0} \int\limits_{-\infty}^{t} K_{ij}(x,x+\tau)\psi_j^{(m+1)}(x+\tau)dxd\tau +$$

$$+ \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \int\limits_{-\delta_0}^{0} \int\limits_{-\infty}^{t} K_{ij}(x,x+\tau)\psi_j^{(m+1)}(x+\tau)dxd\tau =$$

$$= \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \eta_j \int\limits_{-\infty}^{t} \gamma_{ij}(x)dx + \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \int\limits_{-\infty}^{-\delta_0} \int\limits_{-\infty}^{t+\tau} K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau +$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\delta_0}^{0}\int_{-\infty}^{t+\tau}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau\leq$$

$$\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{-\delta_0}\int_{-\infty}^{t-\delta_0}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\delta_0}^{0}\int_{-\infty}^{t}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{t-\delta_0}\int_{-\infty}^{t-\delta_0}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{t-\delta_0}^{-\delta_0}\int_{-\infty}^{t-\delta_0}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{t}\psi_j^{(m+1)}(z)\int_{-\delta_0}^{0}K_{ij}(z-\tau,z)d\tau dz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int_{-\infty}^{t-\delta_0}K_{ij}(z-\tau,z)d\tau dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int_{t-\delta_0}^{-\delta_0}K_{ij}(z-\tau,z)d\tau dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{t}\psi_j^{(m+1)}(z)\int_{-\delta_0}^{0}K_{ij}(z-\tau,z)d\tau dz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int_{-\infty}^{-\delta_0}K_{ij}(z-\tau,z)d\tau dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{t}\psi_j^{(m+1)}(z)\int_{-\delta_0}^{0}K_{ij}(z-\tau,z)d\tau dz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int_{-\infty}^{0}K_{ij}(z-\tau,z)d\tau dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int_{-\delta_0}^{0}K_{ij}(z-\tau,z)d\tau dz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int_{z}^{\infty}K_{ij}(y,z)dydz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz\leq$$

$$\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}a_{ij}\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)dz\sum_{i=1}^{n}a_{ji}\eta_i+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz,$$

from which it follows that

$$\sum_{j=1}^{n}\eta_j\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+$$
$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz. \tag{2.4}$$

Observe that

$$a_{ij}-\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dy\geq\int\limits_{z}^{\infty}K_{ij}(y,z)dy-\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dy=$$
$$=\int\limits_{z+\delta_0}^{\infty}K_{ij}(y,z)dy=\int\limits_{\delta_0}^{\infty}K_{ij}(z+u,z)du\geq\varepsilon_{ij}\text{ for }z\leq 0,\ i,j=1,2,...,n\ . \tag{2.5}$$

Considering (2.4) and (2.5), we obtain

$$\sum_{j=1}^{n}\eta_j\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz\leq$$
$$\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\left(\sum_{i=1}^{n}a_{ji}\eta_i-\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\right)dz=$$
$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\eta_j\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz-\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz,$$

which is the same as

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx. \tag{2.6}$$

Let $p<0$ be an arbitrary number. We integrate both sides of (2.6) with respect to t from p to 0. Then, according to $b_1)$, $b_3)$ and Fubini's theorem from (2.6) we obtain

$$\begin{aligned}0\leq\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{p}^{0}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dzdt\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}\int\limits_{-\infty}^{t}\gamma_{ij}(x)dxdt=\\=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx<+\infty.\end{aligned}\tag{2.7}$$

By passing to the limit as $p\to-\infty$, we obtain

$$0\leq\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{-\infty}^{0}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dzdt\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx$$

or

$$\begin{aligned}0\leq\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{-\infty}^{0}\int\limits_{-\delta_0}^{0}\psi_j^{(m+1)}(t+\tau)d\tau dt\leq\\\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx<+\infty,\ m=0,1,2,\dots\ .\end{aligned}\tag{2.8}$$

By changing the order of integration in (2.8), we have

$$\begin{aligned}0\leq\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{-\delta_0}^{0}\int\limits_{-\infty}^{0}\psi_j^{(m+1)}(t+\tau)dtd\tau\leq\\\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx,\ m=0,1,2,\dots\ ,\end{aligned}$$

from which it follows that

$$0 \leq \sum_{j=1}^{n} \sum_{i=1}^{n} \varepsilon_{ij} \eta_i \int_{-\delta_0}^{0} \int_{-\infty}^{-\delta_0} \psi_j^{(m+1)}(y) dy d\tau \leq$$

$$\leq \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \eta_j \int_{-\infty}^{0} (-x) \gamma_{ij}(x) dx, \;\; m = 0, 1, 2, ...$$

or

$$0 \leq \sum_{j=1}^{n} \sum_{i=1}^{n} \varepsilon_{ij} \eta_i \int_{-\infty}^{-\delta_0} \psi_j^{(m+1)}(y) dy \leq$$
$$\leq \frac{1}{\delta_0} \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} \eta_j \int_{-\infty}^{0} (-x) \gamma_{ij}(x) dx, \;\; m = 0, 1, 2, ... \;. \tag{2.9}$$

Due to 1)-3) we have

$$0 \leq \int_{-\delta_0}^{0} \psi_j^{(m+1)}(y) dy \leq \eta_j \delta_0, \;\; j = 1, 2, ..., n, \;\; m = 0, 1, 2, ... \;. \tag{2.10}$$

We denote

$$\mu := \min_{1 \leq j \leq n} \sum_{i=1}^{n} \varepsilon_{ij} \eta_i. \tag{2.11}$$

Then, from (2.9), in particular, it follows that

$$0 \leq \int_{-\infty}^{-\delta_0} \psi_j^{(m+1)}(y) dy \leq \frac{1}{\mu \delta_0} \cdot \sum_{i=1}^{n} \eta_i \sum_{i=1}^{n} \eta_j \int_{-\infty}^{0} (-x) \gamma_{ij}(x) dx,$$
$$m = 0, 1, 2, ..., \;\; j = 1, 2, ..., n. \tag{2.12}$$

Therefore, inequalities (2.10) and (2.12) entail the following two-sided estimate

$$0 \leq \int_{-\infty}^{0} \psi_j^{(m+1)}(y) dy \leq (\max_{1 \leq j \leq n} \eta_j) \delta_0 + \frac{1}{\mu \delta_0} \sum_{i=1}^{n} \eta_i \sum_{i=1}^{n} \eta_j \int_{-\infty}^{0} (-x) \gamma_{ij}(x) dx < +\infty,$$
$$j = 1, 2, ..., n, \;\; m = 0, 1, 2, ... \;. \tag{2.13}$$

From 1)-4) and (2.11) it follows that the sequence of measurable on $\mathbb{R}$ vector-functions $\psi^{(m)}(x) = \left(\psi_1^{(m)}(x), ..., \psi_n^{(m)}(x)\right)^T$, $m = 0, 1, 2, ...$ has a pointwise limit when $m \to \infty$:

$$\lim_{m \to \infty} \psi^{(m)}(x) = \psi(x),$$

additionally, the limit vector-function $\psi(x) = (\psi_1(x), ..., \psi_n(x))^T$ according to B. Levi's theorem (see [11]) satisfies system (2.1). Once again using 1)-4) and (2.11), we can state that

$$g_i(x) \leq \psi_i(x) \leq \eta_i, \;\; x \in \mathbb{R}, \;\; i = 1, 2, ..., n, \tag{2.14}$$

$$0 \leq \int\limits_{-\infty}^{0} \psi_j(x)dx \leq (\max_{1\leq j\leq n} \eta_j)\delta_0 + \frac{1}{\mu\delta_0}\sum_{i=1}^{n}\eta_i\sum_{i=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx < +\infty, \quad i=1,2,\ldots\,. \tag{2.15}$$

We now consider the second auxiliary linear nonhomogeneous system of integral equations on $\mathbb{R}$:

$$\psi_i^*(x) = g_i^*(x) + \sum_{j=1}^{n}\int\limits_{-\infty}^{x} K_{ij}(x,t)\psi_j^*(t)dt, \ \ x\in\mathbb{R}, \ \ i=1,2,...,n \tag{2.16}$$

with respect to the unknown vector function $\psi^*(x) = (\psi_1^*(x), ..., \psi_n^*(x))^T$, where

$$g_i^*(x) = \sum_{j=1}^{n}\eta_j\gamma_{ij}(x), \ \ i=1,2,...,n, \ \ x\in\mathbb{R}. \tag{2.17}$$

Repeating the same reasoning as for system (2.1), wherein taking $\psi_i(x)$, $i=1,2,...,n$ as the zero approximation, we can prove that system of integral equations (2.16) has a componentwise nonnegative and bounded solution $\psi^*(x) = (\psi_1^*(x), ..., \psi_n^*(x))^T$, and, besides that

$$g_i(x) \leq \psi_i(x) \leq \psi_i^*(x) \leq \eta_i, \ \ x\in\mathbb{R}, \ \ i=1,2,...,n, \tag{2.18}$$

$$0 \leq \int\limits_{-\infty}^{0} \psi_j^*(x)dx \leq (\max_{1\leq j\leq n} \eta_j)\delta_0 + \frac{1}{\mu\delta_0}\sum_{i=1}^{n}\eta_i\sum_{i=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx < +\infty, \quad i=1,2,\ldots\,. \tag{2.19}$$

On the other hand, note that system of integral equations (2.16) also has a trivial solution $\eta = (\eta_1, ..., \eta_n)^T$. Indeed, considering b_1), (2.17) and (1.3), we obtain

$$g_i^*(x) + \sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{x} K_{ij}(x,t)dt = \sum_{j=1}^{n}\eta_j(a_{ij} - \gamma_{ij}(x)) + \sum_{j=1}^{n}\eta_j\gamma_{ij}(x) = \sum_{j=1}^{n}a_{ij}\eta_j = \eta_i, \quad i=1,2,...,n.$$

From (2.18) and (2.19) it follows that $\psi_i^*(x) \not\equiv \eta_i$, $x\in\mathbb{R}$, $i=1,2,...,n$. Therefore,

$$\Phi_i(x) := \eta_i - \psi_i^*(x) \geq 0, \ \ \Phi_i(x) \not\equiv 0, \ \ x\in\mathbb{R}, \ \ i=1,2,...,n$$

and also satisfies the homogeneous system of integral equations

$$\Phi_i(x) = \sum_{j=1}^{n}\int\limits_{-\infty}^{x} K_{ij}(x,t)\Phi_j(t)dt, \ \ x\in\mathbb{R}, \ \ i=1,2,...,n. \tag{2.20}$$

We now prove that there exists

$$\lim_{x\to-\infty}\Phi_i(x) = \eta_i, \ \ i=1,2,...,n.$$

Indeed, for negative values of x from (2.16) due to a) and b_1) we conclude that

$$0 \leq \psi_i^*(x) \leq \sum_{j=1}^{n}\eta_j\gamma_{ij}(x) + \sum_{j=1}^{n}\sup_{(x,t)\in\mathbb{R}^2}(K_{ij}(x,t))\cdot\int\limits_{-\infty}^{x}\psi_j^*(t)dt \to 0, \text{ when } x\to-\infty,$$

from which we obtain that there exists $\lim\limits_{x\to-\infty}\psi_i^*(x)=0,\ i=1,2,...,n$. Therefore, there exists $\lim\limits_{x\to-\infty}\Phi_i(x)=\eta_i,\ i=1,2,...,n$. Since $\psi_i^*\in L_1(-\infty,0),\ i=1,2,...,n$, hence $\eta_i-\Phi_i\in L_1(-\infty,0)$, $i=1,2,...,n$.

Finally, we consider the following family of successive approximations for the system (1.1):

$$f_{i,\gamma}^{(m+1)}(x)=\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\{f_{j,\gamma}^{(m)}(t)+\omega_{ij}(t,f_{j,\gamma}^{(m)}(t))\}dt,$$
$$f_{i,\gamma}^{(0)}(x)=\gamma\Phi_i(x),\ m=0,1,2,...,\ i=1,2,...,n,\ x\in\mathbb{R}, \tag{2.21}$$

where $\gamma\in(0,+\infty)$ is an arbitrary parameter.

By using mathematical induction it is not hard to verify that for every $\gamma\in(0,+\infty)$

$\Gamma_1)$
$$f_{i,\gamma}^{(m)}(x)\text{ are measurable on }\mathbb{R},\ i=1,2,...,n,\ m=0,1,2,..., \tag{2.22}$$

$\Gamma_2)$
$$f_{i,\gamma}^{(m)}(x)\uparrow\text{ with respect to }m,\ i=1,2,...,n,\ x\in\mathbb{R}. \tag{2.23}$$

We will now prove that

$\Gamma_3)$
$$f_{i,\gamma}^{(m)}(x)\le\gamma\Phi_i(x)+\psi_i(x),\ i=1,2,...,n,\ x\in\mathbb{R}. \tag{2.24}$$

For $m=0$ the given inequality directly follows from the definition of the zero approximation with consideration of nonnegativity of the functions $\{\psi_i(x)\}_{i=1}^n$ on $\mathbb{R}$. Assume that (2.24) holds for some $m\in\mathbb{N}$. Then, taking into account (2.1), (2.20) and A)-C), from (2.21) we get

$$f_{i,\gamma}^{(m+1)}(x)\le\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\{\gamma\Phi_j(t)+\psi_j(t)+\omega_{ij}(t,\gamma\Phi_j(t)+\psi_j(t))\}dt\le$$
$$\le\gamma\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\Phi_j(t)dt+\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\psi_j(t)dt+\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\beta_{ij}(t)dt\le$$
$$\le\gamma\Phi_i(x)+\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\psi_j(t)dt+\sum_{j=1}^{n}\beta_{ij}(x)(a_{ij}-\gamma_{ij}(x))=\gamma\Phi_i(x)+\psi_i(x),$$
$$i=1,2,...,n,\ x\in\mathbb{R}.$$

Let us prove that

$\Gamma_4)$ If $\gamma_1,\gamma_2\in(0,+\infty)$ are arbitrary parameters and $\gamma_1>\gamma_2$, then

$$f_{i,\gamma_1}^{(m)}(x)-f_{i,\gamma_2}^{(m)}(x)\ge(\gamma_1-\gamma_2)\Phi_i(x),\ x\in\mathbb{R},\ i=1,2,...,n,\ m=0,1,2,... \tag{2.25}$$

Indeed, in the case of $m=0$ inequalities (2.25) are transformed to equalities by the definition of the zero approximation in iterations (2.21). Let (2.25) hold for some natural m. Then, from (2.21) due to conditions B) and (2.20) we will obtain

$$f_{i,\gamma_1}^{(m+1)}(x) - f_{i,\gamma_2}^{(m+1)}(x) =$$

$$= \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{f_{j,\gamma_1}^{(m)}(t) - f_{j,\gamma_2}^{(m)}(t) + \omega_{ij}(t, f_{j,\gamma_1}^{(m)}(t)) - \omega_{ij}(t, f_{j,\gamma_2}^{(m)}(t))\}dt \geq$$

$$\geq (\gamma_1 - \gamma_2) \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{\Phi_j(t) + \omega_{ij}(t, f_{j,\gamma_2}^{(m)}(t) + (\gamma_1 - \gamma_2)\Phi_j(t)) - \omega_{ij}(t, f_{j,\gamma_2}^{(m)}(t))\}dt \geq$$

$$\geq (\gamma_1 - \gamma_2) \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\Phi_j(t)dt = (\gamma_1 - \gamma_2)\Phi_i(x), \ i = 1, 2, ..., n, \ x \in \mathbb{R}.$$

So, from Γ_1) - Γ_4) it follows that the sequence of measurable vector functions $f_\gamma^{(m)}(x) = (f_{1,\gamma}^{(m)}(x), ..., f_{n,\gamma}^{(m)}(x))^T$, $m = 0, 1, 2, ...$, for every $\gamma \in (0, +\infty)$ has a pointwise limit when $m \to \infty$: $\lim\limits_{m\to\infty} f_\gamma^{(m)}(x) = f^\gamma(x) = (f_1^\gamma(x), ..., f_n^\gamma(x))^T$, moreover,

$$\gamma\Phi_j(x) \leq f_j^\gamma(x) \leq \gamma\Phi_j(x) + \psi_j(x), \ j = 1, 2, ..., n, \ x \in \mathbb{R}, \tag{2.26}$$

$$f_j^{\gamma_1}(x) - f_j^{\gamma_2}(x) \geq (\gamma_1 - \gamma_2)\Phi_j(x), \ j = 1, 2, ..., n, \ x \in \mathbb{R}, \tag{2.27}$$

where $\gamma_1, \gamma_2 \in (0, +\infty)$, $\gamma_1 > \gamma_2$ are arbitrary parameters. Considering conditions D), b) according to B. Levi's theorem for every $\gamma \in (0, +\infty)$ the vector function $f^\gamma(x) = (f_1^\gamma(x), ..., f_n^\gamma(x))^T$ satisfies system of NIE (1.1).

Since $\lim\limits_{x\to-\infty} \psi_i^*(x) = 0$, $i = 1, 2, ..., n$, from (2.18) it follows that

$$\lim_{x\to-\infty} \psi_i(x) = 0, \ i = 1, 2, ..., n. \tag{2.28}$$

From (2.15), (2.26) and (2.28) directly follows that

$$\lim_{x\to-\infty} \{f_i^\gamma(x) - \gamma\Phi_i(x)\} = 0, \ i = 1, 2, ..., n, \gamma \in (0, +\infty), \tag{2.29}$$

$$0 \leq f_i^\gamma - \gamma\Phi_i \in L_1(-\infty, 0), \ i = 1, 2, ..., n, \ \gamma \in (0, +\infty). \tag{2.30}$$

Since $\lim\limits_{x\to-\infty} (\eta_i - \Phi_i(x)) = 0$, $\eta_i - \Phi_i \in L_1(-\infty, 0)$, $i = 1, 2, ..., n$, hence there exists $\lim\limits_{x\to-\infty} f_i^\gamma(x) = \gamma\eta_i$, and from the estimate

$$0 \leq |\gamma\eta_i - f_i^\gamma(x)| \leq \gamma(\eta_i - \Phi_i(x)) + f_i^\gamma(x) - \gamma\Phi_i(x) \in L_1(-\infty, 0), \ i = 1, 2, ..., n$$

it follows that $\gamma\eta_i - f_i^\gamma \in L_1(-\infty, 0)$, $i = 1, 2, ..., n$, $\gamma \in (0, +\infty)$. □

3 Solvability of system of NIE (1.2). Examples

In the current section with the use of the results of Theorem 2.1 and some geometrical inequalities for concave functions, we will deal with the problem of solvability for system of NIE (1.2).

Theorem 3.1. *Under conditions a) - c), I) - III) and A) - D) system of NIE* (1.2) *has componentwise nonnegative (nontrivial) and bounded on* $\mathbb{R}$ *solution* $\varphi(x) = (\varphi_1(x), ..., \varphi_n(x))^T$*, such that*

$$\lim_{x\to-\infty} \varphi_j(x) = \eta_j^*$$

and $\eta_j^* - \varphi_j \in L_1(-\infty, 0)$*,* $j = 1, 2, ..., n$ *where* $\eta*$ *is defined in III).*

Proof. Due to Theorem 2.1 for the number $\gamma^* = \alpha$ corresponds a solution $f^{\gamma^*}(x) = (f_1^{\gamma^*}(x), ..., f_n^{\gamma^*}(x))^T$ of system (1.1) with the properties

$$\alpha\Phi_j(x) \leq f_j^{\gamma^*}(x) \leq \alpha\Phi_j(x) + \psi_j(x), \ j = 1, 2, ..., n, \ x \in \mathbb{R}, , \tag{3.1}$$

$$\lim_{x\to-\infty} f_j^{\gamma^*}(x) = \alpha \cdot \eta_j = \eta_j^*, \ \eta_j^* - f_j^{\gamma^*} \in L_1(-\infty, 0), \ j = 1, 2, ..., n. \tag{3.2}$$

Consider the following iterations for system (1.2):

$$\begin{gathered}\varphi_i^{(m+1)}(x) = \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{G_j(\varphi_j^{(m)}(t)) + \omega_{ij}(t, \varphi_j^{(m)}(t))\}dt, \\ \varphi_i^{(0)}(x) = f_i^{\gamma^*}(x), \ m = 0, 1, 2, ..., \ i = 1, 2, ..., n, \ x \in \mathbb{R}.\end{gathered} \tag{3.3}$$

Using I), II), B), D) and *a*) with induction on m it is easy to check that

E_1) $\varphi_i^{(m)}(x)$ are measurable with respect to x on $\mathbb{R}$, $m = 0, 1, 2, ...,$ $i = 1, 2, ..., n$,

E_2) $\varphi_i^{(m)}(x) \uparrow$ with respect to m, $x \in \mathbb{R}$, $i = 1, 2, ..., n$.

Below we will prove that

E_3) $\varphi_i^{(m)}(x) \leq \eta_i^* + \psi_i(x)$, $x \in \mathbb{R}$, $m = 0, 1, 2, ...,$ $i = 1, 2, ..., n$.

In the case when $m = 0$ inequalities E_3) directly folow from (3.1) and III), by taking into account the estimates $\Phi_i(x) \leq \eta_i$, $i = 1, 2, ..., n$, $x \in \mathbb{R}$. Assume that E_3) holds for some natural m. Then, using the following inequalities

$$G_j(\eta_j^* + u) \leq \eta_j^* + u, \ u \geq 0, \ j = 1, 2, ..., n$$

(which follow from the concaveness of the functions $\{G_j(u)\}_{j=1}^n$ (see Fig. 1.)), and also C), III), II), (1.3), (2.1) and (2.14), from (3.3) we have

$$\begin{gathered}\varphi_i^{(m+1)}(x) \leq \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{G_j(\eta_j^* + \psi_j(t)) + \omega_{ij}(t, \eta_j^* + \psi_j(t))\}dt \leq \\ \leq \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)(\eta_j^* + \psi_j(t) + \beta_{ij}(t))dt \leq \sum_{j=1}^{n} \eta_j^*(a_{ij} - \gamma_{ij}(x)) + \\ + \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\psi_j(t)dt + g_i(x) \leq \eta_i^* + \psi_i(x), \ i = 1, 2, ..., n, \ x \in \mathbb{R}.\end{gathered}$$

So, from E_1)-E_3) we conclude that the sequence of measurable vector functions $\varphi^{(m)}(x) = (\varphi_1^{(m)}(x), ..., \varphi_n^{(m)}(x))^T$, $m = 0, 1, 2, ...$ has a pointwise limit when $m \to \infty$: $\lim\limits_{m\to\infty} \varphi^{(m)}(x) = \varphi(x) = (\varphi_1(x), ..., \varphi_n(x))^T$, moreover,

$$f_i^{\gamma^*}(x) \leq \varphi_i(x) \leq \eta_i^* + \psi_i(x), \ i = 1, 2, ..., n, \ x \in \mathbb{R}. \tag{3.4}$$

Using conditions I) and D) due to B. Levi's theorem we obtain that $\varphi(x) = (\varphi_1(x), ..., \varphi_n(x))^T$ is a solution to system of NIE (1.2). From (3.4), (3.2), (2.15) and (2.28) it follows that $\lim\limits_{x\to-\infty} \varphi_i(x) = \eta_i^*$ and $\eta_i^* - \varphi_i \in L_1(-\infty, 0)$, $i = 1, 2, ..., n$. □

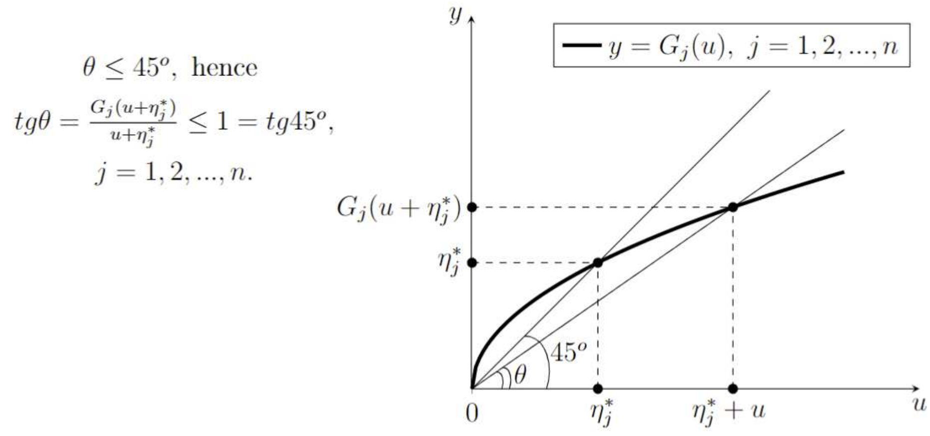

Figure 1:

At the end we will present specific examples of monotonous kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$ and nonlinearities $\{G_j(u)\}_{j=1}^{n}$, $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ that satisfy the conditions of the proven Theorems 2.1 and 3.1.

Firstly, we will give examples of matrix kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$. Let functions $\{\lambda_{ij}(x)\}_{i,j=1}^{n\times n}$ be defined and continuous on the set $\mathbb{R}$ and satisfy the following conditions

$F_1)$ $0<\rho_{ij}:=\inf\limits_{x\in\mathbb{R}}\lambda_{ij}(x)\leq\lambda_{ij}(x)\leq 1,\ \lambda_{ij}(x)\not\equiv 1,\ x\in\mathbb{R},\ i,j=1,2,...,n,$

$F_2)$ $\lim\limits_{x\to-\infty}\lambda_{ij}(x)=1,\ x(1-\lambda_{ij}(x))\in L_1(-\infty,0),\ i,j=1,2,...,n.$

Also, let functions $\{\overset{\circ}{K}_{ij}(x)\}_{i,j=1}^{n\times n}$ be continuous on $\mathbb{R}$ and satisfy the following conditions:

$H_1)$ $\overset{\circ}{K}_{ij}(x)>0,\ x\in\mathbb{R},\ \overset{\circ}{K}_{ij}(-t)=\overset{\circ}{K}_{ij}(t),\ t\geq 0,\ i,j=1,2,...,n,$

$H_2)$ $\overset{\circ}{K}_{ij}\in L_\infty(\mathbb{R}),\ a_{ij}=\int\limits_0^\infty\overset{\circ}{K}_{ij}(x)dx,\ i,j=1,2,...,n.$

Then we can choose the following classes of matrix functions as matrix kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$:

$W_1)$ $K_{ij}(x,t)=\lambda_{ij}(x)\cdot\overset{\circ}{K}_{ij}(x-t),\ (x,t)\in\mathbb{R}^2,\ i,j=1,2,...,n,$

$W_2)$ $K_{ij}(x,t)=\frac{\lambda_{ij}(t)+\lambda_{ij}(x)}{2}\cdot\overset{\circ}{K}_{ij}(x-t),\ (x,t)\in\mathbb{R}^2,\ i,j=1,2,...,n,$

$W_3)$ $K_{ij}(x,t)=\lambda_{ij}(x+t)\cdot\overset{\circ}{K}_{ij}(x-t),\ (x,t)\in\mathbb{R}^2,\ i,j=1,2,...,n.$

Let us take a look at example $W_3)$. Condition $a)$ directly follows from $F_1)$ and $H_1), H_2)$. We will now verify condition $b)$. We have

$$\gamma_{ij}(x)=a_{ij}-\int\limits_{-\infty}^{x}\lambda_{ij}(x+t)\overset{\circ}{K}_{ij}(x-t)dt\geq a_{ij}-\int\limits_{-\infty}^{x}\overset{\circ}{K}_{ij}(x-t)dt=0, i,j=1,2,...,n, x\in\mathbb{R}.$$

On the other hand, considering equations $H_2)$, $F_2)$, $F_1)$ and $H_1)$, we obtain

$$\int_{-\infty}^{0}(-x)\gamma_{ij}(x)dx=\int_{-\infty}^{0}(-x)\int_{-\infty}^{x}(1-\lambda_{ij}(x+t))\overset{\circ}{K}_{ij}(x-t)dtdx=$$

$$=\int_{-\infty}^{0}(-x)\int_{0}^{\infty}(1-\lambda_{ij}(2x-y))\overset{\circ}{K}_{ij}(y)dydx=\int_{0}^{\infty}\overset{\circ}{K}_{ij}(y)\int_{-\infty}^{0}(-x)(1-\lambda_{ij}(2x-y))dxdy=$$

$$=\frac{1}{2}\int_{0}^{\infty}\overset{\circ}{K}_{ij}(y)\int_{-\infty}^{-y}\left(\frac{-t-y}{2}\right)(1-\lambda_{ij}(t))dtdy\le\frac{1}{4}\int_{0}^{\infty}\overset{\circ}{K}_{ij}(y)\int_{-\infty}^{0}(-t-y)\,(1-\lambda_{ij}(t))dtdy\le$$

$$\le\frac{1}{4}\int_{0}^{\infty}\overset{\circ}{K}_{ij}(y)dy\int_{-\infty}^{0}(-t)\,(1-\lambda_{ij}(t))dt=\frac{a_{ij}}{4}\int_{-\infty}^{0}(-t)\,(1-\lambda_{ij}(t))dt<+\infty,\ i,j=1,2,...,n.$$

Now, let us verify that $\lim\limits_{x\to-\infty}\gamma_{ij}(x)=0,\ i,j=1,2,...,n.$ Due to conditions $a)$, $F_1)$, and $F_2)$, $H_2)$ we have

$$0\le\gamma_{ij}(x)=\int_{-\infty}^{x}\overset{\circ}{K}_{ij}(x-t)(1-\lambda_{ij}(x+t))dt\le M\int_{-\infty}^{x}(1-\lambda_{ij}(x+t))dt=$$

$$=M\int_{-\infty}^{2x}(1-\lambda_{ij}(y))dy\to0,\quad\text{when } x\to-\infty,\ \text{where } M:=\max_{1\le i,j\le n}(\sup_{\tau\in\mathbb{R}}\overset{\circ}{K}_{ij}(\tau)).$$

Finally, let us verify condition $c)$. Due to $F_1)$ and $H_2)$ we obtain

$$\int_{\delta_0}^{\infty}K_{ij}(x+y,x)dy=\int_{\delta_0}^{\infty}\lambda_{ij}(2x+y)\overset{\circ}{K}_{ij}(y)dy\ge\rho_{ij}\cdot\widetilde{a}_{ij},\ \text{where } \widetilde{a}_{ij}=\int_{\delta_0}^{\infty}\overset{\circ}{K}_{ij}(y)dy,$$

$$i,j=1,2,...,n,\ x\in\mathbb{R}.$$

Therefore $\varepsilon_{ij}\ge\rho_{ij}\cdot\widetilde{a}_{ij}>0, i,j=1,2,...,n.$ Let us now give examples of nonlinearities $\{G_j(u)\}_{j=1}^{n}$ and $\{\omega_{ij}(u)\}_{i,j=1}^{n\times n}$.

Examples of $\{G_j(u)\}_{j=1}^{n}$:

$Q_1)$ $G_j(u)=\left(\eta_j^*\right)^{\frac{p-1}{p}}\sqrt[p]{u},\ j=1,2,...,n,$ where $p\ge2$ is a natural number, $u\in[0,+\infty)$,

$Q_2)$ $G_j(u)=\dfrac{\eta_j^*}{1-e^{-\eta_j^*}}\left(1-e^{-u}\right),\ j=1,2,...,n,\ u\in[0,+\infty)$,

$Q_3)$ $G_j(u)=\dfrac{1}{2}\left(\sqrt[p]{u}\left(\eta_j^*\right)^{\frac{p-1}{p}}+\dfrac{\eta_j^*}{1-e^{-\eta_j^*}}\left(1-e^{-u}\right)\right),\ j=1,2,...,n,\ u\in[0,+\infty)$.

Examples of $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$:

$V_1)$ $\omega_{ij}(t,u)=\beta_{ij}(t)(1-e^{-u}), u\in[0,+\infty),\ t\in\mathbb{R}, i,j=1,2,...,n,$

$V_2)$ $\omega_{ij}(t,u)=\beta_{ij}(t)\frac{u}{u+1},\ u\in[0,+\infty),\ t\in\mathbb{R},\ i,j=1,2,...,n,$

$V_3)$ $\omega_{ij}(t,u)=\beta_{ij}(t)\cdot th(u),\ u\in[0,+\infty),\ t\in\mathbb{R},\ i,j=1,2,...,n,$

where

$$th(u):=\frac{e^u-e^{-u}}{e^u+e^{-u}}.$$

Note that in all examples $V_1)-V_3)$ it is assumed that $\beta_{ij}\in C(\mathbb{R}),\ i,j=1,2,...,n.$ Let us verify conditions I)-III) on the example $Q_2)$. Firstly, it is obvious that $G_j\in C[0,+\infty)$, $G_j(0)=0$, $j=1,2,...,n.$ Since $G_j''(u)=-\frac{\eta_j^*}{1-e^{-\eta_j^*}}\cdot e^{-u}<0,\ u\in[0,+\infty)$, $j=1,2,...,n$, therefore, the functions $\{G_j(u)\}_{j=1}^n$ are concave. $G_j'(u)=\frac{\eta_j^*}{1-e^{-\eta_j^*}}\cdot e^{-u}>0$, $u\in[0,+\infty)$, $j=1,2,...,n$, $G_j(u)\uparrow$ with respect to u on $[0,+\infty)$, $j=1,2,...,n.$ Obviously, $G_j(\eta_j^*)=\eta_j^*$, $j=1,2,...,n.$ It remains to show that $G_j(u)\geq u$, $u\in[0,\eta_j^*]$, $j=1,2,...,n.$ Let us consider the following functions on the segment $[0,\eta_j^*]$:

$$\chi_j(u)=\frac{\eta_j^*}{1-e^{-\eta_j^*}}(1-e^{-u})-u,\ u\in[0,\eta_j^*],\ j=1,2,...,n.$$

Note that $\chi_j(0)=0,\ \chi_j(\eta_j^*)=0$, $\chi_j''(u)=-\frac{\eta_j^*}{1-e^{-\eta_j^*}}\cdot e^{-u}<0$, $j=1,2,...,n.$ Therefore $\chi_j(u)\geq 0,\ u\in[0,\eta_j^*],\ j=1,2,...,n.$

Let us now verify the conditions A) - D) for the example $V_2)$. Firstly, it is obvious that $\omega_{ij}(t,0)=0,\ t\in\mathbb{R},\ i,j=1,2,...,n.$ Since

$$\frac{\partial\omega_{ij}(t,u)}{\partial u}=\beta_{ij}(t)\frac{1}{(u+1)^2}>0,\ u\in[0,+\infty),\ t\in\mathbb{R},\ i,j=1,2,...,n,$$

$\omega_{ij}(t,u)\uparrow$ with respect to u on the set $[0,+\infty)$, $i,j=1,2,...,n.$ From the representation of $V_2)$ it follows that

$$\sup_{u\in[0,+\infty)}(\omega_{ij}(t,u))=\beta_{ij}(t),\ t\in\mathbb{R},\ i,j=1,2,...,n.$$

For the rest of examples $Q_1)$, $Q_3)$, $V_1)$ and $V_2)$ the verification of the corresponding conditions is made similarly.

For the sake of completeness, let us also give specific examples of $\{\overset{\circ}{K}_{ij}(x)\}_{i,j=1}^{n\times n}$, $\{\lambda_{ij}(x)\}_{i,j=1}^{n\times n}$ and $\{\beta_{ij}(x)\}_{i,j=1}^{n\times n}$.

Examples of $\{\overset{\circ}{K}_{ij}(x)\}_{i,j=1}^{n\times n}$:

$T_1)$ $\overset{\circ}{K}_{ij}(x)=\frac{2a_{ij}}{\sqrt{\pi}}e^{-x^2},\ x\in\mathbb{R},\ i,j=1,2,...,n,$

$T_2)$ $\overset{\circ}{K}_{ij}(x)=\int\limits_a^b e^{-|x|s}d\sigma_{ij}(s),\ x\in\mathbb{R},\ i,j,=1,2,...,n,$

where $\sigma_{ij}(s),\ i,j=1,2,...,n$ are nondecreasing and continuous functions on the set $[a,b),\ 0<a<b\leq+\infty$, moreover,

$$\int\limits_a^b\frac{1}{s}d\sigma_{ij}(s)=a_{ij},\ i,j=1,2,...,n.$$

Examples of $\{\lambda_{ij}(x)\}_{i,j=1}^{n\times n}$:

$S_1)$ $\lambda_{ij}(x)=1-(1-\rho_{ij})D(x),\ x\in\mathbb{R},\ i,j=1,2,...,n,$ where $D(x):=\begin{cases}e^x, & x<0\\ 1, & x\geq 0\end{cases},$

$S_2)$ $\lambda_{ij}(x)=1-\dfrac{(1-\rho_{ij})}{2}\cdot(th(x)+1),\ x\in\mathbb{R},\ i,j=1,2,...,n.$

Examples of $\{\beta_{ij}(x)\}_{i,j=1}^{n\times n}$:

$J_1)$ $\beta_{ij}(x)=\dfrac{\eta_j}{a_{ij}}\gamma_{ij}(x),\ x\in\mathbb{R},\ i,j=1,2,...,n,$ given that $\gamma_{ij}(x)\uparrow$ with respect to x on $\mathbb{R},\ i,j=1,2,...,n,$

$J_2)$ $\beta_{ij}(x)=\dfrac{\eta_j\gamma_{ij}(x)}{a_{ij}-\gamma_{ij}(x)},\ x\in\mathbb{R},\ i,j=1,2,...,n,$ given that $\gamma_{ij}(x)\uparrow$ with respect to x on $\mathbb{R},\ i,j=1,2,...,n.$

Let us take a look at example J_2). First of all let us give examples of functions $\{\gamma_{ij}(x)\}_{i,j=1}^{n}$ that satisfy the condition in J_2). For example in the case of W_1) the functions $\gamma_{ij}(x)$ allow the following representation:

$$\gamma_{ij}(x)=\int\limits_{-\infty}^{x}\overset{\circ}{K}_{ij}(x-t)(1-\lambda_{ij}(x))dt=a_{ij}(1-\lambda_{ij}(x)),\ x\in\mathbb{R},\ i,j=1,2,...,n.$$

Note that in examples S_1) and S_2) the functions $(1-\lambda_{ij}(x)),\ i,j=1,2,...,n$ are increasing on $\mathbb{R}$. Therefore, if as a $\lambda_{ij}(x),\ i,j=1,2,...,n$ we choose examples S_1) and S_2) we will obtain the monotonicity of the functions $\{\gamma_{ij}(x)\}_{i,j=1}^{n}$ on the set $\mathbb{R}$. But in that case the functions $\{\beta_{ij}(x)\}_{i,j=1}^{n\times n}$ in examples J_2) also will be nondecreasing on the set $\mathbb{R}$. For example J_2) inequality (1.4) is automatically satisfied. The corresponding conditions on the functions $\{\beta_{ij}(x)\}_{i,j=1}^{n\times n}$ for example J_1) are verified similarly.

It is interesting to note, that the problem of uniqueness of the solution for system (1.2) in conical segments $\{[0,\eta_j^*]\}_{j=1}^{n}$ still remains open. For system (1.1) the uniqueness of the solution (in the class of bounded on $\mathbb{R}$ vector-functions) fails, since, according to the results of Theorem 2.1, system (1.1) has a one-parameter family of nonnegative (nontrivial) and bounded (on $\mathbb{R}$) solutions.

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CLASSES OF KERNELS AND CONTINUITY PROPERTIES OF THE TANGENTIAL GRADIENT OF AN INTEGRAL OPERATOR IN HÖLDER SPACES ON A MANIFOLD

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Abstract. We prove multiplication and embedding theorems for classes of kernels of integral operators in subsets of metric spaces with a measure. Then we prove a tangential differentiation theorem with respect to a semi-tangent vector for integral operators that are defined on an upper-Ahlfors regular subset of the Euclidean space and a continuity theorem for the corresponding integral operator in Hölder spaces in the specific case of a differentiable manifold.

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1 Introduction

Volume and layer potentials are integrals on a subset Y of the Euclidean space $\mathbb{R}^n$ that depend on a variable in a subset X of $\mathbb{R}^n$. Typically, X and Y are either measurable subsets of $\mathbb{R}^n$ with the n-dimensional Lebesgue measure, or manifolds that are embedded in $\mathbb{R}^n$, or boundaries of open subsets of $\mathbb{R}^n$ with the surface measure and X may well be different from Y.

For many relevant results in Hölder spaces, one can introduce a unified approach by assuming that X and Y are subsets of a metric space (M, d) and that Y is equipped with a measure ν that satisfies an upper Ahlfors growth condition that includes non-doubling measures (cf. (4.2)). With this respect we mention the works of García-Cuerva and Gatto [6], [7], Gatto [8] who have considered the case $X = Y = M$ and proved $T1$ Theorems for integral operators. Then one can also consider a stronger growth condition. Namely, the strong upper Ahlfors growth condition (4.9) that has been introduced in [15] to treat the dependence of singular and weakly singular integral operators both upon the variation of the density and of the kernel, when the kernel belongs to certain classes of kernels that generalize those of Giraud [10], Gegelia [9], Kupradze, Gegelia, Basheleishvili and Burchuladze [13, Chapter IV] and the so-called standard kernels.

In this paper, we first introduce some basic multiplication and embedding theorems for such classes of kernels (see Section 3).

In Section 4, we summarize and complement some results of [15].

In Section 5, we prove the tangential differentiation Theorem 5.1 with respect to a semi-tangent vector for integral operators defined on an upper-Ahlfors regular subset of the Euclidean space.

In Section 6, we consider the case in which Y is a compact manifold of codimension 1 in $\mathbb{R}^n$, and we show application of the results of [15], of the above mentioned properties of the kernel classes and of Theorem 5.1 by proving Theorem 6.3 on the continuity of the tangential gradient of a weakly singular integral operator that is defined in Y upon variation both of the kernel and of the density in Hölder spaces. Here we mention that Theorem 6.3 applies to relevant integral operators such as the layer potentials. In a forthcoming paper, we plan to apply the multiplication and embedding theorems of the classes of kernels of Section 3 and of Theorem 6.3 to analyze the continuity properties of the double layer potential that is associated with the fundamental solution of a second order elliptic operator with constant coefficients.

2 Notation

Let X be a set. Then we set

$$B(X) \equiv \left\{ f \in \mathbb{C}^X : f \text{ is bounded} \right\}, \quad \|f\|_{B(X)} \equiv \sup_X |f| \qquad \forall f \in B(X),$$

where $\mathbb{C}^X$ denotes the set of all functions from X to $\mathbb{C}$. If (M,d) is a metric space, we set

$$B(\xi, r) \equiv \{\eta \in M : d(\xi, \eta) < r\} \tag{2.1}$$

for all $(\xi, r) \in M \times]0, +\infty[$ and

$$\operatorname{diam}(X) \equiv \sup\{d(x_1, x_2) : x_1, x_2 \in X\}$$

for all subsets X of M. Then $C^0(M)$ denotes the set of all continuous functions from M to $\mathbb{C}$ and we introduce the subspace $C^0_b(M) \equiv C^0(M) \cap B(M)$ of $B(M)$. Let ω be a function from $[0, +\infty[$ to itself such that

$$\begin{aligned} &\omega(0) = 0, \qquad \omega(r) > 0 \qquad \forall r \in]0, +\infty[\,, \\ &\omega \text{ is increasing}, \ \lim_{r \to 0^+} \omega(r) = 0\,, \\ &\text{and} \quad \sup_{(a,t) \in [1,+\infty[\times]0,+\infty[} \frac{\omega(at)}{a\omega(t)} < +\infty\,. \end{aligned} \tag{2.2}$$

If f is a function from a subset $\mathbb{D}$ of a metric space (M,d) to $\mathbb{C}$, then we denote by $|f : \mathbb{D}|_{\omega(\cdot)}$ the $\omega(\cdot)$-Hölder constant of f, which is delivered by the formula

$$|f : \mathbb{D}|_{\omega(\cdot)} \equiv \sup \left\{ \frac{|f(x) - f(y)|}{\omega(d(x,y))} : x, y \in \mathbb{D}, x \neq y \right\}.$$

If $|f : \mathbb{D}|_{\omega(\cdot)} < \infty$, we say that f is $\omega(\cdot)$-Hölder continuous. Sometimes, we simply write $|f|_{\omega(\cdot)}$ instead of $|f : \mathbb{D}|_{\omega(\cdot)}$. The subset of $C^0(\mathbb{D})$ whose functions are $\omega(\cdot)$-Hölder continuous is denoted by $C^{0,\omega(\cdot)}(\mathbb{D})$ and $|f : \mathbb{D}|_{\omega(\cdot)}$ is a semi-norm on $C^{0,\omega(\cdot)}(\mathbb{D})$. Then we consider the space $C^{0,\omega(\cdot)}_b(\mathbb{D}) \equiv C^{0,\omega(\cdot)}(\mathbb{D}) \cap B(\mathbb{D})$ with the norm

$$\|f\|_{C^{0,\omega(\cdot)}_b(\mathbb{D})} \equiv \sup_{x \in \mathbb{D}} |f(x)| + |f|_{\omega(\cdot)} \qquad \forall f \in C^{0,\omega(\cdot)}_b(\mathbb{D}).$$

In the case in which $\omega(\cdot)$ is the function r^α for some fixed $\alpha \in]0, 1]$, a so-called Hölder exponent, we simply write $|\cdot : \mathbb{D}|_\alpha$ instead of $|\cdot : \mathbb{D}|_{r^\alpha}$, $C^{0,\alpha}(\mathbb{D})$ instead of $C^{0,r^\alpha}(\mathbb{D})$, $C^{0,\alpha}_b(\mathbb{D})$ instead of $C^{0,r^\alpha}_b(\mathbb{D})$, and we say that f is α-Hölder continuous provided that $|f : \mathbb{D}|_\alpha < +\infty$.

3 Special classes of potential type kernels in metric spaces

If X and Y are sets, then we denote by $\mathbb{D}_{X \times Y}$ the diagonal of $X \times Y$, i.e., we set

$$\mathbb{D}_{X \times Y} \equiv \{(x, y) \in X \times Y : x = y\} \tag{3.1}$$

and if $X = Y$, then we denote by $\mathbb{D}_X$ the diagonal of $X \times X$, i.e., we set

$$\mathbb{D}_X \equiv \mathbb{D}_{X\times X}\,.$$

An off-diagonal function in $X\times Y$ is a function from $(X\times Y)\setminus \mathbb{D}_{X\times Y}$ to $\mathbb{C}$. We now wish to consider a specific class of off-diagonal kernels in a metric space (M,d).

Definition 1. Let X and Y be subsets of a metric space (M,d). Let $s\in\mathbb{R}$. We denote by $\mathcal{K}_{s,X\times Y}$, the set of all continuous functions K from $(X\times Y)\setminus\mathbb{D}_{X\times Y}$ to $\mathbb{C}$ such that

$$\|K\|_{\mathcal{K}_{s,X\times Y}} \equiv \sup_{(x,y)\in(X\times Y)\setminus\mathbb{D}_{X\times Y}} |K(x,y)|\, d(x,y)^s < +\infty\,.$$

The elements of $\mathcal{K}_{s,X\times Y}$ are said to be kernels of potential type s in $X\times Y$.

We plan to consider 'potential type' kernels as in the following definition. See also Dondi and the author [5], where such classes have been introduced in a form that generalizes those of Giraud [10], Gegelia [9], Kupradze, Gegelia, Basheleishvili and Burchuladze [13, Chapter IV].

Definition 2. Let X and Y be subsets of a metric space (M,d). Let $s_1,\ s_2,\ s_3\in\mathbb{R}$. We denote by $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ the set of all continuous functions K from $(X\times Y)\setminus\mathbb{D}_{X\times Y}$ to $\mathbb{C}$ such that

$$\begin{aligned}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \equiv \sup&\Big\{ d(x,y)^{s_1}|K(x,y)| :\ (x,y)\in X\times Y, x\neq y\Big\}\\ +\sup&\Big\{\frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}|K(x',y)-K(x'',y)| :\\ &x',x''\in X, x'\neq x'', y\in Y\setminus B(x',2d(x',x''))\Big\} < +\infty\,.\end{aligned}$$

One can easily verify that $(\mathcal{K}_{s_1,s_2,s_3}(X\times Y), \|\cdot\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)})$ is a normed space. By our definition, if $s_1,\ s_2,\ s_3\in\mathbb{R}$, we have

$$\mathcal{K}_{s_1,s_2,s_3}(X\times Y)\subseteq \mathcal{K}_{s_1,X\times Y}$$

and

$$\|K\|_{\mathcal{K}_{s_1,X\times Y}} \leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \qquad \forall K\in\mathcal{K}_{s_1,s_2,s_3}(X\times Y)\,.$$

We note that if we choose $s_2 = s_1 + s_3$ we have the so-called class of standard kernels. We now turn to prove a series of statements in a metric space setting that extend the validity of corresponding statements for the classes that had been introduced in Giraud [10], Gegelia [9], Kupradze, Gegelia, Basheleishvili and Burchuladze [13, Chapter IV]. We start with the following elementary known embedding lemma.

Lemma 3.1. *Let X and Y be subsets of a metric space (M,d). Let $s_1,\ s_2,\ s_3\in\mathbb{R}$. If $a\in]0,+\infty[$, then $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ is continuously embedded into $\mathcal{K}_{s_1,s_2-a,s_3-a}(X\times Y)$.*

Proof. It suffices to note that if $x',x''\in X$, $x'\neq x''$, then

$$\begin{aligned}\frac{d(x',y)^{s_2-a}}{d(x',x'')^{s_3-a}} = \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}\frac{d(x',x'')^a}{d(x',y)^a} &\leq \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}\left(\frac{\frac{1}{2}d(x',y)}{d(x',y)}\right)^a\\ &= \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}2^{-a} \qquad \forall y\in Y\setminus B(x',2d(x',x''))\,.\end{aligned}$$

□

Next we introduce the following known elementary lemma, which we exploit later and which can be proved by the triangular inequality.

Lemma 3.2. *Let* (M,d) *be a metric space. Then*

$$\frac{1}{2}d(x',y) \leq d(x'',y) \leq 2d(x',y)\,,$$

for all $x', x'' \in M$, $x' \neq x''$, $y \in M \setminus B(x', 2d(x',x''))$.

Next we prove the following product rule for kernels.

Theorem 3.1. *Let* X *and* Y *be subsets of a metric space* (M,d). *Let* s_1, s_2, s_3, t_1, t_2, $t_3 \in \mathbb{R}$.

(i) *If* $K_1 \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ *and* $K_2 \in \mathcal{K}_{t_1,t_2,t_3}(X\times Y)$, *then the following inequality holds*

$$\begin{aligned}&|K_1(x',y)K_2(x',y) - K_1(x'',y)K_2(x'',y)|\\ &\quad \leq \|K_1\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}\|K_2\|_{\mathcal{K}_{t_1,t_2,t_3}(X\times Y)}\left(\frac{d(x',x'')^{s_3}}{d(x',y)^{s_2+t_1}} + \frac{2^{|s_1|}d(x',x'')^{t_3}}{d(x',y)^{t_2+s_1}}\right)\end{aligned}$$

for all $x', x'' \in X$, $x' \neq x''$, $y \in Y \setminus B(x', 2d(x',x''))$.

(ii) *The pointwise product is bilinear and continuous from*

$$\mathcal{K}_{s_1,s_1+s_3,s_3}(X\times Y)\times\mathcal{K}_{t_1,t_1+s_3,s_3}(X\times Y) \quad \text{to} \quad \mathcal{K}_{s_1+t_1,s_1+s_3+t_1,s_3}(X\times Y)\,.$$

Proof. (i) By the triangular inequality and by the definition of the norm for kernels, we have

$$\begin{aligned}&|K_1(x',y)K_2(x',y) - K_1(x'',y)K_2(x'',y)|\\ &\quad \leq |K_1(x',y) - K_1(x'',y)|\,|K_2(x',y)| + |K_1(x'',y)|\,|K_2(x',y) - K_2(x'',y)|\\ &\quad \leq \|K_1\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}\|K_2\|_{\mathcal{K}_{t_1,t_2,t_3}(X\times Y)}\left(\frac{d(x',x'')^{s_3}}{d(x',y)^{s_2+t_1}} + \frac{d(x',x'')^{t_3}}{d(x',y)^{t_2}d(x'',y)^{s_1}}\right)\end{aligned}$$

If $s_1 \geq 0$, Lemma 3.2 implies that

$$\frac{1}{d(x'',y)^{s_1}} \leq \frac{1}{d(x',y)^{s_1}2^{-s_1}} = \frac{2^{s_1}}{d(x',y)^{s_1}}\,.$$

If instead $s_1 < 0$, Lemma 3.2 implies that

$$\frac{1}{d(x'',y)^{s_1}} \leq \frac{1}{d(x',y)^{s_1}2^{s_1}} = \frac{2^{-s_1}}{d(x',y)^{s_1}}\,.$$

Hence, the validity of the inequality of statement (i) follows.

(ii) Since

$$|K_1(x,y)K_2(x,y)| \leq \frac{\|K_1\|_{\mathcal{K}_{s_1,X\times Y}}\|K_2\|_{\mathcal{K}_{t_1,X\times Y}}}{d(x,y)^{s_1}d(x,y)^{t_1}} \qquad \forall x,y \in X\times Y, x\neq y\,,$$

statement (ii) is an immediate consequence of the inequality of statement (i) with $s_3 = t_3$, $s_2 = s_1+s_3$, $t_2 = t_1 + s_3$. □

Then we have the following product rule of a kernel and of a function of either $x \in X$ or $y \in Y$.

Proposition 3.1. *Let X and Y be subsets of a metric space (M,d). Let s_1, s_2, $s_3 \in \mathbb{R}$, $\alpha \in]0,1]$. Then the following statements hold.*

(i) *If $K \in \mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ and $f \in C_b^{0,\alpha}(X)$, then*

$$|K(x,y)f(x)|\, d(x,y)^{s_1} \leq \|K\|_{\mathcal{K}_{s_1,X\times Y}} \sup_X |f| \quad \forall (x,y) \in (X \times Y) \setminus \mathbb{D}_{X\times Y}$$

and

$$\begin{aligned} &|K(x',y)f(x') - K(x'',y)f(x'')| \\ &\qquad \leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \|f\|_{C_b^{0,\alpha}(X)} \left\{ \frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}} + 2^{|s_1|} \frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1}} \right\} \end{aligned}$$

for all $x', x'' \in X$, $x' \neq x''$, $y \in Y \setminus B(x', 2d(x',x''))$.

(ii) *If $s_2 \geq s_1$ and X and Y are both bounded, then the map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C_b^{0,s_3}(X) \quad \text{to} \quad \mathcal{K}_{s_1,s_2,s_3}(X \times Y)$$

that takes (K,f) to the kernel $K(x,y)f(x)$ of the variable $(x,y) \in (X \times Y) \setminus \mathbb{D}_{X\times Y}$ is bilinear and continuous.

(iii) *The map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C_b^{0}(Y) \quad \text{to} \quad \mathcal{K}_{s_1,s_2,s_3}(X \times Y)$$

that takes (K,f) to the kernel $K(x,y)f(y)$ in the variable $(x,y) \in (X \times Y) \setminus \mathbb{D}_{X\times Y}$ is bilinear and continuous.

Proof. (i) The first inequality is an obvious consequence of the definition of the norm in $\mathcal{K}_{s_1,X\times Y}$. To prove the second one, we note that

$$\begin{aligned} &|K(x',y)f(x') - K(x'',y)f(x'')| \\ &\qquad \leq |K(x',y) - K(x'',y)|\,|f(x')| + |K(x'',y)|\,|f(x') - f(x'')| \\ &\qquad \leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \|f\|_{C_b^{0,\alpha}(X)} \left\{ \frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}} + \frac{d(x',x'')^{\alpha}}{d(x'',y)^{s_1}} \right\} \end{aligned}$$

If $s_1 \geq 0$, Lemma 3.2 implies that

$$\frac{d(x',x'')^{\alpha}}{d(x'',y)^{s_1}} \leq \frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1} 2^{-s_1}} = 2^{s_1} \frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1}}.$$

If instead $s_1 < 0$, Lemma 3.2 implies that

$$\frac{d(x',x'')^{\alpha}}{d(x'',y)^{s_1}} \leq \frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1} 2^{s_1}} = 2^{-s_1} \frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1}}.$$

Hence, the second inequality in statement (i) holds true. To prove (ii), it suffices to note that

$$\frac{d(x',x'')^{s_3}}{d(x',y)^{s_1}} = \frac{d(x',x'')^{s_3} d(x',y)^{s_2-s_1}}{d(x',y)^{s_1} d(x',y)^{s_2-s_1}} \leq (\operatorname{diam}(X \cup Y))^{s_2-s_1} \frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}},$$

to apply the second inequality of statement (i) and to invoke the first inequality of statement (i). Statement (iii) is obvious. □

We also point out the validity of the following elementary remark that holds if both X and Y are bounded.

Remark 1. Let (M,d) be a metric space. Let X, Y be bounded subsets of M. Let s_1, s_2, $s_3 \in [0,+\infty[$. If $a \in]0,+\infty[$, then Lemma 3.2 implies the validity of the following inequality

$$\sup\Big\{ \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}} |K(x',y)-K(x'',y)| : \\ x',x'' \in X, a \le d(x',x''), y \in Y \setminus B(x', 2d(x',x''))\Big\} \\ \le \frac{(\mathrm{diam}\,(X\cup Y))^{s_2}}{a^{s_3}} \|K\|_{\mathcal{K}_{s_1,X\times Y}} \left((2a)^{-s_1} + (2^{-1}2a)^{-s_1}\right)$$

for all $K \in \mathcal{K}_{s_1,X\times Y}$ and accordingly the norm on $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ defined by setting

$$\|K\|_{a;\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \equiv \|K\|_{\mathcal{K}_{s_1,X\times Y}} + \sup\Big\{ \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}} |K(x',y)-K(x'',y)| : \\ x',x'' \in X, 0 < d(x',x'') < a, y \in Y \setminus B(x', 2d(x',x''))\Big\}$$

is equivalent to the norm $\|\cdot\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}$ on $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$.

Next we prove the following embedding statement that holds for bounded sets.

Proposition 3.2. *Let (M,d) be a metric space. Let X, Y be bounded subsets of M. Let s_1, s_2, s_3, t_1, t_2, $t_3 \in \mathbb{R}$. Then the following statements hold.*

(i) *If $t_1 \ge s_1$ then $\mathcal{K}_{s_1,X\times Y}$ is continuously embedded into $\mathcal{K}_{t_1,X\times Y}$.*

(ii) *If $t_1 \ge s_1$, $t_3 \le s_3$ and $(t_2-t_3) \ge (s_2-s_3)$, then $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ is continuously embedded into $\mathcal{K}_{t_1,t_2,t_3}(X\times Y)$.*

(iii) *If $t_1 \ge s_1$, $t_3 \le s_3$, then $\mathcal{K}_{s_1,s_1+s_3,s_3}(X\times Y)$ is continuously embedded into $\mathcal{K}_{t_1,t_1+t_3,t_3}(X\times Y)$.*

Proof. Statement (i) is an immediate corollary of the following elementary inequality

$$d(x,y)^{t_1}|K(x,y)| \le d(x,y)^{t_1-s_1} d(x,y)^{s_1}|K(x,y)| \\ \le (\mathrm{diam}\,(X\cup Y))^{t_1-s_1}\|K\|_{\mathcal{K}_{s_1,X\times Y}} \qquad \forall (x,y) \in X\times Y \setminus \mathbb{D}_{X\times Y}\,,$$

which holds for all $K \in \mathcal{K}_{s_1,X\times Y}$. To prove (ii), it suffices to invoke (i) and to note that

$$\frac{d(x',y)^{t_2}}{d(x',x'')^{t_3}}|K(x',y)-K(x'',y)| \\ = \frac{d(x',y)^{t_2-s_2}}{d(x',x'')^{t_3-s_3}} \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}|K(x',y)-K(x'',y)| \\ \le d(x',y)^{t_2-s_2} d(x',x'')^{s_3-t_3}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \\ \le d(x',y)^{t_2-s_2}(2^{-1}d(x',y))^{s_3-t_3}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \\ \le d(x',y)^{(t_2-t_3)-(s_2-s_3)}2^{t_3-s_3}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \\ \le (\mathrm{diam}\,(X\cup Y))^{(t_2-t_3)-(s_2-s_3)}2^{t_3-s_3}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}$$

for all $x',x'' \in X$, $x' \neq x''$, $y \in Y \setminus B(x', 2d(x',x''))$ and $K \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y)$. Finally, statement (iii) is an immediate corollary of statement (ii). □

We now show that we can associate a potential type kernel to all Hölder continuous functions.

Lemma 3.3. *Let X and Y be subsets of a metric space (M,d). Let $\alpha \in]0,1]$. Let $C^{0,\alpha}(X \cup Y)$ be endowed with the Hölder seminorm $|\cdot : X \cup Y|_\alpha$. Then the following statements hold.*

(i) *If $\mu \in C^{0,\alpha}(X \cup Y)$, then the map $\Xi[\mu]$ defined by*

$$\Xi[\mu](x,y) \equiv \mu(x) - \mu(y) \qquad \forall (x,y) \in (X \times Y) \setminus \mathbb{D}_{X \times Y} \tag{3.2}$$

belongs to $\mathcal{K}_{-\alpha,0,\alpha}(X \times Y)$.

(ii) *The operator Ξ from $C^{0,\alpha}(X \cup Y)$ to $\mathcal{K}_{-\alpha,0,\alpha}(X \times Y)$ that takes μ to $\Xi[\mu]$ is linear and continuous.*

Proof. It suffices to observe that

$$|\mu(x) - \mu(y)| \leq |\mu : X \cup Y|_\alpha d(x,y)^\alpha \qquad \forall (x,y) \in (X \times Y) \setminus \mathbb{D}_{X \times Y}$$

and that

$$|(\mu(x') - \mu(y)) - (\mu(x'') - \mu(y))| = |\mu(x') - \mu(x'')| \leq |\mu : X \cup Y|_\alpha \frac{d(x',x'')^\alpha}{d(x',y)^0}$$

for all $x', x'' \in X$, $x' \neq x''$, $y \in Y \setminus B(x', 2d(x',x''))$. □

Sometimes the kernel has a special form which we need later on. Thus we introduce the following preliminary lemma for standard kernels.

Lemma 3.4. *Let X and Y be subsets of a metric space (M,d). Let $s_1 \in \mathbb{R}$, $s_3 \in]-\infty,1]$, $\theta \in]0,1]$. Let $C^{0,\theta}(X \cup Y)$ be endowed with the Hölder seminorm $|\cdot : X \cup Y|_\theta$. Then the following statements hold.*

(i) *The map H from $\mathcal{K}_{s_1,X \times Y} \times C^{0,\theta}(X \cup Y)$ to $\mathcal{K}_{s_1-\theta,X \times Y}$, which takes (Z,g) to the function from $(X \times Y) \setminus \mathbb{D}_{X \times Y}$ to $\mathbb{C}$ defined by*

$$H[Z,g](x,y) \equiv (g(x) - g(y))Z(x,y) \qquad \forall (x,y) \in (X \times Y) \setminus \mathbb{D}_{X \times Y} \tag{3.3}$$

is bilinear and continuous.

(ii) *The map H from*

$$\mathcal{K}_{s_1,s_1+s_3,s_3}(X \times Y) \times C^{0,\theta}(X \cup Y) \quad \text{to} \quad \mathcal{K}_{s_1-\theta,s_1+s_3-1,s_3-(1-\theta)}(X \times Y),$$

which takes (Z,g) to the function defined by (3.3) is bilinear and continuous.

Proof. (i) It suffices to note that the Hölder continuity of g implies that

$$|H[Z,g](x,y)| \leq \frac{|g : X \cup Y|_\theta}{d(x,y)^{s_1-\theta}} \|Z\|_{\mathcal{K}_{s_1,X \times Y}} \qquad \forall (x,y) \in (X \times Y) \setminus \mathbb{D}_{X \times Y}. \tag{3.4}$$

(ii) By Lemma 3.3, the linear operator from

$$\mathcal{K}_{s_1,s_1+s_3,s_3}(X \times Y) \times C^{0,\theta}(X \cup Y) \quad \text{to} \quad \mathcal{K}_{s_1,s_1+s_3,s_3}(X \times Y) \times \mathcal{K}_{-\theta,0,\theta}(X \times Y)$$

that takes (Z,g) to $(Z,\Xi[g])$ is linear and continuous. By the elementary embedding Lemma 3.1, the inclusion map from

$$\mathcal{K}_{s_1,s_1+s_3,s_3}(X \times Y) \times \mathcal{K}_{-\theta,0,\theta}(X \times Y)$$

to $\mathcal{K}_{s_1,s_1+s_3-(1-\theta),s_3-(1-\theta)}(X \times Y) \times \mathcal{K}_{-\theta,-(1-s_3),\theta-(1-s_3)}(X \times Y)$ is linear and continuous. Then the product Theorem 3.1 (ii) for standard kernels implies that the product is continuous from

$$\mathcal{K}_{s_1,s_1+s_3-(1-\theta),s_3-(1-\theta)}(X \times Y) \times \mathcal{K}_{-\theta,-(1-s_3),\theta-(1-s_3)}(X \times Y)$$

to $\mathcal{K}_{s_1-\theta,s_1+s_3-1,s_3-(1-\theta)}(X \times Y)$ and thus the proof is complete. □

4 Preliminaries on upper υ_Y-Ahlfors regular sets

We plan to consider integral operators in subsets X and Y of a metric space (M,d) when Y is endowed of a measure as follows.

$$\begin{aligned}&\text{Let } \mathcal{N} \text{ be a } \sigma\text{-algebra of parts of } Y\,, \mathcal{B}_Y\subseteq\mathcal{N}\,.\\&\text{Let } \nu \text{ be measure on } \mathcal{N}\,.\\&\text{Let } \nu(B(x,r)\cap Y)<+\infty \qquad \forall (x,r)\in X\times]0,+\infty[\,.\end{aligned}\tag{4.1}$$

Here $\mathcal{B}_Y$ denotes the σ-algebra of all Borel subsets of Y.

Definition 3. Let X and Y be subsets of a metric space (M,d). Let $\upsilon_Y\in]0,+\infty[$. Let ν be as in (4.1). We say that Y is upper υ_Y-Ahlfors regular with respect to X provided that the following condition holds

$$\begin{aligned}&\text{there exist } r_{X,Y,\upsilon_Y}\in]0,+\infty]\,,\ c_{X,Y,\upsilon_Y}\in]0,+\infty[\text{ such that}\\&\nu(B(x,r)\cap Y)\leq c_{X,Y,\upsilon_Y}r^{\upsilon_Y}\\&\text{for all } x\in X \text{ and } r\in]0,r_{X,Y,\upsilon_Y}[\,.\end{aligned}\tag{4.2}$$

In the case $X=Y$, we say that Y is upper υ_Y-Ahlfors regular.

One could show that if $n\in\mathbb{N}$, $n\geq 2$ and if Y is a compact embedded differential manifold in $\mathbb{R}^n$ of codimension 1, then Y is upper $(n-1)$-Ahlfors regular with respect to $\mathbb{R}^n$. Then one can prove the following basic inequalities for the integral on an upper Ahlfors regular set Y and on the intersection of Y with balls with center at a point x of X of the powers of $d(x,y)^{-1}$ with exponent $s\in]-\infty,\upsilon_Y[$, that are variants of those proved by Gatto [8, page 104] in the case $X=Y$ (for a proof see [15, Lemmas 3.2, 3.4]).

Lemma 4.1. *Let X and Y be subsets of a metric space (M,d). Let $\upsilon_Y\in]0,+\infty[$. Let ν be as in (4.1). Let Y be upper υ_Y-Ahlfors regular with respect to X. Then the following statements hold.*

(i) $\nu(\{x\})=0\ \forall x\in X\cap Y$.

(ii) *Let $\nu(Y)<+\infty$. If $s\in]0,\upsilon_Y[$, then*

$$c'_{s,X,Y}\equiv\sup_{x\in X}\int_Y\frac{d\nu(y)}{d(x,y)^s}\leq\nu(Y)a^{-s}+c_{X,Y,\upsilon_Y}\frac{\upsilon_Y}{\upsilon_Y-s}a^{\upsilon_Y-s}$$

for all $a\in]0,r_{X,Y,\upsilon_Y}[$. If $s=0$, then

$$c'_{0,X,Y}\equiv\sup_{x\in X}\int_Y\frac{d\nu(y)}{d(x,y)^0}=\nu(Y)\,.$$

(iii) *Let $\nu(Y)<+\infty$ whenever $r_{X,Y,\upsilon_Y}<+\infty$. If $s\in]-\infty,\upsilon_Y[$, then*

$$c''_{s,X,Y}\equiv\sup_{(x,t)\in X\times]0,+\infty[}t^{s-\upsilon_Y}\int_{B(x,t)\cap Y}\frac{d\nu(y)}{d(x,y)^s}<+\infty\,.$$

By the Hölder inequality one can prove the following statement of Hille-Tamarkin (see [15, Proposition 4.1]).

Proposition 4.1. *Let X and Y be subsets of a metric space (M,d). Let $\upsilon_Y \in]0,+\infty[$, $s \in [0,\upsilon_Y[$. Let ν be as in (4.1). Let $\nu(Y) < +\infty$. Let Y be upper υ_Y-Ahlfors regular with respect to X. Then the following statements hold.*

(i) *If $(K,\varphi) \in \mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$, then the function $K(x,\cdot)\varphi(\cdot)$ is integrable in Y for all $x \in X$ and the function $A[K,\varphi]$ defined by*

$$A[K,\varphi](x) \equiv \int_Y K(x,y)\varphi(y)\,d\nu(y) \qquad \forall x \in X \tag{4.3}$$

is bounded.

(ii) *The bilinear map from $\mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$ to $B(X)$, which takes (K,φ) to $A[K,\varphi]$ is continuous and the following inequality holds*

$$\sup_X |A[K,\varphi]| \leq c'_{s,X,Y} \|K\|_{\mathcal{K}_{s,X\times Y}} \|\varphi\|_{L^\infty_\nu(Y)} \tag{4.4}$$

for all $(K,\varphi) \in \mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$ (see Lemma 4.1 (ii) for $c'_{s,X,Y}$).

Under the assumptions of the previous proposition, one can actually prove that the function $A[K,\varphi]$ is continuous. To do so, we first introduce the following result for potential type operators.

Proposition 4.2. *Let X and Y be subsets of a metric space (M,d). Let ν be as in (4.1). Let $\nu(Y) < +\infty$. Let $s \in \mathbb{R}$. Let $K \in \mathcal{K}_{s,X\times Y}$. Let $d(x,\cdot)^{-s}$ belong to $L^1_\nu(Y \setminus \{x\})$ for all $x \in X$. Let*

$$\sup_{x\in X} \int_{Y\setminus\{x\}} d(x,y)^{-s}\,d\nu(y) < +\infty\,. \tag{4.5}$$

If $\nu(\{x\}) = 0$ for all $x \in X \cap Y$ and if for each $\epsilon \in]0,+\infty[$ there exists $\delta \in]0,+\infty[$ such that

$$\sup_{x\in X} \int_{F\setminus\{x\}} d(x,y)^{-s}\,d\nu(y) \leq \epsilon \qquad \text{if } F \in \mathcal{N},\ \nu(F) \leq \delta\,, \tag{4.6}$$

and if $\varphi \in L^\infty_\nu(Y)$, then the function $A[K,\varphi]$ from X to $\mathbb{C}$ defined by (4.3) is continuous.

Proof. Let $\tilde{x} \in X$. It suffices to show that if $\{x_j\}_{j\in\mathbb{N}}$ is a sequence in X which converges to $\tilde{x}$, then

$$\lim_{j\to\infty} \int_Y K(x_j,y)\varphi(y)\,d\nu(y) = \int_Y K(\tilde{x},y)\varphi(y)\,d\nu(y)\,.$$

We now turn to prove such a limiting relation by exploiting the Vitali Convergence Theorem. To do so, we prove the validity of the following two statements.

(j) There exists $N_{\tilde{x}} \in \mathcal{N}$ such that $\nu(N_{\tilde{x}}) = 0$ and

$$\lim_{j\to\infty} K(x_j,y)\varphi(y) = K(\tilde{x},y)\varphi(y) \qquad \forall y \in Y \setminus N_{\tilde{x}}\,.$$

(jj) For each $\epsilon \in]0,+\infty[$, there exists $\delta \in]0,+\infty[$ such that

$$\sup_{j\in\mathbb{N}} \int_F |K(x_j,y)\varphi(y)|\,d\nu(y) \leq \epsilon \qquad \text{if } F \in \mathcal{N}\,, \nu(F) \leq \delta\,.$$

Since $\nu(\{\tilde{x}\} \cap Y) = 0$, we can take $N_{\tilde{x}} \equiv \{\tilde{x}\} \cap Y$ and statement (j) follows by our continuity assumption on K that follows by the membership of K in $\mathcal{K}_{s,X\times Y}$. We now turn to prove (jj). By our assumptions on K, we have

$$\int_F |K(x_j, y)\varphi(y)|\, d\nu(y) \le \|K\|_{\mathcal{K}_{s,X\times Y}} \int_F d(x_j, y)^{-s}\, d\nu(y) \|\varphi\|_{L^\infty_\nu(F)}$$

for all $j \in \mathbb{N}$. Thus it suffices to choose $\delta \in]0, +\infty[$ such that

$$\sup_{x\in X} \int_F d(x, y)^{-s}\, d\nu(y) \le \epsilon(1 + \|K\|_{\mathcal{K}_{s,X\times Y}} \|\varphi\|_{L^\infty_\nu(Y)})^{-1} \quad \text{if } F \in \mathcal{N},\ \nu(F) \le \delta\,,$$

and statement (jj) holds true and the proof is complete. □

In order to apply Proposition 4.2 in the case Y is upper Ahlfors regular, we need to prove the following lemma.

Lemma 4.2. *Let X and Y be subsets of a metric space (M, d). Let $\upsilon_Y \in]0, +\infty[$, $s \in [0, \upsilon_Y[$. Let ν be as in (4.1). Let $\nu(Y) < +\infty$. Let Y be upper υ_Y-Ahlfors regular with respect to X. Then for each $\epsilon \in]0, +\infty[$ there exists $\delta \in]0, +\infty[$ such that*

$$\sup_{x\in X} \int_F d(x, y)^{-s}\, d\nu(y) \le \epsilon \qquad \text{if } F \in \mathcal{N},\ \nu(F) \le \delta\,, \tag{4.7}$$

Proof. We first note that if $F \in \mathcal{N}$, then F is a subset of Y. Accordingly F is also upper υ_Y-Ahlfors regular with respect to X and we can choose $r_{X,F,\upsilon_Y} = r_{X,Y,\upsilon_Y}$, $c_{X,F,\upsilon_Y} = c_{X,Y,\upsilon_Y}$. If $s > 0$, then Lemma 4.1 (ii) implies that

$$\sup_{x\in X} \int_F d(x, y)^{-s}\, d\nu(y) \le \nu(F)a^{-s} + c_{X,Y,\upsilon_Y} \frac{\upsilon_Y}{\upsilon_Y - s} a^{\upsilon_Y - s} \qquad \forall a \in]0, r_{X,Y,\upsilon_Y}[\,.$$

Thus if $\epsilon \in]0, +\infty[$, then we choose $a_\epsilon \in]0, r_{X,Y,\upsilon_Y}[$ such that

$$c_{X,Y,\upsilon_Y} \frac{\upsilon_Y}{\upsilon_Y - s} a_\epsilon^{\upsilon_Y - s} < \frac{\epsilon}{2}$$

and we can set $\delta \equiv \frac{\epsilon}{2} a_\epsilon^s$. Then we have

$$\sup_{x\in X} \int_F d(x, y)^{-s}\, d\nu(y) \le \delta a_\epsilon^{-s} + \frac{\epsilon}{2} = \epsilon$$

whenever $F \in \mathcal{N}$ and $\nu(F) \le \delta$. If instead $s = 0$, then condition (4.7) holds trivially with $\delta = \epsilon$. □

Proposition 4.3. *Let X and Y be subsets of a metric space (M, d). Let ν be as in (4.1). Let ν be finite. Let $s \in [0, \upsilon_Y[$. Let Y be upper υ_Y-Ahlfors regular with respect to X. If $(K, \varphi) \in \mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$, then the function $A[K, \varphi]$ from X to $\mathbb{C}$ defined by (4.3) is continuous.*

Proof. We plan to deduce the continuity of $A[K, \varphi]$ by the continuity Proposition 4.2. To do so, it suffices to note that Lemma 4.1 (i), (ii) imply that $\nu(\{x\}) = 0$ for all $x \in X \cap Y$ and that condition (4.5) is satisfied. Moreover, Lemma 4.2 implies that condition (4.6) is satisfied. □

Next we plan to introduce a result on the integral operator

$$Q[Z, g, 1](x) \equiv \int_Y Z(x, y)(g(x) - g(y))\, d\nu(y) \qquad \forall x \in X\,. \tag{4.8}$$

when Z belongs to the class $\mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ as in Definition 2 and g is a $\mathbb{C}$-valued function in $X \cup Y$. We exploit the operator in (4.8) in the next section and we note that operators as in (4.8) appear in the applications (cf. *e.g.*, Colton and Kress [3, page 56], and Dondi and the author [5, § 8]). In order to estimate the Hölder quotient of $Q[Z, g, 1]$, we need to introduce a further norm for kernels.

Definition 4. Let X and Y be subsets of a metric space (M,d). Let ν be as in (4.1). Let s_1, s_2, $s_3 \in \mathbb{R}$. We set

$$\begin{aligned}\mathcal{K}^{\sharp}_{s_1,s_2,s_3}(X\times Y) \equiv \Big\{ & K \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y): \\ & K(x,\cdot) \text{ is } \nu - \text{integrable in } Y\setminus B(x,r) \text{ for all } (x,r)\in X\times]0,+\infty[\,, \\ & \sup_{x\in X}\sup_{r\in]0,+\infty[}\left|\int_{Y\setminus B(x,r)} K(x,y)\,d\nu(y)\right| < +\infty \Big\}\end{aligned}$$

and

$$\begin{aligned}\|K\|_{\mathcal{K}^{\sharp}_{s_1,s_2,s_3}(X\times Y)} \equiv & \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \\ & + \sup_{x\in X}\sup_{r\in]0,+\infty[}\left|\int_{Y\setminus B(x,r)} K(x,y)\,d\nu(y)\right| \qquad \forall K\in \mathcal{K}^{\sharp}_{s_1,s_2,s_3}(X\times Y)\,.\end{aligned}$$

Clearly, $(\mathcal{K}^{\sharp}_{s_1,s_2,s_3}(X\times Y), \|\cdot\|_{\mathcal{K}^{\sharp}_{s_1,s_2,s_3}(X\times Y)})$ is a normed space. By definition, the space $\mathcal{K}^{\sharp}_{s_1,s_2,s_3}(X\times Y)$ is continuously embedded into the space $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$. Then we consider a stronger version of the upper Ahlfors regularity. Namely, we assume that Y is strongly upper υ_Y-Ahlfors regular with respect to X, *i.e.*, that

$$\begin{aligned}&\text{there exist } r_{X,Y,\upsilon_Y} \in]0,+\infty]\,,\ c_{X,Y,\upsilon_Y} \in]0,+\infty[\text{ such that} \\ &\nu((B(x,r_2)\setminus B(x,r_1))\cap Y) \leq c_{X,Y,\upsilon_Y}(r_2^{\upsilon_Y} - r_1^{\upsilon_Y}) \\ &\text{for all } x\in X \text{ and } r_1,r_2\in[0,r_{X,Y,\upsilon_Y}[\text{ with } r_1<r_2\,,\end{aligned} \tag{4.9}$$

where we mean that $B(x,0)\equiv\emptyset$ (in the case $X=Y$, we just say that Y is strongly upper υ_Y-Ahlfors regular). So, for example, if Y is a compact manifold of class C^1 that is embedded in $M=\mathbb{R}^n$, then Y can be proved to be strongly upper $(n-1)$-Ahlfors regular with respect to Y. Next we introduce a function that we need for a generalized Hölder norm. For each $\theta\in]0,1]$, we define the function $\omega_\theta(\cdot)$ from $[0,+\infty[$ to itself by setting

$$\omega_\theta(r) \equiv \begin{cases} 0 & r=0\,, \\ r^\theta|\ln r| & r\in]0,r_\theta]\,, \\ r_\theta^\theta|\ln r_\theta| & r\in]r_\theta,+\infty[\,, \end{cases}$$

where $r_\theta \equiv e^{-1/\theta}$ for all $\theta\in]0,1]$. Obviously, $\omega_\theta(\cdot)$ is concave and satisfies condition (2.2). We also note that if $\mathbb{D}\subseteq M$, then the continuous embedding

$$C_b^{0,\theta}(\mathbb{D}) \subseteq C_b^{0,\omega_\theta(\cdot)}(\mathbb{D}) \subseteq C_b^{0,\theta'}(\mathbb{D})$$

holds for all $\theta'\in]0,\theta[$ (cf. Section 2). We are now ready to state the following statement of [15, Proposition 6.3] on the Hölder continuity of $Q[Z,g,1]$ that extends some work of Gatto [8, Proof of Theorem 3, Theorem 4]. Here where we mean that $C^{0,\beta}(X\cup Y)$ is endowed with the semi-norm $|\cdot : X\cup Y|_\beta$.

Proposition 4.4. *Let X and Y be subsets of a metric space (M,d). Let*

$$\upsilon_Y\in]0,+\infty[\,,\ \beta\in]0,1]\,,\ s_1\in[\beta,\upsilon_Y+\beta[\,,\ s_2\in[\beta,+\infty[\,,\ s_3\in]0,1]\,.$$

Let ν be as in (4.1), $\nu(Y)<+\infty$.

(i) *If $s_1<\upsilon_Y$, then the following statements hold.*

(a) *If* $s_2 - \beta > \upsilon_Y$, $s_2 < \upsilon_Y + \beta + s_3$ *and* Y *is upper* υ_Y*-Ahlfors regular with respect to* X, *then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\min\{\beta,\upsilon_Y+s_3+\beta-s_2\}}(X),$$

which takes (Z, g) *to* $Q[Z, g, 1]$ *is continuous.*

(aa) *If* $s_2 - \beta = \upsilon_Y$ *and* Y *is strongly upper* υ_Y*-Ahlfors regular with respect to* X, *then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\max\{r^\beta,\omega_{s_3}(r)\}}(X),$$

which takes (Z, g) *to* $Q[Z, g, 1]$ *is continuous.*

(ii) *If* $s_1 = \upsilon_Y$, *then the following statements hold.*

(b) *If* $s_2 - \beta > \upsilon_Y$, $s_2 < \upsilon_Y + \beta + s_3$ *and* Y *is upper* υ_Y*-Ahlfors regular with respect to* X, *then the bilinear map from*

$$\mathcal{K}^\sharp_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\min\{\beta,\upsilon_Y+s_3+\beta-s_2\}}(X),$$

which takes (Z, g) *to* $Q[Z, g, 1]$ *is continuous.*

(bb) *If* $s_2 - \beta = \upsilon_Y$ *and* Y *is strongly upper* υ_Y*-Ahlfors regular with respect to* X, *then the bilinear map from*

$$\mathcal{K}^\sharp_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\max\{r^\beta,\omega_{s_3}(r)\}}(X),$$

which takes (Z, g) *to* $Q[Z, g, 1]$ *is continuous.*

(iii) *If* $s_1 > \upsilon_Y$, *then the following statements hold.*

(c) *If* $s_2 - \beta > \upsilon_Y$, $s_2 < \upsilon_Y + \beta + s_3$ *and* Y *is upper* υ_Y*-Ahlfors regular with respect to* X, *then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\min\{\upsilon_Y+\beta-s_1,\upsilon_Y+s_3+\beta-s_2\}}(X),$$

which takes (Z, g) *to* $Q[Z, g, 1]$ *is continuous.*

(cc) *If* $s_2 - \beta = \upsilon_Y$ *and* Y *is strongly upper* υ_Y*-Ahlfors regular with respect to* X, *then the bilinear map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\max\{r^{\upsilon_Y+\beta-s_1},\omega_{s_3}(r)\}}(X),$$

which takes (Z, g) *to* $Q[Z, g, 1]$ *is continuous.*

5 A differentiation theorem for integral operators on upper Ahlfors regular subsets of $\mathbb{R}^n$

We first introduce some preliminaries.

Definition 5. Let $n \in \mathbb{N} \setminus \{0\}$. Let X be a subset of $\mathbb{R}^n$, $p \in X$. We say that a vector $w \in \mathbb{R}^n$ is semi-tangent to X at the point p provided that either $w = 0$ or there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ in $X \setminus \{p\}$ which converges to p and such that

$$\frac{w}{|w|} = \lim_{j\to\infty} \frac{x_j - p}{|x_j - p|} .$$

We say that a vector $w \in \mathbb{R}^n$ is tangent to X at the point p provided that both w and $-w$ are semi-tangent to X at the point p.

Here $|\cdot|$ denotes the Euclidean modulus in $\mathbb{R}^n$. We denote by T_pX the set of all semi-tangent vectors to X at p. One can easily check that T_pX is a cone of $\mathbb{R}^n$, i.e., that

$$\lambda w \in T_pX \qquad \text{whenever} \quad (\lambda, w) \in]0, +\infty[\times T_pX .$$

We say that T_pX is the cone of semi-tangent vectors to X at p. If T_pX is also a subspace of $\mathbb{R}^n$, then we say that X has a tangent space at p, that T_pX is the tangent space to X at p and that $p + T_pX$ is the affine tangent space to X at p. Next we state the definition of directional derivative for a function defined on an arbitrary subset of $\mathbb{R}^n$.

Definition 6. Let Z be a real or complex normed space. Let X be a subset of $\mathbb{R}^n$. Let ϕ be a function from X to Z. Let $p \in X$, $v \in T_pX$, $|v| = 1$.

We say that ϕ has a derivative at p with respect to the direction v provided that there exists an element $D_{X,v}\varphi(p) \in Z$ such that

$$D_{X,v}\varphi(p) = \lim_{j\to\infty} \frac{\phi(x_j) - \phi(p)}{|x_j - p|} \qquad \text{in } Z$$

for all sequences $\{x_j\}_{j\in\mathbb{N}}$ in $X \setminus \{p\}$ which converge to p and such that

$$v = \lim_{j\to\infty} \frac{x_j - p}{|x_j - p|} .$$

Then we say that $D_{X,v}\varphi(p)$ is the derivative of ϕ at p with respect to the direction v.

We note that if there exist an open neighborhood W of p in $\mathbb{R}^n$ and if $\tilde{\phi}$ is a continuously (real) differentiable function from W to Z and satisfies the equality $\tilde{\phi}_{|X\cap W} = \phi_{|X\cap W}$, then ϕ has a derivative at p with respect to the direction v and

$$D_{X,v}\varphi(p) = D_v\tilde{\phi}(p) = d\tilde{\phi}(p)[v] .$$

Indeed, $d\tilde{\phi}(p)[v] = \lim_{j\to\infty} d\tilde{\phi}(p)\left[\frac{x_j-p}{|x_j-p|}\right]$ in Z and

$$\begin{aligned} 0 = \lim_{j\to\infty} \frac{\|\tilde{\phi}(x_j) - \tilde{\phi}(p) - d\tilde{\phi}(p)[x_j - p]\|_Z}{|x_j - p|} \\ &= \lim_{j\to\infty} \left\| \frac{\tilde{\phi}(x_j) - \tilde{\phi}(p)}{|x_j - p|} - d\tilde{\phi}(p)\left[\frac{x_j - p}{|x_j - p|}\right] \right\|_Z \\ &= \lim_{j\to\infty} \left\| \frac{\phi(x_j) - \phi(p)}{|x_j - p|} - d\tilde{\phi}(p)\left[\frac{x_j - p}{|x_j - p|}\right] \right\|_Z \end{aligned}$$

and accordingly

$$\lim_{j\to\infty} \frac{\phi(x_j) - \phi(p)}{|x_j - p|} = \lim_{j\to\infty} d\tilde{\phi}(p)\left[\frac{x_j - p}{|x_j - p|}\right] = d\tilde{\phi}(p)[v] \qquad \text{in } Z$$

for all sequences $\{x_j\}_{j\in\mathbb{N}}$ as in Definition 5 of a semi-tangent vector. Then we can prove the following differentiation theorem for integral operators that are defined on upper Ahlfors regular subsets of $\mathbb{R}^n$. To do so, we set

$$\mathbb{B}_n(x,\rho) \equiv \{y \in \mathbb{R}^n : |x-y| < \rho\}$$

for all $\rho > 0$, $x \in \mathbb{R}^n$.

Theorem 5.1. *Let X, $Y \subseteq \mathbb{R}^n$. Let $\upsilon_Y \in]0,+\infty[$. Let $(Y,\mathcal{N},\nu)$ be a measured space such that $\mathcal{B}_Y \subseteq \mathcal{N}$. Let ν be finite. Let Y be upper υ_Y-Ahlfors regular with respect to X. Let $s_1 \in [0,\upsilon_Y[$. Let $x \in X$, $v \in T_xX$, $|v| = 1$. Let a kernel $K \in \mathcal{K}_{s_1,s_1+1,1}(X\times Y)$ satisfy the following assumptions*

$$D_{X,v}K(x,y) \text{ exists in } \mathbb{C} \qquad \forall y \in Y\setminus\{x\}\,,$$
$$D_{X,v}\int_Y K(x,y)\,d\nu(y) \text{ exists in } \mathbb{C}\,.$$

Let $\mu \in C^1_b(\mathbb{R}^n)$. Then the function $\int_Y K(\cdot,y)\mu(y)\,d\nu(y)$ admits a derivative with respect to v at the point x, the function $D_{X,v}K(x,y)(\mu(y)-\mu(x))$ is ν-integrable in the variable $y \in Y$ and the following formula holds

$$D_{X,v}\int_Y K(x,y)\mu(y)\,d\nu(y) \tag{5.1}$$
$$= \int_Y [D_{X,v}K(x,y)](\mu(y)-\mu(x))\,d\nu(y) + \mu(x)D_{X,v}\int_Y K(x,y)\,d\nu(y)$$

(see Definition 6 for $D_{X,v}$).

Proof. By the existence of $D_{X,v}\int_Y K(x,y)\,d\nu(y)$ and by the elementary equality

$$\int_Y K(x,y)\mu(y)\,d\nu(y)$$
$$= \int_Y K(x,y)(\mu(y)-\mu(x))\,d\nu(y) + \mu(x)\int_Y K(x,y)\,d\nu(y)$$

(cf. Proposition 4.1), the existence of $D_{X,v}\int_Y K(x,y)\mu(y)\,d\nu(y)$ is equivalent to the existence of $D_{X,v}\int_Y K(x,y)(\mu(y)-\mu(x))\,d\nu(y)$ and in the case of existence, we have

$$D_{X,v}\int_Y K(x,y)\mu(y)\,d\nu(y) = D_{X,v}\int_Y K(x,y)(\mu(y)-\mu(x))\,d\nu(y) \tag{5.2}$$
$$+D_{X,v}\mu(x)\int_Y K(x,y)\,d\nu(y) + \mu(x)D_{X,v}\int_Y K(x,y)\,d\nu(y)\,.$$

We now turn to show the existence of

$$D_{X,v}\int_Y K(x,y)(\mu(y)-\mu(x))\,d\nu(y) \tag{5.3}$$

and to compute it. Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in $X\setminus\{x\}$ such that

$$\lim_{j\to\infty} x_j = x\,, \qquad v = \lim_{j\to\infty}\frac{x_j - x}{|x_j - x|}\,.$$

By the existence of $D_{X,v}K(x,y)$, $D_{X,v}\mu(x)$ and by the continuity of $K(\cdot,y)$ and μ at x, we have

$$\lim_{j\to\infty}\frac{1}{|x_j-x|}\left[K(x_j,y)(\mu(y)-\mu(x_j)) - K(x,y)(\mu(y)-\mu(x))\right] \tag{5.4}$$
$$= D_{X,v}K(x,y)(\mu(y)-\mu(x)) - K(x,y)D_{X,v}\mu(x) \qquad \forall y \in Y\setminus\{x\}\,.$$

We now turn to show the existence of the limit associated to the directional derivative of the integral in (5.3) by applying the Vitali Convergence Theorem. If $E \in \mathcal{N}$, the Lipschitz continuity of μ, Lemma 3.4 with $\theta = 1$ and Lemma 4.1 imply that

$$\begin{aligned}
&\int_E \frac{1}{|x_j - x|} \left| K(x_j, y)(\mu(y) - \mu(x_j)) - K(x, y)(\mu(y) - \mu(x)) \right| \, d\nu(y) \\
&\quad \leq \int_{E \cap \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \left| K(x_j, y)(\mu(y) - \mu(x_j)) \right| \\
&\qquad + \int_{E \cap \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \left| K(x, y)(\mu(y) - \mu(x)) \right| \\
&\qquad + \int_{E \setminus \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \Big| K(x_j, y)(\mu(y) - \mu(x_j)) \\
&\qquad - K(x, y)(\mu(y) - \mu(x)) \Big| \, d\nu(y) \\
&\quad \leq \|H[K, \mu]\|_{\mathcal{K}_{s_1 - 1, s_1 + 1 - 1, 1 - (1 - 1)}} \left\{ \int_{E \cap \mathbb{B}_n(x_j, 3|x_j - x|)} \frac{1}{|x_j - x|} \frac{d\nu(y)}{|x_j - y|^{s_1 - 1}} \right. \\
&\qquad + \int_{E \cap \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \frac{d\nu(y)}{|x - y|^{s_1 - 1}} \\
&\qquad \left. + \int_{E \setminus \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \frac{|x_j - x|^1}{|x - y|^{s_1}} \, d\nu(y) \right\} \\
&\quad \leq \|H[K, \mu]\|_{\mathcal{K}_{s_1 - 1, s_1, 1}} \left\{ 3^{\upsilon_Y - (s_1 - 1)} c''_{s_1 - 1, X, Y} |x_j - x|^{\upsilon_Y - s_1} \right. \\
&\qquad + 2^{\upsilon_Y - (s_1 - 1)} c''_{s_1 - 1, X, Y} |x_j - x|^{\upsilon_Y - s_1} \\
&\qquad \left. + \nu(E) a^{-s_1} + c_{X, Y, \upsilon_Y} \frac{\upsilon_Y}{\upsilon_Y - s_1} a^{\upsilon_Y - s_1} \right\}
\end{aligned}$$

for all $a \in]0, r_{X,Y,\upsilon_Y}[$ and $j \in \mathbb{N}$, where the last summand in the braces is absent if $s_1 = 0$. Now let $\epsilon \in]0, +\infty[$. Then we choose $a \in]0, r_{X,Y,\upsilon_Y}[$ such that

$$\|H[K, \mu]\|_{\mathcal{K}_{s_1 - 1, s_1, 1}} c_{X, Y, \upsilon_Y} \frac{\upsilon_Y}{\upsilon_Y - s_1} a^{\upsilon_Y - s_1} \leq \epsilon / 3$$

and $j_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned}
\|H[K, \mu]\|_{\mathcal{K}_{s_1 - 1, s_1, 1}} &\left\{ 3^{\upsilon_Y - (s_1 - 1)} c''_{s_1 - 1, X, Y} |x_j - x|^{\upsilon_Y - s_1} \right. \\
&\qquad \left. + 2^{\upsilon_Y - (s_1 - 1)} c''_{s_1 - 1, X, Y} |x_j - x|^{\upsilon_Y - s_1} \right\} \leq \epsilon / 3
\end{aligned}$$

for all $j \in \mathbb{N}$ such that $j \geq j_\epsilon$. Thus if $E \in \mathcal{N}$ satisfies the inequality

$$\|H[K, \mu]\|_{\mathcal{K}_{s_1 - 1, s_1, 1}} \nu(E) a^{-s_1} \leq \epsilon / 3$$

we have

$$\int_E \frac{1}{|x_j - x|} \left| K(x_j, y)(\mu(y) - \mu(x_j)) - K(x, y)(\mu(y) - \mu(x)) \right| \, d\nu(y) \leq \epsilon$$

for all $j \in \mathbb{N}$ such that $j \geq j_\epsilon$. Then the pointwise convergence of (5.4) and the Vitali Convergence Theorem imply that the pointwise limit of (5.4) is integrable in $y \in Y \setminus \{x\}$ and that

$$\begin{aligned}\lim_{j\to\infty} \int_Y \frac{1}{|x_j - x|} \left[K(x_j, y)(\mu(y) - \mu(x_j)) - K(x, y)(\mu(y) - \mu(x))\right] d\nu(y) \\ = \int_Y D_{X,v} K(x, y)(\mu(y) - \mu(x)) - K(x, y) D_{X,v}\mu(x)\, d\nu(y)\,. \qquad (5.5)\end{aligned}$$

By our assumptions, $K(x, y)$ is integrable in $y \in Y \setminus \{x\}$ and $D_{X,v}\mu$ is bounded. Hence, $D_{X,v}K(x, y)(\mu(y) - \mu(x))$ is integrable in $y \in Y \setminus \{x\}$ and the right hand side of (5.5) equals

$$\int_Y D_{X,v} K(x, y)(\mu(y) - \mu(x))\, d\nu(y) - \int_Y K(x, y) D_{X,v}\mu(x)\, d\nu(y)\,.$$

Hence,

$$\begin{aligned}D_{X,v} \int_Y K(x, y)(\mu(y) - \mu(x))\, d\nu(y) \\ = \int_Y D_{X,v} K(x, y)(\mu(y) - \mu(x))\, d\nu(y) - \int_Y K(x, y)\, d\nu(y) D_{X,v}\mu(x)\end{aligned}$$

and formula (5.2) implies the validity of the formula of the statement. □

6 Tangential derivatives of weakly singular integral operators on embedded manifolds of $\mathbb{R}^n$ whose kernels have singular derivatives

Since a compact manifold Y of class C^1 that is embedded in $\mathbb{R}^n$ can be proved to be $(n-1)$-upper Ahlfors regular and each C^1 function on Y can be extended to a C^1 function in $\mathbb{R}^n$ with compact support (cf. *e.g.*, proof of Theorem 2.85 of Dalla Riva, the author and Musolino [4]), the differentiation Theorem 5.1 implies the validity of the following theorem, which is a variant of a known result. For the definition of tangential gradient grad_Y, we refer *e.g.*, to Kirsch and Hettlich [11, A.5], Chavel [2, Chapter 1].

Theorem 6.1. *Let $n \in \mathbb{N}$, $n \geq 2$. Let Y be a compact manifold of class C^1 that is embedded in $\mathbb{R}^n$. Let $s_1 \in [0, (n-1)[$. Let the kernel $K \in \mathcal{K}_{s_1, s_1+1, 1}(Y \times Y)$ satisfy the following assumptions*

$$K(\cdot, y) \in C^1(Y \setminus \{y\}) \qquad \forall y \in Y\,, \qquad \int_Y K(\cdot, y)\, d\sigma_y \in C^1(Y)\,.$$

Let $\text{grad}_{Y,x} K(\cdot, \cdot)$ denote the tangential gradient of $K(\cdot, \cdot)$ with respect to the first variable. Let $\mu \in C^1(Y)$. Then the function $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ is of class $C^1(Y)$, the function $[\text{grad}_{Y,x} K(x, y)](\mu(y) - \mu(x))$ is integrable in the variable $y \in Y$ for all $x \in Y$ and the following formula holds for the tangential gradient of $\int_Y K(\cdot, y)\mu(y)\, d\nu(y)$

$$\begin{aligned}\text{grad}_Y \int_Y K(x, y)\mu(y)\, d\sigma_y \qquad (6.1) \\ = \int_Y [\text{grad}_{Y,x} K(x, y)](\mu(y) - \mu(x))\, d\sigma_y + \mu(x)\text{grad}_Y \int_Y K(x, y)\, d\sigma_y\,,\end{aligned}$$

for all $x \in Y$.

Next we prove formula (6.1) for the tangential gradient under weaker assumptions for μ. To do so, however we must strenghten our assumptions on the kernel.

Theorem 6.2. *Let $n \in \mathbb{N}$, $n \geq 2$. Let Y be a compact manifold of class C^1 that is embedded in $\mathbb{R}^n$. Let $s_1 \in [0,(n-1)[$. Let $\beta \in]0,1]$, $t_1 \in]0,(n-1)+\beta[$. Let the kernel $K \in \mathcal{K}_{s_1,s_1+1,1}(Y \times Y)$ satisfy the following assumptions*

$$K(\cdot,y) \in C^1(Y \setminus \{y\}) \qquad \forall y \in Y\,, \qquad \int_Y K(\cdot,y)\,d\sigma_y \in C^1(Y)\,,$$
$$\mathrm{grad}_{Y,x} K(\cdot,\cdot) \in (\mathcal{K}_{t_1,Y\times Y})^n\,,$$

where $\mathrm{grad}_{Y,x}K(\cdot,\cdot)$ denotes the tangential gradient of $K(\cdot,\cdot)$ with respect to the first variable. Let $\mu \in C^{0,\beta}_b(Y)$. Then the function $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ is of class $C^1(Y)$, the function $[\mathrm{grad}_{Y,x}K(x,y)](\mu(y)-\mu(x))$ is integrable in $y \in Y$ for all $x \in Y$ and formula (6.1) for the tangential gradient of $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ holds true.

Proof. We plan to prove the statement by approximating μ by functions of class $C^1(Y)$ for which we know that the statement is true by Theorem 6.1. By the Mc Shane extension Theorem, there exists $\tilde{\mu} \in C^{0,\beta}_b(\mathbb{R}^n)$ that extends μ (cf. *e.g.*, Mc Shane [16], Björk [1, Prop. 1] Kufner, John and Fučik [12, Thm. 1.8.3]). Possibly multiplying $\tilde{\mu}$ by a function of class $C^\infty_c(\mathbb{R}^n)$, we can assume that $\tilde{\mu}$ has a compact support. Next we wish to approximate $\tilde{\mu}$ by functions of class $C^\infty_c(\mathbb{R}^n)$ by means of a standard family of mollifiers $\{\eta_\epsilon\}_{\epsilon\in]0,+\infty[}$ with

$$\mathrm{supp}\,\eta_\epsilon \subseteq \overline{\mathbb{B}_n(0,\epsilon)}\,, \qquad \eta_\epsilon \geq 0\,, \qquad \int_{\mathbb{R}^n} \eta_\epsilon\,dx = 1 \qquad \forall \epsilon \in]0,+\infty[$$

(cf. *e.g.*, Dalla Riva, the author and Musolino [4, A. 11]). Thus we set

$$\mu_l(x) \equiv \tilde{\mu} * \eta_{2^{-l}}(x) \qquad \forall x \in \mathbb{R}^n\,,$$

for all $l \in \mathbb{N}$. By known properties of the convolution, we have $\mu_l \in C^\infty_c(\mathbb{R}^n)$ for each $l \in \mathbb{N}$. Moreover,

$$\lim_{l\to\infty} \mu_l = \tilde{\mu} \qquad \text{uniformly in } \mathbb{R}^n\,.$$

We also observe that the Young inequality for the convolution implies that

$$\sup_{\mathbb{R}^n} |\mu_l| \leq \sup_{\mathbb{R}^n} |\tilde{\mu}| \int_{\mathbb{R}^n} |\eta_{2^{-l}}(y)|\,dy = \sup_{\mathbb{R}^n} |\tilde{\mu}| \qquad \forall l \in \mathbb{N}\,.$$

Then we note that $|\mu_l : \mathbb{R}^n|_\beta \leq |\tilde{\mu} : \mathbb{R}^n|_\beta$ for all $l \in \mathbb{N}$. Indeed, if x', $x'' \in \mathbb{R}^n$, then

$$\begin{aligned} |\mu_l(x') - \mu_l(x'')| &\leq \int_{\mathbb{R}^n} |\tilde{\mu}(x'-y) - \tilde{\mu}(x''-y)|\eta_{2^{-l}}(y)\,dy \\ &\leq |\tilde{\mu} : \mathbb{R}^n|_\beta |x'-x''|^\beta \int_{\mathbb{R}^n} \eta_{2^{-l}}(y)\,dy = |\tilde{\mu} : \mathbb{R}^n|_\beta |x'-x''|^\beta\,. \end{aligned}$$

Then the sequence $\{\mu_{l|Y}\}_{l\in\mathbb{N}}$ is bounded in $C^{0,\beta}(Y)$ and converges uniformly to μ in Y. Now let $\beta' \in]0,\beta[$, $0 < t_1 - \beta' < n-1$. By the compactness of the embedding of $C^{0,\beta}(Y)$ into $C^{0,\beta'}(Y)$, possibly selecting a subsequence, we can assume that

$$\lim_{l\to\infty} \mu_{l|Y} = \mu \qquad \text{in } C^{0,\beta'}(Y)\,.$$

By Lemma 3.4 (i), we have

$$\lim_{l\to\infty} \mathrm{grad}_{Y,x} K(x,y)(\mu_l(y)-\mu(x)) = \mathrm{grad}_{Y,x} K(x,y)(\mu(y)-\mu(x))$$

in $\mathcal{K}_{t_1-\beta',Y\times Y}$. Since $0 < t_1-\beta' < n-1$, the Hille-Tamarkin Proposition 4.1 and Proposition 4.3 imply that

$$\lim_{l\to\infty} \int_Y \mathrm{grad}_{Y,x} K(x,y)(\mu_l(y)-\mu_l(x))\,d\sigma_y = \int_Y \mathrm{grad}_{Y,x} K(x,y)(\mu(y)-\mu(x))\,d\sigma_y \qquad \text{uniformly in } x \in Y$$

and that $\int_Y \mathrm{grad}_{Y,x} K(\cdot,y)(\mu_l(y)-\mu_l(\cdot))\,d\sigma_y$ is continuous in Y for each $l \in \mathbb{N}$. Then the validity of formula (6.1) for μ_l implies that

$$\lim_{l\to\infty} \mathrm{grad}_{Y,x} \int_Y K(x,y)\mu_l(y)\,d\sigma_y = \int_Y [\mathrm{grad}_{Y,x} K(x,y)](\mu(y)-\mu(x))\,d\nu(y) + \mu(x)\mathrm{grad}_Y \int_Y K(x,y)\,d\sigma_y \tag{6.2}$$

uniformly in $x \in Y$. Since $K \in \mathcal{K}_{s_1,Y\times Y}$ and $s_1 < n-1$, again Proposition 4.1 and Proposition 4.3 imply that

$$\lim_{l\to\infty} \int_Y K(x,y)\mu_l(y)\,d\sigma_y = \int_Y K(x,y)\mu(y)\,d\sigma_y \tag{6.3}$$

uniformly in $x \in Y$ and that $\int_Y K(\cdot,y)\mu_l(y)\,d\sigma_y$ is continuous in Y for each $l \in \mathbb{N}$. By (6.2) and (6.3), we deduce that $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ belongs to $C^1(Y)$ and that formula (6.1) for its tangential gradient holds true. □

By combining Proposition 4.4 and the previous theorem, we can now prove a continuity theorem for the integral operator with kernel K and with values into a Schauder space on a compact manifold Y of class C^1. For the definition of the Schauder spaces $C^{1,\beta}(Y)$ and $C^{1,\omega(\cdot)}(Y)$ of all functions μ of class C^1 on Y such that the tangential gradient of μ is β-Hölder continuous and $\omega(\cdot)$-Hölder continuous, respectively or for an equivalent definition based on a finite family of parametrizations of Y, we refer for example to Dondi and the author [5, § 2], Dalla Riva, the author and Musolino [4, § 2.20].

Theorem 6.3. *Let $n \in \mathbb{N}$, $n \geq 2$. Let Y be a compact manifold of class C^1 that is embedded in $\mathbb{R}^n$. Let $s_1 \in [0,(n-1)[$. Let $\beta \in]0,1]$, $t_1 \in [\beta,(n-1)+\beta[$, $t_2 \in [\beta,+\infty[$, $t_3 \in]0,1]$. Let the kernel $K \in \mathcal{K}_{s_1,s_1+1,1}(Y\times Y)$ satisfy the following assumption*

$$K(\cdot,y) \in C^1(Y\setminus\{y\}) \quad \forall y \in Y\,.$$

Let $\mathrm{grad}_{Y,x}K(\cdot,\cdot)$ denote the tangential gradient of $K(\cdot,\cdot)$ with respect to the first variable.

(i) *If $t_1 < (n-1)$ and $\mathrm{grad}_{Y,x}K(\cdot,\cdot) \in (\mathcal{K}_{t_1,t_2,t_3}(Y\times Y))^n$, then the following statements hold.*

(a) *If $t_2-\beta > (n-1)$, $t_2 < (n-1)+\beta+t_3$ and*

$$\int_Y K(\cdot,y)\,d\sigma_y \in C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ to $C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)$ that takes μ to the function $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ is linear and continuous.

(aa) *If* $t_2 - \beta = (n-1)$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\max\{r^\beta, \omega_{t_3}(\cdot)\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

(ii) *If* $t_1 = (n-1)$ *and* $\mathrm{grad}_{Y,x} K(\cdot,\cdot) \in \left(\mathcal{K}^\sharp_{t_1,t_2,t_3}(Y \times Y)\right)^n$, *then the following statements hold.*

(b) *If* $t_2 - \beta > (n-1)$, $t_2 < (n-1) + \beta + t_3$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

(bb) *If* $t_2 - \beta = (n-1)$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\max\{r^\beta, \omega_{t_3}(\cdot)\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

(iii) *If* $t_1 > (n-1)$ *and* $\mathrm{grad}_{Y,x} K(\cdot,\cdot) \in (\mathcal{K}_{t_1,t_2,t_3}(Y \times Y))^n$, *then the following statements hold.*

(c) *If* $t_2 - \beta > (n-1)$, $t_2 < (n-1) + \beta + t_3$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

(cc) *If* $t_2 - \beta = (n-1)$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\max\{r^\beta, r^{(n-1)+\beta-t_1}, \omega_{t_3}(\cdot)\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\max\{r^\beta,r^{(n-1)+\beta-t_1},\omega_{t_3}(\cdot)\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

Proof. Since Y is a compact manifold of class C^1 that is embedded in $\mathbb{R}^n$, Y can be proved to be strongly upper $(n-1)$-Ahlfors regular with respect to Y. By Theorem 6.2, the function $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ is of class $C^1(Y)$ for all $\mu \in C^{0,\beta}(Y)$ and formula (6.1) for the tangential gradient of $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ holds true under any of the assumptions of (i)–(iii). Next, we consider statement (i). Under the assumptions of (a), Proposition 4.4 (i) implies that map

$$\text{from} \qquad C^{0,\beta}(Y) \qquad \text{to} \qquad C^{0,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

which takes μ to the function $\int_Y [\mathrm{grad}_{Y,x} K(x,y)](\mu(y)-\mu(x))\,d\sigma_y$ is linear and continuous. By our assumption on $\int_Y K(\cdot,y)\,d\sigma_y$, the map

$$\text{from} \qquad C^{0,\beta}(Y) \qquad \text{to} \qquad C^{0,\min\{\beta,\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

which takes μ to $\mu(\cdot)\mathrm{grad}_Y \int_Y K(\cdot,y)\,d\sigma_y$ is linear and continuous. Then formula (6.1) implies that the map

$$\text{from} \qquad C^{0,\beta}(Y) \qquad \text{to} \qquad C^{0,\min\{\beta,\beta,(n-1)+t_3+\beta-t_2\}}(Y)$$

that takes μ to $\mathrm{grad}_{Y,x} \int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ is linear and continuous. By Propositions 4.1 and 4.3, the map from $C^{0,\beta}(Y)$ to $C^0(Y)$ that takes μ to $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ is linear and continuous. Hence, we deduce the validity of (a) of statement (i). The proof of (aa) follows the lines of the proof of statement (a) by invoking statement (aa) instead of statement (a) of (i) of Proposition 4.4.

The proofs of statements (ii) and (iii) follow the lines of that of statement (i) by invoking statements (ii) and (iii) instead of statement (i) of Proposition 4.4. In case of statement (iii) (c) we also observe that the pointwise product is bilinear and continuous

$$\text{from} \qquad C^{0,\beta}(Y) \times C^{0,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y) \qquad \text{to} \qquad C^{0,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y)\,.$$

In the case of statement (iii) (cc) we also observe that the pointwise product is bilinear and continuous

$$\text{from} \qquad C^{0,\beta}(Y) \times C^{0,\max\{r^{\beta},r^{(n-1)+\beta-t_1},\omega_{t_3}(\cdot)\}}(Y) \qquad \text{to} \qquad C^{0,\max\{r^{\beta},r^{(n-1)+\beta-t_1},\omega_{t_3}(\cdot)\}}(Y)$$

(cf. *e.g.*, Dondi and the author [5, § 2]). □

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NEW 2-MICROLOCAL BESOV AND TRIEBEL–LIZORKIN SPACES VIA THE LITLLEWOOD – PALEY DECOMPOSITION

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Key words: wavelet, Besov space, Triebel–Lizorkin space, pseudo-differential operator, Calderón–Zygmund operator, atomic and molecular decomposition, 2-microlocal space, φ–transform.

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Abstract. In this paper we introduce and investigate new 2-microlocal Besov and Triebel–Lizorkin spaces via the Littlewood - Paley decomposition. We establish characterizations of these function spaces by the φ–transform, the atomic and molecular decomposition and the wavelet decomposition. As applications we prove boundedness of the the Calderón–Zygmund operators and the pseudo–differential operators on the function spaces. Moreover, we give characterizations via oscillations and differences.

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1 Introduction

It is well known that function spaces have increasing applications in many areas of modern analysis, in particular, harmonic analysis and partial differential equations. The most general function spaces, probably, are the Besov spaces and the Triebel–Lizorkin spaces which cover many classical concrete function spaces such as Lebesgue spaces, Lipschitz spaces, Sobolev spaces, Hardy spaces and BMO spaces ([37], [38]).

D. Yang and W. Yuan in [41], [42] and W. Sickel, D. Yang and W. Yuan in [36], introduced a class of Besov type and Triebel–Lizorkin type spaces which generalized many classical function spaces such as Besov spaces, Triebel-Lizorkin spaces, Morrey spaces and Q-type spaces. Recently the Besov type and Triebel-Lizorkin type space with variable exponents was investigated by many authors (e.g. [43], [44]).

The 2-microloal space is due to Bony [3] in order to study the propagation of singularities of the solutions of nonlinear evolution equations. It is an appropriate instrument to describe the local regularity and the oscillatory behavior of functions near singularity (Meyer [32]). The theory has been elaborated and widely used in fractal analysis and signal processing. For systematic discussions of the concept and further references of 2-microlocal spaces, we refer to Meyer[31], [32], Levy-Vehel and Seuret [30], Jaffard ([17], [18], [19], [20]), Jaffard and Mélot [21], and Jaffard and Meyer [22].

The 2-microlocal spaces have been generalized by Jaffard as a general pointwise regularity associated with Banach or quasi-Banach spaces [19], [20]. In this paper we introduce new inhomogeneous 2-microlocal spaces based on Jaffard's idea (See [33] for the homogeneous 2-microlocal spaces) and we will investigate the properties and the characterizations of these new 2-microlocal Besov and Triebel–Lizorkin spaces which unify many classical function spaces such as the Besov type and Triebel–Lizorkin type spaces, the 2–microlocal spaces in the sense of Meyer [32], the Morrey space and the local Morrey spaces. These new function space are very similar to the classical 2-microlocal

Besov and Triebel-Lizorkin spaces studied recently by many authors ([1], [6], [8], [13], [14], [15], [16], [25], [26], [39], [40]).

The plan of the remaining sections in the paper is as follows:

In Section 2 we give the definitions of our new 2-microlocal spaces via the Littlewood-Paley decomposition and the notations which are used later and we give examples for these spaces.

In Section 3 we define corresponding sequence spaces for our function spaces. Furthermore, we give some auxiliary lemmas which are needed in later sections.

In Section 4 we will characterize our function spaces via the corresponding sequence spaces by the φ–transform in the sense of Fraizer–Jarwerth [10], the atomic and molecular decomposition and the wavelet decomposition. Moreover, we investigate the properties for these function spaces and we also study relations between our 2-microlocal spaces and the classical 2-microlocal spaces.

In Section 5, as applications, we give the conditions under which the Calderón–Zygmund operators and the pseudo–differential operators are bounded on the function spaces.

In Section 6 we give the characterizations via differences and oscillations.

Throughout the paper, we use C to denote a positive constant. But the same notation C are not necessarily the same on any two occurrences. We use the notations $i \vee j = \max\{i,\ j\}$, $i \wedge j = \min\{i,\ j\}$, and $a_+ = a \vee 0$. The symbol $X \sim Y$ means that there exist positive constants C_1 and C_2 such that $X \leq C_1 Y$ and $Y \leq C_2 X$.

2 Definitions

We consider the dyadic cubes in $\mathbb{R}^n$ of the form $Q = [0,\ 2^{-l})^n + 2^{-l}k$ for $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}$, and use the notation $l(Q) = 2^{-l}$ for the side length and $x_Q = 2^{-l}k$ for the corner point. Throughout the paper, we use the notations $P,\ Q,\ R$ for the dyadic cubes of the form $[0,\ 2^{-l})^n + 2^{-l}k$ in $\mathbb{R}^n$, and when the dyadic cubes Q appear as indices, it is understood that Q runs over all dyadic cubes of this form in $\mathbb{R}^n$. We denote by $\mathcal{D}$ the set of all dyadic cubes of this form. For a dyadic cube Q and a constant $c > 1$, cQ denotes the cube of same center as Q and c times larger. We denote by χ_E the characteristic function of a set E in $\mathbb{R}^n$.

We set $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'$ its dual.

We use $\langle f,\ g\rangle$ for the standard inner product $\int f\bar{g}$ of two functions and the same notation is employed for the action of a distribution $f \in \mathcal{S}'$ on $\bar{g} \in \mathcal{S}$.

Let ϕ_0 be a Schwartz function and $\hat{\phi}_0$ its Fourier transform satisfying

(1.1) $\operatorname{supp} \hat{\phi}_0 \subset \{\xi \in \mathbb{R}^n :\ |\xi| \leq 2\}$,

(1.2) $\hat{\phi}_0(\xi) = 1$ if $|\xi| \leq 1$.

We set

$\phi(x) = \phi_0(x) - 2^{-n}\phi_0(2^{-1}x)$, $\phi_0^j = 2^{jn}\phi_0(2^j x)$, $S_j f = f * \phi_0^j$ for $j \in \mathbb{N}_0$, and $\phi_j(x) = 2^{jn}\phi(2^j x)$ for $j \in \mathbb{N}$.

Then we have

(1.3) $\operatorname{supp} \hat{\phi} \subset \{\xi \in \mathbb{R}^n :\ \frac{1}{2} \leq |\xi| \leq 2\}$, and

(1.4) there exist positive numbers c and a sufficiently small ϵ such that $\hat{\phi}(\xi) \geq c$ in $1 - \epsilon \leq |\xi| \leq 1 + \epsilon$.

It holds that $\sum_{j \in \mathbb{N}_0} \hat{\phi}_j = 1$. Let $f \in \mathcal{S}'$, then we have the Littlewood-Paley decomposition $f = \sum_{j \in \mathbb{N}_0} f * \phi_j$ (convergence in $\mathcal{S}'$) [36, Triebel 2.3.1(6)].

Let $s \in \mathbb{R}$. For $f \in \mathcal{S}'$, we define some sequences indexed by dyadic cubes P:

$c(B^s_{pq})(P) = (\sum_{i \geq (-\log_2 l(P)) \vee 0} ||2^{is} f * \phi_i||^q_{L^p(P)})^{1/q}$, $0 < p, q \leq \infty$,

$c(F^s_{pq})(P) = ||\{\sum_{i \geq (-\log_2 l(P)) \vee 0} (2^{is}|f * \phi_i|)^q\}^{1/q}||_{L^p(P)}$,

$0 < p < \infty$, $0 < q \leq \infty$,
$c(F^s_{\infty q})(P) = l(P)^{-\frac{n}{q}} ||\{\sum_{i \geq (-\log_2 l(P)) \vee 0} (2^{is}|f * \phi_i|)^q\}^{1/q}||_{L^q(P)}$,
$0 < q \leq \infty$,
with the usual modification for $q = \infty$.

We shall use the notation E^s_{pq} with either B^s_{pq} or F^s_{pq}. We say the B-type case when $E^{s'}_{pq} = B^{s'}_{pq}$, and the F-type case when $E^{s'}_{pq} = F^{s'}_{pq}$.

Definition 1. Let $s,\ s',\ \sigma\ \in \mathbb{R},\ 0 < p, q \leq \infty$ and $x_0 \in \mathbb{R}^n$.
The space $A^s(E^{s'}_{pq})^\sigma_{x_0}$ is defined to be the space of all $f \in \mathcal{S}'$ such that

$$||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} \equiv \sup_{\mathcal{D} \ni Q \ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D} \ni P \subset 3Q} l(P)^{-s} c(E^{s'}_{pq})(P) < \infty.$$

The following abbreviation $A^0(E^{s'}_{pq})^\sigma_{x_0} \equiv (E^{s'}_{pq})^\sigma_{x_0}$, $A^s(E^{s'}_{pq})^0_{x_0} \equiv A^s(E^{s'}_{pq})$ and $A^0(E^{s'}_{pq})^0_{x_0} \equiv E^{s'}_{pq} \equiv E^{s'}_{pq}(\mathbb{R}^n)$ will be used in the sequel. We note that the space $A^s(E^{s'}_{pq})$ is the inhomogeneous Besov type space or the inhomogeneous Triebel–Lizorkin type space in the sense of Yang–Sickel–Yuan [26] and the space $E^{s'}_{pq} \equiv E^{s'}_{pq}(\mathbb{R}^n)$ is the classical inhomogeneous Besov or inhomogeneous Triebel–Lizorkin space.

Let $f \in \mathcal{S}'$, then we define some sequences indexed by dyadic cubes P:

$c(\tilde{B}^{s'}_{pq})^\sigma_{x_0}(P) =$
$(\sum_{i \geq (-\log_2 l(P)) \vee 0} ||2^{is'}|f * \phi_i(x)|(2^{-i} + |x_0 - x|)^{-\sigma}||^q_{L^p(P)})^{1/q}$,
$0 < p,\ q \leq \infty$,
$c(\tilde{F}^{s'}_{pq})^\sigma_{x_0}(P) =$
$||\{\sum_{i \geq (-\log_2 l(P)) \vee 0} (2^{is'}|f * \phi_i(x)|(2^{-i} + |x_0 - x|)^{-\sigma})^q\}^{1/q}||_{L^p(P)}$,
$0 < p < \infty$, $0 < q \leq \infty$,
$c(\tilde{F}^{s'}_{\infty q})^\sigma_{x_0}(P) =$
$l(P)^{-\frac{n}{q}} ||\{\sum_{i \geq (-\log_2 l(P)) \vee 0} (2^{is'}|f * \phi_i(x)|(2^{-i} + |x_0 - x|)^{-\sigma})^q\}^{1/q}||_{L^q(P)}$,
$0 < q \leq \infty$,
with the usual modification for $q = \infty$.

We shall use the notation $\tilde{E}^{s'}_{pq}$ with either $\tilde{B}^{s'}_{pq}$ or $\tilde{F}^{s'}_{pq}$. We say the B-type case when $\tilde{E}^{s'}_{pq} = \tilde{B}^{s'}_{pq}$, and the F-type case when $\tilde{E}^{s'}_{pq} = \tilde{F}^{s'}_{pq}$.

Definition 2. Let $s,\ s',\ \sigma\ \in \mathbb{R},\ 0 < p, q \leq \infty$ and $x_0 \in \mathbb{R}^n$.
The space $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ is defined to be the space of all $f \in \mathcal{S}'$ such that

$$||f||_{A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}} \equiv \sup_{\mathcal{D} \ni P} l(P)^{-s} c(\tilde{E}^{s'}_{pq})^\sigma_{x_0}(P) < \infty.$$

The space $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ is the classical 2–microlocal Besov or Triebel–Lizorkin space.
We use the abbreviation $A^0(\tilde{E}^{s'}_{pq})^\sigma_{x_0} \equiv (\tilde{E}^{s'}_{pq})^\sigma_{x_0}$.

Examples.

(i) The spaces $A^0(E^{s'}_{pq})^0_{x_0} = A^0(\tilde{E}^{s'}_{pq})^0_{x_0} = E^{s'}_{pq}(\mathbb{R}^n)$ are the inhomogeneous Besov spaces or inhomogeneous Triebel–Lizorkin spaces [37], [38].

(ii) The Besov type spaces $B^{s,\tau}_{pq}(\mathbb{R}^n)$ and the Triebel–Lizorkin type spaces $F^{s,\tau}_{pq}(\mathbb{R}^n)$ introduced by D. Yang , W. Sickel and W. Yuan [36] , are contained in our definition as

$E^{s',s}_{pq}(\mathbb{R}^n) = A^{ns}(E^{s'}_{pq})^0_{x_0} = A^{ns}(\tilde{E}^{s'}_{pq})^0_{x_0}$.

(iii) The Besov-Morrey spaces $\mathcal{N}^s_{uqp}$, and the Triebel–Lizorkin-Morrey spaces $\mathcal{E}^s_{uqp}$ studied by Y. Sawano and H. Tanaka [34], or Y. Sawano, D. Yang and W. Yuan [35] are realized in our definition as

$\mathcal{N}^s_{uqp} \subset A^{n(\frac{1}{p}-\frac{1}{u})}(B^s_{pq})^0_{x_0}$ if $0 < p \le u \le \infty$ and $0 < q \le \infty$,

$\mathcal{E}^s_{uqp} = A^{n(\frac{1}{p}-\frac{1}{u})}(F^s_{pq})^0_{x_0}$ if $0 < p \le u \le \infty$ and $0 < q \le \infty$.

The Morrey space $\mathcal{M}^u_p$ is realized as

$\mathcal{M}^u_p = A^{n(\frac{1}{p}-\frac{1}{u})}(F^0_{p2})^0_{x_0}$ if $1 < p < u < \infty$.

(iv) The $\dot{B}_\sigma$-Morrey spaces $\dot{B}_\sigma(L_{p,\lambda})$ studied by Y. Komori-Furuya et al. [28], are contained in our definition as

$\dot{B}_\sigma(L_{p,\lambda}) = A^{\lambda+\frac{n}{p}}(F^0_{p2})^\sigma_0,\ \ 1 < p < \infty.$

(v) The 2-microlocal Besov spaces $B^{s,s'}_{pq}(U)$ studied in H. Kempka [23, 24], are realized in our definition as

$B^{s,s'}_{pq}(U) = (\tilde{B}^{s+s'}_{pq})^{-s'}_{x_0}$ when $U = \{x_0\}$.

(vi) The local Morrey spaces $LM_{p,\lambda}$ introduced by V.I. Burenkov and H.V. Guliyes [6] and studied in Ts. Batbold and Y. Sawano [2] and a number of papers, are realized in our definition as

$LM_{p,\lambda} = (F^0_{p2})^{\lambda/p}_0,\ \ 1 < p < \infty.$

(vii) The spaces $C^{s,s'}_{x_0}$ studied in Y. Meyer [31], [32], are realized in our definition as

$C^{s,s'}_{x_0} = (\tilde{B}^{s+s'}_{\infty\infty})^{-s'}_{x_0} = (B^{s+s'}_{\infty\infty})^{-s'}_{x_0}$.

3 Sequence spaces

For a sequence $c = (c(R))$ with $l(R) \le 1$ we define some sequences indexed by dyadic cubes P:

$c(b^s_{pq})(P) = (\sum_{i \ge (-\log_2 l(P)) \vee 0} || \sum_{l(R)=2^{-i}} 2^{is}|c(R)|\chi_R||^q_{L^p(P)})^{1/q}$,
$0 < p,\ q \le \infty$,
$c(f^s_{pq})(P) = ||\{\sum_{i \ge (-\log_2 l(P)) \vee 0} (\sum_{l(R)=2^{-i}} 2^{is}|c(R)|\chi_R)^q\}^{1/q}||_{L^p(P)}$,
$0 < p < \infty,\ 0 < q \le \infty$, and
$c(f^s_{\infty q})(P) = l(P)^{-\frac{n}{q}}||\{\sum_{i \ge (-\log_2 l(P)) \vee 0} (\sum_{l(R)=2^{-i}} 2^{is}|c(R)|\chi_R)^q\}^{1/q}||_{L^q(P)}$,
$0 < q \le \infty$, with the usual modification for $q = \infty$.

The notation e^s_{pq} is used to denote either b^s_{pq} or f^s_{pq}. We say the B-type case when $e^{s'}_{pq} = b^{s'}_{pq}$, and the F-type case when $e^{s'}_{pq} = f^{s'}_{pq}$.

Definition 3. Let $s,\ s',\ \sigma\ \in \mathbb{R},\ 0 < p, q \le \infty$ and $x_0 \in \mathbb{R}^n$.

We define the sequence space $a^s(e^{s'}_{pq})^\sigma_{x_0}$ to be the space of all sequences $c = (c(R))_{l(R) \le 1}$ such that

$$||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \equiv \sup_{\mathcal{D} \ni Q \ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D} \ni P \subset 3Q} l(P)^{-s} c(e^{s'}_{pq})(P) < \infty.$$

We use the abbreviation $a^0(e^{s'}_{pq})^{\sigma}_{x_0} \equiv (e^{s'}_{pq})^{\sigma}_{x_0}$, $a^s(e^{s'}_{pq})^0_{x_0} \equiv a^s(e^{s'}_{pq})$ and $a^0(e^{s'}_{pq})^0_{x_0} \equiv e^{s'}_{pq} \equiv e^{s'}_{pq}(\mathbb{R}^n)$. We note that the space $a^s(e^{s'}_{pq})$ is the sequence space of the inhomogeneous Besov type space or the inhomogeneous Triebel–Lizorkin type space in the sense of Yang–Sickel–Yuan [36] and the space $e^{s'}_{pq} \equiv e^{s'}_{pq}(\mathbb{R}^n)$ is the sequence space of the classical inhomogeneous Besov or inhomogeneous Triebel–Lizorkin space.

Remark 1. It is easy that when $\sigma < 0$, we have $A^s(E^{s'}_{pq})^{\sigma}_{x_0} = \{0\}$ and $a^s(e^{s'}_{pq})^{\sigma}_{x_0} = \{0\}$ for $0 < p, q \leq \infty$ (See Proposition 4.1 below).

We define that for a sequence $(c(R))_{l(R)\leq 1}$,

$c(\tilde{b}^{s'}_{pq})^{\sigma}_{x_0}(P) =$
$(\sum_{i\geq(-\log_2 l(P))\vee 0} || \sum_{l(R)=2^{-i}} 2^{is'}|c(R)|(2^{-i}+|x_0-x|)^{-\sigma}\chi_R||^q_{L^p(P)})^{1/q}$,
$0 < p,\ q \leq \infty$,
$c(\tilde{f}^{s'}_{pq})^{\sigma}_{x_0}(P) =$
$||\{\sum_{i\geq(-\log_2 l(P))\vee 0}(\sum_{l(R)=2^{-i}} 2^{is'}|c(R)|(2^{-i}+|x_0-x|)^{-\sigma}\chi_R)^q\}^{1/q}||_{L^p(P)}$,
$0 < p < \infty$, $0 < q \leq \infty$,
$c(\tilde{f}^{s'}_{\infty q})^{\sigma}_{x_0}(P) = l(P)^{-\frac{n}{q}} \times$
$||\{\sum_{i\geq(-\log_2 l(P))\vee 0}(\sum_{l(R)=2^{-i}} 2^{is'}|c(R)|(2^{-i}+|x_0-x|)^{-\sigma}\chi_R)^q\}^{1/q}||_{L^q(P)}$,
$0 < q \leq \infty$,
with the usual modification for $q = \infty$.

The notation $\tilde{e}^{s'}_{pq}$ is used to denote either $\tilde{b}^{s'}_{pq}$ or $\tilde{f}^{s'}_{pq}$. We say the B-type case when $\tilde{e}^{s'}_{pq} = \tilde{b}^{s'}_{pq}$, and the F-type case when $\tilde{e}^{s'}_{pq} = \tilde{f}^{s'}_{pq}$.

Definition 4. Let $s,\ s',\ \sigma \in \mathbb{R}$, $0 < p, q \leq \infty$ and $x_0 \in \mathbb{R}^n$.
We define the sequence space $a^s(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}$ to be the space of all sequences $c = (c(R))_{l(R)\leq 1}$ such that

$$||c||_{a^s(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}} \equiv \sup_{\mathcal{D}\ni P} l(P)^{-s} c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P) < \infty.$$

We use the abbreviation $a^0(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0} \equiv (\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}$.

Definition 5. Let $r_1,\ r_2 \geq 0$ and $L > 0$. We say that a matrix operator $A = \{a_{QP}\}_{QP}$, indexed by dyadic cubes Q and P, is (r_1, r_2, L)-almost diagonal if the matrix $\{a_{QP}\}$ satisfies

$|a_{QP}| \leq C\big(\frac{l(Q)}{l(P)}\big)^{r_1}(1+l(P)^{-1}|x_Q-x_P|)^{-L}$ if $l(Q) \leq l(P)$,
$|a_{QP}| \leq C\big(\frac{l(P)}{l(Q)}\big)^{r_2}(1+l(Q)^{-1}|x_Q-x_P|)^{-L}$ if $l(Q) > l(P)$.

The results about the boundedness of almost diagonal operators in [9: Theorem 3.3], also hold in our cases.

Lemma 3.1. *Suppose that $s,\ s',\ \sigma \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $0 < p,\ q \leq \infty$. Then,*

(i) *an (r_1, r_2, L)–almost diagonal matrix operator A is bounded on $a^s(e^{s'}_{pq})^{\sigma}_{x_0}$ for $r_1 > \max(s',\ \sigma + s + s' - \frac{n}{p})$, $r_2 > J - s'$ and $L > J$ where $J = n/\min(1,\ p,\ q)$ in the case $e^{s'}_{pq} = f^{s'}_{pq}$, and $J = n/\min(1,\ p)$ in the case $e^{s'}_{pq} = b^{s'}_{pq}$, respectively,*

(ii) *an (r_1, r_2, L)-almost diagonal matrix operator A is bounded on $a^s(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}$ for $r_1 > \max(s' + (\sigma\vee 0),\ (\sigma\vee 0) + s + s' - \frac{n}{p})$, $r_2 > J - s' + (\sigma\wedge 0)$ and $L > J$ where $J = n/\min(1,\ p,\ q)$ in the case $\tilde{e}^{s'}_{pq} = \tilde{f}^{s'}_{pq}$, and $J = n/\min(1,\ p)$ in the case $\tilde{e}^{s'}_{pq} = \tilde{b}^{s'}_{pq}$, respectively.*

Proof : (i) We may assume $\sigma \geq 0$ by Remark 1. We assume that $A = (a_{RR'})$ is (r_1, r_2, L)–almost diagonal. Let $c = (c(R)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. For dyadic cubes P and R with $R \subset P$, we write $Ac = A_0c + A_1c + A_2c$ with

$$(A_0c)(R) = \sum_{l(R)\leq l(R')\leq l(P)} a_{RR'}c(R'),$$
$$(A_1c)(R) = \sum_{l(R')<l(R)\leq l(P)} a_{RR'}c(R'),$$
$$(A_2c)(R) = \sum_{l(R)\leq l(P)<l(R')\leq 1} a_{RR'}c(R').$$

We claim that

$$||A_ic||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \leq C||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}, \quad i = 0, 1, 2.$$

We will consider the case of F-type for $0 < p < \infty$, $0 < q \leq \infty$. Since A is almost diagonal, we see that for dyadic cubes P with $l(P) = 2^{-j}$,

$$\begin{aligned}
(A_0c)(f^{s'}_{pq})(P) &= ||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} \big(2^{is'}|(A_0c)(R)|\big)^q\chi_R\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q}\big(\sum_{i\geq k\geq j\vee 0}\sum_{l(R')=2^{-k}} |a_{RR'}||c(R')|\big)^q\chi_R\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q}\times \\
&\big(\sum_{i\geq k\geq j\vee 0}\sum_{l(R')=2^{-k}} 2^{-(i-k)r_1}(1+2^k|x_R - x_{R'}|)^{-L}|c(R')|\big)^q\chi_R\}^{1/q}||_{L^p(P)}.
\end{aligned}$$

Using the maximal function $M_tf(x)$, $0 < t \leq 1$, defined by

$$M_tf(x) = \sup_{x\in Q}\big(\frac{1}{l(Q)^n}\int_Q |f(y)|^t\, dy\big)^{1/t}$$

(cf. [28: Lemma 7.1] or [9: Remark A.3]), we have for $L > n/t$,

$$\begin{aligned}
(A_0c)(f^{s'}_{pq})(P) &\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q}2^{-ir_1q}\times \\
&\Big(\sum_{i\geq k\geq j\vee 0} 2^{kr_1}2^{(k-i)+n/t}M_t\big(\sum_{l(R')=2^{-k}} |c(R')|\chi_{R'}\big)\Big)^q\chi_R\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0} 2^{-i(r_1-s')q}\Big(\sum_{i\geq k\geq j\vee 0} 2^{kr_1}M_t\big(\sum_{l(R')=2^{-k}} |c(R')|\chi_{R'}\big)\Big)^q\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0} 2^{is'q}M_t\big(\sum_{l(R')=2^{-i}} |c(R')|\chi_{R'}\big)^q\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0} 2^{is'q}\big(\sum_{l(R')=2^{-i}} |c(R')|\chi_{R'}\big)^q\}^{1/q}||_{L^p(P)} = Cc(f^{s'}_{pq})(P),
\end{aligned}$$

where these inequalities follow from Hardy's inequality if $r_1 > s'$ and the Fefferman-Stein inequality if $0 < t < \min(p, q)$.

For the B-type case we have the same estimate for $r_1 > s'$ and $0 < t < \min(1, p)$.

Therefore, we get the estimate

$$A_0 c(e^{s'}_{pq})(P) \leq Cc(e^{s'}_{pq})(P)$$

if $r_1 > s'$, $0 < p < \infty$, $0 < q \leq \infty$, $L > J$.

In the sane way we will get the estimate for $(A_1 c)(f^{s'}_{pq})(P)$. We have that for dyadic cubes P with $l(P) = 2^{-j}$,

$$\begin{aligned} (A_1 c)(f^{s'}_{pq})(P) &= ||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} \big(2^{is'}|(A_1 c)(R)|\big)^q \chi_R\}^{1/q}||_{L^p(P)} \\ &\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q}\big(\sum_{i\leq k}\sum_{l(R')=2^{-k}} |a_{RR'}||c(R')|\big)^q \chi_R\}^{1/q}||_{L^p(P)} \\ &\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q} \times \\ \big(\sum_{i\leq k}\sum_{l(R')=2^{-k}} & 2^{-(k-i)r_2}(1+2^i|x_R - x_{R'}|)^{-L}|c(R')|\big)^q \chi_R\}^{1/q}||_{L^p(P)}. \end{aligned}$$

Using the maximal function $M_t f(x)$ as above, we have

$$\begin{aligned} (A_1 c)(f^{s'}_{pq})(P) &\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q}2^{ir_2 q}\times \\ &\Big(\sum_{i\leq k} 2^{-kr_2}2^{(k-i)+n/t}M_t\big(\sum_{l(R')=2^{-k}} |c(R')|\chi_{R'}\big)\Big)^q \chi_R\}^{1/q}||_{L^p(P)} \\ &\leq C||\{\sum_{i\geq j\vee 0} 2^{i(r_2+s'-n/t)q} \times \\ &\Big(\sum_{i\leq k} 2^{-k(r_2-n/t)}M_t\big(\sum_{l(R')=2^{-k}} |c(R')|\chi_{R'}\big)\Big)^q\}^{1/q}||_{L^p(P)} \\ &\leq C||\{\sum_{i\geq j\vee 0} 2^{is'q}M_t\big(\sum_{l(R')=2^{-i}} |c(R')|\chi_{R'}\big)^q\}^{1/q}||_{L^p(P)} \\ &\leq C||\{\sum_{i\geq j\vee 0} 2^{is'q}\big(\sum_{l(R')=2^{-i}} |c(R')|\chi_{R'}\big)^q\}^{1/q}||_{L^p(P)} = Cc(f^{s'}_{pq})(P), \end{aligned}$$

where these inequalities follow from Hardy's inequality if $r_2 + s' - n/t > 0$ and the Fefferman-Stein inequality if $0 < t < \min(p, q)$.

In the same way we get the same estimate for the B-type case that

$$(A_1 c)(b^{s'}_{pq})(P) \leq Cc(b^{s'}_{pq})(P)$$

if $r_2 + s' - n/t > 0$, $0 < t < \min(1, p)$. Therefore, we get the estimate

$$A_1 c(e^{s'}_{pq})(P) \leq Cc(e^{s'}_{pq})(P)$$

if $r_2 > J - s'$, $0 < p < \infty$, $0 < q \leq \infty$, $L > J$.

When $p = \infty$, we get the same estimate. Thus, we get

$$||A_i c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \leq C||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}, \quad i = 0, 1$$

if $r_1 > s'$, $r_2 > J - s'$, $L > J$, $0 < p \le \infty$ and $0 < q \le \infty$.

Next, we will give the estimates for the A_2 case.

We note that if $L > n$,

$$\sum_{l(P)=2^{-j}} (1 + 2^j|x_R - x_P|)^{-L} < \infty$$

(cf. [4, Lemma 3.4]), and if $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$, then

$$|c(R)| \le C(|x_0 - x_R| + l(R))^\sigma l(R)^{s+s'-n/p}||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}$$

for a dyadic cube $R \subset 3Q$ and $x_0 \in Q$. Hence, we obtain, for dyadic cubes P with $l(P) = 2^{-j}$, $0 < p < \infty$ and $0 < q \le \infty$,

$$\begin{aligned}
&(A_2c)(f^{s'}_{pq})(P) = ||\{\sum_{i\ge j} \sum_{l(R)=2^{-i}} \left(2^{is'}|(A_2c)(R)|\right)^q \chi_R\}^{1/q}||_{L^p(P)} \\
&\le C||\{\sum_{i\ge j} \sum_{l(R)=2^{-i}} 2^{is'q}\times \\
&(\sum_{j\ge k\ge 0} \sum_{l(R')=2^{-k}} 2^{-(i-k)r_1}(1 + 2^k|x_R - x_{R'}|)^{-L}|c(R')|)^q \chi_R\}^{1/q}||_{L^p(P)} \\
&\le C||\{\sum_{i\ge j} 2^{-i(r_1-s')q}\times \\
&\quad (\sum_{j\ge k\ge 0} 2^{kr_1}2^{-k(\sigma+s+s'-n/p)}(1 + 2^k|x_0 - x_P|)^\sigma ||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}})^q\}^{1/q}||_{L^p(P)} \\
&\le C2^{-j(r_1-s')}2^{-jn/p} \sum_{j\ge k\ge 0} 2^{k(r_1-\sigma-s-s'+n/p)}(1 + 2^j|x_0 - x_P|)^\sigma||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}} \\
&\le C2^{-j(r_1-s'+n/p)}2^{j(r_1-\sigma-s-s'+n/p)}(1 + 2^j|x_0 - x_P|)^\sigma||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}} \\
&\le C2^{-js}(2^{-j} + |x_0 - x_P|)^\sigma||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}}
\end{aligned}$$

where these inequalities follow if $r_1 > \sigma + s + s' - \frac{n}{p}$, $r_1 > s'$, $L > n$ and $\sigma \ge 0$.

In the same way for the B-type case we have the same estimate.

Hence, we have,

$$||A_2c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \le C||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}$$

if $r_1 > \sigma + s + s' - n/p$, $r_1 > s'$, $0 < p < \infty$ and $0 < q \le \infty$.

We get the same estimate for the case $p = \infty$. Therefore, we obtain the desired conclusion.

(ii) We put $w_i = (2^{-i} + |x_0 - x|)^{-\sigma}$. We see that $w_i \le 2^{(i-k)+\sigma}w_k$ if $0 \le \sigma$, and $w_i \le 2^{(k-i)+\sigma}w_k$ if $0 > \sigma$. Then, using these inequalities we can prove the desired result by using the same way in the above proof of (i). □

Lemma 3.2. *Let $r_1, r_2 \in \mathbb{N}_0$, $L > n$ and $L_1 > n + r_1, L_2 > n + r_2$. Assume that for dyadic cubes P*

and R, ϕ_P and φ_R are functions on $\mathbb{R}^n$ satisfying following properties:

$$(2.1)\quad \int_{\mathbb{R}^n} \phi_P(x)x^\gamma dx = 0 \quad for \quad |\gamma| < r_1,$$

$$(2.2)\quad |\phi_P(x)| \le C(1+l(P)^{-1}|x-x_P|)^{-\max(L,L_1)},$$

$$(2.3)\quad |\partial^\gamma \phi_P(x)| \le Cl(P)^{-|\gamma|}(1+l(P)^{-1}|x-x_P|)^{-L}$$

$$for \quad 0 < |\gamma| \le r_2,$$

$$(2.4)\quad \int_{\mathbb{R}^n} \varphi_R(x)x^\gamma dx = 0 \quad for \quad |\gamma| < r_2,$$

$$(2.5)\quad |\varphi_R(x)| \le C(1+l(R)^{-1}|x-x_R|)^{-\max(L,L_2)},$$

$$(2.6)\quad |\partial^\gamma \varphi_R(x)| \le Cl(R)^{-|\gamma|}(1+l(R)^{-1}|x-x_R|)^{-L}$$

$$for \quad 0 < |\gamma| \le r_1,$$

where (2.1) *and* (2.6) *are void when $r_1 = 0$, and* (2.3) *and* (2.4) *are void when $r_2 = 0$. Then, we have that*

$$l(P)^{-n}|\langle \phi_P \ , \ \varphi_R\rangle| \le C\big(\tfrac{l(P)}{l(R)}\big)^{r_1}(1+l(R)^{-1}|x_P-x_R|)^{-L}$$
$$if \quad l(P) \le l(R),$$
$$l(R)^{-n}|\langle \phi_P \ , \ \varphi_R\rangle| \le C\big(\tfrac{l(R)}{l(P)}\big)^{r_2}(1+l(P)^{-1}|x_P-x_R|)^{-L}$$
$$if \quad l(R) < l(P).$$

Proof. We refer to [10: Corollary B.3] , [5: Lemma 6.3] or [29: Lemma 3.1]. □

Lemma 3.3. *Suppose that $s,\ s',\ \sigma \in \mathbb{R},\ x_0 \in \mathbb{R}^n$ and $0 < p,\ q\ \le \infty$. Let $r_1, r_2 \in \mathbb{N}_0$ and $L > n$. Assume that functions ϕ_P and φ_P satisfy* (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) *in Lemma* 3.2. *Let J as in Lemma* 3.1. *Then we have*

(i) *for a dyadic cube R and a sequence $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$,*
$\sum_{\mathcal{D}\ni P,\ l(P)\le 1} c(P)\langle \phi_P \ , \ \varphi_R\rangle$ *is convergent if $r_1 > J-n-s'$ and $L > J$,*
(ii) *for a dyadic cube R and a sequence $c \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$,*
$\sum_{\mathcal{D}\ni P,\ l(P)\le 1} c(P)\langle \phi_P \ , \ \varphi_R\rangle$ *is convergent if $r_1 > J-n-s'-(\sigma\wedge 0)$ and $L > J+\sigma$.*

Proof : (i) We may assume that $\sigma \ge 0$ by Remark 1.
We write $\sum_{\mathcal{D}\ni P} c(P)\langle \phi_P \ , \ \varphi_R\rangle = I = I_0 + I_1$ with

$$I_0 = \sum_{l(R)\le l(P)\le 1} c(P)\langle \phi_P \ , \ \varphi_R\rangle,$$

$$I_1 = \sum_{l(P)<l(R)} c(P)\langle \phi_P \ , \ \varphi_R\rangle$$

for $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. We claim that $I_i < \infty,\quad i = 0, 1$.

For a dyadic cube R with $l(R) = 2^{-i}$ we have, by Lemma 3.2 that

$$\begin{aligned} |I_0| &\le C \sum_{i\ge j\ge 0}\ \sum_{l(P)=2^{-j}} |c(P)||\langle \phi_P \ , \ \varphi_R\rangle| \\ &\le\ C \sum_{i\ge j\ge 0}\ \sum_{l(P)=2^{-j}} |c(P)|2^{-in}2^{(j-i)r_2}(1+2^j|x_R-x_P|)^{-L} \\ &\le\ C \sum_{i\ge j\ge 0} 2^{-i(r_2+n)}2^{jr_2}M_t(\sum_{l(P)=2^{-j}} |c(P)|\chi_P)(x), \end{aligned}$$

for $L > n/t$, $0 < t < 1$ and $x \in R$. Taking $L^1(R)$ norm and using the Fefferman-Stein inequality, we have,

$$\begin{aligned} |I_0|2^{-in} &= ||I_0||_{L^1(R)} \\ &\leq C2^{-in}||\sum_{i\geq j\geq 0} M_t(\sum_{l(P)=2^{-j}} |c(P)|\chi_P)||_{L^1(R)} \\ &\leq C2^{-in}||\sum_{i\geq j\geq 0}\sum_{l(P)=2^{-j}} |c(P)|\chi_P||_{L^1(R)} \\ &\leq C\sum_{1\geq l(P),\ R\subset P} |c(P)|2^{-2in} < \infty. \end{aligned}$$

In the same way we obtain the estimate of I_1:

$$\begin{aligned} |I_1| &\leq C\sum_{j\geq i\vee 0}\sum_{l(P)=2^{-j}} |c(P)||\langle \phi_P\ ,\ \varphi_R\rangle| \\ &\leq C\sum_{j\geq i\vee 0}\sum_{l(P)=2^{-j}} |c(P)|2^{-jn}2^{(i-j)r_1}(1+2^i|x_R-x_P|)^{-L} \\ &\leq C\sum_{j\geq i\vee 0} 2^{-j(r_1+n)}2^{ir_1}\sum_{l(P)=2^{-j}} |c(P)|(1+2^i|x_R-x_P|)^{-L} \\ &\leq C\sum_{j\geq i\vee 0} 2^{-j(r_1+n-n/t+s')}2^{ir_1}2^{-in/t}M_t(\sum_{l(P)=2^{-j}} 2^{js'}|c(P)|\chi_P)(x) \end{aligned}$$

if $0 < t \leq 1,\ L > n/t$ and $x \in R$ with $l(R) = 2^{-i}$.

By using the monotonicity of l^q-norm and Hölder's inequality, we get the following result,

$$|I_1| \leq C2^{-i(n+s')}\{\sum_{j\geq i\vee 0}(M_t(\sum_{l(P)=2^{-j}} 2^{js'}|c(P)|\chi_P)(x))^q\}^{1/q}$$

if $r_1 + n - n/t + s' > 0,\ 0 < q \leq \infty$ and $x \in R$.

Taking $L^p(R)$ norm and using the Fefferman-Stein inequality, we have, for a dyadic cube R with $l(R) = 2^{-i}$ and $c \in a^s(f^{s'}_{pq})^\sigma_{x_0}$,

$$\begin{aligned} |I_1|2^{-in/p} &= ||I_1||_{L^p(R)} \leq C2^{-i(n+s')}c(f^{s'}_{pq})(R) \\ &\leq C2^{-i(n+s'+\sigma+s)}||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}} < \infty \end{aligned}$$

if $0 < t < \min(p, q),\ 0 < p < \infty,\ 0 < q \leq \infty$. In the same way we get the same estimate for the case $p = \infty$. Furthermore, we obtain the same estimate for the B-type case if $0 < t < p$, $0 < p \leq \infty,\ 0 < q \leq \infty$. Therefore, we obtain that I_1 is convergent if $r_1 > J - n - s'$ and $L > J$.

(ii) Let I_0 and I_1 be as in the proof of (i). Then by arguing as in the proof of (i), we have $I_0 < \infty$ for $L > n$. We put $w_j(P) = (2^{-j} + |x_P - x_0|)^{-\sigma}$ for a dyadic cube P with $l(P) = 2^{-j}$.

Note that

$$|c(P)| \leq Cl(P)^{s+s'-n/p}w_j(P)^{-1}||c||_{a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}}$$

for $c \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$. We have, by Lemma 3.2 for a dyadic cube R with $l(R) = 2^{-i}$ and $\sigma \geq 0$,

$$
\begin{aligned}
|I_1| &\leq C \sum_{j\geq i\vee 0} \sum_{l(P)=2^{-j}} |c(P)||\langle \phi_P \ , \ \varphi_R\rangle| \\
&\leq C \sum_{j\geq i\vee 0} \sum_{l(P)=2^{-j}} |c(P)|2^{-jn}2^{(i-j)r_1}(1+2^i|x_R-x_P|)^{-L} \\
&\leq C \sum_{j\geq i\vee 0} \sum_{l(P)=2^{-j}} |c(P)|2^{-jn}2^{(i-j)r_1}w_j(P)w_j(P)^{-1} \times \\
&\quad (1+2^i|x_R-x_P|)^{-L} \\
&\leq C \sum_{j\geq i\vee 0} 2^{-j(r_1+n)}2^{ir_1}2^{-i\sigma} \sum_{l(P)=2^{-j}} |c(P)|w_j(P) \times \\
&\quad (1+2^i|x_R-x_P|)^{-(L-\sigma)} \\
&\leq C \sum_{j\geq i\vee 0} 2^{-j(r_1+n-n/t+s')}2^{i(r_1+\sigma-n/t)} \times \\
&\quad M_t(\sum_{l(P)=2^{-j}} 2^{js'}w_j(P)|c(P)|\chi_P)(x).
\end{aligned}
$$

By using the same way as in the proof of (i), we get

$$
\begin{aligned}
|I_1|2^{-ip/n} &\leq C2^{-i(n+s'+\sigma)}c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(R) \\
&\leq C2^{-i(n+s'+\sigma+s)}||c||_{a^s(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}} < \infty
\end{aligned}
$$

if $r_1 > J-n-s'$ and $L > \sigma + J$. We also obtain the same estimate for the case $\sigma < 0$. □

For a sequence $c(P)$ with $l(P) = 2^{-j}$, we define the sequence $c^*(P)$ by

$$
c^*(P) = \sum_{l(R)=2^{-j}} |c(R)|(1+2^j|x_P-x_R|)^{-L}
$$

for $L > J$ where J is as in Lemma 3.1.

We define for $f \in \mathcal{S}'$, $\gamma \in \mathbb{N}_0$ and a dyadic cube P with $l(P) = 2^{-j}$, the sequence $\inf_\gamma(f)(P)$ and $t_\gamma(P)$ by

$\inf_\gamma(f)(P) = \max\{\inf_{R\ni y}|\phi_j * f(y)| : R \subset P, l(R) = 2^{-(\gamma+j)}\}$,

$$
t_\gamma(P) = \inf_{P\ni y} |\phi_{j-\gamma} * f(y)|.
$$

Lemma 3.4. *For $s', \ \sigma \in \mathbb{R}, \ x_0 \in \mathbb{R}^n, \ 0 < p,q \leq \infty, \ f \in \mathcal{S}'$ and a dyadic cube P with $l(P) = 2^{-j}$, we have*

(i)

$$
c(e^{s'}_{pq})(P) \sim c^*(e^{s'}_{pq})(P), \quad c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P) \sim c^*(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P),
$$

(ii)

$$
\inf_\gamma(f)(P)\chi_P \leq C2^{\gamma L}\textstyle\sum_{R\subset P, l(R)=2^{-(\gamma+j)}} t^*_\gamma(R)\chi_R.
$$

for γ sufficient large.

Proof. (i) It suffices to prove

$$
c^*(e^{s'}_{pq})(P) \leq Cc(e^{s'}_{pq})(P)
$$

since $|c(P)| \le c^*(P)$.

Using the Fefferman-Stein inequality, we have

$$\begin{aligned} c^*(f^{s'}_{pq})(P) &= ||\{\sum_{i\ge j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}|c^*(R)|\chi_R)^q\}^{1/q}||_{L^p(P)} \\ &\le C||\{\sum_{i\ge j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}\sum_{l(R')=2^{-i}}|c(R')|\times \\ &\quad (1+2^i|x_R-x_{R'}|)^{-L}\chi_R)^q\}^{1/q}||_{L^p(P)} \\ &\le C||\{\sum_{i\ge j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}M_t(\sum_{l(R')=2^{-i}}|c(R')|\chi_{R'})\chi_R)^q\}^{1/q}||_{L^p(P)} \\ &\le C||\{\sum_{i\ge j\vee 0}(\sum_{l(R')=2^{-i}}2^{is'}|c(R')|\chi_{R'})^q\}^{1/q}||_{L^p(P)} = Cc(f^{s'}_{pq})(P) \end{aligned}$$

if $0 < t < \min(p,q)$, $L > n/t$ and $0 < p < \infty, 0 < q \le \infty$. Moreover, for the $p = \infty$ case, we have the same result. For the B-type case , we obtain the same result by the same argument as above. We also obtain the same result for the other case.

(ii) Let R_0 and R in P be cubes with $l(R_0) = l(R) = 2^{-(\gamma+j)}$. It suffices to show

$$t_\gamma(R_0) \le C2^{\gamma L}t^*_\gamma(R).$$

Since

$$1 \le 2^L 2^{\gamma L}(1+2^{\gamma+j}|x_R - x_{R_0}|)^{-L},$$

we have

$$\begin{aligned} t_\gamma(R_0) &\le Ct_\gamma(R_0)2^{\gamma L}(1+2^{\gamma+j}|x_R-x_{R_0}|)^{-L} \\ &\le C2^{\gamma L}\sum_{l(R')=2^{-(\gamma+j)}}t_\gamma(R')(1+2^{\gamma+j}|x_R-x_{R'}|)^{-L} = C2^{\gamma L}t^*_\gamma(R). \end{aligned}$$

□

4 Characterizations

Remark 2. (See [11: (3.20)]). Let ϕ_0 be a Schwartz function satisfying (1.1) and (1.2) and let ϕ be a Schwartz function satisfying (1.3) and (1.4). Then there exist a Schwartz function φ_0 satisfying the same conditions (1.1) and (1.2) and a Schwartz function φ satisfying the same conditions (1.3) and (1.4) such that

$\sum_{j\in\mathbb{N}_0}\hat{\varphi}_j(\xi)\hat{\phi}_j(\xi) = 1$ for any ξ where $\varphi_j(x) = 2^{jn}\varphi(2^jx),\ j \in \mathbb{N}$.

Hence we have the φ-transform [8; Lemma 2.1] for $f \in \mathcal{S}'$ such that

$$f = \sum_{l(Q)\le 1} l(Q)^{-n}\langle f,\ \varphi_Q\rangle\phi_Q,$$

where $\phi_Q(x) = \phi(l(Q)^{-1}(x-x_Q))$ and $\varphi_Q(x) = \varphi(l(Q)^{-1}(x-x_Q))$ for a dyadic cube Q with $l(Q) < 1$, and $\phi_Q(x) = \phi_0(l(Q)^{-1}(x-x_Q))$ and $\varphi_Q(x) = \varphi_0(l(Q)^{-1}(x-x_Q))$ for a dyadic cube Q with $l(Q) = 1$.

Theorem 4.1. *For $s,\ s',\ \sigma \in \mathbb{R},\ 0 < p,\ q \le \infty,\ x_0 \in \mathbb{R}^n$ and $\phi_0,\ \phi \in \mathcal{S}$ as in Remark 2, we have*

(i)

$$A^s(E^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q : \quad (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\},$$

and

(ii)

$$A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q : \quad (c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}.$$

Remark 3. (1) We see that $\sum_{l(Q)\le 1} c(Q)\phi_Q$ is convergent in $\mathcal{S}'$ for each sequence $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ or $c \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$ by Lemma 3.3.

(2) We notice that $D \equiv \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q : \quad c \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$ is independent of the choice of $\phi_0,\ \phi \in \mathcal{S}$ as in Remark 2. Indeed, suppose $\{\phi^1_0, \phi^1\}$ and $\{\phi^2_0, \phi^2\}$ are Schwartz functions as in Remark 2, and the spaces D^1 and D^2 are defined by using $\{\phi^1_0, \phi^1\}$ and $\{\phi^2_0, \phi^2\}$ in the place of $\{\phi_0, \phi\}$ respectively. We consider the φ-transform

$$\phi^1_P = \sum_{l(R)\le 1} l(R)^{-n} \langle \phi^1_P\ ,\ \ \varphi^2_R\rangle \phi^2_R.$$

Then for $D^1 \ni f = \sum_{l(P)\le 1} c(P)\phi^1_P,\ c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$, we have

$$f = \sum_{l(P)\le 1} c(P)\phi^1_P = \sum_{l(R)\le 1} Ac(R)\phi^2_R$$

where $A = \{l(R)^{-n}\langle \phi^1_P\ ,\ \varphi^2_R\rangle\}_{RP}$. From Lemma 3.1 and Lemma 3.2, we see that for $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$, $Ac \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. This shows that $D^1 \subset D^2$. By the same argument, we see that $D^2 \subset D^1$. That is, $D^1 = D^2$. These imply that the space D is independent of the choice of $\{\phi_0, \phi\}$. In the same way $\tilde{D} = \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q : \quad (c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}$ is independent of the choice of $\{\phi_0, \phi\}$.

Proof of Theorem 4.1. (i) We may assume $\sigma \ge 0$ by Remark 1. We put $D \equiv \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q :$ $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$. In order to prove $D \subset A^s(E^{s'}_{pq})^\sigma_{x_0}$ we claim for a dyadic cube P, and for $f = \sum_Q c(Q)\phi_Q \in D$,

$$c(E^{s'}_{pq})(P) \le Cc(e^{s'}_{pq})(P) \qquad \text{(a)}$$

if $0 < p, q \le \infty$. Let $(c(P)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. Since $\mathcal{S}$ is closed under the convolution, we have, for $i \ge 0$,

$$\begin{aligned} |\phi_i * f(x)| &= |\sum_{l(P)\le 1} c(P)\phi_i * \phi_P(x)| \\ &= \ |\sum_{j=(i-1)\vee 0}^{i+1} \sum_{l(P)=2^{-j}} c(P)\phi_i * \phi_P(x)| \\ &\le \ C \sum_{j=(i-1)\vee 0}^{i+1} \sum_{l(P)=2^{-j}} |c(P)|(1 + 2^j|x - x_P|)^{-L} \end{aligned}$$

for a sufficiently large number L. Hence we have, using the maximal function $M_t f(x)$, $0 < t \le 1$, as in the proof of Lemma 3.1

$$\begin{aligned}\{\sum_{i\geq j\vee 0}(2^{is'}|\phi_i*f|)^q\}^{1/q} &\leq C\{\sum_{i\geq j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}|\phi_i*f|\chi_R)^q\}^{1/q}\\ &\leq C\{\sum_{i\geq j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}(\sum_{k=(i-1)\vee 0}^{i+1}\sum_{l(R')=2^{-k}}|c(R')|(1+2^k|x-x_{R'}|)^{-L})\chi_R)^q\}^{1/q}\\ &\leq C\{\sum_{i\geq j\vee 0}(\sum_{l(R)=2^{-i}}M_t(\sum_{k=(i-1)\vee 0}^{i+1}\sum_{l(R')=2^{-k}}2^{is'}|c(R')|\chi_{R'})\chi_R)^q\}^{1/q}\end{aligned}$$

if $0<t\leq 1$ and $L>n/t$. Taking $L^p(P)$-norm and using the Fefferman-Stein inequality, we have for a dyadic cube P with $l(P)=2^{-j}$

$$\begin{aligned}c(F^{s'}_{pq})(P) &= ||\{\sum_{i\geq j\vee 0}(2^{is'}|\phi_i*f|)^q\}^{1/q}||_{L^p(P)}\\ &\leq C||\{\sum_{i\geq j\vee 0}(M_t(\sum_{k=(i-1)\vee 0}^{i+1}\sum_{l(R')=2^{-k}}2^{is'}|c(R')|\chi_{R'}))^q\}^{1/q}||_{L^p(P)}\\ &\leq C||\{\sum_{i\geq j\vee 0}(\sum_{k=(i-1)\vee 0}^{i+1}\sum_{l(R')=2^{-k}}2^{is'}|c(R')|\chi_{R'})^q\}^{1/q}||_{L^p(P)}\\ &\leq C||\{\sum_{i\geq j\vee 0}(\sum_{l(R')=2^{-i}}2^{is'}|c(R')|\chi_{R'})^q\}^{1/q}||_{L^p(P)}=Cc(f^{s'}_{pq})(P)\end{aligned}$$

if $0<t<\min(p,q)$ and $0<p<\infty$. For the $p=\infty$ case , we obtain the same result. In the same way for the B-type case we have the same estimate

$$c(B^{s'}_{pq})(P)\leq Cc(b^{s'}_{pq})(P)\ \text{ if }\ 0<p\leq\infty.$$

This implies $D\subset A^s(E^{s'}_{pq})^\sigma_{x_0}$.

In order to complete the proof of Theorem 4.1 (i), we will show the inverse. We consider the φ-transform $f=\sum_{l(P)\leq 1}c(f)(P)\varphi_P$, $c(f)(P)=l(P)^{-n}\langle f\ ,\ \phi_P\rangle$ where ϕ_P and φ_P as in Remark 2. It suffices to show that $c(f)(P)\in a^s(e^{s'}_{pq})^\sigma_{x_0}$ for $f\in A^s(E^{s'}_{pq})^\sigma_{x_0}$. More precisely, we claim that for a dyadic cube P with $l(P)=2^{-j}$,

$$c(f)(e^{s'}_{pq})(P)\leq Cc(E^{s'}_{pq})(P) \qquad \text{(b)}$$

where $c(f)(e^{s'}_{pq})(P)$ is a sequence defined by replacing the sequence $c(P)$ by the sequence $c(f)(P)$ in the definition of $c(e^{s'}_{pq})(P)$. For $f\in\mathcal{S}'$ and a dyadic cube P with $l(P)=2^{-j}$, we define the sequence $\sup(f)(P)$ by setting

$$\sup(f)(P)=\sup_{P\ni y}|\phi_j*f(y)|.$$

For $\gamma\in\mathbb{N}_0$ the sequences $\inf_\gamma(f)(P)$, $t_\gamma(P)$ are defined previously and for a sequence $c(P)$, we also define a sequence $c^*(P)$ previously (See Lemma 3.4). We have, from the fact in [9, Lemma A.4] that $\sup(f)^*(P)\sim\inf_\gamma(f)^*(P)$ for γ sufficiently large.

Thus, we have

$$|c(f)(P)|=l(P)^{-n}|\langle f\ ,\ \phi_P\rangle|=|\phi_j*f(x_P)|\leq\sup(f)(P)\leq\sup(f)^*(P)\sim\inf_\gamma(f)^*(P)$$

for γ sufficiently large. Therefore, from Lemma 3.4 (i) and (ii) we have

$$
\begin{aligned}
|c(f)(f^{s'}_{pq})(P)| &\leq C\inf_\gamma(f)^*(f^{s'}_{pq})(P) \leq C\inf_\gamma(f)(f^{s'}_{pq})(P)\\
&\leq C||\{\sum_{i\geq j\vee 0}(\sum_{l(R)=2^{-i}} 2^{is'}\inf_\gamma(f)(R)\chi_R)^q\}^{1/q}||_{L^p(P)}\\
&\leq C||\{\sum_{i\geq j\vee 0}(2^{is'}2^{\gamma L}\sum_{l(R')=2^{-(\gamma+i)}} t^*_\gamma(R')\chi_{R'})^q\}^{1/q}||_{L^p(P)}\\
&\leq C2^{\gamma L}||\{\sum_{i\geq (j\vee 0)+\gamma}(2^{is'}2^{-\gamma s'}\sum_{l(R')=2^{-i}} t_\gamma(R')\chi_{R'})^q\}^{1/q}||_{L^p(P)}\\
&\leq C2^{\gamma(L-s')}||\{\sum_{i\geq (j\vee 0)+\gamma}(2^{is'}\sum_{l(R')=2^{-i}} |\phi_{i-\gamma}*f(y)|\chi_{R'})^q\}^{1/q}||_{L^p(P)}\\
&\leq C2^{\gamma(L-s')}||\{\sum_{i\geq j\vee 0}(2^{is'}2^{s'\gamma}\sum_{l(R')=2^{-(i+\gamma)}} |\phi_i*f(y)|\chi_{R'})^q\}^{1/q}||_{L^p(P)}\\
&\leq C2^{\gamma L}||\{\sum_{i\geq j\vee 0}(2^{is'}|\phi_i*f(y)|)^q\}^{1/q}||_{L^p(P)} = Cc(F^{s'}_{pq})(P)
\end{aligned}
$$

if $0 < p < \infty$. For $p = \infty$, we obtain the same result. For the B-type case we can prove the same result by the same argument as above,

$$c(f)(b^{s'}_{pq})(P) \leq Cc(B^{s'}_{pq})(P)$$

if $0 < p \leq \infty$. Thus, we obtain

$$c(f)(e^{s'}_{pq})(P) \leq Cc(E^{s'}_{pq})(P).$$

By Remark 3 (2) this implies that, $A^s(E^{s'}_{pq})^\sigma_{x_0} \subset D$, $0 < p \leq \infty$. Hence, we obtain $A^s(E^{s'}_{pq})^\sigma_{x_0} = D$.

(ii) We can prove (ii) in the same way as (i). □

We have the following properties from Theorem 4.1.

Proposition 4.1. *Suppose that* $s,\ s,'\ \sigma \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$.

(i) *When* $\sigma < 0$, *we have* $A^s(E^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p, q \leq \infty$,

(ii) *When* $\sigma + s < 0$, *we have* $A^s(B^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p, q \leq \infty$, *and* $A^s(F^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p < \infty$, $0 < q \leq \infty$,

(iii) When $s < 0$, *we have* $A^s(\tilde{B}^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p, q \leq \infty$, *and* $A^s(\tilde{F}^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p < \infty$, $0 < q \leq \infty$.

Proof. These properties are shown easily. □

Proposition 4.2. *Suppose that* $s,\ s,'\ \sigma \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$.

(i) *When* $s \leq 0$, *we have*

$A^s(B^{s'}_{pq})^\sigma_{x_0} = (B^{s'}_{pq})^{s+\sigma}_{x_0}$ *for* $0 < p, q \leq \infty$, *and* $A^s(F^{s'}_{pq})^\sigma_{x_0} = (F^{s'}_{pq})^{s+\sigma}_{x_0}$ *for* $0 < p < \infty$, $0 < q \leq \infty$,

In particular, when $\sigma \geq 0$ *and* $\sigma + s = 0$, *we have*

$A^s(B^{s'}_{pq})^\sigma_{x_0} = B^{s'}_{pq}(\mathbb{R}^n)$ *for* $0 < p, q \leq \infty$, *and* $A^s(F^{s'}_{pq})^\sigma_{x_0} = F^{s'}_{pq}(\mathbb{R}^n)$ *for* $0 < p < \infty$, $0 < q \leq \infty$.

(ii) *When* $\sigma \geq 0$, *we have*

$A^s(E^{s'+\sigma}_{pq}) \subset A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} \subset A^s(E^{s'}_{pq})^\sigma_{x_0}$,

and when $\sigma < 0$, *we have*

$A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} \subset A^s(E^{s'+\sigma}_{pq})$.

(iii) *If $\sigma \geq 0$, then we have*

$$(E^{s'}_{\infty\infty})^{\sigma}_{x_0} = (\tilde{E}^{s'}_{\infty\infty})^{\sigma}_{x_0}.$$

Proof. The property (i) can be proved from the fact that

$$Cl(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s} c(e^{s'}_{pq})(P) \geq l(Q)^{-(\sigma+s)} c(e^{s'}_{pq})(Q)$$

and

$$l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s} c(e^{s'}_{pq})(P) \leq Cl(Q)^{-(\sigma+s)} \sup_{\mathcal{D}\ni P\subset 3Q} c(e^{s'}_{pq})(P),$$

if $s \leq 0$.

We obtain the property (ii) from the fact that

$$c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P) \leq Cc(e^{\sigma+s'}_{pq})(P),$$

and

$$l(Q)^{-\sigma} l(P)^{-s} c(e^{s'}_{pq})(P) \leq Cl(P)^{-s} c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P)$$

since $l(Q)^{-\sigma} \leq C(l(P) + |x_0 - x_P|)^{-\sigma}$ for $P \subset 3Q$ if $\sigma \geq 0$. The last half of property (ii) can be proved since

$$c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P) \geq c(e^{s'+\sigma}_{pq})(P)$$

if $\sigma < 0$. To prove the property (iii), it suffices to see from property (ii),

$$(e^{s'}_{\infty\infty})^{\sigma}_{x_0} \subset (\tilde{e}^{s'}_{\infty\infty})^{\sigma}_{x_0}.$$

We consider any dyadic cube R with $l(R) = 2^{-i}$ and dyadic cubes Q_l with $x_0 \in Q_l$ and $l(Q_l) = 2^{-l},\ i \geq l$ such that $Q_i \subset \cdots \subset Q_l \subset Q_{l-1} \subset \cdots$ and $\cup_{i\geq l} Q_l = \mathbb{R}^n$. We set $Q^0_l \equiv 3Q_l \setminus 3Q_{l+1},\ i > l$ and $Q^0_i \equiv 3Q_i$. We divide the proof into two cases:

Case (a): $R \subset Q^0_l,\ i > l$ case. Then we have $2^{-i} + |x_0 - x_R| \geq C2^{-l}$,

Case (b): $R \subset Q^0_i$. Then we have $2^{-i} + |x_0 - x_R| \geq 2^{-i}$.

In the case (a) we have

$$\begin{aligned} 2^{is'}|c(R)|(2^{-i} + |x_0 - x_R|)^{-\sigma} &\leq C2^{is'}2^{l\sigma}|c(R)| \\ &\leq C \sup_{x_0\in Q} 2^{l\sigma} \sup_{R\subset 3Q} 2^{is'}|c(R)| < \infty. \end{aligned}$$

In the case (b) we have

$$\begin{aligned} 2^{is'}|c(R)|(2^{-i} + |x_0 - x_R|)^{-\sigma} &\leq C2^{is'}2^{i\sigma}|c(R)| \\ &\leq C \sup_{x_0\in Q} 2^{l\sigma} \sup_{R\subset 3Q} 2^{is'}|c(R)| < \infty. \end{aligned}$$

□

Proposition 4.3. *Suppose that $s,\ s,'\ \sigma \in \mathbb{R}$, and $x_0 \in \mathbb{R}^n$.*

When $0 < q_1 \leq q_2 \leq \infty$, $0 < p \leq \infty$, we have

$$A^s(B^{s'}_{pq_1})^{\sigma}_{x_0} \subset A^s(B^{s'}_{pq_2})^{\sigma}_{x_0}, \quad A^s(\tilde{B}^{s'}_{pq_1})^{\sigma}_{x_0} \subset A^s(\tilde{B}^{s'}_{pq_2})^{\sigma}_{x_0},$$

and when $0 < q_1 \leq q_2 \leq \infty$, $0 < p < \infty$, we have

$$A^s(F^{s'}_{pq_1})^{\sigma}_{x_0} \subset A^s(F^{s'}_{pq_2})^{\sigma}_{x_0}, \quad A^s(\tilde{F}^{s'}_{pq_1})^{\sigma}_{x_0} \subset A^s(\tilde{F}^{s'}_{pq_2})^{\sigma}_{x_0}.$$

Proof. These inclusions are corollaries of the monotonicity of the l^p-norm. □

Proposition 4.4. *Suppose that $s,\ s,'\ \sigma \in \mathbb{R}$, $0 < \epsilon$ and $x_0 \in \mathbb{R}^n$. We have*

(i) $A^s(B^{s'+\epsilon}_{pq_1})^{\sigma-\epsilon}_{x_0} \subset A^s(B^{s'}_{pq_2})^{\sigma}_{x_0}$ *for* $0 < p \le \infty,\ 0 < q_1,\ q_2 \le \infty$, *and*
$A^s(F^{s'+\epsilon}_{pq_1})^{\sigma-\epsilon}_{x_0} \subset A^s(F^{s'}_{pq_2})^{\sigma}_{x_0}$ *for* $0 < p < \infty,\ 0 < q_1,\ q_2 \le \infty$, *and*

(ii) $A^{s+\epsilon}(E^{s'}_{pq})^{\sigma-\epsilon}_{x_0} \subset A^s(E^{s'}_{pq})^{\sigma}_{x_0}$ *for* $0 < p,\ q \le \infty$, *and*

(iii) $A^{s-\epsilon}(B^{s'+\epsilon}_{pq_1})^{\sigma}_{x_0} \subset A^s(B^{s'}_{pq_2})^{\sigma}_{x_0}$, *and* $A^{s-\epsilon}(\tilde{B}^{s'+\epsilon}_{pq_1})^{\sigma}_{x_0} \subset A^s(\tilde{B}^{s'}_{pq_2})^{\sigma}_{x_0}$ *for* $0 < p,\ q_1,\ q_2 \le \infty$, *and*
$A^{s-\epsilon}(F^{s'+\epsilon}_{pq_1})^{\sigma}_{x_0} \subset A^s(F^{s'}_{pq_2})^{\sigma}_{x_0}$, *and* $A^{s-\epsilon}(\tilde{F}^{s'+\epsilon}_{pq_1})^{\sigma}_{x_0} \subset A^s(\tilde{F}^{s'}_{pq_2})^{\sigma}_{x_0}$ *for* $0 < p < \infty,\ 0 < q_1,\ q_2 \le \infty$.

Proof. (ii) is obvious. (i) and (iii) are corollaries of Hölder's inequality and the monotonicity of the l^p-norm. □

Proposition 4.5. *Suppose that* $s,\ s,'\ \sigma \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$.

(i) *If* $0 < p_2 \le p_1 \le \infty$ *and* $0 < q \le \infty$, *then*
$A^{s+\frac{n}{p_1}}(B^{s'}_{p_1q})^{\sigma}_{x_0} \subset A^{s+\frac{n}{p_2}}(B^{s'}_{p_2q})^{\sigma}_{x_0}$, $A^{s+\frac{n}{p_1}}(\tilde{B}^{s'}_{p_1q})^{\sigma}_{x_0} \subset A^{s+\frac{n}{p_2}}(\tilde{B}^{s'}_{p_2q})^{\sigma}_{x_0}$,
and, if $0 < p_2 \le p_1 < \infty$ *and* $0 < q \le \infty$, *then*
$A^{s+\frac{n}{p_1}}(F^{s'}_{p_1q})^{\sigma}_{x_0} \subset A^{s+\frac{n}{p_2}}(F^{s'}_{p_2q})^{\sigma}_{x_0}$, $A^{s+\frac{n}{p_1}}(\tilde{F}^{s'}_{p_1q})^{\sigma}_{x_0} \subset A^{s+\frac{n}{p_2}}(\tilde{F}^{s'}_{p_2q})^{\sigma}_{x_0}$.

(ii) *If* $0 < q \le \infty,\ 0 < p \le \infty,\ \frac{n}{p} < s$, *then*
$A^s(E^{s'}_{pq})^{\sigma}_{x_0} = (E^{s+s'-\frac{n}{p}}_{\infty\infty})^{\sigma}_{x_0}$ *and* $A^s(\tilde{E}^{s'}_{pq})^{\sigma}_{x_0} = (\tilde{E}^{s+s'-\frac{n}{p}}_{\infty\infty})^{\sigma}_{x_0}$.
In particular, if $0 \le \sigma,\ 0 < q \le \infty,\ 0 < p \le \infty,\ \frac{n}{p} < s$, *then*
$A^s(E^{s'}_{pq})^{\sigma}_{x_0} = A^s(\tilde{E}^{s'}_{pq})^{\sigma}_{x_0}$.

(iii) *If* $0 < p_1,\ p_2,\ q \le \infty$, *then*
$A^{\frac{n}{p_1}}(E^{s'}_{p_1\infty})^{\sigma}_{x_0} = A^{\frac{n}{p_2}}(E^{s'}_{p_2\infty})^{\sigma}_{x_0} = (E^{s'}_{\infty\infty})^{\sigma}_{x_0}$,
$A^{\frac{n}{p_1}}(\tilde{E}^{s'}_{p_1\infty})^{\sigma}_{x_0} = A^{\frac{n}{p_2}}(\tilde{E}^{s'}_{p_2\infty})^{\sigma}_{x_0} = (\tilde{E}^{s'}_{\infty\infty})^{\sigma}_{x_0}$,
$A^{\frac{n}{p_1}}(F^{s'}_{p_1q})^{\sigma}_{x_0} = A^{\frac{n}{p_2}}(F^{s'}_{p_2q})^{\sigma}_{x_0}$, $A^{\frac{n}{p_1}}(\tilde{F}^{s'}_{p_1q})^{\sigma}_{x_0} = A^{\frac{n}{p_2}}(\tilde{F}^{s'}_{p_2q})^{\sigma}_{x_0}$.

Proof. The properties (i) are corollaries of Hölder's inequality. We will prove the properties (ii). We see that

$$a^{s+\frac{n}{p}}(e^{s'}_{pq})^{\sigma}_{x_0} \subset (e^{s'+s}_{\infty\infty})^{\sigma}_{x_0},$$

since

$$l(P)^{-(s+\frac{n}{p})}c(e^{s'}_{pq})(P) \ge l(P)^{-(s'+s)}|c(P)|.$$

Hence in order to prove (ii), it suffices to prove

$$(e^{s'+s}_{\infty\infty})^{\sigma}_{x_0} \subset a^{s+\frac{n}{p}}(e^{s'}_{pq})^{\sigma}_{x_0}.$$

Since

$$c(\dot{e}^{s'}_{pq})(P) \le C(e^{s'+s}_{\infty\infty})(P) \times l(P)^{s+\frac{n}{p}}$$

if $s > 0$ and $0 < q < \infty$, we get the desired result. Similarly, for the other case, we can prove.

The first part of properties (iii) is obtained in the same way in the proof of (ii) and the last part is just [10: Corollary 5.7]. □

Proposition 4.6. (Embedding) *Let* $s,\ s',\ \sigma \in \mathbb{R},\ 0 < p,\ q \le \infty$ *and* $x_0 \in \mathbb{R}^n$. *We have*

(i) $A^s(E^{s'_1}_{p\xi})^\sigma_{x_0} \subset A^s(E^{s'_2}_{p\eta})^\sigma_{x_0}$, $A^s(\tilde{E}^{s'_1}_{p\xi})^\sigma_{x_0} \subset A^s(\tilde{E}^{s'_2}_{p\eta})^\sigma_{x_0}$, *for* $s'_1 > s'_2$ *and* $0 < \xi, \eta \leq \infty$,

(ii) $A^s(B^{s'_1}_{p_1q})^\sigma_{x_0} \subset A^s(B^{s'_2}_{p_2q})^\sigma_{x_0}$, $A^s(\tilde{B}^{s'_1}_{p_1q})^\sigma_{x_0} \subset A^s(\tilde{B}^{s'_2}_{p_2q})^\sigma_{x_0}$, *for* $s'_1 - s'_2 = n(\frac{1}{p_1} - \frac{1}{p_2})$ *and* $0 < p_1 \leq p_2 \leq \infty$,

$A^s(F^{s'_1}_{p_1\xi})^\sigma_{x_0} \subset A^s(F^{s'_2}_{p_2\eta})^\sigma_{x_0}$, $A^s(\tilde{F}^{s'_1}_{p_1\xi})^\sigma_{x_0} \subset A^s(\tilde{F}^{s'_2}_{p_2\eta})^\sigma_{x_0}$, *for* $s'_1 - s'_2 = n(\frac{1}{p_1} - \frac{1}{p_2})$ *and* $0 < p_1 < p_2 < \infty$, $0 < \xi, \eta \leq \infty$,

(iii) $A^s(B^{s'}_{pq})^\sigma_{x_0} \subset A^s(F^{s'}_{pq})^\sigma_{x_0}$, $A^s(\tilde{B}^{s'}_{pq})^\sigma_{x_0} \subset A^s(\tilde{F}^{s'}_{pq})^\sigma_{x_0}$, *for* $0 < q \leq p \leq \infty$,

$A^s(F^{s'}_{pq})^\sigma_{x_0} \subset A^s(B^{s'}_{pq})^\sigma_{x_0}$, $A^s(\tilde{F}^{s'}_{pq})^\sigma_{x_0} \subset A^s(\tilde{B}^{s'}_{pq})^\sigma_{x_0}$, *for* $0 < p \leq q \leq \infty$.

Proof. The embedding properties (i) and the first embedding of (ii) are corollaries of Hölder's inequality and the monotonicity property of the l^p-norm. For the second embedding of (ii), see [37; Proposition 2.5] (cf. [38; Theorem 2.7.1]). (iii) is a corollary of Minkowski's inequality (cf. Triebel[38: 2.3.2 Proposition 2]).

□

Remark 4. Let $0 < p, q \leq \infty$, $s, \sigma \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $s' > n(\frac{1}{p} - 1)_+$. If $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$, then f is locally integrable (and locally L^p integrable). Indeed, we consider the Littlewood-Paley decomposition

$$f = \sum_{i \geq 0} f * \phi_i.$$

It suffices to show that $\sum_{i\geq 0} f * \phi_i$ is locally integrable and locally L^p integrable. We may consider any dyadic cube P with $l(P) \geq 1$. Then we have if $1 \leq p < \infty$,

$$\begin{aligned} ||\sum_{i\geq 0} f * \phi_i||_{L^1(P)} &\leq C||\sum_{i\geq 0} f * \phi_i||_{L^p(P)} \\ &\leq C||\{\sum_{i\geq 0}(2^{is'}|f * \phi_i|)^q\}^{1/q}||_{L^p(P)} \leq Cc(F^{s'}_{pq})(P) < \infty \end{aligned}$$

by using Hölder inequality if $1 \leq q \leq \infty$ and the monotonicity property of the l^p-norm if $0 < q \leq 1$. In the same way we have

$$\begin{aligned} ||\sum_{i\geq 0} f * \phi_i||_{L^1(P)} &\leq C||\sum_{i\geq 0} f * \phi_i||_{L^p(P)} \leq C\sum_{i\geq 0}||f * \phi_i||_{L^p(P)} \\ &\leq C\{\sum_{i\geq 0}(2^{is'}||f * \phi_i||_{L^p(P)})^q\}^{1/q} \leq Cc(B^{s'}_{pq})(P) < \infty. \end{aligned}$$

If $0 < p \leq 1$, in the same way we have

$$\begin{aligned} ||\sum_{i\geq 0} f * \phi_i||_{L^p(P)} &\leq C||\sum_{i\geq 0} f * \phi_i||_{L^1(P)} \\ &\leq C\{\sum_{i\geq 0}(2^{i(s'-n(\frac{1}{p}-1))}||f * \phi_i||_{L^1(P)})^q\}^{1/q} \\ &= Cc(B^{s'-n(\frac{1}{p}-1)}_{1q})(P) \leq Cc(B^{s'}_{pq})(P) < \infty, \end{aligned}$$

where we use Proposition 4.6 in the last inequality. Similarly, by using the fact that $c(B^{s'}_{p\ p\vee q})(P) \leq c(F^{s'}_{pq})(P)$ we have the same estimate for the F-type case if $0 < p \leq 1$. Therefore, we obtain the desired result for $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$. But we note that it holds for $0 < p < \infty$ in the F-type case and for $0 < p \leq \infty$ in the B-type case. We note that it holds an analogous result for $f \in A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ with the weight $w_i = (2^{-i} + |x_0 - x|)^{-\sigma}$.

We recall the definitions of smooth atoms and molecules.

Definition 6. Let $r_1,\ r_2 \in \mathbb{N}_0, L > n$. A family of functions $m = (m_Q)$ indexed by dyadic cubes Q with $l(Q) \leq 1$ is called a family of (r_1, r_2, L)– smooth molecules if

(3.1) $|m_Q(x)| \leq C(1 + l(Q)^{-1}|x - x_Q|)^{-\max(L,L_2)}$ for some $L_2 > n + r_2$ when $l(Q) < 1$,

(3.2) $|\partial^\gamma m_Q(x)| \leq Cl(Q)^{-|\gamma|}(1 + l(Q)^{-1}|x - x_Q|)^{-L}$ for $0 < |\gamma| \leq r_1$, when $l(Q) < 1$ and

(3.3) $\int_{\mathbb{R}^n} x^\gamma m_Q(x)dx = 0$ for $|\gamma| < r_2$ when $l(Q) < 1$,

where (3.2) is void when $r_1 = 0$, and (3.3) is void when $r_2 = 0$,

(3.4) $|\partial^\gamma m_Q(x)| \leq Cl(Q)^{-|\gamma|}(1 + l(Q)^{-1}|x - x_Q|)^{-L},\quad |\gamma| \leq r_1$ when $l(Q) = 1$,

(3.5) we do not assume the vanishing moment condition (3.3) when $l(Q) = 1$.

A family of functions $a = (a_Q)$ indexed by dyadic cubes Q with $l(Q) \leq 1$ is called a family of (r_1, r_2)– smooth atoms if

(3.6) supp $a_Q \subset 3Q$ for each dyadic cube Q when $l(Q) \leq 1$,

(3.7) $|\partial^\gamma a_Q(x)| \leq Cl(Q)^{-|\gamma|}$ for $|\gamma| \leq r_1$ when $l(Q) \leq 1$, and

(3.8) $\int_{\mathbb{R}^n} x^\gamma a_Q(x)dx = 0$ for $|\gamma| < r_2$ when $l(Q) < 1$,

where (3.8) is void when $r_2 = 0$,

(3.9) we do not assume the vanishing moment condition (3.8) when $l(Q) = 1$.

Theorem 4.2. *Let $s,\ s',\ \sigma \in \mathbb{R}$, $0 < p,\ q \leq \infty$ and $x_0 \in \mathbb{R}^n$. Let $r_1, r_2 \in \mathbb{N}_0$, J as in Lemma* 3.1 *and $L > n$.*

(i) *We assume that r_1, r_2 and L satisfy the following condition:*

(4.1) $r_1 > \max(s',\ \sigma + s + s' - \frac{n}{p})$,

(4.2) $r_2 > J - n - s'$,

(4.3) $L > J$.

Then we have

$$A^s(E^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{l(Q)\leq 1} c(Q)m_Q :$$
$$(r_1, r_2, L)\text{– smooth molecules } (m_Q),\quad (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$$
$$= \{f = \sum_{l(Q)\leq 1} c(Q)a_Q :$$
$$(r_1, r_2)\text{– smooth atoms } (a_Q),\quad (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}.$$

(ii) *We assume that r_1, r_2 and L satisfy*

(4.1)′ $r_1 > \max(s' + (\sigma \vee 0),\ (\sigma \vee 0) + s + s' - \frac{n}{p})$,

(4.2)′ $r_2 > J - n - s' - (\sigma \wedge 0)$,

(4.3)′ $L > J + \sigma$

Then we have

$$A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{l(Q)\leq 1} c(Q)m_Q :$$
$$(r_1, r_2, L)\text{– smooth molecules } (m_Q),\quad (c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}$$
$$= \{f = \sum_{l(Q)\leq 1} c(Q)a_Q :$$
$$(r_1, r_2)\text{– smooth atoms } (a_Q),\quad (c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}.$$

Remark 5. From Lemma 3.3, we remark that $f=\sum_{l(Q)\le 1}c(Q)m_Q$ and $f=\sum_{l(Q)\le 1}c(Q)a_Q$ are convergent in $\mathcal{S}'$ for each $c\in a^s(e^{s'}_{pq})^\sigma_{x_0}$ or $a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$.

Proof of Theorem 4.2. (i) We may assume $\sigma\ge 0$ by Remark 1. We put
$A\equiv\{f=\sum_{l(Q)\le 1}c(Q)a_Q:\ \ (r_1,r_2)-$smooth atoms (a_Q), $(c(Q))\in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$,
$M\equiv\{f=\sum_{l(Q)\le 1}c(Q)m_Q:\ \ (r_1,r_2,L)-$smooth molecules (m_Q),
$(c(Q))\in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$.

Since an $(r_1,r_2)-$ atom is an $(r_1,r_2,L)-$ molecule, it is easy to see that $A\subset M$. Let $M\ni f=\sum_{l(Q)\le 1}c(Q)m_Q$ and we consider the φ-transform

$$m_Q=\sum_{l(P)\le 1}l(P)^{-n}\langle m_Q\ ,\ \varphi_P\rangle\phi_P,$$

where ϕ_P and φ_P as in Remark 2. Then we have

$$f=\sum_{l(Q)\le 1}c(Q)m_Q=\sum_{l(P)\le 1}(Ac)(P)\phi_P,$$

where $A=\{l(P)^{-n}\langle m_Q\ ,\ \varphi_P\rangle\}_{PQ}$. Lemma 3.1 and Lemma 3.2 yield that A is $(r_1,\ r_2+n,\ L)-$ almost diagonal and $Ac\in a^s(e^{s'}_{pq})^\sigma_{x_0}$ for $c\in a^s(e^{s'}_{pq})^\sigma_{x_0}$. Hence, if we put $D\equiv\{f=\sum_{l(Q)\le 1}c(Q)\phi_Q:\ \ c\in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$, then we see that $M\subset D$. From Theorem 4.1 we see $D=A^s(E^{s'}_{pq})^\sigma_{x_0}$. Hence, we obtain $A\subset M\subset A^s(E^{s'}_{pq})^\sigma_{x_0}$.

Using the argument similar to the proof of [10: Theorem 4.1] (cf. [4: Theorem 5.9] or [5: Theorem 5.8]), for $D\ni f=\sum_{l(Q)\le 1}c(Q)\phi_Q,\ \ c\in a^s(e^{s'}_{pq})^\sigma_{x_0}$, we see that there exist a family of $(r_1,r_2)-$ atoms $\{a_Q\}$ and a sequence of coefficients $\{c'(Q)\}\in a^s(e^{s'}_{pq})^\sigma_{x_0}$ such that $f=\sum_{l(Q)\le 1}c(Q)\phi_Q=\sum_{l(Q)\le 1}c'(Q)a_Q$. Hence, we see that $D\subset A$. Therefore, we have $A^s(\dot{E}^{s'}_{pq})^\sigma_{x_0}=M=A$. We can prove (ii) by the same way in (i). $\square$

We recall the definition of smooth wavelets.

Definition 7. Let $r\in\mathbb{N}_0$ and $L>n$. A family of $\{\psi_0,\ \psi^{(i)}\}$ is called $(r,\ L)-$ smooth wavelets if $\{\psi_0(x-k)\ (k\in\mathbb{Z}^n),\ 2^{nj/2}\psi^{(i)}(2^jx-k)\ (i=1,\cdots,2^n-1, j\in\mathbb{N}_0,\ \ k\in\mathbb{Z}^n)\}$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$, and $\psi^{(i)}$ satisfies (5.1), (5.2) and (5.3), and a scaling function ψ_0 satisfies (5.4)

(5.1) $|\psi^{(i)}(x)|\le C(1+|x|)^{-\max(L,L_0)}$ for some $L_0>n+r$,
(5.2) $|\partial^\gamma\psi^{(i)}(x)|\le C(1+|x|)^{-L}$ for $0<|\gamma|\le r$,
(5.3) $\int_{\mathbb{R}^n}\psi^{(i)}(x)x^\gamma dx=0$ for $|\gamma|<r$
where (5.2) and (5.3) are void when $r=0$.
(5.4) $|\partial^\gamma\psi_0(x)|\le C(1+|x|)^{-L}$ for $|\gamma|\le r$,
but ψ_0 does not satisfy the vanishing moment condition (5.3). We will forget to write the index i of the wavelet, which is of no consequence.

We put $\psi_{0,k}(x)=\psi_0(x-k),\ k\in\mathbb{Z}^n$, $\psi_Q(x)=\psi(l(Q)^{-1}(x-x_Q))$ for a dyadic cube Q with $l(Q)\le 1$.

Theorem 4.3. *Let* $s,\ s',\ \sigma\in\mathbb{R},\ x_0\in\mathbb{R}^n$ *and* $0<p,\ q\le\infty$.
(i) *For a family of* $(r,L)-$ *smooth wavelets* $\{\psi_0,\ \psi\}$ *satisfying*
(6.1) $r>\max(s',\ \sigma+s+s'-\frac{n}{p},\ J-n-s')$ *and*
(6.2) $L>J$, *where* J *as in Lemma* 3.1,
we have

$A^s(E^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q : (c_k) \in a^s(e^{s'}_{pq})^\sigma_{x_0},$
$(c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\},$

where $(c_k)_{k\in\mathbb{Z}^n} \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ *means that* $(c^0(Q))_{l(Q)\le 1} \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ *such as* $c^0(Q) = c_k$ *if* $Q = Q_{0,k} = [0,1)^n + k,\ k \in \mathbb{Z}^n$ *and* $c^0(Q) = 0$ *if* $l(Q) < 1$.

(ii) *For a family of* $(r,L)-$ *smooth wavelets* $\{\psi_0,\ \psi\}$ *satisfying*

(6.1)′ $r > \max(s' + (\sigma\vee 0),\ (\sigma \vee 0) + s + s' - \frac{n}{p},\ J - n - s' - (\sigma\wedge 0))$ *and*

(6.2)′ $L > J + \sigma$

we have

$A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q : (c_k) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0},$
$(c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}.$

Remark 6. We see that by Lemma 3.3, $\sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k}$ and $\sum_{l(Q)\le 1} c(Q)\psi_Q$ are convergent in $\mathcal{S}'$ for $(c_k),\ (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ or $a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$.

Proof of Theorem 4.3. (i) We may assume $\sigma \ge 0$ by Remark 1. We put $W = \{f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q : (c_k), (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$.

Let $W \ni f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q$ and we consider the φ–transform

$$\psi_{0,k} = \sum_{l(P)\le 1} l(P)^{-n}\langle \psi_{0,k}\ ,\ \varphi_P\rangle\phi_P$$

$$\psi_Q = \sum_{l(P)\le 1} l(P)^{-n}\langle \psi_Q\ ,\ \varphi_P\rangle\phi_P$$

where ϕ_P and φ_P as in Remark 2. Then we have

$$f = \sum_{l(P)\le 1} (B_1c_k)(P)\phi_P + \sum_{l(P)\le 1} (A_1c)(P)\phi_P$$

where $B_1 = \{l(P)^{-n}\langle\psi_{0,k}\ ,\ \varphi_P\rangle\}_{Pk}$ and $A_1 = \{l(P)^{-n}\langle\psi_Q\ ,\ \varphi_P\rangle\}_{PQ}$. Lemma 3.1 and Lemma 3.2 yield that B_1 and A_1 are almost diagonal and $B_1c_k,\ A_1c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ for $c_k,\ c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. Hence, by Theorem 4.1, we see that $W \subset D = A^s(E^{s'}_{pq})^\sigma_{x_0}$ where D is as in the proof of Theorem 4.1.

Conversely, let $D \ni f = \sum_{l(Q)\le 1} c(Q)\phi_Q$ and we consider the wavelet expansion

$$\phi_Q = \sum_{k\in\mathbb{Z}^n} \langle\phi_Q\ ,\ \psi_{0,k}\rangle\psi_{0,k} + \sum_{l(P)\le 1} l(P)^{-n}\langle\phi_Q\ ,\ \psi_P\rangle\psi_P.$$

Then we have

$$f = \sum_{k\in\mathbb{Z}^n} (B_2c)(k)\psi_{0,k} + \sum_{l(Q)\le 1} (A_2c)(Q)\phi_Q$$

where $B_2 = \{\langle\phi_Q\ ,\ \psi_{0,k}\rangle\}_{kQ}$ and $A_2 = \{l(P)^{-n}\langle\phi_Q\ ,\ \psi_P\rangle\}_{PQ}$. Lemma 3.1 and Lemma 3.2 yield that B_2 and A_2 are almost diagonal and $B_2c\ A_2c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ for $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. Hence, by Theorem 4.1, we see that $A^s(\dot{E}^{s'}_{pq})^\sigma_{x_0} = D \subset W$.

We can prove (ii) by the same way in (i). Hence we obtain the result of Theorem 4.3. □

Remark 7. (1) we see that Theorem 4.3 is independent of the choice of smooth wavelets $\{\psi_0,\ \psi^{(i)}\}$ (see Remark 3 (2)).

(2) For $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$ or $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ the parings $\langle f\ ,\ \psi_{0,k}\rangle$ and $\langle f\ ,\ \psi_Q\rangle$ are well-defined. More explicitly, we see that for any $\{\phi_Q,\ \ \varphi_Q\}$ as in Remark 2,

$$\langle f\ ,\ \psi_{0,k}\rangle = \sum_{l(P)\le 1} l(P)^{-n}\langle f\ ,\ \phi_P\rangle\langle\psi_{0,k}\ ,\ \varphi_P\rangle \equiv \sum_{l(P)\le 1} c(f)(P)\langle\psi_{0,k}\ ,\ \varphi_P\rangle$$

and

$$\langle f\ ,\ \psi_Q\rangle = \sum_{l(P)\le 1} l(P)^{-n}\langle f\ ,\ \phi_P\rangle\langle\psi_Q\ ,\ \varphi_P\rangle \equiv \sum_{l(P)\le 1} c(f)(P)\langle\psi_Q\ ,\ \varphi_P\rangle$$

are convergent by Lemma 3.3 and (b) in the proof of Theorem 4.1. Thus, for $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$ or $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ we have a wavelet expansion $f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q$ in $\mathcal{S}'$ and its representation is unique in $\mathcal{S}'$, that is, $c_k = \langle f\ ,\ \psi_{0,k}\rangle$ and $c(Q) = l(Q)^{-n}\langle f\ ,\ \psi_Q\rangle$. Hence, we have that by Lemma 3.1, Lemma 3.2 and (b) in the proof of Theorem 4.1,

$$\begin{aligned} ||(c_k)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} &= ||\langle f\ ,\ \psi_{0,k}\rangle||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\ &\le\ ||\sum_{l(P)\le 1} c(f)(P)\langle\psi_{0,k}\ ,\ \varphi_P\rangle||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\ &\le\ C||c(f)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \le C||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} \end{aligned}$$

and

$$\begin{aligned} ||(c(Q))||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} &= ||l(Q)^{-n}\langle f\ ,\ \psi_Q\rangle||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\ &\le\ C||\sum_{l(P)\le 1} c(f)(P)l(Q)^{-n}\langle\psi_Q\ ,\ \varphi_P\rangle||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\ &\le\ C||c(f)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \le C||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}}. \end{aligned}$$

Conversely, we consider the φ-transform

$$\psi_{0,k} = \sum_P l(P)^{-n}\langle\psi_{0,k}\ ,\ \varphi_P\rangle\phi_P$$

and

$$\psi_Q = \sum_P l(P)^{-n}\langle\psi_Q\ ,\ \varphi_P\rangle\phi_P.$$

Then we have

$$f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_Q c(Q)\psi_Q = \sum_{k\in\mathbb{Z}^n}(Bc_k)(P)\phi_P + \sum_Q Ac(P)\phi_P$$

where $B = \{l(P)^{-n}\langle\psi_{0,k}\ ,\ \varphi_P\rangle\}$ and $A = \{l(P)^{-n}\langle\psi_Q\ ,\ \varphi_P\rangle\}$. Hence we have by Lemma 3.1, Lemma 3.2 and (a) in the proof of Theorem 4.1,

$$\begin{aligned} ||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} &\le C||(Bc_k) + (Ac)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\ &\le\ C||Bc_k||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} + C||Ac||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\ &\le\ C||c_k||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} + C||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}. \end{aligned}$$

Therefore, we have

$$||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} \sim ||(c_k)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} + ||(c(Q))||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}.$$

Similarly, we also obtain

$$||f||_{A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}} \sim ||(c_k)||_{a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}} + ||(c(Q))||_{a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}}.$$

5 Applications

Definition 8. Let $\mathcal{T}$ be the space of Schwartz test functions (C^∞-functions with compact support) and $\mathcal{T}'$ its dual. For arbitrary $r_1,\ r_2 \in \mathbb{N}_0$ the Calderón–Zygmund operator T with an exponent $\epsilon > 0$ is a continuous linear operator $\mathcal{T} \to \mathcal{T}'$ such that its kernel K off the diagonal $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ satisfies

(7.1) $|\partial_1^\gamma K(x, y)| \le C|x-y|^{-(n+|\gamma|)}$ for $|\gamma| \le r_1$,

(7.2) $|K(x,\ y) - K(x,\ y')| \le C|y-y'|^{r_2+\epsilon}|x-y|^{-(n+r_2+\epsilon)}$ if $2|y'-y| \le |x-y|$,

(7.3) $|\partial_1^\gamma K(x,\ y) - \partial_1^\gamma K(x,\ y')| \le C|y-y'|^{\epsilon}|x-y|^{-(n+|\gamma|+\epsilon)}$
if $2|y'-y| \le |x-y|$ for $0 < |\gamma| \le r_1$
(where this statement is void when $r_1 = 0$),
$|\partial_1^\gamma K(x,\ y) - \partial_1^\gamma K(x',\ y)| \le C|x'-x|^{\epsilon}|x-y|^{-(n+|\gamma|+\epsilon)}$
if $2|x'-x| \le |x-y|$ for $|\gamma| \le r_1$,
(where the subindex 1 stands for derivatives in the first variable)

(7.4) T is bounded on $L^2(\mathbb{R}^n)$.

We obtain the following theorem.

Theorem 5.1. *Let $s,\ s',\ \sigma \in \mathbb{R},\ x_0 \in \mathbb{R}^n,\ 0 < p,\ q \le \infty,\ r_1,\ r_2 \in \mathbb{N}_0$ and J as in Lemma* 3.1.

(i) *The Calderón–Zygmund operator T with an exponent $\epsilon > J - n$ satisfying $T(x^\gamma) = 0$ for $|\gamma| \le r_1$ and $T^*(x^\gamma) = 0$ for $|\gamma| < r_2$, is bounded on $A^s(E^{s'}_{pq})^\sigma_{x_0}$ if r_1 and r_2 satisfy* (4.1) *and* (4.2) *as in Theorem* 4.2 *respectively.*

(ii) *The Calderón–Zygmund operator T with an exponent $\epsilon > J - n + \sigma$ satisfying $T(x^\gamma) = 0$ for $|\gamma| \le r_1$ and $T^*(x^\gamma) = 0$ for $|\gamma| < r_2$, is bounded on $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ if r_1 and r_2 satisfy* (4.1)' *and* (4.2)' *as in Theorem* 4.2 *respectively.*

Proof. The proof is similar to ones of [12].

(i) We may assume $\sigma \ge 0$ by Remark 1. Let $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$. Then we consider a wavelet expansion $f = \sum_k c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q$: $(c_k),\ (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ from Theorem 4.3. We may suppose that smooth wavelets $\{\psi_0\ \psi\}$ are compactly supported by Remark 7 (1). Then there exists a positive constant c such that supp $\psi_{0,k} \subset cQ_{0,k}$ where $Q_{0,k} = [0, 1)^n + k$ and supp $\psi_Q \subset cQ$ for every dyadic cube Q with $l(Q) = 2^{-l} \le 1$.

We claim that $Tf = \sum_k c_k(T\psi_{0,k}) + \sum_{l(Q)\le 1} c(Q)(T\psi_Q) \equiv \sum_k c_k m_k + \sum_{l(Q)\le 1} c(Q)m_Q$ is convergent in $\mathcal{S}'$ and $||Tf||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} \le C||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}}$.

More precisely , we will show that m_k and m_Q satisfy following properties:

(8.1) $|m_k(x)| \le C(1 + l(Q)^{-1}|x - x_k|)^{-L}$ with $L > J$,

(8.2) $|m_Q(x)| \le C(1 + l(Q)^{-1}|x - x_Q|)^{-(n+r_2+\epsilon)}$,

(8.3) $|\partial^\gamma m_Q(x)| \le Cl(Q)^{-|\gamma|}(1 + l(Q)^{-1}|x - x_Q|)^{-(n+\epsilon)}$ for $0 < |\gamma| \le r_1$, and

(8.4) $\int_{\mathbb{R}^n} x^\gamma m_Q(x)dx = 0$ for $|\gamma| < r_2$.

From the assumption $T^*x^\gamma = 0$ for $|\gamma| < r_2$ we have $\int_{\mathbb{R}^n} x^\gamma m_Q(x)dx = 0$ for $|\gamma| < r_2$, that is, (8.4) holds.

We choose a suitable large constant C_0. From Fraizer–Torres–Weiss [12: Corollary 2.14], when $|x - x_Q| < 2C_0 2^{-l}$, we have

$$|m_k(x)| \le ||m_k||_\infty \le C \sum_{|\beta|\le 1} ||\partial^\beta \psi_{0,k}||_\infty \le C \le C(1 + |x - x_k|)^{-L}$$

and

$$|\partial^\gamma m_Q(x)| \le ||\partial^\gamma m_Q||_\infty \le C \sum_{|\alpha|\le|\gamma|+1} 2^{l(|\gamma|-|\alpha|)}2^{l|\alpha|}||\partial^\alpha \psi_Q||_\infty$$

$$\leq C2^{l|\gamma|} \leq Cl(Q)^{-|\gamma|}(1 + l(Q)^{-1}|x - x_Q|)^{-L}$$

for any $L \geq 0$ and $|\gamma| \leq r_1$. When $|x - x_Q| \geq 2C_0 2^{-l}$, using (7.1) and (7.2) in Definition 8, we obtain

$$\begin{aligned} |m_k(x)| &= |\int_{\mathbb{R}^n} K(x,y)\psi_{0,k}(y)dy| \leq C\int_{\mathbb{R}^n} |K(x,y)||\psi_{0,k}(y)|dy \\ &\leq C\int_{|y-x_k|\leq C_0} |x-y|^{-n}(1+|y-x_k|)^{-L}dy \leq C(1+|x-x_k|)^{-(L+n)}. \end{aligned}$$

Moreover, using (7.3) in Definition 8 for $0 < |\gamma| \leq r_1$, we have

$$\begin{aligned} |\partial^\gamma m_Q(x)| &\leq C\int_{|y-x_Q|\leq C_0 2^{-l}} |\partial_1^\gamma K(x,y) - \partial_1^\gamma K(x,x_Q)||\psi_Q(y)|dy \\ &\leq C\int_{|y-x_Q|\leq C_0 2^{-l}} |y-x_Q|^\epsilon |x-x_Q|^{-(n+|\gamma|+\epsilon)}dy \\ &\leq C2^{-l(n+\epsilon)}|x-x_Q|^{-(n+|\gamma|+\epsilon)} \leq C2^{l|\gamma|}(1+2^l|x-x_Q|)^{-(n+\epsilon)}. \end{aligned}$$

Therefore, we obtain (8.1), (8,2) and (8.3). Hence by Lemma 3.3, $Tf = \sum_k c_k m_k + \sum_Q c(Q)m_Q$ is convergent in $\mathcal{S}'$ from (8.1), (8.2), (8.3) and (8.4). For the wavelet expansion

$$m_k = \sum_k \langle m_k \ , \ \psi_{0,k}\rangle\psi_{0,k} + \sum_P l(P)^{-n}\langle m_k \ , \ \psi_P\rangle\psi_P,$$

$$m_Q = \sum_k \langle m_Q \ , \ \psi_{0,k}\rangle\psi_{0,k} + \sum_P l(P)^{-n}\langle m_Q \ , \ \psi_P\rangle\psi_P,$$

we have

$$\begin{aligned} Tf &= \sum_k c_k m_k + \sum_{l(Q)\leq 1} c(Q)m_Q = \\ &\sum_k((B_1c_k) + (B_2c_k))\psi_{0,k} + \sum_{l(P)\leq 1}((A_1c) + (A_2c))(P)\psi_P \end{aligned}$$

where $B_1 = \{\langle m_k \ , \ \psi_{0,k'}\rangle\}_{k'k},\ B_2 = \{\langle m_Q \ , \ \psi_{0,k'}\rangle\}_{k'Q}$,
$A_1 = \{l(P)^{-n}\langle m_k \ , \ \psi_P\rangle\}_{Pk},\ A_2 = \{l(P)^{-n}\langle m_Q \ , \ \psi_P\rangle\}_{PQ}$. By Lemma 3.1, Lemma 3.2, (8.1), (8,2), (8,3) and (8.4) the operators $B_1,\ B_2,\ A_1,\ A_2$ are bounded on $a^s(e_{pq}^{s'})_{x_0}^\sigma$ if r_1 and r_2 satisfy (4.1) and (4.2) respectively. By Remark 7 (2), it follows that

$$\begin{aligned} ||Tf||_{A^s(E_{pq}^{s'})_{x_0}^\sigma} &\sim ||(B_1c_k + B_2c_k)||_{a^s(e_{pq}^{s'})_{x_0}^\sigma} + ||(A_1c + A_2c)||_{a^s(e_{pq}^{s'})_{x_0}^\sigma} \\ &\leq C(||c_k||_{a^s(e_{pq}^{s'})_{x_0}^\sigma} + ||c||_{a^s(e_{pq}^{s'})_{x_0}^\sigma}) \sim C||f||_{A^s(E_{pq}^{s'})_{x_0}^\sigma}. \end{aligned}$$

Similarly, we obtain (ii). □

Definition 9. Let $\mu \in \mathbb{R}$. A smooth function a defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to belong to the class $S_{1,1}^\mu(\mathbb{R}^n)$ if a satisfies the following differential inequalities that for all $\alpha,\ \beta \in \mathbb{N}_0^n$,

$$\sup_{x,\xi}(1+|\xi|)^{-\mu-|\alpha|+|\beta|}|\partial_x^\alpha\partial_\xi^\beta a(x,\ \xi)| < \infty.$$

$a(x,D)$ is the corresponding pseudo-differential operator such that

$$a(x,D)f(x) = \int_{\mathbb{R}^n} e^{ix\xi}a(x,\xi)\hat{f}(\xi)\ d\xi$$

for $f \in \mathcal{S}$.

Theorem 5.2. *Let $s,\ s',\ \sigma \in \mathbb{R},\ x_0 \in \mathbb{R}^n,\ 0 < p,\ q \le \infty$. Let $\mu \in \mathbb{R}$, J as in Lemma 3.1 and $a \in S^{\mu}_{1,1}(\mathbb{R}^n)$.*

(i) *$a(x,D)$ is a continuous linear mapping from $A^s(E^{s'}_{pq})^{\sigma}_{x_0}$ to $A^s(E^{s'-\mu}_{pq})^{\sigma}_{x_0}$ if $s' > J - n + \mu$ or $a(x,\xi) = a(\xi)$.*

(ii) *$a(x,D)$ is a continuous linear mapping from $A^s(\tilde{E}^{s'}_{pq})^{\sigma}_{x_0}$ to $A^s(\tilde{E}^{s'-\mu}_{pq})^{\sigma}_{x_0}$ if $s' > J-n+\mu+\sigma\wedge 0$ or $a(x,\xi) = a(\xi)$.*

Proof. (i) We may assume $\sigma \ge 0$ by Remark 1. We write $T \equiv a(x,D)$. Let $f \in A^s(E^{s'}_{pq})^{\sigma}_{x_0}$. By Theorem 4.1, we consider the φ-transform $f = \sum_P c(P)\phi_P$ where $c(P) = c(f)(P) = l(P)^{-n}\langle f,\ \varphi_P\rangle$ and $\phi_P,\ \varphi_P$ as in Remark 2. Then we see $(c(P)) \in a^s(e^{s'}_{pq})^{\sigma}_{x_0}$. We write that $Tf = \sum_P c(P)m_P$ where $m_P = T\phi_P$. We see for a dyadic cube P with $l(P) = 2^{-j}$

$$m_P = \int e^{ix\xi}a(x,\xi)\hat{\phi}_P(\xi)\ d\xi.$$

Then we have, using a change of variables,

$$m_P(x) = \int e^{i(x-x_P)(2^j\xi)}a(x,2^j\xi)\hat{\phi}(\xi)\ d\xi.$$

By the fact that $(1-\triangle_\xi)^L(e^{ix\xi}) = (1+|x|^2)^L e^{ix\xi}$ for the Laplacian $\triangle$ and using an integration by parts, we obtain for $\gamma \in \mathbb{N}^n_0$ and $l(P) < 1$,

$$\begin{aligned}
&\partial^\gamma_x m_P(x)\\
&= \int (1-\triangle_\xi)^L(e^{i2^j(x-x_P)\xi})(1+(2^j|x-x_P|)^2)^{-L} \times\\
&\quad \sum_{\delta\le\gamma}(2^j i\xi)^\delta \partial^{\gamma-\delta}_x a(x,2^j\xi)\hat{\phi}(\xi)\ d\xi\\
&= C(1+(2^j|x-x_P|)^2)^{-L}\int e^{i2^j(x-x_P)\xi}(1-\triangle_\xi)^L \times\\
&\quad \sum_{\delta\le\gamma}(2^j i\xi)^\delta \partial^{\gamma-\delta}_x a(x,2^j\xi)\hat{\phi}(\xi)\ d\xi.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&|\partial^\gamma_x m_P(x)|\\
&\le C(1+2^j|x-x_P|)^{-2L}\int \sum_{\delta\le\gamma}\ \sum_{|\alpha+\beta+\tau|\le 2L,\alpha\le\delta} \times\\
&\quad 2^{j|\delta|}2^{j|\beta|}|\partial^\alpha_\xi(\xi)^\delta||\partial^\beta_\xi\partial^{\gamma-\delta}_x a(x,2^j\xi)||\partial^\tau_\xi\hat{\phi}(\xi)|\ d\xi\\
&\le C(1+2^j|x-x_P|)^{-2L}\int \sum_{\delta\le\gamma}\ \sum_{|\alpha+\beta+\tau|\le 2L,\alpha\le\delta} \times\\
&\quad 2^{j|\delta|}2^{j|\beta|}|\xi|^{|\delta|-|\alpha|}(1+2^j|\xi|)^{\mu+|\gamma|-|\delta|-|\beta|}|\partial^\tau_\xi\hat{\phi}(\xi)|\ d\xi\\
&\le C2^{j\mu}2^{j|\gamma|}(1+2^j|x-x_P|)^{-2L}
\end{aligned}$$

and similarly, for P with $l(P)=1$,

$$\begin{aligned}&|\partial_x^\gamma m_P(x)|\\ &\le\ C(1+|x-x_P|)^{-2L}\times\\ &\qquad\int\sum_{|\alpha+\beta+\tau|\le 2L}(1+|\xi|)^{\mu+|\gamma|-|\alpha|-|\beta|}|\partial_\xi^\tau\hat{\phi}_0(\xi)|\,d\xi\\ &\le\ C(1+|x-x_P|)^{-2L}.\end{aligned}$$

Hence, $m_P(x)$ satisfies

$$|2^{-j\mu}\partial^\gamma m_P(x)|\le C2^{j|\gamma|}(1+2^j|x-x_P|)^{-2L}$$

for P with $l(P)\le 1$, any $\gamma\in\mathbb{N}_0$ and any $L\ge 0$. We choose a suitable large L. For the $\varphi-$ transform

$$2^{-j\mu}m_P=\sum_{l(R)\le 1}l(R)^{-n}\langle 2^{-j\mu}m_P\ ,\ \varphi_R\rangle\phi_R,$$

we have

$$Tf=\sum_{l(P)\le 1}2^{j\mu}c(P)(2^{-j\mu}m_P)=\sum_{l(R)\le 1}A(2^{j\mu}c)(R)\phi_R,$$

where $A=\{l(R)^{-n}\langle 2^{-j\mu}m_P\ ,\ \varphi_R\rangle\}_{RP}$. From Lemma 3.1 and Lemma 3.2, A is bounded on $a^s(e_{pq}^{s'-\mu})_{x_0}^\sigma$ if $s'>J-n+\mu$ or $a(x,\xi)=a(\xi)$. We remark that in the case $s'>J-n+\mu$, we do not assume the vanishing moment condition for m_P. But in the case $a(x,\xi)=a(\xi)$, we have the vanishing moment condition for m_P, indeed, for any P with $l(P)<1$, $\int x^\gamma m_P(x)\,dx=C\partial^\gamma\hat{m}_P(0)=C\partial^\gamma(\hat{\phi}_P)\cdot a)(0)=0$ for any $\gamma\in\mathbb{N}_0$. From (a) and (b) in the proof of Theorem 4.1, it follows that

$$\begin{aligned}&||Tf||_{A^s(E_{pq}^{s'-\mu})_{x_0}^\sigma}\le C||A(2^{j\mu}c)||_{a^s(e_{pq}^{s'-\mu})_{x_0}^\sigma}\\ &\le\ C||2^{j\mu}c||_{a^s(e_{pq}^{s'-\mu})_{x_0}^\sigma}\le C||c||_{a^s(e_{pq}^{s'})_{x_0}^\sigma}\le C||f||_{A^s(E_{pq}^{s'})_{x_0}^\sigma}.\end{aligned}$$

(ii) Similarly, we can prove for this case. □

Corollary . *Let $s,\ s',\ \sigma\in\mathbb{R},\ x_0\in\mathbb{R}^n,\ 0<p,\ q\le\infty$.*

(i) *Let $\mu\in\mathbb{R}$. Then the Bessel potential $(1-\triangle)^{\mu/2}$ is a continuous isomorphisms from $A^s(E_{pq}^{s'})_{x_0}^\sigma$ onto $A^s(E_{pq}^{s'-\mu})_{x_0}^\sigma$, and from $A^s(\tilde{E}_{pq}^{s'})_{x_0}^\sigma$ onto $A^s(\tilde{E}_{pq}^{s'-\mu})_{x_0}^\sigma$.*

(ii) *Let $\gamma\in\mathbb{N}_0^n$. Then the differential operator ∂^γ is continuous from $A^s(E_{pq}^{s'})_{x_0}^\sigma$ to $A^s(E_{pq}^{s'-|\gamma|})_{x_0}^\sigma$, and from $A^s(\tilde{E}_{pq}^{s'})_{x_0}^\sigma$ to $A^s(\tilde{E}_{pq}^{s'-|\gamma|})_{x_0}^\sigma$.*

Proof. These are immediate corollaries of Theorem 5.2. To finish the proof of (i) we need to show the mapping is surjective and one to one. For $h\in A^s(E_{pq}^{s'-\mu})_{x_0}^\sigma$, we set $f=(1-\triangle)^{-\mu/2}h$. Then $h=(1-\triangle)^{\mu/2}f$. □

6 Characterizations via differences and oscillations

Definition 10. Let $k\in\mathbb{N}_0$. We define the differences of functions

$\triangle_u^1 f(x)=f(x+u)-f(x)$ and $\triangle^{k+1}=\triangle^1\triangle^k$.

We set

$d_i^k f(y) = \frac{1}{|B_i(y)|}\int_{k|u|\le 2^{-i}} |\triangle_u^k f(y)|\, du$

where $B_i(x)$ is the ball with a center x and a radius 2^{-i}, and $|B_i(x)|$ means its volume. It is obvious that $|d_i^k f(y)| \le C \sup_{k|u|\le 2^{-i}} |\triangle_u^k f(y)|$.

We define the oscillation of locally L^p integrable functions f $(0 < p \le \infty)$ by

$$\mathrm{osc}_p^k f(x,i) = \inf(\frac{1}{|B_i(x)|}\int_{B_i(x)} |f(y) - P(y)|^p\, dy)^{1/p}$$

with the suitable modification for $p = \infty$, where the infimum is taken over all polynomials $P(x) \in \mathcal{P}_k$, the space of all polynomials with deg $\le k$ on $\mathbb{R}^n$. By $P_B f$ for a ball B we denote the unique polynomial in $\mathcal{P}_k$ such that $\int_B (f(x) - P_B f(x)) x^\alpha\, dx = 0$ for all $|\alpha| \le k$. We see that $||P_B f||_{L^\infty(B)} \le \frac{1}{|B|}\int_B |f(x)|\, dx$ and $P_B f = f$ for $f \in \mathcal{P}_k$. We put

$$\Omega_p^k f(x,i) = (\frac{1}{|B_i(x)|}\int_{B_i(x)} |f(y) - P_{B_i(x)} f(y)|^p\, dy)^{1/p}.$$

Then we see $\mathrm{osc}_p^k f(x,i) \sim \Omega_p^k f(x,i)$ if $1 \le p \le \infty$ (cf. [19]).

Lemma 6.1. (i) *Let* $s \in \mathbb{R}$, $\sigma \ge 0$ *and let* $k \in \mathbb{N}$, $k > s' > 0$, $1 \le p \le \infty$, $0 < q \le \infty$ *and let* f *be locally* L^p *integrable.*

Then we have

$$\begin{aligned}&\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times\\ &\quad(\sum_{i\ge(-\log_2 l(P))\vee 0} (2^{is'} \sup_{k|u|\le 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q}\\ &\le C \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times\\ &\quad(\sum_{i\ge(-\log_2 l(P))\vee 0} (2^{is'} ||\mathrm{osc}_p^{k-1} f(x,i)||_{L^p(P)})^q)^{1/q},\end{aligned}$$

and

$$\begin{aligned}&\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times\\ &\quad(\sum_{i\ge(-\log_2 l(P))\vee 0} (2^{is'} ||\mathrm{osc}_p^{k-1} f(x,i)||_{L^p(P)})^q)^{1/q}\\ &\le C \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times\\ &\quad(||f||_{L^p(P)} + (\sum_{i\ge(-\log_2 l(P))\vee 0} (2^{is'} ||d_i^k f||_{L^p(P)})^q)^{1/q}).\end{aligned}$$

(ii) *Let* $s \in \mathbb{R}$, $\sigma \ge 0$ *and let* $k \in \mathbb{N}$, $k > s' > 0$, $1 \le p < \infty$, $0 < q \le \infty$ *and let* f *be locally* L^p *integrable . Then we have*

$$\begin{aligned}
&\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times \\
&\qquad ||(\sum_{i\geq(-\log_2 l(P))\vee 0} (2^{is'} \sup_{k|u|\leq 2^{-i}} |\triangle_u^k f|)^q)^{1/q}||_{L^p(P)} \\
&\leq C \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times \\
&\qquad ||(\sum_{i\geq(-\log_2 l(P))\vee 0} (2^{is'} \mathrm{osc}_p^{k-1} f(x,i))^q)^{1/q}||_{L^p(P)},
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times \\
&\qquad ||(\sum_{i\geq(-\log_2 l(P))\vee 0} (2^{is'} \mathrm{osc}_p^{k-1} f(x,i))^q)^{1/q}||_{L^p(P)} \\
&\leq C \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times \\
&\qquad (||f||_{L^p(P)} + ||(\sum_{i\geq(-\log_2 l(P))\vee 0} (2^{is'} d_i^k f)^q)^{1/q})||_{L^p(P)}.
\end{aligned}$$

Proof. We will see that for $k|u| \leq 2^{-i}$,

$|\triangle_u^k f(x)| \leq C(\sum_{e=0}^k |f(x+eu) - P_{B_i(x+eu)} f(x+eu)|)$
$\leq C \sum_{e=0}^k \sum_{l\geq i} \Omega_p^{k-1} f(x+eu,l).$

We consider a sequence for $i < \cdots < m \to \infty$,

$$B_i(x+eu) \supset \cdots \supset B_m(x+eu) \supset \cdots \to x+eu.$$

Then we have

$$\begin{aligned}
&\frac{1}{|B_m|}\int_{B_m} |f - P_{B_i} f|\, dy \leq \frac{1}{|B_m|}\int_{B_m} |f - P_{B_m} f|\, dy \\
&+ \frac{1}{|B_m|}\sum_{l=i+1}^m \int_{B_m} |P_{B_l} f - P_{B_{l-1}} f|\, dy \\
&\leq \frac{1}{|B_m|}\int_{B_m} |f - P_{B_m} f|\, dy + C\sum_{l=i+1}^m \frac{1}{|B_l|}\int_{B_l} |f - P_{B_{l-1}} f|\, dy \\
&\leq \frac{1}{|B_m|}\int_{B_m} |f - P_{B_m} f|\, dy + C\sum_{l=i}^m \frac{1}{|B_l|}\int_{B_l} |f - P_{B_l} f|\, dy.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&|f(x+eu) - P_{B_i}(x+eu)| = \lim_{m\to\infty} \frac{1}{|B_m|}\int_{B_m} |f - P_{B_i} f|\, dy \\
&\leq \lim_{m\to\infty} \frac{1}{|B_m|}\int_{B_m} |f - P_{B_m} f|\, dy + C\sum_{l=i}^\infty \frac{1}{|B_l|}\int_{B_l} |f - P_{B_l} f|\, dy \\
&\leq C\sum_{l=i}^\infty \Omega_p^{k-1} f(x+eu,l).
\end{aligned}$$

Therefore, we have for a dyadic cube P with $l(P) = 2^{-j}$,

$$
\begin{aligned}
&(\sum_{i\geq j\vee 0}(2^{is'}\sup_{k|u|\leq 2^{-i}}||\triangle_u^k f||_{L^p(P)})^q)^{1/q}\\
&\leq C(\sum_{i\geq j\vee 0}(2^{is'}\sum_{l\geq i}||\Omega_p^{k-1}f(x,l)||_{L^p(3P)})^q)^{1/q}\\
&\leq C(\sum_{i\geq j\vee 0}(2^{is'}||\Omega_p^{k-1}f(x,i)||_{L^p(3P)})^q)^{1/q}\\
&\leq C(\sum_{i\geq j\vee 0}(2^{is'}||\mathrm{osc}_p^{k-1}f(x,i)||_{L^p(3P)})^q)^{1/q}\\
&\leq C(|x_0-x_P|+2^{-j})^\sigma 2^{-js}\sup_{\mathcal{D}\ni Q\ni x_0}l(Q)^{-\sigma}\sup_{\mathcal{D}\ni P\subset 3Q}l(P)^{-s}\times\\
&\quad(\sum_{i\geq j\vee 0}(2^{is'}||\mathrm{osc}_p^{k-1}f(x,i)||_{L^p(P)})^q)^{1/q}
\end{aligned}
$$

by using Hardy's inequality if $s' > 0$. This completes the proof of the first half of (i).

Next, we will prove the last half of (i).

We consider a function $\theta \in \mathcal{S}$ such that supp $\theta \subset \{k|u| \leq 1\}$ and $\int \theta(u)\, du = 1$. We put

$$h_i(x) = \int (f(x) - \triangle_u^k f(x))\theta_i(u)\, du$$

where $\theta_i(u) = 2^{ni}\theta(2^i u)$. We claim that

$$\mathrm{osc}_p^{k-1}f(x,i) \leq C(\tfrac{1}{|B_i(x)|}\int_{B_i(x)}|d_i^k f(y)|^p\, dy)^{1/p} + C\mathrm{osc}_p^{k-1}h_i(x,\ i).$$

We see that

$$
\begin{aligned}
&\mathrm{osc}_p^{k-1}f(x,i)\sim\Omega_p^{k-1}f(x,i)=\\
&\quad(\frac{1}{|B_i(x)|}\int_{B_i(x)}|f(y)-P_{B_i(x)}f(y)|^p\,dy)^{1/p}\\
&\leq (\frac{1}{|B_i(x)|}\int_{B_i(x)}|f(y)-h_i(y)|^p\,dy)^{1/p}\\
&+ (\frac{1}{|B_i(x)|}\int_{B_i(x)}|h_i(y)-P_{B_i(x)}h_i(y)|^p\,dy)^{1/p}\\
&+ (\frac{1}{|B_i(x)|}\int_{B_i(x)}|P_{B_i(x)}h_i(y)-P_{B_i(x)}f(y)|^p\,dy)^{1/p}\\
&\leq C(\frac{1}{|B_i(x)|}\int_{B_i(x)}|f(y)-h_i(y)|^p\,dy)^{1/p}\\
&+ C(\frac{1}{|B_i(x)|}\int_{B_i(x)}|h_i(y)-P_{B_i(x)}h_i(y)|^p\,dy)^{1/p}\\
&\leq C(\frac{1}{|B_i(x)|}\int_{B_i(x)}(\int_{k|u|\leq 2^{-i}}|\triangle_u^k f(y)||\theta_i(u)|\,du)^p\,dy)^{1/p}\\
&\quad+C\Omega_p^{k-1}h_i(x,i)\\
&\leq C(\frac{1}{|B_i(x)|}\int_{B_i(x)}|d_i^k f(y)|^p\,dy)^{1/p}+C\mathrm{osc}_p^{k-1}h_i(x,i).
\end{aligned}
$$

Next, we will estimate $\mathrm{osc}_p^{k-1} h_i(x,i)$. We consider the $(k-1)$th Taylor polynomial $q(x)$ of h_i at x. Then we have

$$\begin{aligned}
&h_i(y)-q(y)\\
&= \int_0^1 \sum_{|\beta|=k} \frac{k}{\beta!}\partial^\beta h_i(x+t(y-x))(x-y)^\beta(1-t)^{k-1}\,dt\\
&= \int_0^1 \sum_{|\beta|=k} \frac{k}{\beta!}\int\sum_{m=1}^{k}\begin{pmatrix} k\\ m\end{pmatrix}(-1)^{k-m}\partial^\beta f(x+t(y-x)+mu)\times\\
&\quad \theta_l(u)\,du(x-y)^\beta(1-t)^{k-1}\,dt\\
&= \int_0^1 \sum_{|\beta|=k} \frac{k}{\beta!}\int\sum_{m=1}^{k}\begin{pmatrix} k\\ m\end{pmatrix}(-1)^{k-m}m^k f(x+t(y-x)+m2^{-i}u)\times\\
&\quad \partial^\beta\theta(u)\,du(x-y)^\beta(1-t)^{k-1}\,dt.
\end{aligned}$$

Hence, we see by using Minkowski's inequality

$$\begin{aligned}
&||\mathrm{osc}_p^{k-1}h_i(x,i)||_{L^p(P)} \le ||(\frac{1}{|B_i(x)|}\int_{B_i(x)}|h_i(y)-q(y)|^p\,dy)^{1/p}||_{L^p(P)}\\
&\le C(\int_P\frac{1}{|B_i(x)|}\int_{B_i(x)}(\int_0^1\int_{k|u|\le 1}\sum_{m=1}^{k}|f(x+t(y-x)+m2^{-i}u)|\times\\
&\quad |\partial^\beta\theta(u)|du|x-y|^k(1-t)^{k-1}\,dt)^p\,dydx)^{1/p}\\
&\le C\int_0^1\int_{k|u|\le 1}\sum_{m=1}^{k}(\frac{1}{|B_i(x)|}\int_{B_i(0)}\int_P|f(x+ty+m2^{-i}u)|^p\,dxdy)^{1/p}\times\\
&\quad 2^{-ik}(1-t)^{k-1}\,dudt\\
&\le C\int_0^1\int_{k|u|\le 1}\sum_{m=1}^{k}(\frac{1}{|B_i(x)|}\int_{B_i(0)}\int_{P+ty+m2^{-i}u}|f(x)|^p\,dxdy)^{1/p}\times\\
&\quad 2^{-ik}(1-t)^{k-1}\,dudt\\
&\le C2^{-ik}(\int_{5P}|f(x)|^p\,dx)^{1/p}\le C2^{-ik}||f||_{L^p(5P)}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&||(\frac{1}{|B_i(x)|}\int_{B_i(x)}|d_i^k f(y)|^p\,dy)^{1/p}||_{L^p(P)}\\
&\le C(\int_P(\frac{1}{|B_i(x)|}\int_{B_i(0)}|d_i^k f(x+y)|^p\,dydx)^{1/p}\\
&\le C(\frac{1}{|B_i(x)|}\int_{B_i(0)}\int_{P+y}|d_i^k f(x)|^p\,dxdy)^{1/p}\\
&\le C(\int_{3P}|d_i^k f(x)|^p\,dx)^{1/p}\le C||d_i^k f||_{L^p(3P)}.
\end{aligned}$$

Thus, we have for a dyadic cube P with $l(P)=2^{-j}$

$$(\sum_{i\ge j\vee 0}(2^{is'}||\mathrm{osc}_p^{k-1}f(x,i)||_{L^p(P)})^q)^{1/q}$$

$$\leq C(\sum_{i\geq j\vee 0}(2^{is'}||d_i^k f||_{L^p(3P)})^q)^{1/q} + C(\sum_{i\geq j\vee 0} 2^{-i(k-s')q})^{1/q}||f||_{L^p(5P)}$$

$$\leq C(\sum_{i\geq j\vee 0}(2^{is'}||d_i^k f||_{L^p(3P)})^q)^{1/q} + C||f||_{L^p(5P)}$$

$$\leq C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}(\sum_{i\geq j\vee 0}(2^{is'}||d_i^k f||_{L^p(P)})^q)^{1/q}$$

$$+C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}$$

if $k > s'$. The proof of (i) is complete. In the same way we can prove (ii). □

Theorem 6.1. (i) *Let* $s',\ s,\ \sigma\in\mathbb{R}$ *with* $0<s',\ \ 0\leq\sigma$, *and let* $x_0\in\mathbb{R}^n$, $1\leq p\leq\infty,\ 0<q\leq\infty$. *Let* $k\in\mathbb{N}$ *with* $k>s'>0$. *Then we have following equivalences for* $f\in\mathcal{S}'$

$$\begin{aligned}
&||f||_{A^s(B^{s'}_{pq})^\sigma_{x_0}} + \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}\\
&\sim \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
&+ (\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}\sup_{k|u|\leq 2^{-i}}||\triangle_u^k f||_{L^p(P)})^q)^{1/q})\\
&\sim \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
&+ (\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}||\mathrm{osc}_p^{k-1} f||_{L^p(P)})^q)^{1/q})\\
&\sim \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
&+ (\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}||d_i^k f||_{L^p(P)})^q)^{1/q}).
\end{aligned}$$

(ii) *Let* $s,\ s',\ \sigma\in\mathbb{R}$ *with* $0<s',\ \ 0\leq\sigma,\ \ x_0\in\mathbb{R}^n$, $1\leq p<\infty,\ 1\leq q\leq\infty$. *Let* $k\in\mathbb{N}$ *with* $k>s'>0$. *Then we have following equivalences for* $f\in\mathcal{S}'$

$$\begin{aligned}
&||f||_{A^s(F^{s'}_{pq})^\sigma_{x_0}} + \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}\\
&\sim \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
&+ ||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}\sup_{k|u|\leq 2^{-i}}|\triangle_u^k f|)^q)^{1/q}||_{L^p(P)})\\
&\sim \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
&+ ||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}\mathrm{osc}_p^{k-1} f)^q)^{1/q}||_{L^p(P)})\\
&\sim \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
&+ ||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}d_i^k f)^q)^{1/q}||_{L^p(P)}).
\end{aligned}$$

Proof. (i) It suffices to prove the first part of (i) by Lemma 6.1. We consider the Littlewood-Paley decomposition $f = S_i f + \sum_{l>i} f * \phi_l$. Then we have for $k|u| \leq 2^{-i}$ and a dyadic cube P with $l(P) = 2^{-j}$, $i \geq j$

$$\begin{aligned} ||\triangle_u^k f||_{L^p(P)} &\leq ||\triangle_u^k (f - S_i f)||_{L^p(P)} + ||\triangle_u^k S_i f||_{L^p(P)} \\ &\leq \ C \sum_{l>i} ||\triangle_u^k (f * \phi_l)||_{L^p(P)} + C||\triangle_u^k S_i f||_{L^p(P)}. \end{aligned}$$

We will estimate $||\triangle_u^k S_i f||_{L^p(P)}$. Note the following formula

$$\triangle_u^k S_i f(x) = \int_{-\infty}^{\infty} \sum_{|\nu|=k} \frac{k!}{\nu!} u^\nu \partial^\nu S_i f(x + \xi u) N_k(\xi) \ \ d\xi$$

where N_k is the B-spline of order k (e.g. See [27]). Therefor we have for $k|u| \leq 2^{-i}$

$$||\triangle_u^k S_i f||_{L^p(P)} \leq C \textstyle\sum_{|\nu|=k} |u|^k ||\partial^\nu S_i f||_{L^p(2P)}.$$

Next, we will estimate $||\partial^\nu S_i f||_{L^p(2P)}$:

$$\begin{aligned} &||\partial^\nu S_i f||_{L^p(2P)} = || \int f(x - 2^{-i} y) \ \partial^\nu \phi_0(y) \ dy||_{L^p(2P)} \\ &\leq \ C \int (\int_{2P+2^{-i}y} |f(x)|^p \ dx)^{1/p} |\partial^\nu \phi_0(y)| \ dy \\ &\leq \ C 2^{-js} \int (|x_0 - x_P| + 2^{-j}(1 + |y|))^\sigma |\partial^\nu \phi_0(y)| \ dy \\ &\times \ \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)} \\ &\leq \ C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &||\triangle_u^k f||_{L^p(P)} \\ &\leq C \sum_{l>i} ||\triangle_u^k (f * \phi_l)||_{L^p(P)} \\ &+ C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} 2^{-ik} \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)}. \end{aligned}$$

Moreover, we obtain by using Hardy's inequality if $s' > 0$

$$\begin{aligned} &(\sum_{i \geq j \vee 0} (2^{is'} \sup_{k|u| \leq 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q} \\ &\leq \ C(\sum_{i \geq j \vee 0} (2^{is'} \sum_{l>i} ||\triangle_u^k (f * \phi_l)||_{L^p(P)})^q)^{1/q} \\ &+ \ C(\sum_{i \geq j \vee 0} (2^{-i(k-s')} (|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \\ &\times \ \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)})^q)^{1/q} \\ &\leq \ C(\sum_{i > j \vee 0} (2^{is'} ||\triangle_u^k (f * \phi_i)||_{L^p(P)})^q)^{1/q} \\ &+ \ C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)}. \end{aligned}$$

This implies that

$$\begin{aligned}
&\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(\sum_{i\geq j\vee 0}(2^{is'} \sup_{k|u|\leq 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q} \\
\leq\ & C\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(\sum_{i>j\vee 0}(2^{is'}||f*\phi_i||_{L^p(P)})^q)^{1/q} \\
+\ & C\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)} \\
\leq\ & C||f||_{A^s(B^{s'}_{pq})^\sigma_{x_0}} + C\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}.
\end{aligned}$$

We will show the converse statement. It is easy to see that there exist $\phi^m \in \mathcal{S}$ $m = 1, \cdots n$ such that $\phi = \sum_{m=1}^n \triangle_{ce_m}^k \phi^m$ for enough small c where $e_1, \cdots, e_n$ are the canonical basis vectors in $\mathbb{R}^n$. Then we have for $i \in \mathbb{N}$

$$f*\phi_i = \sum_{m=1}^n f*\triangle^k_{c2^{-i}e_m}\phi_i^m = \sum_{m=1}^n \triangle^k_{c2^{-i}e_m} f*\phi_i^m.$$

Therefore, we have for a dyadic cube P with $l(P) = 2^{-j}$ and $i \geq j$

$$\begin{aligned}
&||f*\phi_i||_{L^p(P)} \\
\leq\ & C||\sum_{m=1}^n \triangle^k_{c2^{-i}e_m} f*\phi_i^m||_{L^p(P)} \\
\leq\ & C\int \sum_{m=1}^n (\int_{P+2^{-i}y} |\triangle^k_{c2^{-i}e_m} f(x)|^p\ dx)^{1/p}|\phi^m(y)|\ dy \\
\leq\ & C\int \sum_{m=1}^n (\int_{P+2^{-i}y} \sup_{k|u|\leq 2^{-i}} |\triangle^k_u f(x)|^p\ dx)^{1/p}|\phi^m(y)|\ dy.
\end{aligned}$$

Hence, we have if $l(P) < 1$

$$\begin{aligned}
&(\sum_{i\geq j}(2^{is'}||f*\phi_i||_{L^p(P)})^q)^{1/q} \\
\leq\ & C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \times \\
& \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(\sum_{i\geq j}(2^{is'} \sup_{k|u|\leq 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q}
\end{aligned}$$

and if $l(P) \geq 1$

$$\begin{aligned}
&(\sum_{i\geq 0}(2^{is'}||f*\phi_i||_{L^p(P)})^q)^{1/q} \\
\leq\ & (\sum_{i>0}(2^{is'}||f*\phi_i||_{L^p(P)})^q)^{1/q} + ||f*\phi_0||_{L^p(P)} \\
\leq\ & C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s} \times \\
& (\sum_{i>0}(2^{is'} \sup_{k|u|\leq 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q} \\
+\ & C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} (\sum_{i \geq 0} (2^{is'} ||f * \phi_i||_{L^p(P)})^q)^{1/q} \\
\leq \;& C \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} \times \\
& (\sum_{i \geq 0} (2^{is'} \sup_{k|u| \leq 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q} \\
+ \;& \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)}.
\end{aligned}$$

This completes the proof of Theorem 6.1 (i). In the same way we can prove (ii).

□

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