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- compliance of the title of the paper to its content;

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- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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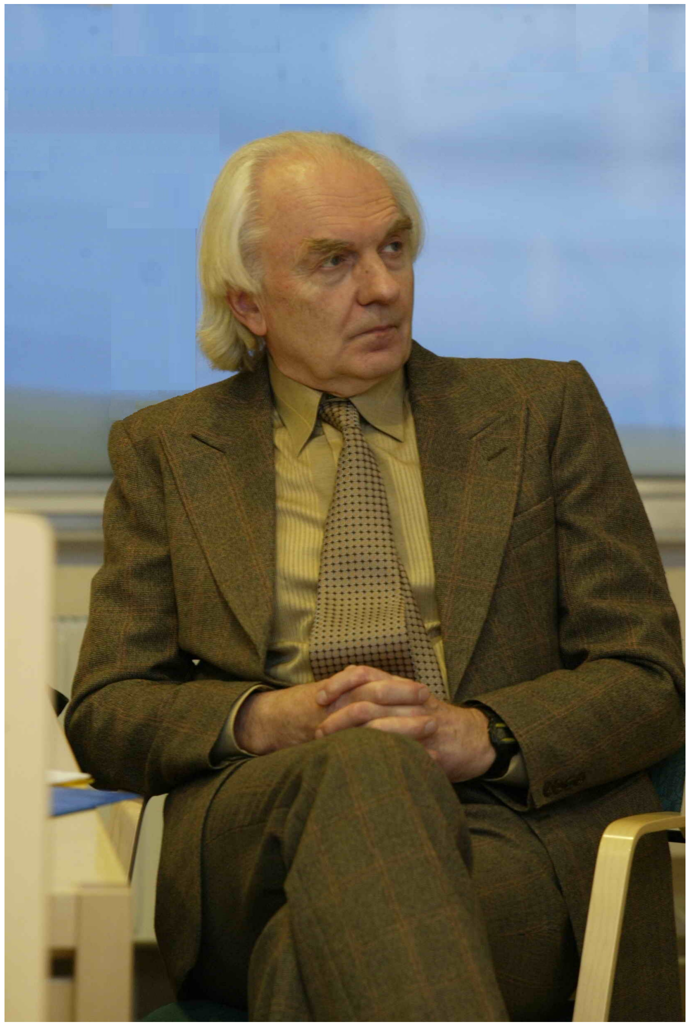
On the 90th birthday of Professor Oleg Vladimirovich Besov



This issue of the Eurasian Mathematical Journal is dedicated to the 90th birthday of Oleg Vladimirovich Besov, an outstanding mathematician, Doctor of Sciences in physics and mathematics, corresponding member of the Russian Academy of Sciences, academician of the European Academy of Sciences, leading researcher of the Department of the Theory of Functions of the V.A. Steklov Institute of Mathematics, honorary professor of the Department of Mathematics of the Moscow Institute of Physics and Technology.

Oleg started scientific research while still a student of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University. His research interests were formed under the influence of his scientific supervisor, the great Russian mathematician Sergei Mikhailovich Nikol'skii.

In the world mathematical community O.V. Besov is well known for introducing and studying the spaces $B^r_{p\theta}(\mathbb{R}^n)$, $1 \leq p, \theta \leq \infty$, of differentiable functions of several real variables, which are now named Besov spaces (or Nikol'skii–Besov spaces, because for $\theta = \infty$ they coincide with Nikol'skii spaces $H^r_p(\mathbb{R}^n)$).

The parameter r may be either an arbitrary positive number or a vector $r = (r_1, ..., r_n)$ with positive components r_j. These spaces consist of functions having common smoothness of order r in the isotropic case (not necessarily integer) and smoothness of orders r_j in variables $x_j, j = 1, ..., n$, in the anisotropic case, measured in L_p-metrics, and θ is an additional parameter allowing more refined classification in the smoothness property.

O.V. Besov published more than 150 papers in leading mathematical journals most of which are dedicated to further development of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$. He considered the spaces $B^r_{p\theta}(\Omega)$ on regular and irregular domains $\Omega \subset \mathbb{R}^n$ and proved for them embedding, extension, trace, approximation and interpolation theorems. He also studied integral representations of functions, density of smooth functions, coercivity, multiplicative inequalities, error estimates in cubature formulas, spaces with variable smoothness, asymptotics of Kolmogorov widths, etc.

The theory of Besov spaces had a fundamental impact on the development of the theory of differentiable functions of several variables, the interpolation of linear operators, approximation theory, the theory of partial differential equations (especially boundary value problems), mathematical physics (Navier–Stokes equations, in particular), the theory of cubature formulas, and other areas of mathematics.

Without exaggeration, one can say that Besov spaces have become a recognized and extensively applied tool in the world of mathematical analysis: they have been studied and used in thousands of articles and dozens of books. This is an outstanding achievement.

The first expositions of the basics of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$ were given by O.V. Besov in [2], [3].

Further developments of the theory of Besov spaces were discussed in a series of survey papers, e.g. [18], [12], [15]. The most detailed exposition of the theory of Besov spaces was given in the book by S.M. Nikol'skii [19] and in the book by O.V. Besov, V.P. Il'in, S.M. Nikol'skii [11], which in 1977 was awarded a State Prize of the USSR. Important further developments of the theory of Besov spaces were given in a series of books by Professor H. Triebel [21], [22], [23]. Many books on real analysis and the theory of partial differential equations contain chapters dedicated to various aspects of the theory of Besov spaces, e.g. [16], [1], [13]. Recently, in 2011, Professor Y. Sawano published the book "Theory of Besov spaces" [20] (in Japanese, in 2018 it was translated into English).

A survey of the main facts of the theory of Besov spaces was given in the dedication to the 80th birthday of O.V. Besov [14].

We would that like to add that during the last 10 years Oleg continued active research and published around 25 papers (all of them without co-authors) on various aspects of the theory of function spaces, namely, on the following topics:

Kolmogorov widths of Sobolev classes on an irregular domain (see, for example, [4]),

embedding theorems for weighted Sobolev spaces (see, for example, [5]),

the Sobolev embedding theorem for the limiting exponent (see, for example, [7]),

multiplicative estimates for norms of derivatives on a domain (see, for example, [8]),

interpolation of spaces of functions of positive smoothness on a domain (see, for example, [9]),

embedding theorems for spaces of functions of positive smoothness on irregular domains (see, for example, [10]).

In 1954 S.M. Nikol'skii organized the seminar "Differentiable functions of several variables and applications", which became the world recognized leading seminar on the theory of function spaces. Oleg participated in this seminar from the very beginning, first as the secretary and later, for more than 30 years, as the head of the seminar first jointly with S.M. Nikol'skii and L.D. Kudryavtsev, then up to the present time on his own.

O.V. Besov participated in numerous research projects supported by grants of several countries, led many of them, and currently is the head of one of them: "Contemporary problems of the theory of function spaces and applications" (project 19-11-00087, Russian Science Foundation).

He takes active part in the international mathematical life, participates in and contributes to organizing many international conferences. He has given more than 100 invited talks at conferences and has been invited to universities in more than 20 countries.

For more than 50 years O.V. Besov has been a professor at the Department of Mathematics of the Moscow Institute of Physics and Technology. He is a celebrated and sought-after lecturer who is able to develop the student's independent thinking. On the basis of his lectures he wrote a popular text-book on mathematical analysis [6].

He spends a lot of time on supervising post-graduate students. One of his former post-graduate students H.G. Ghazaryan, now a distinguished professor, plays an active role in the mathematical life of Armenia and has many post-graduate students of his own.

Professor Besov has close academic ties with Kazakhstan mathematicians. He has many times visited Kazakhstan, is an honorary professor of the Shakarim Semipalatinsk State University and a member of the editorial board of the Eurasian Mathematical Journal. He has been awarded a medal for his meritorious role in the development of science of the Republic of Kazakhstan.

Oleg is in good physical and mental shape, leads an active life, and continues productive research on the theory of function spaces and lecturing at the Moscow Institute of Physics and Technology.

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Oleg Vladimirovich Besov on occasion of his 90th birthday, wishes him good health and further productive work in mathematics and mathematical education.

On behalf of the Editorial Board

V.I. Burenkov, T.V. Tararykova

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ON ESTIMATES OF NON-INCREASING REARRANGEMENT OF GENERALIZED FRACTIONAL MAXIMAL FUNCTION

N.A. Bokayev, A. Gogatishvili, A.N. Abek

Communicated by V.I. Burenkov

Key words: generalized fractional maximal function, non-increasing rearrangements, generalized Riesz potential.

AMS Mathematics Subject Classification: 42B25, 46E30, 47B38.

Abstract. We give a sharp pointwise estimate of the non-increasing rearrangement of the generalized fractional maximal function $(M_\Phi f)(x)$ via an expression involving the non-increasing rearrangement of f. It is shown that the obtained estimate is more sharp than the inequality which follows from the estimate for the generalized Riesz potential.

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1 Introduction

In this paper, we consider the generalized fractional maximal function

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)|dy,$$

for locally integrable functions f under certain assumptions on the function Φ, where $B(x,r)$ is the ball with the center at the point $x \in \mathbb{R}^n$ and radius $r > 0$. When $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0;n)$, $n \in \mathbb{N}$ we get the classical fractional maximal function $(M_\alpha f)(x)$. When $\alpha = 0$ we get the Hardy-Littlewood maximal function. Other types of generalized fractional maximal functions were considered in [6], [11-13].

Let $L_0 = L_0(\mathbb{R}^n)$ be the set of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ and μ_n be the Lebesgue measure on $\mathbb{R}^n$. By L_0^+ we denote the subset of the set L_0 consisting of all non-negative functions:

$$L_0^+ = \{f \in L_0 : f \geq 0\}.$$

By $L_0^+(0,\infty;\downarrow)$ we denote the set of all non-increasing functions belonging to L_0^+. The non-increasing rearrangement f^* is defined by the equality:

$$f^*(t) = \inf\{y \in [0,\infty) : \lambda_f(y) \leq t\}, \ \ t \in \mathbb{R}_+ := (0,\infty),$$

where

$$\lambda_f(y) = \mu_n \{x \in \mathbb{R}^n : |f(x)| > y\}, \ \ y \in [0,\infty)$$

is the Lebesgue distribution function. It is known that f^* is a non-negative, non-increasing and right-continuous function on $\mathbb{R}_+$; f^* is equimeasurable with $|f|$, i.e.

$$\mu_1 \{t \in \mathbb{R}_+ : \ f^*(t) > y\} = \mu_n \{x \in \mathbb{R}^n : \ |f(x)| > y\}.$$

Let $f^{\#} : \mathbb{R}^n \to \mathbb{R}^n$ denote the symmetric rearrangement of f, i.e. a radially symmetric non-negative non-increasing right-continuous function (as a function of $r = |x|$, $x \in \mathbb{R}^n$) which is equimeasurable with f. That is

$$f^{\#}(r) = f^*(v_n r^n); \quad f^*(t) = f^{\#}\left(\left(\frac{t}{v_n}\right)^{\frac{1}{n}}\right), \ r, t \in \mathbb{R}_+,$$

here v_n is the volume of the n-dimensional unit ball.

The function $f^{**} : (0, \infty) \to [0, \infty]$ is defined as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau; \quad t \in \mathbb{R}_+.$$

It is known that f^{**} is a non-increasing function on $\mathbb{R}_+$. For the classical Hardy-Littlewood maximal operator M the rearrangement inequality

$$cf^{**}(t) \leq (Mf)^*(t) \leq Cf^{**}(t), \ \ t \in (0, \infty)$$

holds for some $0 < c \leq C < \infty$ [2, Chapter 3, Theorem 3.8]. For the classical fractional maximal operator

$$(M_\gamma f)(x) := \sup_{r>0} |B(x, r)|^{\frac{\gamma}{n}-1} \int_{B(x,r)} |f(y)|dy, \quad 0 < \gamma < n,$$

in [5] the following estimate was obtained for some $C > 0$

$$(M_\gamma f)^*(t) \leq C \sup_{t<\tau<\infty} \tau^{\gamma/n} f^{**}(\tau), \quad t \in (0, \infty)$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$. Moreover, this estimate is sharp on the class of all non-negative, radially symmetric non-increasing functions.

Definition 1. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *quasi-decreasing (quasi-increasing)* if there exists $C > 1$, such that

$$f(t_2) \leq Cf(t_1) \text{ if } t_1 < t_2$$
$$\big(f(t_1) \leq Cf(t_2) \text{ if } t_1 < t_2\big).$$

Throughout this work we will denote by C, C_1, C_2 positive constants, generally speaking, different in different places.

By the notation $f(x) \cong g(x)$ we mean that there are constants $C_1 > 0$, $C_2 > 0$ such that

$$C_1 f(t) \leq g(t) \leq C_2 f(t), \ \ t \in \mathbb{R}_+.$$

2 The generalized fractional maximal function and estimate of its non-increasing rearrangement

We define the following classes of functions $A_n(R)$, $B_n(R)$, $D(R)$.

Definition 2. Let $n \in \mathbb{N}$ and $R \in (0; \infty]$. We say that a function $\Phi : (0; R) \to \mathbb{R}_+$ belongs to the class $A_n(R)$ if:

(1) Φ is non-increasing and continuous on $(0; R)$;

(2) the function $\Phi(r)r^n$ is quasi-increasing on $(0, R)$.

For example, $\Phi(t) = t^{\alpha-n} \in A_n(\infty), \ \ 0 < \alpha < n.$

Definition 3. [8] Let $n \in \mathbb{N}$ and $R \in (0;\infty]$. A function $\Phi : (0;R) \to \mathbb{R}_+$ belongs to the class $B_n(R)$ if the following conditions hold:

(1) Φ is non-increasing and continuous on $(0;R)$;
(2) there exists $C = C(\Phi, n) > 0$ such that

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \leq C\Phi(r)r^n, \ \ r \in (0,R). \tag{2.1}$$

For example,

$$\Phi(\rho) = \rho^{\alpha-n} \in B_n(\infty) \ (0 < \alpha < n); \ \ \Phi(\rho) = \ln\frac{eR}{\rho} \in B_n(R).$$

For $\Phi \in B_n(R)$ the following estimate also holds

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \geq n^{-1}\Phi(r)r^n, \ r \in (0,R).$$

Therefore

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \cong \Phi(r)r^n, \ r \in (0,R), \tag{2.2}$$

$$\Phi \in B_n(R) \Rightarrow \{ \ \Phi(r)r^n \text{ is quasi-increasing}, \ r \in (0,R)\}. \tag{2.3}$$

It follows from (2.3) that for any $\alpha \in [1;\infty)$ there exists $\beta = \beta(\alpha, C, n) \in [1;\infty)$ (where C is the constant from (2.1)) such that [7]:

$$\left\{\rho, r \in (0;R); \alpha^{-1} \leq \frac{\rho}{r} \leq \alpha\right\} \Rightarrow \beta^{-1} \leq \frac{\Phi(\rho)}{\Phi(r)} \leq \beta. \tag{2.4}$$

Note the well-known equivalence result of N.K. Bari and S.B. Stechkin [1]:

$$(2.1) \Leftrightarrow \exists\gamma \in (0;n) \ \text{ such that } \ \Phi(r)r^\gamma \text{ is quasi-increasing on } (0;R).$$

Definition 4. Let $R \in (0;\infty]$. We say that $\Phi : (0;R) \to \mathbb{R}_+$ belongs to the class $D(R)$ if for some $C = C(\Phi) > 0$

$$\int_0^r \frac{dt}{\Phi(t)t} \leq \frac{C}{\Phi(r)} \ \ r \in (0;R). \tag{2.5}$$

Note that relation (2.5) is equivalent to the inequality:

$$\int_0^{r^n} \frac{ds}{\Phi(s^{1/n})s} \leq \frac{nC}{\Phi(r)}, \ \ r \in (0;R). \tag{2.6}$$

For example the function $\Phi(t) = t^{\alpha-n} \in D(\infty) \ (0 < \alpha < n)$. Indeed,

$$\int_0^r \frac{dt}{\Phi(t)t} = \int_0^r \frac{dt}{t^{\alpha-n+1}} = \frac{1}{n-\alpha}r^{n-\alpha} = \frac{1}{n-\alpha}\frac{1}{\Phi(r)}, \ \ r \in \mathbb{R}_+.$$

Lemma 2.1. *Let $n \in \mathbb{N}$, $R \in (0,\infty]$. Then $B_n(R) \subsetneqq A_n(R)$.*

Proof. Let $\Phi \in B_n(R)$ and $r_1 < r_2$. Then by (2.2) for some C_1, $C_2 > 0$, depending on Φ and n,

$$\Phi(r_1)r_1^n \le C_1 \int\limits_0^{r_1} \Phi(t)t^{n-1}dt \le C_1 \int\limits_0^{r_2} \Phi(t)t^{n-1}dt \le C_2\Phi(r_2)r_2^n,$$

so the function $\Phi(r)r^n$ is quasi-increasing, hence $\Phi \in A_n(R)$.

The function $\Phi(t) = t^{-n}\ln(1+t)^\alpha$, with $\alpha > 0$, belongs to $A_n(R)$ and $\Phi \notin B_n(R)$. Indeed

$$\begin{aligned}\sup_{r>0} \frac{1}{\Phi(r)} \int\limits_0^r \Phi(t)t^{n-1}dt &= \sup_{r>0} \frac{1}{\ln(1+r)^\alpha} \int\limits_0^r \ln(1+t)^\alpha t^{-1}dt \\ &\ge \sup_{r>0} \frac{1}{\ln(1+r)^\alpha} \int\limits_0^r \ln(1+t)^\alpha (1+t)^{-1}dt \\ &= \frac{1}{1+\alpha} \sup_{r>0} \ln(1+r) = \infty.\end{aligned}$$

□

Definition 5. Let $\Phi \in A_n(\infty)$. The *generalized fractional maximal function $M_\Phi f$* is defined for a function $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the open ball with the center at the point $x \in \mathbb{R}^n$ and radius $r > 0$.

In the case $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0;n)$ we obtain the classical fractional maximal function $M_\alpha f$:

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{B(x,r)} |f(y)|dy.$$

Let $E \equiv E(\mathbb{R}^n)$ be a rearrangement invariant space. We introduce the space of generalized fractional maximal functions $M_E^\Phi = M_E^\Phi(\mathbb{R}^n)$ as the set of all functions u, for which there is a function $f \in E(\mathbb{R}^n)$ such that for almost all $x \in \mathbb{R}^n$

$$u(x) = (M_\Phi f)(x),$$

$$\|u\|_{M_E^\Phi} = \inf\{\|f\|_E : f \in E(\mathbb{R}^n);\ \ M_\Phi f = u \quad \text{a.e.}\} < \infty.$$

The generalized Riesz potential was considered in [3-4], [7-10] as the convolution operator

$$(I_G f)(x) = (G * f)(x) = \int\limits_{\mathbb{R}^n} G(x-y)f(y)dy, \ \ f \in E(\mathbb{R}^n),$$

where the kernel $G(x)$ satisfies the following condition: for some $\Phi \in B_n(\infty)$

$$G(x) \cong \Phi(|x|), \ \ x \in \mathbb{R}^n, \tag{2.7}$$

where the equivalence constants depend only on Φ and on n. The kernel of the classical Riesz potential has the form

$$G(x) = |x|^{\alpha-n}, \ \ \alpha \in (0;n).$$

In the following lemma, we prove that the generalized fractional maximal function $M_\Phi f(x)$ is estimated by the generalized Riesz potential.

Lemma 2.2. *Let $\Phi \in B_n(0,\infty)$ and $G(x) = \Phi(|x|)$, $x \in \mathbb{R}^n$. Then*

$$(M_\Phi f)(x) \le (I_G|f|)(x), \quad x \in \mathbb{R}^n$$

for all $f \in E(\mathbb{R}^n)$.

Proof. Indeed,

$$\begin{aligned}(I_G|f|)(x) &= (G * |f|)(x) = \int\limits_{R^n} \Phi(|x-y|)|f(y)|dy = \sup_{r>0} \int\limits_{B(x,r)} \Phi(|x-y|)|f(y)|dy \\ &\ge \sup_{r>0} \operatorname*{ess\,inf}_{y\in B(x,r)} \Phi(|x-y|) \int\limits_{B(x,r)} |f(y)|dy \\ &= \sup_{r>0} \operatorname*{ess\,inf}_{z\in B(0,r)} \Phi(|z|) \int\limits_{B(x,r)} |f(y)|dy = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)|dy = (M_\Phi f)(x).\end{aligned}$$

□

Lemma 2.3. (Hardy-Littlewood inequality, [2]). *If f and g belong to $L_0(\mathbb{R}^n)$, then*

$$\int\limits_{\mathbb{R}^n} |fg|d\mu_n \le \int\limits_0^\infty f^*(s)g^*(s)ds.$$

Lemma 2.4. *Let $\Phi \in B_n(\infty)$, $f \in L^1_{loc}(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$*

$$(M_\Phi f)(x) \le C \sup_{r>0} r\Phi(r^{1/n}) f^{**}(r),$$

where $C > 0$ depends only on Φ and n.

Proof. By using Lemma 2.3 and (2.4) we have

$$\begin{aligned}(M_\Phi f)(x) &= \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)|dy \le \sup_{r>0} \Phi(r) \int\limits_0^{|B(x,r)|} f^*(t)dt \\ &= \sup_{r>0} \Phi(r) \int\limits_0^{v_n r^n} f^*(t)dt = \sup_{s>0} s\Phi\left(\left(\frac{s}{v_n}\right)^{\frac{1}{n}}\right)\frac{1}{s}\int\limits_0^s f^*(t)dt \\ &\le C \sup_{s>0} s\Phi(s^{1/n}) f^{**}(s),\end{aligned}$$

where $C > 0$ depends only on Φ and n. □

Theorem 2.1. *Let $\Phi \in B_n(\infty)$. Then there exists a positive constant C, depending only on Φ and n, such that*

$$(M_\Phi f)^*(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \quad t \in (0,\infty), \tag{2.8}$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$.

Theorem 2.2. *Let $\Phi \in A_n(\infty)$. Inequality (2.8) is sharp in the sense that for every $\varphi \in L_0^+(0,\infty;\downarrow)$ there exists a function $f \in L^+(\mathbb{R}^n)$ such that $f^* = \varphi$ almost everywhere on $(0,\infty)$ and*

$$(M_\Phi f)^*(t) \geq C_1 \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \quad t \in (0,\infty), \tag{2.9}$$

where C_1 is a positive constant which depends only on Φ and n.

Remark 1. For $\Phi(r) = r^{\alpha-n}$, $0<\alpha<n$, hence for the fractional maximal operator M_α, Theorems 2.1 *and* 2.2 *were proved in* [5].

Theorem 2.3. *Let $\Phi \in B_n(\infty)$. Then there exists a positive constant C, depending only on Φ and n, such that*

$$(M_\Phi f)^{**}(t) \leq C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \quad t \in (0,\infty) \tag{2.10}$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$.

Remark 2. It is known that the generalized Riesz potential satisfies the O'Neil estimate for non-increasing rearrangement of the convolution

$$(G*f)^{**}(t) \leq C_0 \left(\frac{1}{t} \int_0^t G^*(s)ds \int_0^t f^*(\tau)d\tau + \int_t^\infty G^*(\tau)f^*(\tau)d\tau \right),$$

where $C_0>0$ depends only on Φ and n. [14, Lemma 1.5].

Then by Lemma 2.2 for some $C>0$ depending only on Φ and n we get

$$(M_\Phi f)^{**}(t) \leq C \left(\frac{1}{t} \int_0^t G^*(s)ds \int_0^t f^*(\tau)d\tau + \int_t^\infty G^*(\tau)f^*(\tau)d\tau \right). \tag{2.11}$$

Assume that $\Phi \in B_n(\infty) \bigcap D(\infty)$ and a function f on $\mathbb{R}^n$ is such that

$$f^*(t) \cong \frac{1}{t\Phi(t^{\frac{1}{n}})}.$$

We show that for such function f the right-hand side of (2.10) is finite while the right-hand side of (2.11) is not. Indeed, by (2.6) we have

$$\begin{aligned} \sup_{t<s<\infty} s\Phi(s^{\frac{1}{n}}) f^{**}(s) &\leq C_1 \sup_{t<s<\infty} s\Phi(s^{\frac{1}{n}}) \frac{1}{s} \int_0^s \frac{1}{t\Phi(t^{\frac{1}{n}})} dt \\ &\leq C_2 \sup_{t<s<\infty} s\Phi(s^{\frac{1}{n}}) \frac{1}{s} \frac{1}{\Phi(s^{\frac{1}{n}})} < \infty, \end{aligned}$$

where $C_1, C_2 > 0$ depends only on Φ and n.

For the second term on the right-hand side of inequality (2.11) we get

$$\int_t^\infty G^*(\tau)f^*(\tau)d\tau \geq C_3 \int_t^\infty \Phi\Big(\big(\frac{\tau}{v_n}\big)^{\frac{1}{n}}\Big) \frac{1}{\tau\Phi\Big(\big(\frac{\tau}{v_n}\big)^{\frac{1}{n}}\Big)} d\tau = C_3 \int_t^\infty \frac{1}{\tau} d\tau = \infty,$$

where $C_3>0$ depends only on Φ and n.

Theorem 2.4. *Let $\Phi \in B_n(\infty) \cap D(\infty)$, then for every $f \in L^1_{loc}(\mathbb{R}^n)$ there exists a positive constant C, depending only on Φ and n, such that*

$$(M_\Phi f)^*(t) \leq C \Bigg(t\Phi(t^{1/n}) f^{**}(t) + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n}) f^*(\tau) \Bigg), \quad t \in (0,\infty). \tag{2.12}$$

3 Proofs of the results of Section 2

3.1 Proof of Theorem 2.1

Fix $t \in (0;\infty)$ and let $f \in L^1_{loc}(\mathbb{R}^n)$. We may assume that

$$\sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s) < \infty,$$

otherwise (2.8) holds trivially. Then by Lemma 2.3

$$\int_E |f(x)|dx \le \int_0^t f^*(y)dy < \infty$$

for every set $E \subset \mathbb{R}^n$ of measure at most t. In particular, if we put

$$E = \{x \in \mathbb{R}^n : |f(x)| > f^*(t)\},$$

then $|E| \le t$ since $\lambda_f(f^*(t)) \le t$ ([2], Chapter 2, (1.18)) and so f is integrable over E. We define the functions:

$$g_t(x) = \max\{|f(x)| - f^*(t), 0\} \operatorname{sgn} f(x), \quad x \in \mathbb{R}^n,$$
$$h_t(x) = \min\{|f(x)|, f^*(t)\} \operatorname{sgn} f(x), \quad x \in \mathbb{R}^n.$$

Then $f = g_t + h_t$ and

$$g_t^*(\tau) = \chi_{(0,t)}(\tau)(f^*(\tau) - f^*(t)), \quad \tau \in (0,\infty),$$
$$h_t^*(\tau) = \min\{f^*(\tau), f^*(t)\}, \quad \tau \in (0,\infty). \tag{3.1}$$

Therefore

$$\|g_t\|_1 = \int_0^\infty g_t^*(\tau)d\tau = \int_0^t \Big(f^*(\tau) - f^*(t)\Big)d\tau \le \int_0^t f^*(\tau)d\tau. \tag{3.2}$$

From inequality (3.2) it follows that

$$(M_\Phi g_t)^*(\tau) \le \Phi(\tau^{1/n})\|g_t\|_1, \quad \tau \in (0;\infty). \tag{3.3}$$

By Lemma 2.4 and by (3.1), we have

$$\begin{aligned}(M_\Phi h_t)^*(\tau) &\le C \sup_{0<\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) h_t^{**}(\tau) \\ &= C \max\Big\{ \sup_{0<\tau<t} \tau \cdot \Phi(\tau^{1/n}) f^{**}(t), \sup_{t\le\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) f^{**}(\tau)\Big\} \\ &= C \max\Big\{ t \cdot \Phi(t^{1/n}) f^{**}(t), \sup_{t\le\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) f^{**}(\tau)\Big\} \\ &\le C \sup_{t<\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) f^{**}(\tau). \end{aligned} \tag{3.4}$$

Hence

$$\sup_{0<\tau<\infty} (M_\Phi h_t)^*(\tau) \le C \sup_{t<\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) f^{**}(\tau).$$

Using inequality ([2])

$$\big(M_\Phi f\big)^*(t) \le \big(M_\Phi g_t\big)^*\Big(\frac{t}{2}\Big) + \big(M_\Phi h_t\big)^*\Big(\frac{t}{2}\Big)$$

and by (3.4), based on (3.3), (3.2) and (2.3) we get

$$\begin{aligned}
\big(M_\Phi f\big)^*(t) &\leq C\Bigg(\Phi\Big((\frac{t}{2})^{1/n}\Big)\|g_t\|_1 + (M_\Phi h_t)^*(\tau)\Bigg) \\
&\leq C_1\Bigg(\Phi(t^{1/n})\int\limits_0^t f^*(u)du + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau)\Bigg) \\
&\leq C_1\Bigg(t\Phi(t^{1/n})f^{**}(t) + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau)\Bigg) \\
&\leq C_2 \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau).
\end{aligned}$$

□

3.2 Proof of Theorem 2.2

Let $\varphi \in L_0^+(0,\infty;\downarrow)$, we put

$$f(x) = \varphi(v_n|x|^n), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Then $f^* = \varphi$ almost everywhere on $(0,\infty)$. For given $y \in \mathbb{R}^n$ we denote

$$B(|y|) = B(0,|y|),$$

for every $x, y \in \mathbb{R}^n$ such that $|y| > |x|$ we have

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(t) \int\limits_{B(x,t)} f(z)dz \geq C_1\Phi(|y|) \int\limits_{B(|y|)} f(z)dz. \tag{3.5}$$

Since the definition of f and spherical coordinates give

$$\int\limits_{B(|y|)} f(z)dz = \int\limits_0^{|y|} \int\limits_{\{|z|=r\}} \varphi(v_n r^n)dv dr = \int\limits_0^{|y|} \varphi(v_n r^n) v_n n r^{n-1} dr = \int\limits_0^{v_n|y|^n} \varphi(\tau)d\tau. \tag{3.6}$$

From (3.5) and (3.6) we have

$$(M_\Phi f)(x) \geq C_1\Phi(|y|) \int\limits_0^{v_n|y|^n} f^*(\tau)d\tau = C_1 H(v_n|y|^n),$$

where $H(t) = \Phi(|t|) \int\limits_0^{v_n|t|^n} f^*(\tau)d\tau$. Consequently,

$$(M_\Phi f)^*(x) \geq C_1 \sup_{\tau > v_n|x|^n} H(\tau),$$

whence (2.9) follows on taking rearrangements. □

3.3 Proof of Theorem 2.3

By using Theorem 2.1 and Lemma 2.1 we get

$$\begin{aligned}
(M_\Phi f)^{**}(t) &= \frac{1}{t}\int_0^t (M_\Phi f)^*(s)ds \le \frac{C}{t}\int_0^t \Big(\sup_{s<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau)\Big)ds \\
&\le \frac{C}{t}\int_0^t \Big(\sup_{s<\tau<t} \tau\Phi(\tau^{1/n})f^{**}(\tau) + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau)\Big)ds \\
&= \frac{C}{t}\int_0^t \left(\sup_{s<\tau<t} \Phi(\tau^{1/n})\int_0^\tau f^*(u)du\right)ds + C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau) \\
&\le \frac{C}{t}\int_0^t \Phi(s^{1/n})ds\int_0^t f^*(u)du + C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau) \\
&= Cf^{**}(t)\int_0^{t^{1/n}} \Phi(s)s^{n-1}ds + C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau) \\
&\le Ct\Phi(t^{1/n})f^{**}(t) + C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau) \le 2C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau).
\end{aligned}$$

□

3.4 Proof of Theorem 2.4

It is clear that

$$f^{**}(s) = \frac{1}{s}\int_0^s f^*(\tau)d\tau = \frac{1}{s}\Big(\int_0^t f^*(\tau)d\tau + \int_t^s f^*(\tau)d\tau\Big)$$

holds for $t < s < \infty$. Then by Theorem 2.1 and taking into account that Φ is non-increasing we have

$$\begin{aligned}
(M_\Phi f)^*(t) &\le C\sup_{t<s<\infty} s\Phi(s^{1/n})f^{**}(s) \\
&= C\sup_{t<s<\infty} \Phi(s^{1/n})\left(\int_0^t f^*(\tau)d\tau + \int_t^s f^*(\tau)d\tau\right) \\
&\le C\left(\Phi(t^{1/n})\int_0^t f^*(\tau)d\tau + \sup_{t<s<\infty} \Phi(s^{1/n})\int_t^s f^*(\tau)d\tau\right) \\
&\le C\left(t\Phi(t^{1/n})f^{**}(t) + \sup_{t<s<\infty} \Phi(s^{1/n})\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^*(\tau)\int_t^s \frac{d\tau}{\tau\Phi(\tau^{1/n})}\right) \\
&= C\left(t\Phi(t^{1/n})f^{**}(t) + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^*(\tau)\sup_{t<s<\infty} \Phi(s^{1/n})\int_0^s \frac{d\tau}{\tau\Phi(\tau^{1/n})}\right),
\end{aligned}$$

therefore (2.12) follows from (2.6).

□

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CAFFARELLI-KOHN-NIRENBERG INEQUALITIES FOR BESOV AND TRIEBEL-LIZORKIN-TYPE SPACES

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Abstract. We present some Caffarelli-Kohn-Nirenberg-type inequalities for Herz-type Besov-Triebel-Lizorkin spaces, Besov-Morrey and Triebel-Lizorkin-Morrey spaces. More precisely, we investigate the inequalities

$$\|f\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} \leq c\,\|f\|^{1-\theta}_{\dot{K}^{\alpha_2,\delta}_u}\,\|f\|^{\theta}_{\dot{K}^{\alpha_3,\delta_1}_p A^s_\beta}$$

and

$$\|f\|_{\mathcal{E}^{\sigma}_{p,2,u}} \leq c\,\|f\|^{1-\theta}_{M^{\delta}_{\mu}}\,\|f\|^{\theta}_{\mathcal{N}^{s}_{q,\beta,v}},$$

with some appropriate assumptions on the parameters, where $\dot{k}^{\alpha_1,r}_{v,\sigma}$ are the Herz-type Bessel potential spaces, which are just the Sobolev spaces if $\alpha_1 = 0, 1 < r = v < \infty$ and $\sigma \in \mathbb{N}_0$, and $\dot{K}^{\alpha_3,\delta_1}_p A^s_\beta$ are Besov or Triebel-Lizorkin spaces if $\alpha_3 = 0$ and $\delta_1 = p$. The usual Littlewood-Paley technique, Sobolev and Franke embeddings are the main tools of this paper. Some remarks on Hardy-Sobolev inequalities are given.

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1 Introduction

Major results in harmonic analysis and partial differential equations invoke some inequalities. Some examples can be mentioned such as: Caffarelli, Kohn and Nirenberg in [7]. They proved the following useful inequality:

$$\left\||x|^{\gamma} f\right\|_{\tau} \leq c\left\||x|^{\beta} f\right\|^{\theta}_{q}\left\||x|^{\alpha}\nabla f\right\|^{1-\theta}_{p}, \quad f \in C^{\infty}_{0}(\mathbb{R}^n), \tag{1.1}$$

where $1 \leq p, q < \infty, \tau > 0, 0 \leq \theta \leq 1, \alpha, \beta, \gamma \in \mathbb{R}$ satisfy some suitable conditions and $c > 0$ depends only on these numerical parameters. This inequality plays an important role in theory of PDE's. It was extended to fractional Sobolev spaces in [32]. This estimate can be rewritten in the following form:

$$\|f\|_{\dot{K}^{\gamma,\tau}_{\tau}} \leq c\big\|f\big\|^{\theta}_{\dot{K}^{\beta,q}_{q}}\big\|\nabla f\big\|^{1-\theta}_{\dot{K}^{\alpha,p}_{p}}, \quad f \in C^{\infty}_{0}(\mathbb{R}^n),$$

where $\dot{K}^{\alpha,p}_{q}$ is the Herz space, see Definition 1 below. These function spaces play an important role in Harmonic Analysis. After they have been introduced in [21], the theory of these spaces had a remarkable development, in particular, due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [2], in the semilinear parabolic equations [13], in the summability of Fourier transforms [16], and in the Cauchy problem for Navier-Stokes equations

[45]. For important and latest results for Herz spaces, we refer the reader to the papers [34], [52] and to the monograph [25].

Inequality (1.1) with $\alpha=\beta=\gamma=0$, takes the form

$$\big\|f\big\|_{L^\tau} \leq c\big\|f\big\|_{L^q}^{\theta}\big\|\nabla f\big\|_{L^p}^{1-\theta}, \quad f\in C_0^\infty(\mathbb{R}^n),$$

where L^p, $1\leq p\leq\infty$ is the Lebesgue space.

The main purpose of this paper is to present a more general version of such inequalities. More precisely, we extend this estimate to Herz-type Besov-Triebel-Lizorkin spaces, called $\dot{K}_q^{\alpha,p}B_\beta^s$ and $\dot{K}_q^{\alpha,p}F_\beta^s$, which generalize the usual Besov and Triebel-Lizorkin spaces. We mean that

$$\dot{K}_p^{0,p}B_\beta^s = B_{p,\beta}^s \quad \text{and} \quad \dot{K}_p^{0,p}F_\beta^s = F_{p,\beta}^s.$$

In addition $\dot{K}_q^{\alpha,p}F_2^0$ are just the Herz spaces $\dot{K}_q^{\alpha,p}$ when $1<p,q<\infty$ and $-\frac{n}{q}<\alpha<n(1-\frac{1}{q})$. In the same manner, we extend these inequalities to Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Our approach based on the Littlewood-Paley technique of Triebel [44] and some results obtained by the author in [9, 10, 11].

The structure of this paper needs some notation. As usual, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{N}$ the set of all natural numbers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For any $u>0, k\in\mathbb{Z}$ we set $C(u)=\{x\in\mathbb{R}^n : \frac{u}{2}<|x|\leq u\}$ and $C_k=C(2^k)$. χ_k, for $k\in\mathbb{Z}$, denote the characteristic function of the set C_k. The expression $f\approx g$ means that $Cg\leq f\leq c\,g$ for some $c,C>0$ independent of non-negative functions f and g.

For any measurable subset $\Omega\subseteq\mathbb{R}^n$ the Lebesgue space $L^p(\Omega)$, $0<p\leq\infty$ consists of all measurable functions for which

$$\big\|f\big\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p\,dx\Big)^{1/p} < \infty, \quad 0<p<\infty$$

and

$$\big\|f\big\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega}|f(x)| < \infty.$$

If $\Omega=\mathbb{R}^n$, then we put $L^p(\mathbb{R}^n)=L^p$ and $\big\|f\big\|_{L^p(\mathbb{R}^n)}=\big\|f\big\|_p$. The symbol $\mathcal{S}(\mathbb{R}^n)$ is used to denote the set of all Schwartz functions on $\mathbb{R}^n$ and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f\in\mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)dx, \quad \xi\in\mathbb{R}^n.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way. The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on L^1_{loc} by

$$\mathcal{M}f(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy, \quad x\in\mathbb{R}^n$$

and $\mathcal{M}_\tau f = (\mathcal{M}\,|f|^\tau)^{1/\tau}$, $0<\tau<\infty$.

Given two quasi-Banach spaces X and Y, we write $X\hookrightarrow Y$ if $X\subset Y$ and the natural embedding of X in Y is continuous. We use c as a generic positive constant, i.e. a constant whose value may be different in different inequalities.

2 Function spaces

We start by recalling the definition and some of the properties of the homogenous Herz spaces $\dot{K}_q^{\alpha,p}$.

Definition 1. Let $\alpha \in \mathbb{R}, 0 < p, q \leq \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}$ is defined by

$$\dot{K}_q^{\alpha,p} = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|_q^p\Big)^{1/p}$$

with the usual modifications made when $p = \infty$ and/or $q = \infty$.

The spaces $\dot{K}_q^{\alpha,p}$ are quasi-Banach spaces and if $\min(p, q) \geq 1$ then $\dot{K}_q^{\alpha,p}$ are Banach spaces. When $\alpha = 0$ and $0 < p = q \leq \infty$ the space $\dot{K}_p^{0,p}$ coincides with the Lebesgue space L^p. In addition

$$\dot{K}_p^{\alpha,p} = L^p(\mathbb{R}^n, |\cdot|^{\alpha p}), \quad \text{(Lebesgue space equipped with power weight)},$$

where

$$\|f\|_{L^p(\mathbb{R}^n,|\cdot|^{\alpha p})} = \Big(\int_{\mathbb{R}^n} |f(x)|^p\, |x|^{\alpha p} dx\Big)^{1/p}.$$

Note that

$$\dot{K}_q^{\alpha,p} \subset \mathcal{S}'(\mathbb{R}^n)$$

for any $\alpha < n(1-\frac{1}{q})$, $1 \leq p, q \leq \infty$ or $\alpha = n(1-\frac{1}{q})$, $p = 1$ and $1 \leq q \leq \infty$. We mean that,

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), f \in \dot{K}_q^{\alpha,p}$$

generates a distribution $T_f \in \mathcal{S}'(\mathbb{R}^n)$. A detailed discussion of the properties of these spaces can be found in [20, 24, 27], and references therein.

The following lemma is the $\dot{K}_q^{\alpha,p}$-version of the Plancherel-Polya-Nikolskij inequality.

Lemma 2.1. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < s, \tau, q, r \leq \infty$. We suppose that $\alpha_1 + \frac{n}{s} > 0$, $0 < q \leq s \leq \infty$ and $\alpha_2 \geq \alpha_1$. Then there exists a positive constant $c > 0$ independent of R such that for all $f \in \dot{K}_q^{\alpha_2,\delta} \cap \mathcal{S}'(\mathbb{R}^n)$ with* supp $\mathcal{F}f \subset \{\xi : |\xi| \leq R\}$, *we have*

$$\|f\|_{\dot{K}_s^{\alpha_1,r}} \leq c\, R^{\frac{n}{q}-\frac{n}{s}+\alpha_2-\alpha_1}\|f\|_{\dot{K}_q^{\alpha_2,\delta}},$$

where

$$\delta = \begin{cases} r, & \text{if} \quad \alpha_2 = \alpha_1, \\ \tau, & \text{if} \quad \alpha_2 > \alpha_1. \end{cases}$$

Remark 1. We would like to mention that Lemma 2.1 improves the classical Plancherel-Polya-Nikolskij inequality if $\alpha_1 = \alpha_2 = 0, r = s$ due to the continuous embedding $\ell^q \hookrightarrow \ell^s$.

In the previous lemma we have not treated the case $s < q$. The next lemma gives a positive answer.

Lemma 2.2. *Let* $\alpha_1, \alpha_2 \in \mathbb{R}$ *and* $0 < s, \tau, q, r \leq \infty$. *We suppose that* $\alpha_1 + \frac{n}{s} > 0$, $0 < s \leq q \leq \infty$ *and* $\alpha_2 \geq \alpha_1 + \frac{n}{s} - \frac{n}{q}$. *Then there exists a positive constant c independent of R such that for all* $f \in \dot{K}_q^{\alpha_2,\delta} \cap \mathcal{S}'(\mathbb{R}^n)$ *with* $\operatorname{supp} \mathcal{F}f \subset \{\xi : |\xi| \leq R\}$, *we have*

$$\big\|f\big\|_{\dot{K}_s^{\alpha_1,r}} \leq c\, R^{\frac{n}{q}-\frac{n}{s}+\alpha_2-\alpha_1}\big\|f\big\|_{\dot{K}_q^{\alpha_2,\delta}},$$

where

$$\delta = \begin{cases} r, & \text{if} \quad \alpha_2 = \alpha_1 + \frac{n}{s} - \frac{n}{q}, \\ \tau, & \text{if} \quad \alpha_2 > \alpha_1 + \frac{n}{s} - \frac{n}{q}. \end{cases}$$

The proof of these inequalities is given in [9], Lemmas 3.10 and 3.14. Let $1 < q < \infty$ and $0 < p \leq \infty$. If f is a locally integrable function on $\mathbb{R}^n$ and $-\frac{n}{q} < \alpha < n(1-\frac{1}{q})$, then

$$\big\|\mathcal{M}f\big\|_{\dot{K}_q^{\alpha,p}} \leq c\big\|f\big\|_{\dot{K}_q^{\alpha,p}}, \tag{2.1}$$

see [24]. We need the following lemma, which is basically a consequence of Hardy's inequality in the sequence Lebesgue space ℓ^q.

Lemma 2.3. *Let* $0 < a < 1$ *and* $0 < q \leq \infty$. *Let* $\{\varepsilon_k\}_{k\in\mathbb{N}_0}$ *be a sequence of positive real numbers, such that*

$$\big\|\{\varepsilon_k\}_{k\in\mathbb{N}_0}\big\|_{\ell^q} = I < \infty.$$

Then the sequences $\Big\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\Big\}_{k\in\mathbb{N}_0}$ *and* $\Big\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\Big\}_{k\in\mathbb{N}_0}$ *belong to* ℓ^q, *and*

$$\big\|\,\{\delta_k\}_{k\in\mathbb{N}_0}\,\big\|_{\ell^q} + \big\|\,\{\eta_k\}_{k\in\mathbb{N}_0}\,\big\|_{\ell^q} \leq c\, I,$$

with $c > 0$ *only depending on a and q.*

Some of our results of this paper are based on the following result, see Tang and Yang [40].

Lemma 2.4. *Let* $1 < \beta < \infty, 1 < q < \infty$ *and* $0 < p \leq \infty$. *If* $\{f_j\}_{j=0}^{\infty}$ *is a sequence of locally integrable functions on* $\mathbb{R}^n$ *and* $-\frac{n}{q} < \alpha < n(1-\frac{1}{q})$, *then*

$$\Big\|\Big(\sum_{j=0}^{\infty}(\mathcal{M}f_j)^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_q^{\alpha,p}} \lesssim \Big\|\Big(\sum_{j=0}^{\infty}|f_j|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_q^{\alpha,p}}.$$

Now, we present the Fourier analytic definition of Herz-type Besov and Triebel-Lizorkin spaces and recall their basic properties. We first need the concept of a smooth dyadic partition of the unity. Let φ_0 be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\varphi_0(x) = 1$ for $|x| \leq 1$ and $\varphi_0(x) = 0$ for $|x| \geq \frac{3}{2}$. We put $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{1-j}x)$ for $j = 1, 2, 3, \ldots$. Then $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a partition of unity, $\sum_{j=0}^{\infty}\varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{j=0}^{\infty}\mathcal{F}^{-1}\varphi_j * f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We are now in a position to state the definition of Herz-type Besov and Triebel-Lizorkin spaces.

Definition 2. Let $\alpha, s \in \mathbb{R}, 0 < p, q \leq \infty$ and $0 < \beta \leq \infty$.

(i) The Herz-type Besov space $\dot{K}_q^{\alpha,p} B_\beta^s$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s} = \Big(\sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1}\varphi_j * f\|_{\dot{K}_q^{\alpha,p}}^\beta \Big)^{1/\beta} < \infty,$$

with the obvious modification if $\beta = \infty$.

(ii) Let $0 < p, q < \infty$. The Herz-type Triebel-Lizorkin space $\dot{K}_q^{\alpha,p} F_\beta^s$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{K}_q^{\alpha,p} F_\beta^s} = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} \left|\mathcal{F}^{-1}\varphi_j * f\right|^\beta \Big)^{1/\beta} \Big\|_{\dot{K}_q^{\alpha,p}} < \infty,$$

with the obvious modification if $\beta = \infty$.

Remark 2. Let $s \in \mathbb{R}, 0 < p, q \leq \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{q}$. The spaces $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are independent of the particular choice of the smooth dyadic partition of the unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms). In particular $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are quasi-Banach spaces and if $p, q, \beta \geq 1$, then they are Banach spaces. Further results, concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in [8, 9, 10, 11, 12, 48, 49, 47].

Now we give the definitions of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$.

Definition 3. (i) Let $s \in \mathbb{R}$ and $0 < p, \beta \leq \infty$. The Besov space $B_{p,\beta}^s$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,\beta}^s} = \Big(\sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1}\varphi_j * f\|_p^\beta \Big)^{1/\beta} < \infty,$$

with the obvious modification if $\beta = \infty$.

(ii) Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leq \infty$. The Triebel-Lizorkin space $F_{p,\beta}^s$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p,\beta}^s} = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} \left|\mathcal{F}^{-1}\varphi_j * f\right|^\beta \Big)^{1/\beta} \Big\|_p < \infty,$$

with the obvious modification if $\beta = \infty$.

The theory of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$ has been developed in detail in [42, 43] but has a longer history already including many contributors; we do not want to discuss this here. Clearly, for $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leq \infty$,

$$\dot{K}_p^{0,p} B_\beta^s = B_{p,\beta}^s \quad \text{and} \quad \dot{K}_p^{0,p} F_\beta^s = F_{p,\beta}^s.$$

Let w denote a positive, locally integrable function and $0 < p < \infty$. Then the weighted Lebesgue space $L^p(\mathbb{R}^n, w)$ consists of all measurable functions such that

$$\|f\|_{L^p(\mathbb{R}^n,w)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \, w(x) dx \Big)^{1/p} < \infty.$$

For $\varrho \in [1, \infty)$ we denote by $\mathcal{A}_\varrho$ the Muckenhoupt class of weights, and $\mathcal{A}_\infty = \cup_{\varrho\geq 1}\mathcal{A}_\varrho$. We refer to [17] for the general properties of these classes. Let $w \in \mathcal{A}_\infty$, $s \in \mathbb{R}$, $0 < \beta \leq \infty$ and $0 < p < \infty$. We define weighted Besov spaces $B_{p,\beta}^s(\mathbb{R}^n, w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,\beta}^s(\mathbb{R}^n,w)} = \Big(\sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1}\varphi_j * f\|_{L^p(\mathbb{R}^n,w)}^\beta \Big)^{1/\beta}$$

is finite. In the limiting case $\beta = \infty$ the usual modification is required.

Let $w \in \mathcal{A}_\infty$, $s \in \mathbb{R}$, $0 < \beta \leq \infty$ and $0 < p < \infty$. We define weighted Triebel-Lizorkin spaces $F^s_{p,\beta}(\mathbb{R}^n, w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{F^s_{p,\beta}(\mathbb{R}^n,w)} = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} \big|\mathcal{F}^{-1}\varphi_j * f\big|^\beta \Big)^{1/\beta} \Big\|_{L^p(\mathbb{R}^n,w)}$$

is finite. In the limiting case $\beta = \infty$ the usual modification is required.

The spaces $B^s_{p,\beta}(\mathbb{R}^n, w) = B^s_{p,\beta}(w)$ and $F^s_{p,\beta}(\mathbb{R}^n, w) = F^s_{p,\beta}(w)$ are independent of the particular choice of the smooth dyadic partition of the unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, \beta \geq 1$). Moreover, for $w \equiv 1 \in \mathcal{A}_\infty$ we obtain the usual (unweighted) Besov and Triebel-Lizorkin spaces. We refer, in particular, to the papers [3, 4, 22] for a comprehensive investigation consists of the weighted spaces. Let w_γ be a power weight, i.e., $w_\gamma(x) = |x|^\gamma$ with $\gamma > -n$. Then we have

$$B^s_{p,\beta}(w_\gamma) = \dot{K}^{\frac{\gamma}{p},p}_p B^s_\beta \quad \text{and} \quad F^s_{p,\beta}(w_\gamma) = \dot{K}^{\frac{\gamma}{p},p}_p F^s_\beta,$$

in the sense of equivalent quasi-norms.

Definition 4. (i) Let $1 < q < \infty, 0 < p < \infty, -\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$ and $s \in \mathbb{R}$. Then the Herz-type Bessel potential space $\dot{k}^{\alpha,p}_{q,s}$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{\dot{k}^{\alpha,p}_{q,s}} = \big\|(1 + |\xi|^2)^{\frac{s}{2}} * f\big\|_{\dot{K}^{\alpha,p}_q} < \infty.$$

(ii) Let $1 < q < \infty, 0 < p < \infty, -\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$ and $m \in \mathbb{N}$. The homogeneous Herz-type Sobolev space $\dot{W}^{\alpha,p}_{q,m}$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{\dot{W}^{\alpha,p}_{q,m}} = \sum_{|\beta|\leq m} \Big\|\frac{\partial^\beta f}{\partial^\beta x}\Big\|_{\dot{K}^{\alpha,p}_q} < \infty,$$

where the derivatives must be understood in the sense of distribution.

In the following, we will present the connection between the Herz-type Triebel-Lizorkin spaces and the Herz-type Bessel potential spaces; see [26, 48]. Let $1 < q < \infty, 1 < p < \infty$ and $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$. If $s \in \mathbb{R}$, then

$$\dot{K}^{\alpha,p}_q F^s_2 = \dot{k}^{\alpha,p}_{q,s} \tag{2.2}$$

with equivalent norms. If $s = m \in \mathbb{N}$, then

$$\dot{K}^{\alpha,p}_q F^m_2 = \dot{W}^{\alpha,p}_{q,m} \tag{2.3}$$

with equivalent norms. In particular

$$\dot{K}^{0,p}_p F^m_2 = W^p_m \quad \text{(Sobolev spaces)}$$

and

$$\dot{K}^{\alpha,p}_q F^0_2 = \dot{K}^{\alpha,p}_q \tag{2.4}$$

with equivalent norms. Let $0 < \theta < 1$,

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{\beta} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}$$

and

$$s = (1-\theta)s_0 + \theta s_1.$$

For simplicity, in what follows, we use $\dot{K}_q^{\alpha,p}A_\beta^s$ to denote either $\dot{K}_q^{\alpha,p}B_\beta^s$ or $\dot{K}_q^{\alpha,p}F_\beta^s$. As an immediate consequence of Hölder's inequality we have the so-called interpolation inequalities:

$$\|f\|_{\dot{K}_q^{\alpha,p}A_\beta^s} \leq \|f\|^{1-\theta}_{\dot{K}_{q_0}^{\alpha_0,p_0}A_{\beta_0}^{s_0}} \|f\|^{\theta}_{\dot{K}_{q_1}^{\alpha_1,p_1}A_{\beta_1}^{s_1}} \tag{2.5}$$

which hold for all $f \in \dot{K}_{q_0}^{\alpha_0,p_0}A_{\beta_0}^{s_0} \cap \dot{K}_{q_1}^{\alpha_1,p_1}A_{\beta_1}^{s_1}$.

We collect some embeddings on these function spaces as obtained in [9]-[10]. First we have elementary embeddings within these spaces. Let $s \in \mathbb{R}, 0 < p,q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{q}$. Then

$$\dot{K}_q^{\alpha,p}B^s_{\min(\beta,p,q)} \hookrightarrow \dot{K}_q^{\alpha,p}F_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p}B^s_{\max(\beta,p,q)}. \tag{2.6}$$

Theorem 2.1. *Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s,p,q,r,\beta \leq \infty, \alpha_1 > -\frac{n}{s}$ and $\alpha_2 > -\frac{n}{q}$. We suppose that*

$$s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2.$$

Let $0 < q \leq s \leq \infty$ and $\alpha_2 \geq \alpha_1$ or $0 < s \leq q \leq \infty$ and

$$\alpha_2 + \frac{n}{q} \geq \alpha_1 + \frac{n}{s}. \tag{2.7}$$

(i) *We have the embedding*

$$\dot{K}_q^{\alpha_2,\theta}B_\beta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,r}B_\beta^{s_1},$$

where

$$\theta = \begin{cases} r, & \text{if} \quad \alpha_2 + \frac{n}{q} = \alpha_1 + \frac{n}{s}, s \leq q \text{ or } \alpha_2 = \alpha_1, q \leq s, \\ p, & \text{if} \quad \alpha_2 + \frac{n}{q} > \alpha_1 + \frac{n}{s}, s \leq q \text{ or } \alpha_2 > \alpha_1, q \leq s. \end{cases}$$

(ii) *Let $0 < q, s < \infty$. The embedding*

$$\dot{K}_q^{\alpha_2,r}F_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p}F_\beta^{s_1}$$

holds if $0 < r \leq p < \infty$, where

$$\theta = \begin{cases} \beta, & \text{if} \quad 0 < s \leq q < \infty \text{ and } \alpha_2 + \frac{n}{q} = \alpha_1 + \frac{n}{s}; \\ \infty, & \text{otherwise.} \end{cases}$$

We now present an immediate corollary of the Sobolev embeddings, which are called Hardy-Sobolev inequalities.

Corollary 2.1. *Let $1 < q \leq s < \infty$, $1 < q < n$ and $\alpha = \frac{n}{q} - \frac{n}{s} - 1$. There is a constant $c > 0$ such that for all $f \in \dot{W}_q^1$*

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{|x|^{-\alpha}} \right)^s dx \leq c \left(\sum_{|\beta|=1} \left\| \frac{\partial^\beta f}{\partial^\beta x} \right\|_{\dot{K}_q^{0,s}} \right)^s \leq c \left(\sum_{|\beta|=1} \left\| \frac{\partial^\beta f}{\partial^\beta x} \right\|_q \right)^s.$$

Now we recall the Franke embedding, see [12].

Theorem 2.2. *Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, p, q < \infty, 0 < \theta \leq \infty, \alpha_1 > -\frac{n}{s}$ and $\alpha_2 > -\frac{n}{q}$. We suppose that*

$$s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2.$$

Let

$$0 < q < s < \infty \quad \text{and} \quad \alpha_2 \geq \alpha_1,$$

or

$$0 < s \leq q < \infty \quad \text{and} \quad \alpha_2 + \frac{n}{q} > \alpha_1 + \frac{n}{s}.$$

Then

$$\dot{K}_q^{\alpha_2,p} B_p^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} F_\theta^{s_1}.$$

Corollary 2.2. *Let $1 < q \leq s < \infty$ with $1 < q < n$. Let $\alpha = \frac{n}{q} - \frac{n}{s} - 1$. There is a constant $c > 0$ such that for all $f \in B^1_{q,s}$*

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{|x|^{-\alpha}} \right)^s dx \leq c \big\| f \big\|^s_{\dot{K}_q^{0,s} B_s^1} \leq c \big\| f \big\|^s_{B^1_{q,s}}.$$

Remark 3. We would like to mention that in Theorem 2.1 and Theorem 2.2 the assumptions $s_1 - \frac{n}{s} - \alpha_1 \leq s_2 - \frac{n}{q} - \alpha_2$, (2.7) and $0 < r \leq p < \infty$ are necessary, see [9, 10, 12].

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a partition of unity. For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we denote, Peetre maximal function,

$$(\mathcal{F}^{-1}\varphi_j)^{*,a} f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\mathcal{F}^{-1}\varphi_j * f(y)|}{(1 + 2^j |x-y|)^a}, \quad j \in \mathbb{N}_0.$$

We now present a fundamental characterization of the above spaces, which plays an essential role in this paper, see [46, Theorem 1].

Theorem 2.3. *Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{q}$. Let $a > \frac{n}{\min\left(q, \frac{n}{\alpha + \frac{n}{q}}\right)}$. Then*

$$\big\| f \big\|^\star_{\dot{K}_q^{\alpha,p} B_\beta^s} = \left(\sum_{j=0}^{\infty} 2^{js\beta} \big\| (\mathcal{F}^{-1}\varphi_j)^{*,a} f \big\|^\beta_{\dot{K}_q^{\alpha,p}} \right)^{1/\beta},$$

is an equivalent quasi-norm in $\dot{K}_q^{\alpha,p} B_\beta^s$. Let $a > \frac{n}{\min\left(\min(q,\beta), \frac{n}{\alpha + \frac{n}{q}}\right)}$. Then

$$\big\| f \big\|^\star_{\dot{K}_q^{\alpha,p} F_\beta^s} = \Big\| \left(\sum_{j=0}^{\infty} 2^{js\beta} (\mathcal{F}^{-1}\varphi_j)^{*,a} f)^\beta \right)^{1/\beta} \Big\|_{\dot{K}_q^{\alpha,p}},$$

is an equivalent quasi-norm in $\dot{K}_q^{\alpha,p} F_\beta^s$.

Let $0 < p, q \leq \infty$. For later use we introduce the following abbreviations:

$$\sigma_q = n \max(\frac{1}{q} - 1, 0) \quad \text{and} \quad \sigma_{p,q} = n \max(\frac{1}{p} - 1, \frac{1}{q} - 1, 0).$$

In the sequel we shall interpret L^1_{loc} as the set of regular distributions.

Theorem 2.4. *Let $0 < p, q, \beta \leq \infty, \alpha > -\frac{n}{q}$, $\alpha_0 = n - \frac{n}{q}$ and $s > \max(\sigma_q, \alpha - \alpha_0)$. Then*

$$\dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow L^1_{\text{loc}},$$

where $0 < p, q < \infty$ in the case of Herz-type Triebel-Lizorkin spaces.

Proof. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a smooth dyadic partition of unity. We set

$$\varrho_k = \sum_{j=0}^{k} \mathcal{F}^{-1}\varphi_j * f, \quad k \in \mathbb{N}_0.$$

For technical reasons, we split the proof into two steps.

Step 1. We consider the case $1 \le q \le \infty$. In order to prove we additionally do it into the four Substeps 1.1, 1.2, 1.3 and 1.4.

Substep 1.1. $-\frac{n}{q} < \alpha < \alpha_0$. Since $s > 0$ and $\dot{K}_q^{\alpha,p} \hookrightarrow \dot{K}_q^{\alpha,\max(1,p)}$, we have

$$\sum_{j=0}^{\infty} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}_q^{\alpha,\max(1,p)}} \lesssim \big\|f\big\|_{\dot{K}_q^{\alpha,p}A_\beta^s}.$$

Then, the sequence $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to $g \in \dot{K}_q^{\alpha,\max(1,p)}$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\langle f-g,\varphi\rangle = \langle f-\varrho_N,\varphi\rangle + \langle g-\varrho_N,\varphi\rangle, \quad N \in \mathbb{N}_0.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. Clearly, the first term tends to zero as $N \to \infty$, while by Hölder's inequality there exists a constant $C > 0$ independent of N such that

$$|\langle g-\varrho_N,\varphi\rangle| \le C\big\|g-\varrho_N\big\|_{\dot{K}_q^{\alpha,\max(1,p)}},$$

which tends to zero as $N \to \infty$. From this and $\dot{K}_q^{\alpha,\max(1,p)} \hookrightarrow L^1_{\mathrm{loc}}$, because of $\alpha < \alpha_0$, we deduce the desired result. In addition, we have

$$\dot{K}_q^{\alpha,p}A_\beta^s \hookrightarrow \dot{K}_q^{\alpha,\max(1,p)}.$$

Substep 1.2. $\alpha \ge \alpha_0$ and $1 < q \le \infty$. Let $1 < q_1 < \infty$ be such that

$$s > \alpha + \frac{n}{q} - \frac{n}{q_1}.$$

We distinguish two cases:

• $q_1 = q$. By Theorem 2.1/(i), we obtain

$$\dot{K}_q^{\alpha,p}B_\beta^s \hookrightarrow \dot{K}_q^{0,q}B_\beta^{s-\alpha} = B_{q,\beta}^{s-\alpha} \hookrightarrow L^1_{\mathrm{loc}}.$$

where the last embedding follows by the fact that

$$B_{q,\beta}^{s-\alpha} \hookrightarrow L^q, \tag{2.8}$$

because of $s-\alpha > 0$. The Herz-type Triebel-Lizorkin case follows by the second embeddings of (2.6).

• $1 < q_1 < q \le \infty$ or $1 < q < q_1 < \infty$. If we assume the first possibility then Theorem 2.1/(i) and Substep 1.1 yield

$$\dot{K}_q^{\alpha,p}B_\beta^s \hookrightarrow \dot{K}_{q_1}^{0,p}B_\beta^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}} \hookrightarrow L^1_{\mathrm{loc}},$$

since $\alpha + \frac{n}{q} > \frac{n}{q_1}$. The latter possibility follows again by Theorem 2.1/(i). Indeed, we have

$$\dot{K}_q^{\alpha,p}B_\beta^s \hookrightarrow \dot{K}_q^{\alpha_0,p}B_\beta^{s+\alpha_0-\alpha} \hookrightarrow \dot{K}_{q_1}^{0,q_1}B_\beta^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}} = B_{q_1,\beta}^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}} \hookrightarrow L^1_{\mathrm{loc}},$$

where the last embedding follows by the fact that

$$B_{q_1,\beta}^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}} \hookrightarrow L^{q_1}. \tag{2.9}$$

Therefore from (2.6) we obtain the desired embeddings.

Substep 1.3. $q = 1$ and $\alpha > 0$. We have

$$\dot{K}_1^{\alpha,p} B_\beta^s \hookrightarrow \dot{K}_1^{0,1} B_\beta^{s-\alpha} = B_{1,\beta}^{s-\alpha} \hookrightarrow L^1,$$

since $s > \alpha$.

Substep 1.4. $q = 1$ and $\alpha = 0$. Let α_3 be a real number such that $\max(-n, -s) < \alpha_3 < 0$. From Theorem 2.1, we get

$$\dot{K}_1^{0,p} A_\beta^s \hookrightarrow \dot{K}_1^{\alpha_3,p} A_\beta^{s+\alpha_3}.$$

We have

$$\sum_{k=0}^{\infty} \big\| \mathcal{F}^{-1}\varphi_k * f \big\|_{\dot{K}_1^{\alpha_3,\max(1,p)}} \lesssim \big\| f \big\|_{\dot{K}_1^{\alpha_3,p} A_\beta^{s+\alpha_3}} \lesssim \big\| f \big\|_{\dot{K}_1^{0,p} A_\beta^s},$$

since $\alpha_3 + s > 0$. Using the same type of arguments as in Substep 1.1 it is easy to see that

$$\dot{K}_1^{\alpha_3,p} A_\beta^{s+\alpha_3} \hookrightarrow \dot{K}_1^{\alpha_3,\max(1,p)} \hookrightarrow L^1_{\mathrm{loc}}.$$

Step 2. We consider the case $0 < q < 1$.

Substep 2.1. $-\frac{n}{q} < \alpha < 0$. By Lemma 2.1, we obtain

$$\sum_{j=0}^{\infty} \big\| \mathcal{F}^{-1}\varphi_j * f \big\|_{\dot{K}_1^{\alpha,\max(1,p)}} \lesssim \sum_{j=0}^{\infty} 2^{j(\frac{n}{q}-n)} \big\| \mathcal{F}^{-1}\varphi_j * f \big\|_{\dot{K}_q^{\alpha,p}} \lesssim \big\| f \big\|_{\dot{K}_q^{\alpha,p} A_\beta^s},$$

since $s > \frac{n}{q} - n$. The desired embedding follows by the fact that $\dot{K}_1^{\alpha,\max(1,p)} \hookrightarrow L^1_{\mathrm{loc}}$ and the arguments in Substep 1.1. In addition

$$\dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow \dot{K}_1^{\alpha,\max(1,p)}. \tag{2.10}$$

Substep 2.2. $\alpha \geq 0$. Let α_4 be a real number such that $\max(-n, -s + \frac{n}{q} - n + \alpha) < \alpha_4 < 0$. By Theorem 2.1, we get

$$\dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow \dot{K}_1^{0,p} A_\beta^{s-\frac{n}{q}+n-\alpha} \hookrightarrow \dot{K}_1^{\alpha_4,p} A_\beta^{s-\frac{n}{q}+n-\alpha+\alpha_4} \hookrightarrow \dot{K}_1^{\alpha_4,\max(1,p)} A_\beta^{s-\frac{n}{q}+n-\alpha+\alpha_4}.$$

As in Substep 1.4, we easily obtain that

$$\dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow\hookrightarrow L^1_{\mathrm{loc}}.$$

Therefore, under the hypothesis of this theorem, every $f \in \dot{K}_q^{\alpha,p} A_\beta^s$ is a regular distribution. □

Let f be an arbitrary function on $\mathbb{R}^n$ and $x, h \in \mathbb{R}^n$. Then

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^{M+1} f(x) = \Delta_h(\Delta_h^M f)(x), \quad M \in \mathbb{N}.$$

These are the well-known differences of functions which play an important role in the theory of function spaces. Using mathematical induction one can show the explicit formula

$$\Delta_h^M f(x) = \sum_{j=0}^{M} (-1)^j \, C_M^j f(x + (M-j)h), \quad x \in \mathbb{R}^n,$$

where C_M^j are the binomial coefficients. By ball means of differences we mean the quantity

$$d_t^M f(x) = t^{-n} \int_{|h| \le t} \big|\Delta_h^M f(x)\big| \, dh = \int_B \big|\Delta_{th}^M f(x)\big| \, dh, \quad x \in \mathbb{R}^n.$$

Here $B = \{y \in \mathbb{R}^n : |h| \leq 1\}$ is the unit ball of $\mathbb{R}^n$ and $t > 0$ is a real number. We set

$$\|f\|^*_{\dot{K}_q^{\alpha,p}B_\beta^s} = \|f\|_{\dot{K}_q^{\alpha,p}} + \Big(\int_0^\infty t^{-s\beta}\|d_t^M f\|^\beta_{\dot{K}_q^{\alpha,p}}\frac{dt}{t}\Big)^{1/\beta}$$

and

$$\|f\|^*_{\dot{K}_q^{\alpha,p}F_\beta^s} = \|f\|_{\dot{K}_q^{\alpha,p}} + \Big\|\Big(\int_0^\infty t^{-s\beta}(d_t^M f)^\beta\frac{dt}{t}\Big)^{1/\beta}\Big\|_{\dot{K}_q^{\alpha,p}}.$$

The following theorem play a central role in our paper.

Theorem 2.5. *Let $0 < p, q, \beta \leq \infty, \alpha > -\frac{n}{q}$, $\alpha_0 = n - \frac{n}{q}$ and $M \in \mathbb{N}\backslash\{0\}$.*

(i) *Assume that*

$$\max(\sigma_q, \alpha - \alpha_0) < s < M.$$

Then $\|\cdot\|^*_{\dot{K}_q^{\alpha,p}B_\beta^s}$ is an equivalent quasi-norm on $\dot{K}_q^{\alpha,p}B_\beta^s$.

(ii) *Let $0 < p < \infty$ and $0 < q < \infty$. Assume that*

$$\max(\sigma_{q,\beta}, \alpha - \alpha_0) < s < M.$$

Then $\|\cdot\|^*_{\dot{K}_q^{\alpha,p}F_\beta^s}$ is an equivalent quasi-norm on $\dot{K}_q^{\alpha,p}F_\beta^s$.

Proof. We split the proof into three steps.

Step 1. We will prove that

$$\|f\|_{\dot{K}_q^{\alpha,p}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p}A_\beta^s}$$

for all $f \in \dot{K}_q^{\alpha,p}A_\beta^s$. We employ the same notations as in Theorem 2.4. Recall that

$$\varrho_k = \sum_{j=0}^k \mathcal{F}^{-1}\varphi_j * f, \quad k \in \mathbb{N}_0.$$

Obviously $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to f in $\mathcal{S}'(\mathbb{R}^n)$ and $\{\varrho_k\}_{k\in\mathbb{N}_0} \subset \dot{K}_q^{\alpha,p}$ for any $0 < p, q \leq \infty$ and any $\alpha > -\frac{n}{q}$. Furthermore, $\{\varrho_k\}_{k\in\mathbb{N}_0}$ is a Cauchy sequence in $\dot{K}_q^{\alpha,p}$ and hence it converges to a function $g \in \dot{K}_q^{\alpha,p}$, and

$$\|g\|_{\dot{K}_q^{\alpha,p}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p}A_\beta^s}.$$

Let us prove that $g = f$ almost everywhere. We will do this in four cases.
Case 1. $-\frac{n}{q} < \alpha < \alpha_0$ and $1 \leq q \leq \infty$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We write

$$\langle f - g, \varphi\rangle = \langle f - \varrho_N, \varphi\rangle + \langle g - \varrho_N, \varphi\rangle, \quad N \in \mathbb{N}_0.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. Clearly, the first term tends to zero as $N \to \infty$, while by Hölder's inequality there exists a constant $C > 0$ independent of N such that

$$|\langle g - \varrho_N, \varphi\rangle| \leq C\|g - \varrho_N\|_{\dot{K}_q^{\alpha,\max(1,p)}},$$

which tends to zero as $N \to \infty$. Then, with the help of Substep 1.1 of the proof of Theorem 2.4, we have $g = f$ almost everywhere.

Case 2. $\alpha \geq \alpha_0$ and $1 < q \leq \infty$. Let $1 < q_1 < \infty$ be as in Theorem 2.4. From (2.8) and (2.9), we derive in this case, that every $f \in \dot{K}_q^{\alpha,p}A_\beta^s$ is a regular distribution, $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to f in L^{q_1} and

$$\|f\|_{q_1} \lesssim \|f\|_{\dot{K}_q^{\alpha,p}A_\beta^s}.$$

Indeed, from the embeddings (2.9) and since $f \in B^{\frac{n}{q_1}-\alpha-\frac{n}{q}+s}_{q_1,\beta}$, it follows that $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to a function $h \in L^{q_1}$. Similarly as in Case 1, we conclude that $f = h$ almost everywhere. It remains to prove that $g = f$ almost everywhere. We have

$$\|f - g\|^{\sigma}_{\dot{K}^{\alpha,p}_q} \leq \|f - \varrho_k\|^{\sigma}_{\dot{K}^{\alpha,p}_q} + \|g - \varrho_k\|^{\sigma}_{\dot{K}^{\alpha,p}_q}, \quad k \in \mathbb{N}_0$$

and

$$\|f - \varrho_k\|^{\sigma}_{\dot{K}^{\alpha,p}_q} \leq \sum_{j=k+1}^{\infty} \|\mathcal{F}^{-1}\varphi_j * f\|^{\sigma}_{\dot{K}^{\alpha,p}_q} \leq \|f\|^{\sigma}_{\dot{K}^{\alpha,p}_q A^s_\beta} \sum_{j=k+1}^{\infty} 2^{-js\sigma},$$

where $\sigma = \min(1, p, q)$. Letting k tends to infinity, we get $g = f$ almost everywhere. For the latter case $1 < q_1 < q \leq \infty$, we have

$$\dot{K}^{\alpha,p}_q A^s_\beta \hookrightarrow \dot{K}^{0,\max(1,p)}_{q_1} A^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}}_\beta.$$

As in Case 1, $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to a function $h \in \dot{K}^{0,\max(1,p)}_{q_1}$. Then again, similarly to the arguments in Case 1 it is easy to check that $f = h$ almost everywhere. Therefore, we can conclude that $g = f$ almost everywhere.

Case 3. $q = 1$ and $\alpha \geq 0$.

Subcase 3.1. $q = 1$ and $\alpha > 0$. We have

$$\dot{K}^{\alpha,p}_1 B^s_\beta \hookrightarrow L^1,$$

since $s > \alpha$, see Theorem 2.4, Substep 1.3. Now one can continue as in Case 2.

Subcase 3.2. $q = 1$ and $\alpha = 0$. Let α_3 be a real number such that $\max(-n, -s) < \alpha_3 < 0$. By Theorem 2.1, we get

$$\dot{K}^{0,p}_1 A^s_\beta \hookrightarrow \dot{K}^{\alpha_3,p}_1 A^{s+\alpha_3}_\beta.$$

We have

$$\sum_{k=0}^{\infty} \|\mathcal{F}^{-1}\varphi_k * f\|_{\dot{K}^{\alpha_3,\max(1,p)}_1} \lesssim \|f\|_{\dot{K}^{\alpha_3,p}_1 A^{s+\alpha_3}_\beta} \lesssim \|f\|_{\dot{K}^{0,p}_1 A^s_\beta},$$

since $\alpha_3 + s > 0$. Hence the sequence $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to f in $\dot{K}^{\alpha_3,\max(1,p)}_1$, see Case 1. As in Case 2, we obtain $g = f$ almost everywhere.

Case 4. $0 < q < 1$.

Subcase 4.1. $-\frac{n}{q} < \alpha < 0$. From the embedding (2.10) and the fact that $s > \frac{n}{q} - n$, the sequence $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converge to f in $\dot{K}^{\alpha,\max(1,p)}_1$. As above we prove that $g = f$ almost everywhere.

Subcase 4.2. $\alpha \geq 0$. Recall that

$$\dot{K}^{\alpha,p}_q A^s_\beta \hookrightarrow \dot{K}^{\alpha_4,\max(1,p)}_1 A^{s-\frac{n}{q}+n-\alpha+\alpha_4}_\beta,$$

see Substep 2.2 of the proof of Theorem 2.4. As in Subcase 3.2 the sequence $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to f in $\dot{K}^{\alpha_4,\max(1,p)}_1$. By the same arguments as above one can conclude that: $g = f$ almost everywhere.

Step 2. In this step we prove that

$$\|f\|^{**}_{\dot{K}^{\alpha,p}_q F^s_\beta} = \Big\| \Big(\int_0^\infty t^{-s\beta}(d^M_t f)^\beta \frac{dt}{t}\Big)^{1/\beta} \Big\|_{\dot{K}^{\alpha,p}_q} \lesssim \|f\|_{\dot{K}^{\alpha,p}_q F^s_\beta}, \quad f \in \dot{K}^{\alpha,p}_q F^s_\beta.$$

Thus, we need to prove that

$$\Big\| \Big(\sum_{k=-\infty}^{\infty} 2^{sk\beta} |d^M_{2^{-k}} f|^\beta\Big)^{1/\beta} \Big\|_{\dot{K}^{\alpha,p}_q}$$

does not exceed $c\big\|f\big\|_{\dot{K}_q^{\alpha,p}F_\beta^s}$. In order to prove this we additionally consider two Substeps 2.1 and 2.2. The estimate for the space $\dot{K}_q^{\alpha,p}B_\beta^s$ is similar.

Substep 2.1. We will estimate

$$\Big\|\Big(\sum_{k=0}^{\infty}2^{sk\beta}|d_{2^{-k}}^{M}f|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_q^{\alpha,p}}.$$

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a smooth dyadic partition of unity. Obviously we need to estimate

$$\Big\{2^{ks}\sum_{j=0}^{k}d_{2^{-k}}^{M}(\mathcal{F}^{-1}\varphi_j*f)\Big\}_{k\in\mathbb{N}_0} \tag{2.11}$$

and

$$\Big\{2^{ks}\sum_{j=k+1}^{\infty}d_{2^{-k}}^{M}(\mathcal{F}^{-1}\varphi_j*f)\Big\}_{k\in\mathbb{N}_0}. \tag{2.12}$$

Recall that

$$d_{2^{-k}}^{M}(\mathcal{F}^{-1}\varphi_j*f)\lesssim 2^{(j-k)M}(\mathcal{F}^{-1}\varphi_j)^{*,a}f\,(x)$$

if $a>0$, $0\le j\le k, k\in\mathbb{N}_0$ and $x\in\mathbb{R}^n$, see, e.g., [13], where the implicit constant is independent of j,k and x. We choose $a>\frac{n}{\min\left(\min(q,\beta),\frac{n}{\alpha+\frac{n}{q}}\right)}$. Since $s<M$, (2.11) in ℓ^β-quasi-norm does not exceed

$$\Big(\sum_{j=0}^{\infty}2^{js\beta}((\mathcal{F}^{-1}\varphi_j)^{*,a}f)^{\beta}\Big)^{1/\beta}. \tag{2.13}$$

By Theorem 2.3, the $\dot{K}_q^{\alpha,p}$-quasi-norm of (2.13) is bounded by $c\big\|f\big\|_{\dot{K}_q^{\alpha,p}F_\beta^s}$.

Now, we estimate (2.12). We can distinguish two cases as follows:

- *Case* 1. $\min(q,\beta)\le 1$. If $-\frac{n}{q}<\alpha<n(1-\frac{1}{q})$, then $s>\frac{n}{\min(q,\beta)}-n$. We choose

$$\max\Big(0,1-\frac{s\min(q,\beta)}{n}\Big)<\lambda<\min(q,\beta), \tag{2.14}$$

which is possible because of

$$s>\frac{n}{\min(q,\beta)}-n=\frac{n}{\min(q,\beta)}\left(1-\min(q,\beta)\right).$$

Let $\frac{n}{\min(q,\beta)}<a<\frac{s}{1-\lambda}$. Then $s>a(1-\lambda)$. Now, assume that $\alpha\ge n(1-\frac{1}{q})$. Therefore

$$s>\max\Big(\frac{n}{\min(q,\beta)}-n,\frac{n}{q}+\alpha-n\Big).$$

If $\min(q,\beta)\le\frac{n}{\frac{n}{q}+\alpha}$, then we choose λ as in (2.14). If $\min(q,\beta)>\frac{n}{\frac{n}{q}+\alpha}$, then we choose

$$\max\Big(0,1-\frac{s}{\frac{n}{q}+\alpha}\Big)<\lambda<\frac{n}{\frac{n}{q}+\alpha}, \tag{2.15}$$

which is possible because of

$$s > \frac{n}{q} + \alpha - n = \left(\frac{n}{q} + \alpha\right)\left(1 - \frac{n}{\frac{n}{q} + \alpha}\right).$$

In this case, we choose $\frac{n}{q} + \alpha < a < \frac{s}{1-\lambda}$. We set

$$J_{2,k}(f) = 2^{ks} \sum_{j=k+1}^{\infty} d^M_{2^{-k}}(\mathcal{F}^{-1}\phi_j * f), \quad k \in \mathbb{N}_0.$$

Recalling the definition of $d^M_{2^{-k}}(\phi_j * f)$, we have

$$\begin{aligned} d^M_{2^{-k}}(\mathcal{F}^{-1}\phi_j * f) &= \int_B \left|\Delta^M_{2^{-k}h}(\mathcal{F}^{-1}\phi_j * f)\right| dh \\ &\leq \int_B \left|\Delta^M_{2^{-k}h}(\mathcal{F}^{-1}\phi_j * f)\right|^{\lambda} dh \sup_{h\in B} \left|\Delta^M_{2^{-k}h}(\mathcal{F}^{-1}\phi_j * f)\right|^{1-\lambda}. \end{aligned} \tag{2.16}$$

Observe that

$$\left|\mathcal{F}^{-1}\phi_j * f(x + (M-i)2^{-k}h)\right| \leq c2^{(j-k)a}\phi_j^{*,a} f(x), \quad |h| \leq 1 \tag{2.17}$$

and

$$\int_B \left|\mathcal{F}^{-1}\phi_j * f(x + (M-i)2^{-k}h)\right|^{\lambda} dh \leq c\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda})(x). \tag{2.18}$$

if $j > k, i \in \{0, ..., M\}$ and $x \in \mathbb{R}^n$. Therefore

$$d^M_{2^{-k}}(\mathcal{F}^{-1}\phi_j * f) \leq c2^{(j-k)a(1-\lambda)}(\phi_j^{*,a} f)^{1-\lambda}\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda})$$

for any $j > k$, where the positive constant c is independent of j and k. Hence

$$J_{2,k}(f) \leq c2^{ks} \sum_{j=k+1}^{\infty} 2^{(j-k)a(1-\lambda)}(\phi_j^{*,a} f)^{1-\lambda}\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda}).$$

Using Lemma 2.3, we obtain that (2.12) in ℓ^{β}-quasi-norm can be estimated from above by

$$\begin{aligned} & c\left(\sum_{j=0}^{\infty} 2^{js\beta}(\phi_j^{*,a} f)^{(1-\lambda)\beta}(\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda}))^{\beta}\right)^{1/\beta} \\ & \lesssim \left(\sum_{j=0}^{\infty} 2^{js\beta}(\phi_j^{*,a} f)^{\beta}\right)^{(1-\lambda)/\beta} \left(\sum_{j=0}^{\infty} 2^{js\beta}(\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda}))^{\beta/\lambda}\right)^{\lambda/\beta}. \end{aligned}$$

Applying the $\dot{K}^{\alpha,p}_q$-quasi-norm and using Hölder's inequality we obtain that

$$\left\| \left(\sum_{j=0}^{\infty} (J_{2,k}(f))^{\beta}\right)^{1/\beta} \right\|_{\dot{K}^{\alpha,p}_q}$$

is bounded by

$$
\begin{aligned}
& c\Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} (\phi_j^{*,a} f)^{\beta} \Big)^{(1-\lambda)/\beta} \Big\|_{\dot{K}^{\alpha(1-\lambda),\frac{p}{1-\lambda}}_{\frac{q}{1-\lambda}}} \\
\times & \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} \left(\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda}) \right)^{\beta/\lambda} \Big)^{\lambda/\beta} \Big\|_{\dot{K}^{\alpha\lambda,\frac{p}{\lambda}}_{\frac{q}{\lambda}}} \\
\lesssim & \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} (\phi_j^{*,a} f)^{\beta} \Big)^{1/\beta} \Big\|^{1-\lambda}_{\dot{K}^{\alpha,p}_{q}} \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1}\phi_j * f|^{\beta} \Big)^{1/\beta} \Big\|^{\lambda}_{\dot{K}^{\alpha,p}_{q}} \\
\lesssim & \|f\|_{\dot{K}^{\alpha,p}_{q} F^{s}_{\beta}},
\end{aligned}
$$

where we have used Lemma 2.4 and Theorem 2.3.

- *Case 2.* $\min(q,\beta) > 1$. Assume that $\alpha \geq n(1-\frac{1}{q})$. Then we choose λ as in (2.15) and $\frac{n}{q} + \alpha < a < \frac{s}{1-\lambda}$. If $-\frac{n}{q} < \alpha < n(1-\frac{1}{q})$, then we choose $\lambda = 1$. The desired estimate can be done in the same manner as in Case 1.

Substep 2.2. We will estimate

$$
\Big\| \Big(\sum_{k=-\infty}^{-1} 2^{sk\beta} |d^{M}_{2^{-k}} f|^{\beta} \Big)^{1/\beta} \Big\|_{\dot{K}^{\alpha,p}_{q}}.
$$

We employ the same notations as in Subtep 1.1. Define

$$
H_{k,2}(f)(x) = \int_B \Big| \sum_{j=0}^{\infty} \Delta^{M}_{z2^{-k}} (\mathcal{F}^{-1}\varphi_j * f)(x) \Big| dz, \quad k \leq 0, x \in \mathbb{R}^n.
$$

As in the estimation of $J_{2,k}$, we obtain that

$$
H_{2,k}(f) \lesssim 2^{-ka(1-\lambda)} \sup_{j\in\mathbb{N}_0} \Big(\big(2^{js} (\mathcal{F}^{-1}\varphi_j)^{*,a} f \big)^{1-\lambda} \mathcal{M} \left(2^{js} |\mathcal{F}^{-1}\varphi_j * f| \right)^{\lambda} \Big)
$$

and this yields that

$$
\Big(\sum_{k=-\infty}^{-1} 2^{sk\beta} |H_{2,k}|^{\beta} \Big)^{1/\beta} \lesssim \sup_{j\in\mathbb{N}_0} \Big(\big(2^{js} (\mathcal{F}^{-1}\varphi_j)^{*,a} f \big)^{1-\lambda} \mathcal{M} \left(2^{js} |\mathcal{F}^{-1}\varphi_j * f| \right)^{\lambda} \Big).
$$

By the same arguments as used in Subtep 2.1 we obtain the desired estimate.

Step 3. Let $f \in \dot{K}^{\alpha,p}_{q} A^{s}_{\beta}$. We will prove that

$$
\|f\|_{\dot{K}^{\alpha,p}_{q} A^{s}_{\beta}} \lesssim \|f\|^{*}_{\dot{K}^{\alpha,p}_{q} A^{s}_{\beta}}.
$$

As the proof for $\dot{K}^{\alpha,p}_{q} B^{s}_{\beta}$ is similar, we only consider $\dot{K}^{\alpha,p}_{q} F^{s}_{\beta}$. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq \frac{3}{2}$, and in addition radially symmetric. We use an observation made by Nikol'skij [33] (see also [37] and [42, Section 3.3.2]). We put

$$
\psi(x) = (-1)^{M+1} \sum_{i=0}^{M-1} (-1)^{i} C^{M}_{i} \Psi(x\,(M-i)).
$$

The function ψ satisfies $\psi(x) = 1$ for $|x| \le \frac{1}{M}$ and $\psi(x) = 0$ for $|x| \ge \frac{3}{2}$. Then, taking $\varphi_0(x) = \psi(x), \varphi_1(x) = \psi(\frac{x}{2}) - \psi(x)$ and $\varphi_j(x) = \varphi_1(2^{-j+1}x)$ for $j = 2, 3, ...$, we obtain that $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a smooth dyadic partition of unity. This yields that

$$\Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1}\varphi_j * f|^{\beta} \Big)^{1/\beta} \Big\|_{\dot{K}_q^{\alpha,p}}$$

is a quasi-norm equivalent in $\dot{K}_q^{\alpha,p} F_\beta^s$. Let us prove that the last expression is bounded by

$$C\big\|f\big\|^*_{\dot{K}_q^{\alpha,p} F_\beta^s}. \tag{2.19}$$

We observe that

$$\mathcal{F}^{-1}\varphi_0 * f(x) = (-1)^{M+1} \int_{\mathbb{R}^n} \mathcal{F}^{-1}\Psi(z)\, \Delta_{-z}^M f(x) dz + f(x) \int_{\mathbb{R}^n} \mathcal{F}^{-1}\Psi(z)\, dz$$

Moreover, it holds for $x \in \mathbb{R}^n$ and $j = 1, 2, ...$

$$\mathcal{F}^{-1}\varphi_j * f(x) = (-1)^{M+1} \int_{\mathbb{R}^n} \Delta_{2^{-j}y}^M f(x)\, \widetilde{\Psi}(y)\, dy,$$

with $\widetilde{\Psi} = \mathcal{F}^{-1}\Psi - 2^{-n}\mathcal{F}^{-1}\Psi(\cdot/2)$. Now, for $j \in \mathbb{N}_0$ we have

$$\begin{aligned} &\int_{\mathbb{R}^n} |\Delta_{2^{-j}y}^M f(x)||\widetilde{\Psi}(y)|dy \\ &= \int_{|y|\le 1} |\Delta_{2^{-j}y}^M f(x)||\widetilde{\Psi}(y)|dy + \int_{|y|>1} |\Delta_{2^{-j}y}^M f(x)||\widetilde{\Psi}(y)|dy. \end{aligned} \tag{2.20}$$

Thus, we need only to estimate the second term of (2.20). We write

$$\begin{aligned} &2^{sj} \int_{|y|>1} |\Delta_{2^{-j}y}^M f(x)||\widetilde{\Psi}(y)|dy \\ &= 2^{sj} \sum_{k=0}^{\infty} \int_{2^k<|y|\le 2^{k+1}} |\Delta_{2^{-j}y}^M f(x)||\widetilde{\Psi}(y)|dv \\ &\le c2^{sj} \sum_{k=0}^{\infty} 2^{nj-Nk} \int_{2^{k-j}<|h|\le 2^{k-j+1}} |\Delta_h^M f(x)|dh \end{aligned} \tag{2.21}$$

where $N > 0$ is at our disposal and we have used the properties of the function $\widetilde{\Psi}$, $|\widetilde{\Psi}(x)| \le c(1+|x|)^{-N}$, for any $x \in \mathbb{R}^n$ and any $N > 0$. Without lost of generality, we may assume $1 \le \beta \le \infty$. Now, the right-hand side of (2.21) in ℓ^β-norm is bounded by

$$c \sum_{k=0}^{\infty} 2^{-Nk} \Big(\sum_{j=0}^{\infty} 2^{(s+n)j\beta} \Big(\int_{|h|\le 2^{k-j+1}} |\Delta_h^M f(x)|dh \Big)^{\beta} \Big)^{1/\beta}. \tag{2.22}$$

After a change of variable $j - k - 1 = v$, we estimate (2.22) by

$$c \sum_{k=0}^{\infty} 2^{(s+n-N)k} \Big(\sum_{v=-k-1}^{\infty} 2^{sv\beta} \big(d_{2^{-v}}^M f(x) \big)^{\beta} \Big)^{1/\beta} \lesssim \Big(\sum_{v=-\infty}^{\infty} 2^{sv\beta} \big(d_{2^{-v}}^M f(x) \big)^{\beta} \Big)^{1/\beta},$$

where we choose $N > n + s$. Taking the $\dot{K}_q^{\alpha,p}$-quasi-norm we obtain the desired estimate (2.19). $\square$

We would like to mention that

$$\|f(\lambda\cdot)\|^*_{\dot{K}_q^{\alpha,p}B_\beta^s} \approx \lambda^{-\alpha-\frac{n}{q}}\big\|f\big\|_{\dot{K}_q^{\alpha,p}} + \lambda^{s-\alpha-\frac{n}{q}}\Big(\int_0^\infty t^{-s\beta}\big\|d_t^M f\big\|^\beta_{\dot{K}_q^{\alpha,p}}\frac{dt}{t}\Big)^{\frac{1}{\beta}} \tag{2.23}$$

and

$$\|f(\lambda\cdot)\|^*_{\dot{K}_q^{\alpha,p}F_\beta^s} \approx \lambda^{-\alpha-\frac{n}{q}}\big\|f\big\|_{\dot{K}_q^{\alpha,p}} + \lambda^{s-\alpha-\frac{n}{q}}\Big\|\Big(\int_0^\infty t^{-s\beta}(d_t^M f)^\beta\frac{dt}{t}\Big)^{\frac{1}{\beta}}\Big\|_{\dot{K}_q^{\alpha,p}}$$

for any $\lambda > 0, 0 < p \leq \infty, 0 < q \leq \infty, \alpha > -\frac{n}{q}, \max(\sigma_q, \alpha - \alpha_0) < s < M$ $(0 < p, q < \infty$ and $\max(\sigma_{q,\beta}, \alpha-\alpha_0) < s < M$ in the $\dot{K}F$-case) and $M \in \mathbb{N}$.

Let $\varphi^j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{1-j}x)$ for $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. In view of [48] we have the following equivalent norm of $\dot{K}_q^{\alpha,p}$. Let $1 < p, q < \infty$ and $-\frac{n}{q} < \alpha < n - \frac{n}{q}$. Then

$$\Big\|\Big(\sum_{j=-\infty}^{\infty}\big|\mathcal{F}^{-1}\varphi^j * f\big|^2\Big)^{1/2}\Big\|_{\dot{K}_q^{\alpha,p}} \approx \big\|f\big\|_{\dot{K}_q^{\alpha,p}}, \tag{2.24}$$

holds for all $f \in \dot{K}_q^{\alpha,p}$.

Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{q}$. We set

$$\big\|f\big\|_{\dot{K}_q^{\alpha,p}\dot{B}_\beta^s} = \Big(\sum_{j=-\infty}^{\infty} 2^{js\beta}\big\|\mathcal{F}^{-1}\varphi^j * f\big\|^\beta_{\dot{K}_q^{\alpha,p}}\Big)^{1/\beta}$$

and

$$\big\|f\big\|_{\dot{K}_q^{\alpha,p}\dot{F}_\beta^s} = \Big\|\Big(\sum_{j=-\infty}^{\infty} 2^{js\beta}\big|\mathcal{F}^{-1}\varphi^j * f\big|^\beta\Big)^{1/\beta}\Big\|_{\dot{K}_q^{\alpha,p}}.$$

Proposition 2.1. *Let* $s > \max(\sigma_q, \alpha - n + \frac{n}{q}), 0 < p, q < \infty, 0 < \beta \leq \infty$ *and* $\alpha > -\frac{n}{q}$.

(i) *Let* $s > \max(\sigma_q, \alpha - n + \frac{n}{q})$ *and* $f \in \dot{K}_q^{\alpha,p}B_\beta^s$. *Then*

$$\big\|f\big\|_{\dot{K}_q^{\alpha,p}B_\beta^s} \approx \big\|f\big\|_{\dot{K}_q^{\alpha,p}} + \big\|f\big\|_{\dot{K}_q^{\alpha,p}\dot{B}_\beta^s},$$

(ii) *Let* $s > \max(\sigma_{q,\beta}, \alpha - n + \frac{n}{q})$ *and* $f \in \dot{K}_q^{\alpha,p}F_\beta^s$. *Then*

$$\big\|f\big\|_{\dot{K}_q^{\alpha,p}F_\beta^s} \approx \big\|f\big\|_{\dot{K}_q^{\alpha,p}} + \big\|f\big\|_{\dot{K}_q^{\alpha,p}\dot{F}_\beta^s}.$$

Proof. As the proof for (i) is similar, we only consider (ii). We use the following Marschall's inequality which is given in [28, Proposition 1.5], see also [14]. Let $A > 0, R \geq 1$. Let $b \in \mathcal{D}(\mathbb{R}^n)$ and a function $g \in C^\infty(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\mathcal{F}g \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq AR\} \quad \text{and} \quad \operatorname{supp} b \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq A\}.$$

Then

$$\big|\mathcal{F}^{-1}b * g(x)\big| \leq c(AR)^{\frac{n}{t}-n}\,\|b\|_{\dot{B}_{1,t}^{\frac{n}{t}}}\,\mathcal{M}_t(g)(x)$$

for any $0 < t \leq 1$ and any $x \in \mathbb{R}^n$, where c is independent of A, R, x, b, j and g. Here $\dot{B}_{1,t}^{\frac{n}{t}}$ denotes the homogeneous Besov spaces. We have

$$\mathcal{F}^{-1}\varphi^j * f = \mathcal{F}^{-1}\varphi^j * \mathcal{F}^{-1}\varphi_0 * f, \quad -j \in \mathbb{N}.$$

Therefore,

$$\left|\mathcal{F}^{-1}\varphi^j * f(x)\right| \leq c\left\|\varphi^j\right\|_{\dot{B}^{\frac{n}{t}}_{1,t}} \mathcal{M}_t(\mathcal{F}^{-1}\varphi_0 * f)(x) \leq c2^{j(n-\frac{n}{t})}\mathcal{M}_t(\mathcal{F}^{-1}\varphi_0 * f)(x), \quad x \in \mathbb{R}^n,$$

where the positive constant c is independent of j and x. If we choose $\frac{n}{s+n} < t < \min(1, q, \beta, \frac{n}{\alpha+\frac{n}{q}})$ then

$$\Big(\sum_{j=-\infty}^{-1} 2^{js\beta}\left|\mathcal{F}^{-1}\varphi^j * f\right|^{\beta}\Big)^{1/\beta} \lesssim \mathcal{M}_t(\mathcal{F}^{-1}\varphi_0 * f).$$

Taking the $\dot{K}^{\alpha,p}_q$-quasi-norm and using (2.1) we obtain

$$\Big\|\Big(\sum_{j=-\infty}^{\infty} 2^{js\beta}\left|\mathcal{F}^{-1}\varphi^j * f\right|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}^{\alpha,p}_q} \lesssim \|f\|_{\dot{K}^{\alpha,p}_q F^s_\beta}.$$

Because of $s > \max(\sigma_q, \alpha - n + \frac{n}{q})$ the series $\sum_{j=0}^{\infty} \mathcal{F}^{-1}\varphi_j * f$ converges not only in $\mathcal{S}'(\mathbb{R}^n)$ but almost everywhere in $\mathbb{R}^n$. Then

$$\|f\|_{\dot{K}^{\alpha,p}_q} \lesssim \left\|\mathcal{F}^{-1}\varphi_0 * f\right\|_{\dot{K}^{\alpha,p}_q} + \Big(\sum_{j=1}^{\infty}\left\|\mathcal{F}^{-1}\varphi_j * f\right\|^{\min(1,p,q)}_{\dot{K}^{\alpha,p}_q}\Big)^{1/\min(1,p,q)}.$$

Therefore $\|f\|_{\dot{K}^{\alpha,p}_q} + \|f\|_{\dot{K}^{\alpha,p}_q \dot{F}^s_\beta}$ can be estimated from above by $c\|f\|_{\dot{K}^{\alpha,p}_q F^s_\beta}$. Obviously

$$\mathcal{F}^{-1}\varphi_0 * f = \sum_{j=0}^{N} \mathcal{F}^{-1}\varphi_j * f - \sum_{j=1}^{N} \mathcal{F}^{-1}\varphi_j * f = g_N + h_N, \quad N \in \mathbb{N}.$$

We have

$$\|h_N\|_{\dot{K}^{\alpha,p}_q} \leq \Big(\sum_{j=1}^{\infty}\left\|\mathcal{F}^{-1}\varphi^j * f\right\|^{\min(1,p,q)}_{\dot{K}^{\alpha,p}_q}\Big)^{1/\min(1,p,q)}, \quad N \in \mathbb{N}.$$

By Lebesgue's dominated convergence theorem, it follows that $\|g_N - f\|_{\dot{K}^{\alpha,p}_q}$ tends to zero as N tends to infinity. Therefore $\left\|\mathcal{F}^{-1}\varphi_0 * f\right\|_{\dot{K}^{\alpha,p}_q}$ can be estimated from above by the quasi-norm

$$c\|f\|_{\dot{K}^{\alpha,p}_q} + c\|f\|_{\dot{K}^{\alpha,p}_q \dot{F}^s_\beta}.$$

□

Proposition 2.2. *Let $s > 0, 1 < p, q < \infty$ and $-\frac{n}{q} < \alpha < n - \frac{n}{q}$. Let*

$$\mathcal{S}_0(\mathbb{R}^n) = \big\{f \in \mathcal{S}(\mathbb{R}^n) : \operatorname{supp}\mathcal{F}f \subset \mathbb{R}^n\backslash\{0\}\big\}.$$

Then $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\dot{k}^{\alpha,p}_{q,s}$.

Proof. Let $\varphi_0 = \varphi$ be as above. As in [44] it suffices to approximate $f \in \mathcal{S}(\mathbb{R}^n)$ in $\dot{W}^{\alpha,p}_{q,k}$, $k \in \mathbb{N}$, by functions belonging to $\mathcal{S}_0(\mathbb{R}^n)$. We have

$$|D^\alpha \mathcal{F}^{-1}(\varphi(2^j\cdot)\mathcal{F}f)| = 2^{-jn}|\tilde{\varphi}_j * D^\alpha f| \leq 2^{-jn}\mathcal{M}(\tilde{\varphi}_j),$$

where $\tilde{\varphi}_j = \mathcal{F}^{-1}\varphi(2^{-j}\cdot), j \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. From (2.1) we obtain

$$\left\|D^\alpha \mathcal{F}^{-1}(\varphi(2^j\cdot)\mathcal{F}f)\right\|_{\dot{K}^{\alpha,p}_q} \leq c2^{-jn}\|\tilde{\varphi}_j\|_{\dot{K}^{\alpha,p}_q} \leq c2^{j(\frac{n}{q}-n+\alpha)},$$

where the positive constant c is independent of j. Since $\alpha < n - \frac{n}{q}$, we obtain that $f - \mathcal{F}^{-1}(\varphi(2^j\cdot)\mathcal{F}f)$ approximate $f \in \mathcal{S}(\mathbb{R}^n)$ in $\dot{W}^{\alpha,p}_{q,k}$, $k \in \mathbb{N}$. □

Proposition 2.3. *Let $s>0, 1<p,q<\infty$ and $-\frac{n}{q}<\alpha<n-\frac{n}{q}$. Let $f\in \dot{k}_{q,s}^{\alpha,p}$. Then*

$$\|f\|_{\dot{k}_{q,s}^{\alpha,p}} \approx \|f\|_{\dot{K}_q^{\alpha,p}} + \|(-\Delta)^{\frac{s}{2}} f\|_{\dot{K}_q^{\alpha,p}},$$

where

$$(-\Delta)^{\frac{s}{2}} f = \mathcal{F}^{-1}(|\xi|^s \mathcal{F} f).$$

Proof. Let $f\in \mathcal{S}_0(\mathbb{R}^n)$. We apply Marschall's inequality to $g_j=\mathcal{F}^{-1}(\varphi^j|x|^s\mathcal{F}f), j\in\mathbb{Z}$ and $b_j(x)=2^{js}|x|^{-s}\psi^j(x), j\in\mathbb{Z}, x\in\mathbb{R}^n$ where

$$\varphi^j(x)=\varphi_0(2^{-j}x)-\varphi_0(2^{1-j}x), \quad \psi^j=\varphi^{j-1}+\varphi^j+\varphi^{j+1}, \quad j\in\mathbb{Z}, x\in\mathbb{R}^n.$$

Then

$$\left|\mathcal{F}^{-1}b_j * g_j(x)\right| \le c\,\|b_j\|_{B_{1,1}^n} \mathcal{M}(\mathcal{F}^{-1}(\varphi^j|\xi|^s\mathcal{F}f))(x) \le c\mathcal{M}(\mathcal{F}^{-1}(\varphi^j|\xi|^s\mathcal{F}f))(x)$$

for any $j\in\mathbb{Z}$ and any $x\in\mathbb{R}^n$, where c is independent of j. Let $j\in\mathbb{Z}$. In view of the fact that

$$\mathcal{F}^{-1}\varphi^j * f = \mathcal{F}^{-1}(\varphi^j\mathcal{F}f) = 2^{-js}\mathcal{F}^{-1}(2^{js}|\xi|^{-s}\psi^j|x|^s\varphi^j\mathcal{F}f) = 2^{-js}\mathcal{F}^{-1}(b_j|\xi|^s\varphi^j\mathcal{F}f),$$

by Lemma 2.4 and (2.24) we obtain

$$\begin{aligned}\Big\|\Big(\sum_{j=-\infty}^{\infty} 2^{2sj}\left|\mathcal{F}^{-1}\varphi^j * f\right|^2\Big)^{1/2}\Big\|_{\dot{K}_q^{\alpha,p}} &\lesssim \Big\|\Big(\sum_{j=-\infty}^{\infty}\left|\mathcal{F}^{-1}(\varphi^j|\xi|^s\mathcal{F}f)\right|^2\Big)^{1/2}\Big\|_{\dot{K}_q^{\alpha,p}} \\ &\lesssim \left\|\mathcal{F}^{-1}(|\xi|^s\mathcal{F}f)\right\|_{\dot{K}_q^{\alpha,p}}.\end{aligned}$$

The same arguments can be used to prove the opposite inequality in view of the fact that

$$\mathcal{F}^{-1}(\varphi^j|\xi|^s\mathcal{F}f) = \mathcal{F}^{-1}(2^{-js}\psi^j|\xi|^s2^{js}\varphi^j\mathcal{F}f) = \mathcal{F}^{-1}(b_j2^{js}\varphi^j\mathcal{F}f), \quad j\in\mathbb{Z}.$$

The rest follows by Propositions 2.1 and 2.2. □

Definition 5. Let $0<u\le p<\infty$. The Morrey space M_u^p is defined to be the set of all u-locally Lebesgue-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{M_u^p} = \sup |B|^{\frac{1}{p}-\frac{1}{u}} \left\|f\chi_B\right\|_u < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Remark 4. The Morrey spaces M_u^p which are quasi-Banach spaces, Banach spaces for $u\ge 1$, were introduced by Morrey to study the regularity of solutions to some PDE's, see [31]. For the theory of Morrey spaces, general Morrey-type spaces, and their applications see the book [1] and survey papers [5, 6, 18, 23, 35, 38, 39].

One can easily see that $M_p^p=L^p$ and that for $0<u\le v\le p<\infty$,

$$M_v^p \hookrightarrow M_u^p.$$

The Sobolev-Morrey spaces are defined as follows.

Definition 6. Let $1<u\le p<\infty$ and $m=1,2,....$ The Sobolev-Morrey space $M_u^{m,p}$ is defined to be the set of all u-locally Lebesgue-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{M_u^{m,p}} = \|f\|_{M_u^p} + \sum_{|\alpha|\le m} \|D^\alpha f\|_{M_u^p} < \infty.$$

Let now recall the definition of Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a partition of the unity, see Section 2.

Definition 7. Let $s \in \mathbb{R}, 0 < u \le p < \infty$ and $0 < q \le \infty$.The Besov-Morrey space $\mathcal{N}^s_{p,q,u}$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{N}^s_{p,q,u}} = \Big(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|^q_{M^p_u}\Big)^{1/q} < \infty.$$

In the limiting case $q = \infty$ the usual modification is required.
The Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{p,q,u}$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{E}^s_{p,q,u}} = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq} \big|\mathcal{F}^{-1}\varphi_j * f\big|^q\Big)^{1/q}\Big\|_{M^p_u} < \infty.$$

In the limiting case $q = \infty$ the usual modification is required.

We have

$$\mathcal{E}^m_{p,2,u} = M^{m,p}_u, \quad m \in \mathbb{N}, \quad 1 < u \le p < \infty$$

and the norms of these spaces are equivalent, see [38, Theorem 3.1]. In particular, we have that

$$\mathcal{E}^0_{p,2,u} = M^p_u, \quad 1 < u \le p < \infty, \tag{2.25}$$

also in the sense of with equivalent norms, see [29, Proposition 4.1].

Theorem 2.6. *Let $s_i \in \mathbb{R}, 0 < q_i \le \infty, 0 < u_i \le p_i < \infty, i = 1, 2$. There is a continuous embedding*

$$\mathcal{E}^{s_1}_{p_1,q_1,u_1} \hookrightarrow \mathcal{E}^{s_2}_{p_2,q_2,u_2}$$

if, and only if,

$$p_1 \le p_2 \quad \text{and} \quad \frac{u_2}{p_2} \le \frac{u_1}{p_1}$$

and

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{or} \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2} \quad \text{and} \quad p_1 \ne p_2.$$

For the proof of these Sobolev embeddings, see [19, Theorem 3.1].

Remark 5. A detailed study of Besov-Morrey and Triebel-Lizorkin-Morrey spaces including their history and properties can be found in [19, 29, 30, 38, 51] and references therein.

3 Caffarelli-Kohn-Nirenberg inequalities

As mentioned in the introduction, Caffarelli-Kohn-Nirenberg inequalities play a crucial role to study regularity and integrability for solutions of nonlinear partial differential equations, see [15, 50]. The main aim of this section is to extend these inequalities to more general function spaces. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a partition of unity and

$$Q_J f = \sum_{j=0}^{J} \mathcal{F}^{-1}\varphi_j * f, \quad J \in \mathbb{N}, f \in \mathcal{S}'(\mathbb{R}^n).$$

3.1 CKN inequalities in Herz-type Besov and Triebel-Lizorkin spaces

In this section, we investigate the Caffarelli, Kohn and Nirenberg inequalities in the spaces $\dot{K}_q^{\alpha,p}A_\beta^s$. The main results of this section are based on the following proposition.

Proposition 3.1. *Let $\alpha_1,\alpha_2\in\mathbb{R},\sigma\geq 0, 1<r,v<\infty, 0<\tau,u\leq\infty$ and*

$$-\frac{n}{v}<\alpha_1<n-\frac{n}{v}.$$

(i) *Assume that $1<u\leq v<\infty$ and $\alpha_2\geq\alpha_1$. Then for all $f\in\dot{K}_u^{\alpha_2,\delta}\cap\mathcal{S}'(\mathbb{R}^n)$ and all $J\in\mathbb{N}$,*

$$\big\|Q_Jf\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}\leq c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}, \tag{3.1}$$

where

$$\delta=\begin{cases} r, & \text{if} \quad \alpha_2=\alpha_1,\\ \tau, & \text{if} \quad \alpha_2>\alpha_1\end{cases}$$

and the positive constant c is independent of J.

(ii) *Assume that $1<v\leq u<\infty$ and $\alpha_2\geq\alpha_1+\frac{n}{v}-\frac{n}{u}$. Then for all $f\in\dot{K}_u^{\alpha_2,\delta}\cap\mathcal{S}'(\mathbb{R}^n)$ and all $J\in\mathbb{N}$, (3.1) holds where the positive constant c is independent of J and*

$$\delta=\begin{cases} r, & \text{if} \quad \alpha_2=\alpha_1+\frac{n}{v}-\frac{n}{u},\\ \tau, & \text{if} \quad \alpha_2>\alpha_1+\frac{n}{v}-\frac{n}{u}.\end{cases}$$

Proof. We only give the proof for (i), the case of (ii) being similar. Let $\sigma=\theta m+(1-\theta)0$, $\alpha\in\mathbb{N}^n$ with $0<\theta<1$ and $|\alpha|\leq m$. From (2.5) we have

$$\big\|Q_Jf\big\|_{\dot{K}_v^{\alpha_1,r}A_2^\sigma}\leq\big\|Q_Jf\big\|_{\dot{K}_v^{\alpha_1,r}A_2^0}^{1-\theta}\big\|Q_Jf\big\|_{\dot{K}_v^{\alpha_1,r}A_2^m}^{\theta}.$$

Observe that

$$\dot{K}_v^{\alpha_1,r}A_2^\sigma=\dot{k}_{v,\sigma}^{\alpha_1,r},\quad \dot{K}_v^{\alpha_1,r}A_2^m=\dot{W}_{v,m}^{\alpha_1,r},\quad\text{and}\quad \dot{K}_v^{\alpha_1,r}A_2^0=\dot{K}_v^{\alpha_1,r},$$

see (2.2), (2.3) and (2.4). It follows that

$$\big\|Q_Jf\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}\leq\big\|Q_Jf\big\|_{\dot{K}_v^{\alpha_1,r}}^{1-\theta}\big\|Q_Jf\big\|_{\dot{W}_{v,m}^{\alpha_1,r}}^{\theta},$$

where the positive constant c is independent of J. Observe that

$$Q_Jf=2^{Jn}\mathcal{F}^{-1}\varphi_0(2^J\cdot)\ast f.$$

Therefore,

$$D^\alpha(Q_Jf)=2^{J(|\alpha|+n)}\omega_J\ast f=2^{J|\alpha|}\tilde{Q}_Jf,\quad |\alpha|\leq m$$

with $\omega_J(x)=D^\alpha(\mathcal{F}^{-1}\varphi_0)(2^Jx)$, $x\in\mathbb{R}^n$. Recall that

$$|\tilde{Q}_Jf|\lesssim\mathcal{M}(f).$$

Applying Lemma 2.1 and estimate (2.1), we obtain

$$\begin{aligned}\big\|D^\alpha(Q_Jf)\big\|_{\dot{K}_v^{\alpha_1,r}}&\leq c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+|\alpha|)}\big\|\tilde{Q}_Jf\big\|_{\dot{K}_u^{\alpha_2,\delta}}\\ &\leq c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+m)}\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}\end{aligned}$$

for any $|\alpha|\leq m$. □

Remark 6. With $\alpha_1 = \alpha_2 = 0$ and $r = v$ estimate (3.1) can be rewritten as

$$\begin{aligned} \big\| Q_J f \big\|_{H_v^\sigma} &\leq c 2^{J(\frac{n}{u} - \frac{n}{v} + \sigma)} \big\| f \big\|_{\dot{K}_u^{0,v}} \\ &\leq c 2^{J(\frac{n}{u} - \frac{n}{v} + \sigma)} \big\| f \big\|_u, \end{aligned}$$

where the second estimate follows by the embedding $L^u \hookrightarrow \dot{K}_u^{0,v}$, for $1 < u \leq v < \infty$, which has been proved by Triebel in [44, Proposition 4.5].

Now we are in position to state the main results of this section.

Theorem 3.1. *Let* $0 < p, \tau, \beta, \varrho < \infty$, $1 < r, v, u < \infty, \sigma \geq 0$,

$$-\frac{n}{v} < \alpha_1 < n - \frac{n}{v}, \quad -\frac{n}{u} < \alpha_2 < n - \frac{n}{u}, \quad \alpha_3 > -\frac{n}{p}, \quad v \geq \max(p, u), \tag{3.2}$$

$$s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > \sigma - \frac{n}{v} + \alpha_2 - \alpha_1 + \frac{n}{u} > 0 \tag{3.3}$$

and

$$\sigma - \frac{n}{v} = -(1-\theta)\frac{n}{u} + \theta\Big(s - \frac{n}{p}\Big) + \alpha_1 - \big((1-\theta)\alpha_2 + \theta\alpha_3\big), \quad 0 < \theta < 1. \tag{3.4}$$

Assume that $s > \sigma_{p,\beta}$ *in the* $\dot{K}F$*-case.*

(i) *Let* $\alpha_1 \leq \alpha_2 \leq \alpha_3$. *There is a constant* $c > 0$ *such that for all* $f \in \dot{K}_u^{\alpha_2,\delta} \cap \dot{K}_p^{\alpha_3,\delta_1} B_\beta^s$,

$$\big\| f \big\|_{\dot{K}_v^{\alpha_1,r} \dot{F}_2^\sigma} \leq c \big\| f \big\|_{\dot{K}_u^{\alpha_2,\delta}}^{1-\theta} \big\| f \big\|_{\dot{K}_p^{\alpha_3,\delta_1} \dot{B}_\beta^s}^\theta \tag{3.5}$$

with

$$\delta = \begin{cases} r, & \text{if} \quad \alpha_2 = \alpha_1, \\ \tau, & \text{if} \quad \alpha_2 > \alpha_1. \end{cases} \quad \text{and} \quad \delta_1 = \begin{cases} r, & \text{if} \quad \alpha_3 = \alpha_1, \\ \varrho, & \text{if} \quad \alpha_3 > \alpha_1. \end{cases}$$

(ii) *Let* $\frac{1}{r} \leq (1-\theta)\frac{n}{u} + \theta\frac{n}{p}$ *and*

$$\alpha_1 = (1-\theta)\alpha_2 + \theta\alpha_3.$$

There is a constant $c > 0$ *such that for all* $f \in \dot{K}_u^{\alpha_2,u} F_\infty^0 \cap \dot{K}_p^{\alpha_3,p} A_\infty^s$,

$$\big\| f \big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq c \big\| f \big\|_{\dot{K}_u^{\alpha_2,u} F_\infty^0}^{1-\theta} \big\| f \big\|_{\dot{K}_p^{\alpha_3,p} A_\infty^s}^\theta.$$

Proof. Proof of (i). For technical reasons, we split the proof into two steps.

Step 1. We consider the case $p \leq u$. Let

$$f = \sum_{j=0}^{\infty} \mathcal{F}^{-1}\varphi_j * f, \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Then it follows that

$$\begin{aligned} f &= \sum_{j=0}^{J} \mathcal{F}^{-1}\varphi_j * f + \sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f \\ &= Q_J f + \sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f, \quad J \in \mathbb{N}. \end{aligned}$$

Hence

$$\big\| f \big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq \big\| Q_J f \big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} + \Big\| \sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f \Big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}. \tag{3.6}$$

Using Proposition 3.1, it follows that

$$\big\|Q_J f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} \lesssim 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_2,\delta}_u}. \tag{3.7}$$

From the embedding

$$\dot{K}^{\alpha_1,r}_v B^\sigma_1 \hookrightarrow \dot{k}^{\alpha_1,r}_{v,\sigma}, \tag{3.8}$$

see (2.6), the last norm in (3.6) can be estimated by

$$\begin{aligned} c\sum_{j=J+1}^{\infty} 2^{j\sigma}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}^{\alpha_1,r}_v} &\lesssim \sum_{j=J+1}^{\infty} 2^{j(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1+\sigma)}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p} \\ &\lesssim 2^{J(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1-s+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}, \end{aligned} \tag{3.9}$$

by Lemma 2.1, where the last estimate follows by (3.3). By substituting (3.7) and (3.9) into (3.6) we obtain

$$\begin{aligned} \big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} &\lesssim 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_2,\delta}_u} + 2^{J(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1-s+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta} \\ &= c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\Big(\big\|f\big\|_{\dot{K}^{\alpha_2,\delta}_u} + 2^{J(\frac{n}{p}-\frac{n}{u}-s-\alpha_2+\alpha_3)}\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}\Big), \end{aligned}$$

with some positive constant c independent of J. Again from, Lemma 2.1, it follows that

$$\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta \hookrightarrow \dot{K}^{\alpha_2,\delta}_u, \tag{3.10}$$

since $s-\frac{n}{p}+\frac{n}{u}+\alpha_2-\alpha_3>0$. We choose $J\in\mathbb{N}$ such that

$$2^{J(\frac{n}{p}-\frac{n}{u}-s-\alpha_2+\alpha_3)} \approx \big\|f\big\|_{\dot{K}^{\alpha_2,\delta}_u}\big\|f\big\|^{-1}_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}.$$

We obtain

$$\big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} \lesssim \big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,\delta}_u}\big\|f\big\|^{\theta}_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}.$$

By (3.3) one has $s>\max\big(\sigma_p,\alpha_3-n+\frac{n}{p}\big)$ and by the fact that $-\frac{n}{u}<\alpha_2<n-\frac{n}{u}$,

$$\sigma > \max\Big(0,\alpha_1+\frac{n}{v}-n\Big)$$

and Theorem 2.5, or Proposition 2.1, can be used. Therefore

$$\big\|f\big\|_{\dot{K}^{\alpha_1,r}_v \dot{F}^\sigma_2} \lesssim \big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}}$$

and

$$\big\|f\big\|_{\dot{K}^{\alpha_1,r}_v \dot{F}^\sigma_2} \lesssim \big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,\delta}_u}\Big(\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p} + \big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p \dot{B}^s_\beta}\Big)^\theta.$$

By replacing $f(\cdot)$ by $f(\lambda\cdot)$ we obtain

$$\big\|f\big\|_{\dot{K}^{\alpha_1,r}_v \dot{F}^\sigma_2} \lesssim \big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,\delta}_u}\Big(\lambda^{-s}\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p} + \big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p \dot{B}^s_\beta}\Big)^\theta.$$

Taking λ large enough we obtain (3.5) but with $p\le u$.

Step 2. We consider the case $u<p$. We choose $\lambda>0$ large enough such that

$$\frac{\big\|f(\lambda\cdot)\big\|_{\dot{K}^{\alpha_2,\delta}_u}}{\big\|f(\lambda\cdot)\big\|_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}} \le 1, \tag{3.11}$$

which is possible because of $s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > 0$, see (2.23). As in Step 1, with $f(\lambda\cdot)$ in place of $f(\cdot)$ and (3.11) in place of (3.10), we obtain the desired estimate. The proof of (i) is complete.

Proof of (ii). Observe that

$$\frac{n}{v_1} = \frac{n}{v} + \theta s - \sigma = (1-\theta)\frac{n}{u} + \theta\frac{n}{p}$$

and $\frac{\sigma}{s} \leq \theta < 1$. Therefore

$$\dot{K}^{\alpha_1,r}_{v_1} F^{\theta s}_{\infty} \hookrightarrow \dot{k}^{\alpha_1,r}_{v,\sigma},$$

see Theorems 2.1. From (2.2), (2.4) and (2.5), we obtain

$$\big\|f\big\|_{\dot{K}^{\alpha_1,r}_{v_1} F^{\theta s}_{\infty}} \leq \big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,u}_{u} F^{0}_{\infty}} \big\|f\big\|^{\theta}_{\dot{K}^{\alpha_3,p}_{p} F^{\theta s}_{\infty}}.$$

We have

$$\dot{K}^{\alpha_3,p}_{p} A^{s}_{\beta} \hookrightarrow \dot{K}^{\alpha_3,p}_{p} F^{\theta s}_{\infty}.$$

This completes the proof of (ii). □

Remark 7. (i) Taking $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $r = v$ we obtain

$$\begin{aligned}\big\|f\big\|_{\dot{H}^{\sigma}_{v}} &\leq c\big\|f\big\|^{1-\theta}_{\dot{K}^{0,v}_{u}} \big\|f\big\|^{\theta}_{\dot{K}^{0,v}_{p} \dot{B}^{s}_{\beta}} \\ &\leq c\big\|f\big\|^{1-\theta}_{u} \big\|f\big\|^{\theta}_{\dot{B}^{s}_{p,\beta}}\end{aligned}$$

for all $f \in L_u \cap B^{s}_{p,\beta}$, because of $L_u \hookrightarrow \dot{K}^{0,v}_{u}$ and $\dot{B}^{s}_{p,\beta} = \dot{K}^{0,p}_{p} \dot{B}^{s}_{p,\beta} \hookrightarrow \dot{K}^{0,v}_{p} \dot{B}^{s}_{\beta}$, which has been proved by Triebel in [44, Theorem 4.6].

(ii) Under the hypothesis of Theorem 3.1/(ii), with $0 < p < \frac{n}{s - \frac{\sigma}{\theta}}$ and $\frac{1}{r} \leq (1-\theta)\frac{n}{u} + \theta(\frac{n}{p} - s + \frac{\sigma}{\theta})$, we have

$$\big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} \leq c\big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,u}_{u} F^{0}_{2}} \big\|f\big\|^{\theta}_{\dot{K}_{p}^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}} A^{s}_{\kappa}}$$

for all $f \in \dot{K}^{\alpha_2,u}_{u} F^{0}_{2} \cap \dot{K}_{p}^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}} A^{s}_{\kappa}$, where

$$\kappa = \begin{cases} \frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}, & \text{if} \quad A = B, \\ \infty, & \text{if} \quad A = F. \end{cases}$$

Indeed, observe that

$$\frac{n}{v} = (1-\theta)\frac{n}{u} + \theta\Big(\frac{n}{p} - s + \frac{\sigma}{\theta}\Big) = (1-\theta)\frac{n}{u} + \theta\frac{n}{u_1}$$

and $\frac{\sigma}{\theta} - s \leq 0$. Therefore, from (2.2), (2.4) and (2.5), we obtain

$$\big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} \leq \big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,u}_{u} F^{0}_{2}} \big\|f\big\|^{\theta}_{\dot{K}_{u_1}^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}} F^{\frac{\sigma}{\theta}}_{2}}.$$

The result follows by the embedding

$$\dot{K}_{p}^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}} A^{s}_{\kappa} \hookrightarrow \dot{K}_{u_1}^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}} F^{\frac{\sigma}{\theta}}_{2},$$

see Theorems 2.1 and 2.2.

Theorem 3.2. *Let* $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}, 0<p,\tau,\beta,\varrho\leq\infty,\ 1<r,v,u<\infty,$

$$s-\frac{n}{p}+\frac{n}{u}+\alpha_2-\alpha_3>-\frac{n}{v}+\alpha_2-\alpha_1+\frac{n}{u}>0$$

and

$$\frac{n}{v}=(1-\theta)\frac{n}{u}+\theta\Big(\frac{n}{p}-s\Big)-\alpha_1+(1-\theta)\alpha_2+\theta\alpha_3,\quad 0<\theta<1.$$

Assume that $0<p,\tau<\infty$ *and* $s>\sigma_{p,\beta}$ *in the* $\dot{K}F$*-case.*

Let δ *and* δ_1 *be as in Theorem* 3.1/(*i*). *Let* $\alpha_1\leq\alpha_2\leq\alpha_3$, $v\geq\max(u,p)$, $\alpha_1>-\frac{n}{v}$, $-\frac{n}{u}<\alpha_2<n-\frac{n}{u}$ *and* $\alpha_3>-\frac{n}{p}$. *We have*

$$\big\|f\big\|_{\dot{K}_v^{\alpha_1,r}}\lesssim\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,\delta_1}A_\beta^s}^{\theta},$$

for all $f\in\dot{K}_u^{\alpha_2,\delta}\cap\dot{K}_p^{\alpha_3,\delta_1}A_\beta^s$.

Proof. We employ the same notation and conventions as in Theorem 3.1. As in Proposition 3.1

$$\big\|Q_Jf\big\|_{\dot{K}_v^{\alpha_1,r}}\lesssim 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1)}\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}},\quad J\in\mathbb{N}.$$

Therefore,

$$\big\|f\big\|_{\dot{K}_v^{\alpha_1,r}}\lesssim 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1)}\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}+\sum_{j=J+1}^{\infty}\big\|\mathcal{F}^{-1}\varphi_j*f\big\|_{\dot{K}_v^{\alpha_1,r}},\quad J\in\mathbb{N}.$$

Repeating the same arguments of Theorem 3.1 we obtain the desired estimate. □

Remark 8. Under the same hypothesis of Theorem 3.2, with $1<p<\infty, -\frac{n}{p}<\alpha_3<n-\frac{n}{p}, r=v$ and $\beta=2$, we obtain

$$\begin{aligned}\big\||\cdot|^{\alpha_1}f\big\|_v&\lesssim\big\|f\big\|_{\dot{K}_u^{\alpha_2,v}}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,v}F_2^s}^{\theta}\\&\lesssim\big\||\cdot|^{\alpha_2}f\big\|_u^{1-\theta}\big\|f\big\|_{\dot{k}_{p,s}^{\alpha_3,v}}^{\theta}\\&\lesssim\big\||\cdot|^{\alpha_2}f\big\|_u^{1-\theta}\big\|f\big\|_{\dot{k}_{p,s}^{\alpha_3,p}}^{\theta}\end{aligned}$$

for all $f\in L^u(\mathbb{R}^n,|\cdot|^{\alpha_2u})\cap\dot{k}_{p,s}^{\alpha_3,p}$, because of

$$\dot{K}_u^{\alpha_2,u}\hookrightarrow\dot{K}_u^{\alpha_2,v}\quad\text{and}\quad\dot{k}_{p,s}^{\alpha_3,p}\hookrightarrow\dot{k}_{p,s}^{\alpha_3,v}.$$

In particular, if $s=m\in\mathbb{N}$, we obtain

$$\begin{aligned}\big\||\cdot|^{\alpha_1}f\big\|_v&\lesssim\big\|f\big\|_{\dot{K}_u^{\alpha_2,v}}^{1-\theta}\Big(\sum_{|\beta|\leq m}\Big\|\frac{\partial^\beta f}{\partial^\beta x}\Big\|_{\dot{K}_p^{\alpha_3,v}}\Big)^{\theta}\\&\lesssim\big\||\cdot|^{\alpha_2}f\big\|_u^{1-\theta}\Big(\sum_{|\beta|\leq m}\Big\||\cdot|^{\alpha_3}\frac{\partial^\beta f}{\partial^\beta x}\Big\|_p\Big)^{\theta}\end{aligned}\tag{3.12}$$

for all $f\in L^u(\mathbb{R}^n,|\cdot|^{\alpha_2u})\cap W_p^m(\mathbb{R}^n,|\cdot|^{\alpha_3u})$. As in [44, Theorem 4.6] replace f in (3.12) by $f(\lambda\cdot)$ with $\lambda>0$, the sum $\sum_{|\beta|\leq m}\cdots$ can be replaced by $\sum_{0<|\beta|\leq m}\cdots$.

By Proposition 2.3 and Theorem 3.1/(i) we obtain the following statement.

Theorem 3.3. *Let* $1 < p, \varrho < \infty, 0 < \tau \leq \infty$, $1 < r, v, u < \infty, \sigma \geq 0$, (3.2), (3.3) *and* (3.4) *with* $\alpha_3 < n - \frac{n}{p}$. *Let* $\alpha_1 \leq \alpha_2 \leq \alpha_3$. *There is a constant* $c > 0$ *such that for all* $f \in \dot{K}_u^{\alpha_2,\delta} \cap \dot{k}_{p,s}^{\alpha_3,\delta_1}$,

$$\big\|(-\Delta^{\frac{\sigma}{2}})f\big\|_{\dot{K}_v^{\alpha_1,r}} \leq c\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}^{1-\theta}\big\|(-\Delta^{\frac{s}{2}})f\big\|_{\dot{K}_p^{\alpha_3,\delta_1}}^{\theta}$$

with

$$\delta = \begin{cases} r, & \text{if } \alpha_2 = \alpha_1, \\ \tau, & \text{if } \alpha_2 > \alpha_1. \end{cases} \quad \text{and} \quad \delta_1 = \begin{cases} r, & \text{if } \alpha_3 = \alpha_1, \\ \varrho, & \text{if } \alpha_3 > \alpha_1. \end{cases}$$

Further we study the case when $p \leq v < u$ in Theorem 3.1.

Theorem 3.4. *Let* $0 < p, \tau < \infty, 0 < \beta, \kappa \leq \infty, 1 < r, v < \infty, \sigma \geq 0, 1 < u < \infty$,

$$-\frac{n}{v} < \alpha_1 < n - \frac{n}{v}, \quad -\frac{n}{u} < \alpha_2 < n - \frac{n}{u}, \quad \alpha_3 > -\frac{n}{p},$$

$$s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > \sigma - \frac{n}{v} + \alpha_2 - \alpha_1 + \frac{n}{u} > 0$$

and

$$\sigma - \frac{n}{v} = -(1-\theta)\frac{n}{u} + \theta\Big(s - \frac{n}{p}\Big) + \alpha_1 - ((1-\theta)\alpha_2 + \theta\alpha_3), \quad 0 < \theta < 1.$$

(i) *Let* $p \leq v < u, \alpha_2 - \alpha_1 > \frac{n}{v} - \frac{n}{u}$ *and* $\alpha_3 = \alpha_2$. *There is a constant* $c > 0$ *such that for all* $f \in \dot{K}_u^{\alpha_2,\tau} \cap \dot{K}_p^{\alpha_3,\tau}F_\beta^s$,

$$\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq c\big\|f\big\|_{\dot{K}_u^{\alpha_2,\tau}}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,\tau}F_\beta^s}^{\theta}. \tag{3.13}$$

(ii) *Let* $p \leq v < u, \alpha_2 - \alpha_1 > \frac{n}{v} - \frac{n}{u}$ *and* $\alpha_3 > \alpha_2$. *There is a constant* $c > 0$ *such that* (3.13) *holds for all* $f \in \dot{K}_u^{\alpha_2,\tau} \cap \dot{K}_p^{\alpha_3,\kappa}F_\beta^s$ *with* $\dot{K}_p^{\alpha_3,\kappa}F_\beta^s$ *in place of* $\dot{K}_p^{\alpha_3,\tau}F_\beta^s$.

Proof. Recall that, as in Theorem 3.1, one has the estimate

$$\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq \big\|Q_J f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} + \Big\|\sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f\Big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}, \quad J \in \mathbb{N}.$$

From Proposition 3.1/(ii),

$$\big\|Q_J f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|f\big\|_{\dot{K}_u^{\alpha_2,\tau}},$$

which is possible since

$$\frac{n}{v} + \alpha_1 - \alpha_2 \leq \frac{n}{u} < \frac{n}{v}.$$

Using again embedding (3.8) and Lemma 2.1, we get

$$\begin{aligned}\Big\|\sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f\Big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} &\lesssim \sum_{j=J+1}^{\infty} 2^{j\sigma}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}_v^{\alpha_1,r}} \\ &\lesssim \sum_{j=J+1}^{\infty} 2^{j(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1+\sigma)}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}_p^{\alpha_3,\vartheta}},\end{aligned}$$

where

$$\vartheta = \begin{cases} \tau, & \text{if } \alpha_3 = \alpha_2, \\ \kappa, & \text{if } \alpha_3 > \alpha_2. \end{cases}$$

Therefore, $\big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}}$ can be estimated by

$$
\begin{aligned}
&c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_2,\tau}_u}+2^{J(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1-s+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_3,\vartheta}_p F^s_\beta}\\
&=c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\Big(\big\|f\big\|_{\dot{K}^{\alpha_2,\tau}_u}+2^{J(\frac{n}{p}-\frac{n}{u}-s-\alpha_2+\alpha_3)}\big\|f\big\|_{\dot{K}^{\alpha_3,\vartheta}_p F^s_\beta}\Big),
\end{aligned}
$$

where the positive constant $c>0$ is independent of J. Observe that

$$\dot{K}^{\alpha_3,\vartheta}_p F^s_\beta \hookrightarrow \dot{K}^{\alpha_2,\tau}_u,$$

since $s-\frac{n}{p}+\frac{n}{u}+\alpha_2-\alpha_3>0$. We choose $J\in\mathbb{N}$ such that

$$2^{J(\frac{n}{p}-\frac{n}{u}-s-\alpha_2+\alpha_3)}\approx\big\|f\big\|_{\dot{K}^{\alpha_2,\tau}_u}\big\|f\big\|^{-1}_{\dot{K}^{\alpha_3,\vartheta}_p F^s_\beta},$$

we obtain the desired estimate. □

By combining Theorem 3.2 with Theorem 3.4 we obtain the following statement.

Theorem 3.5. *Under the hypothesis of Theorem* 3.4 *with* $\alpha_1>-\frac{n}{v}$ *and* $\sigma=0$, *the estimates of Theorem* 3.4 *hold with* $\dot{K}^{\alpha_1,r}_v$ *replaced by* $\dot{k}^{\alpha_1,r}_{v,\sigma}$.

Finally we study the case of $v\le\min(p,u)$.

Theorem 3.6. *Let* $1<r<\infty, 0<p,\beta,\tau\le\infty, 1<v\le\min(p,u), \alpha_2-\alpha_1>\frac{n}{v}-\frac{n}{\max(p,u)}, \alpha_3\ge\alpha_2, \sigma\ge 0$,

$$-\frac{n}{v}<\alpha_1<n-\frac{n}{v},\qquad -\frac{n}{u}<\alpha_2<n-\frac{n}{u},\qquad \alpha_3>-\frac{n}{p}$$

and

$$s-\frac{n}{p}+\frac{n}{u}+\alpha_2-\alpha_3>\sigma-\frac{n}{v}+\alpha_2-\alpha_1+\frac{n}{u}>0.$$

Assume that $0<p,\tau<\infty$ *and* $s>\sigma_{p,\beta}$ *in the* $\dot{K}F$*-case. There is a constant* $c>0$ *such that for all* $f\in\dot{K}^{\alpha_2,\tau}_u\cap\dot{K}^{\alpha_3,\tau}_p A^s_\beta$,

$$\big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}}\le c\big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,\tau}_u}\big\|f\big\|^{\theta}_{\dot{K}^{\alpha_3,\tau}_p A^s_\beta}$$

with

$$\sigma-\frac{n}{v}=-(1-\theta)\frac{n}{u}+\theta\Big(s-\frac{n}{p}\Big)+\alpha_1-\big((1-\theta)\alpha_2+\theta\alpha_3\big).$$

Proof. By similarity, we only consider the case of the spaces $\dot{K}^{\alpha_3,\tau}_p B^s_\beta$. We split the proof into two steps.

Step 1. We consider the case $p\le u$. We employ the same notation as in Theorem 3.1. In view of Theorem 3.4 we need only to estimate

$$\Big\|\sum_{j=J+1}^{\infty}\mathcal{F}^{-1}\varphi_j*f\Big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}},\qquad J\in\mathbb{N}.$$

Using embedding (3.8) and Lemma 2.2, we obtain

$$
\begin{aligned}
\Big\|\sum_{j=J+1}^{\infty}\mathcal{F}^{-1}\varphi_j*f\Big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}}&\lesssim\sum_{j=J+1}^{\infty}2^{j\sigma}\big\|\mathcal{F}^{-1}\varphi_j*f\big\|_{\dot{K}^{\alpha_1,r}_v}\\
&\lesssim\sum_{j=J+1}^{\infty}2^{j(\frac{n}{p}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|\mathcal{F}^{-1}\varphi_j*f\big\|_{\dot{K}^{\alpha_3,\tau}_p}.
\end{aligned}
$$

which is possible since

$$\frac{n}{v} + \alpha_1 - \alpha_2 < \frac{n}{p} \le \frac{n}{v}.$$

Repeating the same arguments as in the proof of Theorem 3.1 we obtain the desired estimate.

Step 2. We consider the case $u < p$. Applying a combination of the arguments used in the corresponding step of the proof of Theorem 3.1 and those used in the first step above, we arrive at the desired estimate. □

Similarly we obtain the following statement.

Theorem 3.7. *Under the hypothesis of Theorem 3.6 with $\sigma = 0$, we have*

$$\|f\|_{\dot{K}_v^{\alpha_1,r}} \lesssim \|f\|_{\dot{K}_u^{\alpha_2,\tau}}^{1-\theta} \|f\|_{\dot{K}_p^{\alpha_3,\tau} A_\varrho^s}^{\theta}$$

for all $f \in \dot{K}_u^{\alpha_2,\tau} \cap \dot{K}_p^{\alpha_2,\tau} A_\varrho^s$.

Remark 9. Under the same hypothesis of Theorems 3.5 and 3.7, with $r = v$, $\sigma = 0$, $\tau = \max(u, p)$ and $\beta = 2$, we, to a certain extent, improve Caffarelli-Kohn-Nirenberg inequality (1.1).

3.2 CKN inequalities in Besov-Morrey and Triebel-Lizorkin-Morrey spaces

In this section, we investigate the Caffarelli, Kohn and Nirenberg inequalities in $\mathcal{E}_{p,q,u}^s$ and $\mathcal{N}_{p,q,u}^s$ spaces. The main results of this section are based on the following statement.

Lemma 3.1. *Let $1 < u \le p < \infty, 1 < s \le q < \infty$ and $R > 0$.*

(i) *Assume that $1 \le v \le u$. There exists a constant $c > 0$ independent of R such that for all $f \in M_v^{\frac{v}{u}p} \cap M_s^q$ with* supp $\mathcal{F}f \subset \{\xi : |\xi| \le R\}$, *we have*

$$\|f\|_{M_u^p} \le cR^{\frac{n}{q} - \frac{vn}{qu}} \|f\|_{M_s^q}^{1-\frac{v}{u}} \|f\|_{M_v^{\frac{v}{u}p}}^{\frac{v}{u}}.$$

(ii) *Assume that $\frac{u}{p} \le \frac{s}{q}$ and $q \le p$. There exists a constant $c > 0$ independent of R such that for all $f \in M_s^q$ with* supp $\mathcal{F}f \subset \{\xi : |\xi| \le R\}$, *we have*

$$\|f\|_{M_u^p} \le cR^{\frac{n}{q} - \frac{n}{p}} \|f\|_{M_s^q}.$$

Proof. We split the proof in two steps.

Step 1. We will prove (i). Let B be a ball in $\mathbb{R}^n$. Then

$$\big\| |B|^{\frac{1}{p} - \frac{1}{u}} f\chi_B \big\|_u^u = u \int_0^\infty t^{u-1} |\{x \in B : |f(x)|\, |B|^{\frac{1}{p} - \frac{1}{u}} > t\}| dt < \infty.$$

We have

$$|f(x)| \le cR^{\frac{n}{q}} \|f\|_{M_s^q}, \quad x \in \mathbb{R}^n,$$

see [36, Proposition 2.1] where $c > 0$ independent of R. Let $p_0 = \frac{v}{u}$. Clearly

$$\begin{aligned} |f(x)| &= |f(x)|^{p_0} |f(x)|^{1-p_0} \\ &\lesssim |f(x)|^{p_0} \big(R^{\frac{n}{q}} \|f\|_{M_s^q}\big)^{1-p_0} \\ &= c|f(x)|^{p_0} d^{1-p_0}, \end{aligned}$$

which yields that

$$\big\| |B|^{\frac{1}{p}-\frac{1}{u}} f\chi_B\big\|_u^u \le u\int_0^\infty t^{u-1}|\{x\in B: |f(x)|\,|B|^{\frac{1}{pp_0}-\frac{1}{v}} > cd^{1-\frac{1}{p_0}}t^{\frac{1}{p_0}}\}|dt$$

$$= cud^{u-v}\int_0^\infty \lambda^{v-1}|\{x\in B: |f(x)|\,|B|^{\frac{u}{pv}-\frac{1}{v}} > \lambda\}|d\lambda,$$

after the change the variable $\lambda^{p_0}c^{-p_0}d^{1-p_0}=t$. The last expression is clearly bounded by

$$cd^{u-v}\big\|f\big\|^v_{M_v^{\frac{pv}{u}}} \le cR^{n\frac{u-v}{q}}\big\|f\big\|^v_{M_v^{\frac{pv}{u}}}\big\|f\big\|^{u-v}_{M_s^q}.$$

Step 2. We will prove (ii). If $p=q$, then $u\le s$ and the estimate follows by Hölder's inequality. Assume that $q<p$ and we choose $v>0$ such that $\max(1,\frac{qu}{p})<v\le u<\frac{pu}{q}$. By Step 1

$$R^{\frac{n}{q}-\frac{vn}{qu}}\big\|f\big\|^{\frac{v}{u}}_{M_v^{\frac{v}{u}p}} = R^{\frac{n}{q}-\frac{n}{p}}\big\|R^{\frac{nu}{pv}-\frac{n}{q}}f\big\|^{\frac{v}{u}}_{M_v^{\frac{v}{u}p}}.$$

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a partition of the unity. Observe that

$$\mathcal{F}^{-1}\varphi_j * f = 0 \quad \text{if} \quad R<2^{j-1}, \quad j\in\mathbb{N}_0.$$

This observation together with (2.25) yield

$$\big\|R^{\frac{nu}{pv}-\frac{n}{q}}f\big\|_{M_v^{\frac{v}{u}p}} \approx \Big\|\Big(\sum_{j\in\mathbb{N}_0,2^{j-1}\le R}^{\infty} R^{\frac{2nu}{pv}-\frac{2n}{q}}\left|\mathcal{F}^{-1}\varphi_j * f\right|^2\Big)^{1/2}\Big\|_{M_v^{\frac{v}{u}p}}$$

$$\lesssim \big\|f\big\|_{\mathcal{E}^{\frac{nu}{pv}-\frac{n}{q}}_{\frac{v}{u}p,2,v}} \lesssim \big\|f\big\|_{M_s^q},$$

which follows by Sobolev embedding, see Theorem 2.6,

$$M_s^q = \mathcal{E}^0_{q,2,s} \hookrightarrow \mathcal{E}^{\frac{nu}{pv}-\frac{n}{q}}_{\frac{v}{u}p,2,v},$$

since

$$-\frac{n}{q} = \frac{nu}{pv}-\frac{n}{q}-\frac{nu}{pv}, \quad q<\frac{vp}{u} \quad \text{and} \quad \frac{u}{p}\le\frac{s}{q}.$$

□

Proposition 3.2. *Let $1<u\le p<\infty, 1<q<\infty$ and $s>0$.*

(i) *Let $f\in\mathcal{N}^s_{p,q,u}$. Then*

$$\big\|f\big\|_{\mathcal{N}^s_{p,q,u}} \approx \big\|f\big\|_{M_u^p} + \big\|f\big\|_{\dot{\mathcal{N}}^s_{p,q,u}}, \tag{3.14}$$

where

$$\big\|f\big\|_{\dot{\mathcal{N}}^s_{p,q,u}} = \Big\|\Big(\sum_{j=-\infty}^{\infty} 2^{qjs}\left|\mathcal{F}^{-1}\varphi^j * f\right|^q\Big)^{1/q}\Big\|_{M_u^p}.$$

(ii) *Let $f\in\mathcal{E}^s_{p,q,u}$. Then*

$$\big\|f\big\|_{\mathcal{E}^s_{p,q,u}} \approx \big\|f\big\|_{M_u^p} + \big\|f\big\|_{\dot{\mathcal{E}}^s_{p,q,u}} \tag{3.15}$$

where

$$\big\|f\big\|_{\dot{\mathcal{E}}^s_{p,q,u}} = \Big\|\Big(\sum_{j=-\infty}^{\infty} 2^{qjs}\left|\mathcal{F}^{-1}\varphi^j * f\right|^q\Big)^{1/q}\Big\|_{M_u^p}.$$

Proof. By similarity, we prove only (ii). We have as in the proof of Proposition 2.1 that

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,u}} \lesssim \|f\|_{\mathcal{E}^s_{p,q,u}}.$$

The only distinction of the proof of Proposition 2.1 is the fact that we use [41, Lemma 2.5]. Since $s>0$ we observe

$$\|f\|_{M^p_u} \approx \|f\|_{\mathcal{E}^0_{p,2,u}} \lesssim \|f\|_{\mathcal{E}^s_{p,q,u}}.$$

Now we prove the opposite inequality. Obviously $\|\mathcal{F}^{-1}\varphi_0 * f\|_{M^p_u}$ can be estimated from above by $\|f\|_{M^p_u}$. ☐

Theorem 3.8. *Let $1<u\le p<\infty$ and $1<v\le q<\infty$. Assume that $\frac{u}{p}\le\frac{v}{q}, q\le p$ and $\sigma\ge 0$. Then for all $f\in M^q_v$ and all $J\in\mathbb{N}$,*

$$\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \le c2^{Jn(\frac{1}{q}-\frac{1}{p})+\sigma}\|f\|_{M^q_v},$$

where c is a positive constant independent of f and J.

Proof. Let $\sigma=\theta m+(1-\theta)0$, $\alpha\in\mathbb{N}^n$ with $0<\theta<1$ and $|\alpha|\le m$. We have

$$\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \le \|Q_J f\|^{1-\theta}_{\mathcal{E}^0_{p,2,u}}\|Q_J f\|^{\theta}_{\mathcal{E}^m_{p,2,u}}.$$

Observe that

$$\mathcal{E}^m_{p,2,u}=M^{m,p}_u \quad\text{and}\quad \mathcal{E}^0_{p,2,u}=M^p_u,$$

which yield that

$$\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \le \|Q_J f\|^{1-\theta}_{M^p_u}\|Q_J f\|^{\theta}_{M^{m,p}_u},$$

where the positive constant c is independent of J. Lemma 3.1 yields that

$$\|D^\alpha(Q_J f)\|_{M^p_u} \lesssim 2^{Jn(\frac{1}{q}-\frac{1}{p})+|\alpha|}\|f\|_{M^q_v}.$$

Therefore,

$$\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \lesssim 2^{Jn(\frac{1}{q}-\frac{1}{p})+\sigma}\|f\|_{M^q_v}.$$

☐

Now we are in position to state the main result of this section.

Theorem 3.9. *Let $1<u\le p<\infty, 1<\mu\le\delta<\infty, 1<\beta<\infty$, $\sigma\ge 0$ and $1<v\le q<\infty$. Assume that*

$$\frac{u}{p}\le\frac{\mu}{\delta}\le\frac{v}{q},\quad s>0\quad\text{and}\quad p\ge\delta\ge q.$$

Let

$$s-\frac{n}{q}>\sigma-\frac{n}{p}\quad\text{and}\quad \sigma-\frac{n}{p}=-(1-\theta)\frac{n}{\delta}+\theta\Big(s-\frac{n}{q}\Big),\quad 0<\theta<1.$$

Then

$$\|f\|_{\dot{\mathcal{E}}^\sigma_{p,2,u}} \lesssim \|f\|^{1-\theta}_{M^\delta_\mu}\|f\|^{\theta}_{\dot{\mathcal{N}}^s_{q,\beta,v}},\quad \sigma>0 \tag{3.16}$$

and

$$\|f\|_{M^p_u} \lesssim \|f\|^{1-\theta}_{M^\delta_\mu}\|f\|^{\theta}_{\dot{\mathcal{N}}^s_{q,\beta,v}} \tag{3.17}$$

for all $f\in M^\delta_\mu\cap\mathcal{N}^s_{q,\beta,v}$.

Proof. We have

$$f = Q_J f + \sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f, \quad J \in \mathbb{N}.$$

Hence

$$\big\|f\big\|_{\mathcal{E}^{\sigma}_{p,2,u}} \le \big\|Q_J f\big\|_{\mathcal{E}^{\sigma}_{p,2,u}} + \Big\|\sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f\Big\|_{\mathcal{E}^{\sigma}_{p,2,u}}. \tag{3.18}$$

By applying Theorem 3.8, it follows that

$$\big\|Q_J f\big\|_{\mathcal{E}^{\sigma}_{p,2,u}} \lesssim 2^{Jn(\frac{1}{\delta}-\frac{1}{p})+\sigma J}\big\|f\big\|_{M^{\delta}_{\mu}}.$$

From the embedding $\mathcal{N}^{\sigma}_{p,1,u} \hookrightarrow \mathcal{N}^{\sigma}_{p,\min(2,u),u} \hookrightarrow \mathcal{E}^{\sigma}_{p,2,u}$ and Lemma 3.1 the last term in (3.18) can be estimated by

$$\begin{aligned} c\sum_{j=J+1}^{\infty} 2^{j\sigma}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{M^{p}_{u}} &\lesssim \sum_{j=J+1}^{\infty} 2^{jn(\frac{1}{q}-\frac{1}{p})+j\sigma}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{M^{q}_{v}} \\ &\lesssim 2^{J(\frac{n}{q}-\frac{n}{p}+\sigma-s)}\big\|f\big\|_{\mathcal{N}^{s}_{q,\infty,v}}, \end{aligned}$$

since $s-\frac{n}{q} > \sigma-\frac{n}{p}$. Therefore,

$$\begin{aligned} \big\|f\big\|_{\mathcal{E}^{\sigma}_{p,2,u}} &\le c2^{J(\frac{n}{\delta}-\frac{n}{p})+\sigma J}\big\|f\big\|_{M^{\delta}_{\mu}} + 2^{J(\frac{n}{q}-\frac{n}{p}+\sigma-s)}\big\|f\big\|_{\mathcal{N}^{s}_{q,\infty,v}} \\ &= c2^{J(\frac{n}{\delta}-\frac{n}{p})+\sigma J}\left(\big\|f\big\|_{M^{\delta}_{\mu}} + 2^{J(\frac{n}{q}-\frac{n}{\delta}-s)}\big\|f\big\|_{\mathcal{N}^{s}_{q,\infty,v}}\right), \end{aligned}$$

where the positive constant c is independent of J. We wish to choose $J \in \mathbb{N}$ such that

$$\big\|f\big\|_{M^{\delta}_{\mu}} \approx 2^{J(\frac{n}{q}-\frac{n}{\delta}-s)}\big\|f\big\|_{\mathcal{N}^{s}_{q,\infty,v}},$$

which is possible since $\mathcal{N}^{s}_{q,\infty,v} \hookrightarrow M^{\delta}_{\mu}$. Indeed, from Theorem 2.6 and (2.25), we get

$$\mathcal{N}^{s}_{q,\infty,v} \hookrightarrow \mathcal{E}^{s}_{q,\infty,v} \hookrightarrow \mathcal{E}^{0}_{\delta,2,\mu} = M^{\delta}_{\mu},$$

becuase of $s-\frac{n}{q} > \sigma-\frac{n}{p} \ge -\frac{n}{\delta}$. Thus

$$\big\|f\big\|_{\mathcal{E}^{\sigma}_{p,2,u}} \lesssim \big\|f\big\|^{1-\theta}_{M^{\delta}_{\mu}}\big\|f\big\|^{\theta}_{\mathcal{N}^{s}_{q,\infty,v}}.$$

Using (3.14) and (3.15) we arrive at the inequality

$$\big\|f\big\|_{\dot{\mathcal{E}}^{\sigma}_{p,2,u}} \lesssim \big\|f\big\|^{1-\theta}_{M^{\delta}_{\mu}}\left(\big\|f\big\|_{M^{q}_{v}} + \big\|f\big\|_{\dot{\mathcal{N}}^{s}_{q,\infty,v}}\right)^{\theta}.$$

In this estimate replacing $f(\cdot)$ by $f(\lambda\cdot)$ and using (3.14) we obtain

$$\big\|f\big\|_{\dot{\mathcal{E}}^{\sigma}_{p,2,u}} \lesssim \big\|f\big\|^{1-\theta}_{M^{\delta}_{\mu}}\left(\lambda^{-s}\big\|f\big\|_{M^{q}_{v}} + \big\|f\big\|_{\dot{\mathcal{N}}^{s}_{q,\infty,v}}\right)^{\theta}.$$

Taking λ sufficiently large we obtain (3.16)-(3.17). □

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THREE WEIGHT HARDY INEQUALITY ON MEASURE TOPOLOGICAL SPACES

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Abstract. For the Hardy inequality to hold on a Hausdorff topological space, we obtain necessary and sufficient conditions on the weights and measures. As in the recent paper by G. Sinnamon (2022), we assume total orderedness of the family of sets that generate the Hardy operator. Sinnamon's method consists in the reduction of the problem to an equivalent one-dimensional problem. We provide a different, direct proof which develops the approach suggested by D. Prokhorov (2006) in the one-dimensional case.

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1 Introduction

Consider the inequality

$$\left[\int_{[a,b]}\left(\int_{[a,x]} fud\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \le C\left(\int_{[a,b]} f^p wd\nu\right)^{1/p} \tag{1.1}$$

for all non-negative functions f. Here $[a,b]$ is a finite or infinite segment on the extended real line, u,v and w are non-negative measurable weight functions and λ,μ,ν are Borel measures. The problem is to find a functional of the weights and measures $\Phi(u,v,w,\lambda,\mu,\nu)$ such that for the best constant C one has

$$c_1\Phi \le C \le c_2\Phi, \tag{1.2}$$

where the positive constants c_1, c_2 do not depend on the weights and measures. The characterizations of weights and measures for which (1.1) holds are very different for the cases $p \le q$ and $q < p$. In particular, the proofs for the case $p \le q$ are a lot simpler. The inequality has a long history described in several books [6], [7], [8], [10].

In [3] and [21] spherical coordinates in R^n were used to obtain the first results for the Euclidean space. Other multidimensional generalizations followed. Results for Banach function spaces and mixed L^p spaces given in [4] and [1] covered only the case $p \le q$, when specified to usual L^p spaces. The two-dimensional result by Sawyer [19] turned out to be difficult to generalize to higher dimensions, unless under additional restrictions on the weights [23]. We believe this is caused by the fact that his domains are not totally ordered, see the definition below.

In the last several years there was a wave of new generalizations. In [16], [17] and [18] the results have been formulated in abstract settings (for homogeneous groups, hyperbolic spaces, Cartan-Hadamard manifolds, and connected Lie groups). All of them are based on the assumption of the existence of a polar decomposition, which for calculational purposes is the same as spherical coordinates. Thus, methodologically, the last three papers return to [3].

G. Sinnamon [22] made a significant contribution by providing a single framework for all Hardy inequalities, regardless of the domain dimension and covering both continuous and discrete cases. His method consists in reducing the general Hardy inequality to a special, one-dimensional one, called a normal form. In addition to being universally applicable, this approach has other advantages. The functional Φ for the normal form is relatively simple, because the weights are constant and only the upper limit of integration changes. (Note that in general there exist many functionals satisfying (1.1), see [8]). This simplicity allows Sinnamon to improve the best constants c_1, c_2 in (1.2) due to Hardy and Bliss [2].

The reduction to the one-dimensional case in Sinnamon's approach requires an additional calculation to obtain the functional in terms of the original weights and measures. The present paper is different in that we give a direct proof leading to the required expressions. See Remark 1 below for a more detailed comparison.

Now we mention the contributions that directly influenced the methods employed here. Everywhere we assume that $1 < p < \infty$, $0 < q < \infty$.

For the case $q < p$ several functionals equivalent to (1.1) have been suggested. The one proposed by Maz'ya and Rozin [10] and used here has the advantage that it works both for $0 < q < 1$ and $1 \leq q < p$.

D. Prokhorov [12] investigated the Hardy inequality on the real line but the merits of his measure-theoretical analysis go beyond the one-dimensional case. We follow his ideas and along the way mention some of his innovations. One of them is that he allowed the weights to be infinite on sets of positive measure and analyzed the implications.

The purpose of this paper is to obtain a criterion for the multidimensional inequality

$$\begin{aligned} & \left[\int_\Omega \left(\int_{\{y\in\Omega:\tau(y)\leq\tau(x)\}} f(y)\, u(y)\, d\lambda(y)\right)^q v(x) d\mu(x)\right]^{1/q} \\ \leq \; & C\left(\int_\Omega f^p w d\nu\right)^{1/p} \end{aligned} \tag{1.3}$$

(the function τ is defined in Section 2), Ω is an open set in a Hausdorff topological space X. The main restriction on the open subsets $\Omega(t)$ of Ω is that they are parameterized by real t and satisfy the monotonicity (total orderedness) condition

$$\Omega(t_1) \subset \Omega(t_2) \text{ if } t_1 < t_2. \tag{1.4}$$

Alternatively, instead of expanding, $\Omega(t)$ may be contracting but the unidirectionality is required for our method. As in Sinnamon's paper, the results can be called dimension-agnostic, because in X the dimension notion is generally not defined, and when X is a linear space, no convexity or connectedness are imposed on $\Omega(t)$ or Ω. The existing results for R^n or measure metric spaces from [1], [3], [16], [17], [18] are special cases of ours. Results of [19] (where rectangles do not satisfy the monotonicity condition) are not covered by ours. In [4] domains of integration are more general than ours and Banach function spaces are considered.

In the multidimensional case generalizations of our results in several directions are possible. For Hardy type integrals with variable kernels extensions can be obtained under the Oinarov condition [9], [11]. The generality of measures in our results may lead to their consequences for discrete problems [24] in the spirit of Sinnamon. It would be interesting to cover the Riemann–Liouville operators [13], although the lack of the derivative notion certainly makes unlikely generalizations of the results in [5].

2 Main assumptions and statements

Description of measures. The phrase "μ is a measure on Ω" means that there is a σ-algebra $\mathfrak{M}$ that contains the σ-algebra $\mathfrak{B}$ of Borel subsets of Ω and such that μ is a σ-finite and σ-additive (non-negative) function on $\mathfrak{M}$, with values in the extended real half-line $[0,+\infty]=\{0\le x\le+\infty\}$. $\mathfrak{M}_\mu$ denotes the domain of the measure μ. Everywhere λ,μ,ν are measures on Ω and λ,ν have a common domain $\mathfrak{M}_{\lambda,\nu}$.

Description of functions. The notation $f\in\{\mathfrak{M}\}^+$ means that f is defined in Ω, takes values in $[0,+\infty]$ and is $\mathfrak{M}$-measurable. The weights u,v,w satisfy $u,w\in\{\mathfrak{M}_{\lambda,\nu}\}^+$, $v\in\{\mathfrak{M}_\mu\}^+$.

Description of sets $\Omega(t)$. Ω is an open set in a Hausdorff topological space X and $\{\Omega(t):t\in[a,b]\}$, $-\infty\le a<b\le\infty$, is a one-parameter family of open subsets of Ω that satisfy monotonicity (1.4) for $a\le t_1<t_2\le b$, start at the empty set and eventually cover λ-almost all Ω :

$$\Omega(a)=\bigcap_{t>a}\Omega(t)=\varnothing,\ \lambda\left(\Omega\backslash\bigcup_{a<t<b}\Omega(t)\right)=0.$$

Let $\omega(t)=\overline{\Omega(t)}\cap\overline{(\Omega\backslash\Omega(t))}$ be the boundary of $\Omega(t)$ in the relative topology. We require the boundaries to be disjoint and cover λ-almost all Ω :

$$\omega(t_1)\cap\omega(t_2)=\varnothing,\ t_1\ne t_2,\ t_1,t_2\in(a,b);\ \lambda\left(\Omega\backslash\bigcup_{a<t<b}\omega(t)\right)=0.$$

The last condition implies that, up to a set of λ-measure zero, for each $x\in\Omega$ there exists a unique $\tau(x)\in(a,b)$ such that $x\in\omega(\tau(x))$, which allows us to define a Hardy type operator

$$Tf(x)=\int_{\{y\in\Omega:\tau(y)\le\tau(x)\}}f\left(y\right)d\lambda\left(y\right).$$

More generally, for any $E\subset[a,b]$ such that $\Omega\left(E\right)\equiv\cup_{t\in E}\omega(t)\in\mathfrak{M}_\lambda$ we can consider the integral

$$\int_{\Omega(E)}fd\lambda.$$

In particular, we denote $\Omega\left[c,d\right]=\cup_{c\le t\le d}\omega(t)$, $\Omega[c,d)=\cup_{c\le t<d}\omega(t)$, etc. for $a\le c<d\le b$.

Remark 1. 1) In comparison with [22], our setup is closer to the classical one, where the domains $\Omega\left(t\right)=(0,t)$ are indexed by their boundaries $\omega\left(t\right)=t$ and can be represented as unions of boundaries $\Omega\left(t\right)=\{s:\omega\left(s\right)<\omega\left(t\right)\}$. 2) Let (S,Σ,λ) and (Y,μ) be σ-finite measure spaces and let $B:Y\to\Sigma$ be a map such that the range of B is a totally ordered subset of Σ (these are assumptions from [22]). Thus the family $\{B\left(y\right):y\in Y\}$ of subsets of Σ is indexed by $y\in Y$. Since the number $t=\lambda\left(B\left(y\right)\right)$ is unique for each $y\in Y$, we can write $\widetilde{B}(t)=B\left(y\right)$ if $t=\lambda\left(B\left(y\right)\right)$. This re-indexing is one-to-one if, as usual, we do not distinguish between two sets which differ by a set of measure zero. Hence the family $\{B\left(y\right):y\in Y\}$ can be indexed by elements from $[0,\infty)$, the total order being preserved. 3) As we mentioned in the Introduction, Sinnamon's result covers more different cases. On the other hand, our proofs are direct (they don't rely on the one-dimensional case as an intermediate step) and we have statements stemming from the assumption that weights may take values in the extended half-axis $[0,\infty]$ (see Lemmas 2.7 and 2.8); Sinnamon does not have them. Our method generalizes [12].

Conventions on improper numbers. $0+(+\infty)=a+(+\infty)=a\cdot(+\infty)=+\infty$ if $0<a\le+\infty$; $0\cdot(+\infty)=0$; $(+\infty)^\alpha=0^{-\alpha}=+\infty$, $(+\infty)^{-\alpha}=0^\alpha=0$, $\alpha\in(0,+\infty)$.

Lowercase c, with or without subscripts, denote constants that do not depend on weights and measures.

The few results in dimensions higher than 1, reviewed in the Introduction, excluding [22], employed tools of one-dimensional analysis along the radial variable, of type

$$\int_a^b \left(\int_a^x f\right)^\gamma f(x)dx = \frac{1}{\gamma+1}\int_a^b \frac{d}{dx}\left(\int_a^x f\right)^{\gamma+1} dx = \frac{1}{\gamma+1}\left(\int_a^b f\right)^{\gamma+1},$$

where dx is the Lebesgue measure. Further advancement has been held back by the lack of a truly multidimensional replacement of such tools. The significance of the following Lemmas 2.1 and 2.2 is that they are such a replacement. See [12] for the argument on the straight line.

Lemma 2.1. *Denote* $\Lambda_f(t) = \int_{\Omega[a,t]} f d\lambda$, $a \le t \le b$, $f \in \{\mathfrak{M}_\lambda\}^+$.

a) If $\gamma > 0$, *then*

$$\frac{\Lambda_f(b)^{\gamma+1}}{\max\{1,\gamma+1\}} \le \int_{\Omega[a,b]} f(x)\Lambda_f(\tau(x))^\gamma d\lambda(x) \le \frac{\Lambda_f(b)^{\gamma+1}}{\min\{1,\gamma+1\}}. \tag{2.1}$$

b) In the case $\gamma \in (-1,0)$, (2.1) *holds if* $\Lambda_f(b) < \infty$.

Proof. a) Let $\gamma > 0$. The second inequality in (2.1) follows from $\Lambda_f(\tau(x)) \le \Lambda_f(b)$, $x \in \Omega$. Let us prove the first inequality. Without loss of generality we assume that

$$\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma d\lambda(x) < \infty.$$

Then for any $t \in [a,b]$

$$\begin{aligned}\left(\int_{\omega(t)} f d\lambda\right)^{\gamma+1} &\le \int_{\omega(t)} f d\lambda \left(\int_{\Omega[a,t]} f d\lambda\right)^\gamma \le \int_{\Omega[t,b]} f(x)\Lambda_f(\tau(x))^\gamma d\lambda(x)\\ &\le \int_\Omega f(x)\Lambda_f(\tau(x))^\gamma d\lambda(x) < \infty,\end{aligned}$$

so

$$\int_{\omega(t)} f d\lambda < \infty \text{ for any } t \in [a,b]. \tag{2.2}$$

Suppose $\Lambda_f(b) = \infty$. Denote

$$E = \{t \in [a,b) : \Lambda_f(t) = \infty\}, \quad e = \begin{cases} \inf E, & \text{if } E \ne \varnothing;\\ b, & \text{if } E = \varnothing.\end{cases}$$

If there is $\xi \in (e,b]$ such that $\int_{\Omega(\xi,b]} f d\lambda \ne 0$, then

$$\infty > \int_\Omega f(x)\Lambda_f(\tau(x))^\gamma d\lambda(x) \ge \Lambda_f(\xi)^\gamma \int_{\Omega(\xi,b]} f d\lambda = \infty,$$

which is impossible. Hence, for any $\xi \in (e,b]$ one has $\int_{\Omega(\xi,b]} f d\lambda = 0$. By the monotone covergence theorem $\int_{\Omega(e,b]} f d\lambda = 0$ and thus $\Lambda_f(e) = \infty$. Recalling (2.2) we see that $e > a$ and $\int_{\Omega[a,e)} f d\lambda = \infty$. By the definition of e,

$$\infty = \int_{\Omega[a,e)} f d\lambda = \int_{\Omega[a,t]} f d\lambda + \int_{\Omega(t,e)} f d\lambda$$

for $t\in[a,e)$, where $\Lambda_f(t)\in(0,\infty)$. Then, again by the definition of e,

$$\infty>\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma d\lambda(x)\geq\Lambda_f(t)^\gamma\int_{\Omega(t,e)}fd\lambda=\infty.$$

The contradiction arises from the assumption $\Lambda_f(b)=\infty$, so without loss of generality we can suppose that $\Lambda_f(b)<\infty$.

Changing the integration order gives

$$\begin{aligned}\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma d\lambda(x) &= \gamma\int_\Omega f(x)\left(\int_0^{\Lambda_f(\tau(x))}s^{\gamma-1}ds\right)d\lambda(x)\\ &= \gamma\int_0^{\Lambda_f(b)}s^{\gamma-1}\left(\int_\Omega f(x)\chi_{[0,\Lambda_f(\tau(x))]}(s)\,d\lambda(x)\right)ds.\end{aligned}$$

For $s\geq 0$ put $E_s=\{t\in[a,b]:\Lambda_f(t)<s\}$. If $E_s=\varnothing$, then $\Lambda_f(\tau(x))\geq s$ for any $x\in\Omega$ and

$$\int_\Omega f(x)\chi_{[0,\Lambda_f(\tau(x))]}(s)\,d\lambda(x)=\Lambda_f(b)\geq\Lambda_f(b)-s.$$

Suppose $E_s\neq\varnothing$ and let $e_s=\sup E_s$. Take a sequence $\left\{t_n^{(s)}\right\}\subset E_s$ such that $t_n^{(s)}\uparrow e_s$ as $n\to\infty$. Then in the case $e_s\in E_s$ we have $\Lambda_f(\tau(x))<s$ for $\tau(x)\leq e_s$, $\Lambda_f(\tau(x))\geq s$ for $\tau(x)>e_s$ and

$$\int_\Omega f(x)\chi_{[0,\Lambda_f(\tau(x))]}(s)\,d\lambda(x)=\int_{\Omega(e_s,b]}fd\lambda=\Lambda_f(b)-\Lambda_f(e_s)\geq\Lambda_f(b)-s,$$

while in the case $e_s\notin E_s$

$$\int_\Omega f(x)\chi_{[0,\Lambda_f(\tau(x))]}(s)\,d\lambda(x)=\int_{\Omega[e_s,b]}fd\lambda=\Lambda_f(b)-\lim_{n\to\infty}\Lambda_f\left(t_n^{(s)}\right)\geq\Lambda_f(b)-s.$$

So, summarizing,

$$\begin{aligned}\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma d\lambda(x) &\geq \gamma\int_0^{\Lambda_f(b)}s^{\gamma-1}(\Lambda_f(b)-s)\,ds\\ &= \Lambda_f(b)^{\gamma+1}-\frac{\gamma}{\gamma+1}\Lambda_f(b)^{\gamma+1}=\frac{\Lambda_f(b)^{\gamma+1}}{\gamma+1},\end{aligned}$$

which completes the argument for $\gamma>0$.

b) Now let $\gamma\in(-1,0)$ and $\Lambda_f(b)<\infty$. Then the first inequality in (2.1) follows from $\Lambda_f(b)^\gamma\leq\Lambda_f(\tau(x))^\gamma$, $x\in\Omega$. Let us prove the second inequality. Start with

$$\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma d\lambda(x)=-\gamma\int_\Omega f(x)\left(\int_{\Lambda_f(\tau(x))}^\infty s^{\gamma-1}ds\right)d\lambda(x)$$

$$=-\gamma\int_\Omega f(x)\left(\int_0^{\Lambda_f(b)}s^{\gamma-1}\chi_{[\Lambda_f(\tau(x)),\Lambda_f(b)]}(s)\,ds+\int_{\Lambda_f(b)}^\infty s^{\gamma-1}ds\right)d\lambda(x)$$

$$=-\gamma\int_0^{\Lambda_f(b)}s^{\gamma-1}\left(\int_\Omega f(x)\chi_{[\Lambda_f(\tau(x)),\Lambda_f(b)]}(s)\,d\lambda(x)\right)ds+\Lambda_f(b)^{\gamma+1}.$$

For $s\geq 0$ define $E_s=\{t\in[a,b]:\Lambda_f(t)\leq s\}$. In case $E_s=\varnothing$ we have $\Lambda_f(\tau(x))>s$ for all $x\in\Omega$ and

$$\int_\Omega f(x)\chi_{[\Lambda_f(\tau(x)),\Lambda_f(b)]}(s)\,d\lambda(x)=0\leq s.$$

Suppose $E_s \neq \varnothing$, let $e_s = \sup E_s$ and take a sequence $\left\{t_n^{(s)}\right\} \subset E_s$ such that $t_n^{(s)} \uparrow e_s$ as $n \to \infty$. Then in case $e_s \in E_s$ we have $e_s \leq b$, $\Lambda_f(\tau(x)) \leq \Lambda_f(e_s) \leq s$ for $\tau(x) \leq e_s$ and $\Lambda_f(\tau(x)) > s$ for $\tau(x) > e_s$, so that

$$\int_\Omega f(x)\chi_{\left[\Lambda_f(\tau(x)),\Lambda_f(b)\right]}(s)\,d\lambda(x) = \int_{\Omega[a,e_s]} f d\lambda = \Lambda_f(e_s) \leq s.$$

On the other hand, in the case $e_s \notin E_s$

$$\int_\Omega f(x)\chi_{\left[\Lambda_f(\tau(x)),\Lambda_f(b)\right]}(s)\,d\lambda(x) = \int_{\Omega[a,e_s)} f d\lambda = \lim_{n\to\infty} \Lambda_f\left(t_n^{(s)}\right) \leq s.$$

As a result,

$$\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma\,d\lambda(x) \leq -\gamma\int_0^{\Lambda_f(b)} s^\gamma ds + \Lambda_f(b)^{\gamma+1} = \frac{\Lambda_f(b)^{\gamma+1}}{\gamma+1}.$$

□

A similar statement holds for the integral with a variable lower limit of integration.

Lemma 2.2. *Let* $\bar\Lambda_f(t) = \int_{\Omega[t,b]} f d\lambda$. *a) If* $\gamma > 0$, *then*

$$\frac{\bar\Lambda_f(a)^{\gamma+1}}{\max\{1,\gamma+1\}} \leq \int_{\Omega[a,b]} f(x)\bar\Lambda_f(\tau(x))^\gamma\,d\lambda(x) \leq \frac{\bar\Lambda_f(a)^{\gamma+1}}{\min\{1,\gamma+1\}}. \tag{2.3}$$

b) For $\gamma \in (-1,0)$ (2.3) *holds if* $\bar\Lambda_f(a) < \infty$.

The proof of the next lemma can be found in [15] (it is dimensionless).

Lemma 2.3. *Let* $1 < p < \infty$, $u \in \{\mathfrak{M}_\lambda\}^+$, $E \in \mathfrak{M}_\lambda$. *If* $\int_E u^{p'} d\lambda = \infty$, *then there exists* $f \in \{\mathfrak{M}_\lambda\}^+$ *such that* $\int_E f^p d\lambda < \infty$ *and* $\int_E f u d\lambda = \infty$.

Lemma 2.4. *a) Let* $E \subset [a,b]$ *be such that* $\Omega(E) \in \mathfrak{M}_\lambda$. *Define* $E_t = E \cap [a,t]$, $\bar E_t = E \cap [t,b]$, $t \in E$. *If* $\lambda(\Omega(E_t)) = 0$ *for any* $t \in E$ *or* $\lambda\left(\Omega\left(\bar E_t\right)\right) = 0$ *for any* $t \in E$, *then* $\lambda(\Omega(E)) = 0$.

b) Alternative formulation. Take a set $E \subset \Omega$ *that belongs to* $\mathfrak{M}_\lambda$ *and define* $E_y = E \cap \Omega[a,\tau(y)]$, $\bar E_y = E \cap \Omega[\tau(y),b]$. *If* $\lambda(E_y) = 0$ *for any* $y \in E$ *or* $\lambda\left(\bar E_y\right) = 0$ *for any* $y \in E$, *then* $\lambda(E) = 0$.

Proof. If E is empty, the statement is obvious. Let $E \neq \varnothing$, put $s = \sup E$ and take a sequence $\{s_n\}$ such that $s_n \uparrow s$, $s_n \in E$ for all n. If $s \in E$, then $E = E \cap [a,s] = E_s$ and $\lambda(\Omega(E)) = \lambda(\Omega(E_s)) = 0$. If $s \notin E$, then $E = \cup_n (E \cap [a,s_n]) = \cup_n E_{s_n}$ and $\lambda(\Omega(E)) = \lim \lambda(\Omega(E_{s_n})) = 0$. This proves a). The proof of b) is similar. □

In the next lemma we look at the special case of (1.3) with $\nu = \lambda$ and $w \equiv 1$. The lemma is a long way of saying that replacing $g = fh$ is all it takes to pass from (2.4) to (2.5).

Lemma 2.5. *Consider weights* $u, h \in \{\mathfrak{M}_\lambda\}^+$ *and* $v \in \{\mathfrak{M}_\mu\}^+$. *The inequalities*

$$\left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]} u f h d\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p d\lambda\right)^{1/p}, \ f \in \{\mathfrak{M}_\lambda\}^+, \tag{2.4}$$

and

$$\left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]} u g d\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega g^p h^{-p} d\lambda\right)^{1/p}, \quad g \in \{\mathfrak{M}_\lambda\}^+, \tag{2.5}$$

are equivalent.

Proof. Fix $f \in \{\mathfrak{M}_\lambda\}^+$ and let (2.5) be true. Plugging $g = fh$ in (2.5) we get

$$\begin{aligned}\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} ufhd\lambda\right)^q v(x)d\mu(x)\right]^{1/q} &\leq C\left(\int_\Omega (fh)^p h^{-p} d\lambda\right)^{1/p} \\ &\leq C\left(\int_\Omega f^p d\lambda\right)^{1/p}\end{aligned}$$

because $h^p h^{-p} \leq 1$, where in case $h = \infty$ or $h = 0$ we have $(\infty)^p (\infty)^{-p} = \infty \cdot 0 = 0 < 1$.

Conversely, let (2.4) hold. Put

$$F_t = \{x \in \Omega[a,t] : h(x) = \infty,\ u(x) \neq 0\}, \quad E = \{t \in [a,b] : \lambda(F_t) > 0\}.$$

We want to show that

$$\int_{\Omega(E)} vd\mu = 0. \tag{2.6}$$

If $t_1 < t_2$, then $F_{t_1} \subset F_{t_2}$ by monotonicity of $\{\Omega(t)\}$ and $\lambda(F_{t_1}) \leq \lambda(F_{t_2})$. Hence, $\Omega(E)$ is Borel measurable. If E is empty, (2.6) is obvious. Let $E \neq \varnothing$ and fix $t \in E$. By Lemma 6.9 from [15] there is a function $f \in \mathfrak{M}_\lambda$ such that $\int_\Omega f^p d\lambda < \infty$ and $0 < f(x) < 1$ on Ω. Then

$$\int_{F_t} ufhd\lambda = \infty,$$

because $u(x) f(x) > 0$, $h(x) = \infty$ on F_t and $\lambda(F_t) > 0$. Plugging $f\chi_{F_t}$ in (2.4) we obtain $F_t \subset \Omega[a,t] \subset \Omega[a,\tau(x)]$ for $\tau(x) \geq t$ and

$$\begin{aligned}\left(\int_{\Omega[t,b]} vd\mu\right)^{1/q} \int_{F_t} ufhd\lambda &\leq \left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} uf\chi_{F_t} hd\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \\ &\leq C\left(\int_\Omega f^p d\lambda\right)^{1/p} < \infty.\end{aligned}$$

This shows that

$$\int_{E\cap\Omega[t,b]} vd\mu \leq \int_{\Omega[t,b]} vd\mu = 0 \text{ for all } t \in E$$

and by Lemma 2.4 (2.6) follows. Hence, to prove (2.5) it suffices to prove that

$$\left[\int_{\Omega([a,b]\setminus E)} \left(\int_{\Omega[a,\tau(x)]} ugd\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega g^p h^{-p} d\lambda\right)^{1/p}, \quad g \in \{\mathfrak{M}_\lambda\}^+. \tag{2.7}$$

Note that

$$\lambda\left(F_{\tau(x)}\right) = 0 \text{ for any } x \in \Omega([a,b]\setminus E) \tag{2.8}$$

by the definition of E.

Now take any $g \in \{\mathfrak{M}_\lambda\}^+$. If $\int_\Omega g^p h^{-p} d\lambda = \infty$, then (2.7) is trivial. Suppose $\int_\Omega g^p h^{-p} d\lambda < \infty$. Then $g^p h^{-p}$ is finite λ-almost everywhere. In particular, for the set $E_1 = \{x \in \Omega : g(x) \neq 0,\ h(x) = 0\}$, where $g^p h^{-p} = \infty$, we have

$$\lambda(E_1) = 0. \tag{2.9}$$

Using (2.6) and $f(y) = g(y)h(y)^{-1}$, $y \in \Omega$, in (2.4) we get

$$\left[\int_{\Omega([a,b]\backslash E)} \left(\int_{\Omega[a,\tau(x)]} ugh^{-1}hd\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \qquad (2.10)$$
$$\leq C\left(\int_\Omega \left(gh^{-1}\right)^p d\lambda\right)^{1/p}.$$

If we show that

$$\lambda(\{y \in \Omega[a,\tau(x)] : u(y)g(y) \neq u(y)g(y)h(y)^{-1}h(y)\}) = 0 \qquad (2.11)$$
$$\text{for any } x \in \Omega([a,b]\backslash E),$$

then (2.10) will imply (2.7). Using the definitions of $F_{\tau(x)}$ and E_1 we see that

$$\begin{aligned}
& \{y \in \Omega[a,\tau(x)] : u(y)g(y) \neq u(y)g(y)h(y)^{-1}h(y)\} \\
\subset\ & \{y \in \Omega[a,\tau(x)] : u(y)g(y) \neq 0,\ h(y)^{-1}h(y) \neq 1\} \\
=\ & \{y \in \Omega[a,\tau(x)] : u(y)g(y) \neq 0,\ h(y) = 0\} \\
& \cup \{y \in \Omega[a,\tau(x)] : u(y)g(y) \neq 0,\ h(y) = \infty\} \\
\subset\ & \{y \in \Omega[a,\tau(x)] : g(y) \neq 0,\ h(y) = 0\} \\
& \cup \{y \in \Omega[a,\tau(x)] : u(y) \neq 0,\ h(y) = \infty\} \subset E_1 \cup F_{\tau(x)}.
\end{aligned}$$

We can use (2.8) and (2.9). This implies (2.11) and finishes the proof. □

We give the proof of the next well-known fact [10] just because we consider a more general situation.

Lemma 2.6. *Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of ν with respect to λ, that is, ν_a is absolutely continuous with respect to λ and ν_s is singular with respect to λ. Then the inequalities*

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} fd\lambda\right)^q d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p w d\nu\right)^{1/p}, \quad f \in \{\mathfrak{M}_\lambda\}^+, \qquad (2.12)$$

and

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} fd\lambda\right)^q d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p w d\nu_a\right)^{1/p}, \quad f \in \{\mathfrak{M}_\lambda\}^+, \qquad (2.13)$$

are equivalent.

Proof. Since $\nu = \nu_a + \nu_s$, (2.13) obviously implies (2.12). Suppose (2.12) is true. Since ν_s and λ are mutually singular, there exists a set $A_s \in \mathfrak{M}_\lambda$ such that $\lambda(A_s) = 0$ and ν_s is concentrated on A_s, implying

$$\nu_s(\Omega\backslash A_s) = 0,\ \lambda(\Omega[a,\tau(x)] \cap A_s) = 0. \qquad (2.14)$$

By absolute continuity of ν_a with respect to λ

$$\nu_a(A_s) = 0. \qquad (2.15)$$

Defining $\tilde{f} = f\chi_{\Omega\backslash A_s}$ we have by (2.14)

$$\int_{\Omega[a,\tau(x)]} fd\lambda = \int_{\Omega[a,\tau(x)]\cap A_s} fd\lambda + \int_{\Omega[a,\tau(x)]\backslash A_s} fd\lambda = \int_{\Omega[a,\tau(x)]} \tilde{f}d\lambda$$

and by (2.14), (2.15)

$$\int_\Omega \tilde{f}^p w d\nu = \int_{\Omega\setminus A_s} f^p w d\nu_a + \int_{\Omega\setminus A_s} f^p w d\nu_s$$
$$= \int_{\Omega\setminus A_s} f^p w d\nu_a + \int_{A_s} f^p w d\nu_a = \int_\Omega f^p w d\nu_a.$$

(2.12) and the last two equations give the desired result:

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} f d\lambda\right)^q d\mu(x)\right]^{\frac{1}{q}} = \left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} \tilde{f} d\lambda\right)^q d\mu(x)\right]^{\frac{1}{q}}$$
$$\le C\left(\int_\Omega \tilde{f}^p w d\nu\right)^{\frac{1}{p}} = C\left(\int_\Omega f^p w d\nu_a\right)^{\frac{1}{p}}.$$

□

Denote

$$I_0 = \left\{x \in \Omega : \int_{\Omega[a,\tau(x)]} u d\lambda = 0\right\},\ I_\infty = \left\{x \in \Omega : \int_{\Omega[a,\tau(x)]} u^{p'} d\lambda = \infty\right\}.$$

By monotonicity I_0 is adjacent to point a and I_∞ is adjacent to point b. See Lemma 2.8 for more information on the structure of these sets. Consider a version of inequality (1.3) with $\nu = \lambda$ and $w \equiv 1$:

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} f u d\lambda\right)^q v(x) d\mu(x)\right]^{1/q} \le C\left(\int_\Omega f^p d\lambda\right)^{1/p},\ f \in \{\mathfrak{M}_\lambda\}^+. \tag{2.16}$$

The next lemma tells us that I_0, I_∞ do not influence the validity of (2.16). The values of f on I_0 should not matter because, as it will be shown,

$$\int_{I_0} u d\lambda = 0. \tag{2.17}$$

By Hölder's inequality for $x \in I_\infty$

$$\int_{\Omega[a,\tau(x)]} f u d\lambda \le \left(\int_{\Omega[a,\tau(x)]} f^p d\lambda\right)^{1/p} \left(\int_{\Omega[a,\tau(x)]} u^{p'} d\lambda\right)^{1/p'}$$

where the last integral on the right is infinite. Hence, by Lemma 2.3 the integral on the left may be infinite. For (2.16) to hold, such values must be suppressed and for this it should be true that

$$\int_{I_\infty} v d\mu = 0. \tag{2.18}$$

That is why the values of the integral $\int_{\Omega[a,\tau(x)]} f u d\lambda$ on I_∞ should not matter. (2.17) and (2.18) arise from allowing weights and measures with improper values and have been discovered in [12].

Lemma 2.7. *a)* (2.17) *is true. b) Put* $I = \Omega \setminus [I_0 \cup I_\infty]$. (2.16) *holds with* $C < \infty$ *if and only if* (2.18) *holds and*

$$\left[\int_I \left(\int_{\Omega[a,\tau(x)]} f u d\lambda\right)^q v(x) d\mu(x)\right]^{1/q} \le C\left(\int_\Omega f^p d\lambda\right)^{1/p},\ f \in \{\mathfrak{M}_\lambda\}^+. \tag{2.19}$$

Proof. a) First note that $I_0, I_\infty, I \in \mathfrak{M}_\lambda$. Define the measure $d\lambda_u = ud\lambda$ and note that for any $x \in I_0$ the set $E_x = I_0 \cap \Omega[a, \tau(x)] \subset \Omega[a, \tau(x)]$ satisfies

$$\lambda_u(E_x) = \int_{E_x} d\lambda_u \le \int_{\Omega[a,\tau(x)]} ud\lambda = 0.$$

Hence, (2.17) follows by Lemma 2.4.

b) It is obvious that (2.16) implies (2.19) and that (2.18) holds in case of an empty I_∞. Let us derive (2.18) in case $I_\infty \neq \varnothing$. Take any $x \in I_\infty$. By Lemma 2.3 there is a function $f \in \{\mathfrak{M}_\lambda\}^+$ such that

$$\int_{\Omega[a,\tau(x)]} f^p d\lambda < \infty, \quad \int_{\Omega[a,\tau(x)]} fud\lambda = \infty.$$

Plugging $f\chi_{\Omega(\tau(x))}$ in (2.16) we get

$$\begin{aligned} &\left(\int_{\Omega[\tau(x),b]} vd\mu\right)^{1/q} \int_{\Omega[a,\tau(x)]} fud\lambda \\ \le\ &\left[\int_\Omega \left(\int_{\Omega[a,\tau(y)]} f\chi_{\Omega[a,\tau(x)]} ud\lambda\right)^q v(y)d\mu(y)\right]^{1/q} \\ \le\ &C\left(\int_{\Omega[a,\tau(x)]} f^p d\lambda\right)^{1/p} < \infty. \end{aligned}$$

Thus, we should have $\int_{\Omega[\tau(x),b]} vd\mu = 0$ for any $x \in I_\infty$ and by Lemma 2.4 (2.18) follows.

Conversely, if (2.18) and (2.19) are true, then, taking into account also (2.17), we see that (2.19) implies (2.16). □

Denote

$$\begin{aligned} U(x) &= \int_{\Omega[a,\tau(x)]} u^{p'} d\lambda, \ V(x) = \int_{\Omega[\tau(x),b]} vd\mu, \\ A(x) &= V(x)^{1/q} U(x)^{1/p'}, \ x \in \Omega, \ A = \sup_{x\in\Omega} A(x), \\ A' &= \sup_{x\in I} A(x) \ \text{ if } I \neq \varnothing, \ A' = 0 \text{ if } I = \varnothing. \end{aligned}$$

Lemma 2.8. *a) Define $s = \sup\{\tau(x) : x \in I_0\}$, $i = \inf\{\tau(x) : x \in I_\infty\}$. Then $I_0 = \Omega[a, s]$, $I_\infty = \Omega[i, b]$*

b) The inequality $A < \infty$ is equivalent to the combination of (2.18) *and*

$$A' < \infty. \tag{2.20}$$

Besides, $A = A'$.

Proof. a) If $x \in I_0$, then the monotonicity $\tau(y) < \tau(x)$ implies

$$\int_{\Omega[a,\tau(y)]} ud\lambda \le \int_{\Omega[a,\tau(x)]} ud\lambda = 0.$$

Hence, with any $x \in I_0$, I_0 contains $\Omega[a, \tau(x)]$ and $\Omega[a, \tau(x)] \cap I_0 = \Omega[a, \tau(x)]$. Choose $\{s_n\} \subset \{\tau(x) : x \in I_0\}$ so that $s_n \uparrow s$. Then

$$\int_{\Omega[a,s]} ud\lambda = \lim \int_{\Omega[a,s_n]} ud\lambda = 0,$$

so $I_0 = \Omega\left[a, s\right]$. Similarly, if $\tau\left(y\right) > \tau\left(x\right)$ and $x \in I_\infty$ then by (2.21)

$$\int_{\Omega[\tau(y),b]} v d\mu \leq \int_{\Omega[\tau(x),b]} v d\mu = 0.$$

Hence, $\Omega\left[\tau\left(x\right), b\right] \cap I_\infty = \Omega\left[\tau\left(x\right), b\right]$. Choosing $\{i_n\} \subset \{\tau\left(x\right) : x \in I_\infty\}$ so that $i_n \downarrow i$ and using the above equation we see that $I_\infty = \Omega\left[i, b\right]$.

a) Let $A < \infty$. For any $x \in I_\infty$ we have $U\left(x\right) = \int_{\Omega[a,\tau(x)]} u^{p'} d\lambda = \infty$, so for $A < \infty$ it is necessary that

$$V\left(x\right) = \int_{\Omega[\tau(x),b]} v d\mu = 0, \ x \in I_\infty. \tag{2.21}$$

In Lemma 2.4 put $E = I_\infty$, $d\mu_v = v d\mu$. Then $\bar{E}_x = I_\infty \cap \Omega\left[\tau\left(x\right), b\right] \subset \Omega\left[\tau\left(x\right), b\right]$ and

$$\mu_v\left(\bar{E}_x\right) \leq \int_{\Omega[\tau(x),b]} v d\mu = 0, \ x \in I_\infty.$$

By Lemma 2.4 (2.18) follows. Besides, from the definition of I_0 we see that

$$U\left(x\right) = \int_{\Omega[a,\tau(x)]} u^{p'} d\lambda = 0, \ x \in I_0. \tag{2.22}$$

By (2.21) and (2.22) $A\left(x\right) = 0$ on $I_0 \cup I_\infty$, so $A' = A$.

Conversely, let (2.18) and (2.20) hold. By (2.22) $A\left(x\right) = 0$ on I_0. Besides, part a) and (2.18) imply

$$V\left(x\right) = \int_{\Omega[\tau(x),b]} v d\mu \leq \int_{I_\infty} v d\mu = 0, \ x \in I_\infty.$$

Thus, $A\left(x\right) = 0$ on I_∞ and $A = A' < \infty$. □

Theorem 2.1. *Let $1 < p \leq q < \infty$. Inequality (2.16) holds if and only if $A < \infty$, with the equivalence $c_1 A \leq C \leq c_2 A$.*

Proof. We want to show that $A = C = 0$ in case $I = \varnothing$. By monotinicity $V\left(x\right) = 0$ on I_∞ and by definition $\int_{\Omega[a,\tau(x)]} u d\lambda = 0$ on I_0. Hence $A = A' = 0$. On the other hand, $\Omega = I_0 \cup I_\infty$ implies

$$\int_\Omega \left(\int_{\Omega[a,\tau(x)]} f u d\lambda\right)^q v(x) d\mu(x) = \int_{I_\infty} \left(\int_{\Omega[a,\tau(x)]\setminus I_0} f u d\lambda\right)^q v(x) d\mu(x) = 0.$$

Thus, $C = 0$. In what follows we can safely assume that $I \neq \varnothing$.

Sufficiency. Let $x \in I$. Since $\Omega\left[a, \tau\left(x\right)\right] \subseteq \Omega \setminus I_\infty$ and u is λ-everywhere zero on I_0, by Hölder's inequality

$$\begin{aligned} \int_{\Omega[a,\tau(x)]} f u d\lambda &= \int_{\Omega[a,\tau(x)]\cap I} f u d\lambda \\ &\leq \left(\int_{\Omega[a,\tau(x)]\cap I} f^p U^{1/p'} d\lambda\right)^{1/p} \left(\int_{\Omega[a,\tau(x)]\cap I} u^{p'} U^{-1/p} d\lambda\right)^{1/p'}. \end{aligned}$$

By Lemma 2.1 with $\gamma = -1/p$ and $I \cap \Omega\left[a, \tau\left(x\right)\right]$ in place of Ω and using the definition of I_0

$$
\begin{aligned}
&\int_{I\cap\Omega[a,\tau(x)]} u^{p'}(y)\left(\int_{\Omega[a,\tau(y)]} u^{p'}d\lambda\right)^{-1/p} d\lambda(y)\\
=\;&\int_{I\cap\Omega[a,\tau(x)]} u^{p'}(y)\left(\int_{I\cap\Omega[a,\tau(y)]} u^{p'}d\lambda\right)^{-1/p} d\lambda(y)\\
\leq\;& c\left(\int_{I\cap\Omega[a,\tau(x)]} u^{p'}d\lambda\right)^{1-1/p} = cU(x)^{1/p'},
\end{aligned}
$$

where $U(x)<\infty$ because $x\notin I_\infty$. Then for $x\in I$

$$
\begin{aligned}
\int_{\Omega[a,\tau(x)]} fud\lambda &= \int_{I\cap\Omega[a,\tau(x)]} fud\lambda \leq c\left(\int_{I\cap\Omega[a,\tau(x)]} f^pU^{1/p'}d\lambda\right)^{1/p} U(x)^{1/(p')^2}\\
&\leq cA^{1/p'}\left(\int_{I\cap\Omega[a,\tau(x)]} f^pU^{1/p'}d\lambda\right)^{1/p} V(x)^{-1/(qp')}
\end{aligned}
$$

where $V(x)>0$ by (2.18) and $V(x)<\infty$ because $U(x)>0$, see (2.17). Using this inequality we bound the left-hand side of (2.19) as follows:

$$
\begin{aligned}
&\left[\int_I\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \qquad (2.23)\\
\leq\;& cA^{1/p'}\left[\int_I\left(\int_{I\cap\Omega[a,\tau(x)]} f^pU^{1/p'}d\lambda\right)^{q/p} V(x)^{-1/p'}v(x)d\mu(x)\right]^{1/q}\\
=\;& cA^{1/p'}\left[\int_I\left(\int_I f^pU^{1/p'}\chi_{\Omega[a,\tau(x)]}d\lambda\right)^{q/p} V(x)^{-1/p'}v(x)d\mu(x)\right]^{1/q}\\
\leq\;& cA^{1/p'}\left[\int_\Omega f(y)^pU(y)^{1/p'}\left(\int_{I\cap\Omega[\tau(y),b]} V^{-1/p'}vd\mu\right)^{p/q} d\lambda(y)\right]^{1/p}.
\end{aligned}
$$

The last transition is by Minkowsky's inequality.

By Lemma 2.8 $x\in I$ implies $\Omega[\tau(x),b]\subseteq\Omega\backslash I_0$. Using also (2.18) we see that $V(x)=\int_{I\cap\Omega[\tau(x),b]} vd\mu$ for $x\in I$. Now by Lemma 2.2 for $y\in I$

$$
\begin{aligned}
&\int_{I\cap\Omega[\tau(y),b]} vV^{-1/p'}d\mu\\
=\;&\int_{I\cap\Omega[\tau(y),b]} v(x)\left(\int_{I\cap\Omega[\tau(x),b]} vd\mu\right)^{-1/p'} d\mu(x)\\
\leq\;& c\left(\int_{I\cap\Omega[\tau(y),b]} vd\mu\right)^{1-1/p'} = cV(y)^{1/p},
\end{aligned}
$$

where $V(y)<\infty$ because $A<\infty$ and $U(x)>0$ on I.

Continuing (2.23) and applying $A<\infty$ together with the last bound we get

$$
\begin{aligned}
&\left[\int_I\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)d\mu(x)\right]^{1/q}\\
\leq\;& c_1A^{1/p'}\left(\int_I f^pU^{1/p'}V^{1/q}d\lambda\right)^{1/p} \leq c_1A\left(\int_\Omega f^pd\lambda\right)^{1/p}.
\end{aligned}
$$

Necessity. Suppose (2.16) is true. By Lemma 2.8 we have $\Omega\left[a,\tau\left(x\right)\right]\supset I_0$ for $x\in I$ and

$$\left(\Omega\backslash I_\infty\right)\cap\Omega\left[\tau\left(x\right),b\right]\subset\left(\Omega\backslash I_\infty\right)\cap\left(\Omega\backslash I_0\right)\subset I.$$

By (2.18)

$$V\left(x\right)=\int_{\Omega[\tau(x),b]}vd\mu=\int_{(\Omega\backslash I_\infty)\cap\Omega[\tau(x),b]}vd\mu\leq\int_I vd\mu,\ x\in I.$$

For $x\in I$ put $f=u^{p'-1}\chi_{\Omega[a,\tau(x)]}$. If $y\in\Omega\left[\tau\left(x\right),b\right]$, then $\tau\left(y\right)\geq\tau\left(x\right)$ and

$$U\left(x\right)=\int_{\Omega[a,\tau(x)]}u^{p'}d\lambda\leq\int_{\Omega[a,\tau(y)]}\left(f\chi_{\Omega[a,\tau(x)]}\right)ud\lambda.$$

Thus, applying also (2.19), we get for $x\in I$

$$\begin{aligned}V\left(x\right)^{1/q}U\left(x\right) &= \left[\int_{\Omega[\tau(x),b]}\left(\int_{\Omega[a,\tau(x)]}u^{p'}d\lambda\right)^q vd\mu\right]^{1/q}\\ &\leq \left[\int_{\Omega[\tau(x),b]}\left(\int_{\Omega[a,\tau(y)]}\left(f\chi_{\Omega[a,\tau(x)]}\right)ud\lambda\right)^q v\left(y\right)d\mu\left(y\right)\right]^{1/q}\\ &\leq \left[\int_I\left(\int_{\Omega[a,\tau(y)]}\left(f\chi_{\Omega[a,\tau(x)]}\right)d\lambda\right)^q v(y)d\mu(y)\right]^{1/q}\\ &\leq C\left(\int_{\Omega[a,\tau(x)]}u^{p'}d\lambda\right)^{1/p}=CU\left(x\right)^{1/p}\end{aligned}$$

which gives $A'\leq C$. By Lemma 2.7, (2.18) is true and Lemma 2.8 gives $A=A'\leq C$. □

Next we consider the case $q<p$ and define r from $1/r=1/q-1/p$. Denote

$$\begin{aligned}B &= \left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]}u^{p'}d\lambda\right)^{r/p'}\left(\int_{\Omega[\tau(x),b]}vd\mu\right)^{r/p}v\left(x\right)d\mu\left(x\right)\right]^{1/r}\\ &= \left(\int_\Omega U^{r/p'}V^{r/p}vd\mu\right)^{1/r}.\end{aligned}$$

Lemma 2.9. *a) (2.17) is true. b) $B<\infty$ is equivalent to the combination of (2.18) and*

$$B'=\left(\int_I U^{r/p'}V^{r/p}vd\mu\right)^{1/r}<\infty.$$

Besides, $B=B'$.

Proof. a) The proof of Lemma 2.7 a) does not rely on the inequality $p\leq q$ and is valid in the current situation.

b) Let $B<\infty$. Then the fact that $U\left(x\right)=\infty$ on I_∞ implies

$$\int_{I_\infty}\left(\int_{\Omega[\tau(x),b]}vd\mu\right)^{r/p}vd\mu=0.$$

We represent

$$I_\infty=\left\{x\in I_\infty:\int_{\Omega[\tau(x),b]}vd\mu=0,\ v\left(x\right)\neq0\right\}\cup\left\{x\in I_\infty:v\left(x\right)=0\right\}=F\cup G.$$

Further, in Lemma 2.4 b) put $E=F$, $d\mu_v = vd\mu$. Then $\bar{E}_x = E\cap\Omega\left[\tau(x),b\right]\subset\Omega\left[\tau(x),b\right]$,

$$\mu_v\left(\bar{L}(x)\right)=\int_{\bar{L}(x)} vd\mu\le\int_{\Omega[\tau(x),b]} vd\mu=0,\ x\in F.$$

By Lemma 2.4 $\int_F vd\mu=0$. Since also $\int_G vd\mu=0$, (2.18) holds. The definition of I_0 and (2.18) give $B=B'$.

Conversely, let $B'<\infty$ and (2.18) hold. Then in view of (2.17) and (2.18) $B=B'<\infty$. □

Lemma 2.10. *If $B<\infty$, then $A\le B$ and* (2.18) *is true.*

Proof. By Lemma 2.2

$$\int_{\Omega[\tau(x),b]}\left(\int_{\Omega[\tau(y),b]} vd\mu\right)^{r/p} v(y)\,d\mu(y)\ge c\left(\int_{\Omega[\tau(x),b]} vd\mu\right)^{r/q}.$$

Hence, for any $\tau(x)\in[a,b]$

$$\begin{aligned}
B &\ge \left(\int_{\Omega[\tau(x),b]} U^{r/p'}V^{r/p}vd\mu\right)^{1/r}\\
&\ge U(x)^{1/p'}\left[\int_{\Omega[\tau(x),b]}\left(\int_{\Omega[\tau(y),b]} vd\mu\right)^{r/p} v(y)\,d\mu(y)\right]^{1/r}\\
&\ge c^{1/r}U(x)^{1/p'}V(x)^{1/q}=c^{1/r}A(x).
\end{aligned}$$

Hence, $c^{1/r}A\le B<\infty$ and (2.18) follows by Lemma 2.8. □

Denote

$$h(x)=\chi_I(x)\left(\int_{\Omega[a,\tau(x)]} U^{r/q'}V^{r/p}u^{p'}d\lambda\right)^{q/r}$$

and define a measure on $\mathfrak{M}_\mu$ by $d\tilde{\mu}=\chi_I v h^{-p/q}d\mu$. Assuming that $B<\infty$ we plan to derive the bound (2.19) from

$$\begin{aligned}
&\left[\int_I\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)\,d\mu(x)\right]^{1/q}\\
&\le c_1B\left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^p d\tilde{\mu}(x)\right]^{1/p}
\end{aligned}\tag{2.24}$$

and

$$\left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^p d\tilde{\mu}(x)\right]^{1/p}\le c_2\left(\int_\Omega f^p d\lambda\right)^{1/p}.\tag{2.25}$$

Lemma 2.11. *If $B<\infty$ and $I\ne\varnothing$, then*

$$\int_{\{x\in I:h(x)=0\}} vd\mu=\int_{\{x\in I:h(x)=\infty\}} vd\mu=0\tag{2.26}$$

and (2.24) *holds.*

Proof. h is Borel measurable. For $x \in I$ by Lemma 2.1

$$\begin{aligned} h(x) &\geq V(x)^{q/p}\left[\int_{\Omega[a,\tau(x)]}\left(\int_{\Omega[a,\tau(y)]}u^{p'}d\lambda\right)^{r/q'}u(y)^{p'}d\lambda(y)\right]^{q/r} \\ &\geq cV(x)^{q/p}U(x)^{q/p'}. \end{aligned}$$

Here $U(x) \neq 0$, so $h(x) = 0$ implies $\int_{\Omega[\tau(x),b]} v d\mu = 0$ and by Lemma 2.4 we get the first equation in (2.26).

Changing the order of integration we see that

$$\begin{aligned} &\int_I vh^{r/q}d\mu = \int_\Omega vh^{r/q}d\mu \qquad (2.27) \\ \leq\ & \int_\Omega\left(\int_{\Omega[a,\tau(x)]}U^{r/q'}V^{r/p}u^{p'}\chi_{\Omega[a,\tau(x)]}d\lambda\right)v(x)\,d\mu(x) \\ =\ & \int_\Omega U(y)^{r/q'}V(y)^{r/p}u(y)^{p'}\left(\int_{\Omega[\tau(y),b]}vd\mu\right)d\lambda(y) \\ =\ & \int_\Omega U^{r/q'}V^{r/q}u^{p'}d\lambda \\ & \text{(using the left inequality in (2.3) and changing integration order)} \\ \leq\ & c_1\int_\Omega U(y)^{r/q'}u(y)^{p'}\left(\int_{\Omega[\tau(y),b]}vV^{r/p}d\mu\right)d\lambda(y) \\ =\ & c_1\int_\Omega\left(\int_{\Omega[a,\tau(x)]}U^{r/q'}u^{p'}d\lambda\right)V(x)^{r/p}v(x)\,d\mu(x) \\ & \text{(by Lemma 2.1)} \\ \leq\ & c_2\int_\Omega U^{r/p'}V^{r/p}vd\mu = c_2B^r. \end{aligned}$$

This bound implies, in particular, the second equality in (2.26).

Using (2.26), (2.27) and Hölder's inequality with the exponents r/q and p/q we complete the proof of (2.24):

$$\begin{aligned} &\left[\int_I\left(\int_{\Omega[a,\tau(x)]}fud\lambda\right)^q v(x)\,d\mu(x)\right]^{1/q} \\ =\ & \left[\int_I h(x)\,h(x)^{-1}\left(\int_{\Omega[a,\tau(x)]}fud\lambda\right)^q v(x)\,d\mu(x)\right]^{1/q} \\ \leq\ & \left(\int_\Omega vh^{r/q}d\mu\right)^{1/r}\left[\int_I h(x)^{-p/q}\left(\int_{\Omega[a,\tau(x)]}fud\lambda\right)^p v(x)\,d\mu(x)\right]^{1/p} \\ \leq\ & c_2^{1/r}B\left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]}fud\lambda\right)^p d\tilde{\mu}(x)\right]^{1/p}. \end{aligned}$$

□

Lemma 2.12. *If $B < \infty$ and $I \neq \varnothing$, then*

$$\sup_{x\in I}\tilde{\mu}\left(\Omega\left[\tau(x),b\right]\right)^{1/p}U(x)^{1/p'} \leq c \qquad (2.28)$$

and (2.25) *is true.*

Proof. Let $x \in I$. By Lemma 2.10 we know that $A < \infty$. Therefore $U(x) > 0$ implies that the integral $\int_{\Omega[\tau(x),b]} v d\mu$ is finite. If it is zero, then $\tilde{\mu}(\Omega[\tau(x), b]) = 0$. Suppose that integral is not zero. Using the inequalities $h(y) \geq h(x)$ for $\tau(y) \geq \tau(x)$ and $V(z) \geq V(x)$ for $\tau(z) \leq \tau(x)$, we have for $x \in I$

$$\begin{aligned}
\tilde{\mu}(\Omega[\tau(x), b]) &= \int_{\Omega[\tau(x),b]} \chi_I v h^{-p/q} d\mu \leq \int_{\Omega[\tau(x),b]} v d\mu h(x)^{-p/q} \\
&\leq V(x) \left(\int_{\Omega[a,\tau(x)]} U^{r/q'} V^{r/p} u^{p'} d\lambda \right)^{-p/r} \\
&\quad \text{(applying Lemma 2.1 with } \gamma = r/q'\text{)} \\
&\leq V(x) \left(V(x)^{r/p} \int_{\Omega[a,\tau(x)]} U^{r/q'} u^{p'} d\lambda \right)^{-p/r} \leq cU(x)^{-p/p'} < \infty.
\end{aligned}$$

This proves (2.28).

Further, put $i = \inf\{\tau(x) : x \in I\}$. If the infimum is attained on I, then $I \subset \Omega[i, b]$, $U(i) > 0$ and

$$\tilde{\mu}(\Omega) = \int_I v h^{-p/q} d\mu = \int_{\Omega[i,b]} \chi_I v h^{-p/q} d\mu = \tilde{\mu}(\Omega[i, b]) < \infty$$

by (2.28). If $i \notin I$, then $I \subseteq \Omega(i, b]$ and $\tilde{\mu}(\Omega[a, i] \cap I) = 0$. Take any sequence $\{i_n\} \subset I$ such that $i_n \downarrow i$ and put $E_n = \Omega[a, i] \cup \Omega[i_n, b]$. Then $\Omega = \cup E_n$ and $\tilde{\mu}(E_n) < \infty$ for all n. Hence $\tilde{\mu}$ is σ-finite on Ω. Besides, (2.18) implies $\tilde{\mu}(I_\infty) = 0$. Therefore (2.28) and Theorem 2.1 give for any n

$$\begin{aligned}
&\left[\int_{E_n} \left(\int_{\Omega[a,\tau(x)]} f u d\lambda \right)^p d\tilde{\mu}(x) \right]^{1/p} \\
&= \left[\int_{E_n \cap I} \left(\int_{\Omega[a,\tau(x)]} f u d\lambda \right)^p d\tilde{\mu}(x) \right]^{1/p} \leq c \left(\int_\Omega f^p d\lambda \right)^{1/p}.
\end{aligned}$$

Here c does not depend on n. With fixed f, we can let $n \to \infty$ and finish the proof. □

Theorem 2.2. *Let* $0 < q < p < \infty$, $p > 1$, $1/r = 1/q - 1/p$. (2.16) *holds if and only if* $B < \infty$, *and* $c_1 B \leq C \leq c_2 B$.

Proof. Sufficiency. Let $B < \infty$. If $I = \varnothing$, by Lemma 2.9 we see that in fact $B = B' = 0$. On the other hand, the best constant in (2.16) in this case is also 0 because of (2.17) and (2.18). If $I \neq \varnothing$, the sufficiency follows by Lemmas 2.11 and 2.12.

Necessity. Suppose (2.16) is true with $C < \infty$ and, hence, (2.17) and (2.18) hold. Since μ is σ-finite, there is a sequence $\{E_n\}$ of sets such that $\Omega = \cup E_n$ and $\mu(E_n) < \infty$. We can assume that $E_n \subset E_{n+1}$ and $E_n \cap I \neq \varnothing$ for all n. Let $\{s_n\} \subset I$ be such that $s_n \uparrow s = \sup\{\tau(x) : x \in I\}$ and for $n \in N$ define

$$F_n = \begin{cases} E_n \cap I, & \text{if } s \in \tau(I) \\ E_n \cap I \cap \Omega[a, s_n], & \text{if } s \notin \tau(I). \end{cases}$$

Then $\{F_n\}$ satisfies $\cup F_n = I$, $F_n \subset F_{n+1}$, $\mu(F_n) < \infty$.

Put $v_n = \min\{v, n\}$, $d\mu_n = v_n \chi_{F_n} d\mu$,

$$\begin{aligned}
B_n &= \left(\int_I U(x)^{r/p'} \mu_n(\Omega[\tau(x), b])^{r/p} d\mu_n(x) \right)^{1/r} \\
&= \left[\int_{F_n} U(x)^{r/p'} \left(\int_{\Omega[\tau(x),b]} v_n \chi_{F_n} d\mu \right)^{r/p} v_n(x) d\mu(x) \right]^{1/r}.
\end{aligned}$$

If $F_n = \varnothing$, then $B_n = 0$. If $F_n \neq \varnothing$, then with $\alpha_n = \sup F_n \in I$ we have

$$B_n \leq \left(\int_{\Omega(\alpha_n)} u^{p'} d\lambda\right)^{1/p'} \left[\int_{F_n} \left(\int_{\Omega[\tau(x),b]} v_n \chi_{F_n} d\mu\right)^{r/p} v_n(x)\, d\mu(x)\right]^{1/r}.$$

By Lemma 2.2 and the definition of v_n

$$B_n \leq \left(\int_{\Omega(\alpha_n)} u^{p'} d\lambda\right)^{1/p'} \mu_n(F_n)^{1/q} \leq \left(\int_{\Omega(\alpha_n)} u^{p'} d\lambda\right)^{1/p'} (n\mu(F_n))^{1/q} < \infty,$$

because $I \cap I_\infty = \varnothing$.

Put

$$f(y) = \mu_n(\Omega[\tau(x), b])^{r/(pq)} U(y)^{r/(pq')} u(y)^{p'-1} \chi_I(y).$$

Then

$$\begin{aligned}
&\left[\int_I \left(\int_{\Omega[a,\tau(x)]} f u d\lambda\right)^q d\mu_n(x)\right]^{1/q} \qquad (2.29)\\
= \;& \left[\int_I \left(\int_{\Omega[a,\tau(x)]\cap I} \mu_n(\Omega[\tau(y), b])^{\frac{r}{pq}} U(y)^{\frac{r}{pq'}} u(y)^{p'} d\lambda(y)\right)^q d\mu_n(x)\right]^{\frac{1}{q}}\\
\geq \;& \left[\int_I \mu_n(\Omega[\tau(x), b])^{r/p} \left(\int_{\Omega[a,\tau(x)]\cap I} U^{r/(pq')} u^{p'} d\lambda\right)^q d\mu_n(x)\right]^{1/q}\\
& \text{(applying (2.17) and Lemma 2.1)}\\
= \;& \left[\int_I \mu_n(\Omega[\tau(x), b])^{r/p} \left(\int_{\Omega[a,\tau(x)]} U^{r/(pq')} u^{p'} d\lambda\right)^q d\mu_n(x)\right]^{1/q}\\
\geq \;& c\left(\int_I \mu_n(\Omega[\tau(x), b])^{r/p} U(x)^{r/p'} d\mu_n(x)\right)^{1/q} = cB_n^{r/q}.
\end{aligned}$$

On the other hand, by Lemma 2.2

$$\begin{aligned}
&\left(\int_\Omega f^p d\lambda\right)^{1/p} = \left[\int_I \mu_n(\Omega[\tau(y), b])^{r/q} U(y)^{r/q'} u(y)^{p'} d\lambda(y)\right]^{1/q} \qquad (2.30)\\
\leq \;& c_2 \left[\int_\Omega \left(\int_{\Omega[\tau(y),b]} \mu_n(\Omega[\tau(x), b])^{\frac{r}{p}} d\mu_n(x)\right) U(y)^{\frac{r}{q'}} u(y)^{p'} \chi_I(y)\, d\lambda(y)\right]^{\frac{1}{p}}\\
& \text{(changing integration order)}\\
= \;& c_2 \left[\int_\Omega \mu_n(\Omega[\tau(x), b])^{r/p} \left(\int_{\Omega[a,\tau(x)]\cap I} U^{r/q'} u^{p'} d\lambda\right) d\mu_n(x)\right]^{1/p}\\
& \text{(using supp}\mu_n \subset F_n \subset I)\\
\leq \;& c_2 \left[\int_I \mu_n(\Omega[\tau(x), b])^{r/p} \left(\int_{\Omega[a,\tau(x)]\cap I} U^{r/q'} u^{p'} d\lambda\right) d\mu_n(x)\right]^{1/p}\\
& \text{(by Lemma 2.1)}\\
\leq \;& c_3 \left(\int_I \mu_n(\Omega[\tau(x), b])^{r/p} U(x)^{r/p'} d\mu_n(x)\right)^{1/p} = B_n^{r/p}.
\end{aligned}$$

Putting together (2.29), (2.19) and (2.30) we see that

$$\begin{aligned} B_n^{r/q} &\le c_4 \left[\int_I \left(\int_{\Omega[a,\tau(x)]} f u d\lambda \right)^q d\mu_n(x) \right]^{1/q} \\ &\le c_5 C \left(\int_\Omega f^p d\lambda \right)^{1/p} \le c_5 C B_n^{r/p} \end{aligned}$$

or $B_n \le c_5 C$. Since $\cup F_n = I$, $F_n \subset F_{n+1}$, we have $v_n \chi_{F_n} \uparrow v$ as $n \to \infty$, so the statement follows by the monotone convergence theorem. □

The general result is stated next.

Theorem 2.3. *Suppose* $1 < p < \infty$, $0 < q < \infty$, $1/r = 1/q - 1/p$. *Let* $\nu = \nu_a + \nu_s$ *be the Lebesgue decomposition of* ν *with respect to* λ, *where* ν_a *is absolutely continuous with respect to* λ *and* ν_s *and* λ *are mutually singular. Denote* $\frac{d\nu_a}{d\lambda}$ *the Radon-Nikodym derivative of* ν_a *with respect to* λ.

a) If $p \le q$, *then the inequality*

$$\begin{aligned} \left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} f u d\lambda \right)^q v(x)\, d\mu(x) \right]^{1/q} &\le C \left(\int_\Omega f^p w d\nu \right)^{1/p}, \\ f &\in \{\mathfrak{M}_\lambda\}^+, \end{aligned} \tag{2.31}$$

holds if and only if $\mathcal{A} = \sup_{x\in\Omega} \mathcal{A}(x) < \infty$, *where*

$$\mathcal{A}(x) = \left[\int_{\Omega[a,\tau(x)]} u^{p'} \left(w \frac{d\nu_a}{d\lambda} \right)^{1-p'} d\lambda \right]^{1/p'} \left(\int_{\Omega[\tau(x),b]} v d\mu \right)^{1/q}.$$

Moreover, $c_1 \mathcal{A} \le C \le c_2 \mathcal{A}$.

b) If $q < p$, *then* (2.31) *is true if and only if* $\mathcal{B} < \infty$, *where*

$$\mathcal{B} = \left\{ \int_\Omega \left[\int_{\Omega[a,\tau(x)]} u^{p'} \left(w \frac{d\nu_a}{d\lambda} \right)^{1-p'} d\lambda \right]^{r/p'} \left(\int_{\Omega[\tau(x),b]} v d\mu \right)^{r/p} v(x)\, d\mu(x) \right\}^{1/r}.$$

Besides, $c_1 \mathcal{B} \le C \le c_2 \mathcal{B}$.

Proof. By Lemma 2.6, (2.31) is equivalent to

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} f u d\lambda \right)^q v(x)\, d\mu(x) \right]^{1/q} \le C \left(\int_\Omega f^p w \frac{d\nu_a}{d\lambda} d\lambda \right)^{1/p}, \quad f \in \{\mathfrak{M}_\lambda\}^+.$$

This inequality, in turn, by Lemma 2.5 is equivalent to

$$\begin{aligned} \left\{ \int_\Omega \left[\int_{\Omega[a,\tau(x)]} f u \left(w \frac{d\nu_a}{d\lambda} \right)^{-1/p} d\lambda \right]^q v(x)\, d\mu(x) \right\}^{1/q} &\le C \left(\int_\Omega f^p d\lambda \right)^{1/p}, \\ f &\in \{\mathfrak{M}_\lambda\}^+. \end{aligned}$$

Application of Theorems 2.1 and 2.2 completes the proof. □

3 Results for the dual operator

The dual operator is defined by

$$T^*f(x)=\int_{\{y\in\Omega:\tau(y)\geq\tau(x)\}} f\,(y)\,d\lambda\,(y)\,,\ x\in\Omega.$$

The analogues of Theorems 2.1 and 2.2 are in the next theorem.

Let

$$\begin{aligned} U^*\,(x) &= \int_{\Omega[\tau(x),b]} u^{p'}d\lambda,\ V^*\,(x)=\int_{\Omega[a,\tau(x)]} vd\mu, \\ A^*\,(x) &= V^*\,(x)^{1/q}\,U^*\,(x)^{1/p'}\,,\ x\in\Omega,\ A^*=\sup_{x\in\Omega}A^*(x), \\ B^* &= \left[\int_\Omega\left(\int_{\Omega[\tau(x),b]} u^{p'}d\lambda\right)^{r/p'}\left(\int_{\Omega[a,\tau(x)]} vd\mu\right)^{r/p} v\,(x)\,d\mu\,(x)\right]^{1/r} \end{aligned}$$

and consider the inequality

$$\left[\int_\Omega\left(\int_{\Omega[\tau(x),b]} fud\lambda\right)^q v(x)d\mu(x)\right]^{1/q}\leq C\left(\int_\Omega f^pd\lambda\right)^{1/p},\ f\in\{\mathfrak{M}_\lambda\}^+. \tag{3.1}$$

Theorem 3.1. *Suppose* $1<p<\infty$, $0<q<\infty$. *a) If* $p\leq q$, *then inequality* (3.1) *holds if and only if* $A^*<\infty$, *with the equivalence* $c_1A^*\leq C\leq c_2A^*$.

b) If $q<p$, *then* (3.1) *holds if and only if* $B^*<\infty$, *and* $c_1B^*\leq C\leq c_2B^*$.

The general result looks as follows.

Theorem 3.2. *Suppose* $1<p<\infty$, $0<q<\infty$, $1/r=1/q-1/p$. *Let* $\nu=\nu_a+\nu_s$ *be the Lebesgue decomposition of* ν *with respect to* λ, *where* ν_a *is absolutely continuous with respect to* λ *and* ν_s *and* λ *are mutually singular. Denote* $\frac{d\nu_a}{d\lambda}$ *the Radon-Nikodym derivative of* ν_a *with respect to* λ.

a) If $p\leq q$, *then the inequality*

$$\left[\int_\Omega\left(\int_{\Omega[\tau(x),b]} fud\lambda\right)^q v\,(x)\,d\mu(x)\right]^{1/q}\leq C\left(\int_\Omega f^pwd\nu\right)^{1/p},\ f\in\{\mathfrak{M}_\lambda\}^+, \tag{3.2}$$

holds if and only if $\mathcal{A}^*=\sup_{x\in\Omega}\mathcal{A}^*\,(x)<\infty$, *where*

$$\mathcal{A}^*\,(x)=\left[\int_{\Omega[\tau(x),b]} u^{p'}\left(w\frac{d\nu_a}{d\lambda}\right)^{1-p'}d\lambda\right]^{1/p'}\left(\int_{\Omega[a,\tau(x)]} vd\mu\right)^{1/q}.$$

Moreover, $c_1\mathcal{A}^*\leq C\leq c_2\mathcal{A}^*$.

b) If $q<p$, *then* (3.2) *is true if and only if* $\mathcal{B}^*<\infty$, *where*

$$\mathcal{B}^*=\left\{\int_\Omega\left[\int_{\Omega[\tau(x),b]} u^{p'}\left(w\frac{d\nu_a}{d\lambda}\right)^{1-p'}d\lambda\right]^{r/p'}\left(\int_{\Omega[a,\tau(x)]} vd\mu\right)^{r/p} v\,(x)\,d\mu\,(x)\right\}^{1/r}.$$

Besides, $c_1\mathcal{B}^*\leq C\leq c_2\mathcal{B}^*$.

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n-MULTIPLICITY AND SPECTRAL PROPERTIES FOR (M, k)-QUASI-*-CLASS Q OPERATORS

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Abstract. In the present article, we introduce a new class of operators which will be called the class of (M, k)-quasi-*-class Q operators. An operator $A \in B(H)$ is said to be (M, k)-quasi-*-class Q for certain integer k, if there exists $M > 0$ such that

$$A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^k \geq 0.$$

Some properties of this class of operators are shown. It is proved that the considered class contains the class of k-quasi-*-class $\mathbb{A}$ operators. The decomposition of such operators, their restrictions on invariant subspaces, the n-multicyclicity and some spectral properties are also presented. We also show that if $\lambda \in \mathbb{C}, \lambda \neq 0$ is an isolated point of the spectrum of A, then the Riesz idempotent E for λ is self-adjoint, and verifies $EH = ker(A - \lambda) = ker(A - \lambda)^{\star}$.

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1 Introduction

Let H be an infinite dimensional complex separable Hilbert space, and let $B(H)$ be the Banach algebra of all bounded linear operators on H. Denote, respectively, by $ker(A)$ and $ran(A)$ the null space and the range space of an operator A in $B(H)$. As an easy extension of normal operators, hyponormal operators have been studied by many mathematicians. Though there are many unsolved interesting problems for hyponormal operators (e.g., the invariant subspace problem), one of recent trends in operator theory is to study natural extensions of hyponormal operators. Below we introduce some of these non-hyponormal operators. Recall ([3, 7]) that $A \in B(H)$ is called hyponormal if $A^*A \geq AA^*$, paranormal if $\|A^2x\| \geq \|Ax\|^2$ and $*$-paranormal if $\|A^2x\| \geq \|A^*x\|^2$) for each unit vector $x \in H$. Following [7] and [12], we say that $A \in B(H)$ belongs to class $\mathbb{A}$ if $|A^2| \geq |A|^2$ where $A^*A = |A|^2$. Recently, B.P. Duggal, I.H. Jeon and I.H. Kim [6] considered the following new class of operators: an operator $A \in B(H)$ is said to belong to the $*$-class $\mathbb{A}$ if $|A^2| \geq |A^*|^2$. For brevity, we shall denote classes of hyponormal operators, paranormal operators, $*$-paranormal operators, class $\mathbb{A}$ operators, and $*$-class $\mathbb{A}$ operators by $\mathcal{H}$, $\mathcal{PN}$, $\mathcal{PN}^*$, $\mathcal{A}$ and $\mathcal{A}^*$ respectively. From [3] and [7], it is well known that

$$\mathcal{H} \subset \mathcal{A} \subset \mathcal{PN}$$

and

$$\mathcal{H} \subset \mathcal{A}^* \subset \mathcal{PN}^*.$$

Recently, the authors of [23] have extended $*$-class $\mathbb{A}$ operators to quasi-$*$-class $\mathbb{A}$ operators. An operator $A \in B(H)$ is said to be quasi-$*$-class $\mathbb{A}$ if $A^*|A^2|A \geq A^*|A^*|^2A$, and quasi-$*$-paranormal if

$$\|A^*Ax\|^2 \leq \|A^3x\|\|Ax\|$$

for all $x \in H$. In [19], many results on quasi-$*$-paranormal operators were proved. In particular, quasi-$*$-paranormal operators have Bishop's property (β) [19]. If we denote the class of quasi-$*$-class $\mathbb{A}$ operators by $\mathcal{QA}^*$ and of quasi-$*$-paranormal operators by $\mathcal{QPN}^*$, we have

$$\mathcal{H} \subset \mathcal{A}^* \subset \mathcal{QA}^* \subset \mathcal{QPN}^*.$$

As a further generalization, S.Mecheri in [16, 14] introduced the class of k-quasi-$*$-class $\mathbb{A}$ operators and the class of k-quasi-$*$-paranormal operators [20]. An operator T is said to be a k-quasi-$*$-class $\mathbb{A}$ operator if

$$A^k(|A^2| - |A^*|^2)A^k \geq 0$$

where k is a natural number and k-quasi-$*$-paranormal if

$$\|A^*A^kx\|^2 \leq \|A^{k+2}x\|\|A^kx\|$$

for all unit vector $x \in H$ where k is a natural number. 1-quasi-$*$-class $\mathbb{A}$ is quasi-$*$-class $\mathbb{A}$ and 1-quasi-$*$-paranormal is quasi-$*$-paranormal. It is shown that a k-quasi-$*$-class $\mathbb{A}$ operator is a k-quasi-$*$-paranormal operator [20].

An operator A in $B(H)$ is said to be an M-*-class Q operator [5], if there exists $M > 0$ such that

$$MA^{*2}A^2 - 2AA^* + I \geq 0.$$

$A \in B(H)$ is said to be (M, k)-quasi-*-class Q operator [5], if

$$A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^k \geq 0.$$

For $k = 1$, A is an M-quasi-*-class Q operator. It is clear that

$$M\text{-*-class } Q \subset M\text{-quasi-*-class } Q \subset (M, k)\text{-quasi-*-class } Q$$

and that

$$(M, k)\text{-quasi-*-class } Q \subset (M, k+1)\text{-quasi-*-class } Q.$$

Example Consider on the Hilbert space l_2, equipped with its standard orthonormal basis $(e_n)_n$, the weighted right shift defined by $Se_n = \lambda_n e_{n+1}$, where $(\lambda_n)_n$ is a decreasing complex sequence. Then, S is an (M, k)-quasi-*-class Q operator if and only if

$$M|\lambda_{n+k}|^2|\lambda_{n+k+1}|^2 + 1 \geq 2|\lambda_{n+k-1}|^2$$

for all n. Indeed, we have

$$\begin{aligned}
&\langle S^{*k}(MS^{*2}S^2 - 2SS^* + I)S^k e_n, e_n\rangle \geq 0 \\
&\Leftrightarrow (M|\lambda_{n+k}|^2|\lambda_{n+k+1}|^2 - 2|\lambda_{n+k-1}|^2 + 1)\lambda_n\lambda_{n+1}...\lambda_{n+k-1}\overline{\lambda_{n+k-1}\lambda_{n+k-2}...\lambda_n} \geq 0 \\
&\Leftrightarrow (M|\lambda_{n+k}|^2|\lambda_{n+k+1}|^2 - 2|\lambda_{n+k-1}|^2 + 1)|\lambda_n|^2|\lambda_{n+1}|^2...|\lambda_{n+k-1}|^2 \geq 0 \\
&\Leftrightarrow M|\lambda_{n+k}|^2|\lambda_{n+k+1}|^2 - 2|\lambda_{n+k-1}|^2 + 1 \geq 0.
\end{aligned}$$

In this paper, we are interested in the study of (M, k)-quasi-*-class Q operators. Some properties of this class of operators are shown. It is proved that this class of operators contains the class of k-quasi-*-class $\mathbb{A}$ operators. The decomposition of such operators, their restrictions on invariant subspaces and other related results are also presented.

2 Main results

We will start by the following useful theorem.

Theorem 2.1. *An operator $A \in B(H)$ is an (M, k)-quasi-*-class Q operator if and only if*

$$2\left\|A^*A^kx\right\|^2 \leq M\left\|A^{k+2}x\right\|^2 + \left\|A^kx\right\|^2$$

for all x in H.

Proof. There exists $M > 0$ such that

$$\left\langle A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^kx, x\right\rangle \geq 0$$

for all $x \in H$. Hence,

$$M\left\langle A^{*k+2}A^{k+2}x, x\right\rangle + \left\langle A^{*k}A^kx, x\right\rangle \geq 2\left\langle A^*A^kx, A^*A^kx\right\rangle.$$

Thus,

$$2\left\|A^*A^kx\right\|^2 \leq M\left\|A^{k+2}x\right\|^2 + \left\|A^kx\right\|^2.$$

The converse can be proved in a similar way. □

Remark 1. It is clear that the class of (M, k)-quasi-*-class Q operators is nested with respect to M, i.e.,

$$(M_1, k)\text{-quasi-*-class } Q \subset (M_2, k)\text{-quasi-*-class } Q$$

whenever $M_1 \leq M_2$.

Remark 2. The class of (M, k)-quasi-*-class Q operators is not convex. For example, the operators $A=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ are 4-quasi-*-class Q. However, the operator $C = \frac{1}{3}A + \frac{2}{3}B$ is not a 4-quasi-*-class Q operator since

$$2\|C^*C(0,-1)\|^2 = \frac{20}{81} > 4\|C^3(0,-1)\|^2 + \|C(0,-1)\|^2 = \frac{85}{729}.$$

Remark 3. Also, the operator $A - I$ is not a $(4, k)$-quasi-*-class Q operator. This shows that the above class is not translation invariant.

Theorem 2.2. *If $A \in B(H)$ is an (M, k)-quasi-*-class Q operator with dense range, then A is an M-*-class Q operator.*

Proof. Let $x \in H$. Since A has dense range, there exists a sequence $(x_n)_n$ in H for which $\lim_{n\to\infty} Ax_n = x$. By the continuity of A, $\lim_{n\to\infty} A^kx_n = A^{k-1}x$. Hence, and by the continuity of the inner product,

$$\begin{aligned}
\left\|A^*A^{k-1}x\right\|^2 &= \left\|\lim_{n\to\infty} A^*A^kx_n\right\|^2 = \lim_{n\to\infty}\left\|A^*A^kx_n\right\|^2 \\
&\leq \lim_{n\to\infty}\frac{1}{2}(M\left\|A^{k+1}x_n\right\|^2 + \left\|A^kx_n\right\|^2) \\
&= \frac{1}{2}(M\left\|\lim_{n\to\infty} A^{k+1}x_n\right\|^2 + \left\|\lim_{n\to\infty} A^kx_n\right\|^2) \\
&= \frac{1}{2}(M\left\|A^{k+1}x\right\|^2 + \left\|A^{k-1}x\right\|^2).
\end{aligned}$$

Thus, A is an $(M, k-1)$-quasi-*-class Q operator. Since $ran(A)$ is dense in H, A is an $(M, k-2)$-quasi-*-class Q operator. By iteration, A is an M-*-class Q operator. □

Corollary 2.1. *Let A be a nonzero (M, k)-quasi-*-class Q operator, but not an M-*-class Q operator. Then A admits a non trivial closed invariant subspace.*

Proof. Suppose that A has no non trivial closed invariant subspaces. Since $A \neq 0$, $\overline{ker(A)} \neq H$ and $\overline{ran(A)} \neq \{0\}$ are closed invariant subspaces for A. Thus, necessarily, $ker(A) = \{0\}$ and $\overline{ran(A)} = H$. By Theorem 2.2, A is an M-*-class Q operator. This contradicts the hypothesis. □

Theorem 2.3. *Let $A \in B(H)$ be an (M, k)-quasi-*-class Q operator. If $N \subset H$ is a closed A-invariant subspace, then the restriction $A|N$ is an (M, k)-quasi-*-class Q operator.*

Proof. Let

$$A = \begin{pmatrix} T & S \\ 0 & R \end{pmatrix} \text{ on } H = N \oplus N^{\perp}.$$

Then, for all integer m, $m \geq 2$, we get

$$A^m = \begin{pmatrix} T^m & \sum_{p=0}^{m-1} T^{m-1-p} S R^p \\ 0 & R^m \end{pmatrix}.$$

Since A is (M, k)-quasi-*-class Q, there exists $M > 0$ such that

$$A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^k \geq 0.$$

Then,

$$A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^k = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$$

where, $X = T^{*k}(MT^{*2}T^2 - 2TT^* - 2SS^* + I)T^k$, Y is a bounded operator from N to $N^{\perp}$ and Z is a bounded operator on $N^{\perp}$. According to [4, Theorem 6], $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X \geq 0, Z \geq 0$ and $Y = X^{\frac{1}{2}} C Z^{\frac{1}{2}}$ for some contraction C on H. Therefore,

$$X = T^{*k}(MT^{*2}T^2 - 2TT^* - 2SS^* + I)T^k \geq 0.$$

Since $SS^* \geq 0$,

$$T^{*k}(MT^{*2}T^2 - 2TT^* + I)T^k \geq 0$$

which completes the proof. □

Theorem 2.4. *If $B \in B(H)$ is unitarily equivalent to an (M, k)-quasi-*-class Q operator A on H, then B is also an (M, k)-quasi-*-class Q operator.*

Proof. There exists a unitary operator V on H satisfying $B = VAV^*$. Since A is an (M, k)-quasi-*-class Q operator,

$$\begin{aligned}
&B^{*k}(MB^{*2}B^2 - 2BB^* + I)B^k \\
&= (VAV^*)^{*k}\left[M\,(VAV^*)^{*2}\,(VAV^*)^2 - 2VAV^*\,(VAV^*)^* + I\right](VAV^*)^k \\
&= VA^{*k}V^*\left[MVA^{*2}V^*VA^2V^* - 2VA^2V^* + I\right]VA^kV^* \\
&= VA^{*k}(MA^{*2}A^2 - 2AA^* + I)A^kV^* \geq 0.
\end{aligned}$$

Thus, B is an (M, k)-quasi-*-class Q operator. □

Remark 4. Theorem 2.4 is in general false if the operator U is invertible and not unitary. Indeed, the bilateral weighted shift S defined on the Hilbert space $\ell_2(\mathbb{Z})$ by

$$Se_n = \begin{cases} e_{n+1}, & n \leq 1 \text{ or } n \geq 3 \\ \sqrt{2}e_3 & n = 2 \end{cases}$$

is in particuliar a $(3,k)$-quasi-*-class Q, and the operator

$$Ue_n = \begin{cases} e_{n+1}, & n \leq 1 \text{ or } n \geq 3 \\ \frac{1}{3}e_3 & n = 2 \end{cases}$$

is invertible and not unitary. Nonetheless, the operator $U^{-1}SU$ is not a $(3,k)$-quasi-*-class Q operator.

Theorem 2.5. *Let $A \in B(H)$ be an (M,k)-quasi-*-class Q operator. If A commutes with an isometric operator $S \in B(H)$, then AS is an (M,k)-quasi-*-class Q operator.*

Proof. We have $AS = SA$ and $S^*S = I$. Since A is an (M,k)-quasi-*-class Q operator,

$$\begin{aligned}
&(AS)^{*k}(M(AS)^{*2}(AS)^2 - 2AS(AS)^* + I)(AS)^k \\
&= S^{*k}A^{*k}\left[MS^*A^*S^*A^*ASAS - 2ASS^*A^* + I\right]S^kA^k \\
&= A^{*k}S^{*k}\left[MA^{*2}A^2 - 2ASS^*A^* + I\right]S^kA^k \\
&= A^{*k}S^{*k-1}\left[MS^*A^{*2}A^2S - 2S^*ASS^*A^*S + S^*S\right]S^{k-1}A^k \\
&= A^{*k}S^{*k-1}\left[MA^{*2}A^2 - 2AA^* + I\right]S^{k-1}A^k \\
&= S^{*k-1}A^{*k}\left[MA^{*2}A^2 - 2AA^* + I\right]A^kS^{k-1} \geq 0.
\end{aligned}$$

Thus, AS is an (M,k)-quasi-*-class Q operator.

□

Theorem 2.6. *Let $A \in B(H)$ be an (M,k) -quasi-*-class Q operator. Assume that $\overline{A^kH} \neq H$, and that*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

with respect to the decomposition $H = \overline{ran(A^k)} \oplus ker(A^{*k})$. Then, A_1 is an M --class Q operator and $A_3^k = 0$. Moreover, $\sigma(A) = \sigma(A_1) \cup \{0\}$, where $\sigma(A)$ denotes the spectrum of A.*

Proof. Since A is an (M,k)-quasi-*-class Q operator,

$$\left\langle A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^ky, y\right\rangle \geq 0$$

for all $y \in H$. Hence,

$$\left\langle (MA^{*2}A^2 - 2AA^* + I)A^ky, A^ky\right\rangle \geq 0.$$

Thus, for all $x \in \overline{ran(A^k)}$,

$$\left\langle (MA^{*2}A^2 - 2AA^* + I)x, x\right\rangle = \left\langle (MA_1^{*2}A_1^2 - 2A_1A_1^* + I)x, x\right\rangle \geq 0.$$

Consequently, A_1 is an M-*-class Q operator. Let now, P be the orthogonal projection on $\overline{ran(A^k)}$. For all $x = x_1 + x_2, y = y_1 + y_2 \in H$, we have

$$\left\langle A_3^kx_2, y_2\right\rangle = \left\langle A^k(I-P)x, (I-P)y\right\rangle = \left\langle (I-P)x, A^{*k}(I-P)y\right\rangle = 0.$$

Thus, $A_3^k = 0$. Furthermore,

$$\sigma(A_1) \cup \sigma(A_3) = \sigma(A) \cup \Omega$$

where Ω is the union of the holes in $\sigma(A)$ which happen to be subsets of $\sigma(A_1) \cap \sigma(A_3)$ using [9, Corollary 7], and $\sigma(A_1) \cap \sigma(A_3)$ has no interior points and A_3 is nilpotent. Thus, $\sigma(A) = \sigma(A_1) \cup \{0\}$. □

It is shown in [25] that for $A, B, Q \in B(H)$, the equation $BX - XA = Q$ admits a unique solution whenever $\sigma(A)$ and $\sigma(B)$ are disjoint. For more details, reader can see [13, 18, 24] and [26].

Corollary 2.2. *Let $A \in B(H)$ be an (M, k)-quasi-*-class Q operator. If the restriction $A_1 = A\big|_{\overline{ran(A^k)}}$ is invertible, then A is similar to the sum of an M-*-class Q operator and a nilpotent operator.*

Proof. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \quad \text{on } H = \overline{ran(A^k)} \oplus ker(A^{*k}).$$

Then, A_1 is an M-*-class Q operator by the above Theorem. Since A_1 is invertible, $0 \notin \sigma(A)$. Hence, $\sigma(A_1) \cap \sigma(A_3) = \emptyset$. By Rosenblum's result [18, 25, 27], there exists $C \in B(H)$ for which $A_1C - CA_3 = A_2$. Thus,

$$\begin{aligned} A &= \begin{pmatrix} I & -C \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}. \end{aligned}$$

□

Let $A \in B(H)$. Denote by $\mathcal{R}(\sigma(A))$ the set of all rational analytic functions on $\sigma(A)$. The operator A is said to be n-multicyclic [11], if there exist n (generating) vectors $x_1, x_2, .., x_n$ in H such that

$$\bigvee \{g(A)x_i\ ,\ 1 \leq i \leq n\ ,\ g \in \mathcal{R}(\sigma(A))\} = H$$

where $\bigvee$ denotes the linear span, that is, the set of all finite linear combinations.

We have then

Theorem 2.7. *If A is an n-multicyclic (M, k)-quasi-*-class Q operator, then its restriction on $\overline{ran(A^k)}$ is also n-multicyclic.*

Proof. Put

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

on the decomposition $H = \overline{ran(A^k)} \oplus ker(A^{*k})$. Since $\sigma(A_1) \subset \sigma(A)$ by Theorem 2.6, $\mathcal{R}(\sigma(A_1)) \subset \mathcal{R}(\sigma(A))$. The operator A is n-multicyclic. Then, there exist n generating vectors $x_1, x_2, .., x_n \in H$ for which

$$\bigvee \{g(A)x_i,\ 1 \leq i \leq n,\ g \in \mathcal{R}(\sigma(A))\} = H.$$

Put $y_i = A^k x_i$, $1 \leq i \leq n$. Hence,

$$\begin{aligned}\bigvee\{g(A_1)y_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\} &= \bigvee\left\{g(A_1)A^kx_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\right\}\\ &= \bigvee\left\{g(A)A^kx_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\right\}\\ &= \bigvee\left\{A^kg(A)x_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\right\}\\ &= \overline{ran(A^k)}.\end{aligned}$$

But

$$\bigvee\{g(A_1)y_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\}\subset\bigvee\{g(A_1)y_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A_1))\}.$$

Thus,

$$\overline{ran(A^k)}\subset\bigvee\{g(A_1)y_i,\ 1\le i\le n,\ \ g\in\mathcal{R}(\sigma(A_1))\}.$$

Therefore, $\{y_i\}_{i=1}^n$ are n-generating vectors of A_1, and A_1 is n-multicyclic. □

Recall that an operator $A\in B(H)$ is said to be a class $\mathbb{A}$ operator [8, 20, 23] if $|A^2|-|A|^2\ge 0$. This class was introduced by Furuta-Ito-Yamazaki [8], and it is shown that it contains both p-hyponormal operators and log-hyponormal operators. It is also proved in [8, 28] that the class $\mathbb{A}$ is a subclass of paranormal operators. It is known that p-hyponormal operators are normaloid, i.e., $\|A\|=r(A)$ where $r(A)$ denotes the spectral radius of A. However, a quasi-class $\mathbb{A}$ operator is not normaloid [23], [28]. $A\in B(H)$ is said to be in the -*-class $\mathbb{A}$ if $|A|^2-|A^*|^2\ge 0$, and in the k-quasi-*-class $\mathbb{A}$ if $A^{*k}(|A|^2-|A^*|^2)A^k\ge 0$ for a positive integer k. A 1-quasi-*-class $\mathbb{A}$ operator is quasi-*-class $\mathbb{A}$.

In the sequel, we will show that the (M,k)-quasi-*-class Q operators contains the k-quasi-*-class $\mathbb{A}$. We need first the following result

Lemma 2.1. *If $A\in B(H)$ is a k-quais-*-class $\mathbb{A}$ operator, then*

$$\||A|^2A^kx\|\le\|A^{k+2}x\|$$

for all $x\in H$.

Proof. Let x be any vector in H. Since A is a k-quasi-*-class $\mathbb{A}$, we have

$$\begin{aligned}\||A|^2A^kx\|^2=\|A^*AA^kx\|^2 &= \langle A^*A^{k+1}x,A^*A^{k+1}x\rangle\\ &= \langle x,A^*(A^{*k}AA^*A^k)Ax\rangle\\ &= \langle Ax,(A^{*k}AA^*A^k)Ax\rangle\\ &= \langle (A^{*k}AA^*A^k)Ax,Ax\rangle\\ &\le \langle A^{*k+1}A^{k+1}Ax,Ax\rangle\\ &= \|A^{k+2}x\|^2.\end{aligned}$$

□

Theorem 2.8. *An operator belonging to the k-quasi-*-class $\mathbb{A}$ is an (M,k)-quasi-*-class Q operator.*

Proof. Let A be a k-quasi-*-class $\mathbb{A}$ operator. Then,

$$A^{*k}(|A|^2-|A^*|^2)A^k\ge 0.$$

Hence, for $M\ge 1$ we have

$$A^{*k}(\sqrt{M}\,|A|^2-|A^*|^2)A^k\ge 0.$$

Thus, for all $x \in H$,

$$\begin{aligned}\langle A^{*k} |A^*|^2 A^k x, x\rangle &= \langle A^{*k} AA^* A^k x, x\rangle \\ &= \|A^* A^k x\|^2 \\ &\leq \left\langle \sqrt{M} A^{*k} |A|^2 A^k x, x\right\rangle \\ &= \left\langle \sqrt{M} |A|^2 A^k x, A^k x\right\rangle .\end{aligned}$$

Using the Cauchy-Schwarz inequality and Lemma 2.1,

$$\begin{aligned}\|A^* A^k x\|^2 &\leq \sqrt{M} \left\||A|^2 A^k x\right\| \|A^k x\| \\ &\leq \sqrt{M} \|A^{k+2} x\| \|A^k x\| \\ &\leq \frac{1}{2}(M \|A^{k+2} x\|^2 + \|A^k x\|^2).\end{aligned}$$

This shows that A is an (M, k)-quasi-*-class Q operator.

□

Theorem 2.9. *If $A \in B(H)$ with $\|A\| \leq \frac{1}{\sqrt{2}}$, then A is an (M, k)-quasi-*-class Q operator.*

Proof. Let $x \in H$. We have $\|A^* x\| \leq \frac{1}{\sqrt{2}} \|x\|$. Hence,

$$\begin{aligned}\langle (MA^{*2} A^2 - 2AA^* + I)x, x\rangle &= M \|A^2 x\|^2 - 2\|A^* x\|^2 + \|x\|^2 \\ &\geq M \|A^2 x\|^2 - \|x\|^2 + \|x\|^2 \geq M \|A^2 x\|^2 \\ &\geq 0.\end{aligned}$$

Thus,

$$\langle A^{*k}(MA^{*2} A^2 - 2AA^* + I)A^k x, x\rangle \geq 0.$$

□

Recall that an operator A in $B(H)$ is said to have the *Single Valued Extension Property*, briefly SVEP, at a complex number α, if for each open neighborhood V of α, the unique analytic function $f : V \to H$ that satisfies

$$\forall \lambda \in V : (A - \lambda)f(\lambda) = 0$$

is the function $f \equiv 0$. If furthermore, A has SVEP at every $\alpha \in \mathbb{C}$, we say that A has SVEP. For more details see ([2, 17, 15, 21]).

Also, the *local resolvent set* of A at a vector $x \in H$, denoted by $\rho_A(x)$, is defined to consist of all complex elements z_0 such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, for which $(A - z)f(z) = x$. [2]

The set $\sigma_A(x) = \mathbb{C} \setminus \rho_A(x)$ is called the *local spectrum* of A at x. We've then the following important result.

Theorem 2.10. *Let*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

be an (M, k)-quasi--class Q operator with respect to the decomposition $H = \overline{ran(A^k)} \oplus ker(A^{\star k})$. Then, for all $x = x_1 + x_2 \in H$:*

i. $\sigma_{A_3}(x_2) \subset \sigma_A(x_1 + x_2)$.

ii. $\sigma_{A_1}(x_1) = \sigma_A(x_1 + 0)$.

Proof. i. Let $z_0 \in \rho_A(x_1 + x_2)$. By the definition of the local resolvent set of A at x, there exists a neighborhood U of z_0 and an analytic function $f(z)$ defined on U, with values in H, for which

$$(A - z)f(z) = x, \ z \in U. \tag{2.1}$$

Let $f = f_1 + f_2$ where

$$f_1 : U \to \overline{ran(A^k)}), \quad f_2 : U \to ker(A^{*k})$$

are in the Frechet spaces $O(U, \overline{ran(A^k)}), O(U, ker(A^{*k}))$ respectively, consisting of analytic functions on U with values in H, and equipped with the topology of uniform convergence, [2]. Equality (2.1) can then be written

$$\begin{pmatrix} A_1 - z & A_2 \\ 0 & A_3 - z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then, for all $z \in U$,

$$(A_3 - z)f_2(z) = x_2.$$

Hence,

$$z_0 \in \rho_{A_3}(x_2).$$

Thus, (i) holds by passing to the complement.

ii. For $z_1 \in \rho_A(x_1 + 0)$, there exists a neighborhood V_1 of z_1 and an analytic function g defined on V_1 with values in H verifying

$$(A - z)f(z) = x_1 + 0, \ z \in V_1. \tag{2.2}$$

Let $g = g_1 + g_2$, where

$$g_1 \in O(V_1, \overline{ran(A^k)}), \ g_2 \in O(V_1, ker(A^{*k}))$$

be as in (i). From equation (2.2), we obtain

$$(A_1 - z)g_1(z) + A_2 g_2(z) = x_1$$

and

$$(A_3 - z)g_2(z) = 0, \ z \in V_1$$

Since A_3 is nilpotent by Theorem 2.6, A_3 has SVEP by [2]. Thus,

$$g_2(z) = 0$$

Consequently,

$$(A_1 - z)g_1(z) = x_1$$

Therefore, $z_1 \in \rho_{A_1}(x_1)$, and then

$$\rho_T(x_1 + 0) \subset \rho_{A_1}(x_1)$$

Thus,

$$\sigma_{A_1}(x_1) \subset \sigma_A(x_1 + 0)$$

Now, if $z_2 \in \rho_{A_1}(x_1)$, then, there exists a neighborhood V_2 of z_2 and an analytic function h from V_2 onto H, such that

$$(A_1 - z)h(z) = x_1, \ z \in V_2$$

Thus,

$$(A-z)(h(z)+0)=(A_1-z)h(z)=x_1=x_1+0$$

Hence,

$$z_2\in\rho_A(x_1+0)$$

□

Definition 1. An operator $A\in B(H)$ is said to be (M,k)-quasi-*-paranormal if there exists M and a positive integer k such that

$$A^{*k}(MA^{*2}A^2-2\lambda AA^*+\lambda^2)A^k\geq 0$$

for all $\lambda>0$.

This definition is equivalent to

$$\|A^{\star}A^kx\|^2\leq\sqrt{M}\|A^{k+2}x\|\|A^kx\|$$

for all $x\in H$.

Theorem 2.11. *Let $A\in B(H)$ be an (M,k)-quasi-*-class Q operator such that A^2 is an isometry on H. Then A is (M,k)-quasi-*-paranormal.*

Proof. Since A^2 is an isometry, $A^{\star 2}A^2=I$, and then $\|A^2x\|=\|x\|,\ x\in H$. By iteration, $\|A^{k+2}x\|=\|A^kx\|,\ k\geq 1$. Since A is an (M,k)-quasi-*-class Q operator,

$$\begin{aligned}2\|A^{\star}A^kx\|^2&\leq M\|A^{k+2}x\|^2+\|A^kx\|^2\\&\leq\left(\sqrt{M}\|A^{k+2}x\|-\|A^kx\|\right)^2+2\sqrt{M}\|A^{k+2}x\|\|A^kx\|\\&\leq 2\sqrt{M}\|A^{k+2}x\|\|A^kx\|\end{aligned}$$

□

Definition 2. An operator $A\in B(H)$ is said to be isoloid, if every isolated point of its spectrum is an eigenvalue of A.

We have then the following result.

Theorem 2.12. *Each (M,k)-quasi-*-class Q operator is isoloid.*

Proof. Let A be an (M,k)-quasi-*-class Q operator. Suppose that A has a representation given in Theorem 2.6. Let z be an isolated point in $\sigma(A)$. Since $\sigma(A)=\sigma(A_1)\cup\{0\}$, z is an isolated point in $\sigma(A_1)$ or $z=0$.

If z is an isolated point in $\sigma(A_1)$, then $z\in\sigma_p(A_1)$. Assume that $z=0$ and $z\notin\sigma(A_1)$. Then, for $x\in\ker A_3$, we get $(-A_1^{-1}A_2x\oplus x)\in\ker A$. □

Theorem 2.13. *Let $A\in B(H)$ be an (M,k)-quasi-*-class Q operator, and let $N\subseteq H$ be a closed A-invariant subspace for which the restriction $A|N$ is an injective and normal operator. Then N reduces A, that is, N is invariant for A and A^*.*

Proof. Suppose that P is an orthogonal projection of H onto $\overline{ranA^k}$. Since A is an (M,k)-quasi-*-class Q operator, we have

$$P(MA^{*2}A^2 - AA^*)P \geq 0.$$

By assumption, $A|_N$ is an injective normal operator. Then, $E \leq P$ for the orthogonal projection E of H onto N, and $\overline{ranA^k|_N} = N$ because $A|_N$ has a dense range. Therefore, $N \subseteq \overline{ranA^k}$ and hence

$$E(MA^{*2}A^2 - AA^*)E \geq 0.$$

Let

$$A = \begin{pmatrix} A\,|N & A_2 \\ 0 & A_3 \end{pmatrix},$$

on $N \oplus N^{\perp}$. Then,

$$AA^* = \begin{pmatrix} A\,|N\;A^*\,|N + A_2A_2^* & A_2A_3^* \\ A_3A_2^* & A_3A_3^* \end{pmatrix}$$

and

$$MA^{\star 2}A^2 = \begin{pmatrix} MA^{*2}\,|N\;A^2\,|N & S \\ T & R \end{pmatrix}$$

for some bounded linear operators S, T and R. Thus,

$$\begin{aligned} \begin{pmatrix} A\,|N\;A^*\,|N + A_2A_2^* & 0 \\ 0 & 0 \end{pmatrix} = E(AA^*)E = E|A^*|^2E &\leq E(A^{*2}A^2)^{\frac{1}{2}}E \\ &\leq (E(A^{*2}A^2E))^{\frac{1}{2}} \\ &= \begin{pmatrix} A^{*2}\,|N\;A^2\,|N & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{2}} \end{aligned}$$

This implies that

$$A|_NA^*|_N + A_2A_2^* \leq A|_NA^*|_N.$$

Since $A|N$ is normal and $A_1A_1^*$ is positive, it follows that $A_2 = 0$. Hence N reduces A. □

Remark 5. The previous result is in general false if the restriction $A\,|N$ is not injective. In fact, if A is a nilpotent operator of order k, such that $A^{k-1} \neq 0$, then $A\,\Big|\overline{ranA^{k-1}} = 0$ is a normal operator. Assume that $\overline{ranA^{k-1}}$ reduces A. Then, $A^\star A^{k-1}H \subset \overline{ranA^{k-1}}$. Thus, $A^{*k-1}A^{k-1}H \subset \overline{ranA^{k-1}}$ and $kerA^{*k-1} \subset kerA^{*k-1}A^{k-1} = kerA^{k-1}$. Since $A^{*k} = A^{*k-1}A^* = 0$, $A^{k-1}A^* = 0$. Hence, $A^{k-1}A^{*k-1} = 0$. Therefore, $A^{k-1} = 0$. This contradicts the hypotheses on A.

Theorem 2.14. *Let A be an (M,k)-quasi-*-class Q operator. Equation $(A-\lambda)x = 0$ implies $(A-\lambda)^\star x = 0$ for all non-zero complex scalar λ.*

Proof. Assume that $x \neq 0$. Let $N = \text{span}\{x\}$ and

$$A = \begin{pmatrix} \lambda & T \\ 0 & S \end{pmatrix} \quad \text{on } H = N \oplus N^{\perp}.$$

Let $P : H \to N$ be the orthogonal projection. Then, $A\,|N = \lambda$ is an injective normal operator. Hence, N reduces A by Theorem 2.11. Thus, $T = 0$. □

Theorem 2.15. *Let $A \in B(H)$ be an (M,k)-quasi-*-class Q operator, and let $\lambda \in \mathbb{C}, \lambda \neq 0$ be an isolated point of the spectrum of A. Then, the Riesz idempotent E for λ is self-adjoint, and satisfies the following equality*

$$EH = ker(A-\lambda) = ker(A-\lambda)^\star.$$

Proof. By Theorem 2.12, λ is an eigenvalue of A, and $EH = ker(A-\lambda)$. According to Theorem 2.14, it sufficies to show that $ker(A-\lambda)^\star \subset ker(A-\lambda)$. The subspace $ker(A-\lambda)$ reduces A by Theorem 2.14, and the restriction of A on its reducing subspace is an (M,k)-quasi-*-class Q operator by Theorem 2.3. It follows that

$$A = \lambda \oplus B \text{ on } H = ker(A-\lambda) \oplus (ker(A-\lambda))^\perp$$

where B is (M,k)-quasi-*-class Q and $ker(B-\lambda) = \{0\}$. We've

$$\lambda \in \sigma(A) = \{\lambda\} \cup \sigma(B)$$

and λ is isolated. Then, either $\lambda \notin \sigma(B)$, or λ is an isolated point of $\sigma(B)$, which contradicts the fact that $ker(A-\lambda) = \{0\}$. Since B is invertible on $(ker(A-\lambda))^\perp$,

$$ker(A-\lambda) = ker(A-\lambda)^\star.$$

Furthermore, since $EH = ker(A-\lambda) = ker(A-\lambda)^\star$,

$$((z-A)^\star)^{-1}E = \overline{(z-\lambda)^{-1}}E.$$

Thus,

$$\begin{aligned} E^\star = -\frac{1}{2\pi i}\int_{\partial D} ((z-A)^\star)^{-1}E\, d\overline{z} &= -\frac{1}{2\pi i}\int_{\partial D} \overline{(z-\lambda)^{-1}}E\, d\overline{z} \\ &= \overline{\frac{1}{2\pi i}\int_{\partial D} (z-\lambda)^{-1}\, dz}E = E. \end{aligned}$$

So, E is self-adjoint. □

3 Weyl's Theorem

An operator $A \in B(H)$ is called Fredholm if $R(A)$ is closed, $\alpha(A) = \dim N(A) < \infty$ and $\beta(A) - \dim H \setminus R(A) < \infty$. Moreover if $i(A) = \alpha(A) - \beta(A) = 0$, then A is called Weyl. The Weyl spectrum $w(A)$ of A is defined by

$$w(A) := \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not Weyl}\}.$$

According to [10], we say that Weyl's theorem holds for A if

$$\sigma(A) \setminus w(A) = \pi_{00}(A),$$

where

$$\pi_{00}(A) = \{\lambda \in iso\sigma(A) : \ 0 < \dim N(A - \lambda I < \infty\}.$$

In [22], Patel showed that Weyl's theorem holds for 2-isometric operators, i.e., operators satisfying

$$A^{*2}A^2 - 2A^*A + I = 0$$

[1], which has been extended to many non normal operators [16, 19]. In this section, we prove that Weyl's theorem holds for (M,k)-quasi-$*$-class Q operators without any additional conditions.

Theorem 3.1. *Weyl's theorem holds for any (M,k)-quasi-*-class Q operator.*

Proof. Suppose that A is a (M,k)-quasi-*-class Q operator. Then A has SVEP at zero. Either $\sigma(A_1) \subseteq \partial\mathcal{D}$ or $\sigma(A_1) = \overline{\mathcal{D}}$, where $\mathcal{D}$ denotes the open unit disc, and $\partial\mathcal{D}$ is its boundary. If $\sigma(A_1) \subseteq \partial\mathcal{D}$, then A has SVEP everywhere: else $\sigma(A_1) = \overline{\mathcal{D}}$. The operator A has SVEP on $\sigma(A) \setminus w(A)$, then $< 0 \dim(A-\lambda) < \infty$. We have $\lambda \in \sigma_p(A) \subseteq \partial\mathcal{D} \cup \{0\}$, An operator such that its point spectrum has empty interior has SVEP [2, Remark 2.4(d)]. Hence A has SVEP. Also, if $\sigma(A_1) = \sigma(A) = \overline{\mathcal{D}}$, then $iso\sigma(A) = \emptyset$. If $\sigma(A_1) \subset \partial\mathcal{D}$, then A_1 is polaroid, that is, the isolated points of the spectrum of A_1 are poles of the resolvent. Hence, A is polaroid. This proves Weyl's theorem for A. □

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HARDY INEQUALITIES FOR p-WEAKLY MONOTONE FUNCTIONS

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Abstract. We prove Hardy-type inequalities

$$\left(\int_d^\infty \left|\int_d^s f(x)dx\right|^p s^\beta ds\right)^{1/p} \le C\left(\int_d^\infty |f(s)|^q s^\alpha ds\right)^{1/q}$$

for the class of p-weakly monotone functions with q or p smaller than 1 and $d \ge 0$.

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1 Introduction

The goal of this paper is to extend the results presented in [25] and [5] by proving inequalities of the type

$$\left(\int_d^\infty \left|\int_d^s f(x)dx\right|^p s^\beta ds\right)^{1/p} \le C\left(\int_d^\infty |f(s)|^q s^\alpha ds\right)^{1/q}$$

for p or q smaller than one and for p-weakly monotone f .

Definition 1. [31, 3] Let $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ be a measurable function, then we say that f is p-weakly monotone $\big($and write $f \in WM(K, \lambda, p)$, where $K > 0, \lambda > 1, p > 0\big)$, if the inequality

$$f(x)^p \le K \int_{x/\lambda}^{\lambda x} \frac{f(s)^p}{s} ds \tag{1.1}$$

holds for every $x > 0$. Similarly, let $f : I = [a, b] \to \mathbb{R}^+ \cup \{0\}$ be a measurable function, then we say that $f \in WM(K, \lambda, p)$ on I whenever $f\chi_I$ satisfies inequality (1.1).

Here and throughout the paper by χ_I we denote the characteristic function of I. The next concept was studied in [28] with applications to number series. It appeared in [25] as a quasi-monotonicity.

Definition 2. [28] Let $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ be a function, then we say that f is weakly monotone $\big($and write $f \in WM(K)$, where $K > 0\big)$ if the inequality

$$f(x) \le Kf(y) \tag{1.2}$$

holds for every $2y \ge x \ge y > 0$.

Let us mention that both weakly monotone and p-weakly monotone functions/sequences play an important role in various problems in analysis (see the precise references below). It is worth mentioning that the class of weakly monotone functions contains as a subclass the class of general monotone functions. Recall that for $C>0$, the $GM(C)$ class (see [30, 27]) is defined in the following way:

$$GM(C)=\Big\{f\in BV_{loc}:\operatorname{Var}(f)_{[x;2x]}\le C|f(x)|\quad\text{for all}\quad x\in(0,\infty)\Big\}.$$

Here assuming that f is locally absolutely continuous on $\mathbb{R}^+$, the expression $\operatorname{Var}(f)_{[x;2x]}$ can be replaced by $\int_x^{2x}|f'(t)|dt$. Similarly, any p-general monotone function is always p-weakly monotone (see [3, 27]), that is, $GM(C,\lambda,p)\subsetneq WM(K,\lambda,p)$, where K only depends on p,C and λ, and where

$$GM(C,\lambda,p)=\Big\{f\in BV_{loc}:\operatorname{Var}(f)_{[x;2x]}\le C\left(\int_{x/\lambda}^{\lambda x}\frac{|f(t)|^p}{t}dt\right)^{1/p}\quad\text{for all } x>0\Big\}.$$

It is known that for $p>1$ $GM(C)\subsetneq GM(C',\lambda,1)\subsetneq GM(C'',\lambda,p)$, where C' depends on C and λ; and C'' depends on C' and λ. For the first embedding see [27, 31], for the second one see [3]. We will see in Proposition 1.1 that the scale of weakly monotone functions has a similar structure.

Various applications of both general and weakly monotone sequences can be found in Fourier analysis and approximation theory. In particular, in the study of integrability of Fourier transforms [8, 17, 22] and trigonometric series [3, 4, 12, 14, 18], investigating various problems in approximation theory [11, 15, 20, 19, 26, 30, 31], convergence problems [7, 13, 16, 23, 27, 30], theory of number series [7, 28], and embedding theorems for smooth function spaces [3, 10, 9]. We emphasise that in many problems the consideration of either general monotone or weakly monotone sequences/functions imply completely different answers; see e.g. [3, 15, 27].

Let us present the main properties of weakly monotone and p-weakly monotone functions.

Proposition 1.1. *The following properties hold:*

1. $f\in WM(K,\lambda,p)$ *if and only if, for all* $x\in\mathbb{R}$
$$f(\exp(x))^p\le K\int_{x-\ln\lambda}^{\ln\lambda+x}f(\exp(t))^p dt;$$

2. $WM(K)\subsetneq WM(K',\lambda,p)$, *where* K' *depends only on* K,p *and* λ;

3. *Let* $q>p>0$, *then* $WM(K,\lambda,p)\subsetneq WM(K',\lambda,q)$, *where* K' *depends only on* K,p,q *and* λ;

4. *Let* $f\in WM(K,\lambda,p)$, *then* $g(t)=f(t^{-1})\in WM(K,\lambda,p)$. *However, if* $f\in WM(K)$, *then* g *may not be in* $WM(K')$ *for any* K';

5. *Let* $f\in WM(K,\lambda,p)$ *and* $\alpha\in\mathbb{R}$. *If* $g(t)=f(t)t^\alpha$, *then* $g\in WM(\lambda^{|\alpha|p}K,\lambda,p)$. *If* $f\in WM(K)$, *then* $g\in WM(K')$, *where* K' *depends only on* K *and* α;

6. *Let* $f\in WM(K,\lambda,p)$ *and* $\alpha\in\mathbb{R}$, *then* $g(t)=f(t)^\alpha\in WM(K,\lambda,p/\alpha)$. *If* $f\in WM(K)$, *then* $g\in WM(K')$, *where* K' *depends only on* K *and* α;

7. *Let* $f\in WM(K,\lambda,p)$ *and* $\alpha>0$, *then* $g(t)=f(t/\alpha)\in WM(K,\lambda,p)$. *If* $f\in WM(K)$, *then* $g\in WM(K')$, *where* K' *depends only on* K *and* α.

Proof. To show 1), we use a logarithmic change of variable. Furthermore, if $f \in WM(K)$, we have that
$$\frac{x(\lambda-1)}{\lambda} f(x)^p \leq K' \int_{x/\lambda}^{x} f(y)^p dy \leq K' \int_{x/\lambda}^{\lambda x} f(y)^p dy.$$
Hence, f is p-weakly monotone and 2) follows. To see that the inclusion is proper, consider $f(x) = x^a \chi_{(0,1)\cup(1,+\infty)}(x)$. Since $f(1) = 0$, f cannot be $WM(K)$ for any K and a simple calculation shows that $f \in WM(K(\lambda, a, p), \lambda, p)$, for every $\lambda > 1, p > 0$.

The embedding $WM(K, \lambda, p) \subsetneq WM(K', \lambda, q)$ follows from Hölder's inequality. To see its sharpness, for $c > 1$, consider f such that
$$f^q(\exp(x)) = g(x) = \sum_{n=1}^{\infty} c^n 4^{2^n} \chi_{[n, n+4^{-2^n-1}]}(x).$$

It is easy to see that $f \in WM(1/c, e^2, q)$ but $f \notin WM(K, \mu, q/2)$ for any K or μ. Therefore if $r = p/q < 1$ there must be some $n \geq 0$ such that
$$f^{r^n} \in WM(1/c, e^2, 1) \text{ but } f^{r^{n+1}} \notin WM(K, \mu, 1),$$
therefore $f^{r^n/q} \in WM(1/c, e^2, q)$ but $f^{r^n/q} \notin WM(K, \mu, p)$. For λ other than e^2, we can modify the previous example correspondingly.

To show the first part of 4) we use a change of variables, for the second part, consider $f(x) = x^{-1}\chi_{(0,1)}(x)$. Property 5) follows from the monotonicity of power functions while 6) is obvious. Finally, the first part of 7) follows from a change of variables and the second part is clear.

□

2 Weighted L_p spaces and Hardy inequalities

For $p > 0$ and $\alpha \in \mathbb{R}$ we denote
$$\|f\|_{p,\alpha} = \left(\int_0^{\infty} |f(s)|^p s^{\alpha} ds \right)^{1/p}, \tag{2.1}$$
and for $d > 0$ we denote
$$\|f\|_{p,\alpha}^{(d)} = \left(\int_d^{\infty} |f(s)|^p s^{\alpha} ds \right)^{1/p}. \tag{2.2}$$
Note that if $d > 0$, then $\|f\|_{p,\alpha}^{(d)} \leq d^{-\varepsilon/p} \|f\|_{p,\alpha+\varepsilon}^{(d)}$ for $\varepsilon > 0$.

First we are going to study the embeddings between weighted L_p spaces for p-weakly monotone functions.

From now on, by p and q we will denote positive numbers, by α and β, real numbers; and by C, a constant which depends only on $p, q, \alpha, \beta, K, \lambda$.

Lemma 2.1. *Let $p \geq q > 0$, $\beta \in \mathbb{R}$, and $f \in WM(K, \lambda, q)$. Let $\alpha = \frac{q}{p}(\beta + 1) - 1$. Then*
$$\left(\int_0^{x} f^p(s) s^{\beta} ds \right)^{q/p} \leq C \int_0^{\lambda x} f(s)^q s^{\alpha} ds$$
and
$$\left(\int_x^{\infty} f^p(s) s^{\beta} ds \right)^{q/p} \leq C \int_{x/\lambda}^{\infty} f(s)^q s^{\alpha} ds.$$

Proof. Let $n \in \mathbb{Z}$ be such that $\lambda^{n-1} < x \leq \lambda^n$. For each $j \in \mathbb{Z}$, let $\lambda^j \leq s_j \leq \lambda^{j+1}$ be such that

$$\int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds \leq \left(\lambda^{j+1} - \lambda^j\right) f(s_j)^p s_j^\beta.$$

Note that if for all $\lambda^j \leq t \leq \lambda^{j+1}$,

$$\int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds > \left(\lambda^{j+1} - \lambda^j\right) f(t)^p t^\beta,$$

integrating both sides,

$$\left(\lambda^{j+1} - \lambda^j\right) \int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds = \int_{\lambda^j}^{\lambda^{j+1}} \left(\int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds\right) dt > \left(\lambda^{j+1} - \lambda^j\right) \int_{\lambda^j}^{\lambda^{j+1}} f(t)^p t^\beta dt,$$

we arrive at a contradiction, therefore s_j must exist.

We see that

$$\left(\int_0^x f^p(s) s^\beta ds\right)^{q/p} \leq \left(\int_0^{\lambda^{n-1}} f^p(s) s^\beta ds\right)^{q/p} + \left(\int_{\lambda^{n-1}}^x f^p(s) s^\beta ds\right)^{q/p}.$$

Hence

$$\left(\int_0^{\lambda^{n-1}} f^p(s) s^\beta ds\right)^{q/p} \leq \sum_{j=-\infty}^{n-2} \left(\int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds\right)^{q/p} \leq C \sum_{j=-\infty}^{n-2} s_j^{q\beta/p} f(s_j)^q \lambda^{qj/p}.$$

Now, since $f \in WM(K, \lambda, q)$

$$\sum_{j=-\infty}^{n-2} s_j^{q\beta/p} f(s_j)^q \lambda^{qj/p} \leq C \sum_{j=-\infty}^{n-2} s_j^{q\beta/p} \lambda^{qj/p} \int_{\lambda^{j-1}}^{\lambda^{j+2}} \frac{f(s)^q}{s} ds \leq C \int_0^{\lambda x} f(s)^q s^\alpha ds.$$

Finally, by the same token

$$\left(\int_{\lambda^{n-1}}^x f^p(s) s^\beta ds\right)^{q/p} \leq C \int_{\lambda^{n-2}}^{\lambda x} f(s)^q s^\alpha ds.$$

And the result follows by adding both inequalities up. The proof of the second inequality is analogous. □

Proposition 2.1. *There is a $C > 0$ such that for all $f \in WM(K, \lambda, q)$*

$$\|f\|_{p,\beta} \leq C\|f\|_{q,\alpha} \quad \Longleftrightarrow \quad \frac{\alpha+1}{q} = \frac{\beta+1}{p} \text{ and } q \leq p.$$

Proof. The "if" part is a restatement of Lemma 2.1. The proof of the "only if" part will be given in section 3. □

Remark 1. In the general case, that is, without the assumption that $f \in WM(K, \lambda, p)$, it is not possible to obtain any non-trivial embedding of the type $\|f\|_{p,\beta} \leq C\|f\|_{q,\alpha}$.

Proof. First, let f be a non-negative function which is not zero almost everywhere. For $\lambda > 0$, let $f_\lambda(t) = f(\lambda t)$. Then a change of variables shows that $\|f_\lambda\|_{p,\beta} = \lambda^{-\frac{\beta+1}{p}} \|f\|_{p,\beta}$. Therefore if such a $C > 0$ exists, we derive

$$\|f_\lambda\|_{p,\beta} = \lambda^{-\frac{\beta+1}{p}} \|f\|_{p,\beta} \leq C \|f_\lambda\|_{q,\alpha} = C\lambda^{-\frac{\alpha+1}{q}} \|f\|_{q,\alpha},$$

which implies $\frac{\alpha+1}{q} = \frac{\beta+1}{p}$.

Next, consider $f_n(t) = \chi_{(1,1+1/n)}(t)$. A simple calculation shows that

$$\lim_{n\to\infty} n^{1/p} \|f_n\|_{p,\beta} = 1.$$

Therefore, if such a $C > 0$ exists,

$$1 = \lim_{n\to\infty} \frac{n^{1/p}\|f_n\|_{p,\beta}}{n^{1/q}\|f_n\|_{q,\alpha}} \leq C \lim_{n\to\infty} n^{1/p-1/q},$$

from which it follows that $p \leq q$.

Finally, let $f(x) = x^{-(\beta+1)/p} \ln(x+1)^{-1/p} \chi_{[1,\infty)}(x)$. Then

$$\|f\|_{p,\beta}^p = \int_1^\infty \frac{1}{x \ln(x+1)} dx = \infty,$$

and

$$\|f\|_{q,\alpha}^q = \int_1^\infty \frac{1}{x^{q(\beta+1)/p-\alpha} \ln(x+1)^{q/p}} dx.$$

The last integral is finite when $\frac{\alpha+1}{q} = \frac{\beta+1}{p}$ and $q > p$. Thus the only remaining possibility is the trivial one: $p = q$ and $\alpha = \beta$. □

Proposition 2.2. *Let $d > 0$, then there is a $C > 0$ such that for all $f \in WM(K, \lambda, q)$ on $[d, \infty]$,*

$$\|f\|_{p,\beta}^{(d)} \leq C \, \|f\|_{q,\alpha}^{(d)}$$

if and only if

$$\frac{\alpha+1}{q} > \frac{\beta+1}{p} \quad \text{and} \quad q > p \quad \text{or} \quad \frac{\alpha+1}{q} \geq \frac{\beta+1}{p} \quad \text{and} \quad q \leq p.$$

Proof. For the "if" part, the $q > p$ case follows from Hölder's inequality and the $q \leq p$ case from Lemma 2.1 by considering $f\chi_{[d,\infty]}$ and the following fact:

$$\|f\|_{p,\alpha}^{(d)} \leq d^{-\varepsilon/p} \, \|f\|_{p,\alpha+\varepsilon}^{(d)} \text{ for } \varepsilon > 0 \text{ and } d > 0.$$

The proof of the "only if" part will be given in Section 3. □

We now state and prove Hardy-type inequalities for p-weakly monotone functions.

Let us recall the original Hardy inequality. Denote

$$F(x) = \int_0^x f(s) ds.$$

Theorem A. (see, e.g., [24]) *Let $p > 1$. Then*

$$\|F\|_{p,-p} \leq \frac{p}{p-1} \, \|f\|_{p,0} \, .$$

There are many generalizations of this result in various settings. Let us mention the following classical result by Bradley [5] for power weights.

Theorem B. [5] *Let $1 < q \le p$. Then there is a $C > 0$ such that*

$$\|F\|_{p,\beta} \le C\,\|f\|_{q,\alpha} \qquad \iff \qquad \frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1 \quad \text{and} \quad \beta < -1.$$

For $q < 1$ it is necessary to restrict ourselves to a narrower class of functions, as the following example shows.

Example 1. Let $1 > \varepsilon > q$. Consider the following function

$$f(x) = \sum_{n=1}^{\infty} 4^n \chi_{[n, n+4^{-\varepsilon n}]}(x)$$

An easy calculation shows that

$$\|f\|_{q,\alpha} \le C \left(\sum_{n=1}^{\infty} 4^{(q-\varepsilon)n} n^{\alpha} \right)^{1/q} < \infty$$

and, if $2 \le n \le x < n+1$,

$$\int_0^x f(s)ds \ge \sum_{j=1}^{n-1} 4^{j(1-\varepsilon)} \ge C 4^{n(1-\varepsilon)}.$$

Hence,

$$\|F\|_{p,\beta} \ge C \left(\sum_{n=2}^{\infty} n^{\beta} 4^{(1-\varepsilon)pn} \right)^{1/p} = \infty.$$

We mention that the Hardy inequalities $\|F\|_{p,\alpha-p} \le C\,\|f\|_{p,\alpha}$ for $0 < p < 1$ and $-1 < \alpha < p-1$ under some monotone-type condition of f have been recently studied in [6, 1, 2]. This topic has been originated by Konuyshkov [21], who considered quasi-monotone functions, and Leindler [25], who restricted himself to consideration of functions from the $WM(K)$ class.

In this paper we investigate the (p,q) case and weakly monotone functions.

Theorem 2.1. *Let $p \ge q \le 1$, and $\beta < -1$. Let $f \in WM(K, \lambda, q)$. Then,*

$$\|F\|_{p,\beta} \le C\,\|f\|_{q,\alpha} \qquad \iff \qquad \frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1.$$

Furthermore, if $0 < p < q < \infty$ there is no such C.

Proof. Note that F is monotonically increasing and thus $F \in WM(K, \lambda, p)$ for any λ and p. Hence, applying Proposition 2.1, we obtain

$$\|F\|_{p,\beta} \le C\,\|F\|_{q, \frac{q(\beta+1)}{p} - 1}.$$

Let $\gamma = \alpha - q = \frac{q(\beta+1)}{p} - 1 < -1$. Then, by Lemma 2.1 with $p = 1$,

$$\|F\|_{q,\gamma}^q = \int_0^\infty x^\gamma \left(\int_0^x f(s)ds \right)^q dx \le C \int_0^\infty x^\gamma \int_0^{\lambda x} \frac{f(s)^q}{s^{1-q}} ds dx = C \int_0^\infty \frac{f(s)^q}{s^{1-q}} \int_{s/\lambda}^\infty x^\gamma dx ds.$$

Since $\gamma < -1$, we continue as follows

$$C\left(\int_0^\infty \frac{f(s)^q}{s^{1-q}} s^{1+\gamma} ds\right)^{1/q} = C\left(\int_0^\infty f(s)^q s^{q+\gamma}\right)^{1/q} = C\left\|f\right\|_{q,q+\gamma} = C\left\|f\right\|_{q,\alpha}.$$

The "only if" part as well as the $q > p$ case will be proved in Section 3. □

Remark 2. Note that Theorem 2.1 is optimal with respect to q, that is, for every $1 > q \leq p$ and $q' > q$ there exists $f \in WM(K, \mu, q')$ such that the inequality $\|F\|_{p,\beta} \leq C\|f\|_{q,\alpha}$ does not hold.

Proof. Let $q' > q < 1$ and $\lambda > 1$ such that $q' > \lambda^{-1} > q$. Consider the following function:

$$g = \sum_{n=1}^{\infty} 4^{\lambda^n} \chi_{[n, n+4^{-\lambda^{n-1}}]}$$

and let $f(e^x) = g(x)$. Note that if $1 \leq n \leq x < n+1$, one has

$$g(x)^{q'} \leq 4^{q'\lambda^n} \leq 4^{(\lambda^n)(q'-\lambda^{-1})\lambda^m} \leq \int_{n+m}^{n+m+1} g(s)^{q'} ds \leq \int_{x-m-1}^{x+m+1} g(s)^{q'} ds$$

for $m \in \mathbb{N}$ such that $(q' - \lambda^{-1})\lambda^m > q'$. Thus, from Proposition 1.1 we conclude that $f \in WM(1, e^{m+1}, q')$.

First, we show that

$$\int_0^\infty f(x)^q x^\alpha dx = \int_{-\infty}^\infty g(s)^q e^{s(\alpha+1)} ds \leq C' \sum_{n=1}^\infty 4^{(q-\lambda^{-1})\lambda^n} e^{n(\alpha+1)} < \infty.$$

Now,

$$\int_0^\infty \left(\int_0^x f(y)dy\right)^p x^\beta dx = \int_{-\infty}^\infty \left(\int_{-\infty}^x g(y)e^y dy\right)^p e^{s(\beta+1)} dx$$

and, for $n \in \mathbb{N}$,

$$\int_{n+1}^{n+2} \left(\int_{-\infty}^x g(y)e^y dy\right)^p e^{s(\beta+1)} dx \geq C' e^{(n+1)(\beta+1)} \left(\int_{-\infty}^{n+1} g(y)e^y dy\right)^p$$
$$\geq C' e^{(n+1)(\beta+1)} 4^{\lambda^n p(1-\lambda^{-1})} e^{pn}.$$

Therefore,

$$\int_0^\infty \left(\int_0^x f(y)dy\right)^p x^\beta dx \geq C' \sum_{n=1}^\infty e^{(n+1)(\beta+1)} 4^{\lambda^n p(1-\lambda^{-1})} e^{pn} = \infty$$

and consequently, the inequality $\|F\|_{p,\beta} \leq C\|f\|_{q,\alpha}$ is not valid. □

Now, similarly to F, we define an average of f with a lower limit of the integral being non zero and we will see that in this case the set of admissible parameters α, β becomes wider. For $d > 0$, we denote

$$F_d(x) = \int_d^x f(s)ds.$$

Theorem 2.2. *Let $d > 0$. Let $p \geq q \leq 1$, and $\beta < -1$. Let $f \in WM(K, \lambda, q)$ on $[d, \infty]$. Then,*

$$\|F_d\|_{p,\beta}^{(d)} \leq C\|f\|_{q,\alpha}^{(d)} \iff \frac{\alpha+1}{q} \geq \frac{\beta+1}{p} + 1.$$

Proof. Applying Theorem 2.1 to $f\chi_{[d,\infty]}$ we obtain the result in the case $\frac{\alpha+1}{q} = \frac{\beta+1}{p}+1$. The remaining cases follow from the following fact:

$$\|f\|_{p,\alpha}^{(d)} \leq d^{-\varepsilon/p} \|f\|_{p,\alpha+\varepsilon} ^{(d)} \text{ for } \varepsilon > 0 \text{ and } d > 0.$$

The proof of the "only if" part will be given in Section 3. ☐

Theorem 2.3. *Let $d > 0$. Let $q > p \leq 1$, and $\beta < -1$. Let $f \in WM(K, \lambda, p)$ on $[d, \infty]$ Then,*

$$\|F_d\|_{p,\beta}^{(d)} \leq C \|f\|_{q,\alpha}^{(d)} \qquad \text{if and only if} \qquad \frac{\alpha+1}{q} > \frac{\beta+1}{p} + 1.$$

Proof. Applying Theorem 2.1 to $f\chi_{[d,\infty]}$ we obtain

$$\|F_d\|_{p,\beta}^{(d)} \leq C \|f\|_{p,\beta+p}^{(d)}.$$

Finally, since $q > p$ we can use Proposition 2.2 to obtain

$$\|f\|_{p,\beta+p}^{(d)} \leq C \|f\|_{q,\alpha}^{(d)}$$

for $\frac{\alpha+1}{q} > \frac{\beta+1+p}{p}$. The proof of the "only if" part will be given in Section 3. ☐

Note that since F is non-decreasing, we have $\|F\|_{\infty,\beta} = \sup_{x\in[0,\infty]} F(x) = \int_0^\infty f(s)ds$. Thus,

Theorem 2.4 (Case $p = \infty$). *Let $q \leq 1$ and $f \in WM(K, \lambda, q)$, then*

$$\int_0^\infty f(s)ds \leq C \left(\int_0^\infty f^q(s)s^\alpha ds \right)^{1/q}$$

if and only if $\alpha = q - 1$.

Proof. The "if" part is a restatement of Lemma 2.1. The proof of the "only if" part will be given in Section 3. ☐

As an immediate corollary, we obtain

Theorem 2.5. *Let $d > 0$. Let $q \leq 1$ and $f \in WM(K, \lambda, q)$ on $[d, \infty]$, then*

$$\int_d^\infty f(s)ds \leq C \left(\int_d^\infty f^q(s)s^\alpha ds \right)^{1/q}$$

if and only if $\alpha \geq q - 1$.

For $0 < D \leq \infty$, denote

$$G^*(x) = \int_x^D g(s)ds$$

and

$$\|g\|_{p,\alpha}^{*,(D)} = \left(\int_0^D |g(s)|^p s^\alpha ds \right)^{1/p}.$$

The following result is well known, see for example, [24].

Theorem C. (see, e.g., [24]). *Let $1 < q \leq p$, then there exists C such that*

$$\|G^*\|_{p,\beta}^{*,(\infty)} \leq C \|g\|_{q,\alpha}^{*,(\infty)} \qquad \iff \qquad \frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1 \text{ and } \beta > -1.$$

We obtain the following counterparts of Theorems $2.5, 2.7, 2.8, 2.9$ and 2.10.

Theorem 2.6. *Let $g \in WM(K,\lambda,q)$ on $[0,D]$ for $0 < D \leq \infty$. Let also $\beta > -1$.*

1. *Let $g \in WM(K,\lambda,q)$. If $1 \geq q \leq p \leq \infty$, then $\|G^*\|_{p,\beta}^{*,(\infty)} \leq C\|g\|_{q,\alpha}^{*,(\infty)} \iff \frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1$. Furthermore, if $\infty > q > p > 0$ there is no such C.*

2. *Let $g \in WM(K,\lambda,q)$ on $[0,D]$ for $0 < D < \infty$. If $1 \geq q \leq p \leq \infty$, then $\|G^*\|_{p,\beta}^{*,(D)} \leq C\|g\|_{q,\alpha}^{*,(D)} \iff \frac{\alpha+1}{q} \leq \frac{\beta+1}{p} + 1$.*

3. *Let $g \in WM(K,\lambda,q)$ on $[0,D]$ for $0 < D < \infty$. If $q > p \leq 1$, then $\|G^*\|_{p,\beta}^{*,(D)} \leq C\|g\|_{q,\alpha}^{*,(D)} \iff \frac{\alpha+1}{q} < \frac{\beta+1}{p} + 1$.*

Proof. Let $d = 1/D$. Denote
$$g(t) = f(t^{-1})t^{-2}.$$
Note that $g(t^{-1})t^{-2} = f(t)$. Using the properties of p-weakly monotone functions, we know that $g \in WM(K',\lambda,p)$ on $[0,1/d]$ if and only if $f \in WM(K,\lambda,p)$ on $[d,\infty]$.
We have
$$G^*(x) = \int_x^{1/d} g(s)ds.$$
Since
$$\int_x^{1/d} g(s)ds = \int_d^{1/x} g(t^{-1})t^{-2}dt = \int_d^{1/x} f(t)dt,$$
we have
$$G^*(x) = F(x^{-1}).$$
Similarly, we derive that
$$\|G^*\|_{p,-\beta-2}^{*,(1/d)} = \|F\|_{p,\beta}^{(d)} \quad \text{and} \quad \|g\|_{q,2q-2-\alpha}^{*,(1/d)} = \|f\|_{q,\alpha}^{(d)}.$$
Thus,
$$\|G^*\|_{p,-\beta-2}^{*,(1/d)} \leq C\|g\|_{q,2q-2-\alpha}^{*,(1/d)} \quad \text{if and only if} \quad \|F\|_{p,\beta}^{(d)} \leq C\|f\|_{q,\alpha}^{(d)}.$$
Finally, using Theorems $2.5, 2.7, 2.8, 2.9, 2.10$, the result follows. □

3 Optimality

Note that if we prove the sharpness of Theorems $2.1, 2.2, 2.3, 2.4$, then the sharpness of Propositions 2.1 and 2.2 follows. Remark also that for $\gamma > -1$, $g(x) = x^\gamma \chi_{[0,1]}$ is $WM(K,\lambda,p)$ for every p and λ since
$$x^{p\gamma} \leq K\int_{x/\lambda}^{x} s^{p\gamma-1}ds \leq K\int_{x/\lambda}^{\lambda x} \frac{g(s)^p}{s}ds.$$
Denote $\int_0^x g(s)ds = G(x)$.

Now, $\|g\|_{q,\alpha} < \infty \iff \gamma > \frac{-1-\alpha}{q}$, and, for $\beta < -1$, $\|G\|_{p,\beta} = \infty$ if and only if either $\gamma \leq -1$ or $\gamma \leq \frac{-1-\beta}{p} - 1$. So if $\frac{-1-\alpha}{q} < \gamma < \frac{-1-\beta}{p} - 1$, $\|G\|_{p,\beta} = \infty$ and $\|g\|_{q,\alpha} < \infty$. Thus, the inequality in Theorem 2.1 cannot possibly hold for $\frac{1+\alpha}{q} > \frac{1+\beta}{p} + 1$.

For the $p = \infty$ case (Theorem 2.4), the same considerations for $x^\gamma \chi_{[0,1]}$ suffice to obtain the condition $\frac{1+\alpha}{q} \leq 1$.

Now for $d \geq 0$. Let $p \neq \infty$, define

$$g(x) = x^{-(\beta+1+p)/p} \ln(x+b)^{-1/p} \left(-\frac{\beta+1}{p} - \frac{1}{p} \frac{x}{(x+b)\ln(x+b)} \right).$$

Note that since $\beta < -1$, for large enough b, $f(x) > 0$ for $x > 0$, and

$$Dx^{-(\beta+1+p)/p} \ln(x+b)^{-1/p} > g(x) > Cx^{-(\beta+1+p)/p} \ln(x+b)^{-1/p}$$

for some $C, D > 0$. It is easy to see that

$$G(x) = \int_0^x g(s)ds = x^{-(\beta+1)/p} \ln(x+b)^{-1/p}.$$

Set

$$f(x) = x^{-(\beta+1+p)/p} \ln(x+b)^{-1/p}.$$

It is clear that $f(x)$ and $g(x)$ have the same behaviour at infinity and so do $F(x) = \int_0^x f(s)ds$ and $G(x)$.

Now assume that there is a locally integrable function h and $M > 0$ such that

1. $h(x) = 0$ for $x < d+1$;
2. $h(x) = f(x)$ for $x > M$;
3. $\int_d^x h(s)ds = H(x) = F(x) > D^{-1}G(x)$ for $x > M$;
4. $h \in WM(K, \lambda, r)$ on $[d, \infty]$ for any r.

Then

$$\left(\|H\|_{p,\beta}^{(d)} \right)^p > D^{-1} \int_M^\infty \frac{1}{x \ln(x+b)} dx = \infty$$

and

$$\left(\|h\|_{q,\alpha}^{(d)} \right)^q = \int_{d+1}^M h^q(x) x^\alpha dx + \int_M^\infty \frac{1}{x^{q(\beta+1+p)/p-\alpha} \ln(x+b)^{q/p}} dx,$$

which is finite provided either $\frac{q(\beta+1+p)}{p} - \alpha > 1$ or $\frac{q(\beta+1+p)}{p} - \alpha = 1$ and $q > p$.

So if $\frac{q(\beta+1+p)}{p} - \alpha > 1$ (or, equivalently, $\frac{\alpha+1}{q} < \frac{\beta+1}{p} + 1$) or if $\frac{q(\beta+1+p)}{p} - \alpha = 1$ (or, equivalently, $\frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1$) and $q > p$, the inequalities in Theorems 2.1, 2.2 and 2.3 cannot hold.

All that remains is to build h satisfying the former properties. Since $f(x) = x^{-(\beta+1+p)/p} \ln(x+b)^{-1/p}$, if $(\beta+1+p)/p < 0$, f will be eventually monotonically increasing, say for $x > N$. Now, let $n \in \mathbb{N}$ be such that $(d+1)\lambda^n \geq N$ and $n \geq 4$. For $m \in \mathbb{R}^+$, let

$$h(x,m) = \begin{cases} 0, & x < d+1 \\ m, & d+1 \leq x \leq (d+1)\lambda^2, \\ 0, & (d+1)\lambda^2 < x < \lambda^n(d+1), \\ f(x), & x \geq (d+1)\lambda^n. \end{cases} \tag{3.1}$$

Note that for any $x > 0$, h is monotonic on $[x/\lambda, \lambda x]$, thus $h \in WM(K, \lambda, r)$ for any $r > 0$ and for some K. Furthermore, by construction $h(x) = 0$, for $x < d+1$ and $h(x) = f(x)$ for $x \geq M = (d+1)\lambda^n$.

Finally, since

$$\int_0^M h(x,0)dx \le \int_0^M f(s)ds < \int_0^M h(x,\infty)dx = \infty,$$

by continuity there must be some m^* such that

$$\int_0^M h(x,m^*)dx = \int_0^M f(s)ds.$$

So if $h(x) = h(x,m^*)$, one has

$$\int_0^x h(s,m^*)ds = \int_0^x f(s)ds$$

for $x > M$.

Obviously, if $(\beta+1+p)/p \ge 0$, $f(x)$ will be always decreasing. For $m \in \mathbb{R}^+$, let

$$h(x,m) = \begin{cases} 0, & x < d+1, \\ m, & d+1 \le x \le (d+1)\lambda^2, \\ f(x), & x \ge (d+1)\lambda^2. \end{cases} \tag{3.2}$$

Note that if $m \ge f((d+1)\lambda^2)$, then for any $x > 0$, h is monotonic on $[x/\lambda, \lambda x]$, thus $h \in WM(K,\lambda,r)$ for any $r > 0$ and for some K.

Let $m = \frac{F(\lambda^2(d+1))}{\lambda^2(d+1)-(d+1)} \ge \frac{F(\lambda^2(d+1))}{\lambda^2(d+1)} \ge f(\lambda^2(d+1))$, where the last inequality holds because f is decreasing. Then $h(x,m)$ is the desired counterexample.

The only case that remains is when $p = \infty$. To deal with it, it suffices to use the previously described idea to build a locally monotonic function which agrees with x^γ for large enough x.

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Events

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INTERNATIONAL CONFERENCE "THEORY OF FUNCTIONS OF SEVERAL REAL VARIABLES" DEDICATED TO THE 90TH ANNIVERSARY OF THE BIRTHDAY OF CORRESPONDING MEMBER OF THE RUSSIAN ACADEMY OF SCIENCES OLEG VLADIMIROVICH BESOV

The V.A. Steklov Mathematical Institute of the Russian Academy of Sciences (MIAN) and World-class Mathematical Center of the V.A. Steklov Mathematical Institute (MCMU MIAN) held the international conference «Theory of functions of several real variables» through May 29 – June 2, 2023 (MIAN, 8 Gubkin St, Moscow)

Theory of functions of several real variables is an important integral part of modern analysis. It finds numerous applications in the theory of approximation, theory partial differential equations and calculus of variations. The conference will discuss topical issues of the theory of spaces of differentiable functions on domains of Euclidean spaces and metric spaces: embedding theorems, trace theorems, interpolation theory, extension theorems, properties of differential and integral operators, issues of harmonic analysis, widths of classes of functions, integral representations, and approximation of functions.

The conference is dedicated to the 90th anniversary of the birthday of an outstanding world recognised scientist corresponding member of the Russian Academy of Sciences O.V. Besov. The topics of the conference are related to the directions of research of O.V. Besov and his school.

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Vodopyanov Sergey Konstantinovich
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Programme of the conference

May 29, 2023
09:30–10:00, Opening ceremony (D.V. Treschev, B.S. Kashin)
10:00–10:50, S.V. Kislyakov, *Correction theorems and the uncertainty principle*
11:00–11:50, V.N. Temlyakov, *Rate of convergence of thresholding greedy algorithms*
12:30–13:20, W. Sickel (online), *On the regularity of characteristic functions*
15:00–15:50, V.I. Burenkov, D.J. Joseph, *Inequalities for entire functions of exponential type for Morrey spaces*
16:00–16:50, E.I. Berezhnoi, *Discrete variant of Morrey spaces. New approach*

May 30, 2023
10:00–10:50, A.A. Shkalikov, *Spectral asymptotics for the systems of differential equations and applications*
11:00–11:50, S.K. Vodop'yanov, *Function spaces and geometry of mappings*
12:30–13:20, S.Yu. Tikhonov, *Truncated Besov spaces*
15:00–15:50, Y. Sawano (online), *Maximal regularity for Morrey spaces*
16:00–16:50, A.A. Grigor'yan (online), *Analysis on fractal spaces and heat kernel*

May 31, 2023
10:00–10:50, V.G. Krotov, I.N. Katkovskaya *On the tangential boundary behaviour of functions from Hardy type spaces*
11:00–11:50, A.I. Nazarov, *Hardy-type inequalities with mixed weights*
12:30–13:20, D.M. Stolyarov, *Bourgain–Brezis inequalities and Besov spaces*
15:00–15:50, E.D. Nursultanov, *Interpolation of linear and nonlinear operators and applications*
16:00–16:50, V.I. Bogachev, *Pointwise characterizations of Sobolev functions*

June 1, 2023
10:00–10:50, G.A. Kalyabin (online), *Diverse results concerning Besov and Sobolev spaces*
11:00–11:50, R.O. Oinarov, *Boundedness of one class of Volterra-type integral operators in Lebesgue spaces and applications*
12:30–13:20, D.B. Bazarkhanov (online), *Multilinear pseudo-differential operators on the multidimensional torus*
15:00–15:50, V.D. Stepanov, *Strong and weak associativity and reflexivity of certain function classes*
16:00–16:50, E.P. Ushakova, *Spline wavelets and Riemann–Liouville operators in Besov-type spaces*

June 2, 2023

10:00–10:50, H.G. Ghazaryan (online), *Coercive estimates for multilayer-degenerate differential operators (polynomials)*

11:00–11:50, S.V. Astashkin, *Interpolation properties of K-monotone couples of quasi-Banach spaces*

12:30–13:20, D. Yang (online), *Matrix-weighted Besov-type and Triebel–Lizorkin-type spaces*

15:00–15:50, H. Rafeiro, S.G. Samko (online), *Local grandization of Lebesgue spaces*

16:00–16:50, E.A. Kalita, *Dual Morrey spaces in nonlinear elliptic PDEs*

16:50–17:00, Closing ceremony (B.S. Kashin)

Congratulations to Oleg Vladimirovich Besov from the research group "Function Spaces", Jena

Dear Professor O.V. Besov

The members of the seminar "Function spaces" want to congratulate you with your ninetieth birthday, Almoat all members of this seminar had profit from your remarkable work on function spaces, In 1992 the first meeting of the groups from Moscow and Jena took place in Friedrichroda. It was a pleasure for us that this meeting has been repeated from time to time for over the last 30 years, also thanks to your support. We wish you all the best for the future.

Dorothee Haroske, Winfried Sickel, Hans-Juergen Schmeisser, Hans Triebel, Dann van Dyk, Hans-Gerd Leopold, Jonas Sauer, Zhen Liu. Sergei Artamonov, Simon Murmann, Glenn Byerenheul, T. Ullrich, Guillance Neultriens, Kristof Starvans, Komoric Kono, Henning Kempka.

Video records

On Math-Net.Ru ($https://www.mathnet.ru/php/conference.phtml?confid = 2247\&option_lang = rus$) one can find video records of all talks of the conference.

V.I. Burenkov (RUDN University, MIAN), Editor-in-chief of the EMJ one can find video records
T.V. Tararykova (RUDN University), Deputy editor-in-chief of the EMJ
A.I. Tyulenev (MIAN), member of the Organizing Committee

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