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ON THE LAGRANGE MULTIPLIER RULE FOR MINIMIZING SEQUENCES

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Key words: constraint optimization, Lagrange multiplier rule, optimality condition, minimizing sequence, Caristi-like condition.

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Abstract. In the paper, an optimization problem with equality-type constraints is studied. It is assumed that the minimizing function and the functions defining the constraints are Frechet differentiable, the set of the admissible points is nonempty and the minimizing function is bounded below on the set of admissible points. Under these assumptions we obtain an estimate of the derivative of the Lagrange function. Moreover, we prove the existence of a minimizing sequence $\{x^n\}$ and a sequence of unit Lagrange multipliers $\{\lambda^n\}$ such that the sequence of the values of derivative of the Lagrange function at the point (x^n, λ^n) tends zero. This result is a generalization of the known assertion stating that for a bounded below differentiable function f there exists a minimizing sequence $\{x^n\}$ such that the values of the derivative $f'(x^n)$ tend to zero. As an auxiliary tool,there was introduced and studied the property of the directional covering for mappings between normed spaces. There were obtained sufficient conditions of directional covering for Frechet differentiable mappings.

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1 Introduction

Let X be a Banach space with the norm $\|\cdot\|$. Denote by $B_X(x,r)$ a closed ball centered at $x \in X$ with radius $r \geq 0$. Let X^* stand for the topological dual to X and stand $\|\cdot\|_*$ for the norm of X^*.

Given a positive integer k and Frechet differentiable functions $f_0, f_1, ..., f_k : X \to \mathbb{R}$, consider the optimization problem

$$f_0(x) \to \min, \quad f_1(x) = 0, \quad ..., \quad f_k(x) = 0. \tag{1.1}$$

Define the Lagrange function $L : X \times \mathbb{R}^{k+1} \to \mathbb{R}$ by the formula

$$L(x,\lambda) := \lambda_0 f_0(x) + \lambda_1 f_1(x) + ... + \lambda_k f_k(x), \quad x \in X, \quad \lambda = (\lambda_0, \lambda_1, ..., \lambda_k) \in \mathbb{R}^{k+1}.$$

Denote the set of all admissible points by $\mathcal{M}$, i.e.

$$\mathcal{M} := \{x \in X : f_1(x) = ... = f_k(x) = 0\}.$$

The Lagrange multiplier rule (see, for example, [10, Section 1.2]) states that if a point $\widehat{x} \in X$ is a local solution to problem (1.1), then there exists a nonzero vector $\lambda \in \mathbb{R}^{k+1}$ such that $\dfrac{\partial L}{\partial x}(\widehat{x}, \lambda) = 0$ and $\lambda_0 \geq 0$.

In this paper, we show that if a function f_0 is bounded from below on $\mathcal{M} \neq \emptyset$ then there exist sequences $\{x^n\} \subset \mathcal{M}$ and $\{\lambda^n\} \subset \mathbb{R}^{k+1}$ such that $\left\| \frac{\partial L}{\partial x}(x^n, \lambda^n) \right\|_* \to 0$, $f_0(x^n) \to \inf_{x \in \mathcal{M}} f_0(x)$ as $n \to \infty$ and $\|\lambda^n\| = 1$ for every n. This result is an analog of the known result for unconstrained optimization problem stating that for a bounded below differentiable functional f_0 on X there exists a minimizing sequence $\{x^n\}$ such that $\frac{\partial f_0}{\partial x}(x^n) \to 0$ as $n \to \infty$ (see, for example, [6, Chapter 5, Section 3]).

Moreover, in this paper, we obtain an estimate of the derivative of the Lagrange function. When X is a Hilbert space, similar estimates for the first-order and the second-order derivatives were obtained in [2] and [3]. For the unconstrained optimization problem the estimates of the first-order and the second-order derivatives of the minimizing function were obtained in [7, §2.5.2].

2 Main results

Given $x_0 \in \mathcal{M}$ and $R > 0$, denote

$$\gamma(x_0, R) := \inf\{f_0(x) : x \in \mathcal{M} \cap B_X(x_0, R)\}.$$

Here $\gamma(x_0, R)$ may take the value $-\infty$. However, in what follows, we will assume that $\gamma(x_0, R) > -\infty$. Note also that $f_0(x_0) - \gamma(x_0, R) \geq 0$ for every $x_0 \in \mathcal{M}$ and $R > 0$.

Theorem 2.1. *Given a point $x_0 \in \mathcal{M}$ and a number $R > 0$, assume that*

$$\gamma(x_0, R) > -\infty.$$

Then for every $\varepsilon > 0$ there exist vectors $\lambda = (\lambda_0, \lambda_1, ..., \lambda_k) \in \mathbb{R}^{k+1}$ and $\widehat{x} \in \mathcal{M} \cap B_X(x_0, R)$ such that

$$\|\lambda\| = 1, \qquad \lambda_0 \geq 0, \qquad f_0(\widehat{x}) \leq f_0(x_0),$$
$$\left\| \frac{\partial L}{\partial x}(\widehat{x}, \lambda) \right\|_* \leq (1+\varepsilon)\lambda_0 \frac{f_0(x_0) - \gamma(x_0, R)}{R}. \tag{2.1}$$

Note that if the set $\mathcal{M} \cap B_X(x_0, R)$ contains a point x for which the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0, k}$ are linearly dependent and $f_0(x) \leq f_0(x_0)$, then the proposition of Theorem 2.1 trivially holds. In this case, $\widehat{x} = x$ and the unit vector $\lambda = (\lambda_0, \lambda_1, ..., \lambda_k)$ satisfying the equality $\sum_{i=0}^{k} \lambda_i \frac{\partial f_i}{\partial x}(x) = 0$ and the inequality $\lambda_0 \geq 0$ is the desired one (if $\lambda_0 < 0$ then we take $-\lambda$ instead of λ). In this case, the left-hand side of (2.1) equals zero and the right-hand side is nonnegative.

If the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0, k}$ are linearly independent on the set $\{x \in \mathcal{M} \cap B_X(x_0, R) : f_0(x) \leq f_0(x_0)\}$ then the proposition of Theorem 2.1 is nontrivial. In this case, inequality (2.1) implies that $\lambda_0 > 0$.

Note also that inequality (2.1) implies the following weaker estimate

$$\left\| \frac{\partial L}{\partial x}(\widehat{x}, \lambda) \right\|_* \leq (1+\varepsilon) \frac{f_0(x_0) - \gamma(x_0, R)}{R}, \tag{2.2}$$

since $\|\lambda\| = 1$.

Theorem 2.2. *Assume that the function f_0 is bounded from below on $\mathcal{M}$. Then there exist sequences of vectors $\{x^n\} \subset \mathcal{M}$ and $\{\lambda^n\} \subset \mathbb{R}^{k+1}$ such that*

$$\frac{\partial L}{\partial x}(x^n, \lambda^n) \to 0, \quad f_0(x^n) \to \inf_{x \in \mathcal{M}} f_0(x) \quad \text{as} \quad n \to \infty \quad \text{and} \quad \|\lambda^n\| = 1 \quad \forall n.$$

The proofs of these theorems are presented in Section 4. Let us now discuss the ideas of proofs of these assertions.

Given a point $x_0 \in X$ and a number $R \geq 0$, we put $v := (-1, 0, 0, ..., 0) \in \mathbb{R}^{k+1}$, $F := (f_0, f_1, ..., f_k)$. Since $\gamma(x_0, R)$ is the infimum of f_0 over the admissible set $\mathcal{M}$, the points $F(x_0) + \mu v$, $\mu \geq 0$ do not belong to $F(B_X(x_0, R))$ as $\mu > f_0(x_0) - \gamma(x_0, R)$. If the mapping F is $\bar{\alpha}$-covering in the direction v at every point $x \in \mathcal{M} \cap B_X(x_0, R)$ such that $f_0(x) \leq f_0(x_0)$ (i.e. $\frac{\partial F}{\partial x}(x)X = Y$ and $\sup\left\{\alpha \geq 0 : \ \alpha v \in \frac{\partial F}{\partial x}(x)B_X(0,1)\right\} \geq \bar{\alpha}$) then $F(x_0) + \mu v \in F(B_X(x_0, R))$ for $\mu \in [0, \alpha R)$. This assertion is Lemma 3.1 below. These reasonings imply that there exists a point $\widehat{x}$ such that F is $\bar{\alpha}$-covering with the constant $\bar{\alpha}$ not exceeding the right-hand side of inequality (2.1). Inequality (2.1) simply follows from this fact (see Lemma 3.2 below). To prove Theorem 2.2, it is enough to take an arbitrary minimizing sequence $\{x_0^n\}$ and apply Theorem 2.1 as $x_0 := x_0^n$, $R := 1$ and $\varepsilon := 1$ for every n.

3 Auxiliary assertions

In this section, we prove two auxiliary assertions: Lemmas 3.1 and 3.2. In the proof of Lemma 3.1, we will use the following minimum existence conditions from [1, Theorem 3] (see also [8, Lemma 1]).

Theorem 3.1. *Given a complete metric space (M, ρ), a lower semicontinuous function $U : M \to \mathbb{R}_+$ and a number $\alpha > 0$, assume that the function U satisfies the Caristi-like condition*

$$\forall\, x \in M : \quad U(x) > 0 \quad \exists\, x' \in M \setminus \{x\} : \quad U(x') + \alpha\rho(x, x') \leq U(x). \tag{3.1}$$

Then for every $x_0 \in M$ there exists a point $\bar{x} \in M$ such that $U(\bar{x}) = 0$ and $\rho(x_0, \bar{x}) \leq \alpha^{-1}U(x_0)$.

Let Y be a finite-dimensional linear space with a norm $\|\cdot\|$. Denote by Y^* a dual space to Y. We denote the value of the functional $\lambda \in Y^*$ on the vector $y \in Y$ by $\langle\lambda, y\rangle$. An analogous notation we will use for the functionals from X^*. Denote the unit sphere in Y by S, i.e.

$$S := \{v \in Y : \ \|v\| = 1\}.$$

For an arbitrary linear bounded operator $A : X \to Y$ we denote by $A^* : Y^* \to X^*$ the adjoint operator to A. For an arbitrary vector $v \in S$ we put

$$\mathrm{cov}(A|v) := \sup\{\alpha \geq 0 : \quad \alpha v \in AB_X(0, 1)\}.$$

It is a straightforward task to ensure that $\mathrm{cov}(A|v) > 0$ if and only if $v \in AX$.

Lemma 3.1. *Given a Frechet differentiable mapping $F : X \to Y$, vectors $x_0 \in X$, $v \in S$ and a number $R > 0$, assume that*

(i) $\bar{\alpha} := \inf\left\{\mathrm{cov}\left(\frac{\partial F}{\partial x}(x)\,\middle|\, v\right) : \ x \in B_X(x_0, R), \ \ F(x) \in F(x_0) + \mathrm{cone}\{v\}\right\} > 0;$

(ii) $\frac{\partial F}{\partial x}(x)X = Y \ \forall\, x \in B_X(x_0, R) : F(x) \in F(x_0) + \mathrm{cone}\{v\}.$

Then

$$F(x_0) + \alpha r v \in F(B_X(x_0, r)) \quad \forall\, r \in [0, R], \quad \forall\, \alpha \in (0, \bar{\alpha}).$$

Proof. Fix an arbitrary $r \in [0, R]$ and $\alpha \in (0, \bar\alpha)$. Put

$$M := \{x \in X : F(x) - F(x_0) - s\alpha r v = 0, \ \ \|x - x_0\| \le sr, \ \ s \in [0, 1]\}.$$

Obviously, the set M is nonempty, since it contains the point x_0. Moreover, M is closed, since F is continuous. Define a functional $U : M \to \mathbb{R}$ by the formula

$$U(x) = \|F(x) - F(x_0) - \alpha r v\|, \quad x \in M. \tag{3.2}$$

To prove the lemma it is enough to show that there exists a point $\bar x \in M$ such that $U(\bar x) = 0$. To prove this assertion we will apply Theorem 3.1.

Obviously, the functional U is continuous and nonnegative. So, it is enough to prove that U satisfies the Caristi-like condition (3.1).

Fix an arbitrary $x \in M$ such that $U(x) > 0$ and show that there exists a point $x' \in M \setminus \{x\}$ such that

$$U(x') + \alpha\|x - x'\| \le U(x). \tag{3.3}$$

The definition of M implies that there exists $t \in [0, 1]$ such that

$$F(x) = F(x_0) + t\alpha r v, \quad \|x - x_0\| \le tr. \tag{3.4}$$

Since $U(x) = \|F(x) - F(x_0) - \alpha r v\| > 0$, we have $t < 1$.

Put $A := \dfrac{\partial F}{\partial x}(x)$. It follows from the assumption (**i**) that $\operatorname{cov}(A|v) \ge \bar\alpha > 0$. Hence, $\bar\alpha > (\alpha + \bar\alpha)/2$ by virtue of the choice of α. The definition of $\operatorname{cov}(A|v)$ implies that there exists a vector $e \in B_X(0, 1)$ such that

$$Ae = \frac{\alpha + \bar\alpha}{2} v.$$

Since $AX = Y$ by virtue of (**ii**), we have that there exists a linear operator $R : Y \to X$ such that

$$e = R\left(\frac{\alpha + \bar\alpha}{2} v\right) \quad \text{and} \quad ARy \equiv y. \tag{3.5}$$

Since the mapping F is differentiable, we have

$$F(x + \xi) = F(x) + A\xi + o(\xi), \quad \xi \in X, \tag{3.6}$$

where $o : X \to Y$ is a continuous mapping such that there exists $\delta > 0$, for which the following relation takes place

$$\|o(\xi)\| \le \frac{\bar\alpha - \alpha}{\|R\|(\bar\alpha + \alpha)}\|\xi\| \quad \forall \xi \in B_X(0, \delta). \tag{3.7}$$

Reducing δ we obtain that

$$0 < \delta < r - tr. \tag{3.8}$$

Note that when we reduce δ, relation (3.7) remains true.

Consider the equation

$$\xi = R(\alpha\delta v - o(\xi))$$

with the unknown $\xi \in B_X(0, \delta)$. Define a mapping $\Phi : B_X(0, \delta) \to B_X(0, \delta)$ by the formula

$$\Phi(\xi) := R(\alpha\delta v - o(\xi)), \quad \xi \in B_X(0, \delta).$$

This mapping is well-defined, i.e. $\|R(\alpha\delta v - o(\xi))\| \le \delta$ for every $\xi \in B_X(0, \delta)$, since

$$\|R(\alpha\delta v - o(\xi))\| \le \|\alpha\delta R v\| + \|Ro(\xi)\| \overset{(3.5)}{\le} \frac{2\alpha\delta}{\alpha + \bar\alpha} + \|Ro(\xi)\| \overset{(3.7)}{\le}$$

$$\overset{(3.7)}{\leq} \frac{2\alpha\delta}{\alpha+\bar\alpha} + \frac{\bar\alpha-\alpha}{\bar\alpha+\alpha}\|\xi\| \leq \frac{2\alpha\delta}{\alpha+\bar\alpha} + \frac{\bar\alpha-\alpha}{\bar\alpha+\alpha}\delta = \delta \quad \forall\, \xi \in B_X(0,\delta).$$

Moreover, the mapping Φ is compact and continuous since $o(\cdot)$ is continuous and the linear operator $R: Y \to X$ has a finite-dimensional image (recall that the space Y is finite-dimensional). Thus, the Schauder fixed-point theorem (see, for example, [11, Section 2.1]) implies that there exists a point $\xi' \in B_X(0,\delta)$ such that $\xi' = \Phi(\xi')$. Therefore,

$$\xi' = R(\alpha\delta v - o(\xi')), \quad \|\xi'\| \leq \delta. \tag{3.9}$$

Put

$$x' := x + \xi'. \tag{3.10}$$

Let us show that $x' \in M \setminus \{x\}$. We have

$$F(x') - F(x_0) = \alpha r\left(t + \frac{\delta}{r}\right)v, \quad \|x_0 - x'\| \leq r\left(t + \frac{\delta}{r}\right), \tag{3.11}$$

since

$$F(x') \overset{(3.10)}{=} F(x+\xi') \overset{(3.6)}{=} F(x) + A\xi' + o(\xi') \overset{(3.9)}{=} F(x) + AR(\alpha\delta v - o(\xi')) + o(\xi') \overset{(3.5)}{=}$$
$$\overset{(3.5)}{=} F(x) + \alpha\delta v \overset{(3.4)}{=} t\alpha r v + \alpha\delta v + F(x_0) = \alpha r v\left(t + \frac{\delta}{r}\right) + F(x_0);$$

$$\|x_0 - x'\| \leq \|x_0 - x\| + \|x - x'\| \overset{(3.4)}{\leq} tr + \|x - x'\| \overset{(3.10)}{=} tr + \|\xi'\| \overset{(3.9)}{\leq} r\left(t + \frac{\delta}{r}\right).$$

It follows from (3.8) that the inequality $t + \frac{\delta}{r} < 1$ takes place. Therefore, relation (3.11) and the definition of M implies $x' \in M$. Moreover, $Rv \neq 0$ by virtue of (3.5). Therefore, $\xi' \neq 0$ by virtue of (3.9). So, relation (3.10) implies that $x' \neq x$. Hence, we have $x' \in M \setminus \{x\}$.

Let us prove that (3.3) holds. We have

$$U(x') \overset{(3.2)}{=} \|F(x') - F(x_0) - \alpha r v\| \overset{(3.11)}{=} \|t\alpha r v + \alpha\delta v - \alpha r v\| = \Big\|\big((r - tr) - \delta\big)\alpha v\Big\| \overset{(3.8)}{=}$$
$$\overset{(3.8)}{=} \Big\|(r - tr)\alpha v\Big\| - \|\delta \cdot \alpha v\| = \|t\alpha r v - \alpha r v\| - \|\alpha\delta v\| \overset{(3.4)}{=} \|F(x) - F(x_0) - \alpha r v\| - \|\alpha\delta v\| \overset{(3.9)}{\leq}$$
$$\overset{(3.9)}{\leq} \|F(x) - F(x_0) - \alpha r v\| - \alpha\|\xi'\| \overset{(3.2)}{=} U(x) - \alpha\|\xi'\| \overset{(3.10)}{=} U(x) - \alpha\|x - x'\|.$$

So, it is shown that there exists a point $x' \in M \setminus \{x\}$ such that relation (3.3) holds. Therefore, the Caristi-like condition (3.1) holds for the function U.

It is shown that all the assumptions of Theorem 3.1 hold. This theorem implies that there exists a point $\bar x \in M$ such that $U(\bar x) = 0$. The definitions of the set M and the functional U imply that $\bar x \in B_X(x_0, r)$ and $F(x_0) + \alpha r v = F(\bar x)$. Therefore, $F(x_0) + \alpha r v \in F(B_X(x_0, r))$. □

Lemma 3.2. *Given a linear bounded operator $A: X \to Y$ and a vector $v \in S$, there exists a nonzero functional $\lambda \in Y^*$ such that*

$$\|A^*\lambda\|_* \leq -\langle\lambda, v\rangle \operatorname{cov}(A|v).$$

Here, obviously, $\langle\lambda, v\rangle \leq 0$.

Proof. Put $c := \operatorname{cov}(A|v)$. The point cv does not belong to the interior of the set $AB_X(0,1)$. Otherwise, the inclusion $(\delta + c)v \in AB_X(0,1)$ takes place fo a sufficiently small $\delta > 0$, so $\operatorname{cov}(A|v) > c$ in contradiction to the definition of c. Moreover, the set $AB_X(0,1) \subset Y$ is convex.

By the finite-dimensional separability theorem (see, for example [4, Theorem 4.6]) there exists a nonzero $\lambda \in Y^*$ such that $\langle\lambda,\, Ax\rangle \geq \langle\lambda, v\rangle c$ for any $x \in B_X(0,1)$. Therefore, $\langle A^*\lambda,\, x\rangle \geq \langle\lambda, v\rangle c$ for every $x \in B_X(0,1)$. So, $-\|A^*\lambda\|_* \geq \langle\lambda, v\rangle c$. Therefore, $\|A^*\lambda\|_* \leq -\langle\lambda, v\rangle c$. □

4 Proofs of the main results

Proof of Theorem 2.1. Take an arbitrary $\varepsilon > 0$. Consider the set

$$M := \{x \in B_X(x_0, R) \cap \mathcal{M} : \quad f_0(x) \le f_0(x_0)\}.$$

Two cases may occur: either there exists a point $x \in M$ such that the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0,k}$ are linearly dependent or these vectors are linearly independent for every $x \in M$. In the first case, the point $\widehat{x} = x$ is the desired one (see the comments after the formulation of Theorem 2.1).

Consider the second case: the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0,k}$ are linearly independent for every $x \in M$. Then the Lagrange multiplier rule imply that the point x_0 is not a point of local minimum of f_0 under the constraints $f_1(x) = ... = f_k(x) = 0$ (see, for example, [9] or [5]). Thus,

$$f_0(x_0) > \gamma(x_0, R). \tag{4.1}$$

Put $Y := \mathbb{R}^{k+1}$, $v := (-1, 0, ..., 0) \in Y$. Define a mapping $F : X \to Y$ by the formula

$$F(x) := (f_0(x), f_1(x), ..., f_k(x)), \quad x \in X.$$

Obviously, the mapping F is differentiable and satisfies the assumption **(ii)** of Lemma 3.1. Indeed, if $F(x) \in F(x_0) + \text{cone}\{v\}$ for some $x \in B_X(x_0, R)$, then by virtue of the choice of v we have $f_0(x) \le f_0(x_0)$ and x_0, $x \in \mathcal{M}$, where $\mathcal{M} = \{\xi : f_1(\xi) = ... = f_k(\xi) = 0\}$. Thus, the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0,k}$ are linearly independent.

Put

$$\alpha_0 := (1 + \varepsilon)(f_0(x_0) - \gamma(x_0, R))R^{-1}.$$

It follows from (4.1) that $\alpha_0 > 0$. Let us show that there exists a point $\widehat{x} \in B_X(x_0, R)$ such that

$$F(\widehat{x}) \in F(x_0) + \text{cone}\{v\} \quad \text{and} \quad \text{cov}\left(\frac{\partial F}{\partial x}(\widehat{x}) \,\middle|\, v\right) < \alpha_0. \tag{4.2}$$

Consider to the contrary that

$$\bar{\alpha} := \inf\left\{\text{cov}\left(\frac{\partial F}{\partial x}(x) \,\middle|\, v\right) : x \in B_X(x_0, R), \;\; F(x) \in F(x_0) + \text{cone}\{v\}\right\} \ge \alpha_0. \tag{4.3}$$

Then the assumption **(i)** of Lemma 3.1 holds, since $\alpha_0 > 0$.

Put

$$\alpha := (1 + 2^{-1}\varepsilon)(f_0(x_0) - \gamma(x_0, R))R^{-1}.$$

It follows from relations (4.1) and $\alpha < \alpha_0 \le \bar{\alpha}$ that $\alpha \in (0, \bar{\alpha})$. Therefore, Lemma 3.1 implies that

$$F(x_0) + \alpha R v \in F(B_X(x_0, R)).$$

Therefore, there exists a point $x \in B_X(x_0, R)$ such that $F(x_0) + \alpha R v = F(x)$. Then the definition of the vector v implies that

$$f_0(x_0) - \left(1 + \frac{\varepsilon}{2}\right)(f_0(x_0) - \gamma(x_0, R)) = f_0(x), \quad f_i(x) = 0, \quad i = \overline{1,k}.$$

So, relation (4.1) and the definition of the mapping F imply that $f_0(x) < \gamma(x_0, R)$ and $x \in \mathcal{M} \bigcap B_X(x_0, R)$ which contradicts the definition of γ. Relation (4.2) is proved.

Applying Lemma 3.2 to the linear operator $A := \dfrac{\partial F}{\partial x}(\widehat{x})$ and the vector $v = (-1, 0, ..., 0)$ we obtain that there exists a vector $\lambda \in Y^*$ such that $\|\lambda\| = 1$ and

$$\left\| \left(\frac{\partial F}{\partial x}(\widehat{x}) \right)^* \lambda \right\|_* \leq -\langle \lambda, v \rangle \operatorname{cov}\left(\frac{\partial F}{\partial x}(\widehat{x}) \,\middle|\, v \right) = \lambda_0 \operatorname{cov}\left(\frac{\partial F}{\partial x}(\widehat{x}) \,\middle|\, v \right). \tag{4.4}$$

This inequality and the equality $\dfrac{\partial L}{\partial x}(\widehat{x}, \lambda) = \left(\dfrac{\partial F}{\partial x}(\widehat{x}) \right)^* \lambda$, imply that

$$\left\| \frac{\partial L}{\partial x}(\widehat{x}, \lambda) \right\|_* \leq \lambda_0 \operatorname{cov}\left(\frac{\partial F}{\partial x}(\widehat{x}) \,\middle|\, v \right).$$

The definition of α_0 and the strict inequality in (4.2) imply (2.1). The inclusion (4.2) implies that $f_0(\widehat{x}) \leq f_0(x_0)$. Inequality (4.4) and the relation $A^*\lambda \neq 0$ imply that $\lambda_0 \geq 0$. So, the vectors $\widehat{x}$ and λ are the desired ones. □

Proof of Theorem 2.2. Put

$$\gamma_0 := \inf_{x \in \mathcal{M}} f_0(x), \quad \varepsilon := 1.$$

Take an arbitrary sequence $x_0^n \subset \mathcal{M}$ such that $f_0(x_0^n) \to \gamma_0$. Applying Theorem 2.1 at the point $x_0 = x_0^n$ as $R = 1$ we obtain that there exist sequences $\{x^n\} \subset \mathcal{M}$ and $\{\lambda^n\} \subset \mathbb{R}^{k+1}$ such that $\|\lambda^n\| = 1$ for every n, $f_0(x^n) \leq f_0(x_0^n)$ for every n and

$$\left\| \frac{\partial L}{\partial x}(x^n, \lambda^n) \right\|_* \leq 2\frac{f_0(x_0^n) - \gamma(x_0^n, R)}{R} \leq 2\frac{f_0(x_0^n) - \gamma_0}{R} \to 0$$

as $n \to \infty$. The constructed sequences $\{x^n\} \subset \mathcal{M}$ and $\{\lambda^n\} \subset \mathbb{R}^{k+1}$ are the desired ones. □

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References

[1] A.V. Arutyunov, *Caristi's condition and existence of a minimum of a lower bounded function in a metric space. Applications to the theory of coincidence points.* Proc. Steklov Inst. Math. 291 (2015), no. 1, 24–37.

[2] A.V. Arutyunov, *Necessary extremum conditions and an inverse function theorem without a priori normality assumptions.* Proc. Steklov Inst. Math. 236 (2002), 25–36.

[3] A. Arutyunov, N. Bobylev, S. Korovin, *One remark to Ekeland's variational principle.* Computers Math. Appl., 34 (1997), no. 2–4, 267–271.

[4] A. Arutyunov, V. Obukhovskii, *Convex and set-valued analysis.* De Gruyter, Berlin, 2016.

[5] A.V. Arutyunov, R.B. Vinter, *A simple "finite approximations" proof of the Pontryagin maximum principle under reduced differentiability hypotheses.* Set-Valued Var. Anal. 12 (2004), no. 1–2, 5–24.

[6] J.-P. Aubin, I. Ekeland, *Applied nonlinear anlysis.* J. Wiley & Sons, N.Y. 1984.

[7] N.A. Bobylyev, S.V. Emel'yanov, S.K. Korovin, *Geometrical methods in variational problems.* Magistr, Moscow, 1998.

[8] M.J. Fabian, D. Preiss, *A generalization of the interior mapping theorem of Clarke and Pourciau.* Comment. Math. Univ. Carolin. 28 (1987), no. 2, 311–324.

[9] H. Halkin, *Implicit functions and optimization problems without continuous differentiability of the data.* SIAM J. Control. 12 (1974), 229–236.

[10] A.D. Ioffe, V.M. Tihomirov, *Theory of extremal problems.* North-Holand Publ. Comp., Amsterdam, 1979.

[11] L. Nirenberg, *Topics in nonlinear functional analysis.* Courant Inst. of Math. Sciences, 1974.

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A NOTE ON QUASILINEAR ELLIPTIC SYSTEMS WITH L^∞-DATA

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Abstract. We prove the existence of a weak energy solution for the boundary value problem

$$\begin{aligned} -\mathrm{div}\, a(x,u,Du) = f \quad &\text{in } \Omega,\\ u = 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

where Ω is a smooth bounded open domain in $\mathbb{R}^n$ $(n \geq 3)$ and $f \in L^\infty(\Omega;\mathbb{R}^m)$. The existence result is proved using the concept of Young measures.

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1 Introduction

This article is concerned with the existence of weak energy solutions of the boundary value problems for quasilinear elliptic systems of the form

$$\begin{cases} -\mathrm{div}\, a(x,u,Du) &= f \quad \text{in } \Omega,\\ \qquad\qquad\quad u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ $(n \geq 3)$ with a smooth boundary $\partial\Omega$ and f belongs to $L^\infty(\Omega;\mathbb{R}^m)$. Here $u : \Omega \to \mathbb{R}^m$, $m \in \mathbb{N}^*$, is a vector-valued function and Du is the Jacobian matrix of u given by

$$Du(x) = \big(D_1u(x), D_2u(x), ..., D_nu(x)\big) \quad \text{with} \quad D_i = \partial/\partial_i(x_i).$$

We denote by $\mathbb{M}^{m\times n}$ the real space of all $m \times n$ matrices equipped with the inner product $\xi : \eta = \sum_{i,j} \xi_{ij}\eta_{ij}$ for all $\xi, \eta \in \mathbb{M}^{m\times n}$.

We assume that the function $a : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e., $x \mapsto a(x,s,\xi)$ is measurable for every $(s,\xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(s,\xi) \mapsto a(x,s,\xi)$ is continuous for almost every $x \in \Omega$ and satisfies the following conditions: $\xi \mapsto a(x,u,\xi)$ is continuously differentiable and such that for a convex and C^1-mapping $A : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$, we have

$$a(x,u,\xi) = \frac{\partial}{\partial\xi} A(x,u,\xi) \tag{1.2}$$

and

$$A(x,u,0) = 0 \tag{1.3}$$

for almost every $x \in \Omega$ and all $u \in \mathbb{R}^m$. Moreover, we assume that

$$|a(x,s,\xi)| \leq d_1(x) + |s|^{p-1} + |\xi|^{p-1} \tag{1.4}$$

for almost every $x \in \Omega$ and for every $(s,\xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$, where $0 \le d_1 \in L^{p'}(\Omega)$, with $1/p + 1/p' = 1$ and the exponent p is such that $2 \le p < n$. In addition, the mapping $\xi \to a(x,s,\xi)$ is monotone, i.e.,

$$\big(a(x,s,\xi) - a(x,s,\eta)\big) : (\xi - \eta) \ge 0, \quad \forall \xi, \eta \in \mathbb{M}^{m\times n}. \tag{1.5}$$

Finally, the following inequality holds:

$$|\xi|^p \le a(x,s,\xi) : \xi \le pA(x,s,\xi). \tag{1.6}$$

The concept of Young measure was introduced in [15] to prove the existence of solutions for (1.1) when $p \in (1, 2 - \frac{1}{n}]$ and $f = \mu$ is a measure. The authors used weak monotonicity assumptions on the function a and the weak derivative Du is replaced by the approximate derivative $apDu$. Hungerbühler has studied, in [19], the existence of weak solutions for (1.1) when the right-hand side belongs to the dual of the Sobolev space $W_0^{1,p}(\Omega;\mathbb{R}^m)$. He used also mild monotonicity assumptions and Young measures to achieve the result. The uniqueness and maximal regularity for nonlinear elliptic systems (1.1) have been proved in [16] when $f = \mu$ a Radon measure. Zhou [28] introduced the sign condition:

$$a_i(x,u,\xi)\cdot\xi_i \ge 0 \quad \text{for } i = 1,...,m,$$

instead of the angle condition:

$$a(x,u,\xi) : M\xi \ge 0$$

assumed in [15], to prove the existence and regularity of solutions to (1.1) with $f = \mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$. For more results, we refer the reader to see [14, 20, 21, 22, 23, 24, 26, 27] and [1, 2, 3, 4, 5, 6, 7, 8] where we have used the theory of Young measures for various quasilinear systems.

In [2, 3] we have proved the existence of weak solutions for various kinds of quasilinear elliptic systems similar to (1.1), for $f \in W^{-1,p'}(\Omega;\mathbb{R}^m)$, under various kinds of monotonicity assumptions and based on the theory of Young measures. See also [10, 11, 12, 13] for more results and [25] for different theories and methods used in nonlinear analysis.

In this paper, the source term in (1.1) is assumed to be in $L^\infty(\Omega;\mathbb{R}^m)$ and a to satisfy conditions (1.2)-(1.6). The main objective is to prove the existence of a weak energy solution using the concept of Young measure and energy functionals. Moreover, a is assumed to be the derivative over the third argument of another function A. This assumption is necessary in order to associate with the problem an energy functional, and then to minimize this functional to obtain a weak solution. The main result of the paper consists in justification of sufficient assumptions for such minimization

A prototype example that is covered by our assumptions (1.2)-(1.6) is the following p-Laplacian problem: Consider

$$A(x,u,\xi) = \frac{1}{p}|\xi|^p, \quad a(x,u,\xi) = |\xi|^{p-2}\xi$$

where $p \ge 2$.

The remaining part of this paper is organized as follows: a brief review on Young measures is presented in Section 2, while Section 3 is devoted to state the existence result and its proof.

2 A brief review on Young measures

By $C_0(\mathbb{R}^m)$ we denote the closure of the space of continuous functions on $\mathbb{R}^m$ with compact support with respect to the $\|.\|_\infty$-norm. Its dual can be identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with finite mass. The related duality pairing is given for $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$, by

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda).$$

Lemma 2.1 (See p. 19 in [17]). *Let $\{z_j\}_{j\geq 1}$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exists a subsequence $\{z_k\}_k \subset \{z_j\}_j$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x\in\Omega$, such that for almost each $\varphi\in C(\mathbb{R}^m)$ we have*

$$\varphi(z_k) \rightharpoonup^* \overline{\varphi} \quad \text{weakly in } L^\infty(\Omega;\mathbb{R}^m),$$

where $\overline{\varphi}(x) = \langle \nu_x, \varphi\rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu_x(\lambda)$ for a.e. $x\in\Omega$.

Definition 1. *We call $\{\nu_x\}_{x\in\Omega}$ the family of Young measures associated with the subsequence $\{z_k\}_k$.*

Remark 1.

- *In [9], it is shown that for any Carathéodory function $\varphi:\Omega\times\mathbb{R}^m\to\mathbb{R}$ and $\{z_k\}_k$ a sequence that generates the Young measure ν_x, we then have*

$$\varphi(x,z_k) \rightharpoonup \langle \nu_x, \varphi(x,.)\rangle = \int_{\mathbb{R}^m} \varphi(x,\lambda) d\nu_x(\lambda)$$

  weakly in $L^1(\Omega')$ for all measurable $\Omega'\subset\Omega$, provided that the negative part $\varphi^-(x,z_k)$ is equiintegrable.

- *Ball shows also in [9], that if z_k generates the Young measure ν_x, then for $\varphi\in L^1(\Omega;C_0(\mathbb{R}^m))$*

$$\lim_{k\to\infty}\int_\Omega g(x,z_k(x))dx = \int_\Omega \langle \nu_x, g(x,.)\rangle dx.$$

Lemma 2.2 ([18]). *If $|\Omega|<\infty$ then*

$$z_k\to z \text{ in measure } \Leftrightarrow \nu_x=\delta_{z(x)} \quad \textit{for a.e. } x\in\Omega.$$

Lemma 2.3 ([1]). *If $\{Dz_k\}_k$ is bounded in $L^p(\Omega;\mathbb{M}^{m\times n})$, then the Young measure ν_x generated by Dz_k has the following properties:*

(i) ν_x is a probability measure, i.e. $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} := \int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda) = 1$ for almost every $x\in\Omega$.

(ii) The weak L^1-limit of Dz_k is given by $\langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$.

(iii) ν_x satisfies $\langle \nu_x, id\rangle = Dz(x)$ for almost every $x\in\Omega$.

We conclude this section by recalling the following Fatou-type inequality.

Lemma 2.4 ([15]). *Let $\varphi:\Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n}\to\mathbb{R}$ be a Carathéodory function and $z_k:\Omega\to\mathbb{R}^m$ a sequence of measurable functions such that $z_k\to z$ in measure and such that Dz_k generates the Young measure ν_x, with $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})}=1$ for almost every $x\in\Omega$. Then*

$$\liminf_{k\to\infty}\int_\Omega \varphi(x,z_k,Dz_k)dx \geq \int_\Omega\int_{\mathbb{M}^{m\times n}} \varphi(x,z,\lambda)d\nu_x(\lambda)dx$$

provided that the negative part $\varphi^-(x,z_k,Dz_k)$ is equiintegrable.

For more results and details about Young measures, we refer the reader not familiar with this concept to see for example [9, 17, 18, 25].

3 Existence of weak energy solution

Before we state the main result of this paper, let us introduce the following definition of weak energy solutions of (1.1).

Definition 2. *A weak energy solution of* (1.1) *is a function* $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ *such that*

$$\int_\Omega (a(x,u,Du) : D\varphi)dx = \int_\Omega f(x)\varphi dx, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m).$$

The main result is given in the following.

Theorem 3.1. *Assume* $f \in L^\infty(\Omega;\mathbb{R}^m)$ *and* (1.2)-(1.6) *hold. Then there exists a weak energy solution of* (1.1).

Proof of the main result. Let us define the energy functional $J : W_0^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ by

$$J(u) = \int_\Omega A(x,u,Du)dx - \int_\Omega fudx.$$

Proposition 3.1. *The functional* J *is well-deffined on* $W_0^{1,p}(\Omega;\mathbb{R}^m)$ *and* $J \in C^1(W_0^{1,p}(\Omega;\mathbb{R}^m),\mathbb{R})$ *with the derivative given by*

$$\langle J'(u),\varphi\rangle = \int_\Omega (a(x,u,Du) : D\varphi)dx - \int_\Omega f\varphi dx,$$

for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Proof. For any $x \in \Omega$, $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $\xi \in \mathbb{M}^{m\times n}$, we have

$$A(x,u,\xi) = \int_0^1 \frac{d}{dt}A(x,u,t\xi)dt = \int_0^1 a(x,u,t\xi) : \xi dt.$$

Using (1.4), we get

$$\begin{aligned} A(x,u,\xi) &\leq \int_0^1 \left(d_1(x) + |u|^{p-1} + t^{p-1}|\xi|^{p-1}\right)|\xi|dt \\ &\leq d_1(x)|\xi| + |u|^{p-1}|\xi| + \frac{1}{p}|\xi|^p. \end{aligned} \tag{3.1}$$

This and the Hölder inequality imply that

$$0 \leq \int_\Omega |A(x,u,Du)|dx \leq \|d_1\|_{p'}\|Du\|_p + \|u\|_p^{p-1}\|Du\|_p + \frac{1}{p}\|Du\|_p^p$$

and

$$\int_\Omega |fu|dx \leq \|f\|_{q'}\|u\|_q, \quad \text{where } 1 < q < p.$$

Next we deduce that J is well-defined on $W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Let us fix $x \in \Omega$ and $0 < |r| < 1$. According to the mean value theorem, there exists $\theta \in [0,1]$ such that

$$\begin{aligned} &\left|a(x,u,Du+\theta D\varphi)\right||D\varphi| \\ &= \frac{\left|A(x,u,Du+rD\varphi) - A(x,u,Du)\right|}{|r|} \\ &\leq \left(d_1(x) + |u|^{p-1} + |Du+\theta rD\varphi|^{p-1}\right)|D\varphi| \\ &\leq \Big(d_1(x) + |u|^{p-1} + 2^{p-2}\big(|Du|^{p-1} + (\theta r)^{p-1}|D\varphi|^{p-1}\big)\Big)|D\varphi|. \end{aligned}$$

Hölder's inequality gives that

$$\int_\Omega d_1(x)|D\varphi|dx \le \|d_1\|_{p'}\|D\varphi\|_p,$$

$$\int_\Omega |Du|^{p-1}|D\varphi|dx \le \|Du\|_p^{p-1}\|D\varphi\|_p$$

and

$$\int_\Omega |D\varphi|^{p-1}|D\varphi|dx = \|D\varphi\|_p^p.$$

From these inequalities, we deduce that

$$\Big(d_1(x) + |u|^{p-1} + 2^{p-2}\big(|Du|^{p-1} + (\theta r)^{p-1}|D\varphi|^{p-1}\big)\Big)|D\varphi| \in L^1(\Omega).$$

Thanks to the Lebesgue theorem, it follows that

$$\langle J'(u), \varphi\rangle = \int_\Omega a(x,u,Du) : D\varphi dx - \int_\Omega f\varphi dx.$$

Assume now that $u_k \to u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Then $(u_k)_k$ is a bounded sequence in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. According to Lemma 2.1 there is a Young measure ν_x generated by Du_k in $L^p(\Omega;\mathbb{M}^{m\times n})$ and satisfying the properties of Lemma 2.3. Using (1.5) and [2, Lemma 5.3], we get that

$$\begin{aligned}0 &\le \big(a(x,u,\lambda) - a(x,u,Du+\tau\xi)\big) : (\lambda - Du - \tau\xi)\\ &= a(x,u,Du) : (\lambda - Du) - a(x,u,\lambda) : \tau\xi\\ &- a(x,u,Du+\tau\xi) : (\lambda - Du - \tau\xi),\end{aligned}$$

which gives

$$-a(x,u,\lambda) : \tau\xi \ge -a(x,u,Du) : (\lambda - Du) + a(x,u,Du+\tau\xi) : (\lambda - Du - \tau\xi),$$

for every $\lambda, \xi \in \mathbb{M}^{m\times n}$ and $\tau \in \mathbb{R}$. We have $\xi \mapsto a(x,u,\xi)$ is continuously differentiable, hence we can write

$$\begin{aligned}&a(x,u,Du+\tau\xi) : (\lambda - Du - \tau\xi)\\ &= a(x,u,Du+\tau\xi) : (\lambda - Du) - a(x,u,Du+\tau\xi) : \tau\xi\\ &= a(x,u,Du) : (\lambda - Du)\\ &+ \tau\Big(\big(\nabla a(x,u,Du)\xi\big) : (\lambda - Du) - a(x,u,Du) : \xi\Big) + o(\tau),\end{aligned}$$

where ∇ is the derivative of a with respect to its third variable. Therefore,

$$-a(x,u,\lambda) : \tau\xi \ge \tau\Big(\big(\nabla a(x,u,Du)\xi\big) : (\lambda - Du) - a(x,u,Du) : \xi\Big) + o(\tau)$$

which gives, since τ is arbitrary in $\mathbb{R}$, that

$$a(x,u,\lambda) : \xi = a(x,u,Du) : \xi + \big(\nabla a(x,u,Du)\xi\big) : (Du - \lambda) \tag{3.2}$$

on the support of ν_x. Since $(a(x,u_k,Du_k))_k$ is equiintegrable by (1.4) and $(u_k)_k$ is bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, it follows that its weak L^1-limit $\overline{a}$ is given by

$$\begin{aligned}\overline{a}(x) &:= \int_{\mathbb{M}^{m\times n}} a(x,u,\lambda)d\nu_x(\lambda)\\ &\overset{(3.2)}{=} a(x,u,Du)\underbrace{\int_{\operatorname{supp}\nu_x} d\nu_x(\lambda)}_{:=1} + \big(\nabla a(x,u,Du)\big)^t\underbrace{\int_{\operatorname{supp}\nu_x}(Du-\lambda)d\nu_x(\lambda)}_{:=0}\\ &= a(x,u,Du).\end{aligned}$$

As $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ is reflexive, it follows that $(a(x,u_k,Du_k))_k$ converges in $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ and its weak $L^{p'}$-limit is also $\overline{a}(x)=a(x,u,Du)$. This and the Hölder inequality imply

$$\big|\langle J'(u_k)-J'(u),\varphi\rangle\big| \le \int_\Omega \big|a(x,u_k,Du_k)-a(x,u,Du)\big||D\varphi|dx$$

and so

$$\|J'(u_k)-J'(u)\| \le \|a(x,u_k,Du_k)-a(x,u,Du)\|_{p'} \longrightarrow 0$$

as $k\to\infty$. □

Lemma 3.1. *The functional J is bounded from below, coercive and weakly lower semi-continuous.*

Proof. By (3.1) and Hölder's inequality, it is obvious that J is bounded from below. Using (1.6), we have

$$\begin{aligned} J(u) &= \int_\Omega A(x,u,Du)dx - \int_\Omega fudx \\ &\ge \frac{1}{p}\int_\Omega |Du|^p dx - \|f\|_{q'}\|u\|_q, \quad (\text{with } 1<q<p) \\ &\ge \frac{1}{p}\int_\Omega |Du|^p dx - c\|u\|_{1,p} \longrightarrow +\infty \end{aligned}$$

as $\|u\|_{1,p}\to\infty$, since $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is continuously embedded in $L^q(\Omega;\mathbb{R}^m)$. Then J is coercive. Let $(u_k)\subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ be a sequence which converges weakly to u in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Hence $u_k\to u$ in $L^p(\Omega;\mathbb{R}^m)$ and in measure on Ω (for a subsequence still indexed by k), by the compact embedding of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ in $L^p(\Omega;\mathbb{R}^m)$. Since $\nu_x=\delta_{Du(x)}$ for a.e. $x\in\Omega$ by Lemma 2.3, then Lemma 2.2 implies $Du_k\to Du$ in measure. We have $(A(x,u_k,Du_k))_k$ is equiintegrable by (3.1), it follows then by Lemma 2.4 that

$$\int_\Omega\int_{\mathbb{M}^{m\times n}} A(x,u,\lambda)d\nu_x(\lambda)dx \le \liminf_{k\to\infty}\int_\Omega A(x,u_k,Du_k)dx. \tag{3.3}$$

On the other hand, assumption (1.5) and the relation $a(x,u,\xi)=\frac{\partial}{\partial\xi}A(x,u,\xi)$ imply, in particular, that $\xi\mapsto A(x,u,\xi)$ is convex, i.e.,

$$\underbrace{A(x,u,\lambda)}_{=:F(\lambda)} \ge \underbrace{A(x,u,Du)+a(x,u,Du):(\lambda-Du)}_{=:G(\lambda)}, \quad \forall\lambda\in\mathbb{M}^{m\times n}.$$

Since $\lambda\mapsto F(\lambda)$ is a C^1-function by Proposition 3.1, then for $\tau\in\mathbb{R}$

$$\frac{F(\lambda+\tau\xi)-F(\lambda)}{\tau} \le \frac{G(\lambda+\tau\xi)-G(\lambda)}{\tau} \quad \text{for } \tau<0$$

and

$$\frac{F(\lambda+\tau\xi)-F(\lambda)}{\tau} \ge \frac{G(\lambda+\tau\xi)-G(\lambda)}{\tau} \quad \text{for } \tau>0.$$

Hence $\nabla F=\nabla G$, i.e.,

$$A(x,u,\lambda)=A(x,u,Du) \quad \text{for all } \lambda\in\operatorname{supp}\nu_x. \tag{3.4}$$

Going back to (3.3), it follows by (3.4) that

$$\begin{aligned} \int_\Omega\int_{\mathbb{M}^{m\times n}} A(x,u,\lambda)d\nu_x(\lambda) &= \int_\Omega\int_{\operatorname{supp}\nu_x} A(x,u,Du)d\nu_x(\lambda)dx \\ &= \int_\Omega A(x,u,Du)dx \\ &\le \liminf_{k\to\infty}\int_\Omega A(x,u_k,Du_k)dx. \end{aligned}$$

This fact implies that

$$J(u) \leq \liminf_{k\to\infty} J(u_k).$$

Hence, J is weakly lower semi-continuous and the proof is complete. □

Since J is proper, weakly semi-continuous and coercive, then J has a minimizer which is in fact a weak energy solution of (1.1). The proof of the main result is complete.

References

[1] E. Azroul, F. Balaadich, *Weak solutions for generalized p-Laplacian systems via Young measures*, Moroccan J. of Pure and Appl. Anal. (MJPAA) 4 (2018), no. 2, 76-83.

[2] E. Azroul, F. Balaadich, *Quasilinear elliptic systems in perturbed form*, Int. J. Nonlinear Anal. Appl., 10 (2019), no. 2, 255-266.

[3] E. Azroul, F. Balaadich, *A weak solution to quasilinear elliptic problems with perturbed gradient*, Rend. Circ. Mat. Palermo (2) (2020). https://doi.org/10.1007/s12215-020-00488-4

[4] E. Azroul, F. Balaadich, *On strongly quasilinear elliptic systems with weak monotonicity*, J. Appl. Anal., (2021) https://doi.org/10.1515/jaa-2020-2041

[5] E. Azroul, F. Balaadich, *Existence of solutions for a class of Kirchhoff-type equation via Young measures*, Numer. Funct. Anal. Optim. (2021) https://doi.org/10.1080/01630563.2021.1885044

[6] F. Balaadich, E. Azroul, (2021). *Elliptic systems of p-Laplacian type*. Tamkang Journal of Mathematics, 53. https://doi.org/10.5556/j.tkjm.53.2022.3296

[7] F. Balaadich, *On p-Kirchhoff-type parabolic problems*, Rendiconti del Circolo Matematico di Palermo Series 2 (2022) https://doi.org/10.1007/s12215-021-00705-8

[8] F. Balaadich, E. Azroul, *Existence results for fractional p-Laplacian systems via young measures*, Mathematical Modelling and Analysis 27 (2022), no. 2, 232-241.

[9] J.M. Ball, *A version of the fundamental theorem for Young measures*, In: PDEs and Continuum Models of Phase Transitions (Nice, 1988). Lecture Notes in Phys, vol. 344 (1989), 207-215.

[10] L.C. Berselli, C.R. Grisanti, *On the regularity up to the boundary for certain nonlinear elliptic systems*, Discrete Contin. Dyn. Syst. Ser. S, 9 (2016), no. 1, 53-71.

[11] M. Bulíček, G. Cupini, B. Stroffolini, A. Verde, *Existence and regularity results for weak solutions to (p,q)-elliptic systems in divergence form*, Adv. Calc. Var., 11 (2018), no. 3, 273-288.

[12] J. Chabrowski, K. Zhang, *Quasi-monotonicity and perturbated systems with critical growth*, Indiana Univ. Math. J.,(1992), 483-504.

[13] G. Cupini, P. Marcellini, E. Mascolo, *Existence and regularity for elliptic equations under p,q-growth*, Adv. Differential Equations 19 (2014), 693-724.

[14] G. Dal Maso, F. Murat, *Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems*, Nonlinear Anal., 31 (1998), no. 3/4, 405-412.

[15] G. Dolzmann, N. Hungerbühler, S. Müller, *Nonlinear elliptic systems with measure-valued right hand side. Math. Z.*, 226 (1997), 545-574.

[16] G. Dolzmann, N. Hungerbühler, S. Müller, *Uniqueness and maximal regularity for nonlinear elliptic systems of n-Laplace type with measure valued right hand sid*e. J. Reine Angew. Math. (Crelles J.) 520 (2000), 1-35.

[17] LC. Evans, *Weak convergence methods for nonlinear partial differential equations.* Am. Math. Soc., New York 1990.

[18] N. Hungerbühler, *A refinement of Balle's theorem on Young measures*, N.Y. J. Math. 3 (1997), 48-53.

[19] N. Hungerbühler, *Quasilinear elliptic systems in divergence form with weak monotonicity*, New York J. Math., 5 (1999), 83-90.

[20] T. Kuusi, G. Mingione, *Nonlinear potential theory of elliptic systems*, Nonlinear Anal., 138 (2016), 277-299.

[21] F. Leonetti, E. Rocha, V. Staicu, *Quasilinear elliptic systems with measure data*, Nonlinear Anal., 154 (2017), 210-224.

[22] F. Leonetti, E. Rocha, V. Staicu, *Smallness and cancellation in some elliptic systems with measure data*, J. Math. Anal. Appl., 465 (2018), no. 2, 885-902.

[23] G. Mingione, *Nonlinear measure data problems*, Milan J. Math., 79 (2011), 429-496.

[24] B.Z. Omarova, Z.A. Sartabanov, *On multiperiodic solutions of perturbed nonlinear autonomous systems with the differentiation operator on a vector field*, Eurasian Mathematical Journal, 12 (2021), no. 1, 68-81.

[25] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Nonlinear analysis - theory and methods*, Springer Nature, Cham, 2019.

[26] M.A. Ragusa, *Hölder regularity results for solutions of parabolic equations, Variational Analysis and Application*, Book Series: Nonconvex Optimization and its applications, 79 (2005), 921-934.

[27] A. Vinodkumar, C. Loganathan, S. Vijay, *Approximate controllability of random impulsive quasilinear evolution equation*, Filomat, 34 (2020), no. 5, 1611-1620,

[28] S. Zhou, *A note on nonlinear elliptic systems involving measures*, Electron. J. Differential Equations 2000 (2000), no. 8, 1-6.

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CALDERÓN – LOZANOVSKIĬ CONSTRUCTION FOR A COUPLE OF GLOBAL MORREY SPACES

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Abstract. It is shown that under certain conditions on the parameters included in the definition of global Morrey spaces, the Calderón – Lozanovskiĭ construction on a pair of global Morrey spaces gives a space of the same type. Calculation of the Calderón – Lozanovskiĭ construction for a pair of global Morrey spaces allowed us to obtain new interpolation theorems even for classical global Morrey spaces.

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1 Introduction

In 1938, due to the applications in elliptic partial differential equations, Morrey [28] introduced a class of function spaces, nowadays named after him. In recent years, there is an increasing interest in applications of Morrey spaces in various areas of analysis, such as partial differential equations, potential theory and harmonic analysis; we refer, for example, to [1], [11], [12], [21], [25], [33], [35] and their references.

We begin with some basic notation from the theory of Morrey spaces.

Let μ be Lebesgue measure in $\mathbb{R}^n$, let $S(\mu,\mathbb{R}^n) = S(\mu)$ be the space of all Lebesgue measurable functions $x : \mathbb{R}^n \to \mathbb{R}$ and let $\chi(D)$ stand for the characteristic function of a set $D \subset \mathbb{R}^n$. Along with the Lebesgue spaces $L^p \equiv L^p(\mathbb{R}^n)$, $p \in [1,\infty]$ ideal spaces X are often used in harmonic analysis. Recall their definition (see, for example, [20], [24]).

A Banach space X of measurable functions on Ω is said to be ideal if it follows from the condition $x \in X$, the measurability of y and the validity of the inequality $|y(t)| \le |x(t)|$ for almost all $t \in \Omega$ that $y \in X$ and $\|y|X\| \le \|x|X\|$ (the symbol $\|x|X\|$ denotes the norm of an element x in the space X). Let $v \in S(\mu)$, $v > 0$ almost everywhere (v is a weight). We denote by the symbol X_v a new ideal space in which the norm is given by the equation $\|x|X_v\| = \|x \cdot v|X\|$. When $X = L^p$, our definition of weighted space differs somewhat from the often used one: when the weight is included in the measure.

Along with function spaces we need ideal spaces of sequences. Let $e^i = \{..., 0, 1, 0, ...\}$, ($i \in \mathbb{Z}$, the unit stands in the i-th place) be the standard basis in the space of two-side sequences. We denote by the symbol l an ideal space of sequences $x = \sum_{i=-\infty}^{\infty} x_i e^i$ ($x_i \in \mathbb{R}$) with the norm $\|x|l\|$. All the properties listed above for function spaces are preserved for sequence spaces. For details concerning the theory of sequence spaces, see [23].

The classical Morrey space M_{λ,L^p}, $(\lambda \in \mathbb{R})$ (see [28]), consists of all functions $x \in L^{1,loc}(\mathbb{R}^n)$ for which the following norm is finite:

$$\|x|M_{\lambda,L^p}\| = \sup_{t\in\mathbb{R}^n}\sup_{r>0} r^{-\lambda}\|x(t+.)\chi(B(0,r))|L^p\|.$$

We note that if $\lambda = 0$, then $M_{\lambda,L^p} = L^p$, if $\lambda = \frac{n}{p}$, then $M_{\lambda,L^p} = L^\infty$, if $\lambda < 0$ or $\lambda > \frac{n}{p}$, then M_{λ,L^p} consists only of functions equivalent to zero.

As a natural generalization of Lebesgue spaces, the interpolation properties of Morrey spaces became an interesting question. The first result on this problem is due to Stampacchia [34] and, independently, Campanato and Murthy [17]. They obtained an interpolation property for linear operators from Lebesgue spaces to Morrey spaces on $\mathbb{R}^n$ and showed that, if a linear operator T is bounded from $L^{q_i}(\mathbb{R}^n)$ to Morrey spaces $M_{\lambda_i,L^{p_i}}(\mathbb{R}^n)$ with the operator norm M_i, $i \in \{0,1\}$, then T is also bounded from $L^{q_\theta}(\mathbb{R}^n)$ to $M_{\lambda_\theta,L^{p_\theta}}(\mathbb{R}^n)$ when

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \lambda_\theta = (1-\theta)\lambda_0 + \theta\lambda_1 \tag{1.1}$$

for some $\theta \in (0,1)$ with the operator norm not more than a positive constant multiple of $M_0^{1-\theta}M_1^\theta$. In 1969, Peetre [31] found that the previous conclusion still holds true when $(L^{q_0}(\mathbb{R}^n),\ L^{q_1}(\mathbb{R}^n))$ and $L^{q_\theta}(\mathbb{R}^n)$ are replaced, respectively, by a certain abstract pair (A_0, A_1) and an interpolation space A constructed from (A_0, A_1).

However, the converse result in general is not true. In 1995, Ruiz and Vega [32] proved that, when $n \geq 2$, $u \in (0,n)$, $\theta \in (0,1)$, $1 \leq p_2 < p_3 < \frac{n-1}{u} < p_1 < \infty$ and $\lambda_1 = \frac{1}{p_1} - \frac{1}{u}$, $\lambda_2 = \frac{1}{p_2} - \frac{1}{u}$, $\lambda_3 = \frac{1}{p_3} - \frac{1}{u}$ for any given $C \in (0,\infty)$, there exists a positive continuous linear operator $T : M_{\lambda_i,L^{p_i}}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, $i \in \{1,2,3\}$, with the operator norm satisfying $\|T|M_{\lambda_i,L^{p_i}}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)\| \leq K_i$, $i \in \{1,2\}$, but $\|T|M_{\lambda,L^{p_3}}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)\| \geq CK_0^{1-\theta}K_1^\theta$ for $\frac{1}{p_3} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. This implies the lack of convexity of operators on Morrey spaces.

In the case $n = 1$, Blasco, Ruiz and Vega [9] in 1999 proved that, for a particular u, if $1 < p_0 < p_1 < u < \infty$ and $\lambda_1 = \frac{1}{p_1} - \frac{1}{u}$, $\lambda_2 = \frac{1}{p_2} - \frac{1}{u}$, then there exist $q_0,\ q_1 \in (1,\infty)$ and a positive continuous linear operator T which is bounded from $M_{\lambda_i,L^{p_i}}(\mathbb{R})$ to $L^{q_i}(\mathbb{R})$, $i \in \{0,1\}$, but not bounded from $M_{\lambda_\theta,L^{p_\theta}}(\mathbb{R})$ to $L^{q_\theta}(\mathbb{R})$ when conditions (1.1) are satisfied. These counterexamples show that Morrey spaces have no interpolation property in general.

Nevertheless, under some restriction, Morrey spaces also have some interpolation properties. Let $0 < \lambda_0 < \frac{n}{p_0}$, $0 < \lambda_1 < \frac{n}{p_1}$, $\theta \in (0,1)$ and p_θ, λ_θ be defined by (1.1). Recently, Lemarie-Rieusset [21], [22] showed that for $p_0,\ p_1,\ \lambda_0,\ \lambda_1,\ \theta,\ p_\theta$ and λ_θ as above,

$$[M_{\lambda_0,L^{p_0}}(\mathbb{R}^n), M_{\lambda_1,L^{p_1}}(\mathbb{R}^n)]_\theta = M_{\lambda_\theta,L^{p_\theta}}(\mathbb{R}^n) \tag{1.2}$$

if and only if

$$p_0\lambda_0 = p_1\lambda_1, \tag{1.3}$$

holds, which gives a necessary and sufficient condition ensuring the interpolation property of Morrey spaces on $\mathbb{R}^n$. Here, $[M_{\lambda_0,L^{p_0}}(\mathbb{R}^n), M_{\lambda_1,L^{p_1}}(\mathbb{R}^n)]_\theta$ denotes the space obtained using the first of Calderón interpolation methods [16] for a pair of Morrey spaces $(M_{\lambda_0,L^{p_0}}(\mathbb{R}^n), M_{\lambda_1,L^{p_1}}(\mathbb{R}^n))$.

Note that the situation changes radically for pairs of local Morrey spaces [13], [14], [15], [2], [3], [6]. For example, if in (1.2) global Morrey spaces are replaced by local Morrey spaces, then equality (1.2) will hold without restriction on the indices (1.3).

In this paper, we give a generalization of equality (1.2) to general Morrey spaces. Namely, for any functions $\varphi_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ each of which is concave, positively homogeneous of degree one, nondecreasing and continuous in each variable and such that $\varphi_i(0,0) = 0$, (i=0,1,2), the triple of spaces

$$\{\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}),\quad \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1});\quad \overline{\varphi}(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1})\}.$$

has interpolation properties. Here, $\overline{\varphi}(t,s) \equiv \varphi_2(\varphi_0(t,s),\varphi_1(t,s))$, $(t,s \geq 0)$, $(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1})$ is a pair of general Morrey spaces, and $\varphi(X_0,X_1)$ denotes the space constructed from the pair of ideal spaces (X_0,X_1) using the construction of Calderón – Lozanovskiĭ. In particular, we show that for any concave function φ, the triple of spaces

$$\{M_{\lambda_0,L^{p_0}}(\mathbb{R}^n),\quad M_{\lambda_1,L^{p_1}}(\mathbb{R}^n),\quad \widetilde{\varphi}(M_{\lambda,L^1}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))\} \tag{1.4}$$

has interpolation properties, when condition (1.3) is met. Here, $\widetilde{\varphi}(t,s) \equiv \varphi(t^{\theta_0}s^{1-\theta_0}, t^{\theta_1}s^{1-\theta_1})$, $(\theta_0 = 1/p_0,\ \theta_1 = 1/p_1, \lambda = \lambda_0/\theta_0;\ t,s \geq 0)$.

Note that if instead of the triplet of global Morrey spaces (1.4) we consider the corresponding triplet of local Morrey spaces, then the triplet of local Morrey spaces will have the interpolation property not only when (1.3) is satisfied, but also in a much more general case [3], [6].

2 Basis constructions

We now replace the Lebesgue space L^p in the definition of the classical Morrey space by an ideal space X, the outer sup-norm by the norm in an any ideal space L and replace the balls $B(0,r)$ by homothetic sets $U(0,r) \subset \mathbb{R}^n$. Below, we always assume that $0 \in U(0,1)$ and $\mu(U(0,1))) \in (0,\infty)$. Moreover, we often assume that $U(0,1)$ is star-shaped with respect to the point 0, that is, if $t \in U(0,1)$, then $\gamma t \in U(0,1)$ for all $\gamma \in (0,1)$. In general, the star-shapedness assumption is not necessary, but sometimes is useful.

We also need local Morrey spaces constructed from a family of sets $\{U(0,r_i)\}$ with discretely varying parameter.

We denote by Υ the set of non-negative number sequences $\tau = \{\tau_i\}$ each of which satisfies the conditions

$$\forall i: \qquad \tau_i < \tau_{i+1}, \qquad \bigcup_i (\tau_i,\tau_{i+1}] = R_+.$$

When $\tau_{i+1} = \infty$, we assume that $(\tau_i,\infty] = (\tau_i,\infty)$.

Definition 1. [2]. Let an ideal space X on $\mathbb{R}^n$, an ideal space l of two-sided sequences with the standard basis $\{e^i\}$ and a sequence $\tau \in \Upsilon$ be given. By Morrey space $M^{\tau}_{l,X}$ we mean the set of all functions $x \in L^{1,loc}(\mathbb{R}^n)$ for which the following norm is finite:

$$\|x|M^{\tau}_{l,X}\| = \sup_{t\in\mathbb{R}^n} \|\sum_{i=-\infty}^{\infty} e^i \|x(t+.)\chi(U(0,\tau_i))|X\||l\|.$$

The spaces introduced in Definition 1 are called global discrete Morrey spaces.

Discrete spaces are more convenient to consider at least for the following reasons. Firstly, all classical Morrey spaces can be realized as discrete Morrey spaces (see the example below), and secondly, one does not need to think about the measurability of the function $\|x(t+.)\chi(B(0,r))|X\|$.

Note that all discrete Morrey spaces are ideal.

The following example shows that most recently investigated Morrey spaces can be implemented as discrete Morrey spaces.

Example 1. Let $U(0,1)$ be a star-shaped set of a positive measure, $\lambda > 0$, $p \in [1,\infty]$, the ideal space X and the space $M_{\lambda,p;X}$, the norm in which is given by the equality

$$\|x|M_{\lambda,p;X}\| = \begin{cases} \sup_{t\in\mathbb{R}^n}\left(\int_0^\infty (r^{-\lambda}\|x(t+.)\chi(U(0,r))|X\|)^p \frac{dr}{r}\right)^{1/p}, & \text{for } p\in[1,\infty); \\ \sup_{t\in\mathbb{R}^n}\sup_r\{r^{-\lambda}\|x(t+.)\chi(U(0,r))|X\|\}, & \text{for } p=\infty \end{cases}$$

be given.

If $p \in [1,\infty)$, then for each function $x \in M_{\lambda,p;X}$ the following inequalities hold:

$$\sup_{t\in\mathbb{R}^n} 2^{-\lambda}(ln2)^{1/p}(\sum_i (2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\|)^p)^{1/p} \le \|x|M_{\lambda,p;X}\|$$

$$\le \sup_{t\in\mathbb{R}^n} 2^{\lambda}\cdot(ln2)^{1/p}(\sum_i (2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\|)^p)^{1/p}.$$

Thus, for $p \in [1,\infty)$ on the space $M_{\lambda,p;X}$ we can introduce an equivalent norm

$$\|x|M_{\lambda,p;X}\|_b = \sup_{t\in\mathbb{R}^n}(\sum_i (2^{-\lambda i}\|x(t+.)\chi(U(0,2^i))|X\|)^p)^{1/p}.$$

If $p=\infty$, then for each $x \in M_{\lambda,\infty;X}$ the following inequalities hold:

$$\sup_{t\in\mathbb{R}^n} 2^{-\lambda}\sup_i 2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\| \le \|x|M_{\lambda,\infty;X}\|$$

$$\le \sup_{t\in\mathbb{R}^n} 2^{\lambda}\sup_i 2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\|.$$

So on the space $M_{\lambda,\infty;X}$

$$\|x|M_{\lambda,\infty;X}\|_b = \sup_{t\in\mathbb{R}^n}\{\sup_i 2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\|\}$$

is an equivalent norm.

Put $\tau_i = 2^i$, $(i\in\mathbb{Z})$, for the sequence of points $\{\tau_i\}_{-\infty}^{\infty}$ consider the corresponding partition τ for R_+ and define a weight sequence by setting $\omega_\lambda(i) = 2^{-\lambda i}$, $(i\in\mathbb{Z})$. Then we get that for all $p\in[1,\infty]$ up to equivalence of the norms:

$$M^\tau_{l^p_{\omega_\lambda},X} = M_{\lambda,p;X}.$$

Let C_{cv} denote the set of all functions $\varphi : \mathbb{R}^2_+ \to \mathbb{R}_+$ concave, positively homogeneous of degree one, nondecreasing and continuous in each variable and such that $\varphi(0,0)=0$.

The class C_{cv} is a cone with respect to the operations of addition and multiplication by a non-negative number.

We recall the definition of the construction of Calderón – Lozanovskiĭ.

Definition 2. Let a couple of ideal spaces (X_0,X_1) on Ω and $\varphi\in C_{cv}$ be given. The space $\varphi(X_0,X_1)$ consists of all measurable functions x, for which there is a pair of functions $x_0\in X_0$, $x_1\in X_1$ such that almost everywhere holds the inequality

$$|x(t)| \le \varphi(x_0(t),x_1(t)).$$

On the space $\varphi(X_0,X_1)$ the norm is introduced by the equality

$$\begin{aligned}&\|x|\varphi(X_0,X_1)\| \\ &= \inf\{\lambda>0 : |x(t)|\le\lambda\varphi(x_0(t),x_1(t)) \ \ (\text{for a. e. } t\in\Omega), \\ &\quad x_i\in X_i, \|x_i|X_i\|\le 1; (i=0,1)\}. \end{aligned} \tag{2.1}$$

The space $\varphi(X_0,X_1)$ is an ideal Banach space equipped with this norm.

If $\varphi_\theta(t,s) = t^\theta \cdot s^{1-\theta}$ then the definition of the space $\varphi_\theta(X_0, X_1)$, which is usually denoted by $X_0^\theta \cdot X_1^{1-\theta}$, was proposed by A.P. Calderón [16]; for an arbitrary $\varphi \in C_{cv}$ the space $\varphi(X_0, X_1)$ was defined by G.Ya. Lozanovskiĭ [26].

The equality proposed below is well known

$$\|x|\varphi_\theta(X_0, X_1)\| = \inf\{\|x_0|X_0\|^\theta \cdot \|x_1|X_1\|^{1-\theta} : |x(t)| \leq x_0^\theta(t) \cdot x_1^{1-\theta}(t) \quad \text{a.e. on } \Omega\}.$$

The Calderón – Lozanovskiĭ construction of $\varphi(X_0, X_1)$ has found many applications in the theory of ideal spaces [27], in the theory of interpolation of linear operators [10], [19], [30], in the geometric theory of Banach spaces [8].

In cases in which exact estimates of constants are important, we can introduce on the space $\varphi(X_0, X_1)$ norms different from (2.1) as follows. Let $\psi(a_1, a_2) : \mathbb{R}^2 \to \mathbb{R}_+$ be a norm on $\mathbb{R}^2$. Then on $\varphi(X_0, X_1)$ the norm is defined by the equality

$$\|x|\{\varphi(X_0, X_1), \psi\}\| = \inf\{\psi(a_1, a_2) :$$

$$|x(t)| \leq \varphi(x_0(t), x_1(t)), \quad \text{a.e. on } \Omega, \quad x_i \in X_i, \|x_i|X_i\| = a_i; (i = 0, 1)\}. \tag{2.2}$$

The space $\varphi(X_0, X_1)$ is an ideal Banach space equipped with the norm $\| \,.\, |\{\varphi(X_0, X_1), \psi\}\|$.

Of course all the norms on $\varphi(X_0, X_1)$, defined by equation (2.2), are equivalent. If we put $\psi_\infty(a_1, a_2) = max\{|a_1|, |a_2|\}$, then the norm on the space $\{\varphi(X_0, X_1), \psi_\infty\}$ coincide with the norm defined in (2.1). For example (see [4]), using the introduced norms one can to define the exact dual space $\{\varphi(X_0, X_1), \psi\}'$ and exact dual norm on the space $\{\varphi(X_0, X_1), \psi\}$.

For each $\varphi \in C_{cv}$ for all $a, b, c, d > 0$ the following inequality holds

$$\varphi(a+b, c+d) = (c+d)\varphi(\frac{a+b}{c+d}, 1) = (c+d)\varphi(\frac{a}{c}\frac{c}{c+d} + \frac{b}{d}\frac{d}{c+d}, 1) \geq$$

$$(c+d)\{\frac{c}{c+d}\varphi(\frac{a}{c}, 1) + \frac{d}{c+d}\varphi(\frac{b}{d}, 1)\} = \varphi(a, c) + \varphi(b, d). \tag{2.3}$$

Now we will show that condition (1.3) is equivalent to the fact that the corresponding Morrey spaces are obtained using Calderón's constructions $\varphi_{\theta_0}(.,.)$, $\varphi_{\theta_1}(.,.)$ for one special pair of spaces.

Lemma 2.1. *Let the space $M^\tau_{l,X}$ be constructed from the spaces X, l, the sequence $\tau \in \Upsilon$ and the set $U(0,1)$. Let $\theta \in (0,1)$. Then*

$$(M^\tau_{l,X})^\theta (L^\infty)^{1-\theta} = M^\tau_{l^\theta, X^\theta}$$

and the norms on these spaces coincide.

Proof. Let $x \in (M^\tau_{l,X})^\theta (L^\infty)^{1-\theta}$. This means that there exists $x_0 \in M^\tau_{l,X}$ with $\|x_0|M^\tau_{l,X}\| = 1$ such that the equality $|x(t)| = \lambda x_0^\theta(t) \cdot 1^{1-\theta}$, $(t \in R^n)$ holds and $\lambda = \|x|(M^\tau_{l,X})^\theta\|$. Then the following relations follow

$$\||\frac{x}{\lambda}|^{1/\theta}|(M^\tau_{l,X})^\theta (L^\infty)^{1-\theta}\| = 1 \quad \Leftrightarrow \quad \sup_t \|\Sigma_{-\infty}^\infty \|x^{1/\theta}(t+.)\chi(U(0,r_i))|X\|e^i|l\|^\theta = \lambda$$

$$\Leftrightarrow \quad \sup_t \|\Sigma_{-\infty}^\infty \left((\|x(t+.)\chi(U(0,r_i))|X\|)^\theta\right)^{1/\theta} e^i|l\|^\theta = \lambda \Leftrightarrow \quad \|x|M^\tau_{l^\theta, X^\theta}\| = \lambda.$$

Let us prove the reverse inequality. Let $x \in M^\tau_{l^\theta, X^\theta}$, $x \geq 0$ and $\|x|M^\tau_{l^\theta, X^\theta}\| = 1$. This means that the equality

$$\sup_t \|\Sigma_{-\infty}^\infty \left((\|x^{1/\theta}(t+.)\chi(B(0,r_i))|X\|)^\theta\right)^{1/\theta} e^i|l\|^\theta = 1.$$

Put $x_0(t) = x^{1/\theta}(t)$. Then obvious equality $x(t) = x_0^{\theta}(t) \cdot (1)^{1-\theta}$, $t \in \Omega$ holds. Let us check that the equality $\|x_0|M^{\tau}_{l,X}\| = 1$ holds. Indeed,

$$\|x_0|M^{\tau}_{l,X}\| = \sup_t \|\Sigma^{\infty}_{-\infty}\|x_0(t+.)\chi(U(0,r_i))|X\|e^i|l\|$$

$$= \sup_t \|\Sigma^{\infty}_{-\infty}\|x^{1/\theta}(t+.)\chi(U(0,r_i))|X\|e^i|l\|$$

$$= \sup_t \|\Sigma^{\infty}_{-\infty}\ ((\|x^{1/\theta}(t+.)\chi(U(0,r_i))|X\|)^{\theta})\ ^{1/\theta}e^i|l\| = 1.$$

□

Corollary 2.1. *Let $0 < \lambda < \frac{n}{p}$ and $\theta \in (0,1)$ be given. We define the numbers γ and q by the equalities*

$$\nu = \theta\lambda, \quad q = \frac{p}{\theta}.$$

Then the space $(M^{\tau}_{\lambda,L^p})^{\theta}(L^{\infty})^{1-\theta}$ and the space M^{τ}_{γ,L^q} coincide and the norms in these spaces are equal.

Proof. If the inequalities $|x(t)| \le \gamma|x_0(t)|^{\theta}$ and $\|x_0|M^{\tau}_{\lambda,L^p}\| \le 1$ are satisfied for all t, then the following relations are valid

$$\sup_t\{\sup_{r>0} r^{-\lambda}\|\chi(U(0,r))(\frac{|x(t+.)|}{\gamma})^{1/\theta}|L^p\|\} \le 1$$

$$\Leftrightarrow \sup_t\{\sup_{r>0} r^{-\lambda\theta}\|\chi(U(0,r))|x(t+.)|^{1/\theta}|L^p\|^{\theta}\} \le \gamma$$

$$\Leftrightarrow \sup_t\{\sup_{r>0} r^{-\nu}\||\chi(U(0,r))x(t+.)||L^q\|\} \le \gamma.$$

□

Corollary 2.2. *Let a couple of Morrey spaces $(M_{\lambda_0,L^{p_0}}, M_{\lambda_1,L^{p_1}})$ be given. Condition (1.3) is satisfied if and only if there are numbers λ, p and $\theta_0, \theta_1 \in (0,1)$ for which the following equalities are satisfied*

$$M_{\lambda_0,L^{p_0}} = (M_{\lambda,L^1})^{\theta_0}(L^{\infty})^{1-\theta_0}, \quad M_{\lambda_1,L^{p_1}} = (M_{\lambda,L^1})^{\theta_1}(L^{\infty})^{1-\theta_1}$$

Proof. Define the parameters θ_0, θ_1 by the equalities $\theta_0 = \frac{1}{p_0}, \theta_1 = \frac{1}{p_1}$, $\lambda = \frac{\lambda_0}{\theta_0} \equiv \frac{\lambda_1}{\theta_1}$. Then the following equalities are valid

$$M_{\lambda_0,L^{p_0}} = (M_{\lambda,L^1})^{\theta_0}(L^{\infty})^{1-\theta_0}, \quad M_{\lambda_1,L^{p_1}} = (M_{\lambda,L^1})^{\theta_1}(L^{\infty})^{1-\theta_1}$$

and it suffices to apply Corollary 2.1. □

Let us note the connection between the Calderón – Lozanovskiĭ construction and the generalized Orlicz - Morrey space.

First we recall the definition of Young functions. A function $N : \mathbb{R}_+ \to \mathbb{R}_+$ is called a Young function if N is convex, left-continuous, strictly increases and $\lim_{t\to 0} N(t) = N(0) = 0$, $\lim_{t\to\infty} N(t) = \infty$.

Let a Young function N be given, by which the Orlicz space $L^N(\mathbb{R}^n)$ is constructed. A natural generalization of the Lebesgue-Morrey space is the Orlicz-Morrey space, the norm in which is given by the equality

$$\|x|M^{\tau}_{l,L^N}\| = \sup_t\{\|\sum e^i\|\chi(U(0,r_i))x(t+.)|L^N\||l\|\}$$

$$= \sup_t \{\| \sum e^i \{\inf\{\lambda_i > 0 : \|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda_i})|L^1\| \leq 1\}\}\||l\|.$$

If we put $l = l^\infty_\omega$, then the formula for the norm in the space $M^\tau_{l^\infty_\omega, L^N}$ has the form

$$\|x|M^\tau_{l^\infty_\omega, L^N}\| = \sup_t \{\| \sum e^i \|\chi(U(0,r_i))x(t+.)|L^N\|\||l\|\}$$

$$= \sup_t \{\sup_i \omega(i)\{\inf\{\lambda_i > 0 : \|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda_i})|L^1\| \leq 1\}\}. \tag{2.4}$$

Let $\theta \in (0,1)$, $p = \frac{1}{\theta}$, $N_\theta(t) = t^{\frac{1}{\theta}}$, $(t \in [0,\infty))$. Then from (2.4) follows the equality

$$M^\tau_{l^\infty_\omega, L^{N_\theta}} = M^\tau_{l^\infty_\omega, L^p}$$

and the norms in these spaces coincide.

Another natural generalization of the Lebesgue-Morrey space is the Orlicz-Morrey space $\varphi_N(M^\tau_{l,L^1}, L^\infty)$, constructed by the Calderón – Lozanovskiĭ construction.

Let a Young function N be given. We define the function $\varphi_N(.,1)$ by the equality $\varphi_N(s,1) = N^{-1}(s)$, $(s \in [0,\infty))$, and put $\varphi_N(s,t) = t\varphi_N(s/t,1)$. Then $\varphi_N \in C_{cv}$.

Lemma 2.2. *Let N be a Young function, and the function $\varphi_N \in C_{cv}$ is constructed. Then the following equality is true*

$$\|x|\varphi_N(M^\tau_{l,L^1}, L^\infty)\| = \sup_t \{\inf\{\lambda > 0 : \| \sum e^i \|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda})|L^1\|\||l\| \leq 1\}\}.$$

Proof. If $|x(t)| \leq \gamma\varphi_N(x_0(t),1)$ and $\|x_0|M^\tau_{l,L^1}\| = 1$, then

$$|x(t)| \leq \gamma\varphi_N(x_0(t),1) \Leftrightarrow \frac{|x(t)|}{\gamma} \leq \varphi_N(x_0(t),1) \Leftrightarrow \|N(\frac{|x(t)|}{\gamma})|M^\tau_{l,L^1}\| = \|x_0|M^\tau_{l,L^1}\|$$

$$\Leftrightarrow \sup_t \{\| \sum e^i \|\chi(U(0,2^i))N(\frac{|x(t+.)|}{\gamma})|L^1\|\||l\|\} = \|x_0|M^\tau_{l,L^1}\|$$

$$\Rightarrow \sup_t \{\inf\{\lambda > 0 : \| \sum e^i \|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda})|L^1\|\||l\| \leq 1\}\} \leq \gamma.$$

From here it follows that

$$\|x|\varphi_N(M^\tau_{l,L^1}, L^\infty)\| \geq \sup_t \{\inf\{\lambda > 0 : \| \sum e^i \|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda})|L^1\|\||l\| \leq 1\}\}.$$

On the other hand, if

$$\sup_t \{\inf\{\lambda > 0 : \| \sum e^i \|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda})|L^1\|\||l\| \leq 1\}\} < 1,$$

then

$$\sup_t \{\| \sum e^i \|\chi(U(0,r_i))N(\frac{|x(t+.)|}{1})|L^1\|\||l\|\} \leq 1.$$

Therefore $N(|x(.)|) \in M^\tau_{l,L^1}$, $\|N(|x(.)|)|M^\tau_{l,L^1}\| \leq 1$ and $|x(t)| \equiv \varphi_N(N(|x(t)|),1)$. From here it follows that

$$\|x|\varphi_N(M^\tau_{l,L^1}, L^\infty)\| \leq 1.$$

□

If we put $l = l^\infty_\omega$, then the formula for the norm in the space $\varphi_N(M^\tau_{l^\infty_\omega,L^1}, L^\infty)$ has the form

$$\|x|\varphi_N(M^\tau_{l^\infty_\omega,L^1}, L^\infty)\| = \sup_t\{\inf\{\lambda > 0 : \sup_i\{\omega(i)\|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda})|L^1\| \leq 1\}\}\}$$

$$= \sup_t\{\sup_i\{\inf\{\lambda_i > 0 : \|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda_i})|L^1\| \leq \frac{1}{\omega_i}\}\}\}. \quad (2.5)$$

Let $\theta \in (0,1)$. We define a Young function by the equality $N_\theta(t) = t^{\frac{1}{\theta}}$, $(t \in [0,\infty))$ and put $p = \frac{1}{\theta}$, $\omega_\theta(i) = (\omega(i))^{\frac{1}{\theta}}$, $(i \in Z)$.

Then it follows from (2.5) that

$$\varphi_{N_\theta}(M^\tau_{l^\infty_\omega,L^1}, L^\infty) = M^\tau_{l^\infty_{\omega_\theta},L^p}$$

and the norms in these spaces coincide.

Note that from (2.5) it turns out that the space $\varphi_n(M^\tau_{l^\infty_\omega,L^1}, L^\infty)$ coincides with the Orlicz-Morrey space introduced by E. Nakai [29]. It is for these spaces that interpolation theorems are formulated below.

The following theorem is a basis for obtaining interpolation theorems for global Morrey spaces.

Theorem 2.1. *Let X_0, X_1 be two ideal spaces, φ, φ_0, $\varphi_1 \in C_{cv}$ and*

$$\overline{\varphi}(t,s) = \varphi(\varphi_0(t,s), \varphi_1(t,s)), \quad t,s \in \mathbb{R}_+. \quad (2.6)$$

Then $\overline{\varphi} \in C_{cv}$, the following equality is true

$$\overline{\varphi}(X_0,X_1) = \varphi(\varphi_0(X_0,X_1), \varphi_1(X_0,X_1)),$$

and for each $x \in \overline{\varphi}(X_0,X_1)$ the inequalities

$$\|x|\varphi(\varphi_0(X_0,X_1), \varphi_1(X_0,X_1))\|$$

$$\leq \|x|\overline{\varphi}(X_0,X_1)\| \leq 2\|x|\varphi(\varphi_0(X_0,X_1), \varphi_1(X_0,X_1))\| \quad (2.7)$$

hold.

Proof. Let us prove first that $\overline{\varphi} \in C_{cv}$.

The positive homogeneity of first degree of the function $\overline{\varphi}$ is obvious. Let us check the concavity. Indeed, using inequality (2.3), we obtain

$$\overline{\varphi}(\frac{t_0+t_1}{2},1) = \varphi(\varphi_0(\frac{t_0+t_1}{2},1), \varphi_1(\frac{t_0+t_1}{2},1))$$

$$\geq \varphi(\frac{1}{2}(\varphi_0(t_0,1)+\varphi_0(t_1,1)), \frac{1}{2}(\varphi_1(t_0,1)+\varphi_1(t_1,1)))$$

$$= \frac{1}{2}\varphi(\varphi_0(t_0,1)+\varphi_0(t_1,1), \varphi_1(t_0,1)+\varphi_1(t_1,1))$$

$$\geq \frac{1}{2}\{\varphi(\varphi_0(t_0,1),\varphi_1(t_0,1)) + \varphi(\varphi_0(t_1,1),\varphi_1(t_1,1))\} = \frac{1}{2}\{\overline{\varphi}(t_0,1)+\overline{\varphi}(t_1,1)\}.$$

Let us prove the left inequality in (2.7).

Let $x \in \overline{\varphi}(X_0,X_1)$ and $\|x|\overline{\varphi}(X_0,X_1)\| < 1$. Then there are $x_0 \in X_0$, $x_1 \in X_1$ such that

$$|x(t)| \leq \overline{\varphi}(x_0(t),x_1(t)), \quad (t \in \Omega); \quad \|x_0|X_0\| < 1; \quad \|x_1|X_1\| < 1.$$

Let us define new functions z_0, z_1 by the equalities:

$$z_0(t) = \varphi_0(x_0(t), x_1(t)), \quad z_1(t) = \varphi_1(x_0(t), x_1(t)); \quad (t \in \Omega).$$

Then

$$z_0 \in \varphi_0(X_0, X_1), \quad \|z_0|\varphi_0(X_0, X_1)\| < 1, \quad z_1 \in \varphi_1(X_0, X_1), \quad \|z_1|\varphi_1(X_0, X_1)\| < 1$$

and

$$\varphi(z_0(t), z_1(t)) = \overline{\varphi}(\varphi_0(x_0(t), x_1(t)), \varphi_1(x_0(t), x_1(t))), \quad (t \in \Omega).$$

Therefore

$$x \in \varphi(\varphi_0(X_0, X_1), \varphi_1(X_0, X_1)), \quad \|x|\varphi(\varphi_0(X_0, X_1), \varphi_1(X_0, X_1))\| < 1.$$

These relations prove the left inequality in (2.7).

Let us prove the right inequality in (2.7).

Let

$$x \in \varphi(\varphi_0(X_0, X_1), \varphi_1(X_0, X_1)), \quad \|x|\varphi(\varphi_0(X_0, X_1), \varphi_1(X_0, X_1))\| < 1.$$

Then there are $x_0, x_1 \in X_0$, $y_0, y_1 \in X_1$ such that

$$\|x_0|X_0\| < 1; \quad \|x_1|X_0\| < 1; \quad \|y_0|X_1\| < 1, \quad \|y_1|X_1\| < 1$$

and

$$|x(t)| \leq \varphi(\varphi_0(x_0(t), y_0(t)), \varphi_1(x_1(t), y_1(t))), \quad (t \in \Omega).$$

Let us define new functions by the equalities: $z_0(t) = \max\{x_0(t), x_1(t)\}$, $z_1(t) = \max\{y_0(t), y_1(t)\}$.

Then

$$\varphi_0(x_0(t), y_0(t)) \leq \varphi_0(z_0(t), z_1(t)), \quad (t \in \Omega); \quad \varphi_1(x_1(t), y_1(t)) \leq \varphi_1(z_0(t), z_1(t)), \quad (t \in \Omega);$$

$$\|z_0|X_0\| < 2; \quad \|z_1|X_1\| < 2.$$

For all $t \in \Omega$ holds the inequality

$$|x(t)| \leq \varphi(\varphi_0(z_0(t), z_1(t)), \varphi_1(z_0(t), z_1(t))) = \overline{\varphi}(z_0(t), z_1(t)), \quad (t \in \Omega).$$

Therefore $\|x|\overline{\varphi}(X_0, X_1)\| < 2$. These relations prove the right inequality in (2.7). □

Corollary 2.3. *Let a couple ideal space X_i on $\mathbb{R}^n$, a couple ideal space of sequence l_i, $(i = 0, 1)$, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0, \infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $M^{\tau}_{l_i, X_i}$ be constructed from the spaces X_i, l_i, $(i = 0, 1)$, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$.*

Let φ, φ_0, $\varphi_1 \in C_{cv}$ be fixed, and the function $\overline{\varphi} \in C_{cv}$ is constructed by equality (2.6).

Then

$$\overline{\varphi}(M^{\tau}_{l_0, X_0}, M^{\tau}_{l_1, X_1}) = \varphi(\varphi_0(M^{\tau}_{l_0, X_0}, M^{\tau}_{l_1, X_1}), \varphi_1(M^{\tau}_{l_0, X_0}, M^{\tau}_{l_1, X_1}))$$

and for all $x \in \overline{\varphi}(M^{\tau}_{l_0, X_0}, M^{\tau}_{l_1, X_1})$ the following inequalities are valid

$$\|x|\varphi(\varphi_0(M^{\tau}_{l_0, X_0}, M^{\tau}_{l_1, X_1}), \varphi_1(M^{\tau}_{l_0, X_0}, M^{\tau}_{l_1, X_1}))\| \leq \|x|\overline{\varphi}(M^{\tau}_{l_0, X_0}, M^{\tau}_{l_1, X_1})\|$$

$$\leq 2\, \|x|\varphi(\varphi_0(M^{\tau}_{l_0, X_0}, M^{\tau}_{l_1, X_1}), \varphi_1(M^{\tau}_{l_0, X_0}, M^{\tau}_{l_1, X_1}))\|.$$

Corollary 2.4. *Let $0 \le \theta_0, \theta_1 \le 1$, $\varphi \in C_{cv}$ be fixed, and the function $\varphi_{\theta_0,\theta_1} \in C_{cv}$ is constructed by the equality*

$$\varphi_{\theta_0,\theta_1}(t,s) = \varphi(t^{\theta_0}s^{1-\theta_0}, t^{\theta_1}s^{1-\theta_1}). \tag{2.8}$$

Then

$$\varphi_{\theta_0,\theta_1}(M^{\tau}_{l,X}, L^{\infty}) = \varphi((M^{\tau}_{l,X})^{\theta_0}(L^{\infty})^{1-\theta_0}, (M^{\tau}_{l,X})^{\theta_1}(L^{\infty})^{1-\theta_1})$$

and the following inequalities are valid

$$\|x|\varphi((M^{\tau}_{l,X})^{\theta_0}(L^{\infty})^{1-\theta_0}, (M^{\tau}_{l,X})^{\theta_1}(L^{\infty})^{1-\theta_1})\| \le \|x|\varphi_{\theta_0,\theta_1}(M^{\tau}_{l,X}, L^{\infty})\|$$

$$\le 2\, \|x|\varphi((M^{\tau}_{l,X})^{\theta_0}(L^{\infty})^{1-\theta_0}, (M^{\tau}_{l,X})^{\theta_1}(L^{\infty})^{1-\theta_1})\|.$$

Corollary 2.5. *Let $0 < \lambda_0 < \frac{n}{p_0}$, $0 < \lambda_1 < \frac{n}{p_1}$, $\theta \in (0,1)$ and $\varphi \in C_{cv}$ be given and condition (1.3) be satisfied. Let $\theta_0 = \frac{1}{p_0}$, $\theta_1 = \frac{1}{p_1}$, $\lambda = \frac{\lambda_0}{\theta_0}$, and the function $\varphi_{\theta_0,\theta_1}$ is defined by (2.8).*

Then

$$\varphi_{\theta_0,\theta_1}(M_{\lambda,L^1}, L^{\infty}) = \varphi(M_{\lambda_0,L^{p_0}}, M_{\lambda_1,L^{p_1}})$$

and the following inequalities are valid

$$\|x|\varphi(M_{\lambda_0,L^{p_0}}, M_{\lambda_1,L^{p_1}})\| \le \|x|\varphi_{\theta_0,\theta_1}(M_{\lambda,L^1}, L^{\infty})\| \le 2\, \|x|\varphi(M_{\lambda_0,L^{p_0}}, M_{\lambda_1,L^{p_1}})\|.$$

To obtain interpolation theorems, we need one geometric property of an ideal space.

Definition 3. Say (see, for example, [20], [24]) that an ideal space $X \subset S(\mu,\Omega)$ has the Fatou property if from $0 \le x_n \uparrow x$; $x_n \in X$ and $\sup_n \|x_n|X\| < \infty$ it follows that $x \in X$ and $\|x|X\| = \sup_n \|x_n|X\|$.

It is well known that the Lebesgue spaces L^p_{ω}, (l^p_{ω}) for $p \in [1,\infty]$ have the Fatou property, and the space c^0 has not the Fatou property.

The following theorem is not a very general fact for the Calderón – Lozanovskiĭ construction on a couple of ideal spaces. The question of when the space $\varphi(X_0, X_1)$ has the Fatou property depends on the properties of the couple of ideal spaces (X_0, X_1) and the function φ is discussed in more detail in [5].

Theorem 2.2. *[5]. Let $\varphi \in C_{cv}$ and an interpolation couple of ideal spaces (X_0, X_1) be given. If X_0 and X_1 have the Fatou property, then the space $\varphi(X_0, X_1)$ has the Fatou property too.*

The next theorem shows that if parameters of the global Morrey space have the Fatou property, then the global Morrey space also has the Fatou property.

Theorem 2.3. *[7]. Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the space $M^{\tau}_{l,X}$ be constructed from the spaces X, l, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$.*

If both ideal spaces l and X have the Fatou property, then the space $M^{\tau}_{l,X}$ has the Fatou property too.

We apply Theorems 2.1 - 2.3 to obtain interpolation theorems. Namely, we write out conditions for the coincidence of the Calderón – Lozanovskiĭ construction on a couple of Morrey spaces with the value of the Gustavsson – Peetre – Ovchinnikov interpolation functor on a couple of Morrey spaces.

We recall [10] that a couple of normed spaces (A_0, A_1) is referred to as an interpolation couple if both spaces are embedded in a separable topological linear space V.

Let $\varphi \in C_{cv}$ and a interpolation couple (A_0, A_1) be given. Denote by $(A_0, A_1)_\varphi$ the Gustavsson – Peetre – Ovchinnikov interpolation functor [10], [19], [30] calculated for the couple (A_0, A_1):

$$a \in (A_0, A_1)_\varphi \Leftrightarrow a = \sum_{-\infty}^{\infty} a_i; a_i \in A_0 \bigcap A_1, \text{the series converges in the space} A_0 + A_1;$$

$$\|a|(A_0, A_1)_\varphi\|$$

$$= \inf\{\max\{\sup_n\{\sup_{\varepsilon_i=\pm 1} \|\sum_{-n}^{n} \varepsilon_i \frac{a_i}{\varphi(1, 2^i)}|A_0\|, \sup_{\varepsilon_i=\pm 1} \|\sum_{-n}^{n} \varepsilon_i \frac{a_i}{\varphi(2^i, 1)}|A_1\|\}\} : a = \sum_{-\infty}^{\infty} a_i\} < \infty.$$

Theorem 2.4. *[10], [19], [30]. Let $\varphi \in C_{cv}$ and an interpolation couple of ideal spaces (X_0, X_1) on Ω be given. If X_0 and X_1 have the Fatou property, then*

$$\{\varphi(X_0, X_1), \psi\} = (X_0, X_1)_\varphi,$$

the norms in these spaces are equivalent, and the equivalence constant does not depend on X_0, X_1 and the function φ.

Thus, the triple of spaces $\{X_0,\ X_1;\ \ \varphi(X_0, X_1)\}$ is an interpolation triple.
From Theorems 2.1 – 2.4 we obtain the following interpolation theorem.

Theorem 2.5. *Let a couple of ideal spaces X_i on $\mathbb{R}^n$, a couple of ideal spaces of sequence l_i, $(i = 0, 1)$, a set $U(0, 1) \subset \mathbb{R}^n$, for which $0 \in U(0, 1)$ and $\mu(U(0, 1)) \in (0, \infty)$, and a sequence $\tau \in \Upsilon$ be given. Let all spaces X_0, X_1, l_0, l_1 have the Fatou property. Let the spaces $M^\tau_{l_i,X_i}$ be constructed from the spaces X_i, l_i, $(i = 0, 1)$, the set $U(0, 1)$ and the sequence $\tau \in \Upsilon$. Let $\varphi, \varphi_0, \varphi_1 \in C_{cv}$ be given. Define the function $\overline{\varphi}$ by equality (2.6). We form the triple of spaces*

$$\{\varphi_0(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}),\quad \varphi_1(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1});\quad \overline{\varphi}(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})\}.$$

Let an interpolation couple (A_0, A_1) be given.
1) *If a linear operator S is bounded as an operator*

$$S : A_i \to \varphi_i(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}),\quad (i = 0, 1),$$

then

$$S : (A_0, A_1)_\varphi \to \overline{\varphi}(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})$$

and is bounded.
2) *If a linear operator P is bounded as an operator*

$$P : \varphi_i(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}) \to A_i,\quad (i = 0, 1),$$

then

$$P : \overline{\varphi}(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}) \to (A_0, A_1)_\varphi$$

and is bounded.

Proof. From Theorems 2.2 and 2.3 it follows that all spaces

$$M^\tau_{l_0,X_0},\ M^\tau_{l_1,X_1};\quad \varphi_0(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}),\quad \varphi_1(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1});$$

$$\varphi(\varphi_0(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}), \varphi_1(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}));\quad \overline{\varphi}(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})$$

have the Fatou property. Therefore, it follows from Theorem 2.4, that

$$\varphi(\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}))$$
$$= (\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}))_{\varphi} \tag{2.9}$$

and the norms in these spaces are equivalent.

From Theorem 2.1 it follows that

$$\overline{\varphi}(M^{\tau}_{l_0,X_)}, M^{\tau}_{l_1,X_1}) = \varphi(\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1})) \tag{2.10}$$

and the norms in these spaces are equivalent.

From (2.9) – (2.10) it follows that

$$\overline{\varphi}(M^{\tau}_{l_0,X_)}, M^{\tau}_{l_1,X_1}) = (\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}))_{\varphi}$$

and the norms in these spaces are equivalent.

From the latter relation we obtain statements 1) and 2). □

Corollary 2.6. *(An interpolation theorem for classical Morrey spaces.)*

Let $0 < \lambda_0 < \frac{n}{p_0}$, $0 < \lambda_1 < \frac{n}{p_1}$, $\theta \in (0,1)$ *and* $\varphi \in C_{cv}$ *be given and condition (1.3) be satisfied. Let* $\theta_0 = \frac{1}{p_0}$, $\theta_1 = \frac{1}{p_1}$, $\lambda = \frac{\lambda_0}{\theta_0}$, *and the function* $\varphi_{\theta_0,\theta_1}$ *is defined by (2.8).*

Then statements 1) and 2) in Theorem 2.5 hold for the triple of spaces

$$\{M_{\lambda_0,L^{p_0}}, \quad M_{\lambda_1,L^{p_1}}; \quad \varphi_{\theta_0,\theta_1}(M_{\lambda,L^1}, L^{\infty})\}.$$

Corollary 2.7. *(An interpolation theorem for generalized Orlicz – Morrey spaces.)*

Let two Young functions N_0, N_1 *be given, and the functions* $\varphi_{N_i}(s,1) = N_i^{-1}(s)$ *and* $\varphi_{N_i}(s,t) = t\varphi_{N_i}(s/t,1)$ $(i = 0,1)$ *are constructed.*

Let $\varphi \in C_{cv}$ *be fixed, and the function* $\varphi_{N_0,N_1} \in C_{cv}$ *be defined by the formula*

$$\varphi_{N_0,N_1}(t,s) = \varphi(tN_0^{-1}(\frac{s}{t}),\ tN_1^{-1}(\frac{s}{t})); \quad t,s > 0.$$

Then statements 1) and 2) in Theorem 2.5 hold for the triple of spaces

$$\{\varphi_{N_0}(M^{\tau}_{l^{\infty}_{\omega},L^1}, L^{\infty}),\ \varphi_{N_1}(M^{\tau}_{l^{\infty}_{\omega},L^1}, L^{\infty}); \varphi_{N_0,N_1}(M^{\tau}_{l^{\infty}_{\omega},L^1}, L^{\infty})\}.$$

Remark 1. In the article we considered the Morrey space defined on $\mathbb{R}^n$. If we consider the Morrey space defined on a subset $\Omega \subset \mathbb{R}^n$, $(0 \in \Omega)$ then in Definitions 1 it is necessary to replace $U(0,\tau)$ by $U(0,\tau) \cap \Omega$. All results will remain true.

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References

[1] D.R. Adams, J. Xiao, *Morrey spaces in harmonic analysis.* Arkiv for matematik, 50 (2012), no. 2, 201-230.

[2] E.I. Berezhnoi, *A discrete varsion of local Morrey spaces.* Izvestiya RAN: Ser. Mat. 81 (2017), no. 1, 3–31 (in Russian). English transl.: Izvestiya: Mathematics, 81 (2017), no. 1, 1-28.

[3] E.I. Berezhnoi, *Calculation of the Calderón – Lozanovskiĭ construction for a couple of local Morrey spaces.* Eurasian Math. J., 11 (2020), no. 3, 21-34.

[4] E.I. Berezhnoi, *On a theorem of G.Ya. Lozanovskiĭ.* Izvestiya VUSov. Ser. Mathematics, 1982, no. 2, 81-83 (in Russian). English transl.: Soviet Math. (Iz. VUZ), 26 (1982), no. 2, 107–111.

[5] E.I. Berezhnoi, *Geometric properties of the space $\varphi(X,Y)$.* Func. anal. i ego pril., 18 (1984), no. 1, 59-60 (in Russian). English transl.: Functional Analysis and its Applications, 18 (1984), no. 1, 48-50. https://doi.org/10.1007/BF01076362

[6] E.I. Berezhnoi, *On interpolation and k-monotonicity for discrete local Morrey spaces.* St. Petersburg Math. J., 33(2021), no. 1, 427–447.

[7] E.I. Berezhnoi, *On reflexivity and other geometric properties of Morrey spaces.* Operator Theory and Harmonic Analysis. Springer Proceedings in Mathematics and Statistics, 357 (2021), 119 - 136.

[8] E.I. Berezhnoi, L. Maligranda, *Representation of Banach ideal spaces and factorization of operators.* Canad. J. Math., 57 (2005), no. 5, 897–940.

[9] O. Blasco, A. Ruiz, L. Vega, *Non interpolation in Morrey-Companato and block spaces.* Ann. Scuola Norm. Super Pisa, 28 (1999), no. 1, 31-40.

[10] Yu.A. Brudnyĭ , N.Ya. Krugljak, *Interpolation functors and interpolation spaces.* North-Holland, Amsterdam, 1991, xvi+718 pp.

[11] V.I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. I.* Eurasian Math. J., 3 (2012), no. 3, 11-32.

[12] V.I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. II.* Eurasian Math. J., 4 (2013), no. 1, 21-45.

[13] V.I. Burenkov, E.D. Nursultanov, *Description of interpolation spaces for local Morrey-type spaces.* Trudy MI RAN im. V.A. Steklova, 269 (2010), 46-56 (in Russian). English transl.: Proceedings Steklov Inst. Math. 269 (2010), 52-621, 46-53.

[14] V.I. Burenkov, D.K. Darbaeva, E.D. Nursultanov, *Description of interpolation spaces for general local Morrey-type spaces.* Eurasian Math. J., 4 (2013), no. 1, 46-53.

[15] V.I. Burenkov, E.D. Nursultanov, D.K. Chigambaeva, *Description of interpolation spaces for a couple of local Morrey-type spaces and their generalizations.* Trudy MI RAN im. V.A. Steklova, 284 (2014), 105-137 (in Russian). English transl.: Proceedings Steklov Inst. Math. 284 (2014), 97-128.

[16] A.P. Calderón, *Intermediate spaces and interpolation, the complex method.* Studia Math., 24 (1964), no. 2, 113-190.

[17] S. Campanato, M.K.V. Murthy, *Una generalizzazione del teorema di Riesz-Thorin.* Ann. Scuola Norm. Sup. Pisa, (3) 19 (1965), 87-100.

[18] D.I. Hakim, Y. Sawano, *Interpolation of generalised Morrey spaces.* Rev. Math. Complut., 29 (2015), no. 2, 295-340.

[19] J. Gustavsson, J. Peetre, *Interpolation of Orlicz spaces.* Studia Math., 60 (1977), no. 1, 33 – 59.

[20] L.V. Kantorovich, G.P. Akilov, *Functional analysis.* Nauka, Moscow, 1977 (Russian). English transl. Pergamon Press, Oxford-Elmsford, New York, 1982, xiv+589 pp.

[21] P.G. Lemarie-Rieusset, *Multipliers and Morrey spaces.* Potential Anal., 38 (2013), no. 3, 741-752.

[22] P.G. Lemarie-Rieusset, *Erratum to: Multipliers and Morrey spaces.* Potential Anal., 41 (2014), no. 4, 1359-1362.

[23] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces, I. Sequence spaces.* Springer-Verlag, Berlin– Heidelberg– New-York, 1977.

[24] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces, II. Function spaces.* Springer-Verlag, Berlin– Heidelberg– New-York, 1979.

[25] Y. Lu, D. Yang, W. Yuan, *Interpolation of Morrey spaces on metric measure spaces.* Canad. Math. Bull., 57 (2014), no. 3, 598 – 608.

[26] G.Ya. Lozanovskiĭ, *On some Banach lattices IV.* Siberian Math. J., 14 (1973), 97–108.

[27] L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Math. 5, Univ. of Campinas, Campinas, 1989, iii + 206 pp.

[28] C.B. Morrey, *On the solution of quasi-linear elliptic partial differential equations.* Trans. Amer. Math. Soc., 43 (1938), 126-166.

[29] E. Nakai, *Orlicz– Morrey spaces and the Hardy – Littlewood maximal function.* Studia Math., 188 (2008), n. 3, 193-224.

[30] V.I. Ovchinnikov, *The methods of orbits in interpolation theory.* Math. Reports, 1 (1984), Part 2, 349–516.

[31] J. Peetre, *On the theory of $L_{p\lambda}$ spaces.* J. Funct. Anal., 4 (1969), 71-87.

[32] A. Ruiz, L. Vega, *Corrigendato "Unique continuation for Schrödinger operators" and remark on interpolation in Morrey spaces.* Publ. Math. Barc., 39 (1995), 405-411.

[33] Y. Sawano, *Morrey spaces from various points of view.* In: Rodino L., Toft J. (eds) Mathematical Analysis and Applications - Plenary Lectures. ISAAC 2017. Springer Proceedings in Mathematics & Statistics, vol. 262 (2018), Springer, 1-15

[34] G. Stampacchia, $L^{(p,\lambda)}$ *–spaces and interpolation.* Comm. Pure Appl. Math., 17 (1964), 293-306. http://dx.doi.org/10.1002/cpa.3160170303

[35] W. Yuan, W. Sickel, D. Yang, *Interpolation of Morrey – Campanato and smoothness spaces.* Sci. Math. China, 58 (2015), no. 9, 1835-1908.

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AN INTRODUCTION TO COMPOSITION OPERATORS IN SOBOLEV SPACES

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We propose a survey of the results on the composition operators in classical Sobolev spaces, obtained between 1975 and 2020. A first version of these notes were the subject of a series of lectures, given in Padova University in January 2018.

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1 Introduction

The composition of two maps f and g is defined by $(f \circ g)(x) := f(g(x))$, if the range of g is contained in the definition set of f. We denote by T_f the composition operator $T_f(g) := f \circ g$.

Definition 1. Let E be a set of real valued functions, and let $f : \mathbb{R} \to \mathbb{R}$. We say that f acts on E by composition (or: superposition) if $T_f(E) \subseteq E$.

Here are some elementary examples :

- Let E be a vector space of functions, which means that $g_1 + g_2 \in E$ and $\lambda g_1 \in E$, for all $g_1, g_2 \in E$ and all $\lambda \in \mathbb{R}$. Then every linear function $f : \mathbb{R} \to \mathbb{R}$ acts on E.
- Let E be an algebra of functions, which means that E is a vector space as above, and that $g_1 g_2 \in E$ for all $g_1, g_2 \in E$. Then any polynomial f such that $f(0) = 0$ acts on E.

We have a list of natural problems concerning operators T_f.

In case E is a vector space of functions, a composition operator T_f is said *trivial* if the function f is linear. Then we have the following questions :

$\mathcal{Q}_1$: *Do nontrivial composition operators exist ?*

In case E is an algebra of functions, the answer is positive. We will see that it is negative for certain Sobolev spaces.

$\mathcal{Q}_2$: *Describe explicitly the set of functions which act on E.*

For instance, if E is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, then a function f acts on E if and only if f is itself continuous.

In case E is endowed with a norm, then the following problems make sense :

$\mathcal{Q}_3$: *Determine the functions* f *for which* $T_f : E \to E$ *is bounded.*

$\mathcal{Q}_4$: *Determine the functions* f *for which* $T_f : E \to E$ *is continuous.*

We propose a wide survey on the answers to the above questions, in case E is the classical Sobolev space $W_p^m(\mathbb{R}^n)$. Some results are given together with their proofs. Some proofs are simpler than the original ones.

2 Notation

$\mathbb{N}$ denotes the set of all positive integers, including 0. $\mathbb{Z}$ denotes the set of all integers. For $x \in \mathbb{R}^n$, $|x|$ denotes its euclidean norm.

If E, F are topological spaces, then $E \hookrightarrow F$ means that $E \subseteq F$, as sets, and the natural mapping $E \to F$ is continuous. If B is a Lebesgue measurable subset of $\mathbb{R}^n$, we denote by $|B|$ its Lebesgue measure. We denote by χ_A the characteristic function of a set A.

A *multi-index* is n-tuple $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. For such α, and for all $h := (h_1, \dots, h_n) \in \mathbb{R}^n$, we set $|\alpha| := \alpha_1 + \cdots + \alpha_n$ (this differs from the euclidean norm), $\alpha! := \alpha_1! \cdots \alpha_n!$, $h^\alpha := h_1^{\alpha_1} \cdots h_n^{\alpha_n}$. If f is a function defined on an open subset of $\mathbb{R}^n$, and $\alpha \in \mathbb{N}^n$ as above, we denote by $f^{(\alpha)}$ the partial derivative

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If $h \in \mathbb{R}^n$, the *translation operator* is defined by $(\tau_h f)(x) := f(x - h)$ for all function f on $\mathbb{R}^n$. The *finite difference operator* is defined by $\Delta_h f := \tau_{-h} f - f$. The m-th power of Δ_h satisfies the following formula :

$$(\Delta_h^m f)(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(x + kh) \tag{2.1}$$

(easy proof by induction).

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $L_{1,loc}(\Omega)$ the set of (equivalence classes of) locally integrable functions on Ω, endowed with its natural topology (mean convergence on compact subsets of Ω), and by $\mathcal{D}(\Omega)$ the set of all indefinitely many times differentiable compactly supported functions on Ω, endowed with its natural topology, see [1, 1.56].

Let $Q := [-1/2, 1/2]^n$. We fix some function $\rho \in \mathcal{D}(\mathbb{R}^n)$ such that $\rho(x) = 1$ on Q and $\operatorname{supp} \rho \subseteq 2Q$.

Let E be a subset of $L_{1,loc}(\mathbb{R}^n)$. We say that a function $f \in L_{1,loc}(\mathbb{R}^n)$ belongs *locally* to E if $\varphi f \in E$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$; in case E is endowed with a norm, we say that a function $f \in L_{1,loc}(\mathbb{R}^n)$ belongs *locally uniformly* to E if

$$\sup_{a \in \mathbb{R}^n} \|(\tau_a \varphi) f\|_E < +\infty\,,$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Through the paper, "ball" means "closed ball with nonzero radius" (we exclude balls reduced to one point).

3 Composition operators in Lebesgue spaces

Proposition 3.1. *Let $1 \le p < +\infty$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. Then f acts on $L_p(\mathbb{R}^n)$ if and only if there exists $c > 0$ such that*

$$|f(t)| \le c|t|\,, \quad \text{for all}\, t \in \mathbb{R}\,. \tag{3.1}$$

Proof. 1. If estimate (3.1) holds, it is easily seen that $g \in L_p(\mathbb{R}^n)$ implies $f \circ g \in L_p(\mathbb{R}^n)$. Indeed, the following holds :

$$\|f \circ g\|_p \le c\,\|g\|_p, \quad \text{for all}\, g \in L_p(\mathbb{R}^n)\,. \tag{3.2}$$

2. Assume that f acts on $L_p(\mathbb{R}^n)$. Since the null function belongs to $L_p(\mathbb{R}^n)$, the same holds for the constant function $f(0)$. By condition $p < \infty$ we deduce $f(0) = 0$. Arguing by contradiction, let us assume that estimate (3.1) does not hold. Then, for some sequence $(a_k)_{k\ge1}$, we have $|f(a_k)| > k|a_k|$ for all $k \ge 1$. Consider a sequence $(B_k)_{k\ge1}$ of disjoint measurable sets in $\mathbb{R}^n$ such that

$$|a_k|^p|B_k| = k^{-p-1}\,. \tag{3.3}$$

Let

$$g := \sum_{k\ge1} a_k \chi_{B_k}\,.$$

By (3.3), it follows easily that $g \in L_p(\mathbb{R}^n)$. Since

$$f \circ g = \sum_{k\ge1} f(a_k)\chi_{B_k}\,,$$

(3.3) implies again $f \circ g \notin L_p(\mathbb{R}^n)$, a contradiction. □

Remark 1. The above proof works as well in case of $L_p(A)$, for any measurable subset A of $\mathbb{R}^n$ such that $|A| = +\infty$. For the generalization of Proposition 3.1 to L_p spaces on abstract measure spaces, we refer to [3, Theorem 3.1].

In case of linear operators on normed spaces, it is well known that boundedness is equivalent to continuity. Of course that does not hold for nonlinear ones. In particular, composition operators can be bounded but not continuous.

Proposition 3.2. *Assume $1 \le p \le +\infty$. Let (X, μ) be a measure space. Assume that (X, μ) is non trivial, i.e. there exists a measurable set A in X such that $0 < \mu(A) < +\infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be such that T_f takes $L_p(X, \mu)$ to itself. If T_f is continuous from $L_p(X, \mu)$ to itself, then f is continuous.*

Proof. Assume that T_f is continuous from $L_p(X, \mu)$ to itself. Without loss of generality, assume $f(0) = 0$. Let A be as in the above statement. For all real numbers u, v,

$$f \circ u\chi_A - f \circ v\chi_A = (f(u) - f(v))\chi_A\,,$$

hence

$$\|f \circ u\chi_A - f \circ v\chi_A\|_p = |f(u) - f(v)|\,\mu(A)^{1/p}\,. \tag{3.4}$$

Clearly

$$\lim_{v\to u} v\chi_A = u\chi_A \quad \text{in } L_p\,.$$

By continuity of T_f, and by (3.4), we obtain the continuity of f. □

By Propositions 3.1 and 3.2, it follows that, in case of $L_p(\mathbb{R}^n)$, there exist bounded composition operators which are not continuous. Proposition 3.2 admits a converse statement :

Proposition 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for some constant $c > 0$, $|f(t)| \leq c\,|t|$, for all $t \in \mathbb{R}$. Let (X, μ) be a measure space and let $1 \leq p < +\infty$. Then T_f is continuous from $L_p(X, \mu)$ to itself.*

Proof. It suffices to prove the following : for all sequence (g_j) converging to g in $L_p(X, \mu)$, there exists a subsequence (g_{j_k}) such that $(f \circ g_{j_k})$ converges to $f \circ g$ in $L_p(X, \mu)$. By the classical measure theoretic result (see, for instance, the proof of Theorem 3.11 in [15]), there exists a subsequence (g_{j_k}) and a function $h \in L_p(X, \mu)$ such that

$$g_{j_k} \to g \quad a.e.\,, \quad |g_{j_k}| \leq h\,.$$

By the continuity of f, it holds $f \circ g_{j_k} \to f \circ g$ a.e.. By the assumption on f,

$$|f \circ g_{j_k} - f \circ g| \leq 2c\,h\,.$$

By the Lebesgue dominated convergence Theorem, we conclude that $\|f \circ g_{j_k} - f \circ g\|_p$ tends to 0. □

Remark 2. If $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, T_f is easily seen to be continuous from $L_\infty(X, \mu)$ to itself. The details are left to the reader.

4 Automatic boundedness

Definition 2. Let E be a normed space. A mapping $T : E \to E$ is said *bounded* if, for all bounded set A of E, the set $T(A)$ is bounded.

For instance, according to estimate (3.2), any composition operator, which sends $L_p(\mathbb{R}^n)$ to itself, is bounded on $L_p(\mathbb{R}^n)$. More generally, for all "reasonable" function space, a weak form of boundedness is satisfied by composition operators. Thus we have a kind of automatic boundedness for a large class of function spaces.

Proposition 4.1. *Let E, F be vector subspaces of $L_{1,loc}(\Omega)$. Assume that*

- *E and F are endowed with complete norms such that the embeddings of E and F into $L_{1,loc}(\Omega)$ are continuous.*
- *$\mathcal{D}(\Omega)$ is embedded into E.*
- *$\varphi g \in F$, for all $\varphi \in \mathcal{D}(\Omega)$ and $g \in F$.*

For all $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $T_f(E) \subseteq F$, there exist a closed ball $B \subset \Omega$ and two numbers $c_1, c_2 > 0$ such that, for all $g \in E$,

$$\|g\|_E \leq c_1 \quad \text{and} \quad \operatorname{supp} g \subseteq B \qquad \Rightarrow \qquad \|f \circ g\|_F \leq c_2\,. \tag{4.1}$$

Proof. By contradiction, assume that, for all B, c_1, c_2 there exists $g \in E$ such that

$$\|g\|_E \leq c_1\,, \quad \operatorname{supp} g \subseteq B\,, \quad \|f \circ g\|_F > c_2\,. \tag{4.2}$$

Consider a sequence $(B_j)_{j \geq 1}$ of disjoint closed balls in Ω. Take functions $\varphi_j \in \mathcal{D}(\Omega)$ such that $\varphi_j(x) = 1$ on $\frac{1}{2}B_j$ (the ball of the same center and half radius than B_j) and $\varphi_j(x) = 0$ out of B_j. It is easily seen (Closed Graph Theorem, see [17, Chapter II, §6, Theorem 1] or [14, Theorem 2.15]) that, for $\varphi \in \mathcal{D}(\Omega)$, the linear multiplication operator $g \mapsto \varphi g$ is bounded on F. Thus we can consider

$$M_j := \sup\{\|\varphi_j g\|_F \,:\, \|g\|_F \le 1\}\,.$$

According to (4.2), there exist functions g_j such that

$$\|g_j\|_E \le 2^{-j}\,, \quad \operatorname{supp} g_j \subseteq \frac{1}{2}B_j\,, \quad \|f \circ g_j\|_F > jM_j\,.$$

Let $g := \sum_j g_j$. Clearly $g \in E$ and, by the embedding $E \hookrightarrow L_{1,loc}(\Omega)$,

$$g(x) = \sum_{j\ge 0} g_j(x) \quad \text{a.e.}\,.$$

By considering supports, $\varphi_j(f \circ g) = f \circ g_j$, hence

$$jM_j \le \|\varphi_j(f \circ g)\|_F \le M_j\|f \circ g\|_F$$

for any $j \ge 1$, a contradiction. □

Remark 3. If $\Omega = \mathbb{R}^n$, and if E is translation and dilation invariant, the conclusion of Proposition 4.1 can be improved : indeed *for all* balls or cubes B, there exist $c_1, c_2 > 0$ such that (4.1) holds for all $g \in E$.

As an example of use of Proposition 4.1, we give the following variant of Proposition 3.1 :

Proposition 4.2. *Let $1 \le p < +\infty$, let Ω be an open subset of $\mathbb{R}^n$ such that $|\Omega| < +\infty$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. Then f acts on $L_p(\Omega)$ if and only if there exist $\alpha, \beta > 0$ such that*

$$|f(t)| \le \alpha|t| + \beta\,, \quad \text{for all } t \in \mathbb{R}\,. \tag{4.3}$$

Proof. Since the sufficiency of condition (4.3) is clear, we deal only with necessity. Assume that f acts on $L_p(\Omega)$. Without loss of generality, we can assume that $f(0) = 0$. By Proposition 4.1, there exist a cube $Q' \subset \Omega$ and two numbers $c_1, c_2 > 0$ such that, for all $g \in L_p(\Omega)$,

$$\|g\|_p \le c_1 \quad \text{and} \quad \operatorname{supp} g \subseteq Q' \qquad \Rightarrow \qquad \|f \circ g\|_p \le c_2\,. \tag{4.4}$$

Let $b \in \Omega$ and $r > 0$ be such that $Q' = b + 2rQ$. For any $a \in \mathbb{R}$, and $0 < \varepsilon \le 1$, let

$$g_{a,\varepsilon}(x) := a\rho\left(\frac{x-b}{r\varepsilon}\right)\,.$$

Then the support of $g_{a,\varepsilon}$ is contained in Q'. We choose ϵ depending on a in the following way.

In the case of large a, more precisely if $|a| \ge R := r^{-n/p}c_1\|\rho\|_p^{-1}$, we choose ε such that

$$|a|\, r^{n/p}\|\rho\|_p\, \varepsilon^{n/p} = c_1\,. \tag{4.5}$$

If $|a| < R$, we take $\varepsilon = 1$. In both cases, we obtain $\|g_{a,\varepsilon}\|_p \le c_1$, hence $\|f \circ g_{a,\varepsilon}\|_p \le c_2$. Since

$$\rho\left(\frac{x-b}{r\varepsilon}\right) = 1$$

on the cube $b + \varepsilon rQ$, this implies

$$\int_{b+\varepsilon rQ} |f(a)|^p \,\mathrm{d}x \le c_2^p\,,$$

hence $|f(a)|^p\varepsilon^n \le c_3$, for some constant c_3.

If $|a| \ge R$, by using (4.5), we obtain $|f(a)| \le c_4|a|$, for some constant c_4. If $|a| < R$, we obtain $|f(a)| \le c_3^{1/p}$. □

Remark 4. The above proof can be viewed as a prototype of a number of results on composition operators, as we will see further.

5 Definition and main properties of Sobolev spaces

Definition 3. Let $f \in L_{1,loc}(\mathbb{R}^n)$, and $\alpha \in \mathbb{N}^n$. We say that f has a weak derivative of order α if there exists $g \in L_{1,loc}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} g(x)\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\varphi^{(\alpha)}(x)\,\mathrm{d}x$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

If such g exists, it is easily seen to be unique, up to equality almost everywhere; then we denote it by $f^{(\alpha)}$ and we call it the *weak derivative* of f of order α.

Definition 4. *Let $m \in \mathbb{N}$ and $1 \le p \le +\infty$. The Sobolev space $W^m_p(\mathbb{R}^n)$ is the set of functions $f \in L_{1,loc}(\mathbb{R}^n)$ such that, for all $|\alpha| \le m$, $f^{(\alpha)}$ exists in the weak sense, and $f^{(\alpha)} \in L_p(\mathbb{R}^n)$.*

$W^m_p(\mathbb{R}^n)$ is a vector subspace of $L_p(\mathbb{R}^n)$. It will be endowed with the following norm :

$$\|f\|_{W^m_p(\mathbb{R}^n)} := \sum_{|\alpha|\le m} \|f^{(\alpha)}\|_p\,. \tag{5.1}$$

We give here some useful properties of Sobolev spaces.

First of all, $W^m_p(\mathbb{R}^n)$ is a function space which satisfies the assumptions of Proposition 4.1, see [1, Theorem 3.3].

The behavior of (5.1) with respect to dilations is described in the following assertion, with a simple proof :

Proposition 5.1. *It holds*

$$\|f(\lambda(.))\|_{W^m_p(\mathbb{R}^n)} \le \lambda^{m-(n/p)}\|f\|_{W^m_p(\mathbb{R}^n)}\,,$$

for all $\lambda \ge 1$.

Then we have the so-called *Sobolev embedding theorems*, see [1, Theorem 4.12] :

Proposition 5.2. *If*

$$m_1 - m_2 \ge \frac{n}{p_1} - \frac{n}{p_2} > 0,$$

then $W^{m_1}_{p_1}(\mathbb{R}^n) \hookrightarrow W^{m_2}_{p_2}(\mathbb{R}^n)$.

In particular $W^m_p(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$ if $m > n/p$. In fact, we have a more precise statement, where $C_b(\mathbb{R}^n)$ denotes the space of bounded continuous functions on $\mathbb{R}^n$:

Proposition 5.3. *If $m > n/p$, or $p = 1$ and $m = n$, then $W^m_p(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}^n)$.*

The assumptions on the parameters are sharp : $W^m_p(\mathbb{R}^n)$ is not embedded in $L_\infty(\mathbb{R}^n)$ in the case $m < n/p$, or $m = n/p$ and $p > 1$.

Remark 5. The elements of $W^m_p(\mathbb{R}^n)$ are equivalence classes of functions with respect to the equality almost everywhere. Thus the precise meaning of Proposition 5.3 is the following : all $f \in W^m_p(\mathbb{R}^n)$ contains a (necessarily unique) bounded continuous representative such that $\|f\|_\infty \le c\|f\|_{W^m_p}$, for some constant $c > 0$ depending only on m, p, n.

Proposition 5.4. *If $m > n/p$, or $m = n$ and $p = 1$, the Sobolev space $W^m_p(\mathbb{R}^n)$ is a subalgebra of $C_b(\mathbb{R}^n)$.*

See [1, Theorem 4.39] for the proof.

6 Necessity of Lipschitz continuity

In case $m \geq 1$, any function which acts on W_p^m by composition is necessarily Lipschitz continuous, at least locally. This is a major distinction with the case of L_p. To prove this property, we need some preliminary results.

Lemma 6.1. *Assume that $W_p^m(\mathbb{R}^n)$ is not embedded into $L_\infty(\mathbb{R}^n)$. There exists a sequence $(\theta_j)_{j\geq 1}$ in $\mathcal{D}(\mathbb{R}^n)$ such that*

$$\theta_j(x) = 1 \quad \text{on} \quad 2^{-j}Q\,, \quad \operatorname{supp}\theta_j \subseteq Q\,, \quad \lim_{j\to+\infty} \|\theta_j\|_{W_p^m(\mathbb{R}^n)} = 0\,.$$

Proof. In case $m < n/p$, we take $\theta_j(x) = \rho(2^j x)$. By Proposition 5.1,

$$\|\theta_j\|_{W_p^m(\mathbb{R}^n)} \leq 2^{j(m-(n/p))} \|\rho\|_{W_p^m(\mathbb{R}^n)}\,.$$

Thus the sequence (θ_j) has the desired properties.

Now assume that $m = n/p$ and $1 < p < +\infty$. Let

$$\theta_j(x) := \frac{1}{j}\sum_{k=1}^{j} \rho(2^k x)\,.$$

If $|\alpha| = m$, then the function $x \mapsto \rho^{(\alpha)}(2^k x)$ has support in the set $S_k := 2^{-k+1}Q \setminus 2^{-k}Q$. Thus, for all $1 \leq k \leq j$ and $x \in S_k$,

$$\left|\theta_j^{(\alpha)}(x)\right| = \frac{1}{j} 2^{mk} \left|\rho^{(\alpha)}(2^k x)\right| \leq c j^{-1} 2^{mk}\,.$$

Hence

$$\|\theta_j^{(\alpha)}\|_p^p = \sum_{k=1}^{j} \int_{S_k} \left|\theta_j^{(\alpha)}(x)\right|^p \mathrm{d}x \leq c j^{-p} \sum_{k=1}^{j} 2^{kmp} 2^{-nk} = c j^{1-p}\,.$$

Thus the sequence $(\|\theta_j^{(\alpha)}\|_p)$ tends to 0 for all $|\alpha| = m$. The same holds, with a simple proof, for $|\alpha| < m$. □

Lemma 6.2. *Define the sequence of functions $(B_m)_{m\geq 1}$ in $L_1(\mathbb{R})$ by $B_1 := \mathbf{1}_{[0,1]}$ and $B_{m+1} := B_m * B_1$ for all m. Then*

$$\Delta_h^m f(x) = \int_{-\infty}^{+\infty} B_m(t) \left(\sum_{|\alpha|=m} \frac{m!}{\alpha!} f^{(\alpha)}(x+th)\, h^\alpha \right) \mathrm{d}t\,, \tag{6.1}$$

for almost all $x \in \mathbb{R}^n$, all $h \in \mathbb{R}^n$, all $m \geq 1$ and all $f \in W_p^m(\mathbb{R}^n)$.

Proof. We consider the case of an m times continuously differentiable function f. An approximation procedure will complete the proof in general case.

Step 1 : case $n = 1$. In this case, formula (6.1) reduces to

$$\Delta_h^m f(x) = \int_{-\infty}^{+\infty} B_m(t) f^{(m)}(x+th)\, h^m \mathrm{d}t\,. \tag{6.2}$$

We prove it by induction. The case $m = 1$ is well known. Assuming that (6.2) holds, we obtain

$$\Delta_h^{m+1} f(x) = h^m \int_{-\infty}^{+\infty} B_m(t) \Delta_h f^{(m)}(x+th)\, \mathrm{d}t$$

$$= h^{m+1} \int_{-\infty}^{+\infty} B_m(t) \left(\int_{-\infty}^{+\infty} B_1(s) f^{(m+1)}(x + (t+s)h)\, \mathrm{d}s \right) \mathrm{d}t\,.$$

By Fubini, a change of variable, and the definition of B_m, we obtain formula (6.2) at rank $m+1$.

Step 2 : general case. We fix x, h in $\mathbb{R}^n$, and set $g(t) := f(x + t(h/|h|))$ for all $t \in \mathbb{R}$. Then $\Delta_h^m f(x) = \Delta_{|h|}^m g(0)$. Applying Step 1 to the function g, we obtain (6.1). The details are left to the reader. □

Lemma 6.3. *For all $m \geq 1$, $1 \leq p \leq \infty$, there exists $c > 0$ such that*

$$\left(\int_{\mathbb{R}^n} |\Delta_h^m f(x)|^p \,\mathrm{d}x \right)^{1/p} \leq c|h|^m \|f\|_{W_p^m(\mathbb{R}^n)}$$

for all $h \in \mathbb{R}^n$ and all $f \in W_p^m(\mathbb{R}^n)$.

Proof. By definition of B_m, $B_m \geq 0$ and $\int_{-\infty}^{+\infty} B_m(t)\,\mathrm{d}t = 1$. Applying (6.1), we obtain

$$\|\Delta_h^m f\|_p \leq |h|^m \sum_{|\alpha|=m} \frac{m!}{\alpha!} \|f^{(\alpha)}\|_p\,.$$

□

Theorem 6.1. *Assume that $m \geq 1$ and that $W_p^m(\mathbb{R}^n)$ is not embedded into $L_\infty(\mathbb{R}^n)$. Then any function $f : \mathbb{R} \to \mathbb{R}$, such that T_f sends $W_p^m(\mathbb{R}^n)$ to itself, is Lipschitz continuous on $\mathbb{R}$.*

Proof. Throughout the proof, $\|.\|$ will denote the norm in $W_p^m(\mathbb{R}^n)$.

Step 1 : construction of the comb-shaped function. This construction was first introduced by S. Igari [11]. Let $A_N := \mathbb{Z}^n \cap [-N, N]^n$, for every positive integer N. We fix a real number s such that

$$0 < s < \frac{1}{2m+1}\,. \tag{6.3}$$

Let b, b' be real numbers. Then we consider integers $N, j \geq 1$, and a real number $r > 0$, whose values will be fixed depending on b, b'. Our test function will be defined by

$$g(x) := \sum_{\mu \in A_N} \rho\left(\frac{1}{s} \left(\frac{x}{r} - \mu \right) \right) (b' - b) + \theta_j(x)\, b\,. \tag{6.4}$$

The first condition on parameters will be

$$3rN \leq 2^{-j}\,. \tag{6.5}$$

By inequality $s < 1/2$ and by condition (6.5), we deduce that the cubes $r(2sQ + \mu)$ are disjoint, and that $r(2sQ + \mu) \subset r(Q + \mu) \subset 2^{-j}Q$, if $\mu \in A_N$. Hence

$$g(x) = b'\,, \quad \text{if } x \in r(sQ + \mu) \text{ for some } \mu \in A_N\,, \tag{6.6}$$

$$g(x) = b\,, \quad \text{if } x \in 2^{-j}Q \setminus \bigcup_{\mu \in A_N} r(2sQ + \mu)\,. \tag{6.7}$$

By (6.5), we have $r \leq 1$. Then Proposition 5.1 gives us

$$\Big\| \sum_{\mu \in A_N} \rho\Big(\frac{1}{s} \Big(\frac{\cdot}{r} - \mu \Big) \Big) \Big\| \leq c_1 r^{(n/p)-m} N^{n/p}\,, \tag{6.8}$$

for some constant c_1.

Step 2 : adjustment of parameters. Now we assume that f acts on $W^m_p(\mathbb{R}^n)$ by composition. By Proposition 4.1, we can find constants δ_1, δ_2 such that $\|f \circ u\| \leq \delta_2$ for every function u such that $\|u\| \leq \delta_1$, and u has support in Q. In order to apply this property to $u = g$, we need the following inequalities :

$$|b|\,\|\theta_j\| \leq \frac{\delta_1}{2}\,, \tag{6.9}$$

$$\frac{\delta_1}{3c_1|b-b'|} \leq r^{(n/p)-m} N^{n/p} \leq \frac{\delta_1}{2c_1|b-b'|}\,. \tag{6.10}$$

Now we discuss the choice of j, N, r with respect to b, b', such that conditions (6.5), (6.9) and (6.10) hold. First, we choose $j = j(b) \geq 1$ such that (6.9) holds. This is possible by Lemma 6.1. In the case $m < n/p$, we define

$$r := \left(\frac{\delta_1}{2c_1|b-b'|} N^{-n/p}\right)^{\frac{p}{n-mp}}\,,$$

which ensures condition (6.10) ; since

$$rN = \left(\frac{\delta_1}{2c_1|b-b'|}\right)^{\frac{p}{n-mp}} N^{\frac{mp}{pm-n}}\,,$$

condition (6.5) holds for all sufficiently large N, depending on $|b-b'|$.

In the case $m = n/p$, we take N such that (6.10) holds. Such a choice is possible if $|b-b'| \leq c_2$, where $c_2 > 0$ depends only on p, n, δ_1. Then we put $r := 2^{-j}/3N$.

Step 3 : end of the proof. By combining inequalities (6.8), (6.9) and (6.10), we deduce $\|g\| \leq \delta_1$. Using Lemma 6.3, we obtain

$$\|\Delta^m_h(f \circ g)\|_p \leq \delta_3 |h|^m\,,$$

for all $h \in \mathbb{R}^n$, where δ_3 depends only on δ_2, m, n, p. Let $Q^+ :=]0, 1/2]^n$ and $e_1 := (1, 0, \ldots, 0) \in \mathbb{R}^n$. By condition (6.3) we have

$$x + \ell r s e_1 \in r(Q + \mu) \subset 2^{-j} Q \quad (\ell = 0, \ldots, m)\,,$$

$$x + \ell r s e_1 \notin \bigcup_{\mu' \in A_N} r(2sQ + \mu')\,, \quad (\ell = 1, \ldots, m)\,,$$

for all $x \in r(sQ^+ + \mu)$; for such x, equalities (6.6) and (6.7), and formula (2.1), imply that

$$\left|\Delta^m_{rse_1}(f \circ g)(x)\right| = |f(b') - f(b)|\,.$$

Hence

$$\begin{aligned} \delta_3 &\geq c_3 r^{-m} \left(\sum_{\mu \in A_N} \int_{r(sQ^+ + \mu)} \left|\Delta^m_{rse_1}(f \circ g)(x)\right|^p \mathrm{d}x\right)^{1/p} \\ &\geq c_4 |f(b') - f(b)|\, N^{n/p} r^{(n/p)-m}\,. \end{aligned}$$

By (6.10) we obtain the existence of a constant δ_4 such that $|f(b') - f(b)| \leq \delta_4 |b - b'|$ for all $b, b' \in \mathbb{R}$ satisfying $|b' - b| \leq c_2$. Thus f is uniformly Lipschitz continuous. □

Theorem 6.2. *Assume that $m \geq 1$. Then any function $f : \mathbb{R} \to \mathbb{R}$, such that T_f sends $W^m_p(\mathbb{R}^n)$ to itself, is locally Lipschitz continuous on $\mathbb{R}$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which acts on $W^m_p(\mathbb{R}^n)$. Let $a \in \mathbb{R}$. We introduce a localized version of T_f with the help of the following statement :

Lemma 6.4. *There exists a nonlinear operator U_a which sends $W^m_p(\mathbb{R}^n)$ to itself, such that, for all $g \in W^m_p(\mathbb{R}^n)$,*

$$U_a g(x) = f(a + g(x)) - f(a)\,, \quad \text{for all} \quad x \in Q\,,$$

$$\|g\|_{W^m_p(\mathbb{R}^n)} \le \delta_1 \quad \text{and} \quad \operatorname{supp} g \subseteq Q \qquad \Rightarrow \qquad \|U_a g\|_{W^m_p(\mathbb{R}^n)} \le \delta_2\,.$$

The proof is essentially the same as that of Proposition 4.1, see [6, Lemma 1] for details.

Returning to the proof of Theorem 6.2, we argue in the same way as in the proof of Theorem 6.1, just replacing T_f by U_a. We define g by (6.4), with $\theta_j(x)$ replaced by $\rho(2x)$, $s = 1/4$ and $r = 1/6N$. Inequality (6.9) becomes $|b| \le \delta_3$, for some constant δ_3 depending only on δ_1. The double inequality (6.10) reduces to

$$\frac{\delta_4}{|b - b'|} \le N^m \le \frac{\delta_5}{|b - b'|}\,, \tag{6.11}$$

where δ_4, δ_5 depend on δ_1 and c_1. If $|b - b'| \le \delta_4$, we can choose N satisfying (6.11). We obtain a constant δ_6 such that

$$|f(a + b) - f(a + b')| \le \delta_6 |b - b'|\,,$$

for b, b' satisfying $|b| \le \delta_3$ and $|b - b'| \le \delta_4$. Thus f is Lipschitz continuous in a neighborhood of a. □

An easy modification of the above proof gives us the following statement :

Proposition 6.1. *Let us assume $m \ge 3$, and define p_1 by :*

$$2 - \frac{n}{p_1} := m - \frac{n}{p}\,. \tag{6.12}$$

Then every function $f : \mathbb{R} \to \mathbb{R}$, such that T_f sends $W^m_p(\mathbb{R}^n)$ to $W^2_{p_1}(\mathbb{R}^n)$, is locally Hölder continuous of order $2/m$.

7 A case of degeneracy: Dahlberg Theorem

As announced in Introduction, Sobolev spaces provide simple examples of spaces for which the answer to question $\mathcal{Q}_1$ is negative.

Theorem 7.1. *Assume that m is an integer satisfying*

$$1 + \frac{1}{p} < m < \frac{n}{p}\,. \tag{7.1}$$

Then, for each function $f : \mathbb{R} \to \mathbb{R}$ which acts on $W^m_p(\mathbb{R}^n)$ by composition, there exists $c \in \mathbb{R}$ such that $f(t) = ct$ for all $t \in \mathbb{R}$.

This theorem was first proved by B. Dahlberg [9] for smooth functions f. Indeed, we have a slightly stronger property :

Proposition 7.1. *Under condition (7.1), let us define p_1 by condition (6.12). Then, for each function $f : \mathbb{R} \to \mathbb{R}$ such that T_f sends $W^m_p(\mathbb{R}^n)$ to $W^2_{p_1}(\mathbb{R}^n)$, there exists $c \in \mathbb{R}$ such that $f(t) = ct$ for all $t \in \mathbb{R}$.*

By Proposition 5.2, we have $W^m_p(\mathbb{R}^n) \hookrightarrow W^2_{p_1}(\mathbb{R}^n)$. Thus Theorem 7.1 follows by Proposition 7.1.

Proof. Step 1. We assume first that f is of class C^2. Since $W^m_p(\mathbb{R}^n)$ does not contain nonzero constant functions, we have $f(0)=0$. By Proposition 4.1, there exist two numbers $c_1, c_2 > 0$ such that, for all $g \in W^m_p(\mathbb{R}^n)$,

$$\|g\|_{W^m_p(\mathbb{R}^n)} \le c_1 \quad \text{and} \quad \operatorname{supp} g \subseteq 2Q \qquad \Rightarrow \qquad \|f\circ g\|_{W^2_{p_1}(\mathbb{R}^n)} \le c_2 \,. \tag{7.2}$$

Define the function $u \in \mathcal{D}(\mathbb{R}^n)$ by

$$u(x) := x_1 \rho(x)\,, \tag{7.3}$$

where x_1 denotes the first coordinate of $x \in \mathbb{R}^n$. Let $a > 0$, and $0 < \varepsilon \le 1$ (a number to be determined with respect to a). Let us define $g_a \in \mathcal{D}(\mathbb{R}^n)$ by

$$g_a(x) := a u\left(\frac{x}{\varepsilon}\right) .$$

Then $\operatorname{supp} g_a \subset 2Q$, and $\|g_a\|_{W^m_p(\mathbb{R}^n)} \le c_1$ if

$$a\,\varepsilon^{(n/p)-m}\|u\|_{W^m_p(\mathbb{R}^n)} = c_1 \,. \tag{7.4}$$

Due to the condition $m < n/p$, the above equality determines ε as a function of a, if a is sufficiently large. Hence it holds $\|f\circ g_a\|_{W^2_{p_1}(\mathbb{R}^n)} \le c_2$ for all large numbers a. Since

$$(f\circ g_a)(x) = f\left(\frac{a}{\varepsilon}x_1\right) , \quad x \in \varepsilon Q\,,$$

we deduce that

$$\left(\frac{a}{\varepsilon}\right)^{2p_1} \int_{\varepsilon Q} \left|f''\left(\frac{a}{\varepsilon}x_1\right)\right|^{p_1} \,\mathrm{d}x \le c_2^{p_1} \,.$$

By using (7.4) and a change of variable, we obtain a constant $c_3 > 0$ such that

$$a^{p_1-1}\int_{-a/2}^{+a/2} |f''(t)|^{p_1}\,\mathrm{d}t \le c_3\,, \tag{7.5}$$

for all large numbers a. By the assumption $m > 1 + (1/p)$, we have $p_1 > 1$. If we take a to $+\infty$, we deduce that

$$\int_{-\infty}^{+\infty} |f''(t)|^{p_1}\,\mathrm{d}t = 0\,.$$

Hence $f''(t) = 0$ almost everywhere. Since f'' is continuous, we conclude that $f(t) = ct$, for some constant c.

Step 2. We turn now to the general case. By Theorem 6.2 and Proposition 6.1, we know that f is continuous. Let $\omega \in \mathcal{D}(\mathbb{R})$, with support in $[-1,+1]$, even, such that $\int \omega(t)\,\mathrm{d}t = 1$. Let us set $\omega_j(t) := j\omega(jt)$ for all positive integers j. The convolution $\omega_j * f$ is defined, and it is a smooth function. Let us define

$$f_j(t) := (\omega_j * f)(t) - (\omega_j * f)(0)\,.$$

For all function g with support in Q,

$$(f_j\circ g)(x) = \rho(x)\int_{\mathbb{R}} (f((g(x)+t)\rho(x)) - f(t\rho(x)))\,\omega_j(t)\,\mathrm{d}t$$

for all $x \in \mathbb{R}^n$. In other words :

$$\operatorname{supp} g \subseteq Q \quad \Rightarrow \quad f_j\circ g = \rho\int_{\mathbb{R}} (f\circ((g+t)\rho) - f\circ(t\rho))\,\omega_j(t)\,\mathrm{d}t\,. \tag{7.6}$$

Let $M := \sup\{\|\rho\, h\|_{W^m_p(\mathbb{R}^n)} \,:\, \|h\|_{W^m_p(\mathbb{R}^n)} \leq 1\}$. Let j_0 be the first integer such that

$$j_0 \geq 2c_1^{-1}\|\rho\|_{W^m_p(\mathbb{R}^n)}\,.$$

Let g be such that $\operatorname{supp} g \subseteq Q$ and

$$\|g\|_{W^m_p(\mathbb{R}^n)} \leq \frac{c_1}{2M}\,.$$

Then, for all $j \geq j_0$, and all $|t| \leq 1/j$, it holds

$$\|(g+t)\rho\|_{W^m_p(\mathbb{R}^n)} \leq c_1\,.$$

By (7.2), we obtain

$$\|f_j \circ g\|_{W^2_{p_1}(\mathbb{R}^n)} \leq 2Mc_2$$

for all $j \geq j_0$. All together, we have obtained constants $c_3, c_4 > 0$ such that

$$\|g\|_{W^m_p(\mathbb{R}^n)} \leq c_3 \quad \text{and} \quad \operatorname{supp} g \subseteq Q \qquad \Rightarrow \qquad \|f_j \circ g\|_{W^2_{p_1}(\mathbb{R}^n)} \leq c_4\,, \tag{7.7}$$

for all $j \geq j_0$. Reasoning as in Step 1, we conclude that, for some constants a_j, $j \geq j_0$, we have $f_j(t) = a_j t$ for all $t \in \mathbb{R}$. Thus we obtain

$$(\omega_j * f)(t) = (\omega_j * f)(0) + a_j t$$

for all $t \in \mathbb{R}$. Since f is continuous, we know that $\lim_{j\to+\infty}(\omega_j * f)(t) = f(t)$ for all $t \in \mathbb{R}$. Taking $t = 1$, we obtain $\lim_{j\to+\infty} a_j = f(1)$. We conclude that $f(t) = f(1)t$ for all $t \in \mathbb{R}$. □

8 Composition operators on W^1_p

First of all, we recall a classical result :

Theorem 8.1. *For all $f : \mathbb{R} \to \mathbb{R}$, the following properties are equivalent :*

(1) f is Lipschitz continuous,

(2) f has a weak derivative in $L_\infty(\mathbb{R})$,

(3) There exists $g \in L_\infty(\mathbb{R})$ and a constant $c \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R} \quad f(x) = \int_0^x g(t)\,\mathrm{d}t + c\,.$$

Proof. The implication (3) ⇒ (1) is straightforward. The equivalence (2) ⇔ (3) is easy to prove. Concerning (1) ⇒ (3), we refer to [10, Theorem 7.18] (Alternatively, we can observe that any Lipschitz continuous function is absolutely continuous, then apply [15, Theorem 8.17]). □

Theorem 8.2. *Let $f : \mathbb{R} \to \mathbb{R}$, such that $f(0) = 0$. Then f acts on $W^1_p(\mathbb{R}^n)$ if and only if*

- *f is Lipschitz continuous, if $W^1_p(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R})$,*
- *f is locally Lipschitz continuous, if $W^1_p(\mathbb{R}^n) \subset L_\infty(\mathbb{R})$.*

This theorem is due to Marcus and Mizel [12]. Roughly speaking, sufficiency result relies upon the formula $\partial_j(f \circ g) = (f' \circ g)\partial_j g$. In the case $W^1_p(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$, we just need that f' belongs to L_∞ on the range of g. The necessity of the Lipschitz conditions follows by Theorems 6.1 and 6.2.

9 Full description of acting functions in higher order Sobolev spaces

Let us give a sufficient condition for composition :

Theorem 9.1. *Assume that $m \geq \max(2, n/p)$, or $m = 2$, $p = 1$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$ and $f' \in W_p^{m-1}(\mathbb{R})$, then f acts on $W_p^m(\mathbb{R}^n)$.*

Proof. A preliminary remark : under the assumptions of Theorem 9.1, it holds

$$W_p^{m-1}(\mathbb{R}) \hookrightarrow L_\infty(\mathbb{R}) .$$

That follows by Proposition 5.3.

Here we restrict ourselves to the case $m = 2$. The method that we use is typical of the general case. Also we assume that f is of class C^m, with bounded derivatives up to order m, and that g is smooth, with derivatives tending to 0 at infinity; see [4, 5] and [16, 5.2.4, Theorem 2] for the approximation procedure to cover the general case.

Let $g \in W_p^2(\mathbb{R}^n)$. We have to prove that the second order derivatives of $f \circ g$ belongs to $L_p(\mathbb{R}^n)$. It holds

$$\partial_j \partial_k (f \circ g) = (f'' \circ g)(\partial_j g)(\partial_k g) + (f' \circ g)\partial_j \partial_k g . \tag{9.1}$$

The second term belongs to L_p, because $f' \in L_\infty$. Thus we can concentrate on the first one. By the applying Cauchy-Schwarz inequality, we obtain

$$\|(f'' \circ g)\partial_j g \, \partial_k g\|_p \leq U_j^{1/2p} U_k^{1/2p} , \tag{9.2}$$

where

$$U_j := \int_{\mathbb{R}^n} |(f'' \circ g)(x)|^p |\partial_j g(x)|^{2p} \, \mathrm{d}x .$$

Let us introduce

$$h(x) := \int_x^{+\infty} |f''(t)|^p \, \mathrm{d}t .$$

Then $-U_j$ is equal to

$$\int_{\mathbb{R}^n} (h' \circ g)(x) \partial_j g(x) \partial_j g(x) |\partial_j g(x)|^{2p-2} \, \mathrm{d}x = \int_{\mathbb{R}^n} \partial_j (h \circ g)(x) \, \partial_j g(x) |\partial_j g(x)|^{2p-2} \, \mathrm{d}x .$$

An integration by parts gives

$$U_j = (2p-1) \int_{\mathbb{R}^n} (h \circ g)(x) \, \partial_j^2 g(x) |\partial_j g(x)|^{2p-2} \, \mathrm{d}x .$$

Hence

$$U_j \leq (2p-1) \|f''\|_p^p \int_{\mathbb{R}^n} |\partial_j^2 g(x)| \, |\partial_j g(x)|^{2p-2} \, \mathrm{d}x . \tag{9.3}$$

In case $p = 1$, the above inequality becomes $U_j \leq \|f''\|_1 \, \|\partial_j^2 g\|_1$. That completes the proof of Theorem in the case $m = 2$, $p = 1$.

In case $p > 1$, we use the Hölder inequality to derive

$$U_j \leq (2p-1) \|f''\|_p^p \, \|\partial_j^2 g\|_p \left(\int_{\mathbb{R}^n} |\partial_j g(x)|^{2p} \, \mathrm{d}x \right)^{1-(1/p)} .$$

By Proposition 5.2 and condition $2 \geq n/p$, $W_p^2(\mathbb{R}^n) \hookrightarrow W_{2p}^1(\mathbb{R}^n)$. That completes the proof of Theorem 9.1. □

Remark 6. The above proof shows also that the composition operator is bounded under assumptions of Theorem 9.1. More precisely, there exist a constant $c = c(p, n) > 0$ such that

$$\|f \circ g\|_{W^2_p(\mathbb{R}^n)} \le c\|f''\|_p \left(\|g\|_{W^2_p(\mathbb{R}^n)} + \|g\|^{2-(1/p)}_{W^2_p(\mathbb{R}^n)}\right) . \tag{9.4}$$

We turn now to the complete description of composition operators. Due to Theorems 7.1 and 8.2, we will consider only the case $m \ge 2$, together with the three following subcases :

- $m > n/p$, or $m = n$ and $p = 1$.
- $m = n/p$ and $p > 1$.
- $m = 2$, $p = 1$ and $n \ge 3$.

Theorem 9.2. *Let $m \ge 2$, $1 \le p < +\infty$. If $m > n/p$, or if $m = n$ and $p = 1$, then a function $f : \mathbb{R} \to \mathbb{R}$ acts on $W^m_p(\mathbb{R}^n)$ if and only if $f(0) = 0$ and f belongs locally to $W^m_p(\mathbb{R})$.*

Proof. 1- Assume that f belongs locally to $W^m_p(\mathbb{R})$, and that $g \in W^m_p(\mathbb{R}^n)$. By Proposition 5.3, g is bounded. Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi(t) = 1$ on the range of g. Then $f \circ g = (\varphi\, f) \circ g$. Since $\varphi\, f \in W^m_p(\mathbb{R})$, we can apply Theorem 9.1, and conclude that $f \circ g \in W^m_p(\mathbb{R}^n)$.

2- Assume that T_f sends $W^m_p(\mathbb{R}^n)$ to itself. By considering $f \circ g$, where $g \in \mathcal{D}(\mathbb{R}^n)$ satisfies $g(x) = x_1$ on an arbitrary ball of $\mathbb{R}^n$, we conclude that f, together with all its derivatives up to order m, belongs to L^p on each bounded interval of $\mathbb{R}$. □

Theorem 9.3. *Let $m = n/p \ge 2$ and $p > 1$. Then a function $f : \mathbb{R} \to \mathbb{R}$ acts on $W^m_p(\mathbb{R}^n)$ if and only if $f(0) = 0$ and f' belongs locally uniformly to $W^{m-1}_p(\mathbb{R})$.*

Proof. The sufficiency of the condition on f follows by a modification of the proof of Theorem 9.1, see [4, 5] or [16, 5.2.4, Theorem 2].

To prove the necessity, we use the same ideas as in the proof of Theorem 6.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which acts on $W^m_p(\mathbb{R}^n)$. We introduce constants δ_1, δ_2 as in the proof of Theorem 6.1. Let b be a real number. Let $j = j(b) \ge 1$ such that (6.9) holds. Let us consider the function

$$g_b(x) := \lambda u(2^j x) + \theta_j(x)\, b\,,$$

where u is the function introduced in (7.3), and λ is a constant, to be fixed below. By the assumption $m = n/p$, it holds $\|u(2^j(.))\| \le \|u\|$. Thus, the choice of $\lambda := \delta_1/2\|u\|$ implies $\|g_b\| \le \delta_1$. Hence we have

$$\|f \circ g_b\| \le \delta_2\,. \tag{9.5}$$

On the cube $2^{-j}Q$, it holds $(f \circ g_b)(x) = f(\lambda 2^j x_1 + b)$, hence

$$\partial^m_1 (f \circ g_b)(x) = \lambda^m 2^{jm} f^{(m)}(\lambda 2^j x_1 + b)\,.$$

Then using (9.5), a change of variable, and condition $m = n/p$, we find a constant $\delta_3 > 0$ such that

$$\int_{b-(\lambda/2)}^{b+(\lambda/2)} |f^{(m)}(y)|^p\, \mathrm{d}y \le \delta_3\,,$$

for every $b \in \mathbb{R}$. Thus we have proved that $f^{(m)}$ belongs to $L_p(\mathbb{R})$ locally uniformly. Since we know yet that $f' \in L_\infty$, it follows easily that f' belongs to $W^{m-1}_p(\mathbb{R})$ locally uniformly. □

Theorem 9.4. *If $n \ge 3$, then a function $f : \mathbb{R} \to \mathbb{R}$ acts on $W^2_1(\mathbb{R}^n)$ if and only if $f(0) = 0$ and $f'' \in L_1(\mathbb{R})$.*

Proof. Sufficiency of $f'' \in L_1(\mathbb{R})$ follows by Theorem 9.1. To prove necessity, we proceed as in the proof of Theorem 7.1. Then the estimate (7.5) becomes

$$\int_{-a/2}^{+a/2} |f''(t)|\,\mathrm{d}t \leq c_3\,,$$

for all large a. By taking $a \to +\infty$, we obtain $f'' \in L_1(\mathbb{R})$. □

10 Continuity of composition on Sobolev spaces

The more precise versions of Theorems 9.1, 9.2, 9.3 show that all the composition operators which send $W_p^m(\mathbb{R}^n)$ to itself are bounded. They are also continuous, according to the following :

Theorem 10.1. *Let m be an integer ≥ 1, $1 \leq p < \infty$, and let $f : \mathbb{R} \to \mathbb{R}$. If f acts by composition on $W_p^m(\mathbb{R}^n)$, then the composition operator T_f is continuous from $W_p^m(\mathbb{R}^n)$ to itself.*

This theorem was proved step by step between 1976 and 2019 :

- for $m = 1$ and $p = 2$, by Ancona [2],
- for $m = 1$ and any p, by Marcus and Mizel [13],
- for $m > n/p$ and $1 < p < \infty$, by Lanza de Cristoforis and the author [6],
- in the general case by Moussai and the author [7], who proved also this "automatic" continuity on the so-called Adams-Frazier spaces $W_p^m \cap \dot{W}^1_{mp}(\mathbb{R}^n)$, where $\dot{W}$ denotes the homogeneous Sobolev space, and on the spaces $\dot{W}_p^m \cap \dot{W}^1_{mp}(\mathbb{R}^n)$, conveniently realized.

References

[1] R. Adams, J. Fournier, *Sobolev spaces.* Elsevier (2003).

[2] A. Ancona, *Continuité des contractions dans les espaces de Dirichlet.* C.R.A.S. 282 (1976), 871–873, and Springer LNM 563 (1976), 1–26.

[3] J. Appell, P. Zabrejko, *Nonlinear superposition operators.* Cambridge U.P. (1990).

[4] G. Bourdaud, *Le calcul fonctionnel dans les espaces de Sobolev.* Invent. math. 104 (1991), 435–446.

[5] G. Bourdaud, *Superposition in homogeneous and vector valued Sobolev spaces.* Trans. Amer. Math. Soc. 362 (2010), 6105–6130.

[6] G. Bourdaud, M. Lanza de Cristoforis, *Regularity of the symbolic calculus in Besov algebras.* Studia Math. 184 (2008), 271–298.

[7] G. Bourdaud, M. Moussai, *Continuity of composition operators in Sobolev spaces.* Ann. I. H. Poincaré - AN 36 (2019), 2053–2063.

[8] G. Bourdaud, W. Sickel, *Composition operators on function spaces with fractional order of smoothness.* RIMS Kokyuroku Bessatsu B26 (2011), 93–132.

[9] B.E.J. Dahlberg, *A note on Sobolev spaces.* Proc. Symp. Pure Math. 35 (1979), no. 1, 183–185.

[10] J. Foran, *Fundamentals of real analysis.* Marcel Dekker (1991).

[11] S. Igari, *Sur les fonctions qui opèrent sur l'espace* $\widehat{A}^2$. Ann. Inst. Fourier 15 (1965), 525–536.

[12] M. Marcus, V.J. Mizel, *Complete characterization of functions which act via superposition on Sobolev spaces.* Trans. Amer. Math. Soc. 251 (1979), 187–218.

[13] M. Marcus, V.J. Mizel, *Every superposition operator mapping one Sobolev space into another is continuous.* J. Funct. Anal. 33 (1979), 217–229.

[14] W. Rudin, *Functional analysis.* McGraw-Hill (1973).

[15] W. Rudin, *Real and complex analysis.* 3rd ed., McGraw-Hill (1987).

[16] T. Runst, W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations.* de Gruyter, Berlin, 1996.

[17] K. Yosida. *Functional analysis.* 6th edition, Springer (1980).

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ON A LINEAR INVERSE POTENTIAL PROBLEM WITH APPROXIMATE DATA ON THE POTENTIAL FIELD ON AN APPROXIMATELY GIVEN SURFACE

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Abstract. An approximate solution of the linear inverse problem for the Newtonian potential for bodies of constant thickness is constructed. The solution is stable with respect to the error in the data on the potential field given on an inaccurately known surface. The problem is reduced to an integral equation of the first kind, the proof of the stability of the solution is based on the Tikhonov regularization method.

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1 Introduction

The problem considered here is a linear version of the inverse potential problem, considered in [8]. The paper provides a solution to the problem of restoring the shape of the Newtonian potential density carrier for bodies of constant thickness belonging to the Sretensky class, defined in [9], which ensures the uniqueness of the solution of the inverse potential problem. In [9], the uniqueness of the inverse potential problem is proved for bounded homogeneous bodies having a common secant plane, such that every line perpendicular to it intersects the body at no more than two points lying on different sides of this plane. The problem is formulated in the framework of the odd-periodic model [4], which allows us to obtain a solution in the form of a Fourier series, which is essential for the application of numerical methods for solving the problem. The error of the periodic model with respect to the non-periodic one is studied in [5]. In the problem considered in this paper, information about the potential is given in the form of a potential field on a surface of a general form. Both the field and the surface are given approximately. The idea of the method in [6] is the basis for constructing a solution to the problem. The problem in this case, including for bounded bodies of constant thickness with variable density, is reduced to a linear integral equation of the first kind, the approximate solution of which, stable with respect to the error in data on the potential and the surface, is constructed on the basis of the Tikhonov regularization method [10], [11]. As an approximate solution, we consider the extremal of the Tikhonov functional, obtained as a solution of the Euler equation for this functional. The approximate solution is obtained in the form of a Fourier series with a regularizing factor. The convergence theorem of the approximate solution to the exact one is proved. The linear problem of reconstructing the distribution density function of sources with an infinitely thin carrier in the model of a heat-conducting body with convective heat exchange at the boundary, solved in [1], is closely related to the problem considered here.

2 Problem statement

In an infinite cylinder of rectangular cross-section

$$D^\infty = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, -\infty < z < \infty\} \subset \mathbb{R}^3 \tag{2.1}$$

we consider the following model for the Newtonian potential

$$\begin{aligned} &\Delta v(M) = -4\pi\rho(M), \quad M \in D^\infty, \\ &v|_{x=0,l_x} = 0,\ v|_{y=0,l_y} = 0, \\ &v \to 0 \ \text{ when } \ z \to \pm\infty. \end{aligned} \tag{2.2}$$

We assume that the support of the density ρ is located in the domain $z > H > 0$ in the cylinder D^∞.

Let $\varphi(M,P)$ be the source function of problem (2.2) in the domain D^∞ of form (2.1). The function $\varphi(M,P)$ can be obtained as a series

$$\varphi(M,P) = \frac{2}{l_x l_y} \sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}|z_P - z_M|}}{k_{nm}} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}, \tag{2.3}$$

where

$$k_{nm} = \sqrt{\left(\frac{\pi n}{l_x}\right)^2 + \left(\frac{\pi m}{l_y}\right)^2}.$$

If $z_P > H$, series (2.3) converges uniformly with respect to the variable M in the domain

$$D(-\infty, H-\varepsilon) = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, -\infty < z < H - \varepsilon\},\ \varepsilon > 0. \tag{2.4}$$

In the domain of $D(-\infty, H-\varepsilon)$ the solution of problem (2.2) can be represented as

$$v(M) = 4\pi \int\limits_{\mathrm{supp}\varphi} \rho(P)\varphi(M,P)dV_P = \frac{8\pi}{l_x l_y} \int\limits_{\mathrm{supp}\varphi} dV_P \rho(P) \sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}(z_P - z_M)}}{k_{nm}} \times \sin(\frac{\pi n x_P}{l_x}) \sin(\frac{\pi m y_P}{l_y}) \sin(\frac{\pi n x_M}{l_x}) \sin(\frac{\pi m y_M}{l_y}). \tag{2.5}$$

It can be shown [4] that such a potential corresponds to a Newtonian potential with an odd-periodic source distribution function ρ in $\mathbb{R}^3$.

In the domain of $D(-\infty, H-\varepsilon)$ the field of potential (2.5) has the form

$$\begin{aligned} \mathbf{E}(M) = \mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z = -\nabla v(M) = -\frac{8\pi}{l_x l_y} \int\limits_{\mathrm{supp}\varphi} dV_P \rho(P) \\ \times \sum_{n,m=1}^{\infty} e^{-k_{nm}(z_P - z_M)} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y} \Big(\mathbf{i}\frac{\pi n}{l_x k_{nm}} \cos\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \\ + \mathbf{j}\frac{\pi m}{l_y k_{nm}} \sin\frac{\pi n x_M}{l_x} \cos\frac{\pi m y_M}{l_y} + \mathbf{k} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}\Big). \end{aligned} \tag{2.6}$$

Thus, within the framework of model (2.2), if the density ρ is given, then the potential of density ρ and the potential field can be calculated using formulas (2.5) and (2.6), respectively.

Let us formulate the inverse problem. We assume that the source density ρ in problem (2.2) corresponds to a body of constant thickness h, located on the plane $z = H$:

$$\rho(x,y,z) = \sigma(x,y)\theta(z-H)\theta(H+h-z), \tag{2.7}$$

where $\theta(z)$ is the Heaviside function. According to (2.7), we consider the source distribution density functions as constants along the z axis and variables in the (x,y) plane inside the density carrier.

THE INVERSE PROBLEM. Let in the framework of model (2.2) on the surface

$$S=\{(x,y,z):0<x<l_x,0<y<l_y,z=F(x,y)<H\},\quad F\in C^2(\Pi), \tag{2.8}$$

$$\Pi=\{(x,y):0<x<l_x,0<y<l_y,\}\,. \tag{2.9}$$

the field $\mathbf{E}$ of form (2.6) of potential (2.5) be given as a vector function $\mathbf{E}^0$:

$$\mathbf{E}|_S=\mathbf{E}^0, \tag{2.10}$$

and the density ρ of form (2.7) is unknown. Let us set the problem of restoring the function ρ of form (2.7) for the field $\mathbf{E}^0$ given on S. Assuming that the parameters H and h are known, in fact, the inverse problem consists in reconstructing the function $\sigma(x,y)$ in (2.7) for the known function $\mathbf{E}^0$ on the surface S.

3 Reducing the inverse problem to an integral equation in the case of a flat surface S

Let us consider the $z-$component of a field (2.6) with a density (2.7) in the domain $D(-\infty,H-\varepsilon)$ of form (2.4). The value of ε is arbitrarily small and can be chosen so that the surface S of form (2.8) is located in the domain $D(-\infty,H-\varepsilon)$, that is, $\varepsilon<H-\max\limits_{(x,y)}F(x,y)$.

Given formula (2.7) for the density ρ, and also given that $z_M<z_P-\varepsilon$ if $M\in D(-\infty,H-\varepsilon)$, for the component E_z of field (2.6), we obtain

$$\begin{aligned}E_z(M)=&-\frac{8\pi}{l_xl_y}\int\limits_0^{l_x}\int\limits_0^{l_y}\sigma(x_P,y_P)\int\limits_H^{H+h}dz_P\sum_{n,m=1}^{\infty}e^{-k_{nm}(z_P-z_M)}\\&\times\sin\frac{\pi n x_P}{l_x}\sin\frac{\pi m y_P}{l_y}\sin\frac{\pi n x_M}{l_x}\sin\frac{\pi m y_M}{l_y}dx_Pdy_P\\&=\frac{16\pi}{l_xl_y}\int\limits_0^{l_x}\int\limits_0^{l_y}\sum_{n,m=1}^{\infty}e^{-k_{nm}(H+\frac{h}{2}-z_M)}\frac{\operatorname{sh}k_{nm}\frac{h}{2}}{k_{nm}}\sin\frac{\pi n x_M}{l_x}\sin\frac{\pi m y_M}{l_y}\\&\times\sigma(x,y)\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}dxdy=\int\limits_0^{l_x}\int\limits_0^{l_y}K_z(x_M,y_M,z_M,x,y)\sigma(x,y)dxdy,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}K_z(x_M,y_M,z_M,x,y)=&\frac{16\pi}{l_xl_y}\sum_{n,m=1}^{\infty}e^{-k_{nm}(H+\frac{h}{2}-z_M)}\frac{\operatorname{sh}k_{nm}\frac{h}{2}}{k_{nm}}\\&\times\sin\frac{\pi n x_M}{l_x}\sin\frac{\pi m y_M}{l_y}\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y},\quad k_{nm}=\sqrt{\left(\frac{\pi n}{l_x}\right)^2+\left(\frac{\pi m}{l_y}\right)^2}.\end{aligned} \tag{3.2}$$

So, if the function σ in (2.7) is known, then we obtain the component of the field E_z in form (3.1).

If now, in accordance with the inverse problem, the field $\mathbf{E}$, or only its component E_z on a flat surface (2.8) when $F(x,y)\equiv a<H$, is known, i.e. according to (2.10)

$$E_z\Big|_{z=a}=E_z^0,$$

from (3.1) we obtain an integral equation of the first kind, linear with respect to the desired function σ:

$$\int_0^{l_x}\int_0^{l_y} K(x_M, y_M, x, y)\sigma(x, y)dxdy = E_z^0(x_M, y_M), \quad (x_M, y_M) \in \Pi, \tag{3.3}$$

where the kernel of the integral operator according to representation (3.2) has the form

$$K(x_M, y_M, x, y) = K_z(x_M, y_M, a, x, y)$$
$$= \frac{16\pi}{l_x l_y} \sum_{n,m=1}^{\infty} e^{-k_{nm}(H+\frac{h}{2}-a)} \frac{\operatorname{sh} k_{nm}\frac{h}{2}}{k_{nm}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}. \tag{3.4}$$

We shall now obtain an equation similar to (3.3) in the case when the surface S has form (2.8) with the function F of general form.

4 Reducing the inverse problem to an integral equation in the case of a surface S of general form

We note that the z-component, like every component of field (2.6) of potential (2.5), is a harmonic function in the domain $D(-\infty, H)$. It also follows from (2.6) that the component E_z satisfies the conditions

$$E_z|_{x=0,l_x} = 0 \quad E_z|_{y=0,l_y} = 0,$$
$$E_z \to 0 \text{ when } z \to -\infty.$$

Taking into account condition (2.10) of the inverse problem for E_z of form (2.6), we obtain the problem

$$\begin{array}{l} \Delta E_z(M) = 0, \quad M \in D(-\infty, H), \\ E_z|_S = E_z^0, \\ E_z|_{x=0,l_x} = 0 \quad E_z|_{y=0,l_y} = 0, \\ E_z \to 0 \text{ when } z \to -\infty. \end{array} \tag{4.1}$$

If E_z^0 is z-component of field (2.6) on the surface S of form (2.8), then problem (4.1) is the Dirichlet problem in the domain

$$D(-\infty, F) = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, -\infty < z < F(x, y)\} \tag{4.2}$$

which has an unique solution, represented with formula (2.6).

From condition (2.10) of the inverse problem for field (2.6), an additional condition for the normal derivative on the surface S can be obtained. Indeed, field (2.6) is potential, and in the domain $D(-\infty, H)$ satisfies the equations

$$\operatorname{rot} \mathbf{E}(M) = 0, \quad M \in D(-\infty, H),$$
$$\operatorname{div} \mathbf{E}(M) = 0.$$

For the normal derivative of the component E_z on the surface S of form (2.8), given by the equation $z = F(x, y) < H$, we obtain

$$n_1 \frac{\partial E_z}{\partial n}\Big|_S = (\mathbf{n}_1, \nabla E_z)\big|_S = \left(\frac{\partial E_z}{\partial x} F'_x + \frac{\partial E_z}{\partial y} F'_y - \frac{\partial E_z}{\partial z}\right)\Big|_S,$$

where $\mathbf{n}_1 = (F'_x, F'_y, -1)$ is the inner normal with respect to the domain $D(-\infty, F)$ of form (4.2). Then, extracting from the equation $div\,\mathbf{E} = 0$, valid at the points of the surface $S \subset D(-\infty, H)$, the derivative with respect to the variable z, we obtain

$$n_1 \frac{\partial E_z}{\partial n}\Big|_S = (\mathbf{n}_1, \nabla E_z)\big|_S = (\frac{\partial E_z}{\partial x} F'_x + \frac{\partial E_z}{\partial y} F'_y + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y})\Big|_S. \tag{4.3}$$

Using the equations $rot\,\mathbf{E} = 0$ at the points of the surface $S \subset D(-\infty, H)$, namely

$$\frac{\partial E_z}{\partial x}\Big|_S = \frac{\partial E_x}{\partial z}\Big|_S, \quad \frac{\partial E_z}{\partial y}\Big|_S = \frac{\partial E_y}{\partial z}\Big|_S,$$

from (4.3) we obtain

$$n_1 \frac{\partial E_z}{\partial n}\Big|_S = (\frac{\partial E_x}{\partial z} F'_x + \frac{\partial E_y}{\partial z} F'_y + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y})\Big|_S. \tag{4.4}$$

We shall consider the field $\mathbf{E}^0$ in (2.10), given on S, as a function of the variables x and y on the rectangle Π of form (2.9). Differentiating the components of the field $\mathbf{E}^0$ by the arguments x and y, we obtain

$$\frac{\partial}{\partial x} E^0_x = \frac{\partial}{\partial x} E_x(x, y, F(x, y)) = (\frac{\partial E_x}{\partial x} + \frac{\partial E_x}{\partial z} F'_x)\Big|_S,$$

$$\frac{\partial}{\partial y} E^0_y = \frac{\partial}{\partial y} E_y(x, y, F(x, y)) = (\frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial z} F'_y)\Big|_S.$$

Substituting these derivatives in (4.4), we obtain the expression for the normal derivative in terms of the derivatives of the components of the vector $\mathbf{E}^0$:

$$n_1 \frac{\partial E_z}{\partial n}\Big|_S = (\frac{\partial E_x}{\partial x} + \frac{\partial E_x}{\partial z} F'_x + \frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial z} F'_y)\Big|_S = \frac{\partial}{\partial x} E^0_x + \frac{\partial}{\partial y} E^0_y. \tag{4.5}$$

If we add condition (4.5) to (4.1), then the component E_z of field (2.6) in the domain $D(-\infty, F) \subset D(-\infty, H)$ of form (4.2) is a solution of the problem

$$\begin{aligned} &\Delta E_z(M) = 0, \quad M \in D(-\infty, F), \\ &E_z|_S = E^0_z, \\ &\frac{\partial E_z}{\partial n}|_S = \frac{1}{n_1}(\frac{\partial E^0_x}{\partial x} + \frac{\partial E^0_y}{\partial y}), \quad \mathbf{n}_1 = (F_x, F_y, -1), \\ &E_z|_{x=0,l_x} = 0, \quad E_z|_{y=0,l_y} = 0, \\ &E_z \to 0 \text{ при } z \to -\infty, \end{aligned} \tag{4.6}$$

where the vector $\mathbf{E}^0 = (E^0_x, E^0_y, E^0_z)$ is field (2.10) in the formulation of the inverse problem.

We shall show now that, following the scheme in [6], the inverse problem can be reduced to an integral equation.

The source function $\varphi(M, P)$ of problem (2.2) can be represented as the sum of the fundamental solution and the function $W(M, P)$, harmonic in P:

$$\varphi(M, P) = \frac{1}{4\pi r_{MP}} + W(M, P), \tag{4.7}$$

where r_{MP} is the distance between points M and P. Let us put the point M in the domain

$$D(R, F) = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, R < z < F(x, y), R = Const < 0\}$$

and apply Green formula in the domain $D(R,F)$ to the solution of problem (4.6) $E_z(P)$ and to functions $(4\pi r_{MP})^{-1}$ and $W(M,P)$. Then we obtain

$$E_z(M) = \int\limits_{\partial D(R,F)} \Big[\frac{\partial E_z}{\partial n}(P)\frac{1}{4\pi r_{MP}} - E_z(P)\frac{\partial}{\partial n_P}\frac{1}{4\pi r_{MP}}(M,P)\Big] d\sigma_P,\ M \in D(R,F) \tag{4.8}$$

and

$$0 = \int\limits_{\partial D(R,F)} \Big[\frac{\partial E_z}{\partial n}(P)W(M,P) - E_z(P)\frac{\partial W}{\partial n_P}(M,P)\Big] d\sigma_P, \quad M \in D(R,F) \tag{4.9}$$

Here the normal is external to the domain $D(R,F)$. Summing (4.8) and (4.9) and taking into account (4.7), we obtain

$$E_z(M) = \int\limits_{\partial D(R,F)} \Big[\frac{\partial E_z}{\partial n}(P)\varphi(M,P) - E_z(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P, \quad M \in D(R,F).$$

Given the boundary conditions for E_z and φ in problems (4.6) and (2.2), as well as replacing the external normal with the internal one, we obtain the representation of the component of the field E_z as the sum of the surface integrals

$$\begin{aligned} E_z(M) = \int\limits_S \Big[-\frac{1}{n_1}\Big(\frac{\partial E_x^0}{\partial x}(P) + \frac{\partial E_y^0}{\partial y}(P)\Big)\varphi(M,P) + E_z^0(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P \\ - \int\limits_{\Pi(R)} \Big[\frac{\partial E_z}{\partial n_P}(P)\varphi(M,P) - E_z(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P, \quad M \in D(R,F), \end{aligned} \tag{4.10}$$

where the rectangle $\Pi(R)$ has the form

$$\Pi(R) = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, z = R\}, \quad R < \min_{(x,y)} F(x,y). \tag{4.11}$$

The integral over the rectangle $\Pi(R)$, due to the representation of field (2.6) and the representation of the source function for a fixed point $z_M > z_P = R$ in accordance with (2.3)

$$\varphi(M,P) = \frac{2}{\pi l_x l_y}\sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}(z_M - R)}}{k_{nm}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y},$$

converges to zero when $R \to -\infty$.

The integral over the surface S in (4.10) is reduced to the integral with respect to the variables x_P and y_P, given that $\frac{\partial \varphi}{\partial n}(M,P) = (\mathbf{n}, \nabla_P \varphi(M,P))$, $\mathbf{n} = \frac{\mathbf{n_1}}{n_1}$, $\mathbf{n_1} = (F'_x, F'_y, -1)$, and $d\sigma_P = n_1 dx_P dy_P$,

$$\begin{aligned} E_z(M) = \int\limits_0^{l_x}\int\limits_0^{l_y} \Big[-\Big(\frac{\partial E_x^0}{\partial x_P}(x_P,y_P) + \frac{\partial E_y^0}{\partial y_P}(x_P,y_P)\Big)\varphi(M,P) \\ + E_z^0(x_P,y_P)(\mathbf{n_1}, \nabla_P\varphi(M,P))\Big]_{P\in S} dx_P dy_P. \end{aligned}$$

Integrating by parts, taking into account the boundary conditions for φ, we obtain

$$E_z(M) = \int_0^{l_x}\int_0^{l_y} \Big[E_x^0(x_P,y_P)\frac{\partial}{\partial x_P}\varphi(M,P)\big|_{P\in S} + E_y^0(x_P,y_P)\frac{\partial}{\partial y_P}\varphi(M,P)\big|_{P\in S} + E_z^0(x_P,y_P)(\mathbf{n_1},\nabla_P\varphi(M,P))\big|_{P\in S}\Big]dx_Pdy_P. \quad (4.12)$$

Let us introduce the notation

$$\Phi(x_M,y_M) = E_z(M)\big|_{z_M=a}, \quad a < \min_{(x,y)} F(x,y), \quad (4.13)$$

where E_z is the function of form (4.12). Since the field $\mathbf{E}^0$ is given, Φ is a known function, and the source function $\varphi(M,P)$ for $M\in\Pi(a)$ of form (4.11) where $z=a$ and $P\in S$ of form (2.8) can be represented as an uniformly convergent series (2.3).

On the other hand, since E_z of form (4.12) is a component of field (2.6) of the potential, integral representation (3.1) is valid for E_z. Then, from integral representation (3.1) in order to determine the unknown density of σ, we obtain the Fredholm integral equation of the first kind with respect to the desired function σ, similar to (3.3)

$$\int_0^{l_x}\int_0^{l_y} K(x_M,y_M,x,y)\sigma(x,y)dxdy = \Phi(x_M,y_M), \quad (x_M,y_M)\in\Pi. \quad (4.14)$$

where the kernel of the integral operator has form (3.4) and the rectangle Π has form (2.9).

5 Exact solution of the inverse problem

When solving the inverse potential problem, we assume that the field $\mathbf{E}^0$ in (2.10) is field (2.6) on surface (2.8), so the solution of equation (4.14) exists in $L_2(\Pi)$. Since the system of eigenfunctions of the Dirichlet problem for the Laplace equation in the rectangle Π

$$\Big\{\sin\frac{\pi n x}{l_x}\Big\}\cdot\Big\{\sin\frac{\pi m y}{l_y}\Big\}\Big|_{n,m=1}^{n,m=\infty}$$

is complete, the kernel of integral equation (4.14) is closed and the equation has an unique solution.

The solution of integral equation (4.14) can be obtained as a Fourier series

$$\sigma(x,y) = \sum_{n,m=1}^{\infty}\tilde\sigma_{nm}\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y} = \sum_{n,m=1}^{\infty}\tilde\Phi_{nm}K_{nm}\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}, \quad (5.1)$$

where $\tilde\Phi_{nm}$ are the Fourier coefficients

$$\tilde\Phi_{nm} = \frac{4}{l_x l_y}\int_0^{l_x}\int_0^{l_y}\Phi(x,y)\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}dxdy \quad (5.2)$$

of the function Φ of form (4.13), and

$$K_{nm} = e^{k_{nm}(H+\frac{h}{2}-a)}\frac{k_{nm}}{4\pi\,\mathrm{sh}\,k_{nm}\frac{h}{2}}, \quad k_{nm} = \sqrt{\Big(\frac{\pi n}{l_x}\Big)^2 + \Big(\frac{\pi m}{l_y}\Big)^2}. \quad (5.3)$$

Since, when solving equation (4.14), we consider that the function Φ of form (4.13) corresponds to the density σ of form (2.7), the coefficients $\tilde{\Phi}_{nm}(a) = \sigma_{nm}/K_{nm}$ decrease faster than the value $e^{k_{nm}(H-a)}k_{nm}$ increases and series (5.1) converges to σ in $L_2(\Pi)$.

In the case when $\sigma(M) = \sigma_0\chi_D(M)$, where $\chi_D(M)$ is the characteristic function of some domain $D \subset \Pi$ and σ_0 is a known constant, the solution of the inverse problem is reduced to finding the support D of the source density function. To do this, we can use the formula

$$D = \{(x,y) \in \Pi : \frac{1}{\sigma_0}\sigma(x,y) > \lambda = Const,\ 0 < \lambda < 1\}. \tag{5.4}$$

As it is known [10, 11], the Fredholm equation of the first kind is an ill-posed problem. Its approximate solution is unstable with respect to the error of the right part and requires the use of regularizing algorithms. Let us construct an approximate right-hand side of the integral equation in the case of an inaccurate data on the field $\mathbf{E}^0$ and the surface S and estimate its error.

6 Approximate calculation of the normal to an inaccurately defined surface

As follows from (4.13), (4.12), when forming the right-hand side of integral equation (4.14), it is necessary to calculate the vector function of the normal $\mathbf{n}_1$ to the surface S of form (2.8), which is the gradient of the function $F(x,y) - z$,

$$\mathbf{n}_1 = grad\,(F(x,y) - z) = \nabla_{xy}F - \mathbf{k}. \tag{6.1}$$

Let the surface S is given with an error, namely, instead of the exact function F in (2.8), the function F^μ is known, given on a rectangle Π of form (2.9), such that

$$\|F^\mu - F\|_{L_2(\Pi)} \leqslant \mu. \tag{6.2}$$

For the approximate calculation of integral (4.12), it is necessary to calculate the normal to the surface given approximately, which is also an ill-posed problem, since the calculation of the normal $\mathbf{n}_1$ is associated with the calculation of the derivatives of the function F.

To obtain a stable solution to this problem, we use the approach of [7], that is, we consider the problem of calculating the gradient of a function as the problem of calculating values of an unbounded operator [2].

As an approximation to the function $\nabla_{xy}F$ in (6.1) calculated from the known function F^μ, associated with the function F by condition (6.2), we consider the gradient of the extremal of the functional

$$N^\beta[W] = \left\|W - F^\mu\right\|^2_{L_2(\Pi)} + \beta\left\|\nabla W\right\|^2_{L_2(\Pi)}, \quad \beta > 0. \tag{6.3}$$

For simplicity of calculating the extremal, we consider such surfaces S, for which

$$F|_{x=0,l_x} = 0, \quad F|_{y=0,l_y} = 0.$$

This condition, in particular, occurs in the case when S can be considered as a perturbation of the plane $z = 0$. Then the extremal of functional (6.3) is the solution of the following problem for the Euler equation

$$-\beta\Delta W + W = F^\mu,$$
$$W|_{x=0,l_x} = 0,\ W|_{y=0,l_y} = 0.$$

The solution of this problem is

$$W^\mu_\beta(x,y) = \sum_{n,m=1}^{\infty} \frac{\tilde F^\mu_{nm}}{1+\beta k^2_{nm}} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \tag{6.4}$$

where the Fourier coefficients $\tilde F^\mu_{nm}$ are calculated by formulas of form (5.2) and k_{nm} has form (5.3). It is easy to see that series (6.4) converges uniformly on Π.

As an approximate value of the gradient of the function F^μ, we consider the vector function

$$\begin{aligned}\nabla_{xy} W^\mu_\beta(x,y) = &\sum_{n,m=1}^{\infty} \frac{\tilde F^\mu_{nm}}{1+\beta k^2_{nm}} \\ &\times\left(\mathbf{i}\frac{\pi n}{l_x}\cos\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y} + \mathbf{j}\frac{\pi m}{l_y}\cos\frac{\pi m y}{l_y}\sin\frac{\pi n x}{l_x}\right).\end{aligned} \tag{6.5}$$

Series (6.5) converges in $L_2(\Pi)$.

Let F^- be an odd-periodic continuation of the function F, given on the rectangle Π of form (2.9), with a period of $2l_x$ for the variable x and with a period of $2l_y$ for the variable y, i.e.

$$\begin{aligned}&F^-(x,y) = F(x,y), \quad (x,y)\in\Pi,\\ &F^-(-x,y) = -F(x,y), \quad (x,y)\in\Pi,\\ &F^-(x,-y) = -F(x,y), \quad (x,y)\in\Pi,\\ &F^-(-x,-y) = F(x,y), \quad (x,y)\in\Pi,\\ &F^-(x+2l_x n, y+2l_y m) = F^-(x,y), \quad (x,y)\in\mathbb{R}^2, \quad n,m=\pm1,\pm2,\ldots.\end{aligned}$$

Theorem 6.1. [7] *Let* $F^- \in C^2(\mathbb{R}^2)$, $\beta = \beta(\mu) > 0$, $\beta(\mu)\to 0$ *and* $\mu/\sqrt{\beta(\mu)} \to 0$ *when* $\mu \to 0$. *Then*

$$\left\| \nabla_{xy} W^\mu_{\beta(\mu)} - \nabla_{xy} F \right\|_{L_2(\Pi)} \leqslant \frac{\mu}{2\sqrt{\beta}} + \frac{\sqrt{\beta}}{2}\left\|\Delta F\right\|_{L_2(\Pi)} \to 0 \text{ when } \mu\to 0.$$

Based on the theorem, we can use formula (6.5) to approximate the normal to the surface using formula (6.1):

$$\mathbf{n}^\mu_{1,\beta} = \nabla_{xy} W^\mu_\beta - \mathbf{k}. \tag{6.6}$$

With a known estimate

$$\left\|\Delta F\right\|_{L_2(\Pi)} \leqslant M,$$

it follows from the statement of the theorem that

$$\left\|\mathbf{n}^\mu_{1,\beta} - \mathbf{n}_1\right\|_{L_2(\Pi)} = \left\|\nabla_{xy} W^\mu_\beta - \nabla_{xy} F\right\|_{L_2(\Pi)} \leqslant \frac{\mu}{2\sqrt{\beta}} + \frac{\sqrt{\beta}}{2} M.$$

The maximum for the β expression on the right is achieved when

$$\beta(\mu) = \frac{\mu}{M} \tag{6.7}$$

and, thus denoting in accordance with (6.6) and (6.7)

$$\mathbf{n}^\mu_1 = \mathbf{n}^\mu_{1,\beta(\mu)} = \nabla_{xy} W^\mu_{\beta(\mu)} - \mathbf{k}, \tag{6.8}$$

we shall obtain:

$$\left\|\mathbf{n}^\mu_1 - \mathbf{n}_1\right\|_{L_2(\Pi)} \leqslant \sqrt{M\mu} \xrightarrow[\mu\to 0]{} 0. \tag{6.9}$$

It is also not difficult to obtain the estimate

$$\left\| W^\mu_{\beta(\mu)} - F\right\|_{L_2(\Pi)} \leqslant 2\mu. \tag{6.10}$$

The surface defined by the equation $z = W^\mu_{\beta(\mu)}(x,y)$, we denote as

$$S^\mu = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, z = W^\mu_{\beta(\mu)}(x,y)\}. \tag{6.11}$$

7 Solution of the inverse problem in the case of an approximately given field $\mathbf{E}^0$ on an approximately given surface

Let instead of the exact vector function $\mathbf{E}^0$ in condition (2.10) of the inverse problem, the function $\mathbf{E}^{0,\delta} = (E_x^{0,\delta}, E_y^{0,\delta}, E_z^{0,\delta})$ is known, given as a function on the rectangle Π of form (2.9), such that

$$\|\mathbf{E}^{0,\delta} - \mathbf{E}^0\|_{L_2(\Pi)} \leqslant \delta. \tag{7.1}$$

In this case, we assume that the surface S of form (2.8) is given approximately by condition (6.2).

We assume that we also know that

$$a_1 < F(x,y) < a_2. \tag{7.2}$$

In this case using the results of the previous paragraph, the right part $\Phi(M)$ of form (4.13) in integral equation (4.14) will be calculated approximately on a rectangle

$$\Pi(a) = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, z = a\}, \quad a < \min_{(x,y)} W^{\mu}_{\beta(\mu)}(x,y), \quad a < a_1 \tag{7.3}$$

in accordance with formula (4.12) and (4.13) as a function

$$E_z^{\delta,\mu}(M) = \int_0^{l_x}\int_0^{l_y} [E_x^{0,\delta}(x_P,y_P)\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S^\mu} + E_y^{0,\delta}(x_P,y_P)\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S^\mu}$$
$$+ E_z^{0,\delta}(x_P,y_P)(\mathbf{n}_1^\mu, \nabla_P\varphi(M,P))|_{P\in S^\mu}]dx_P dy_P, \quad M \in \Pi(a), \tag{7.4}$$

where the surface S^μ has form (6.11), the approximate normal $\mathbf{n}_1^\mu$ is calculated by formula (6.8) and the function

$$\varphi(M,P) = \frac{2}{l_x l_y}\sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}(z_P-a)}}{k_{nm}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y}$$

is source function (2.3) of problem (2.2).

Let us estimate the error in calculating the function $E_z^{\delta,\mu}$ of form (7.4) with respect to the function E_z of form (4.12) on the rectangle $\Pi(a)$ – the right-hand side of integral equation (4.14), i.e. we estimate the difference

$$\left|E_z^{\delta,\mu}(M) - E_z(M)\right| \leqslant \left|E_z^{\delta,\mu}(M) - E_z^{\delta,\mu,1}(M)\right| + \left|E_z^{\delta,\mu,1}(M) - E_z^{\delta}(M)\right|$$
$$+ \left|E_z^{\delta}(M) - E_z(M)\right|, \quad M \in \Pi(a). \tag{7.5}$$

where $\Pi(a)$ has form (7.3). In this estimate the function $E_z^{\delta,\mu,1}$ of form (7.4) is introduced, where formally the approximate normal $\mathbf{n}_1^\mu$ is replaced by the exact normal $\mathbf{n}_1$ (note that $\mathbf{n}_1(x_P,y_P)|_{P\in S^\mu} = \mathbf{n}_1(x_P,y_P)|_{P\in S}$):

$$E_z^{\delta,\mu,1}(M) = \int_0^{l_x}\int_0^{l_y} [E_x^{0,\delta}(x_P,y_P)\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S^\mu} + E_y^{0,\delta}(x_P,y_P)\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S^\mu}$$
$$+ E_z^{0,\delta}(x_P,y_P)(\mathbf{n}_1, \nabla_P\varphi(M,P))|_{P\in S^\mu}]dx_P dy_P, \quad \mathbf{n}_1 = (F_x', F_y', -1), \tag{7.6}$$

and is also introduced the function E_z^δ of form (7.4), which is calculated on an exactly specified surface

$$E_z^\delta(M) = \int\limits_0^{l_x}\int\limits_0^{l_y}[E_x^{0,\delta}(x_P,y_P)\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S} + E_y^{0,\delta}(x_P,y_P)\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S}$$
$$+ E_z^{0,\delta}(x_P,y_P)(\mathbf{n}_1,\nabla_P\varphi(M,P))|_{P\in S}]dx_Pdy_P, \quad \mathbf{n}_1 = (F_x^{'},F_y^{'},-1). \quad (7.7)$$

Let us estimate the difference between functions (7.4) and (7.6) in the right-hand side of inequality (7.5):

$$\Big|\, E_z^{\delta,\mu}(M) - E_z^{\delta,\mu,1}(M)\,\Big|_{M\in\Pi(a)}$$
$$= \Big|\int\limits_0^{l_x}\int\limits_0^{l_y} E_z^{0,\delta}(x_P,y_P)((\mathbf{n}_1^\mu - \mathbf{n}_1),\nabla_P\varphi(M,P))|_{P\in S^\mu}dx_Pdy_P\Big|$$
$$\leqslant \int\limits_0^{l_x}\int\limits_0^{l_y}\Big[\,\big|\,E_z^{0,\delta}(x_P,y_P)\,\big|\cdot\big|\,\mathbf{n}_1^\mu(P) - \mathbf{n}_1(P)\,\big|\cdot\big|\,\nabla_P\varphi(M,P)\,\big|\,\Big]_{P\in S^\mu}dx_Pdy_P$$
$$\leqslant \max_{\substack{M\in\Pi(a)\\ P\in S^\mu}}\big|\,\nabla_P\varphi(M,P)\,\big|\int\limits_0^{l_x}\int\limits_0^{l_y}\big|\,E_z^{0,\delta}(x_P,y_P)\,\big|\cdot\big|\,\mathbf{n}_1^\mu(P) - \mathbf{n}_1(P)\,\big|_{P\in S^\mu}dx_Pdy_P.$$

Using the Cauchy-Bunyakovsky inequality, estimate (6.9) and estimate $\|E_z^{0,\delta}\| \leqslant \|\mathbf{E}^0\| + \delta$, we obtain

$$\Big|\, E_z^{\delta,\mu}(M) - E_z^{\delta,\mu,1}(M)\,\Big|_{M\in\Pi(a)} = \max_{\substack{M\in\Pi(a)\\ P\in S^\mu}}\big|\,\nabla_P\varphi(M,P)\,\big|\,\|E_z^{0,\delta}\|\cdot\|\mathbf{n}_1^\mu - \mathbf{n}_1\|$$
$$\leqslant \max_{\substack{M\in\Pi(a)\\ P\in S^\mu}}\big|\,\nabla_P\varphi(M,P)\,\big|\,(\|\mathbf{E}^0\| + \delta)\cdot\sqrt{M\mu} \leqslant C_1\sqrt{\mu}. \quad (7.8)$$

Let us estimate the difference between functions (7.6) and (7.7) in the right-hand side of inequality (7.5) using the Lagrange formula

$$\Big|\, E_z^{\delta,\mu,1}(M) - E_z^\delta(M)\,\Big|_{M\in\Pi(a)}$$
$$= \Big|\int\limits_0^{l_x}\int\limits_0^{l_y}\Big[E_x^{0,\delta}(x_P,y_P)\Big(\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S^\mu} - \frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S}\Big)$$
$$+ E_y^{0,\delta}(x_P,y_P)\Big(\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S^\mu} - \frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S}\Big)$$
$$+ E_z^{0,\delta}(x_P,y_P)\big(\mathbf{n}_1,\nabla_P\varphi(M,P)|_{P\in S^\mu} - \nabla_P\varphi(M,P)|_{P\in S}\big)\Big]dx_Pdy_P\,\Big|$$
$$= \Big|\int\limits_0^{l_x}\int\limits_0^{l_y}\Big[E_x^{0,\delta}(x_P,y_P)\Big(\frac{\partial^2}{\partial x_P z_P}\varphi(M,P_1)(z_P\big|_{P\in S^\mu} - z_P\big|_{P\in S})\Big)$$
$$+ E_y^{0,\delta}(x_P,y_P)\Big(\frac{\partial^2}{\partial y_P z_P}\varphi(M,P_2)(z_P\big|_{P\in S^\mu} - z_P\big|_{P\in S})\Big)$$
$$+ E_z^{0,\delta}(x_P,y_P)\big(\mathbf{n}_1,\frac{\partial}{\partial z_P}\nabla_P\varphi(M,P_3)\big)(z_P\big|_{P\in S^\mu} - z_P\big|_{P\in S})\Big]dx_Pdy_P\,\Big|, \quad M\in\Pi(a).$$

Since according to (6.11) $z_P\Big|_{P\in S^\mu} = W^\mu_{\beta(\mu)}(x_P,y_P)$ and $z_P\Big|_{P\in S} = F(x_P,y_P)$, we obtain

$$\Big| E_z^{\delta,\mu,1}(M) - E_z^\delta(M) \Big|_{M\in\Pi(a)}$$
$$= \Big| \int_0^{l_x}\int_0^{l_y} \Big[E_x^{0,\delta}(x_P,y_P)\Big(\frac{\partial^2}{\partial x_P z_P}\varphi(M,P_1)(W^\mu_{\beta(\mu)}(x_P,y_P) - F(x_P,y_P))\Big)$$
$$+ E_y^{0,\delta}(x_P,y_P)\Big(\frac{\partial^2}{\partial y_P z_P}\varphi(M,P_2)(W^\mu_{\beta(\mu)}(x_P,y_P) - F(x_P,y_P))\Big)$$
$$+ E_z^{0,\delta}(x_P,y_P)\big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P\varphi(M,P_3)\big)(W^\mu_{\beta(\mu)}(x_P,y_P) - F(x_P,y_P))\Big] dx_P dy_P \Big|. \quad (7.9)$$

We introduce the following notation using (7.2)

$$\begin{aligned} z_1(x_P,y_P) &= \min\{W^\mu_{\beta(\mu)}(x_P,y_P), a_1\},\\ z_2(x_P,y_P) &= \max\{W^\mu_{\beta(\mu)}(x_P,y_P), a_2\}. \end{aligned} \quad (7.10)$$

Now from (7.9) using (7.10) we obtain

$$\Big| E_z^{\delta,\mu,1}(M) - E_z^\delta(M) \Big|_{M\in\Pi(a)}$$
$$\leqslant \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big|\frac{\partial^2}{\partial x_P z_P}\varphi(M,P)\Big| \int_0^{l_x}\int_0^{l_y} |E_x^{0,\delta}(x,y)\Big| \cdot \Big| W^\mu_{\beta(\mu)}(x,y) - F(x,y)\Big| dxdy$$
$$+ \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big|\frac{\partial^2}{\partial y_P z_P}\varphi(M,P)\Big| \int_0^{l_x}\int_0^{l_y} |E_y^{0,\delta}(x,y)| \cdot \Big| W^\mu_{\beta(\mu)}(x,y) - F(x,y)\Big| dxdy$$
$$+ \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big|\big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P\varphi(M,P)\big)\Big| \int_0^{l_x}\int_0^{l_y} |E_z^{0,\delta}(x,y)| \cdot \Big| W^\mu_{\beta(\mu)}(x,y) - F(x,y)\Big| dxdy.$$

Applying the Cauchy-Bunyakovsky inequality, assuming that $\delta < \delta_0$, and using the estimate (6.10), we obtain

$$\Big| E_z^{\delta,\mu,1}(M) - E_z^\delta(M) \Big|_{M\in\Pi(a)} = \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big|\frac{\partial^2}{\partial x_P z_P}\varphi(M,P)\Big| \|E_x^{0,\delta}\| \cdot \|W^\mu_{\beta(\mu)} - F\|$$
$$+ \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big|\frac{\partial^2}{\partial y_P z_P}\varphi(M,P)\Big| \|E_y^{0,\delta}\| \cdot \|W^\mu_{\beta(\mu)} - F\|$$
$$+ \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big|\big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P\varphi(M,P)\big)\Big| \|E_z^{0,\delta}\| \cdot \|W^\mu_{\beta(\mu)} - F\|$$
$$\leqslant C\|\mathbf{E}^{0,\delta}\|\mu \leqslant C(\|\mathbf{E}^0\|) + \delta)\mu \leqslant C_2\mu. \quad (7.11)$$

Let us estimate the difference between functions (7.7) and (4.12) in the right-hand side of inequality (7.5):

$$\Big| E_z^\delta(M) - E_z(M) \Big|_{M\in\Pi(a)} = \Big| \int\limits_0^{l_x}\int\limits_0^{l_y} \Big[\big(E_x^{0,\delta}(x_P,y_P) - E_x^0(x_P,y_P)\big)\Big(\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S}\Big)$$
$$+ \big(E_y^{0,\delta}(x_P,y_P) - E_y^0(x_P,y_P)\big)\Big(\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S}\Big)$$
$$+ \big(E_z^{0,\delta}(x_P,y_P) - E_z^0(x_P,y_P)\big)\big(\mathbf{n}_1, \nabla_P\varphi(M,P)\big)_{P\in S}\Big] dx_P dy_P \Big| .$$

Using the Cauchy-Bunyakovsky inequality, as well as (7.1), we obtain from here

$$\Big| E_z^\delta(M) - E_z(M) \Big| = \max_{\substack{M\in\Pi(a)\\ P\in S}} \Big| \frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S} \Big| \int\limits_0^{l_x}\int\limits_0^{l_y} |E_x^{0,\delta}(x,y) - E_x^0(x,y)| dxdy$$
$$+ \max_{\substack{M\in\Pi(a)\\ P\in S}} \Big| \frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S} \Big| \int\limits_0^{l_x}\int\limits_0^{l_y} |E_y^{0,\delta}(x,y) - E_y^0(x,y)| dxdy$$
$$+ \max_{\substack{M\in\Pi(a)\\ P\in S}} \Big| (\mathbf{n}_1, \nabla_P\varphi(M,P))_{P\in S} \Big| \int\limits_0^{l_x}\int\limits_0^{l_y} |E_z^{0,\delta}(x,y) - E_z^0(x,y)| dxdy$$
$$\leqslant C_3\|\mathbf{E}^{0,\delta} - \mathbf{E}^0\| \leqslant C_3\delta, \quad M\in\Pi(a). \quad (7.12)$$

Collecting estimates (7.8), (7.11), (7.12) and assuming that $\mu < \mu_0$, from (7.5) we obtain

$$\big| E_z^{\delta,\mu}(M) - E_z(M) \big|_{M\in\Pi(a)} \leqslant C_1\sqrt{\mu} + C_2\mu + C_3\delta \leqslant C_4\sqrt{\mu} + C_3\delta. \quad (7.13)$$

Denoting, similarly to (4.13), the approximate right-hand side of integral equation (4.14)

$$\Phi^{\delta,\mu}(x_M, y_M) = E_z^{\delta,\mu}(M)\big|_{M\in\Pi(a)}, \quad (7.14)$$

from (7.13) we obtain an estimate in L_2 of the error of the approximate right-hand side of integral equation (4.14)

$$\big\|\Phi^{\delta,\mu} - \Phi\big\|_{L_2(\Pi)} \leqslant \overline{C}_4\sqrt{\mu} + \overline{C}_3\delta = \gamma(\mu,\delta) \xrightarrow[\substack{\mu\to 0\\ \delta\to 0}]{} 0, \quad (7.15)$$

where $\overline{C}_4, \overline{C}_3$ are some constants.

Let us now construct an approximate solution of integral equation (4.14) with right-hand side (7.14) by the Tikhonov regularization method [10, 11]. As an approximate solution, we consider the extremal of the Tikhonov functional

$$M[w] = \big\|Kw - \Phi^{\delta,\mu}\big\|^2_{L_2(\Pi)} + \alpha\,\|w\|^2_{L_2(\Pi)}, \quad \alpha > 0, \quad (7.16)$$

where K is the integral operator in (4.14). The extremal $\sigma_\alpha^{\delta,\mu}$ can be obtained as a solution of the Euler equation

$$K^*Kw + \alpha w = K^*\Phi^{\delta,\mu}$$

for functional (7.16) and has the form

$$\sigma_\alpha^{\delta,\mu}(x,y) = \sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}^{\delta,\mu} K_{nm}}{1 + \alpha K_{nm}^2} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \quad \alpha > 0. \quad (7.17)$$

Here $\tilde{\Phi}^{\delta,\mu}_{nm}$ are the Fourier coefficients

$$\tilde{\Phi}^{\delta,\mu}_{nm} = \frac{4}{l_x l_y} \int\limits_0^{l_x} \int\limits_0^{l_y} \Phi^{\delta,\mu}(x,y) \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y} dxdy \tag{7.18}$$

of the function $\Phi^{\delta,\mu}$ of form (7.14). The value K_{nm} in formula (7.17) has form (5.3).

Let us note that for $\delta = 0$, $\mu = 0$ and $\alpha = 0$, formula (7.17) turns into an explicit representation of exact solution (5.1). When $\delta > 0$, $\mu > 0$ and $\alpha = 0$, (7.17), generally speaking, may diverge in accordance with the fact that the inverse problem is ill-posed. For $\delta > 0$, $\mu > 0$ and $\alpha > 0$, the convergence is provided by the regularizing factor $(1+\alpha K^2_{nm})^{-1}$.

The following theorem proves the convergence of approximate solution (7.17) in $L_2(\Pi)$ to the exact solution of the integral equation.

Theorem 7.1. *For any $\alpha = \alpha(\gamma) > 0$ such that $\alpha(\gamma) \to 0$, $\gamma/\sqrt{\alpha(\gamma)} \to 0$ when $\gamma \to 0$, the function $\sigma^{\delta,\mu}_{\alpha(\gamma)}$ of form (7.17), where according to (7.15) $\gamma = \gamma(\mu,\delta) = \overline{C}_4\sqrt{\mu} + \overline{C}_3\delta$, converges to the exact solution of integral equation (4.14) in $L_2(\Pi)$ when $\delta \to 0$, $\mu \to 0$.*

Proof. Following the general scheme [2] of estimating an approximate solution of a linear integral equation, introducing a function σ_α of form (7.17) when $\delta = 0, \mu = 0$, we obtain

$$\|\sigma^{\delta,\mu}_\alpha - \sigma\|_{L_2} \leqslant \|\sigma^{\delta,\mu}_\alpha - \sigma_\alpha\|_{L_2} + \|\sigma_\alpha - \sigma\|_{L_2}. \tag{7.19}$$

To estimate the first difference in the right-hand side of inequality (7.19), we use estimate (7.15)

$$\|\sigma^{\delta,\mu}_\alpha - \sigma_\alpha\|_{L_2} \leqslant \Big[\frac{l_x l_y}{4} \sum_{n,m=1}^{\infty} \big(\frac{K_{nm}}{1+\alpha K^2_{nm}}\big)^2 |\tilde{\Phi}^{\delta,\mu}_{nm} - \tilde{\Phi}_{nm}|^2\Big]^{1/2} \leqslant \max_x \big(\frac{x}{1+\alpha x^2}\big) \|\Phi^{\delta,\mu} - \Phi\|_{L_2} \leqslant \frac{\gamma}{2\sqrt{\alpha(\gamma)}}. \tag{7.20}$$

We estimate the second difference in the right-hand side of inequality (7.19). We note that according to (5.1) $\tilde{\Phi}_{nm}K_{nm} = \tilde{\sigma}_{nm}$, so we obtain

$$\|\sigma_\alpha - \sigma\|_{L_2} \leqslant \Big[\frac{l_x l_y}{4} \sum_{n,m=1}^{\infty} \big(\frac{\alpha K^2_{nm}}{1+\alpha K^2_{nm}}\big)^2 |\tilde{\Phi}_{nm} K_{nm}|^2\Big]^{1/2} = \Big[\frac{l_x l_y}{4} \sum_{n,m=1}^{\infty} \big(\frac{\alpha K^2_{nm}}{1+\alpha K^2_{nm}}\big)^2 \tilde{\sigma}^2_{nm}\Big]^{1/2}.$$

Since the series depending on the parameter α is majorized by a converging numerical series with coefficients $\tilde{\sigma}^2_{nm}$ it is possible to pass to the limit in α, and hence

$$\|\sigma_\alpha - \sigma\|_{L_2} \to 0, \quad \text{when} \quad \alpha \to 0. \tag{7.21}$$

It follows from (7.19), (7.20), (7.21), and the assumptions of the theorem that

$$\|\sigma^{\delta,\mu}_{\alpha(\gamma)} - \sigma\|_{L_2} \to 0, \quad \text{when} \quad \delta \to 0, \mu \to 0.$$

□

In the case when $\sigma(M) = \sigma_0\chi_D(M)$, where $\chi_D(M)$ is the characteristic function of the domain D in accordance with (5.4), we construct an approximation $D_\lambda^{\delta,\mu}$ to the support D of the density σ based on the approximate density function of sources (7.17)

$$D_\lambda^{\delta,\mu} = \{(x,y) \in \Pi : \frac{1}{\sigma_0}\sigma_{\alpha(\gamma)}^{\delta,\mu}(x,y) > \lambda = Const,\ 0 < \lambda < 1\}. \tag{7.22}$$

A criterion for the quality of the approximation can be the measure of the symmetric difference between domain (7.22) and the domain D of form (5.4).

Theorem 7.2. *Under the conditions of Theorem* 7.1 *the measure of the symmetric difference* $\boldsymbol{mes}(D_\lambda^{\delta,\mu}\Delta D) \to 0$ *when* $\delta \to 0$, $\mu \to 0$.

Proof. It follows from theorem 7.1,

$$\|\frac{1}{\sigma_0}\sigma_\alpha^{\delta,\mu} - \chi_D\|_{L_2(\Pi)} \to 0 \quad \text{when} \quad \delta \to 0,\ \mu \to 0.$$

From the convergence of $\frac{1}{\sigma_0}\sigma_\alpha^\delta$ to χ_D in L_2, the convergence in measure follows (see [3]). Further, the proof repeats verbatim the proof of the theorem in [1]. □

Formulas (7.4), (7.14), (7.17), (7.18), (7.22) give the solution to the inverse problem.

8 Conclusions

The inverse problem of the Newtonian potential for bodies of constant thickness is posed and solved in the case when the potential field on the surface of the general form is known. In this case, the density function of the distribution of potential sources is found as an approximate regularized solution of the linear integral Fredholm equation of the first kind, which is stable both with respect to the error in setting the potential and to the error in the surface.

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References

[1] N.Y. Chernikova, E.B. Laneev, M.N. Muratov, E.Y. Ponomarenko, *On an inverse problem to a mixed problem for the Poisson equation.* In: Pinelas S., Kim A., Vlasov V. (eds) Mathematical Analysis With Applications. CONCORD-90 2018. Springer Proceedings in Mathematics & Statistics, 318 (2020), 141-146. Springer, Cham. https://doi.org/10.1007/978-3-030-42176-2_14

[2] V.K. Ivanov, V.V. Vasin, V.P. Tanana, *The theory of linear ill-posed problems and its applications.* Nauka, Moscow, (1978) (in Russian).

[3] A.N. Kolmogorov, S.V. Fomin, *Elements of function theory and functional analysis.* Nauka, Moscow, (1972) (in Russian).

[4] E.B. Laneev, *On some statements of the continuation problem the potential field.* Bulletin of the RUDN. Physics Series. 8 (2000), no. 1, 21-28 (in Russian).

[5] E.B. Laneev, *On the error of the periodic model of the problem of continuation of the potential field.* Bulletin of the RUDN. Physics Series. 9 (2001), no. 1, 4-16 (in Russian).

[6] E.B. Laneev, *Construction of a Carleman function based on the Tikhonov regularization method in an ill-posed problem for the Laplace equation*, Differential Equations, 54 (2018), no. 4, 476–485 doi: 10.1134/S0012266118040055

[7] E.B. Laneev, M.N. Muratov, *On the stable solution of a mixed boundary value problem for the Laplace equation with an approximately given boundary.* Bulletin of the RUDN. Mathematics Series. 9 (2002), no. 1, 102-111 (in Russian).

[8] A.I. Prilepko, *Inverse problems of potential theory.* Math. Notes, 14 (1973), no. 5, 990–996.

[9] L.N. Sretensky, *On the uniqueness of determining the shape of an attracting body by the values of its external potential.* Dokl. Akad. Nauk SSSR. 99 (1954), no. 1, 21-22 (in Russian).

[10] A.N. Tikhonov, V.Ya. Arsenin, *Methods for solving ill-posed problems.* Nauka, Moscow, 1986 (in Russian).

[11] A.G. Yagola, Wang Yanfei, I.E. Stepanova, V.N. Titarenko, *Inverse problems and methods of their solution, Applications to geophysics.* BINOM. Knowledge Lab, Moscow, 2014 (in Russian).

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APPROXIMATE SOLUTIONS OF THE SWIFT-HOHENBERG EQUATION WITH DISPERSION

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Key words: differential transform method, reduced differential transform method, Swift-Hohenberg equation.

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Abstract. In this paper, the initial and boundary value problems for the Swift-Hohenberg equation as over the finite spatial interval $x \in [0, l]$ and finite time interval $t \in [0, t^*]$ are considered. Approximate solutions for the initial and boundary value problems are obtained via the differential transform method and reduced differential transform method. Finally, several numerical examples are presented in order to demonstrate the effectivity of the methods and clarify the influence of the parameters on the solution.

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1 Introduction

The Swift-Hohenberg equation is a model pattern-forming equation which was introduced by Jack Swift and Pierre Hohenberg as a model for a fluid which is thermally convecting [24]. The Swift-Hohenberg equation is one of important equations for describing localized structures in the modern physics. This equation occurs in fluid dynamics, optical physics and other fields [4, 11, 22]. The Swift-Hohenberg equation with dispersion has the form [9]

$$u_t + 2u_{xx} - \sigma u_{xxx} + u_{xxxx} = \alpha u + \beta u^2 - \gamma u^3, \tag{1.1}$$

where α, β, γ and σ are parameters of the equation. At $\sigma = 0$ equation (1.1) is reduced to the standard Swift-Hohenberg equation. We consider the problem with the boundary conditions

$$\begin{aligned} &u = 0, \quad u_{xx} = 0, \text{ at } x = 0, l, \ \forall t, \, t > 0, \\ &u(x, 0) = u_0(x), \ \forall x, \ \ 0 < x < l, \end{aligned} \tag{1.2}$$

so that solutions can be extended as periodic functions over the real line. For $\sigma = \beta = 0$ and $\alpha = 1 - a$, $a \in \mathbb{R}$, equation (1.1) and (1.2) were solved by the homotopy analysis method in [3] and the differential transform method as time-fractional derivative in [19].

The aim of this paper is to find an approximate analytical solution of (1.1) and (1.2) with the help of powerful analytic methods. We use the differential transform method (DTM) and reduced differential transform method (RDTM) to obtain the solutions and compare them with each other. We know that the DTM is based on the use of Taylor series in all variables, while RDTM does not require Taylor series in all variables and therefore it reduces significantly the numerical computation. For the standard cases, comparing the methodology with some known techniques, shows that these approaches are effective and powerful.

2 Methods

In this section, the techniques are explained for the two-dimensional differential transform.

2.1 The DTM

The DTM was first proposed by Zhou [25], who solved linear and nonlinear initial value problems in electric circuit analysis, then was widely used in the literature and was successfully applied to fractional differential equations [5], integro-differential equations [6], higher-order initial value problems [1], systems of differential equations [2, 7, 12], partial differential equation [10, 13, 21, 23], high index differential-algebraic equations [20].

In [8, 14] the basic definitions and fundamental operations are introduced for the two-dimensional differential transform as the following

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} u(x,t)\right]_{(0,0)}, \tag{2.1}$$

where $u(x,t)$ is the original function and $U(k,h)$ is the transformed function. The differential inverse transform of $U(k,h)$ is of the form

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h, \tag{2.2}$$

and from equations (2.1) and (2.2) can be concluded that

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} u(x,t)\right]_{(0,0)} x^k t^h. \tag{2.3}$$

In Table 2.1 the fundamental mathematical operations of the two-dimensional differential transform are listed. The proofs are available in [8].

Table 2.1. Two-dimensional differential transformation

Original Function	Transformed Function
$u(x,t) \pm v(x,t)$	$U(k,h) \pm V(k,h)$
$cu(x,t)$	$cU(k,h)$
$\frac{\partial u(x,t)}{\partial x}$	$(k+1)U(k+1,h)$
$\frac{\partial u(x,t)}{\partial t}$	$(h+1)U(k,h+1)$
$\frac{\partial^{r+s} u(x,t)}{\partial x^r \partial t^s}$	$\frac{(k+r)!}{k!}\frac{(h+s)!}{h!}U(k+r,h+s)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s)$
$u(x,t)v(x,t)w(x,t)$	$\sum_{r=0}^{k}\sum_{s=0}^{k-r}\sum_{q=0}^{h}\sum_{p=0}^{h-q} U(r,h-q-p)V(s,q)W(k-r-s,p)$

2.2 The RDTM

The basic definitions and operations of the RDTM [15, 16, 17, 18] are defined as follows.

Definition 1. If a function $u(x,t)$ is analytic with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{2.4}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stands for the transformed function.

Definition 2. The reduced differential transform of the sequence $\{U_k(x)\}_{k=0}^{\infty}$ is introduced as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{2.5}$$

By combining equation (2.4) and (2.5), we have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k. \tag{2.6}$$

Some basic properties of the reduced differential transformation obtained from definitions (2.4) and (2.6) are summarized in Table 2.2. The proofs and the basic definitions of the RDTM are available in [15].

Table 2.2. Basic operations of RDTM

Original Function	**Transformed Function**
$u(x,t)$	$U_h(x)$
$u(x,t) \pm v(x,t)$	$U_h(x) \pm V_h(x)$
$cu(x,t)$	$cU_h(x)$ c is a cons.
$x^m t^n$	$x^m\delta(h-n)$
$x^m t^n u(x,t)$	$x^m U_{h-n}(x)$
$\frac{\partial}{\partial x}u(x,t)$	$U_h^{'}(x)$
$\frac{\partial^r}{\partial t^r}u(x,t)$	$\frac{(h+r)!}{h!}U_{h+r}(x)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{h} U_r(x)V_{h-r}(x)$
$u(x,t)v(x,t)w(x,t)$	$\sum_{r=0}^{h}\sum_{s=0}^{h-r} U_r(x)V_s(x)W_{h-r}(x)$

3 The Swift-Hohenberg equation

In this section, we consider two methodologies DTM and RDTM for the Swift-Hohenberg equation. To illustrate the capability, reliability and simplicity of the methods, several different cases for parameters of the equation will be discussed here.

3.1 Solution of the problem by the DTM

We apply the DTM to equation (1.1), the resulting transformed version of equation (1.1) is

$$\begin{aligned}(h+1)U(k,h+1) &= -2\tfrac{(k+2)!}{k!}U(k+2,h) + \sigma\tfrac{(k+3)!}{k!}U(k+3,h) - \tfrac{(k+4)!}{k!}U(k+4,h)\\ &+\alpha U(k,h) + \beta \textstyle\sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)U(k-r,s)\\ &-\gamma \textstyle\sum_{r=0}^{k}\sum_{s=0}^{k-r}\sum_{q=0}^{h}\sum_{p=0}^{h-q} U(r,h-q-p)U(s,q)U(k-r-s,p).\end{aligned} \tag{3.1}$$

From the boundary conditions given by (1.2), we have

$$\begin{aligned}
&U(k,0)=\tfrac{1}{k!}u_0^{(k)}(0), \quad k=0,1,2,\ldots\\
&U(0,h)=0, \quad h=0,1,2,\ldots\\
&U(2,h)=0, \quad h=0,1,2,\ldots\\
&\textstyle\sum_{k=0}^{\infty}U(k,h)l^k=0, \quad h=0,1,2,\ldots\\
&\textstyle\sum_{k=0}^{\infty}\frac{(k+2)!}{k!}U(k+2,h)l^k=0, \quad h=0,1,2,\ldots.
\end{aligned} \tag{3.2}$$

In real applications, the function $u(x,t)$ is given by a finite series of equations (3.1) and (3.2) can be written as follows

$$u(x,t)\approx\tilde{u}(x,t)=\sum_{k=0}^{n-2h}\sum_{h=0}^{m}U(k,h)x^kt^h,$$

where the value of the parameter m should not be greater than $\frac{n}{2}$.

By using equations (3.1) and (3.2), the corresponding $U(k,h)$ can be calculated for arbitrary different selections of n and m. In real applications, we seek obtain an excellent approximate solution of the differential equation. Therefore the selection n and m i.e. iterations continue until the absolute value of the error function defined as follows

$$E_{DTM}(x,t)=|\tilde{u}_t+2\tilde{u}_{xx}-\sigma\tilde{u}_{xxx}+\tilde{u}_{xxxx}-\alpha\tilde{u}-\beta\tilde{u}^2+\gamma\tilde{u}^3|, \tag{3.3}$$

becomes very small for each x,t in the domain, in other words $|E_{DTM}(x,t)|<\textit{tolerance}$ for all $x\in[0,l]$, $t\in[0,t^*]$.

Then the corresponding $U(k,h)$ can be obtained as follows

$$\begin{aligned}
&U(0,0)=u_0(0),U(1,0)=u_0'(0),\ldots,U(n,0)=\tfrac{1}{n!}u_0^{(n)}(0),\ldots,\\
&U(0,0)=0,U(0,1)=0,\ldots,U(0,m)=0,\ldots,\\
&U(2,0)=0,\ U(2,1)=0,\ldots,U(2,m)=0,\ldots.
\end{aligned}$$

If $h=0$, then from (3.1) for $k=1$ *and* $k=3,\ldots,n-4$ we have

$$\begin{aligned}
U(k,1)=&-2\tfrac{(k+2)!}{k!}U(k+2,0)+\sigma\tfrac{(k+3)!}{k!}U(k+3,0)-\tfrac{(k+4)!}{k!}U(k+4,0)\\
&+\alpha U(k,0)+\beta\textstyle\sum_{r=0}^{k}U(r,0)U(k-r,0)\\
&-\gamma\textstyle\sum_{r=0}^{k}\sum_{s=0}^{k-r}U(r,0)U(s,0)U(k-r-s,0),
\end{aligned}$$

and by the final two relations of (3.2) also can obtain

$$\begin{aligned}
U(n-3,1)&=\tfrac{1}{l^{(n-3)}}\textstyle\sum_{k=0}^{n-4}l^kU(k,1),\\
U(n-2,1)&=\tfrac{1}{(n-3)(n-2)l^{(n-4)}}\textstyle\sum_{k=0}^{n-5}(k+1)(k+2)l^kU(k+2,1).
\end{aligned}$$

If $h=1$, then for $k=1$ *and* $k=3,\ldots,n-6$ we have

$$\begin{aligned}
U(k,2)=\tfrac{1}{2}(&-2\tfrac{(k+2)!}{k!}U(k+2,1)+\sigma\tfrac{(k+3)!}{k!}U(k+3,1)-\tfrac{(k+4)!}{k!}U(k+4,1)\\
&+\alpha U(k,1)+\beta\textstyle\sum_{r=0}^{k}\sum_{s=0}^{1}U(r,1-s)U(k-r,s)\\
&-\gamma\textstyle\sum_{r=0}^{k}\sum_{s=0}^{k-r}\sum_{q=0}^{1}\sum_{p=0}^{1-q}U(r,1-q-p)U(s,q)U(k-r-s,p)),
\end{aligned}$$

and

$$U(n-5,2) = \frac{1}{l^{(n-5)}} \sum_{k=0}^{n-6} l^k U(k,2),$$
$$U(n-4,2) = \frac{1}{(n-5)(n-4)l^{(n-6)}} \sum_{k=0}^{n-7} (k+1)(k+2) l^k U(k+2,2).$$

If $h = 2$, then for $k = 1$ *and* $k = 3, \dots, n-8$ we have

$$\begin{aligned} U(k,3) = \tfrac{1}{3}(&-2\tfrac{(k+2)!}{k!}U(k+2,2) + \sigma\tfrac{(k+3)!}{k!}U(k+3,2) - \tfrac{(k+4)!}{k!}U(k+4,2) \\ &+\alpha U(k,2) + \beta \textstyle\sum_{r=0}^{k} \sum_{s=0}^{2} U(r,2-s)U(k-r,s) \\ &-\gamma \textstyle\sum_{r=0}^{k} \sum_{s=0}^{k-r} \sum_{q=0}^{2} \sum_{p=0}^{2-q} U(r,2-q-p)U(s,q)U(k-r-s,p)), \end{aligned}$$

and

$$U(n-7,3) = \frac{1}{l^{(n-7)}} \sum_{k=0}^{n-8} l^k U(k,3),$$
$$U(n-6,3) = \frac{1}{(n-7)(n-6)l^{(n-8)}} \sum_{k=0}^{n-9} (k+1)(k+2) l^k U(k+2,3).$$

By using the recursive scheme of equation (3.1) and conditions (3.2), the rest values of $U(k,h)$ can be obtained.

3.2 Solution of the problem by the RDTM

To solve equation (1.1) by the RDTM, we consider differential transformation of Table 2 and have

$$\begin{aligned} (h+1)U_{h+1}(x) = &-2U_h''(x) + \sigma U_h^{(3)}(x) - U_h^{(4)}(x) + \alpha U_h(x) + \\ &\beta \textstyle\sum_{r=0}^{h} U_r(x)U_{h-r}(x) - \gamma \sum_{r=0}^{h} \sum_{s=0}^{h-r} U_r(x)U_s(x)U_{h-r}(x). \end{aligned} \tag{3.4}$$

We can obtain the initial and boundary conditions as follows

$$\begin{aligned} &U_0(x) = u_0(x), \\ &U_h(0) = 0, \quad h = 0,1,\dots \\ &U_h(l) = 0, \quad h = 0,1,\dots \\ &U_h''(0) = 0, \quad h = 0,1,\dots \\ &U_h''(l) = 0, \quad h = 0,1,\dots. \end{aligned} \tag{3.5}$$

By substituting (3.5) into (3.4) and by a straight forward iterative calculations, we obtain the all required values of $U_h(x)$. Therefore, the inverse transformation of the set of values $\{U_h(x)\}_{h=0}^{m}$ gives the approximate solution as

$$u(x,t) \approx \hat{u}(x,t) = \sum_{h=0}^{m} U_h(x)t^h.$$

Similarly to the previous case, let us consider the error functional for approximate solution

$$E_{RDTM}(x,t) = |\hat{u}_t + 2\hat{u}_{xx} - \sigma\hat{u}_{xxx} + \hat{u}_{xxxx} - \alpha\hat{u} - \beta\hat{u}^2 + \gamma\hat{u}^3|, \tag{3.6}$$

and the iterations continue until $|E_{RDTM}(x,t)| < tolerance$ for all $x \in [0,l]$, $t \in [0,t^*]$.

4 Numerical results and discussion

The convergence of the proposed methods will depend on $\alpha, \beta, \gamma, \sigma, l$ and on the number of terms employed in a series approximation. These methods consist in building a sequence of numerical approximations of $u(x,t)$ via the generated sequence. To find the solution of equation (1.1), an error analysis is performed. Here, $E_{DTM}(x,t)$ and $E_{RDTM}(x,t)$ show the error functions of the proposed method for fixed $n, m, \alpha, \beta, \gamma, \sigma$ and l.

To see the effects of the parameters on the solutions, we fix $u_0(x) = \frac{1}{10}sin(\frac{\pi x}{l})$ and $l = 10$, consider solutions $u(x,t)$ for various values of parameters. To avoid a three-dimensional plot, we plot two-dimensional cross sections. The qualitative properties of such solutions are displayed in figures 1, 3 and 5. A comparison of the figures allows one to see the influence of the parameters on the solution profiles.

A clear conclusion from the numerical results is that the DTM and RDTM provide highly accurate numerical solutions without the need for spatial discretizations in solving the Swift-Hohenberg equation.

Because of memory problem, we only increase the number of iterations until we achieve that the modulus of the error function is less than 0.05 (tolerance). The results show that in the memory problem and boundary conditions the DTM acts better than the RDTM, and in the number of iterations and careful of solutions the RDTM is better than the DTM.

Here, we take different values of the parameters to compare the results of DTM and RDTM in the form of two dimensional figures for each case, we would see that DTM and RDTM solutions are in excellent agreement.

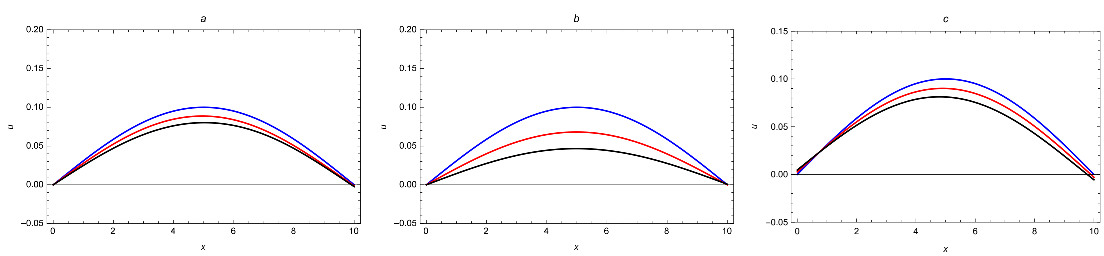

Fig. 1: (a) Profiles of $u(x,t)$ versus x at $\alpha = -0.3, \sigma = -1, \beta = 0.1$ and $\gamma = 0.2$ for $t = 0\,(Upper), 2\,(Middle), 4\,(Lower)$ with $n = 15$ and $m = 4$ by DTM. (b) Profiles of $u(x,t)$ versus x at $\alpha = -0.95, \sigma = -1, \beta = 0.1$ and $\gamma = 0.2$ for $t = 0\,(Upper), 2\,(Middle), 4\,(Lower)$ with $n = 20$ and $m = 5$ by DTM. (c) Profiles of $u(x,t)$ versus x at $\alpha = -0.3, \sigma = -1, \beta = 0.1$ and $\gamma = 0.2$ for $t = 0\,(Upper), 1\,(Middle), 2\,(Lower)$ with $m = 4$ by RDTM.

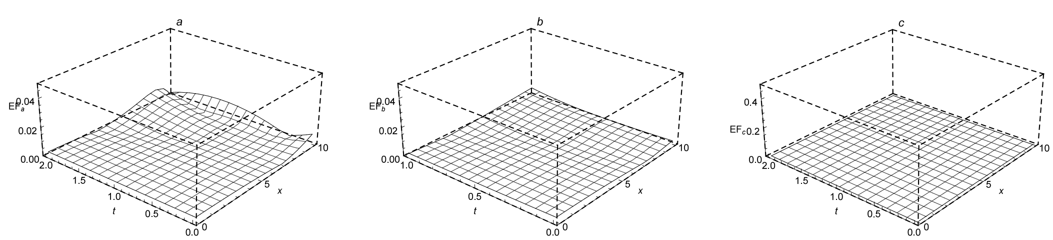

Fig. 2: (a) Profiles of $E_{DTM}(x,t)$ for Fig. 1 (a). (b) Profiles of $E_{DTM}(x,t)$ for Fig. 1 (b). (c) Profiles of $E_{RDTM}(x,t)$ for Fig. 1 (c).

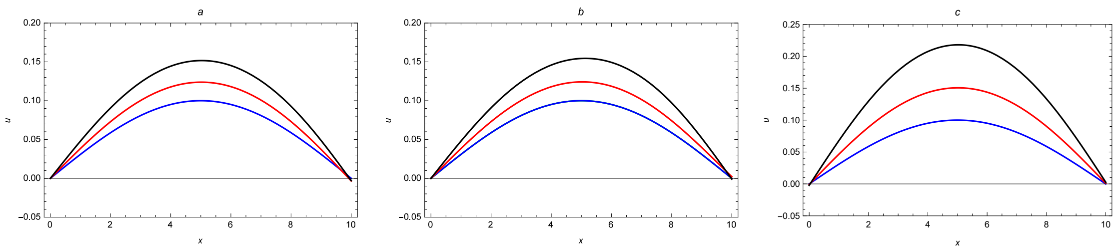

Fig. 3: (a) Profiles of $u(x,t)$ versus x at $\alpha = 0.25, \sigma = 0.2, \beta = -0.04$ and $\gamma = 1.1$ for $t = 0\,(Lower), 2\,(Middle), 4\,(Upper)$ with $n = 14$ and $m = 6$ by DTM. (b) Profiles of $u(x,t)$ versus x at $\alpha = 0.25, \sigma = -0.15, \beta = -0.04$ and $\gamma = 1.1$ for $t = 0\,(Lower), 2\,(Middle), 4\,(Upper)$ with $n = 15$ and $m = 6$ by DTM. (c) Profiles of $u(x,t)$ versus x at $\alpha = 0.25, \sigma = 0.2, \beta = -0.04$ and $\gamma = 1.1$ for $t = 0\,(Lower), 1\,(Middle), 2\,(Upper)$ with $m = 3$ by RDTM.

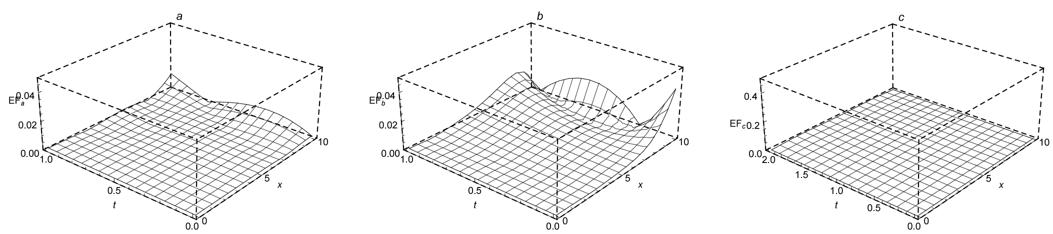

Fig. 4: (a) Profiles of $E_{DTM}(x,t)$ for Fig. 3 (a). (b) Profiles of $E_{DTM}(x,t)$ for Fig. 3 (b). (c) Profiles of $E_{RDTM}(x,t)$ for Fig. 3 (c).

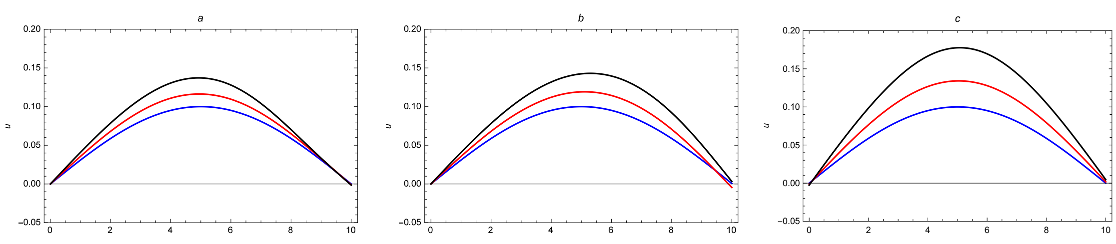

Fig. 5: (a) Profiles of $u(x,t)$ versus x at $\alpha = 0.1, \sigma = 0.4, \beta = 0.1$ and $\gamma = -2.3$ for $t = 0\,(Lower), 2\,(Middle), 4\,(Upper)$ with $n = 16$ and $m = 6$ by DTM. (b) Profiles of $u(x,t)$ versus x at $\alpha = 0.1, \sigma = 0.4, \beta = -0.16$ and $\gamma = -2.3$ for $t = 0\,(Lower), 2\,(Middle), 4\,(Upper)$ with $n = 16$ and $m = 6$ by DTM. (c) Profiles of $u(x,t)$ versus x at $\alpha = 0.1, \sigma = 0.4, \beta = -0.16$ and $\gamma = -2.3$ for $t = 0\,(Lower), 1\,(Middle), 2\,(Upper)$ with $m = 3$ by RDTM.

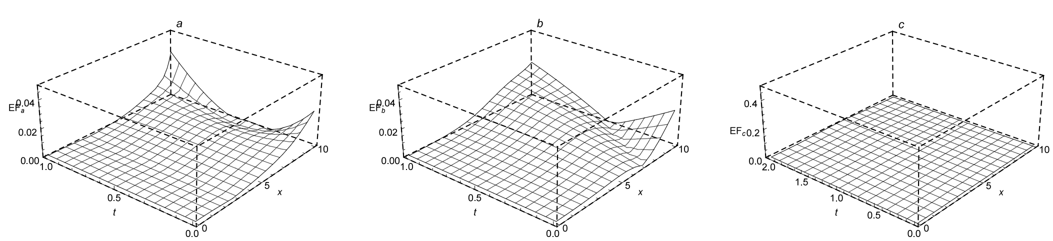

Fig. 6: (a) Profiles of $E_{DTM}(x,t)$ for Fig. 5 (a). (b) Profiles of $E_{DTM}(x,t)$ for Fig. 5 (b). (c) Profiles of $E_{RDTM}(x,t)$ for Fig. 5 (c).

5 Conclusion

Application of the DTM and RDTM to the Swift-Hohenberg equation with dispersion have been presented. The results show that the DTM and RDTM are powerful and efficient methods for finding analytic approximate solutions to the Swift-Hohenberg equation. Also, not many iterations are required to achieve fairly accurate solutions of the equation by the DTM and RDTM.

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References

[1] I.H. Abdel-Halim Hassan, *Differential transformation technique for solving higher-order initial value problems.* Appl. Math. Comput. 154 (2004), 299-311.

[2] I.H. Abdel-Halim Hassan, *Application to differential transformation method for solving systems of differential equations.* Appl. Math. Modell. 32 (2008), no. 12, 2552-2559.

[3] F.T. Akyildiz, D.A. Siginer, K. Vajravelu, R.A. Van Gorder, *Analytical and numerical results for the SwiftвЂ"Hohenberg equation.* Appl. Math. Comput. 216 (2010), no. 1, 221-226.

[4] I.S. Aranson, K.A. Gorshkov, A.S. Lomov, M.I. Rabinovich, *Stable particle-like solutions of multidimensional nonlinear fields.* Phys. D 43 (1990), 435-453.

[5] A. Arikoglu, I. Ozkol, *Solution of fractional differential equations by using differential transform method.* Chaos, Solitons and Fractals 34 (2007), no. 5, 1473-1481.

[6] A. Arikoglu, I. Ozkol, *Solution of boundary value problems for integro-differential equations by using differential transform method.* Appl. Math. Comput. 168 (2005), 1145-1158.

[7] F. Ayaz, *Solutions of the system of differential equations by differential transform method.* Applied Mathematics and Computation 147 (2004), no. 2, 547-567.

[8] F. Ayaz, *On the two-dimensional differential transform method.* Appl. Math. Comput. 143 (2003), 361-374.

[9] J. Burke, S.M. Houghton, E. Knobloch, *Swift-Hohenberg equation with broken reflection symmetry.* Phys Rev E 80 (2009), 036202.

[10] S. Çatal, *Solution of free vibration equations of beamon elastic soil by using differential transform method.* Applied Mathematical Modelling 32 (2008), no. 9, 1744-1757.

[11] L.Yu, Glebsky, L.M. Lerman, *On small stationary localized solutions for the generalized 1-D Swift-Hohenberg equation.* CHAOS 5 (1995), 424-431.

[12] M.J. Jang, C.L. Chen, *Analysis of the response of a strongly nonlinear damped system using a differential transformation technique.* Appl. Math. Comput. 88 (1997), 137-151.

[13] M.J. Jang, C.L. Chen, Y.C. Liu, *Two-dimensional differential transform for partial differential equations.* Applied Mathematics and Computation 121 (2001), no. 2-3, 261-270.

[14] F. Kangalgil, F. Ayaz, *Solitary wave solutions for the KdV and mKdV equations by differential transform method.* Chaos Solitons & Fractals 41 (2009), 464-472.

[15] Y. Keskin, *Ph.D. Thesis.* Selcuk University, 2010 (in Turkish).

[16] Y. Keskin, G. Oturanç *Reduced differential transform method for partial differential equations.* Int. J. of Nonl. Sci. and Num. Simulation 10 (2009), no. 6, 741-749.

[17] Y. Keskin, G. Oturanç, *Reduced differential transform method for generalized KdV equations.* Math. & Computational Applications 15 (2010), no. 3, 382-393.

[18] Y. Keskin, G. Oturanç, *Reduced differential transform method for solving linear and nonlinear wave equations.* Iranian J. of Sci. & Tech. Transaction A 34 (2010), no. A2, 113-122.

[19] N.A. Khan, N.-U. Khan, M. Ayaz, A. Mahmood, *Analytical methods for solving the time-fractional Swift-Hohenberg (S-H) equation.* Computers and Math. with App. 61 (2011), 2182-2185.

[20] H. Liu, S. Yongzhong, *Differential transform method applied to high index differential algebraic equations.* Appl. Math. Comput. 184 (2007), 748-753.

[21] O.O. Ozdemir, M.O. Kaya, *Flapwise bending vibration analysis of a rotating tapered cantilever Bernoulli-Euler beam by differential transform method.* J. of Sound and Vibration 289 (2006), no. 1-2, 413-420.

[22] Y. Pomeau, S. Zaleski, *Dislocation motion in cellular structures.* J. Phys. 42 (1981), 2710-2726.

[23] H. Rouhparvar, *Computational technique of linear partial differential equations by reduced differential transform method.* Int. J. of Industrial Math. 8 (2016), no. 4, 339-346.

[24] J. Swift, P.C. Hohenberg, *Hydrodynamic fluctuations at the convective instability.* Phys. Rev. A 15 (1977), 319-328.

[25] J.K. Zhou, *Differential transformation and its applications for electrical circuIts.* Huazhong University Press, Wuhan, 1986 (in Chinese).

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ITERATED DISCRETE HARDY-TYPE INEQUALITIES

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AMS Mathematics Subject Classification: 26D15, 26D20.

Abstract. In this paper, we discuss new discrete inequalities of Hardy-type involving iterated operators. Under some conditions on weight sequences, we establish necessary and sufficient conditions for the validity of these inequalities.

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1 Introduction

Let $0 < q, p, \theta < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ be a sequence of non-negative numbers, $u = \{u_i\}_{i=1}^{\infty}$ and $w = \{w_i\}_{i=1}^{\infty}$ be sequences of positive numbers, which will be called the weight sequences. We consider the Hardy operator H_φ defined for any $f \in l_1$ by

$$(H_\varphi f)_k := \varphi_k \sum_{i=1}^{k} f_i,$$

where $k \in \mathbb{N}$. Let us denote by $l_{p,u}$ the space of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,u} = \left(\sum_{i=1}^{\infty} |u_i f_i|^p \right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty.$$

For any $f \in l_{p,u}$ we characterize the following iterated discrete Hardy-type inequality with three weights

$$\left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} \left| \varphi_k \sum_{i=1}^{k} f_i \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \le C \left(\sum_{i=1}^{\infty} |u_i f_i|^p \right)^{\frac{1}{p}}, \tag{1.1}$$

where C is a positive constant independent of f. The dual discrete version of inequality (1.1) has the form

$$\left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=n}^{\infty} \left| \varphi_k \sum_{i=k}^{\infty} f_i \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \le C \left(\sum_{i=1}^{\infty} |u_i f_i|^p \right)^{\frac{1}{p}}. \tag{1.2}$$

The continuous analogue of inequality (1.1) can be written as follows

$$\left(\int_0^{\infty} w^{\theta}(x) \left(\int_0^{x} \left| \varphi(t) \int_0^{t} f(s)\, ds \right|^q dt \right)^{\frac{\theta}{q}} dx \right)^{\frac{1}{\theta}} \le C \left(\int_0^{\infty} |u(x) f(x)|^p \, dx \right)^{\frac{1}{p}}. \tag{1.3}$$

The boundedness of the Hardy-type operator in Morrey-type spaces, weighted Sobolev spaces was studied in many papers (see, [6], [9], [15]). In paper [3], the problem of boundedness of the Hardy operator from a Lebesgue space to a local Morrey-type space has been reduced to the validity of inequality (1.3). The results of paper [3] have aroused the interest to study inequalities of form (1.3). We believe that the relation between p and θ is more important than between p and q because we have found out that inequalities of form (1.3) are easier to characterize for $p \le \theta$ rather than for $\theta < p$, as for the standard Hardy inequalities. Paper [14] has covered all possible relations between p, θ and q for characterizations of inequalities of form (1.3), but the obtained results require some auxiliary function and are not given explicitly. Paper [3], where inequality (1.3) was firstly considered and explicitly characterized, has completely covered the case $p \le \theta$, in sense that q can be any positive number, and partially covered the case $\theta < p$ only for $0 < q < \theta$. In paper [12], discrete Hardy-type inequality (1.1) have been characterized for the same relations between p, θ and q, namely, for the cases $p \le \theta < \infty$, $0 < q$ and $\theta < p < \infty$, $0 < q < \theta$. Here we consider the most difficult case $\theta < p < \infty$ and $0 < \theta < q$ or, equivalently, $0 < \theta < \min\{p, q\} < \infty$, which has no explicit characterizations even in the continuous case.

In the relations between p, θ and q listed above, for the continuous case it is assumed that $p > 1$, since for the interval $0 < p < 1$ inequalities of form (1.3) hold only in the trivial cases. For the discrete case the interval $0 < p < 1$ is not excluded, so in this paper we consider the case $0 < \theta < \min\{p, q\} < \infty$ for both $p > 1$ and $0 < p \le 1$. Paper [11] also contains results for inequality (1.2) for the case $0 < p \le 1$, but when $p \le \min\{q, \theta\} < \infty$. In order to complete the relation $p \le \theta$, we include the case $0 < q < p \le \theta < \infty$, $0 < p \le 1$, as an auxiliary result.

The iterated operator $K^+f(x) = \left(\int\limits_0^x \left|\varphi(t)\int\limits_0^t f(s)\,ds\right|^q dt\right)^{\frac{1}{q}}$ in inequality (1.3) has the same types of integrals as well as the operator $K^-f(x) = \left(\int\limits_x^\infty \left|\varphi(t)\int\limits_t^\infty f(s)\,ds\right|^q dt\right)^{\frac{1}{q}}$ in the continuous analogue of dual inequality (1.2). We can also write two inequalities with the iterated operators $T^+f(x) = \left(\int\limits_0^x \left|\varphi(t)\int\limits_t^\infty f(s)\,ds\right|^q dt\right)^{\frac{1}{q}}$ and $T^-f(x) = \left(\int\limits_x^\infty \left|\varphi(t)\int\limits_0^t f(s)\,ds\right|^q dt\right)^{\frac{1}{q}}$, which have different types of integrals. In paper [8], the problems of boundedness of the conjugate Hardy operator from a Lebesgue space to a Morrey-type space and boundedness of the Hardy operator from a Lebesgue space to a complementary Morrey-type space have been reduced to the validity of the inequalities with the operators T^+f and T^-f, respectively. The inequalities with the operators T^+f and T^-f have been studied more fully than the inequalities with the operators K^+f and K^-f (see, [2], [4], [5], [10], [13] and [16]). On the contrary, the study of (1.1) and (1.2), which are discrete analogues of the inequalities for the operators K^+f and K^-f, is almost completed in this paper, while the investigation of inequalities for the discrete versions of the operators T^+f and T^-f has only started.

Note that the interest in inequalities with iterated operators has been caused not only by their applicability to Morrey-type spaces shown in [3] and [8], but also by the fact that their characterizations can be applied to obtain characterizations for the bilinear Hardy inequalities (see, [2] and [7]).

The work is organized as follows. Section 2 contains all statements and definitions, which are needed to characterize inequalities (1.1) and (1.2). The main results for $0 < \theta < \min\{p, q\} < \infty$, $p > 1$, are presented in section 3. The main results for $0 < \theta < \min\{p, q\} < \infty$, $0 < p \le 1$, are given in section 4. Section 5 contains the auxiliary result for $0 < q < p \le \theta < \infty$, $0 < p \le 1$.

2 Preliminaries

In the proofs of our main results for the case $0 < q < p \leq \theta < \infty$, $0 < p \leq 1$, we need the following theorem. This theorem proved in [1, Theorem 1 (iv)] presents characterizations of the following weighted discrete Hardy-type inequality.

Theorem 2.1. *Let $0 < p \leq 1$, $p \leq q < \infty$. The inequality*

$$\left(\sum_{k=1}^{\infty} v_k^q \Big|\sum_{i=1}^{k} f_i\Big|^q\right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p,u}, \tag{2.1}$$

holds for some $C > 0$ if and only if $A < \infty$, where

$$A = \sup_{j\geq 1} \left(\sum_{i=j}^{\infty} v_i^q\right)^{\frac{1}{q}} u_j^{-1}.$$

Moreover, $C \approx A$, where C is the best constant in (2.1).

For the proofs we also need the following lemma.

Lemma 2.1. *Let $r > 0$, $1 \leq n < N \leq \infty$. Then*

$$\sum_{k=n}^{N} a_k \left(\sum_{j=k}^{N} a_j\right)^{r-1} \approx \left(\sum_{i=n}^{N} a_i\right)^{r} \approx \sum_{k=n}^{N} a_k \left(\sum_{j=n}^{k} a_j\right)^{r-1}. \tag{2.2}$$

Convention: The symbol $N \ll M$ means $N \leq CM$ with some positive constant C, depending on the parameters p, θ and q. Moreover, the notation $N \approx M$ means $N \ll M \ll N$.

For the estimations we use various classical inequalities such as the Minkowski inequality, the Hölder inequality and the following elementary inequalities.

If $a_i > 0$, $i = 1, 2, ..., k$, then

$$\left(\sum_{i=1}^{k} a_i\right)^{\alpha} \leq \sum_{i=1}^{k} a_i^{\alpha}, \quad 0 < \alpha \leq 1, \tag{2.3}$$

and

$$\left(\sum_{i=1}^{k} a_i\right)^{\alpha} \geq \sum_{i=1}^{k} a_i^{\alpha}, \quad \alpha \geq 1. \tag{2.4}$$

3 Main results for $0 < \theta < \min\{p, q\} < \infty$, $p > 1$

Theorem 3.1. *Let $0 < \theta < \min\{p, q\} < \infty$, $p > 1$. Then inequality* (1.2) *holds if and only if $B_1 < \infty$, where*

$$B_1 = \left[\sum_{i=1}^{\infty} u_i^{-p'} \left(\sum_{j=i}^{\infty} u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=1}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_1$, where C is the best constant in (1.2).

Proof. Necessity. Suppose that inequality (1.2) holds with the best constant $C > 0$. Let us show that $B_1 < \infty$. For an arbitrary $1 \le r < N < \infty$ we take a test sequence $\widetilde{f}_r = \{\widetilde{f}_{r,i}\}_{i=1}^{\infty}$ such that

$$\widetilde{f}_{r,i} = \begin{cases} 0, & 1 \le i < r, \quad i > N, \\ u_i^{-p'} \left(\sum\limits_{j=i}^{N} u_j^{-p'}\right)^{\frac{\theta-1}{p-\theta}} \left(\sum\limits_{n=r}^{i} w_n^{\theta} \left(\sum\limits_{s=n}^{i} \varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{p-\theta}}, & r \le i \le N < \infty. \end{cases}$$

Then

$$\|\widetilde{f}_r\|_{p,u} = \left(\sum_{i=1}^{\infty} |\widetilde{f}_r \cdot u_i|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=r}^{N} u_i^{-p'} \left(\sum_{j=i}^{N} u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta} \left(\sum_{s=n}^{i} \varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right)^{\frac{1}{p}} =: \widetilde{B}^{\frac{1}{p}} < \infty. \tag{3.1}$$

Substituting $\widetilde{f}_r$ in the left-hand side $I = I(f)$ of inequality (1.2), we derive that

$$I(\widetilde{f}) = \left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=n}^{\infty} \left|\varphi_k \sum_{i=k}^{\infty} \widetilde{f}_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \ge \left(\sum_{n=r}^{N} w_n^{\theta} \left(\sum_{k=n}^{N} \varphi_k^q \left(\sum_{i=k}^{N} \widetilde{f}_i\right)^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}.$$

By applying Lemma 2.1, we obtain

$$I(\widetilde{f}) \gg \left(\sum_{n=r}^{N} w_n^{\theta} \left(\sum_{k=n}^{N} \varphi_k^q \sum_{i=k}^{N} \widetilde{f}_i \left(\sum_{j=i}^{N} \widetilde{f}_j\right)^{(q-1)}\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}.$$

Next, changing the orders of sums and using Lemma 2.1, we get

$$I(\widetilde{f}) \gg \left(\sum_{n=r}^{N} w_n^{\theta} \sum_{i=n}^{N} \widetilde{f}_i \left(\sum_{j=i}^{N} \widetilde{f}_j\right)^{(q-1)}\right.$$

$$\left. \times \sum_{k=n}^{i} \varphi_k^q \left(\sum_{m=i}^{N} \widetilde{f}_m \left(\sum_{s=m}^{N} \widetilde{f}_s\right)^{(q-1)} \sum_{z=n}^{m} \varphi_z^q\right)^{\frac{\theta-q}{q}}\right)^{\frac{1}{\theta}}$$

$$= \left(\sum_{i=r}^{N} \widetilde{f}_i \left(\sum_{j=i}^{N} \widetilde{f}_j\right)^{(q-1)} \left(\sum_{m=i}^{N} \widetilde{f}_m \left(\sum_{s=m}^{N} \widetilde{f}_s\right)^{(q-1)}\right)^{\frac{\theta-q}{q}} \sum_{n=r}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}$$

$$\gg \left(\sum_{i=r}^{N} \widetilde{f}_i \left(\sum_{j=i}^{N} \widetilde{f}_j\right)^{(\theta-1)} \sum_{n=r}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}. \tag{3.2}$$

First we estimate

$$\sum_{j=i}^{N} \widetilde{f}_j = \sum_{j=i}^{N} u_j^{-p'} \left(\sum_{s=j}^{N} u_s^{-p'}\right)^{\frac{\theta-1}{p-\theta}} \left(\sum_{n=r}^{j} w_n^{\theta} \left(\sum_{k=n}^{j} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{p-\theta}}$$

$$\gg \left(\sum_{j=i}^{N} u_j^{-p'}\right)^{\frac{p-1}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{p-\theta}}. \tag{3.3}$$

Now, we put (3.3) into (3.2), then substitute $\widetilde{f}_r$ and find

$$I(\widetilde{f}) \gg \left(\sum_{i=r}^{N} u_i^{-p'} \left(\sum_{j=i}^{N} u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right)^{\frac{1}{\theta}} = \widetilde{B}^{\frac{1}{\theta}}. \tag{3.4}$$

From (3.1), (3.4) and (1.2) it follows that

$$\widetilde{B}^{\frac{p-\theta}{p\theta}} \ll C, \quad \text{for all } \ 1 \le r < N < \infty. \tag{3.5}$$

Since $r \ge 1$ is arbitrary, taking the supremum on both sides of inequality (3.5) with respect to r (C is independent of r) and passing to the limit $N \to \infty$, we get that

$$B_1 \ll C < \infty. \tag{3.6}$$

Sufficiency. Suppose that $B_1 < \infty$. Now, we prove that inequality (1.2) holds. Let $0 \le f \in l_{p,u}$ be such that $\sum\limits_{i=1}^{\infty} f_i < \infty$.

Let

$$k_1 := \sup\{k \in \mathbb{Z} : \sum_{i=1}^{\infty} f_i \le 2^{-k}\},$$

then

$$2^{-k_1-1} < \sum_{i=1}^{\infty} f_i \le 2^{-k_1}.$$

We consider the sequence $\{j_k\}$, where j_k are defined by

$$j_k := \min\{j \ge 1 : \sum_{i=j}^{\infty} f_i \le 2^{-k_1-k+1}\}.$$

We note that

$$j_1 := \min\{j \ge 1 : \sum_{i=j}^{\infty} f_i \le 2^{-k_1}\} = 1.$$

For all $k \ge 1$ it yields that

$$\sum_{i=j_k}^{\infty} f_i \le 2^{-k_1-k+1} < \sum_{i=j_k-1}^{\infty} f_i. \tag{3.7}$$

Therefore, the set of natural numbers $\mathbb{N}$ can be written

$$\mathbb{N} = \bigcup_{k\ge1} [j_k, j_{k+1}-1].$$

Furthermore,

$$2^{-k_1-m+1} < \sum_{i=j_m-1}^{\infty} f_i = \sum_{i=j_m-1}^{j_{m+1}-1} f_i + \sum_{i=j_{m+1}}^{\infty} f_i$$

$$< \sum_{i=j_m-1}^{j_{m+1}-1} f_i + 2^{-k_1-(m+1)+1}, \quad m \geq 2.$$

$$2^{-k_1-m} < \sum_{i=j_m-1}^{j_{m+1}-1} f_i, \quad m \geq 2.$$

$$2^{-k_1-m+2} < 4 \sum_{i=j_m-1}^{j_{m+1}-1} f_i, \quad m \geq 2.$$

Substituting m by $m+1$, we obtain

$$2^{-k_1-m+1} < 4 \sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i, \quad m \geq 1. \tag{3.8}$$

Hence, taking into account (3.7), we get

$$I^\theta(f) := \sum_{n=1}^{\infty} w_n^\theta \left(\sum_{s=n}^{\infty} \left| \varphi_s \sum_{i=s}^{\infty} f_i \right|^q \right)^{\frac{\theta}{q}}$$

$$\leq \sum_{k=1}^{\infty} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{m=k}^{\infty} \sum_{s=\max\{n,j_m\}}^{j_{m+1}-1} \varphi_s^q \left(\sum_{i=j_m}^{\infty} f_i \right)^q \right)^{\frac{\theta}{q}}.$$

Therefore, using (3.7) and (3.8), we have

$$I^\theta(f) \leq 4^\theta \sum_{k=1}^{\infty} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{m=k}^{\infty} \sum_{s=\max\{n,j_m\}}^{j_{m+1}-1} \varphi_s^q \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^q \right)^{\frac{\theta}{q}}.$$

Using inequality (2.3), we get

$$I^\theta(f) \leq 4^\theta \sum_{k=1}^{\infty} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \sum_{m=k}^{\infty} \left(\sum_{s=\max\{n,j_m\}}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta.$$

Next, changing the orders of sums, we have

$$I^\theta(f) \leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \sum_{k=1}^{m} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=\max\{n,j_m\}}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}$$

$$= 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \left(\sum_{k=1}^{m-1} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_m}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right.$$

$$\left. + \sum_{n=j_m}^{j_{m+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)$$

$$\leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \sum_{k=1}^{m} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}.$$

Hence,

$$I^\theta(f) \leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \sum_{n=1}^{j_{m+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}. \tag{3.9}$$

Using the Hölder inequality with powers p and p' in (3.9), we have

$$I^\theta(f) \leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i u_i|^p \right)^{\frac{\theta}{p}} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} u_i^{-p'} \right)^{\frac{\theta}{p'}}$$

$$\times \sum_{n=1}^{j_{m+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}. \tag{3.10}$$

For the outer sum in (3.10) again using the Hölder inequality with the parameters $\frac{p}{\theta}$ and $\frac{p}{p-\theta}$, we get

$$I^\theta(f) \leq 4^\theta \left(\sum_{m=1}^{\infty} \sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i u_i|^p \right)^{\frac{\theta}{p}} \left(\sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} u_i^{-p'} \right)^{\frac{\theta(p-1)}{(p-\theta)}} \right.$$

$$\left. \times \left(\sum_{n=1}^{j_{m+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{p-\theta}{p}}. \tag{3.11}$$

Now, applying Lemma 2.1 to (3.11), we find that

$$I^\theta(f) \ll 2^{\theta(2+\frac{1}{p})} \left(\sum_{i=1}^{\infty} |f_i u_i|^p \right)^{\frac{\theta}{p}} \left(\sum_{m=1}^{\infty} \sum_{i=j_{m+1}-1}^{j_{m+2}-1} u_i^{-p'} \left(\sum_{j=i}^{j_{m+2}-1} u_j^{-p'} \right)^{\frac{p(\theta-1)}{p-\theta}} \right.$$

$$\left. \times \left(\sum_{n=1}^{i} w_n^\theta \left(\sum_{s=n}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{p-\theta}{p}} = 2^{\theta(2+\frac{1}{p})} \left(\sum_{m=1}^{\infty} \left(u_{j_{m+1}-1}^{-p'} \right. \right.$$

$$\times \left(\sum_{j=j_{m+1}-1}^{j_{m+2}-1} u_i^{-p'} \right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=1}^{j_{m+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}}$$

$$\left. \left. + \sum_{i=j_{m+1}}^{j_{m+2}-1} u_i^{-p'} \left(\sum_{j=i}^{j_{m+2}-1} u_j^{-p'} \right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=1}^{i} w_n^\theta \left(\sum_{s=n}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right) \right)^{\frac{p-\theta}{p}} \|f\|_{p,u}^\theta$$

$$\leq 2^{\theta(2+\frac{1}{\theta})} \left(\left(\sum_{i=1}^{\infty} u_i^{-p'} \left(\sum_{j=i}^{\infty} u_j^{-p'} \right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=1}^{i} w_n^\theta \left(\sum_{s=n}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{p-\theta}{p\theta}} \right)^\theta$$

$$\times\|f\|_{p,u}^{\theta} \le 2^{\theta(2+\frac{1}{\theta})} B_1^{\theta} \|f\|_{p,u}^{\theta}.$$

Hence,

$$I(f) \ll B_1 \|f\|_{p,u} \tag{3.12}$$

and $C \ll B_1$, where C is the best constant in (1.2). Inequalities (3.6) and (3.12) give that $C \approx B_1$. □

Theorem 3.2. *Let* $0 < \theta < \min\{p, q\} < \infty$, $p > 1$. *Then inequality* (1.1) *holds if and only if* $B_2 < \infty$, *where*

$$B_2 = \left[\sum_{i=1}^{\infty} u_i^{-p'} \left(\sum_{j=1}^{i} u_j^{-p'} \right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=i}^{\infty} w_n^{\theta} \left(\sum_{k=i}^{n} \varphi_k^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_2$, *where* C *is the best constant in* (1.1).

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1.

4 Main results for $0 < \theta < \min\{p, q\} < \infty$, $0 < p \le 1$

Theorem 4.1. *Let* $0 < \theta < \min\{p, q\} < \infty$, $0 < p \le 1$. *Then inequality* (1.2) *holds if and only if* $B_3 < \infty$, *where*

$$B_3 = \left[\sum_{i=1}^{\infty} u_i^{-\frac{\theta p}{p-\theta}} \left(\sum_{n=1}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_3$, *where* C *is the best constant in* (1.2).

Proof. Necessity. Suppose that inequality (1.2) holds with the best constant $C > 0$. Let $1 \le r < N < \infty$. We take a test sequence $\widetilde{f}_r = \{\widetilde{f}_{r,i}\}_{i=1}^{\infty}$ such that $\widetilde{f}_{r,i} = 0$ for $1 \le i < r$, $i > N$ and $\widetilde{f}_{r,i} = u_i^{-\frac{p}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta} \left(\sum_{s=n}^{i} \varphi_s^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{1}{p-\theta}}$ for $r \le i \le N < \infty$.

Then

$$\|\widetilde{f}_r\|_{p,u} = \left(\sum_{i=1}^{\infty} |\widetilde{f}_r \cdot u_i|^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=r}^{N} u_i^{-\frac{p\theta}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta} \left(\sum_{s=n}^{i} \varphi_s^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{1}{p}} =: \mathcal{B}^{\frac{1}{p}} < \infty. \tag{4.1}$$

In the same way as in the proof of Theorem 3.1, we substitute $\widetilde{f}_r$ in the left-hand side of inequality (1.2) and obtain inequality (3.2). Now, let us estimate

$$\sum_{j=i}^{N} \widetilde{f}_j \ge u_i^{-\frac{p}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^{q} \right)^{\frac{\theta}{q}} \right)^{\frac{1}{p-\theta}}. \tag{4.2}$$

We put (4.2) into (3.2), then we have

$$I(\widetilde{f}) \gg \left(\sum_{i=r}^{N} u_i^{-\frac{p\theta}{p-\theta}} \left(\sum_{n=r}^{i} w_n^\theta \left(\sum_{s=n}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{1}{\theta}} = \mathcal{B}^{\frac{1}{\theta}}. \tag{4.3}$$

From (4.1), (4.3) and (1.2), as a result we get

$$\mathcal{B}^{\frac{p-\theta}{p\theta}} \ll C, \text{for all } \ 1 \le r < N < \infty.$$

Since $r \ge 1$ is arbitrary, passing to the limit $N \to \infty$, we have

$$B_3 \ll C < \infty. \tag{4.4}$$

Sufficiency. We start to prove the sufficient part of Theorem 4.1 in the same way as the sufficient part of Theorem 3.1. Since in this case $0 < p \le 1$, we can not use the Hölder inequality in (3.9). Therefore, we continue the proof in the following way

$$I^\theta(f) \le 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i u_i u_i^{-1} \right)^{p\frac{\theta}{p}} \sum_{n=1}^{j_{m+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}.$$

Applying (2.3) with $0 < p \le 1$, we obtain that

$$I^\theta(f) \le 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i u_i|^p \right)^{\frac{\theta}{p}}$$

$$\times \sup_{j_{m+1}-1 \le k \le j_{m+2}-1} u_k^{-\theta} \sum_{n=1}^{j_{m+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}.$$

Using the Hölder inequality for the outer sum, we get

$$I^\theta(f) \le 2^{\theta(2+\frac{1}{p})} \left(\sum_{i=1}^{\infty} |f_i u_i|^p \right)^{\frac{\theta}{p}}$$

$$\times \left(\sum_{m=1}^{\infty} \sum_{k=j_{m+1}-1}^{j_{m+2}-1} u_k^{-\frac{p\theta}{p-\theta}} \left(\sum_{n=1}^{j_{m+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{p-\theta}{p}}$$

$$\le 2^{\theta(2+\frac{1}{\theta})} \left(\sum_{k=1}^{\infty} u_k^{-\frac{p\theta}{p-\theta}} \left(\sum_{n=1}^{k} w_n^\theta \left(\sum_{s=n}^{k} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{p-\theta}{p}} \|f\|_{p,u}^\theta.$$

Hence,

$$I^\theta(f) \le 2^{\theta(2+\frac{1}{\theta})} B_3^\theta \|f\|_{p,u}^\theta,$$

so that

$$I(f) \ll B_3 \|f\|_{p,u}. \tag{4.5}$$

Therefore, from inequalities (4.4) and (4.5), we get $C \approx B_3$, where C is the best constant in (1.2). $\square$

Theorem 4.2. *Let $0 < \theta < \min\{p, q\} < \infty$, $0 < p \leq 1$. Then inequality* (1.1) *holds if and only if $B_4 < \infty$, where*

$$B_4 = \left[\sum_{i=1}^{\infty} u_i^{-\frac{\theta p}{p-\theta}} \left(\sum_{n=i}^{\infty} w_n^{\theta} \left(\sum_{k=i}^{n} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_4$, where C is the best constant in (1.1).

The proof of Theorem 4.2 is similar to the proof of Theorem 4.1.

Remark 1. Theorems 3.1 and 4.1 mean that inequality (1.2) holds for both cases $0 < \theta < q < p < \infty$ and $0 < \theta < p < q < \infty$.

5 Auxiliary result for $0 < q < p \leq \theta < \infty$, $0 < p \leq 1$

Theorem 5.1. *Let $0 < q < p \leq \theta < \infty, 0 < p \leq 1$. Then inequality* (1.1) *holds if and only if $B = \max\{B_5, B_6\} < \infty$, where*

$$B_5 = \sup_{i \geq 1} \left(\sum_{n=i}^{\infty} w_n^{\theta} \left(\sum_{k=i}^{n} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_i^{-1},$$

$$B_6 = \sup_{i \geq 1} \left(\sum_{n=i}^{\infty} w_n^{\theta} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{i} \varphi_k^q \right)^{\frac{1}{q}} \sup_{j \leq i} u_j^{-1}.$$

Moreover, $C \approx B$, where C is the best constant in (1.1).

Proof. Necessity. Assume that inequality (1.1) holds with the best constant $C > 0$. First, we prove that $B_5 < \infty$. Let $j \geq 1$. We take a test sequence $\widetilde{f}_j = \{\widetilde{f}_{j,i}\}_{i=1}^{\infty}$ such that $\widetilde{f}_{j,i} = u_i^{-1}$ for $i = j$ and $\widetilde{f}_{j,i} = 0$ for $i \neq j$. Then

$$\|\widetilde{f}_j\|_{p,u} = \left(\sum_{i=1}^{\infty} |\widetilde{f}_j \cdot u_i|^p \right)^{\frac{1}{p}} = 1. \tag{5.1}$$

Substituting $\widetilde{f}_j$ in left-hand side of inequality (1.1), we deduce that

$$I(\widetilde{f}) := \left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} \left| \varphi_k \sum_{i=1}^{k} \widetilde{f}_{j,i} \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \geq \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{k=j}^{n} \left| \varphi_k \sum_{i=1}^{k} \widetilde{f}_{j,i} \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}}$$

$$\geq \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{k=j}^{n} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_j^{-1}. \tag{5.2}$$

From (5.1), (5.2) and (1.1) it follows that

$$\left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{k=j}^{n} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_j^{-1} \leq C, \quad \forall j \geq 1.$$

Since $j \geq 1$ is arbitrary, we have

$$B_5 = \sup_{j\geq 1} \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{k=j}^{n} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_j^{-1} \leq C < \infty. \tag{5.3}$$

Now, let us show that $B_6 < \infty$. For $1 < r \leq j < \infty$, we take a test sequence $\widetilde{v}_k = \{\widetilde{v}_{k,r}\}_{r=1}^{\infty}$ such that $\widetilde{v}_{k,r} = u_r^{-1}$ for $r = k$ and $\widetilde{v}_{k,r} = 0$ for $r \neq k$. Then

$$\|\widetilde{v}_r\|_{p,u} = 1. \tag{5.4}$$

Substituting $\widetilde{v}_k$ in the left-hand side of inequality (1.1), we find that

$$I(\widetilde{v}) \geq \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} \left| \varphi_k \sum_{i=1}^{k} \widetilde{v}_{i,r} \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \geq \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{j} \left| \varphi_k \sum_{i=1}^{k} \widetilde{v}_{i,r} \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}}$$

$$\geq \left(\sum_{n=j}^{\infty} w_n^{\theta} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{j} \varphi_k^q \right)^{\frac{1}{q}} u_r^{-1}, \quad \forall r \leq j. \tag{5.5}$$

From (5.4), (5.5) and (1.1), we obtain

$$\left(\sum_{n=j}^{\infty} w_n^{\theta} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{j} \varphi_k^q \right)^{\frac{1}{q}} u_r^{-1} \leq C, \quad \forall r \leq j.$$

$$\left(\sum_{n=j}^{\infty} w_n^{\theta} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{j} \varphi_k^q \right)^{\frac{1}{q}} \sup_{r\leq j} u_r^{-1} \leq C, \quad \forall j \geq 1.$$

Therefore,

$$B_6 = \sup_{j\geq 1} \left(\sum_{n=j}^{\infty} w_n^{\theta} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{j} \varphi_k^q \right)^{\frac{1}{q}} \sup_{r\leq j} u_r^{-1} \leq C < \infty. \tag{5.6}$$

Sufficiency. Let $B < \infty$. Without loss of generality, we assume that $0 \leq f \in l_{p,u}$.
Let $\inf \emptyset = \infty$ and

$$k_{\infty} = \inf \left\{ k \in \mathbb{Z} : \sum_{s=1}^{\infty} \left(\varphi_s \sum_{i=1}^{s} f_i \right)^q < 2^{q(k+1)} \right\}.$$

Assume that $k \leq k_{\infty}$ if $k_{\infty} < \infty$ and

$$j_k = \inf \left\{ j \geq 1 : \sum_{s=1}^{j} \left(\varphi_s \sum_{i=1}^{s} f_i \right)^q \geq 2^{qk} \right\}.$$

Then

$$\sum_{s=1}^{j_k-1} \left(\varphi_s \sum_{i=1}^{s} f_i \right)^q < 2^{qk} \leq \sum_{s=1}^{j_k} \left(\varphi_s \sum_{i=1}^{s} f_i \right)^q.$$

Therefore, the set of natural numbers $\mathbb{N}$ can be written

$$\mathbb{N} = \bigcup_{k\geq 1} \left[j_k, j_{k+1} - 1 \right].$$

Since in this case $0 < q < 1$, we have

$$2^{q(k-1)} = \frac{2^{qk} - 2^{q(k-1)}}{2^q - 1} \leq \frac{1}{2^q - 1} \left(\sum_{s=1}^{j_k} \left(\varphi_s \sum_{i=1}^{s} f_i \right)^q - \sum_{s=1}^{j_{k-1}-1} \left(\varphi_s \sum_{i=1}^{s} f_i \right)^q \right) \leq \frac{1}{2^q - 1} \left(\sum_{s=j_{k-1}}^{j_k} \left(\varphi_s \sum_{i=1}^{s} f_i \right)^q \right)$$

$$\leq \frac{1}{2^q - 1} \left(\sum_{s=j_{k-1}}^{j_k} \left(\varphi_s \sum_{i=1}^{j_{k-1}} f_i \right)^q + \sum_{s=j_{k-1}}^{j_k} \left(\varphi_s \sum_{i=j_{k-1}}^{s} f_i \right)^q \right).$$

Hence,

$$2^{(k-1)} \leq \frac{2^{\frac{1}{q}-1}}{(2^q - 1)^q} \left(\left(\sum_{s=j_{k-1}}^{j_k} \left(\varphi_s \sum_{i=1}^{j_{k-1}} f_i \right)^q \right)^{\frac{1}{q}} + \left(\sum_{s=j_{k-1}}^{j_k} \left(\varphi_s \sum_{i=j_{k-1}}^{s} f_i \right)^q \right)^{\frac{1}{q}} \right). \tag{5.7}$$

For the left-hand side $I(f)$ of inequality (1.1) we have

$$I(f) = \left(\sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=1}^{n} \left(\varphi_s \sum_{i=1}^{s} f_i \right)^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \leq 4 \left(\sum_k 2^{\theta(k-1)} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \right)^{\frac{1}{\theta}}. \tag{5.8}$$

Combining (5.7) with (5.8), we have

$$I(f) \ll \left(\sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\left(\sum_{s=j_{k-1}}^{j_k} \left(\varphi_s \sum_{i=1}^{j_{k-1}} f_i \right)^q \right)^{\frac{1}{q}} + \left(\sum_{s=j_{k-1}}^{j_k} \left(\varphi_s \sum_{i=j_{k-1}}^{s} f_i \right)^q \right)^{\frac{1}{q}} \right)^\theta \right)^{\frac{1}{\theta}}.$$

In both cases $\theta > 1$ and $0 < \theta \leq 1$, we get that

$$I(f) \ll \left(\sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=1}^{j_{k-1}} f_i \right)^\theta \right)^{\frac{1}{\theta}}$$

$$+ \left(\sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q \left(\sum_{i=j_{k-1}}^{s} f_i \right)^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} = I_1 + I_2. \tag{5.9}$$

Let us estimate I_1

$$I_1 = \left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{j} f_i \right)^\theta \mu(j) \right)^{\frac{1}{\theta}}, \tag{5.10}$$

where

$$\mu(j) = \sum_{k} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \delta(j - j_{k-1})$$

and $\delta(\cdot)$ is the Dirac delta-function. By Theorem A from (5.10) we have

$$I_1 \leq \left\{ \sup_{r\geq 1} \left(\sum_{j=r}^{\infty} \mu(j) \right)^{\frac{1}{\theta}} u_r^{-1} \right\} \|f\|_{p,u}. \tag{5.11}$$

Since

$$\sum_{j=r}^{\infty} \mu(j) = \sum_{j_{k-1}\geq r} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \leq \sum_{n=r}^{\infty} w_n^\theta \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}},$$

we have

$$\sup_{r\geq 1} \left(\sum_{n=r}^{\infty} w_n^\theta \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_r^{-1} \ll B_5. \tag{5.12}$$

From (5.11) and (5.12) we obtain

$$I_1 \leq B_5 \|f\|_{p,u}. \tag{5.13}$$

Let us estimate I_2:

$$I_2 \leq \left(\sum_{k} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{k-1}}^{j_k} f_i \right)^{\theta} \right)^{\frac{1}{\theta}}$$

$$\leq \left(\sum_{k} \left(\sum_{i=j_{k-1}}^{j_k} f_i u_i u_i^{-1} \right)^{p\frac{\theta}{p}} \sum_{n=j_k}^{\infty} w_n^\theta \left(\sum_{s=1}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}}.$$

Using the condition (2.3), we get

$$I_2 \ll \left(\sum_{k} \left(\sum_{i=j_{k-1}}^{j_k} |f_i u_i|^p \right)^{\frac{\theta}{p}} \sup_{i\leq j_k} u_i^{-\theta} \sum_{n=j_k}^{\infty} w_n^\theta \left(\sum_{s=1}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}}$$

$$\leq \left(\sum_{k} \left(\sum_{i=j_{k-1}}^{j_k} |f_i u_i|^p \right)^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}} \sup_{k} \left(\sum_{n=j_k}^{\infty} w_n^\theta \right)^{\frac{1}{\theta}} \left(\sum_{s=1}^{j_k} \varphi_s^q \right)^{\frac{1}{q}} \sup_{i\leq j_k} u_i^{-1}.$$

Therefore, by applying (2.4) with $\alpha = \frac{\theta}{p}$, we obtain that

$$I_2 \ll \left(\sum_{i=1}^{\infty} |f_i u_i|^p \right)^{\frac{1}{p}} \sup_{r\geq 1} \left(\sum_{n=r}^{\infty} w_n^\theta \right)^{\frac{1}{\theta}} \left(\sum_{s=1}^{r} \varphi_s^q \right)^{\frac{1}{q}} \sup_{i\leq r} u_i^{-1},$$

so that

$$I_2 \leq B_6 \|f\|_{p,u}. \tag{5.14}$$

From (5.9), (5.13) and (5.14) we have

$$I(f) \ll \max\{B_5, B_6\}\|f\|_{p,u}. \tag{5.15}$$

Therefore, from inequality (5.15), we get $C \ll B$. The latter together with (5.6) gives that $C \approx B$, where C is the best constant in (1.1). □

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References

[1] G. Bennett, *Some elementary inequalities III.* Quart. J. Math. Oxford Ser. 42 (1991), no. 1, 149–174. https://doi.org/10.1093/qmath/42.1.149

[2] A.L. Bernardis, P. Ortega Salvador, *Some new iterated Hardy-type inequalities and applications.* J. Math. Ineq. 11 (2017), no. 2, 577–594. https://doi.org/10.7153/jmi-11-47

[3] V.I. Burenkov, R. Oinarov, *Necessary and sufficient conditions for boundedness of the Hardy-type operator from a weighted Lebesgue space to a Morrey-type space.* Math. Ineq. Appl. 16 (2013), no. 1, 1–19. https://doi.org/10.7153/mia-16-01

[4] A. Gogatishvili, R. Mustafayev, L.-E. Persson, *Some new iterated Hardy-type inequalities: the case $\theta = 1$.* J. Ineq. Appl. 2013 (2013), no. 515. https://doi.org/10.1186/1029-242X-2013-515

[5] A. Gogatishvili, R. Mustafayev, L.-E. Persson, *Some new iterated Hardy-type inequalities.* J. Funct. Spaces Appl. (2012). http://dx.doi.org/10.1155/2012/734194

[6] M.L. Goldman, E.G. Bakhtigareeva, *Some classes of operators in general Morrey-type spaces.* Eurasian mathematical journal 11, (2020), no. 4, 35–44.

[7] P. Jain, S. Kanjilal, V.D. Stepanov, E.P. Ushakova, *Bilinear Hardy-Steklov operators.* Math. Notes 104 (2018), 823–832. https://doi.org/10.1134/S0001434618110275

[8] A. Kalybay, *On boundedness of the conjugate multidimensional Hardy operator from a Lebesque space to a local Morrey-type space.* Int. J. Math. Anal. 11 (2014), no. 8, 539–553.

[9] R. Oinarov, A. Kalybay, *Boundedness of Riemann-Liouville operator from weighted Sobolev space to weighted Lebesgue space.* Eurasian mathematical journal 12 (2021), no. 1, 39–48.

[10] R. Oinarov, A. Kalybay, *Weighted inequalities for a class of semiadditive operators.* Ann. Funct. Anal. 6 (2015), no. 4, 155–171.

[11] R. Oinarov, B.K. Omarbayeva, A.M. Temirkhanova, *Discrete iterated Hardy-type inequalities with three weights.* Journal of Mathematics, Mechanics and Computer Science 105 (2020), no. 1, 19–29.

[12] B.K. Omarbayeva, L.-E. Persson, A.M. Temirkhanova, *Weighted iterated discrete Hardy-type inequalities.* Math. Ineq. Appl. 23 (2020), no. 3, 943–959. https://doi.org/10.7153/mia-2020-23-73

[13] D.V. Prokhorov, *On a class of weighted inequalities containing quasilinear operators.* Proc. Steklov Inst. Math. 293 (2016), 272–287. https://doi.org/10.1134/S0371968516020199

[14] D.V. Prokhorov, V.D. Stepanov, *On weighted Hardy inequalities in mixed norms.* Proc. Steklov Inst. Math. 283 (2013), 149–164 (in Russian).

[15] M.A. Senouci, *Boundedness of Riemann-Liouville fractional integral operator in Morrey spaces.* Eurasian mathematical journal 12 (2021), no. 1, 82–91.

[16] V.D. Stepanov, G.E. Shambilova, *On iterated and bilinear integral Hardy-type operators.* Math. Ineq. Appl. 22 (2019), no. 4, 1505–1533. https://doi.org/10.7153/mia-2019-22-105

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Events

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INTERNATIONAL CONFERENCE "THEORY OF FUNCTIONS OF SEVERAL REAL VARIABLES" DEDICATED TO THE 90TH BIRTHDAY OF PROFESSOR O.V. BESOV

INFORMATION LETTER

Dear Colleagues!

The V.A. Steklov Mathematical Institute of the Russian Academy of Sciences (MIAN) and World-class Mathematical Center of the V.A. Steklov Mathematical Institute (MCMU MIAN) will hold international conference "Theory of functions of several real variables" through May 29 - June 2, 2023 (MIAN, 8 Gubkin St, Moscow).

Theory of functions of several real variables is an important integral part of modern analysis. It finds numerous applications in the theory of approximation, theory partial differential equations and calculus of variations. The conference will be devoted to the topical issues of the theory of spaces of differentiable functions on domains of Euclidean spaces and metric spaces: embedding theorems, trace theorems, interpolation theory, extension theorems, properties of differential and integral operators, issues of harmonic analysis, widths of classes of functions, integral representations, and approximation of functions.

The conference is dedicated to the 90th anniversary of the birthday of an outstanding world recognised scientist corresponding member of the Russian Academy of Sciences Oleg Vladimirovich Besov. The topics of the conference are related to the directions of research of O.V. Besov and his school.

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