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TIME OPTIMAL CONTROL PROBLEM WITH INTEGRAL CONSTRAINT FOR THE HEAT TRANSFER PROCESS

Sh.A. Alimov, G.I. Ibragimov

Communicated by K.N. Ospanov

Key words: heat transfer process, control function, integral constraint, optimal control, optimal time.

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Abstract. In the present paper a mathematical model of thermocontrol processes is studied. Several convectors are installed on the disjoint subsets Γ_k of the wall $\partial\Omega$ of a volume Ω and each convector produces a hot or cold flow with magnitude equal to $\mu_k(t)$, which are control functions, and on the surface $\partial\Omega \setminus \Gamma$, $\Gamma = \cup\Gamma_k$, a heat exchange occurs by the Newton law. The control functions $\mu_k(t)$ are subjected to an integral constraint. The problem is to find control functions to transfer the state of the process to a given state. A necessary and sufficient condition is found for solvability of this problem. An equation for the optimal transfer time is found, and an optimal control function is constructed explicitly.

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1 Introduction

The control problem for evolution equations is a classical problem. The controllability in finite-dimensional linear systems can be described in terms of the rank of a matrix generated by the coefficient matrix and the matrix of the control action.

Controlled systems described by PDEs are typically infinite-dimensional. There are many works on controllability/observability of systems governed by PDEs. The works of Russell [16] and Lions [15] are classical in this area. However, compared with Kalman's classical theory, the theories on controllability of systems governed by PDEs are not very mature. Important researches in this area can be found in the works [6, 3, 20, 19]. For other related works in this direction, we refer to [7, 9, 18, 17, 2, 1].

The time-optimal control problem for PDFs of parabolic type was first concerned in [10]. More detailed information on the optimal control problems for the systems governed by PDEs is given in the monograph [11].

The decomposition method is widely used in studying control and differential game problems for the systems in distributed parameters. This method leads us to a control problem described by an infinite system of ordinary differential equations (see, for example, [12, 6, 8, 9, 18, 5]). The paper [4] is devoted to the control problem for an infinite system of differential equations.

In the present paper, we study a mathematical model of thermocontrol processes. Several convectors are installed on the disjoint subsets Γ_k of the wall $\partial\Omega$ of a volume Ω and each convector produces a hot or cold flow with magnitude equal to $\mu_k(t)$, which are the control functions, and on the surface $\partial\Omega \setminus \Gamma$, $\Gamma = \cup\Gamma_k$, a heat exchange occurs by the Newton law. The control functions $\mu_k(t)$ are subjected to an integral constraint. The problem is to find control functions to transfer the state of the process to a given state. We obtain a necessary and sufficient condition of solvability of the problem. We find an equation for the optimal transfer time, and construct an optimal control function explicitly.

2 Statement of problem

We study the following heat equation [2]

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - p(x)u(x,t), \quad p(x) \geq 0, t > 0, \tag{2.1}$$

with the boundary conditions

$$\frac{\partial u(x,t)}{\partial n} = \mu_k(t)a_k(x), \quad x \in \Gamma_k, \quad t > 0, \tag{2.2}$$

and

$$\frac{\partial u(x,t)}{\partial n} + h(x)u(x,t) = 0, \quad x \in \partial\Omega \setminus \Gamma_k, \quad t > 0, \tag{2.3}$$

and the initial condition

$$u(x,0) = 0, \tag{2.4}$$

where Ω is a subset of $\mathbb{R}^n$ whose boundary $\partial\Omega$ is piecewise smooth, Γ_k are disjoint subsets of $\partial\Omega$ which are convectors (heaters or coolers). It is assumed that the boundaries $\partial\Gamma_k$ of Γ_k are piecewise smooth, $\Gamma = \bigcup_{k=1}^{m} \Gamma_k$. The functions $h(x)$ (the thermal conductivity of the walls), $a_k(x)$ (the power density of the k-th convector) and $p(x)$ are given, $h(x)$ and $a_k(x)$ are assumed to be given piecewise smooth non-negative non-trivial functions, $p(x)$ is a sufficiently smooth function in $\bar{\Omega} = \Omega \cup \partial\Omega$.

The meaning of boundary conditions (2.2) and (2.3) is that each convector produces a hot or cold flow with magnitude of output given by a measurable real-valued function $\mu_k(t)$, and on the surface $\partial\Omega \setminus \Gamma$ a heat exchange occurs by the Newton law.

Let

$$\mu(t) = (\mu_1(t), \mu_2(t), ..., \mu_m(t)), \quad \mu : [0,\infty) \to \mathbb{R}^m, \quad \mu(\cdot) \in L_2[0,\infty). \tag{2.5}$$

Definition 1. We call a function $\mu : [0,\infty) \to \mathbb{R}^m$ with measurable coordinates $\mu_i(t)$, $t \geq 0$, $i = 1, ..., m$, an admissible control if it satisfies the following integral constraint

$$\int_0^\infty |\mu(t)|^2 dt \leq \rho^2, \tag{2.6}$$

where ρ is a given positive number.

We extend the functions $h(x)$ and $a(x)$ to the whole boundary $\partial\Omega$ by setting $h(x) = 0$ for $x \in \Gamma$ and $a_k(x) = 0$ for $x \in \partial\Omega \setminus \Gamma_k$ [2].

Next, consider the following vector-functions

$$a(x) = (a_1(x), a_2(x), ..., a_m(x)), \quad a : \partial\Omega \to \mathbb{R}^m, \tag{2.7}$$

Using (2.5) and (2.7) we can combine conditions (2.2) and (2.3) as follows

$$\frac{\partial u(x,t)}{\partial n} + h(x)u(x,t) = \mu(t) \cdot a(x), \quad x \in \partial\Omega, \quad t > 0, \tag{2.8}$$

We define a generalized solution of the initial-boundary value problem (2.1), (2.4), (2.8) as a function $u(x,t)$ that satisfies the equation

$$\int_0^t ds \int_\Omega [\nabla u(x,s)\nabla \eta(x,s) + p(x)u(x,s)\eta(x,s)]dx$$
$$-\int_0^t ds \int_\Omega [u(x,s)\frac{\partial \eta(x,s)}{\partial s}dx + \int_\Omega u(x,s)\eta(x,s)]dx$$
$$= \int_0^t ds \int_{\partial\Omega} [\mu(s)\cdot a(x)]\eta(x,s)d\sigma(x) - \int_0^t ds \int_{\partial\Omega} h(x)u(x,s)\eta(x,s)d\sigma(x) \tag{2.9}$$

for $0 < t \le T$, for any number $T > 0$ and any function $\eta(x,t) \in W_2^{1,1}(\Omega \times [0,T])$ (see formula (5.5) and Theorem 5.1 in [14], III.5).

Next, we define generalized solution of the eigenvalue problem for the Laplace operator [2]

$$-\Delta v(x) + p(x)v(x) = \lambda v(x), \quad x \in \Omega, \tag{2.10}$$

with the boundary condition

$$\frac{\partial v(x)}{\partial n} + h(x)v(x) = 0, \quad x \in \partial\Omega, \tag{2.11}$$

as a function $v(x)$ in the Sobolev space $W_2^1(\Omega)$ which satisfies the equation

$$\int_\Omega [\nabla v(x)\nabla \eta(x) + p(x)v(x)\eta(x)]dx + \int_{\partial\Omega} h(x)v(x)\eta(x)d\sigma(x) = \lambda \int_\Omega v(x)\eta(x)dx \tag{2.12}$$

for any function $\eta \in W_2^1(\Omega)$ (see [13], Sec. III.6, formula (6.3)).

We consider this problem in the Hilbert space $L_2(\Omega)$ with the inner product $(u,v) = \int_\Omega u(x)v(x)dx$ and norm $||u|| = \sqrt{(u,u)}$. It is well known that under the above assumptions there exists a sequence of positive eigenvalues $\{\lambda_i\}_{i=1}^\infty$ such that

$$0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_i \le ..., \quad \lambda_i \to \infty, \quad i \to \infty,$$

and the corresponding eigenfunctions $v_i(x)$ form an orthonormal basis $\{v_i\}_{i=1}^\infty$ in $L_2(\Omega)$ (see, for example, [13], Sec. III.6).

We will investigate the following problem: Let $u_0(x) \in L_2(\Omega)$. Find a time θ and an admissible control $\mu(t)$, $t \ge 0$, such that the solution $u(x,t)$ of the initial-boundary value problem (2.1), (2.4), (2.6), (2.8) exists, is unique and satisfies the following condition

$$(u(x,\theta), v_i(x)) = (u_0(x), v_i(x)), \quad i = 1, 2, ..., m. \tag{2.13}$$

3 Main result

3.1 Integral equation for $\mu(t)$

We use some properties of the Green function G [2] defined by the following equation:

$$G(x,y,t) = \sum_{i=1}^\infty e^{-\lambda_i t} v_i(x)v_i(y), \quad x, y \in \Omega \cup \partial\Omega, \quad t > 0. \tag{3.1}$$

Since $h(x) \geq 0$, $x \in \partial\Omega$, and $h(x)$ is not identically 0. Then

$$G(x,y,t) \geq 0, \quad (x,y) \in \bar{\Omega} \times \bar{\Omega}, \quad t > 0,$$

and the solution of boundary-value problem (2.1), (2.4), (2.8) can be represented by the Green function as follows

$$u(x,t) = \int_0^t ds \int_{\partial\Omega} G(x,y,t-s)\mu(s) \cdot a(y) d\sigma(y), \tag{3.2}$$

where $a(y)$, $y \in \partial\Omega$, and $\mu(s)$, $s \geq 0$, are defined by (2.5) and (2.7). Since $\mu(t) \cdot a(x) = \sum_{j=1}^{m} \mu_j(t) a_j(x)$, we obtain

$$u(x,t) = \sum_{j=1}^{m} \int_0^t \mu_j(s) ds \int_{\partial\Omega} G(x,y,t-s) a_j(y) d\sigma(y). \tag{3.3}$$

By the condition (2.13) we have

$$\int_\Omega u(x,\theta) v_i(x) dx = \int_\Omega u_0(x) v_i(x) dx \doteq c_i, \quad c_i \in \mathbb{R}, \quad i = 1,2,...,m. \tag{3.4}$$

To evaluate the integral in the left-hand side of (3.4), we substitute (3.3) into (3.4) to obtain

$$\int_\Omega u(x,\theta) v_i(x) dx = \int_\Omega v_i(x) dx \sum_{j=1}^{m} \int_0^\theta \mu_j(s) ds \int_{\partial\Omega} G(x,y,\theta-s) a_j(y) d\sigma(y) \tag{3.5}$$

By (3.1)

$$\int_\Omega G(x,y,t) v_i(x) dx = e^{-\lambda_i t} v_i(y), \quad y \in \Omega \cup \partial\Omega. \tag{3.6}$$

Then equation (3.4) takes the form

$$\sum_{j=1}^{m} \int_0^\theta e^{-\lambda_i(\theta-s)} \mu_j(s) ds \int_{\partial\Omega} v_i(y) a_j(y) d\sigma(y) = c_i, \quad t > 0. \tag{3.7}$$

Denote

$$\int_{\partial\Omega} v_i(y) a_j(y) d\sigma(y) = a_{ij}, \quad i,j = 1,2,...,m. \tag{3.8}$$

We obtain then the following equations

$$\sum_{j=1}^{m} \int_0^\theta e^{-\lambda_i(\theta-s)} a_{ij} \mu_j(s) ds = c_i, \quad i,j = 1,2,...,m, \tag{3.9}$$

which can be written as

$$\int_0^\theta A(\theta-s)\mu(s) ds = c, \quad c^T = (c_1, c_2, ..., c_m), \quad \mu^T(s) = (\mu_1(s), ..., \mu_m(s)), \tag{3.10}$$

(see (2.5)) where

$$A(\theta-s)=\begin{bmatrix} a_{11}e^{-\lambda_1(\theta-s)} & a_{12}e^{-\lambda_1(\theta-s)} & \dots & a_{1m}e^{-\lambda_1(\theta-s)} \\ a_{21}e^{-\lambda_2(\theta-s)} & a_{22}e^{-\lambda_2(\theta-s)} & \dots & a_{2m}e^{-\lambda_2(\theta-s)} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1}e^{-\lambda_m(\theta-s)} & a_{m2}e^{-\lambda_m(\theta-s)} & \dots & a_{mm}e^{-\lambda_m(\theta-s)} \end{bmatrix} = \begin{bmatrix} e^{-\lambda_1(\theta-s)}a_1^T \\ e^{-\lambda_2(\theta-s)}a_2^T \\ \vdots \\ e^{-\lambda_m(\theta-s)}a_m^T \end{bmatrix} \tag{3.11}$$

is a $m\times m$ matrix, where $a_i^T=(a_{i1},a_{i2},...,a_{im})$, $i=1,2,...,m$, are row vectors of the matrix

$$A_0=A(0)=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}.$$

3.2 Important subspaces

Next, we study the problem of finding an admissible control $\mu(t)$ that satisfies equation (3.9) for some time θ. To this end we consider the following operator $L: L_2[0,\infty)\to\mathbb{R}^m$ defined by the equation

$$L\mu=L(\theta)\mu=\int_0^\theta A(\theta-s)\mu(s)ds, \quad \mu(\cdot)\in L_2[0,\infty), \tag{3.12}$$

where $\mu(t)$ does not need to satisfy (2.6), and the Gram matrix

$$W=W(\theta)=\int_0^\theta A(\theta-s)A^T(\theta-s)ds \tag{3.13}$$

where A^T is the transpose of A. Clearly, by (3.11)

$$A^T(\theta-s)=\left[e^{-\lambda_1(\theta-s)}a_1,\ e^{-\lambda_2(\theta-s)}a_2,\ \dots,\ e^{-\lambda_m(\theta-s)}a_m\right].$$

We have

$$A(\theta-s)A^T(\theta-s)=\begin{bmatrix} e^{-\lambda_1(\theta-s)}a_1^T \\ e^{-\lambda_2(\theta-s)}a_2^T \\ \vdots \\ e^{-\lambda_m(\theta-s)}a_m^T \end{bmatrix}\left[\ e^{-\lambda_1(\theta-s)}a_1,\quad e^{-\lambda_2(\theta-s)}a_2,\quad \dots,\quad e^{-\lambda_m(\theta-s)}a_m\ \right]$$

$$=\begin{bmatrix} e^{-2\lambda_1(\theta-s)}a_1^Ta_1 & e^{-(\lambda_1+\lambda_2)(\theta-s)}a_1^Ta_2 & \dots & e^{-(\lambda_1+\lambda_m)(\theta-s)}a_1^Ta_m \\ e^{-(\lambda_2+\lambda_1)(\theta-s)}a_2^Ta_1 & e^{-2\lambda_2(\theta-s)}a_2^Ta_2 & \dots & e^{-(\lambda_2+\lambda_m)(\theta-s)}a_2^Ta_m \\ \dots & \dots & \dots & \dots \\ e^{-(\lambda_m+\lambda_1)(\theta-s)}a_m^Ta_1 & e^{-(\lambda_m+\lambda_2)(\theta-s)}a_m^Ta_2 & \dots & e^{-2\lambda_m(\theta-s)}a_m^Ta_m \end{bmatrix}, \tag{3.14}$$

$$
\begin{aligned}
W &= W(\theta) = \int_0^\theta A(\theta-s)A^T(\theta-s)ds \\
&= \begin{bmatrix}
a_1^Ta_1\int_0^\theta e^{-2\lambda_1(\theta-s)}ds & a_1^Ta_2\int_0^\theta e^{-(\lambda_1+\lambda_2)(\theta-s)}ds & \dots & a_1^Ta_m\int_0^\theta e^{-(\lambda_1+\lambda_m)(\theta-s)}ds \\
a_2^Ta_1\int_0^\theta e^{-(\lambda_2+\lambda_1)(\theta-s)}ds & a_2^Ta_2\int_0^\theta e^{-2\lambda_2(\theta-s)}ds & \dots & a_2^Ta_m\int_0^\theta e^{-(\lambda_2+\lambda_m)(\theta-s)}ds \\
\dots & \dots & \dots & \dots \\
a_m^Ta_1\int_0^\theta e^{-(\lambda_m+\lambda_1)(\theta-s)}ds & a_m^Ta_2\int_0^\theta e^{-(\lambda_m+\lambda_2)(\theta-s)}ds & \dots & a_m^Ta_m\int_0^\theta e^{-2\lambda_m(\theta-s)}ds
\end{bmatrix}
\end{aligned}
\tag{3.15}
$$

Thus, $W(\theta)$ is an $m \times m$ symmetric matrix.

Denote by

$$
R(L) = \left\{ x \mid x = \int_0^\theta A(\theta-s)\mu(s)ds, \quad \mu(\cdot) \in L_2[0,\infty) \right\}
\tag{3.16}
$$

the range of the operator L, and by

$$
R(W(\theta)) = \{x \mid x = W(\theta)\eta, \quad \eta^T = (\eta_1, \eta_2, ..., \eta_m) \in \mathbb{R}^m\}
\tag{3.17}
$$

the range of the Gram matrix $W(\theta)$. Since the matrix $W(\theta)$ is symmetric, therefore $R(W(\theta))$ is a row space as well as a column space of the matrix $W(\theta)$. Note that $R(L)$ and $R(W(\theta))$ are subspaces of $\mathbb{R}^m$. We prove the following statement.

Lemma 3.1. *$R(L) = R(W(\theta))$ for any $\theta > 0$.*

Proof. 1) First, show that $R(W(\theta)) \subset R(L)$ where $\theta > 0$ is any fixed number. Let $x \in R(W)$. Then $x = W\eta$ for some $\eta \in \mathbb{R}^m$. Choose the control

$$
\mu(t) = A^T(\theta-t)\eta, \quad 0 \le t \le \theta.
\tag{3.18}
$$

For this control,

$$
L(\theta)\mu = \int_0^\theta A(\theta-s)\mu(s)ds = \int_0^\theta A(\theta-s)A^T(\theta-s)\eta ds = W(\theta)\eta = x.
\tag{3.19}
$$

Thus, $x \in R(L)$.

2) We show now that $R(L) \subset R(W)$. Let $x \in R(L)$. Then there exists $\mu(\cdot) \in L_2[0,\infty)$ such that

$$
x = \int_0^\theta A(\theta-s)\mu(s)ds.
\tag{3.20}
$$

Assume the contrary, $x \notin R(W)$. Then by the fact that the subspace $ker(W) = \{x \in \mathbb{R}^m \mid Wx = 0\}$ is orthogonal to the row space of W and, hence, to $R(W)$, the vector x can be represented as follows

$$
x = x_1 + x_2, \quad x_1 \in R(W), \quad x_2 \in ker(W), \quad x_2 \ne 0.
\tag{3.21}
$$

Note that $x_2 \neq 0$ since otherwise $x = x_1 \in R(W)$ which contradicts our assumption.

Since $Wx_2 = 0$, we have $x_2^T W x_2 = 0$, hence,

$$x_2^T W x_2 = \int_0^\theta |x_2^T A(\theta - s)|^2 ds = 0, \tag{3.22}$$

and so

$$x_2^T A(\theta - s) = 0, \quad 0 \leq s \leq \theta.$$

Consequently,

$$x_2^T \cdot x = x_2^T \int_0^\theta A(\theta - s)\mu(s)ds = \int_0^\theta x_2^T A(\theta - s)\mu(s)ds = 0.$$

However,

$$x_2^T \cdot x = x_2^T \cdot x_1 + x_2^T \cdot x_2 = |x_2|^2 \neq 0.$$

Contradiction. The proof of the lemma is complete. □

Corollary 3.1. *Equation* (3.10) *is satisfied for some* θ *and* $\mu(t)$, $0 \leq t \leq \theta$, *if and only if* $c \in R(W)$.

Next, we denote $B = [A_0, \Lambda A_0, \Lambda^2 A_0, ..., \Lambda^{m-1} A_0]$ which is an $m \times m^2$ matrix, where

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda_m^k \end{bmatrix}, \quad k = 1, 2, ..., m-1. \tag{3.23}$$

The following lemma shows that the subspace $R(W(\theta))$ does not depend on θ.

Lemma 3.2. $R(B) = R(W(\theta))$ *for any* $\theta > 0$.

Proof. 1) Show that $R(W(\theta)) \subset R(B)$. Indeed, let $x \in R(W(\theta))$ for some θ. By (3.11) we have

$$\begin{aligned} A(\theta - s) &= \begin{bmatrix} a_1^T \left(1 - \frac{\lambda_1(\theta-s)}{1!} + \frac{\lambda_1^2(\theta-s)^2}{2!} - \frac{\lambda_1^3(\theta-s)^3}{3!} + ...\right) \\ a_2^T \left(1 - \frac{\lambda_2(\theta-s)}{1!} + \frac{\lambda_2^2(\theta-s)^2}{2!} - \frac{\lambda_2^3(\theta-s)^3}{3!} + ...\right) \\ \vdots \\ a_m^T \left(1 - \frac{\lambda_m(\theta-s)}{1!} + \frac{\lambda_m^2(\theta-s)^2}{2!} - \frac{\lambda_m^3(\theta-s)^3}{3!} + ...\right) \end{bmatrix} \\ &= \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} - \frac{(\theta - s)}{1!} \begin{bmatrix} \lambda_1 a_1^T \\ \lambda_2 a_2^T \\ \vdots \\ \lambda_m a_m^T \end{bmatrix} + \frac{(\theta - s)^2}{2!} \begin{bmatrix} \lambda_1^2 a_1^T \\ \lambda_2^2 a_2^T \\ \vdots \\ \lambda_m^2 a_m^T \end{bmatrix} - \frac{(\theta - s)^3}{3!} \begin{bmatrix} \lambda_1^3 a_1^T \\ \lambda_2^3 a_2^T \\ \vdots \\ \lambda_m^3 a_m^T \end{bmatrix} + ... \\ &= A_0 - \frac{(\theta - s)}{1!}\Lambda^1 A_0 + \frac{(\theta - s)^2}{2!}\Lambda^2 A_0 - \frac{(\theta - s)^3}{3!}\Lambda^3 A_0 + ... \end{aligned} \tag{3.24}$$

By the Cayley-Hamilton theorem every square matrix satisfies its characteristic equation, therefore Λ^k for $k \geq m$, can be represented as a linear combination of the matrices I, Λ, Λ^2,..., Λ^{m-1}. Using this fact and (3.24) we obtain

$$A(\theta - s) = \sum_{k=0}^{m-1} \beta_k(\theta - s)\Lambda^k A_0$$

for some scalar functions $\beta_k(\theta - s)$. By Lemma 3.1 $R(W(\theta)) = R(L)$, therefore $x \in R(L)$, and, hence, there exists $\mu(t)$, $0 \le t \le \theta$, such that

$$x = \int_0^\theta A(\theta - s)\mu(s)ds = \sum_{k=0}^{m-1} \Lambda^k A_0 \int_0^\theta \beta_k(\theta - s)\mu(s)ds$$
$$= \sum_{k=0}^{m-1} \Lambda^k A_0 \eta_k = B\eta \in R(B),$$

where $\eta^T = [\eta_0^T, \eta_1^T, ..., \eta_{m-1}^T] \in \mathbb{R}^{m^2}$,

$$\eta_k = \int_0^\theta \beta_k(\theta - s)\mu(s)ds \in \mathbb{R}^m, \quad k = 0, 1, 2, ..., m-1.$$

Thus, $x \in R(B)$.

2) Show that $R(B) \subset R(W(\theta))$. Let $x \in R(B)$. Then $x = B\eta$ for some $\eta \in \mathbb{R}^{m^2}$. We show that $x \in R(W(\theta))$. Assume the contrary, $x \notin R(W(\theta))$ for some $\theta > 0$. Then $x = x_1 + x_2$ with $x_1 \in R(W(\theta))$, $x_2 \in ker(W(\theta))$, where $x_2 \neq 0$ since otherwise $x = x_1 \in R(W(\theta))$. Note that

$$x^T \cdot x_2 = x_1^T x_2 + x_2^T \cdot x_2 = |x_2|^2 \neq 0.$$

From the inclusion $x_2 \in ker(R(W(\theta)))$ we obtain $W(\theta)x_2 = 0$ and so

$$x_2^T W(\theta) x_2 = \int_0^\theta |x_2^T A(\theta - s)|^2 ds = 0.$$

This implies that $x_2^T A(\theta - s) = 0$, $0 \le s \le \theta$. Differentiating this equation k times for $k = 0, 1, ..., m-1$ and letting $t = \theta$ we obtain

$$x_2^T A_0 = 0, \quad x_2^T \Lambda A_0 = 0, \quad ..., \quad x_2^T \Lambda^{m-1} A_0 = 0.$$

Thus, $x_2^T B = 0$. Then, $x_2^T x = x_2^T B\eta = 0$. This contradicts the condition $x_2^T x \neq 0$. Therefore, $x \in R(W(\theta))$. □

It should be noted that Lemma 3.2 shows that $R(W(\theta))$, the subspace of $\mathbb{R}^m$, does not depend on θ. Also, this lemma implies that $rank(W(\theta)) = rank(B)$.

Lemma 3.3. *If $rank(B) = m$, then for any $\theta > 0$, the matrix $W(\theta)$ is positive definite.*

Proof. For any $x \in \mathbb{R}^m$, $x \neq 0$, we have

$$x^T W(\theta) x = \int_0^\theta x^T A(\theta - s) A^T(\theta - s) x ds = \int_0^\theta |A^T(\theta - s)x|^2 ds \ge 0,$$

and if we assume that $x^T W(\theta) x = 0$ for some $x \neq 0$, then

$$\int_0^\theta |x^T A(\theta - s)|^2 ds = 0,$$

and so $x^T A(\theta - s)x = 0$ for all $0 \le s \le \theta$. Taking derivatives of this equation with respect to s and evaluating at $s = \theta$ we have

$$x^T \Lambda^k A_0 = 0, \quad k = 0, 1, 2, ...$$

This implies that $x^T B = 0$, and so $rank(B) < m$. Contradiction, since $rank(B) = rank(W(\theta)) = m$. Thus, $x^T W(\theta)x$ cannot equal to 0 for any $x \neq 0$. Hence, $W(\theta)$ is positive definite. □

Further, we assume that $rank(B) = m$. Then $det(W(t)) \neq 0$ for any $t > 0$ and the equation $W(\theta)x = c$ has the unique solution $x = W^{-1}(\theta)c$.

Lemma 3.4. *For any $c \in \mathbb{R}^m$, $c \neq 0$, the function $g(t) = c^T W^{-1}(t)c$, $t > 0$, is non-increasing and $\lim\limits_{t\to+0} g(t) = +\infty$.*

Proof. We show first that $g(t)$, $t > 0$, is non-increasing. Since $det(W(t)) \neq 0$, differentiating the equation $W^{-1}(t)W(t) = I$, where I is the $m \times m$ identity matrix, we obtain

$$\frac{d}{dt}\left(W^{-1}(t)\right) W(t) + W^{-1}(t)\frac{d}{dt}\left(W(t)\right) = 0.$$

Hence, the derivative of the inverse matrix is

$$\frac{d}{dt}\left(W^{-1}(t)\right) = -W^{-1}(t)\frac{d}{dt}\left(W(t)\right) W^{-1}(t).$$

Then,

$$\frac{d}{dt}g(t) = c^T \frac{d}{dt}\left(W^{-1}(t)\right) c = -c^T W^{-1}(t)\frac{d}{dt}\left(W(t)\right) W^{-1}(t)c.$$

It is not difficult to verify that

$$\begin{aligned} \frac{d}{dt}\left(W(t)\right) &= \begin{bmatrix} a_1^T a_1 e^{-2\lambda_1 t} & a_1^T a_2 e^{-(\lambda_1+\lambda_2)t} & \dots & a_1^T a_m e^{-(\lambda_1+\lambda_m)t} \\ a_2^T a_1 e^{-(\lambda_2+\lambda_1)t} & a_2^T a_2 e^{-2\lambda_2 t} & \dots & a_2^T a_m e^{-(\lambda_2+\lambda_m)t} \\ \vdots & \vdots & \vdots & \vdots \\ a_m^T a_1 e^{-(\lambda_m+\lambda_1)t} & a_m^T a_2 e^{-(\lambda_m+\lambda_2)t} & \dots & a_m^T a_m e^{-(\lambda_m+\lambda_m)t} \end{bmatrix} \\ &= A(t)A^T(t), \quad A^T(t) = \left[e^{-\lambda_1 t}a_1,\ e^{-\lambda_2 t}a_2,\ \dots,\ e^{-\lambda_m t}a_m\right], \end{aligned} \tag{3.25}$$

and so

$$\frac{d}{dt}g(t) = -c^T W^{-1}(t)A(t)A^T(t)W^{-1}(t)c = -|A^T(t)W^{-1}(t)c|^2 \le 0.$$

Thus, $g(t)$ is non-increasing.

Next, we show that $\lim\limits_{t\to+0} g(t) = +\infty$. Since $W(t)$ is symmetric, there exists an orthogonal matrix $Q(t)$ (by definition $Q^{-1}(t) = Q^T(t)$) such that

$$W(t) = Q(t)D(t)Q^T(t), \quad D(t) = \begin{bmatrix} \nu_1(t) & 0 & 0 & \dots & 0 \\ 0 & \nu_2(t) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \nu_m(t) \end{bmatrix},$$

where $\nu_1(t), ..., \nu_m(t)$ are eigenvalues of the matrix $W(t)$. Since by Lemma 3.3 the matrix $W(t)$ is positive definite, therefore the eigenvalues of $W(t)$ are positive, that is, $\nu_i(t) > 0$ for all $i = 1, ..., m$.

Recall, $Q(t)$ as an orthogonal matrix has the following properties $Q(t)Q^T(t) = Q^T(t)Q(t) = I$, and for any $x \in \mathbb{R}^m$, $|Q(t)x| = |Q^T(t)x| = |x|$. Then,

$$W^{-1}(t) = Q(t)D^{-1}(t)Q^T(t), \quad D^{-1}(t) = \begin{bmatrix} \frac{1}{\nu_1(t)} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\nu_2(t)} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{\nu_m(t)} \end{bmatrix},$$

Letting $\xi(t) = Q^T(t)c$ we have $|\xi(t)| = |Q^T(t)c| = |c|$ and

$$\begin{aligned} c^T W(t)c &= c^T Q(t)D(t)Q^T(t)c = \xi^T(t)D(t)\xi(t) \\ &= \nu_1(t)\xi_1^2(t) + \dots + \nu_m(t)\xi_m^2(t), \end{aligned} \tag{3.26}$$

where $\xi(t) = (\xi_1(t), ..., \xi_m(t))$, $|\xi(t)|^2 = \xi_1^2(t) + ... + \xi_m^2(t) = |c|^2$.

$$\begin{aligned} g(t) &= c^T W^{-1}(t)c = c^T Q(t)D^{-1}(t)Q^T(t)c \\ &= \xi^T(t)D^{-1}(t)\xi(t) = \frac{\xi_1^2(t)}{\nu_1(t)} + \dots + \frac{\xi_m^2(t)}{\nu_m(t)}. \end{aligned} \tag{3.27}$$

For the entries $w_{ij}(t)$, $i, j \in I$, of the matrix $W(t)$ we have

$$w_{ij}(t) = a_i^T a_j \int_0^t e^{-(\lambda_i+\lambda_j)(t-s)} ds \to 0 \text{ as } t \to +0. \tag{3.28}$$

Therefore, $c^T W(t)c \to 0$ as $t \to +0$, and hence by (3.26)

$$\nu_1(t)\xi_1^2(t) + \dots + \nu_m(t)\xi_m^2(t) \to 0$$

as $t \to +0$. Consequently, $\nu_i(t)\xi_i^2(t) \to 0$ for all $i = 1, 2, ..., m$ as $t \to +0$.

Since the sphere $\xi_1^2(t) + ... + \xi_m^2(t) = |c|^2 \neq 0$ is a compact set, the sequence $\xi(t_n)$, $n = 1, 2, ...,$ where $t_n \to 0$ as $n \to \infty$, contains a convergent subsequence. Without restriction of generality, we assume that the sequence $\xi(t_n)$, $n = 1, 2, ...,$ is convergent and

$$\xi(t_n) \to \xi_0 = (\xi_{10}, ..., \xi_{m0}), \quad |\xi_0| = |c| \text{ as } n \to \infty.$$

Let $\xi_{s0} \neq 0$ for some $1 \le s \le m$. We obtain then from $\nu_s(t_n)\xi_s^2(t_n) \to 0$ and $\xi_s(t_n) \to \xi_{s0} \neq 0$ that $\nu_s(t_n) \to 0$ as $n \to \infty$. Next, letting $t = t_n$ in (3.27) and passing to the limit as $n \to \infty$ we obtain

$$g(t_n) = \frac{\xi_1^2(t_n)}{\nu_1(t_n)} + \dots + \frac{\xi_s^2(t_n)}{\nu_s(t_n)} + \dots + \frac{\xi_m^2(t_n)}{\nu_m(t_n)} \to +\infty$$

since $\xi_s(t_n) \to \xi_{s0} \neq 0$ and $0 < \nu_s(t_n) \to 0$. Hence, $g(t) \to +\infty$ as $t \to +0$. $\square$

Lemma 3.4 implies that $\inf_{t>0} g(t) = \lim_{t\to\infty} g(t)$. Let

$$\rho_0 \doteq \left(\lim_{t\to\infty} g(t)\right)^{1/2} = \left(\lim_{t\to\infty} c^T W^{-1}(t)c\right)^{1/2}.$$

3.3 Necessary and sufficient condition for solvability of Problem 1

Now, we turn to Problem 1, or equivalently, to the problem of finding a control $\mu(t)$ and time θ such that
$$\int_0^\theta A(\theta-s)\mu(s)ds = c, \quad \int_0^\theta |\mu(s)|^2 ds \le \rho^2. \tag{3.29}$$

Theorem 3.1. *Let* $rank(B) = m$. *(i) If* $\rho > \rho_0$, *then Problem 1 is solvable; (ii) if Problem 1 is solvable, then* $\rho \ge \rho_0$.

Proof. (i) Let $\rho > \rho_0$. Since the function $g(t) = c^T W^{-1}(t)c$ is continuous and decreases on $t \in (0, \infty)$ from $+\infty$ to ρ_0^2, therefore, there exists a time $\theta > 0$ such that $c^T W^{-1}(\theta)c = \rho^2$. Assume that θ is the first time that satisfies this equation. Set
$$\mu(t) = A^T(\theta - t)\eta, \quad 0 \le t \le \theta, \quad \eta = W^{-1}(\theta)c.$$

Then $\mu(t)$ is admissible since $W^{-1}(\theta)$ is symmetric and
$$\int_0^\theta |\mu(t)|^2 dt = \int_0^\theta |A^T(\theta-s)\eta|^2 ds = \eta^T W(\theta)\eta = c^T W^{-1}(\theta)c = \rho^2. \tag{3.30}$$

Also, we have
$$\int_0^\theta A(\theta-s)\mu(s)ds \quad = \quad \int_0^\theta A(\theta-s)A^T(\theta-s)\eta ds = W(\theta)\eta = c. \tag{3.31}$$

Hence, Problem 1 is solvable.

We turn to the part (ii) of the theorem. Let Problem 1 be solvable. Then there exists a time τ and a control $\mu(t)$, $0 \le t \le \tau$, such that
$$\int_0^\tau A(\tau-s)\mu(s)ds = c, \quad \int_0^\tau |\mu(s)|^2 ds \le \rho^2. \tag{3.32}$$

We show that $\rho \ge \rho_0$. Clearly, for the control
$$\mu_0(t) = A^T(\tau - t)\eta_0, \quad 0 \le t \le \tau, \quad \eta_0 = W^{-1}(\tau)c,$$
we have
$$\int_0^\tau A(\tau-s)\mu_0(s)ds = c, \quad \int_0^\tau |\mu_0(s)|^2 ds = \eta_0^T W(\tau)\eta_0. \tag{3.33}$$

If we show the inequality
$$\int_0^\tau |\mu(s)|^2 ds \ge \eta_0^T W(\tau)\eta_0, \tag{3.34}$$

then in view of (3.32) and (3.33) we obtain the inequality $\int_0^\tau |\mu_0(s)|^2 ds \le \rho^2$.

To show (3.34), we multiply by η_0 the both sides of equation in (3.32) to obtain

$$\int_0^\tau \eta_0^T A(\tau - s)\mu(s)ds = \eta_0^T c = \eta_0^T W(\tau)\eta_0.$$

Using the Cauchy-Schwartz inequality in the left-hand side yields

$$\begin{aligned} \eta_0^T W(\tau)\eta_0 &= \int_0^\tau \eta_0^T A(\tau - s)\mu(s)ds \\ &\leq \left(\int_0^\tau |\eta_0^T A(\tau - s)|^2 ds\right)^{1/2} \left(\int_0^\tau |\mu(s)|^2 ds\right)^{1/2} \\ &\leq \left(\eta_0^T W \eta_0\right)^{1/2} \left(\int_0^\tau |\mu(s)|^2 ds\right)^{1/2}. \end{aligned} \tag{3.35}$$

This implies (3.34). Hence,

$$\eta_0^T W(\tau)\eta_0 = \int_0^\tau |\mu_0(s)|^2 ds \leq \int_0^\tau |\mu(s)|^2 ds \leq \rho^2.$$

Consequently, we have

$$\rho_0^2 = \inf_{t>0} \eta_0^T W(t)\eta_0 \leq c^T W^{-1}(\tau)c = \eta_0^T W(\tau)\eta_0 \leq \rho^2,$$

which is the desired result. □

3.4 Optimal transfer time and optimal control

Let $\rho > \rho_0$. As denoted above that $t = \theta$ is the minimum root of the equation

$$c^T W^{-1}(t)c = \rho^2. \tag{3.36}$$

Theorem 3.2. *The number θ, the root of equation (3.36), is the optimal transfer time of the state $u(x, t)$ from the state $u(x, 0) = 0$ to the state for which $(u(x, \theta), v_i(x)) = (u_0(x), v_i(x))$, $i = 1, ..., m$.*

Proof. We show that the control

$$\mu(t) = A^T(\theta - t)\eta, \quad 0 \leq t \leq \theta, \quad \eta = W^{-1}(\theta)c,$$

which satisfies equation (3.10), is optimal. Assume the contrary, let for some control $\bar{\mu}(t)$, $0 \leq t \leq \theta_0$, $\theta_0 < \theta$,

$$\int_0^{\theta_0} A(t - s)\bar{\mu}(s)ds = c, \quad \int_0^{\theta_0} |\bar{\mu}(s)|^2 ds \leq \rho^2. \tag{3.37}$$

Then, it is not difficult to verify that the control

$$\mu_0(t) = A^T(\theta_0 - t)\eta_0, \quad 0 \leq t \leq \theta_0, \quad \eta_0 = W^{-1}(\theta_0)c,$$

satisfies the relations

$$\int_0^{\theta_0} A(t-s)\mu_0(s)ds = c, \quad \int_0^{\theta_0} |\mu_0(s)|^2 ds \le \rho^2. \tag{3.38}$$

Thus,

$$\rho^2 \ge \int_0^{\theta_0} |\mu_0(s)|^2 ds = \eta_0^T W(\theta_0)\eta_0 = c^T W^{-1}(\theta_0)c \ge c^T W^{-1}(\theta)c = \rho^2.$$

Hence,

$$c^T W^{-1}(\theta_0)c = \rho^2,$$

which contradicts the fact that θ is the smallest root of equation (3.36). Thus, θ is the optimal transfer time. □

4 Conclusions

In the present paper, we have studied a mathematical model of thermocontrol processes. The control functions $\mu_k(t)$ are subjected to an integral constraint. The problem is to find control functions to transfer the state of the process to a given state. We have found a necessary and sufficient condition for existence of a control function which transfers the state of the system to a given state. Also, we have found an equation for the optimal transfer time, and constructed an optimal control function that transfers the state of the system to a given state.

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APPLICATIONS OF λ-TRUNCATIONS TO THE STUDY OF LOCAL AND GLOBAL SOLVABILITY OF NONLINEAR EQUATIONS

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Abstract. In this paper, we consider the equation $F(x) = y$ in a neighbourhood of a given point $\bar{x}$, where F is a given continuous mapping between finite-dimensional real spaces. We study a class of polynomial mappings and show that these polynomials satisfy certain regularity assumptions. We show that if a λ-truncation of F at $\bar{x}$ belongs to the considered class of polynomial mappings then for every y close to $F(\bar{x})$ there exists a solution to the equation $F(x) = y$ that is close to $\bar{x}$. For polynomial mappings satisfying the regularity conditions we study their stability to bounded continuous perturbations.

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1 Introduction

Let n and m be positive integers, $\bar{x} \in \mathbb{R}^n$ be a given point, $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping continuously differentiable in a neighbourhood of $\bar{x}$.

Numerous classical results of analysis allow to study properties of the mapping F when the point $\bar{x}$ is normal, i.e. the first derivative $F'(\bar{x}) : \mathbb{R}^n \to \mathbb{R}^m$ of the mapping F at the point $\bar{x}$ is a surjective linear operator. For instance, the classical inverse function theorem (see, for example, [4, Theorem 1F.6]) states that the equation

$$F(x) = y \tag{1.1}$$

with the unknown $x \in \mathbb{R}^n$ and the parameter $y \in \mathbb{R}^m$ has a solution $x(y)$ for y close to $F(\bar{x})$ such that $x(y) \to \bar{x}$ as $y \to F(\bar{x})$.

If the point $\bar{x}$ is abnormal, i.e. the linear operator $F'(\bar{x})$ is not surjective, then the study of the behavior of the mapping F in a neighbourhood of the point $\bar{x}$ becomes significantly more complicated. In this case, this problem is studied under certain conditions of non-degeneracy, formulated in terms of the first two derivatives $F'(\bar{x})$ and $F''(\bar{x})$ of the mapping F at the point $\bar{x}$, and under very general and natural assumptions. A detailed overview of the relevant results is given in [1]. The need for the investigation of equation (1.1) in the abnormal case is partly dictated by optimization problems with equality-type constraints that degenerate in one sense or another. For example, some topics related to numerical methods for investigation of optimization problems with abnormality were studied in [6]. Theoretical problems concerning degenerating constraints were discussed and studied in [1].

Another approach, meaningful also in the abnormal case, was proposed in [2]. Unlike the results using the first and second derivatives of the mapping F at the point $\bar{x}$, in [2], there were obtained solvability conditions that use the derivatives of higher orders. The corresponding results are applicable to equation (1.1) when the mentioned conditions of the nondegeneracy in terms of the first two derivatives $F'(\bar{x})$ and $F''(\bar{x})$ are violated. Note that in this case, the methods from [1] are fundamentally inaplicable.

The present work is a natural continuation of the research begun in [2]. In Section 1 we recall the definition of λ-truncation and the inverse function theorem from [2]. In Section 2 we give an example of a wide class of λ-truncations that have the regularity property. In Section 3 we study the stability of the surjectivity property of regular λ-truncations under continuous bounded perturbations.

2 Preliminaries

Below $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$ and $\mathbb{R}^m$ and $|\cdot|$ stands for the corresponding Euclidean norm in these spaces.

Let D be the set of all non-zero n-dimensional vectors $d = (d_1, ..., d_n)$ with the non-negative components, $\widehat{D} \subset D$ be the subset of all vectors with integer components.

For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $s = (s_1, ..., s_n) \in \widehat{D}$ and $d = (d_1, ..., d_n) \in D$, we set

$$x^s := \prod_{j=1}^{n} x_j^{s_j}, \quad |x|^d := \prod_{j=1}^{n} |x_j|^{d_j}.$$

Here and below we assume that $x_j^0 = |x_j|^0 = 1$. The vector s is called the multi-index of the monomial x^s.

Given a vector $\lambda = (\lambda_1, ..., \lambda_n) \in D$ and nonempty finite sets $S_i \subset \widehat{D}$, $i \in \{1, ..., m\}$, assume that

$$\exists\, \alpha_i > 0 : \quad \langle \lambda, s \rangle = \alpha_i \quad \forall\, s \in S_i. \tag{2.1}$$

Given a collection of real numbers $p_{i,s} \neq 0$, $s \in S_i$, $i \in \{1, ..., m\}$, define the mapping $P = (P_1, ..., P_m) : \mathbb{R}^n \to \mathbb{R}^m$ by formula

$$P_i(x) = \sum_{s \in S_i} p_{i,s} x^s, \quad x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

Note that the polynomials P_i have the property of quasihomogeneity, i.e.

$$P_i(t^{\lambda_1} x_1, ..., t^{\lambda_n} x_n) \equiv t^{\alpha_i} P_i(x), \quad x \in \mathbb{R}^n, \quad t > 0.$$

Moreover, since S_i are nonempty and $p_{i,s} \neq 0$ for all $s \in S_i$, all functions P_i are nonzero polynomials. Moreover, it is obvious that $P(0) = 0$.

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping that is continuous in a neighbourhood of the given point $\bar{x}$. The mapping P is said to be a *λ-truncation of the mapping F at the point $\bar{x}$*, if there exist nonempty finite sets $D_i \subset D$, $i \in \{1, ..., m\}$ such that the following properties are satisfied. Firstly, the strict inequalities

$$\langle \lambda, d \rangle > \alpha_i \quad \forall\, d \in D_i, \quad \forall\, i \in \{1, ..., m\}$$

hold. Secondly, the representation

$$F(x) \equiv F(\bar{x}) + P(x - \bar{x}) + \Delta(x - \bar{x}),$$

is valid, in which for the mapping $\Delta = (\Delta_1, ..., \Delta_m) : \mathbb{R}^n \to \mathbb{R}^m$ there exists const ≥ 0 such that

$$|\Delta_i(x - \bar{x})| \leq \text{const} \sum_{d \in D_i} |x - \bar{x}|^d \quad \forall\, i \in \{1, ..., m\}.$$

for every x from a neighbourhood of the point $\bar{x}$.

For a vector $h \in \mathbb{R}^n$ we say that the λ-truncation P is regular in the direction h, if

$$P(h) = 0, \quad P'(h)\mathbb{R}^n = \mathbb{R}^m, \tag{2.2}$$

and $h_j = 0$ for any j such that $\lambda_j = 0$.

In [2, Theorem 1], the following inverse function theorem is obtained. *Let $\lambda > 0$, P be a λ-truncation of the mapping F at $\bar{x}$ and P be regular in a direction $h \in \mathbb{R}^n$. Then there exists a neighbourhood O of the point $F(\bar{x})$ and a number* const > 0 *such that for every $y \in O$ there exists a solution $x(y)$ to equation* (1.1) *such that*

$$|x(y) - \bar{x}| \leq \text{const}|y - F(\bar{x})|^{\theta} \quad \forall\, y \in O. \tag{2.3}$$

Here

$$\theta = \left(\max_{i=\overline{1,m}} \alpha_i \right)^{-1} \min_{i=\overline{1,m}} \lambda_i. \tag{2.4}$$

In connection with the above introduced concept of the regularity in a direction h, the following question arises. Is it essential to assume that $h_j = 0$ for any j such that $\lambda_j = 0$, can the assumption $\lambda > 0$ in [2, Theorem 1] be replaced by the assumption $\lambda \geq 0$, $\lambda \neq 0$, and $h_j = 0$ for any j such that $\lambda_j = 0$? The following example gives a negative answer to this question.

Example 1. Let $n = 2$, $m = 1$, $\bar{x} = 0$, $F(x) = P(x) = x_1^2 + x_1^2 x_2$. We have

$$S_1 = \{(2,0),\, (2,1)\}.$$

Put $\lambda := (1,0)$ and $h := (1,1,1)$. Then (2.1) holds for $\alpha_1 = 2$ and equalities (2.2) take place. At the same time, for a solution (x_1, x_2) to the equation

$$P(x) = y$$

with the unknown $x \in \mathbb{R}^2$ we have the following. If $y < 0$, then we have $x_1^2 + x_1^2 x_2 < 0$. Therefore, $x_2 < -1$. So, in this example there exists no solution $x(y)$ to the equation $F(x) = y$ such that $x(y) \to \bar{x} = 0$ as $y \to F(\bar{x}) = 0$.

This example shows the essentiality of the assumption that $h_j = 0$ for any j such that $\lambda_j = 0$. It also shows that the assumption $\lambda > 0$ of [2, Theorem 1] is essential.

To apply [2, Theorem 1] to a continuous mapping F, we must first find an appropriate vector λ and a polynomial mapping P (if they exist) such that P is a λ-truncation of F in a neighbourhood of the point $\bar{x}$, and then verify that P is regular in some direction $h \in \mathbb{R}^n$.

3 On one type of regular λ-truncations

First, we present a class of mappings $F : \mathbb{R}^n \to \mathbb{R}^m$, for which the assumptions of [2, Theorem 1] are satisfied.

Let n_i be positive integers such that $n = n_1 + ... + n_m + 1$, Q_i be given nonzero symmetric $n_i \times n_i$-matrices, b_i be given nonzero real numbers, $i \in \{1, ..., m\}$.

Consider m functions $P_i : \mathbb{R}^n \to \mathbb{R}$ defined by the formula

$$P_i(x) = \langle Q_i \chi_i, \chi_i \rangle + b_{i+1}(\chi_{i+1}^1)^3, \quad i \in \{1, ..., m\}. \tag{3.1}$$

Here $x = (\chi_1, ..., \chi_m, \chi_{m+1}^1) \in \mathbb{R}^n$, where $\chi_1 = (x_1, ..., x_{n_1}) \in \mathbb{R}^{n_1}$, $\chi_2 = (x_{n_1+1}, ..., x_{n_1+n_2}) \in \mathbb{R}^{n_2}$, etc.; χ_{i+1}^1 is the first component of the vector χ_{i+1}, $i \in \{1, ..., m-1\}$, i.e. $\chi_2^1 = x_{n_1+1}$, $\chi_3^1 = x_{n_1+n_2+1}$, etc.;

and $\chi^1_{m+1} = x_n$ is a real number, which for definiteness we will consider as the first coordinate of the one-dimensional vector χ_{m+1}.

So, for every $i \in \{1, ..., m\}$, the function P_i is the sum of two terms. The first one is a non-zero quadratic form in the variable χ_i, which is defined by the square $n_i \times n_i$-matrix Q_i. The second term is the non-zero cubic form $b_{i+1}(\chi^1_{i+1})^3$ in the variable χ_{i+1} with a given non-zero coefficient b_{i+1}.

Define an n-dimensional vector $\lambda > 0$ as follows. First, we put $\lambda_n = \widehat{\lambda}_{m+1} = \frac{1}{3}$. Then take $\widehat{\lambda}_i = \frac{1}{3}(\frac{3}{2})^{(m-i+1)}$, $i \in \{1, ..., m\}$. We define the remaining coordinates $\lambda_1, ..., \lambda_{n-1}$ of the vector λ so that in the places corresponding to the vector χ_i, all coordinates of the vector λ are equal to $\widehat{\lambda}_i$, $i \in \{1, ..., m\}$. As a result, we have

$$\lambda_1 = ... = \lambda_{n_1} = \frac{1}{3}\left(\frac{3}{2}\right)^m, \quad \lambda_{n_1+1} = ... = \lambda_{n_1+n_2} = \frac{1}{3}\left(\frac{3}{2}\right)^{m-1}, \quad ..., \quad \lambda_n = \frac{1}{3}. \tag{3.2}$$

Let S_i be the set of all the multi-indices of the monomials of P_i, $i \in \{1, ..., m\}$. So, each set S_i is a disjoint union of two finite nonempty subsets $S_{i,1} \sqcup S_{i,2}$.

The vectors $s \in S_{i,1}$ has two ones ore one two in the places corresponding to the vector χ_i, while the remaining components are zeros. The second subset $S_{i,2}$ consists of the only vector s with all the components equal to zero except the component corresponding to the variable χ^1_{i+1}. This component equals to three.

Let us show that (2.1) holds. Put

$$\alpha_i = \left(\frac{3}{2}\right)^{m-i}, \quad i \in \{1, ..., m\}.$$

Let us prove that $\langle \lambda, s\rangle = \alpha_i$ for every $s \in S_i$. Fix an arbitrary $i \in \{1, ..., m\}$. Let $s \in S_i$. If $s \in S_{i,1}$ then

$$\langle \lambda, s\rangle = 2\widehat{\lambda}_i = \frac{2}{3}\left(\frac{3}{2}\right)^{(m-i+1)} = \left(\frac{3}{2}\right)^{(m-i)} = \alpha_i.$$

If $s \in S_{i,2}$ then

$$\langle \lambda, s\rangle = 3\widehat{\lambda}_{i+1} = 3\frac{1}{3}\alpha_i = \alpha_i$$

for $i \le m-1$. Obviously, the last equality also holds for $i = m$. So, we have $\langle \lambda, s\rangle = \alpha_i$ for every $s \in S_i$. Hence, (2.1) holds.

So, it is shown that the polynomial mapping P is a λ-truncation of itself in a neighbourhood of zero.

Let us construct an n-dimensional vector h such that

$$P(h) = 0, \quad P'(h)\mathbb{R}^n = \mathbb{R}^m.$$

We divide the construction into several stages.

First, we construct the vector h_1 by taking $h_1 := \widehat{x}_1$, where $\widehat{x}_1$ is an arbitrary vector such that $\langle Q_1\widehat{x}_1, \widehat{x}_1\rangle \neq 0$. This vector exists since $Q_1 \neq 0$. Now we choose the first component h^1_2 of the vector h_2 satisfying the equality

$$\langle Q_1 h_1, h_1\rangle + b_2(h^1_2)^3 = 0.$$

This real number h^1_2 exists since $b_2 \neq 0$. We have $h^1_2 \neq 0$ since $\langle Q_1\widehat{x}_1, \widehat{x}_1\rangle \neq 0$.

Now we construct the remaining coordinates of the vector h_2. If $n_2 = 1$ then we put $h_2 = h^1_2$. Since $h^1_2 \neq 0$ and $Q_2 \neq 0$, then $\langle Q_2 h_2, h_2\rangle \neq 0$. In this case, the construction of the vector h_2 is completed.

Assume now that $n_2 \geq 2$. Let us show that the already constructed first coordinate h_2^1 can be supplemented with real numbers $h_2^2, ..., h_2^{n_2}$ to the vector $h_2 \in \mathbb{R}^{n_2}$ so that $\langle Q_2 h_2, h_2 \rangle \neq 0$.

Consider the contrary, i.e. $\langle Q_2 \chi_2, \chi_2 \rangle = 0$ for every vector χ_2 such that its first component h_2^1 is not equal zero, i.e. $h_2^1 \neq 0$.

Denote by $M \subset \mathbb{R}^{n_2}$ the set of all vectors $\chi_2 \in \mathbb{R}^{n_2}$ whose first component is not zero.

Take an arbitrary $\chi_2 \in M$. Then there exists $t \neq 0$ such that the first component of $t\chi_2$ equals to h_2^1. Then the assumption made implies

$$t^2 \langle Q_2 \chi_2, \chi_2 \rangle = \langle Q_2 t\chi_2, t\chi_2 \rangle = 0.$$

Therefore, $\langle Q_2 \chi_2, \chi_2 \rangle = 0$. At the same time, the set M is everywhere dense in $\mathbb{R}^{n_2}$. So, since the quadratic form Q_2 is a continuous function, it vanishes over the entire space $\mathbb{R}^{n_2}$.

The latter contradicts the assumption $Q_2 \neq 0$. This means that the first coordinate h_2^1 can be supplemented with the real numbers $h_2^2, ..., h_2^{n_2}$ to the vector h_2 so that $\langle Q_2 h_2, h_2 \rangle \neq 0$. The construction of the vector h_2 in the case under consideration is completed.

Now we take the first component h_3^1 of the vector h_3 so that

$$\langle Q_2 h_2, h_2 \rangle + b_3 (h_3^1)^3 = 0.$$

Obviously, $h_3^1 \neq 0$.

We continue this procedure until the end. At the last stage, we take the vector h_m such that $\langle Q_m h_m, h_m \rangle \neq 0$ and take $h_{m+1}^1 \in \mathbb{R}$ such that

$$\langle Q_m h_m, h_m \rangle + (h_{m+1}^1)^3 = 0.$$

Obviously, $h_{m+1}^1 \neq 0$.

Define an n-dimensional vector h by the formula

$$h = (h_1, h_2, ..., h_m, h_{m+1}^1).$$

By construction we have $\langle Q_i h_i, h_i \rangle + b_{i+1} (h_{i+1}^1)^3 = 0$ for each $i \in \{1, ..., m\}$.

Let us show that the polynomial mapping P is regular in the constructed direction h, i.e. equalities (2.2) hold. The equality $P(h) = 0$ is satisfied due to the above constructions. Therefore, it suffices to verify that the rows of the matrix $P'(h)$ are linearly independent.

The i-th row $P_i'(h)$ of the matrix $P'(h)$ is

$$P_i'(h) = (0, \ ..., \ 2Q_i h_i, \ 3b_{i+1} (h_{i+1}^1)^2, \ 0, \ ..., \ 0), \quad i \in \{1, ..., m\}.$$

Here, $2Q_i h_i$ are on the places corresponding to χ_i and h_{i+1}^1 is the first component of the vector h_{i+1}.

Let the real numbers γ_i be such that

$$\sum_{i=1}^{m} \gamma_i P_i'(h) = 0. \tag{3.3}$$

The last component of the vectors $P_1'(h), ..., P_{m-1}'(h)$ corresponding to χ_{m+1} is zero, while the last component of $P_m'(h)$ equals to $3b_{m+1}(h_{m+1}^1)^2 \neq 0$. So, it follows from (3.3) that

$$\sum_{i=1}^{m-1} \gamma_i P_i'(h) = 0, \quad \gamma_m = 0. \tag{3.4}$$

Similarly, for vectors $P_1'(h), ..., P_{m-2}'(h)$ the second to last component corresponding to χ_{m-1} is zero, while the second to last component of $P_{m-1}'(h)$ equals $3b_m(h_m^1)^2 \neq 0$. So, it follows from (3.3) and (3.4) that

$$\sum_{i=1}^{m-2} \gamma_i P_i'(h) = 0, \quad \gamma_{m-1} = \gamma_m = 0.$$

Carrying out similar reasoning for $i = m-2$, $i = m-3$, etc. up to $i = 1$, as a result we obtain that $\gamma_1 = ... = \gamma_m = 0$. Therefore, the vectors $P_i'(h)$, $i \in \{1, ..., m\}$ are linearly independent.

So, we have shown that under the assumptions $Q_i \neq 0$ and $b_{i+1} \neq 0$ for $i \in \{1, ..., m\}$ the polynomial mapping P is a λ-truncation of itself in a neighbourhood of zero and it is regular in a direction h, i.e. (2.2) holds.

Thus, the following assertion is proved. Let λ be the above constructed n-dimensional vector, i.e. the components of λ are defined by formula (3.2). Let S_i be the finite set of all the multi-indices of the monomials of P_i, $i \in \{1, ..., m\}$, while P_i be defined by (3.1).

Theorem 3.1. *Let the mapping $P = (P_1, ..., P_m) : \mathbb{R}^n \to \mathbb{R}^m$ be defined by formula* (3.1), *all the matrices Q_i and real numbers b_{i+1} be non-zero, $i \in \{1, ..., m\}$.*

Then for every $i \in \{1, ..., m\}$, equality (2.1) *holds and there exists a vector $h \in \mathbb{R}^n$ such that equalities* (2.2) *hold, i.e. $P(h) = 0$ and $P'(h)\mathbb{R}^n = \mathbb{R}^m$.*

The assumptions $Q_i \neq 0$ and $b_{i+1} \neq 0$ for every $i \in \{1, ..., m\}$ in Theorem 3.1 are essential even when $m = 1$. Let us show this.

Let us first assume that $b_2 \neq 0$ and $Q_1 = 0$. Then $P(x) \equiv b_2 x_n^3$. It is obvious that in this case at least one of the equalities in (2.2) is violated for every $h \in \mathbb{R}^n$. Assume now that $b_2 = 0$. Take a positive symmetric $(n-1) \times (n-1)$-matrix Q_1. We have $P(x) \equiv \langle Q_1\chi_1, \chi_1 \rangle \geq 0$ for every $x = (\chi_1, x_n) \in \mathbb{R}^n$. So, the classical inverse function theorem implies that if $P(h) = 0$ for some $h \in \mathbb{R}^n$ then $P'(h)\mathbb{R}^n = \{0\}$. Thus, relation (2.2) is violate for every $h \in \mathbb{R}^n$ in the second case too.

In the special case when $n = 3$, $m = 2$, Q_1 and Q_2 are 1×1-matrices and $b_2 = b_3 = 1$, the mapping $F = P$ was considered in [2, Example 7].

Let us now return to equation (1.1). The following assertion follows from Theorem 3.1 and [2, Theorem 1].

Theorem 3.2. *Let the mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ be continuous in a neighbourhood of the point $\bar{x}$, the mapping $P = (P_1, ..., P_m) : \mathbb{R}^n \to \mathbb{R}^m$, be defined by formula* (3.1) *and P be the λ-truncation of the mapping F in a neighbourhood of $\bar{x}$. Assume that all the matrices Q_i and the real numbers b_{i+1} are non-zero, $i \in \{1, ..., m\}$.*

Then there exists a neighbourhood O of the point $F(\bar{x})$ and a real const > 0 *such that for every $y \in O$ there exists a solution $x = x(y)$ to the equation $F(x) = y$ with the unknown x such that*

$$|x(y) - \bar{x}| \leq \text{const}|y - F(\bar{x})|^\theta \quad \forall\, y \in O.$$

Here $\theta = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^{m-1}$.

Proof. Theorem 3.1 implies that there exists a vector $h \in \mathbb{R}^n$ such that (2.2). Moreover, P is the λ-truncation of the mapping F in a neighbourhood of the point $\bar{x}$. So, applying [2, Theorem 1] we obtain that there exists a neighbourhood O of the point $F(\bar{x})$ and a real number const > 0 such that for every $y \in O$ there exists a solution $x = x(y)$ to the equation (1.1) satisfying the inequality (2.3). Computing the value of θ by formula (2.4) we obtain that $\theta = \frac{1}{3} \cdot (\frac{2}{3})^{m-1}$. □

Note that for the mapping F in the above theorem the equality $F'(\bar{x}) = 0$ holds. Therefore, the classical inverse function theorem is not applicable to the mapping F. In addition, since each matrix Q_i is only non-zero and can be sign-definite, the results from the survey [1] are not applicable as well.

4 The stability of λ-truncations to nonlocal perturbations

Theorem 1 from [2] is local in nature. The main idea of the proof of this theorem is to replace the original equation $F(x) = y$ with the equivalent equation

$$P(h+\xi) + \Phi(t,\xi) = \widetilde{y}(t,\eta)$$

for η from a neighbourhood of zero. Here P is a λ-truncation of the mapping F in a neighbourhood of zero which is regular in a direction h, t and η are parameters, ξ is an unknown variable from a neighbourhood of zero, $\widetilde{y}$ is an auxiliary function. This leads to the problem of the global solvability of the equation

$$P(x) + \Phi(x) = y \tag{4.1}$$

for all $y \in \mathbb{R}^m$ and all the continuous bounded mappings $\Phi : \mathbb{R}^n \to \mathbb{R}^m$.

The solvability of equation (4.1) for all specified Φ and y can be interpreted as the stability of the solvability property of the equation $P(x) = 0$ under the perturbations Φ and y.

Let the following be given: a vector $\lambda = (\lambda_1, ..., \lambda_n) \in D$ nonempty finite sets $S_i \subset \widehat{D}$, $i \in \{1, ..., m\}$ satisfying (2.1), and real numbers $p_{i,s}$, $s \in S_i$, $i \in \{1, ..., m\}$. Define the mapping $P = (P_1, ..., P_m) : \mathbb{R}^n \to \mathbb{R}^m$ by the formula

$$P_i(x) = \sum_{s \in S_i} p_{i,s} x^s, \quad x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

For an arbitrary bounded continuous mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, we denote

$$\|\Phi\| := \sup_{x \in \mathbb{R}^n} |\Phi(x)|.$$

Denote by B^m_δ the closed ball in $\mathbb{R}^m$ centred at zero with the radius $\delta \geq 0$, i.e.

$$B^m_\delta := \{y \in \mathbb{R}^m : |y| \leq \delta\}.$$

Theorem 4.1. *Assume that P satisfies (2.1), $\lambda > 0$ and there exists a vector $h \in \mathbb{R}^n$ such that P is regular in the direction h, i.e. equalities (2.2) hold.*

Then there exists a real number const > 0 such that for every continuous bounded mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ and for every vector $y \in \mathbb{R}^m$ there exists a solution $x = (x_1, ..., x_n) \in \mathbb{R}^n$ to equation (4.1) such that

$$|x_j| \leq \text{const}\left(\max_{i=\overline{1,m}}\left((\|\Phi\| + |y|)^{1/\alpha_i}\right)\right)^{\lambda_j}, \quad j \in \{1, ..., n\}. \tag{4.2}$$

Proof. Take an arbitrary bounded continuous mapping $\Phi = (\Phi_1, ..., \Phi_m) : \mathbb{R}^n \to \mathbb{R}^m$ and an arbitrary vector $y = (y_1, ..., y_m) \in \mathbb{R}^m$. As it is mentioned above, $P(0) = 0$ and so if both $\Phi = 0$ and $y = 0$ then $x = 0$ is the desired solution to equation (4.1). So, we assume that either $\Phi \neq 0$ or $y \neq 0$. Hence,

$$\|\Phi\| + |y| \neq 0.$$

We apply the classical inverse function theorem to the mapping P at the point h. Since the equalities (2.2) hold, i.e.

$$P(h) = 0 \quad \text{and} \quad P'(h)\mathbb{R}^n = \mathbb{R}^m,$$

this theorem implies that there exist reals $\mu > 0$ and $\delta > 0$ and a continuous mapping $g : B^m_\delta \to \mathbb{R}^n$ such that

$$P(h + g(z)) = z, \quad |g(z)| \leq \mu|z| \quad \forall\, z \in B^m_\delta. \tag{4.3}$$

Without loss of generality we will assume that a positive $\delta < 1$.

Denote

$$t := \max_{i=\overline{1,m}} \left(\left(\frac{\|\Phi\| + |y|}{\delta} \right)^{1/\alpha_i} \right). \tag{4.4}$$

Note that if $\|\Phi\| + |y| \le \delta$ then $t = \left(\frac{\|\Phi\|+|y|}{\delta} \right)^{1/\alpha}$, where $\alpha = \max\{\alpha_1, ..., \alpha_m\}$. If $\|\Phi\| + |y| > \delta$ then $t = \left(\frac{\|\Phi\|+|y|}{\delta} \right)^{1/\alpha}$, where $\alpha = \min\{\alpha_1, ..., \alpha_m\}$.

The inequality

$$t^{-\alpha_i} \le \frac{\delta}{\|\Phi\| + |y|} \quad \forall\, i \in \{1, ..., m\} \tag{4.5}$$

takes place. Indeed, fix an arbitrary $i \in \{1, ..., m\}$. It follows from (4.4) that $t \ge \big((\|\Phi\| + |y|)/\delta\big)^{1/\alpha_i}$. Therefore, we have $t^{\alpha_i} \ge (\|\Phi\| + |y|)/\delta$. Hence, inequalities (4.5) take place.

Denote

$$x(\xi) := (t^{\lambda_1}(h_1 + \xi_1), ..., t^{\lambda_n}(h_n + \xi_n)), \quad \xi \in \mathbb{R}^n.$$

Define the mapping $\Gamma = (\Gamma_1, ..., \Gamma_m) : \mathbb{R}^n \to \mathbb{R}^m$ by the formula

$$\Gamma_i(\xi) = t^{-\alpha_i}(y_i - \Phi_i(x(\xi))), \quad \xi \in \mathbb{R}^n, \quad i \in \{1, ..., m\}.$$

Obviously, the mapping Γ is continuous, since $x(\cdot)$ is an affine one by the definition.

For every ξ, we have

$$|\Gamma(\xi)| = \sqrt{\sum_{i=1}^m t^{-2\alpha_i}(y_i - \Phi_i(x(\xi)))^2} \le \sqrt{\sum_{i=1}^m \left(\frac{\delta}{\|\Phi\| + |y|} \right)^2 (y_i - \Phi_i(x(\xi)))^2} =$$

$$= \frac{\delta}{\|\Phi\| + |y|} \sqrt{\sum_{i=1}^m (y_i - \Phi_i(x(\xi)))^2} = \frac{\delta}{\|\Phi\| + |y|} |\Phi(x(\xi)) - y| \le \frac{\delta}{\|\Phi\| + |y|} (\|\Phi\| + |y|) = \delta.$$

Here, the first equality follows from the definition of Γ, the first inequality follows from inequalities (4.5), and the last inequality is the triangle inequality.

The obtained estimate implies that $|\Gamma(\xi)|$ is sufficiently small, i.e. $|\Gamma(\xi)| \le \delta\ \forall\ \xi$. Hence, the composition $g(\Gamma(\xi))$ is well-defined for all $\xi \in B^n_\mu$. Moreover, the inequality

$$|g(\Gamma(\xi))| \overset{(4.3)}{\le} \mu\delta < \mu \quad \forall\, \xi \in B^n_\mu$$

takes place. Therefore, the composition $\xi \mapsto g(\Gamma(\xi))$, $\xi \in B^n_\mu$ of continuous mappings is a continuous self-mapping of the ball B^n_μ. So, Brouwer's fixed-point theorem (see, for example, [7, Theorem 1.6.2]) implies that there exists a point $\widetilde{\xi} = (\widetilde{\xi}_1, ..., \widetilde{\xi}_n) \in B^n_\mu$ such that

$$\widetilde{\xi} = g(\Gamma(\widetilde{\xi})).$$

Let us show that the point $x := x(\widetilde{\xi})$ is the desired solution to equation (4.1).

At first, let us verify the equality $P(x) + \Phi(x) = y$. For every $i \in \{1, ..., m\}$, we have

$$P_i(x(\widetilde{\xi})) = P_i(t^{\lambda_1}(h_1 + \widetilde{\xi}_1), ..., t^{\lambda_n}(h_n + \widetilde{\xi}_n)) = \sum_{s \in S_i} p_{i,s} \prod_{j=1}^n (t^{\lambda_j}(h_j + \widetilde{\xi}_j))^{s_j} =$$

$$= \sum_{s \in S_i} p_{i,s} t^{\langle \lambda, s\rangle} \prod_{j=1}^{n} (h_j + \widetilde{\xi}_j)^{s_j} = t^{\alpha_i} \sum_{s \in S_i} p_{i,s} \prod_{j=1}^{n} (h_j + \widetilde{\xi}_j)^{s_j} = t^{\alpha_i} P_i(h + \widetilde{\xi}).$$

Here, the first equality follows from the definition of $x(\widetilde{\xi})$, the second equality follows from the definition of P_i, the second to last equalities follow from (2.1), and the last equality follows from the definition of P_i.

Moreover, for each $i \in \{1, ..., m\}$, we have

$$P_i(h + \widetilde{\xi}) = P_i(h + g(\Gamma(\widetilde{\xi}))) = \Gamma_i(\widetilde{\xi}) = t^{-\alpha_i}(y_i - \Phi_i(x(\widetilde{\xi}))).$$

Here, the first equality holds since $\widetilde{\xi} = g(\Gamma(\widetilde{\xi}))$, the second equality follows from the identity in (4.3) since $\Gamma(\widetilde{\xi}) \in B_\delta^m$, and the last equality follows from the definition of the mapping Γ_i.

So, it follows from the obtained equalities that

$$P_i(x(\widetilde{\xi})) = t^{\alpha_i} P_i(h + \widetilde{\xi}) = t^{\alpha_i} t^{-\alpha_i}(y_i - \Phi_i(x(\widetilde{\xi}))) = y_i - \Phi_i(x(\widetilde{\xi})) \quad \forall i \in \{1, ..., m\}.$$

Hence, for the vector $x = x(\widetilde{\xi})$ we have $P(x) + \Phi(x) = y$.

Let us prove that the desired inequalities hold for the components of the constructed vector x. For each $j \in \{1, ..., n\}$, we have

$$|x_j(\widetilde{\xi})| = t^{\lambda_j} |h_j + \widetilde{\xi}_j| = \left(\max_{i=\overline{1,m}} \left(\left(\frac{\|\Phi\| + |y|}{\delta} \right)^{1/\alpha_i} \right) \right)^{\lambda_j} |h_j + \widetilde{\xi}_j| \leq$$

$$\leq \frac{|h| + \mu}{(\min\limits_{i=\overline{1,m}} (\delta^{1/\alpha_i}))^{\lambda_j}} \left(\max_{i=\overline{1,m}} \left((\|\Phi\| + |y|)^{1/\alpha_i} \right) \right)^{\lambda_j}.$$

Here, the first equality follows from the definition of $x(\widetilde{\xi})$, the second equality follows from (4.4), and the inequality holds since $\widetilde{\xi} \in B_\mu^n$. Denoting $\dfrac{|h| + \mu}{(\min\limits_{i=\overline{1,m}} (\delta^{1/\alpha_i}))^{\lambda_j}}$ by const we obtain that estimate (4.2) is proved. Thus, since $x := x(\widetilde{\xi})$ is the desired solution to equation (4.1) we complete the proof. $\square$

In the special case when $\Phi(x) \equiv 0$, Theorem 4.1 implies the following assertion on surjectivity of λ-truncations.

Corollary 4.1. *Assume that P satisfies (2.1), $\lambda > 0$ and there exists a vector $h \in \mathbb{R}^n$ such that P is regular in the direction h, i.e. equalities (2.2) hold.*

Then there exists a real number const > 0 such that for every vector $y \in \mathbb{R}^m$ there exists a solution $x = (x_1, ..., x_n) \in \mathbb{R}^n$ to the equation $P(x) = y$ such that

$$|x_j| \leq \text{const} \left(\max_{i=\overline{1,m}} \left(|y|^{1/\alpha_i} \right) \right)^{\lambda_j}, \quad j \in \{1, ..., n\}.$$

Let us briefly discuss the problem on stability under set-valued perturbation. Recall that a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a mapping that corresponds to each $x \in \mathbb{R}^n$ a non-empty closed subset of $\mathbb{R}^m$. This mapping is called bounded if there exists $R > 0$ such that $\Phi(x) \subset B_R^m$ for every $x \in \mathbb{R}^n$. A set-valued mapping Φ is called convex-valued if $\Phi(x)$ is convex for each $x \in \mathbb{R}^n$. A set-valued mapping Φ is called lower semicontinuous if for every $x \in \mathbb{R}^n$, for every open set $W \subset \mathbb{R}^k$ such that $W \cap \Phi(x) \neq \emptyset$ there exists a neighbourhood $V \subset \mathbb{R}^n$ of x such that $W \cap \Phi(\chi) \neq \emptyset$ for each $\chi \in V$.

For an arbitrary bounded set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we denote

$$\|\Phi\| := \sup\{|y| : \, x \in \mathbb{R}^n, \quad y \in \Phi(x)\}.$$

Corollary 4.2. *Assume that P satisfies* (2.1), $\lambda > 0$ *and there exists a vector $h \in \mathbb{R}^n$ such that P is regular in the direction h, i.e. equalities* (2.2) *hold.*

Then there exists a real number const > 0 *such that for every convex-valued bounded lower semicontinuous mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and for every vector $y \in \mathbb{R}^m$ there exists $x = (x_1, ..., x_n) \in \mathbb{R}^n$ such that $y \in P(x) + \Phi(x)$ and* (4.2) *holds.*

Proof. Applying the Michael continuous selection theorem we obtain a continuous mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ such that $\varphi(x) \in \Phi(x)$ for every $x \in \mathbb{R}^n$. Applying Theorem 4.1 to the mapping P, the perturbation φ and a vector $y \in \mathbb{R}^m$ since $\|\varphi\| \leq \|\Phi\|$, we obtain that the desired x exists. $\square$

Here, we consider convex-valued bounded lower semicontinuous perturbations. Another types of perturbations can be considered using different technique based on various fixed point theorems for set-valued mappings (see, for example, [3, 5]).

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A PROBLEM WITH GELLERSTEDT CONDITIONS ON DIFFERENT CHARACTERISTICS FOR A MIXED LOADED EQUATION OF THE SECOND KIND

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Keywords: loaded equation of the second kind, problem with Gellerstedt conditions, representation of the general solutions, energy integral method, extremum principle, integral equation with a shift.

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Abstract. This work is devoted to a formulation and an investigation of a boundary value problem with Gellerstedt conditions on different characteristics for the loaded parabolic-hyperbolic type equation of the second kind.By using the extremum principle and the method of energy integrals, there are proved the uniqueness of solution of the formulated problem, and the existence of a solution to the problem - by the method integral equations.

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1 Introduction

The study of loaded differential equations is one of the actual directions in the theory of ordinary differential equations and partial differential equations.

The first works on loaded equations were devoted to loaded integral equations. These include the works of L. Lichtenstein [28], N.N. Nazarov [35], N.M. Gunter and A.Sh. Gabibzade [12]. In the work of A.M. Nakhushev [34] there is given the most general definition of a loaded equation and a detailed classification of various loaded equations: loaded differential, integral, integro-differential, functional equations, as well as their numerous applications.

At present, the range of problems under consideration for loaded equations of the first kind of hyperbolic-parabolic and elliptic-parabolic types, when the loaded part contains only the trace or derivative of the desired function, has expanded significantly. Note the works [3], [5], [6], [8 -10], [16], [17], [19], [22], [23], [39].The obtained results on fractional differential and integral operators (see [13], [27], [32]) can be useful in the study of local and non-local problems for mixed loaded equations of the first kind, when the loaded part contains integro-differential operators in the sense of Riemann-Liouville and Caputo [7], [18], [25], [26], [40]. This is due to the fact, that the loaded equations describe the problems of optimal control [21], regulation of the soil water layer and ground moisture [33], modeling of particle transfer processes [45], problems of heat and mass transfer at a finite rate, modeling of fluid filtration in porous media [43], the study of inverse problems [29].The monographs [21], [33] contain various applications of loaded equations as a method for studying mathematical problems of biology, mathematical physics, theory of mathematical modeling of non-local processes and phenomena, theory of elastic shells.

The theory of boundary value problem with nonlocal integral condition for loaded equations was studied numerically in research work [1]. Boundary value problems for nonlinear loaded difference equations with multipoint boundary conditions have been studied by many researchers. We note works [2], [4], [36].

Boundary-value problems for mixed type equations of the second kind, in which the line of degeneracy is the envelope of a family of characteristics and is itself also a characteristic, are usually called as the mixed-type equations of the second kind, in the literature.

In works [15], [24], [30], [37], [38], [42], [46], introducing a generalized solution of the class R_2, there were studied the analognes of the Tricomi problem for a model degenerate equation of parabolic-hyperbolic and elliptic-hyperbolic types of the second kind.

Notice, that the boundary value problems for loaded degenerate equations of mixed type of the second kind have not yet been studied (see [20]). This is due, first of all, to the lack of representations of the general solution, on the other hand such problems are reduced to little-studied integral equations with a shift.

Proceeding from this, in this paper general representations of the solution to a degenerate loaded equation of parabolic-hyperbolic type of the second kind are constructed. Using the general representation and the method of energy integrals, the uniqueness of the solution to the problem with the Gellerstedt conditions on different characteristics, which were not previously known, is proved. The existence of a solution to the problem is equivalently reduced to little-studied integral equations with a shift, and a new approach is found for proving the unique solvability of such an equation.

2 Formulation of Problem

We consider the equation

$$0=\begin{cases} u_{xx}-x^p u_y-\mu_1 u\left(x,0\right), & (x,y)\in D_1,\\ u_{xx}-\left(-y\right)^m u_{yy}+\mu_2 u\left(x,0\right), & (x,y)\in D_2,\end{cases} \tag{2.1}$$

where $m,\quad p,\quad \mu_0\quad \mu_1\quad \mu_2$ are arbitrary real constants such that

$$0<m<1,\ p>0,\ \mu_1>0,\ \mu_2<0. \tag{2.2}$$

Let D_1 be the connected domain, bounded by segments AB, AA_0, BB_0, A_0B_0 on the lines $y=0,\quad x=0,\quad x=1,\quad y=h$, respectively;

D_{21} be the characteristic triangle, bounded by the segment $A\left(0,0\right)E\left(x_0,0\right)$ of the x axis and by two characteristics $AC_1: x-\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=0$, $EC_1: x+\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=x_0$ of equation (2.1), going out from the points $A(0;0)$, $E(x_0;0)$ and intersecting at the point $C_1\left[\frac{x_0}{2};\ -\left(\frac{2-m}{4}x_0\right)^{\frac{2}{2-m}}\right]$;

D_{22} be the characteristic triangle, bounded by the segment $E(x_0;0)B(1;0)$ of the x axis and by two characteristics $EC_2: x-\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=x_0$, $BC_2: x+\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=1$ of equation (2.1), going out from the points $E\left(x_0;0\right)$ and $B\left(1;0\right)$ and intersecting at the point $C_2\left[\frac{1+x_0}{2};-\left(\frac{2-m}{4}\left(1-x_0\right)\right)^{\frac{2}{2-m}}\right]$;

D_{23} be the characteristic rectangle, bounded by the characteristics $C_1C: x-\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=0$, EC_1, EC_2 and $C_2C: x+\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=1$ of equation (2.1), intersecting at the points E, C_1, C_2 and $C\left[\frac{1}{2};\ -\left(\frac{2-m}{4}\right)^{\frac{2}{2-m}}\right]$, where $x>0,\ y<0$, and $x_0\in[0,1]$.

We denote: $J=\{(x,y):\ \ 0<x<1,\ \ y=0\}$,

$$J_1=\left\{(x,y):\ \ 0<x<x_0,\ \ y=0\right\},\quad J_2=\left\{(x,y):\ \ x_0<x<1,\ \ y=0\right\},$$

$$D_2=D_{21}\cup D_{22}\cup D_{23}\cup EC_1\cup EC_2,\quad D=D_1\cup D_2\cup J,\qquad 2\beta=m/(m-2)$$

moreover, we assume thet

$$-1 < 2\beta < 0, \tag{2.3}$$

$$\mathbb{D}^{\sigma}_{ax} f(x) = \begin{cases} \frac{sign(x-a)}{\Gamma(-\sigma)} \int_a^x \frac{f(t)dt}{|x-t|^{1+\sigma}}, & \text{at} \quad \sigma < 0, \\ f(x), & \text{at} \quad \sigma = 0, \\ [sign(x-a)]^{n+1} \frac{d^{n+1}}{dx^{n+1}} \mathbb{D}^{\sigma-(n+1)}_{ax} f(x), & \text{at} \quad \sigma > 0, \end{cases} \tag{2.4}$$

is the fractional integro-differential operator of order σ [44, c.16], $\mathbb{D}^{\sigma}_{ax} \equiv D^{\sigma}_{ax}$ at $x > a$ and $\mathbb{D}^{\sigma}_{ax} \equiv D^{\sigma}_{xa}$ at $x < a$, $n = [\sigma]$ is the integer part of the number σ.

In the domain D for equation (2.1) we investigate a boundary value problem with Gellerstedt conditions on the different characteristics.

Problem AG_1. Find in the domain D a function $u(x, y)$, with the following properties:

1) $u(x, y) \in C(\overline{D}) \cap C^1(D)$, besides $u_y(x, 0)$ can tend to infinity of order less than -2β at $x \to x_0$, in addition at $x \to 0$ and $x \to 1$ $u(x, y)$ is bounded;

2) $u(x, y) \in C^{2,1}_{x,y}(D_1)$ and it is a regular solution of equation (2.1) in the domain D_1;

3) $u(x, y)$ is a generalized solution of equation (2.1) belonging to the class R_2 [24] in the domain $D_2 \backslash \{EC_1 \cup EC_2\}$;

4) $u(x, y)$ satisfies the boundary conditions

$$u(x, y)|_{AA_0} = \varphi_1(y), \qquad u(x, y)|_{BB_0} = \varphi_2(y), \ \ 0 \le y \le h, \tag{2.5}$$

$$u|_{EC_1} = \psi_1(x), \ \frac{x_0}{2} \le x \le x_0, \quad u|_{EC_2} = \psi_2(x), \ \ x_0 \le x \le \frac{x_0 + 1}{2}, \tag{2.6}$$

where $\varphi_j(y)$, $\psi_j(x)$ $(j = 1, 2)$ are given functions, satisfyig the following conditions

$$\varphi_1(0) = \varphi_2(0) = 0, \quad \psi_1(x_0) = \psi_2(x_0), \tag{2.7}$$

$$\varphi_1(y), \ \ \varphi_2(y) \in C[0, h] \cap C^1(0, h), \tag{2.8}$$

$$\psi_1(x) \in C^1\left[\frac{x_0}{2}, x_0\right] \cap C^2\left(\frac{x_0}{2}, x_0\right), \psi_2(x) \in C^1\left[x_0, \frac{x_0+1}{2}\right] \cap C^2\left(x_0, \frac{x_0+1}{2}\right). \tag{2.9}$$

3 Investigation of Problem AG_1 for equation (2.1)

If conditions 1) - 3) of AG_1 are satisfied, then any regular solution to equation (2.1) can be represent in the form [16], [41]:

$$u(x, y) = \upsilon(x, y) + \omega(x), \tag{3.1}$$

where

$$\upsilon(x, y) = \begin{cases} \upsilon_1(x, y) & (x, y) \in D_1, \\ \upsilon_{2k}(x, y) & (x, y) \in D_{2k}, \end{cases} \quad (k = \overline{1, 3}), \tag{3.2}$$

$$\omega(x) = \begin{cases} \omega_1(x), & (x, 0) \in \bar{J}, \\ \omega_{2j}(x), & (x, 0) \in \bar{J}_j, \end{cases} \quad (j = 1, 2), \tag{3.3}$$

here $\upsilon_1(x, y)$ and $\upsilon_{2j}(x, y)$ are regular solutions to the equations

$$0 = \begin{cases} L\upsilon_1 \equiv \upsilon_{1\,xx} - x^p \upsilon_{1y}, & (x, y) \in D_1, \\ L\upsilon_{2j} \equiv \upsilon_{2\,jxx} - (-y)^m \upsilon_{2j\,yy}, & (x, y) \in D_{2j}, \end{cases} \tag{3.4}$$

$\omega_1(x)$, $\omega_{2j}(x)$ $(j = 1, 2)$ are arbitrary twice continuously differentiable solutions to the equations

$$\omega_1''(x) - \mu_1 \omega_1(x) = \mu_1 \upsilon_1(x, 0), \quad (x, 0) \in J, \tag{3.5}$$

$$\omega''_{2j}(x) + \mu_2\omega_{2j}(x) = -\mu_2 \upsilon_{2j}(x,0), \quad (x,0) \in J_j. \tag{3.6}$$

Remark 3.1. Taking into account, that the function $ax+b$ satisfies equation (3.3), the functions $\omega_1(x)$ and $\omega_{2i}(x)$ can be defined uniquely if they satisfy the conditions

$$\omega_1(0) = \omega_1(1) = 0, \tag{3.7}$$

$$\omega_{21}(0) = \omega_{21}(x_0) = 0, \tag{3.8}$$

$$\omega_{22}(x_0) = \omega_{22}(1) = 0. \tag{3.9}$$

Solutions to problems (3.5), (3.7) and (3.6), (3.8)((3.9)) have the forms

$$\omega_1(x) = \frac{\sqrt{\mu_1} sh\,(x-1)\sqrt{\mu_1}}{sh\sqrt{\mu_1}} \int\limits_0^1 sht\sqrt{\mu_1}\tau_1(t)dt -$$

$$-\sqrt{\mu_1}\int\limits_0^1 sh\sqrt{\mu_1}(x-t)\tau_1(t)dt, \quad (x,0) \in \overline{J}, \tag{3.10}$$

$$\omega_{2j}(x) = (-1)^j \frac{\sqrt{-\mu_2} sh\sqrt{-\mu_2}(x_0 - x)}{sh\sqrt{-\mu_2}(x_0 - \theta_j)} \int_{\theta_j}^{x_0} \tau_{2j}(t) sh\,\sqrt{-\mu_2}(t-\theta_j)\,dt -$$

$$-(-1)^j\sqrt{-\mu_2}\int_{x_0}^{x} \tau_{2j}(t) sh\sqrt{-\mu_2}\,((-1)^j(x-t))dt, \qquad (x,0) \in \bar{J}_j, \tag{3.11}$$

respectively, where $\theta_j = 0$ at $j=1$, $\theta_j = 1$ at $j=2$, $\tau_1(x) = \upsilon_1(x,0)$, $(x,0) \in \bar{J}$, $\tau_{2j}(x) = \upsilon_{2j}(x,0)$, $(x,0) \in \bar{J}_j$.

By virtue of representation (3.1) owing to (3.7), (3.8), (3.9), Problem AG_1 is reduced to Problem AG_1^* of finding a solution to equation (3.4) in the domain D satisfying the conditions

$$\upsilon_1(x,y)|_{AA_0} = \varphi_1(y), \upsilon_1(x,y)|_{BB_0} = \varphi_2(y), \quad 0 \le y \le h, \tag{3.12}$$

$$\upsilon_{21}|_{EC_1} = \psi_1(x) - \omega_{21}(x), \quad \frac{x_0}{2} \le x \le x_0, \tag{3.13}$$

$$\upsilon_{22}|_{EC_2} = \psi_2(x) - \omega_{22}(x), \quad x_0 \le x \le \frac{x_0+1}{2}, \tag{3.14}$$

where $\omega_{2j}(x)$ $(j=1,2)$ are defined in (3.11).

3.1. Function relations

The generalized solution of the class R_2 [24] of the Cauchy problem with the initial conditions

$$\upsilon_{2j}(x,-0) = \tau_{2j}(x),\ (x,0) \in \bar{J}_j, \quad \upsilon_{2jy}(x,-0) = \nu_{2j}(x),\ (x,0) \in J_j \tag{3.15}$$

for equation (3.4) in the domains $\Delta_{2j}(j=1,2)$ is given by the formula

$$\upsilon_{21}(\xi,\eta) = \int_{\xi}^{x_0} (t-\xi)^{-\beta}(t-\eta)^{-\beta} T_1(t)\,dt + \int_{\eta}^{\xi} (\xi-t)^{-\beta}(t-\eta)^{-\beta} N_1(t)\,dt, \tag{3.16}$$

$$\upsilon_{22}(\xi,\eta) = \int_{x_0}^{\eta} (\xi-t)^{-\beta}(\eta-t)^{-\beta} T_2(t)\,dt + \int_{\eta}^{\xi} (\xi-t)^{-\beta}(t-\eta)^{-\beta} N_2(t)\,dt, \tag{3.17}$$

where $\Delta_{21}=\{(\xi,\eta):\ 0<\eta<\xi, 0<\xi<x_0\}$, $\Delta_{22}=\{(\xi,\eta):\ x_0<\eta<1, \eta<\xi<1\}$,

$$\xi=x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}},\quad \eta=x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}}, \tag{3.18}$$

$$\tau_{2j}(x)=(-1)^j\int_{x_0}^{x}\left[(-1)^j(x-t)\right]^{-2\beta}T_j(t)dt,\quad (x,0)\in J_j, \tag{3.19}$$

$$N_j\left(x\right)=T_j\left(x\right)/2\cos\pi\beta-\gamma_2\nu_{2j}\left(x\right),\ (j=1,2), \tag{3.20}$$

besides, the functions $T_j\left(x\right)$ and $\nu_{2j}\left(x\right)$ are continuous on J_j and integrable on $\bar{J}_j$.

Substituting $\xi=x_0$, $\eta=x$ and $\eta=x_0$, $\xi=x$ into (3.16) and (3.17) respectively, taking into account (2.4), (3.13), (3.14), (3.20), $D_{xx_0}^{1-\beta}\cdot D_{xx_0}^{\beta-1}f(x)=f(x)$, $\quad D_{x_0x}^{1-\beta}\cdot D_{x_0x}^{\beta-1}f(x)=f(x)$ [27], [44] we get

$$T_j(x)=\gamma_3\nu_{2j}(x)+\frac{2\cos\pi\beta}{\Gamma(1-\beta)}\left[(-1)^j(x-x_0)\right]^{\beta}(-1)^jD_{x_0x}^{1-\beta}\Psi_j(x),\quad (x,0)\in J_j, \tag{3.21}$$

where $\gamma_3=2\gamma_2\cos\pi\beta$, $\Psi_j(x)=\psi_j(x)-\omega_{2j}(x), (j=1,2)$.

From (3.21) and (3.19), we find the following functional relation between $\tau_{2j}(x)$ and $\nu_{2j}(x)$, which follows from D_{2i} on the I_j:

$$\tau_{2j}(x)=\gamma_3(-1)^j\int_{x_0}^{x}\left[(-1)^j(x-t)\right]^{-2\beta}\nu_{2j}(t)dt+\Phi_j(x),\quad (x,0)\in\bar{J}_j, \tag{3.22}$$

where

$$\Phi_j(x)=\frac{2\Gamma(1-2\beta)\cos\pi\beta}{\Gamma(1-\beta)}D_{xx_0}^{-(1-2\beta)}\left[(-1)^j(x-x_0)\right]^{\beta}D_{xx_0}^{1-\beta}\Psi_j(x),\ \ (j=1,2). \tag{3.23}$$

According to the conditions 1) - 2) of Problem AG_1, taking into account (3.1), (3.7), passing to the limit in equation (3.4) as $y\to+0$, taking into account (3.12) and

$$\upsilon_1\left(x,+0\right)=\tau_1\left(x\right),\quad (x,0)\in\bar{J},\quad \upsilon_{1y}\left(x,+0\right)=\nu_1\left(x\right),\quad (x,0)\in J \tag{3.24}$$

we get

$$\tau_1''\left(x\right)=x^p\nu_1\left(x\right), \tag{3.25}$$

$$\begin{array}{l}\tau_1(0)=\varphi_1(0),\quad \tau_1(x_0)=\psi_1(x_0),\\ \tau_1(x_0)=\psi_2(x_0),\quad \tau_1\left(1\right)=\varphi_2\left(0\right).\end{array} \tag{3.26}$$

Solving equations (3.25) and (3.26) considering gluing condition (see conditions of Problem AG_1), we get the second functional relation between $\tau_{2j}(x)$ and $\nu_{2j}(x)$, which follows from D_1 on J_j:

$$\tau_{2j}(x)=(-1)^{j-1}\int_{\theta_j}^{x_0}G_j\left(x,t\right)t^p\nu_{2j}(t)dt+f_j(x),\quad (x,0)\in\bar{J}_j, \tag{3.27}$$

where $\theta_j=0$ at $\quad j=1, \quad \theta_j=1\quad$ при $\ j=2$,

$$G_1\ \left(x,t\right)=\begin{cases}\frac{t(x-x_0)}{x_0}, & 0\le t\le x,\\ \frac{x(t-x_0)}{x_0}, & x\le t\le x_0,\end{cases}\quad G_2\ \left(x,t\right)=\begin{cases}\frac{(x-1)(t-x_0)}{1-x_0}, & x_0\le t\le x,\\ \frac{(t-1)(x-x_0)}{1-x_0}, & x\le t\le 1,\end{cases} \tag{3.28}$$

$$f_1(x)=\varphi_1(0)+\frac{x}{x_0}\left[\psi_1(x_0)-\varphi_1(0)\right], f_2(x)=\varphi_2(0)+\frac{1-x}{1-x_0}\left[\psi_2(x_0)-\varphi_2(0)\right]. \tag{3.29}$$

3.2. Uniqueness of a solution to Problem AG_1

To prove the uniqueness of a solution to Problem AG_1, at the first step we prove the uniqueness of a solution to Problem AG_1^* for equation (3.4).

The following lemma plays an important role in proving the uniqueness of a solution to Problem AG_1^* for equation (3.4).

Lemma 3.1. *If conditions* (2.2), (2.3), (2.7) *are satisfied,*

$$p+2\beta>1, \quad (-y)^{-m/2}\upsilon_{21}(E)=0, \quad (-y)^{-m/2}\upsilon_{22}(B)=0, \tag{3.30}$$

and

$$\varphi_1(y)\equiv\varphi_2(y)\equiv 0, \forall y\in[0,h\,], \psi_1(x)\equiv 0, \forall x\in\left[\frac{x_0}{2},x_0\right], \psi_2(x)\equiv 0, \forall x\in\left[x_0,\frac{x_0+1}{2}\right],$$

then

$$\tau_{2j}(x)\equiv 0, \quad \forall x\in\bar{J}_j \quad (j=1,2), \tag{3.31}$$

where $\tau_{2j}(x)$ $(j=1,2)$ *if defined in* (3.15).

Proof. We prove this lemma using the method of energy integrals. Let $\upsilon_{2j}(x,y)$ be a twice continuously differentiable solution of the homogeneous problem AG_1^* in the domain $\bar{D}_{2j}^{\varepsilon}$, here D_{21}^{ε} is a domain with boundaries $\partial D_{21}^{\varepsilon}=\overline{A_\varepsilon C}_{1\varepsilon}\cup\overline{C_{1\varepsilon}E}_\varepsilon\cup\bar{J}_{1\varepsilon}$, strictly lying in the domain D_{21} for $j=1$, and for $j=2$, D_{22}^{ε} is a domain with boundaries $\partial D_{22}^{\varepsilon}=\overline{E_\varepsilon C}_{2\varepsilon}\cup\overline{C_{2\varepsilon}B}_\varepsilon\cup\bar{J}_{2\varepsilon}$, strictly lying in the region D_{22}, ε is a sufficiently small positive number.

Let $j=1$, then, integrating the equality

$$0=x^p(-y)^{-m}\upsilon_{21}(\upsilon_{21\,xx}-(-y)^m\,\upsilon_{21\,yy})=\frac{\partial}{\partial x}(x^p(-y)^{-m}\upsilon_{21}\upsilon_{21x})-\frac{\partial}{\partial y}(x^p\upsilon_{21}\upsilon_{21\,y})-$$

$$-x^p\left[(-y)^{-m}\upsilon_{21x}^2-\upsilon_{21y}^2\right]-px^{p-1}(-y)^{-m}\upsilon_{21}\upsilon_{21\,x} \tag{3.32}$$

over the domain $\bar{D}_{21}^{\varepsilon}$ and applying Green's formula, we have

$$\int_{\overline{A_\varepsilon C}_{1\varepsilon}\cup\overline{C_{1\varepsilon}E}_\varepsilon\cup\bar{J}_{1\varepsilon}} x^p(-y)^{-m}\upsilon_2\upsilon_{2x}dy+x^p\upsilon_2\upsilon_{2\,y}dx=\iint_{D_{21}^{\varepsilon}}x^p\left[(-y)^{-m}\upsilon_{2x}^2-\upsilon_{2y}^2\right]dxdy+$$

$$+p\iint_{D_{21}^{\varepsilon}}x^{p-1}(-y)^{-m}\upsilon_2\upsilon_{2\,x}dxdy.$$

From here, passing to the limit at $\varepsilon\to 0$, taking into account conditions (2.7) and 1)-3) of Problem AG_1^*, we obtain

$$\int_0^{x_0}x^p\tau_{21}(x)\nu_{21}(x)dx=-\int_{\overline{AC_1}}x^p(-y)^{-\frac{m}{2}}\upsilon_{21}d\upsilon_{21}+\int_{\overline{C_1E}}x^p(-y)^{-\frac{m}{2}}\upsilon_{21}d\upsilon_{21}-$$

$$-\iint_{D_{21}}x^p\left[(-y)^{-m}\upsilon_{21x}^2-\upsilon_{21y}^2\right]dxdy-p\iint_{D_{21}}x^{p-1}(-y)^{-m}\upsilon_{21}\upsilon_{21x}dxdy, \tag{3.33}$$

where $\tau_{21}(x)$, $\nu_{21}(x)$ are defined in (3.15) (see [11, Chapter 5, pp. 96-97]).

To calculate the right-hand side of equality (3.32), we move on to the characteristic coordinates $\xi=x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}}$, $\eta=x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}}$. Further, considering (3.13), (3.14) with $\psi_1(x)=0$, $\psi_2(x)=0$ and using in the domain Δ_{21} the canonical form of hyperbolic equation (3.4) in the form: $\upsilon_{21\xi\eta}=\frac{\beta}{\xi-\eta}\left(\upsilon_{21\xi}-\upsilon_{21\eta}\right)$ from the right-hand side of equality (3.33), taking into account (3.30), we find

$$-\int_{\overline{AC_1}} x^p(-y)^{-\frac{m}{2}}\upsilon_{21}d\upsilon_{21}=-\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{2\beta}x_0^{p+2\beta}\left(\omega_{21}\left(\frac{x_0}{2}\right)\right)^2+$$

$$+\frac{p+2\beta}{2}\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{2\beta}\int_0^{x_0}\frac{\upsilon_{21}^2(\xi,0)}{\xi^{1-p-2\beta}}d\xi, \tag{3.34}$$

$$\int_{\overline{C_1E}} x^p(-y)^{-\frac{m}{2}}\upsilon_{21}d\upsilon_{21}=-\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{2\beta}x_0^{p+2\beta}\left(\omega_{21}\left(\frac{x_0}{2}\right)\right)^2-$$

$$-\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{2\beta}p\int_0^{x_0}\frac{(x_0+\eta)^{p-1}}{(x_0-\eta)^{-2\beta}}\upsilon_{21}^2(x_0,\eta)d\eta+$$

$$+\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{2\beta}\beta\int_0^{x_0}\frac{(x_0+\eta)^p}{(x_0-\eta)^{1-2\beta}}\upsilon_{21}^2(x_0,\eta)d\eta, \tag{3.35}$$

$$-\iint_{D_{21}} x^p\left[(-y)^{-m}\upsilon_{21x}^2-\upsilon_{21y}^2\right]dxdy=$$

$$=\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{2\beta}x_0^{p+2\beta}\left(\omega_{21}\left(\frac{1}{2}\right)\right)^2-(\beta+p)\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{2\beta}\int_0^1\xi^{p+2\beta-1}\upsilon_{21}^2(\xi,0)d\xi+$$

$$+\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{2\beta}p\int_0^{x_0}(x_0+\eta)^{p-1}(x_0-\eta)^{2\beta}\upsilon_{21}^2(x_0,\eta)d\eta-$$

$$-\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{2\beta}\beta\int_0^{x_0}(x_0+\eta)^p(x_0-\eta)^{2\beta-1}\upsilon_{21}^2(x_0,\eta)d\eta-$$

$$-\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{2\beta}p(p-1))\iint_{\Delta_{21}}(\xi+\eta)^{p-2}(\xi-\eta)^{2\beta}\upsilon_{21}^2(\xi,\eta)d\xi d\eta, \tag{3.36}$$

$$-p\iint_{D_2}x^{p-1}(-y)^{-m}\upsilon_2\upsilon_{2x}dxdy=\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{2\beta}p\ \times$$

$$\times\left[\int_0^{x_0}\xi^{p+2\beta-1}\upsilon_{21}^2(\xi,0)d\xi-\int_0^{x_0}(x_0+\eta)^{p-1}(x_0-\eta)^{2\beta}\upsilon_{21}^2(x_0,\eta)d\eta\right]+$$

$$+\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{2\beta}p(p-1)\iint_{\Delta_{21}}(\xi+\eta)^{p-2}(\xi-\eta)^{2\beta}\upsilon_{21}^2(\xi,\eta)d\xi d\eta. \tag{3.37}$$

Substituting (3.34)-(3.37) in (3.32) owing to (2.2), (2.3) and $p+2\beta>1$, we get

$$\int_0^{x_0}x^p\tau_{21}(x)\nu_{21}(x)dx=0. \tag{3.38}$$

Let $j=2$, then integrating identity (3.31) over the domain D_{22} in the same way, we obtain

$$\int_{x_0}^1 x^p\tau_{22}(x)\nu_{22}(x)dx=0, \tag{3.39}$$

where $\tau_{22}(x)$, $\nu_{22}(x)$ are defined in (3.15).

Substituting (3.19) in (3.38) and (3.39), taking into account the conditions of Problem AG_1 and Lemma 1, as well as the equalities $\tau_{21}(0) = \tau_{22}(1) = 0$, $\tau_{2j}(x_0) = 0$, $(j = 1, 2)$, we find

$$\int_0^{x_0} x^p \tau_{21}(x)\nu_{21}(x)dx = \int_0^{x_0} \tau_{21}(x)\tau''_{21}(x)dx = -\int_0^{x_0} \tau'^2_{21}(x)dx \le 0, \tag{3.40}$$

$$\int_{x_0}^1 x^p \tau_{22}(x)\nu_{22}(x;\lambda)dx = -\int_{x_0}^1 \tau'^2_{22}(x)dx \le 0. \tag{3.41}$$

Comparing (3.40) and (3.41), we have

$$\int\limits_0^{x_0} x^p \tau_{21}(x)\nu_{21}(x)dx = 0 \quad \text{if} \quad \int\limits_0^{x_0} \tau'^2_{21}(x)dx = 0$$

$$\left(\int\limits_{x_0}^{1} x^p \tau_{22}(x)\nu_{22}(x)dx = 0 \quad \text{if} \quad \int\limits_{x_0}^{1} \tau'^2_{22}(x)dx = 0\right).$$

This implies the validity of equality (3.31). □

By virtue of (3.2), (3.31) and condition 1) of Problem AG_1, due to the equalities $\upsilon_1(x,+0) = \upsilon_{21}(x,-0)$, $(x,0) \in \bar{J}_1$, $\upsilon_1(x,+0) = \upsilon_{22}(x,-0)$, $(x,0) \in \bar{J}_2$, we get

$$\tau_1(x) \equiv 0, \quad (x,0) \in \bar{J}. \tag{3.42}$$

Taking into account (3.3), (3.15), (3.24), (3.31), (3.42) , from (3.10) and (3.11), we get

$$\omega(x) \equiv 0, \qquad \forall\, x \in \bar{J}. \tag{3.43}$$

Theorem 3.1. *If the conditions of Lemma* 3.1 *and* (3.43) *are satisfied, then Problem* AG_1^* *in the domain* D *cannot have more than one solution.*

Proof. According to the maximum principle for parabolic equations [14], boundary value problem AG_1^* for equation (3.4) in domain $\bar{D}_1$ with homogeneous conditions (3.12) and $\upsilon_1(x,0) = 0$, $(x,0) \in \bar{J}$ and (3.43) does not have a non-zero solution, i.e. $\upsilon_1(x,y) \equiv 0$ to $\bar{D}_1$.

Due to the uniqueness of a solution of the Cauchy problem with homogeneous conditions (3.15) for equation (3.4) in the domain D_2, taking into account (3.43), we get $\upsilon_2(x,y) \equiv 0$ in $\bar{D}_2$.

Consequently, from (3.2) we have

$$\upsilon(x,y) \equiv 0, \quad (x,y) \in \bar{D}\,. \tag{3.44}$$

From (3.44) the uniqueness of a solution of Problem AG_1^* for equation (3.4). □

Theorem 3.2. *If the conditions of Theorem* 3.1 *are satisfied, then Problem* AG_1 *in* D *cannot have more than one solution.*

Proof. By virtue (3.42), (3.43) from (3.1) it follows, that

$$u(x,y) \equiv 0, \quad (x,y) \in \bar{D}\,. \tag{3.45}$$

This proves the uniqueness of a solution to Problem AG_1 for equation (2.1). □

3.3. Existence of a solution to Problem AG_1

The existence of a solution to Problem AG_1 is proved by the method integral equations. To prove the existence of a solution to Problem AG_1, first we prove the existence of a solution to Problem AG_1^* for equation (3.4).

Theorem 3.3. *If* $p+2\beta>1$, *and conditions* (2.2), (2.3), (2.8), (2.9) *hold, then a solution to Problem* AG_1^* *in* D *exists.*

Proof. Substituting (3.27) in (3.19), taking into account the properties of operator (2.4) and gluing conditions (see the conditions of Problem AG_1), we find the function $T_i(x)$:

$$T_j(x)=\frac{sin2\beta\pi}{2\beta\pi}(-1)^{j-1}\int_{\theta_j}^{x_0}t^p\nu_{2j}(t)dt\frac{d^2}{dx^2}(-1)^j\int_{x_0}^{x}G_j(z,t)\left((-1)^j(x-z)\right)^{2\beta}dz+$$
$$+\frac{(-1)^j\mathbb{D}^{1-2\beta}_{xx_0}f_j(x)}{\Gamma(1-2\beta)},\quad (j=1,2), \tag{3.46}$$

where $\theta_j=0$ at $j=1$, $\theta_j=1$ at $j=2$, $G_j(z,t)$ and $f_j(x)$ are defined in (3.28) and (3.29) respectively.

Now eliminating $T_j(x)$ from (3.21) and (3.37) owing to (3.7) and the equality $D^{1-2\beta}_{0x}g(x)=D^{-2\beta}_{0x}g'(x)$ we get the integral equation for $\nu_{2j}(x)$:

$$\nu_{2j}(x)-\int_{\theta_j}^{x_0}P_j(x,t)\nu_{2j}(t)dt=F_j(x),\quad (x,0)\in J_j, \tag{3.47}$$

where $\theta_j=0$ at $j=1$, $\theta_j=1$ at $j=2$,

$$P_j(x,t)=\frac{(-1)^{j-1}t^p}{\gamma_3}\left\{\frac{2\cos\pi\beta}{\beta\,\Gamma(1-\beta)}\frac{\mu_2\left[(-1)^{j-1}(x_0-x)\right]^{2\beta}}{sh\sqrt{-\mu_2}\,(x_0-\theta_j)}(-1)^j\times\right.$$
$$\times\int_{\theta_j}^{x_0}G_j(z,t)\,sh\,\sqrt{-\mu_2}(z-\theta_j)dz+\frac{2\cos\pi\beta}{\beta\,\Gamma(1-\beta)}\frac{\mu_2\sqrt{-\mu_2}\,\left[(-1)^{j-1}(x_0-x)\right]^{\beta}}{sh\sqrt{-\mu_2}\,(x_0-\theta_j)}(-1)^{j-1}\times$$
$$\times\int_{\theta_j}^{x_0}G_j(z,t)\,sh\,\sqrt{-\mu_2}(z-\theta_j)\,dz\int_x^{x_0}\left[(-1)^{j-1}(s-x)\right]^{\beta}sh\sqrt{-\mu_2}(x_0-s)ds-$$
$$+\frac{2\mu_2\cos\pi\beta}{\beta\Gamma(1-\beta)}\left[(-1)^{j-1}(x_0-x)\right]^{\beta}(-1)^{j-1}\int_x^{x_0}G_j(z,t)\left[(-1)^{j-1}(z-x)\right]^{\beta}dz+$$
$$+\frac{2\mu_2\sqrt{-\mu_2}\cos\pi\beta\left[(-1)^{j-1}(x_0-x)\right]^{\beta}}{\Gamma(1-\beta)}(-1)^{j-1}\int_x^{x_0}\left[(-1)^{j-1}(s-x)\right]^{\beta}ds\times$$
$$\times\int_s^{x_0}G_j(z,t)\,sh\sqrt{-\mu_2}(z-s)dz+$$
$$\left.+\frac{sin2\beta\pi}{2\beta\pi}\frac{d}{dx}(-1)^{j-1}\int_x^{x_0}\frac{\partial G_j(z,t)}{\partial z}\left[(-1)^{j-1}(z-x)\right]^{2\beta}dz\right\}, \tag{3.48}$$

$$F_j(x)=\frac{2\mu_2\cos\pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)}\frac{\left[(-1)^{j-1}(x_0-x)\right]^{2\beta}}{sh\sqrt{-\mu_2}\,(x_0-\theta_j)}(-1)^j\int_{\theta_j}^{x_0}f_j(t)sh\,\sqrt{-\mu_2}(t-\theta_j)\,dt+$$
$$+\frac{2\mu_2\sqrt{-\mu_2}\cos\pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)}\frac{\left[(-1)^{j-1}(x_0-x)\right]^{\beta}}{sh\sqrt{-\mu_2}\,(x_0-\theta_j)}(-1)^{j-1}\int_{\theta_j}^{x_0}f_j(z)sh\,\sqrt{-\mu_2}(z-\theta_j)\,dz\times$$

$$\times \int_x^{x_0} \left[(-1)^{j-1}(t-x)\right]^\beta sh\sqrt{-\mu_2}(x_0-t)dt-$$

$$-\frac{2\mu_2 \cos \pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)} \left[(-1)^{j-1}(x_0-x)\right]^\beta (-1)^{j-1} \int_x^{x_0} \left[(-1)^{j-1}(t-x)\right]^\beta f_j(t)dt-$$

$$-\frac{2\mu_2 \cos \pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)} \left[(-1)^{j-1}(x_0-x)\right]^\beta (-1)^{j-1} \int_x^{x_0} \left[(-1)^{j-1}(t-x)\right]^\beta dt\times$$

$$\times \int_t^{x_0} f_j(z) sh\sqrt{-\mu_2}\,(z-t)dz + \frac{sin2\pi\beta}{2\pi\beta\gamma_3} \frac{\left[(-1)^{j-1}(x_0-x)\right]^{2\beta}}{x_0-\theta_j}\psi_j(x_0)+$$

$$+\frac{2\cos\pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)} \left[(-1)^{j-1}(x_0-x)\right]^{2\beta} \psi_j'(x_0)+$$

$$\left.+\left[(-1)^{j-1}(x_0-x)\right]^\beta (-1)^{j-1} \int_x^{x_0} \left[(-1)^{j-1}(t-x)\right]^\beta \psi_j''(t)dt\right]. \tag{3.49}$$

By virtue of (2.2), (2.3), (2.8) and (2.9), the properties of the operator of integro-differentiation, Beta-function, hypergeometric functions [44, Chapter 1, §1, 2 and 4, pp. 4-32] and the functions $G_j(x,t)$ (3.48) and (3.49) imply that the kernel and the right-hand side of equation (3.47) admit the following estimates

$$|P_1(x,t)| \le c_1(x_0-x)^{2\beta}, \qquad |P_2(x,t)| \le c_2(x-x_0)^{2\beta}, \tag{3.50}$$

$$|F_1(x)| \le c_3(x_0-x)^{2\beta}, \qquad |F_2(x)| \le c_4(x-x_0)^{2\beta}, \quad c_i = const > 0. \tag{3.51}$$

Based on (2.8), (2.9), taking into account (3.51), we conclude that $F_j(x) \in C^2(J_j)$, and the functions $F_j(x)$ $(j=1,2)$ can go to infinity with order of growth less than -2β for $x \to x_0$, and for $x \to 0$ and $x \to 1$ they are bounded.

By virtue of (2.2), (3.50) and (3.51) equation (3.47) is a Fredholm integral equation of the second kind. According to the theory of Fredholm integral equations [31] and from the uniqueness of a solution to Problem AG_1^* (see Theorems 3.1), we conclude that integral equation (3.47) is uniquely solvable in the class $C^2(J_j)$, and the solutions $\nu_{2j}(x)$ can have the order of singularity less than -2β for $x \to x_0$, and for $x \to 0$ and $x \to 1$ are bounded and have the form:

$$\nu_{2j}(x) = F_j(x) + \int_{\theta_j}^{x_0} P_j^*(x,t)\,F_j(t)dt, \quad (x,0) \in J_j, \tag{3.52}$$

where $P_j^*(x,t)$ is the resolvent kernel.

Substituting (3.52) into (3.22) and (3.27) to the equalities $\upsilon_1(x,+0) = \upsilon_{21}(x,-0)$, $(x,0) \in \bar{J}_1$, $\upsilon_1(x,+0) = \upsilon_{22}(x,-0)$, $\quad (x,0) \in \bar{J}_2$, we find

$$\tau_j(x) \in C(\bar{J}) \cap {}^2(J), \;\; (j=1,2)\,. \tag{3.53}$$

Therefore, Problem AG_1^* is uniquely solvable due to its equivalence to the Fredholm integral equation of the second kind (3.47).

Thus, the solution to Problem AG_1^* can be reconstructed in the domain D_1 as a solution of the first boundary value problem for equation (3.4), and in the domains D_{2j} (D_{23}) $(j=1,2)$ as a solution to the Cauchy (Goursat) problem for equation (3.4). This completes the study of the existence of a solution of Problem AG_1^* for equation (3.4). □

We turn to the proof of the existence of a solution to Problem AG_1.

The following theorem is true.

Theorem 3.4. *If the conditions of Theorem 3.3 are satisfied, then a solution to Problem AG_1 in D exists.*

Proof. By virtue (3.27) (or (3.22)) taking (3.52) into account, from (3.10) and (3.11) we find $\omega_1(x)$ and $\omega_{2j}(x)$ $(j=1,2)$. Then, a solution to Problem AG_1 in the domain can be found as $u_1(x,y)=\upsilon_1(x,y)+\omega_1(x)$, where $\upsilon_1(x,y)$ is a solution of the first boundary value problem for equation (3.4). In the domains D_{2i} and D_{23} it has the form $u_2(x,y)=\upsilon_{2j}(x,y)+\omega_{2i}(x)$, $\quad (j=\overline{1,3})$, $\quad (i=1,2)$, where $\upsilon_{2i}(x,y)\,(\upsilon_{23}(x,y))$ is a solution of the Cauchy problem for equation (3.4) in the domain $D_{2i}\,(D_{23})$.

Thus, in the domain D, a solution to Problem AG_1 exists.

This completes the study of Problem AG_1 for equation (2.1). □

Example illustrating the problem.

Let $m=\frac{1}{2}$, $\quad p=1$, $\quad \mu_1=1$ $\quad \mu_2=-1$, $\quad x_0=0$, $\quad \beta=-\frac{1}{6}$, $\varphi_1(y)\equiv\varphi_2(y)\equiv 0$, $\quad \psi_2(x)=\psi(x)=x$, then the problem posed is reduced to Problem T_1 :

$$0=\begin{cases} u_{xx}-xu_y-u\left(x,0\right), & x>0, \quad y>0,\\ u_{xx}-\sqrt{-y}u_{yy}-u\left(x,0\right), & x>0, \quad y<0,\end{cases} \tag{3.54}$$

$$u(x,y)|_{AA_0}=0, \quad u(x,y)|_{BB_0}=0, \;\; 0\le y\le h,$$

$$u|_{AC}=x, \;\; 0\le x\le\frac{1}{2}.$$

In this case the conditions of Theorems 3.1, 3.2 and 3.3 are satisfied. Then formulas (3.22) and (3.27) take the form

$$\tau_2(x)=\tilde{\gamma}_3\int\limits_0^x(x-t)^{-\frac{1}{3}}\nu_2(t)dt+\Phi_2(x), \quad x\in[0,1]\,, \tag{3.55}$$

$$\tau_1(x)=\int\limits_0^1 G_1\left(x,t\right)t\nu_1(t)dt, \quad x\in[0,1]\,, \tag{3.56}$$

where

$$\tilde{\gamma}_3=16\sqrt{3}\left(\frac{3}{8}\right)^{4/3}\Gamma\left(\frac{1}{3}\right)/\Gamma^2\left(\frac{1}{6}\right),$$

$$G_1\;\,(x,t)=\begin{cases} t(x-1), & 0\le t\le x,\\ x(t-1), & x\le t\le 1,\end{cases} \tag{3.57}$$

$$\Phi_2(x)=\frac{2\sqrt{3}\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}D_{0x}^{-\frac{4}{3}}x^{-\frac{1}{6}}D_{0x}^{\frac{7}{6}}\left[x-\int\limits_0^x\tau_2\left(t\right)sh\left(x-t\right)dt\right], \quad x\in[0,1]\,.$$

From (3.55) and (3.56) taking into account condition 1) of Problem AG_1 and that $D_{0x}^{4/3}\,g(x)=D_{0x}^{1/3}\,g'(x)$, we get the following integral equation for $\nu_2(x)$:

$$\nu_2(x)+\int\limits_0^1\tilde{P}_2(x,t)\nu_2(t)dt=F_2(x), \quad x\in(0,1)\,, \tag{3.58}$$

where $\tilde{P}_2(x,t)$ and $F_2(x)$ are the known functions satisfying the estimates

$$\left|\tilde{P}_2(x,t)\right|\le c_1x^{-\frac{1}{3}}, \qquad |F_2(x)|\le c_2x^{-\frac{1}{3}}, \quad c_1,c_2=const>0.$$

According to the theory of Fredholm integral equations and from the uniqueness of a solution to Problem T_1 (see Theorem 3.2), we conclude that integral equation (3.58) is uniquely solvable in the class $C^2\left(0,1\right)$, and $\nu_2(x)$ has a singularity of order less than $\frac{1}{3}$ and for $x \to 0$, and for $x \to 1$, is bounded.

In the same way as above, the solution of Problem T_1 is restored.

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ASYMPTOTICS OF THE SOLUTION OF PARABOLIC PROBLEMS WITH NONSMOOTH BOUNDARY FUNCTIONS

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Key words: singularity, perturbation, parabolic equation, nonsmooth boundary function, regularized asymptotics.

AMS Mathematics Subject Classification: 39A14, 34E10.

Abstract. In this paper, we construct the asymptotics of the solution to a singularly perturbed parabolic problem with a nonsmooth boundary layer function. In contrast to works devoted to this direction, our asymptotics contains only one boundary layer function, which is the product of parabolic and exponential boundary layer functions. Our approach allows us to construct a classical solution without applying smoothing procedures.

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1 Introduction

Singularly perturbed problems with nonsmooth regular boundary functions were studied in [1]-[8], [10]-[11]. To construct the asymptotics of the solution of such problems, the method of matching asymptotic expansions was used in [7]. In [4], [5] the asymptotics is constructed by using the smoothing procedure. Using the methodology of [2], in [3] the asymptotics of a solution of any order was constructed without the use of matching and smoothing procedures. In this paper, using the method of [10], a regularized asymptotics of the problem posed is constructed, applying for regularization the method of [9], regularizing functions are introduced, which are determined from partial differential equations of the first order and an ordinary differential equation. This choice of regularizing functions made it possible to pass the difficulties related to the non-smoothness of the boundary functions. The asymptotics of the solution constructed by us, in contrast to [2], [5]-[4], [7]-[10], contains only angular boundary layer functions represented as a product of parabolic and exponential boundary layer functions. The parabolic boundary layer function describes the boundary layer along the characteristic $t + B(x)/\sqrt{\varepsilon} = 0$, and the exponential boundary layer function describes the boundary layer along $x = 0$.

2 Statement of the problem

We consider the following singularly perturbed parabolic equation with nonsmooth boundary functions:

$$L_\varepsilon u(x,t,\varepsilon) \equiv -\partial_t u + \varepsilon^2 a(x)\partial_x^2 u + \sqrt{\varepsilon} b(x)\partial_x u - c(x,t)u = f(x,t) \tag{2.1}$$
$$(x,t) \in \Omega, u|_{t=0} = u|_{x=0} = 0,$$

which is studied in [3], where $\varepsilon > 0$ is a small parameter, $a(x), b(x), c(x,t), f(x,t)$ are continuously differentiable and bounded together with their derivatives in Ω, moreover $a(x) > 0, b(x) > 0, \Omega = (0 < x < \infty) \times (0 < t < T]$.

3 Regularization of the problem

Following [9], [10] we introduce the regularizing variables:

$$\xi = \varphi(x,t,\varepsilon), \eta = \psi(x,\varepsilon) \tag{3.1}$$

and extended function $u(M,\varepsilon), M = (x,t,\xi,\eta)$ such that:

$$\tilde{u}(M,\varepsilon)|_{\theta=\gamma(x,t,\varepsilon)} \equiv u(x,t,\varepsilon), \theta = (\xi,\eta), \gamma(x,t,\varepsilon) = (\varphi(x,t,\varepsilon), \psi(x,\varepsilon)). \tag{3.2}$$

Based on (3.1) we find the derivatives:

$$\partial_t u \equiv (\partial_t \tilde{u} + \partial_t \varphi(x,t,\varepsilon)\partial_\xi \tilde{u})_{\theta=\gamma(x,t,\varepsilon)},$$

$$\partial_x u \equiv \left(\partial_x \tilde{u} + \partial_x \varphi(x,t,\varepsilon)\partial_\xi \tilde{u} + \psi^{'}(x,\varepsilon)\partial_\eta \tilde{u}\right)_{\theta=\gamma(x,t,\varepsilon)},$$

$$\partial_x^2 \equiv \left(\partial_x^2 \tilde{u} + (\partial_x \varphi(x,t,\varepsilon))^2 \partial_\xi^2 \tilde{u} + (\psi^{'}(x,\varepsilon))^2 \partial_\eta^2 \tilde{u} + L_\xi \tilde{u} + L_\eta \tilde{u}\right)_{\theta=\gamma(x,t,\varepsilon)},$$

$$L_\xi \equiv 2\partial_x \varphi \partial_{x,\xi}^2 + \partial_x^2 \varphi \partial_\xi,$$

$$L_\eta \equiv 2\psi^{'}\partial_{x,\eta}^2 + \psi^{''}\partial_\eta,$$

then, instead of problem (2.1), we pose the extended problem:

$$\begin{gathered}
\tilde{L}_\varepsilon \tilde{u}(M,\varepsilon) \equiv -\left(\partial_t \tilde{u} + \partial_t \varphi(x,t,\varepsilon)\partial_\xi \tilde{u}\right) + \\
\varepsilon^2 a(x)\left(\partial_x^2 \tilde{u} + (\partial_x \varphi(x,t,\varepsilon))^2 \partial_\xi^2 \tilde{u} + (\psi^{'}(x,\varepsilon))^2 \partial_\eta^2 \tilde{u} + L_\xi \tilde{u} + L_\eta \tilde{u}\right) + \\
\sqrt{\varepsilon} b(x)\left(\partial_x \tilde{u} + \partial_x \varphi(x,t,\varepsilon)\partial_\xi \tilde{u} + \psi^{'}(x,\varepsilon)\partial_\eta \tilde{u}\right) - \\
c(x,t)\tilde{u} = f(x,t), M \in Q, \\
\tilde{u}|_{t=0} = \tilde{u}|_{x=\xi=\eta=0} = 0.
\end{gathered} \tag{3.3}$$

Let us choose the regularizing functions $\varphi(x,t,\varepsilon), \psi(x,\varepsilon)$ as solutions of the problems:

$$-\partial_t \varphi + \sqrt{\varepsilon} b(x)\partial_x \varphi = 0, \varphi(0,t,\varepsilon) = 0,$$

$$\sqrt{\varepsilon} b(x)\psi^{'} = 1, \psi(0,\varepsilon) = 0.$$

The solution to these problems will be:

$$\varphi(x,t,\varepsilon) = \Phi\left(t + \frac{1}{\sqrt{\varepsilon}}B(x)\right), \psi(x,\varepsilon) = \frac{1}{\sqrt{\varepsilon}}B(x),$$

$$B(x) = \int_0^x \frac{dt}{b(t)},$$

where $\Phi\left(t + \frac{1}{\sqrt{\varepsilon}}B(x)\right)$ is an arbitrary function such that $\Phi(0) = 0$. Taking into account the found functions, extended equation (3.3) can be rewritten as:

$$\tilde{L}_\varepsilon \tilde{u}(M,\varepsilon) \equiv -(\partial_t \tilde{u}) + \varepsilon^2 a(x)\left[\partial_x^2 \tilde{u} + \left(\Phi_x^{'}\left(t + \frac{1}{\sqrt{\varepsilon}}B(x)\right)\right)^2 \partial_\xi^2 \tilde{u} + L_\xi \tilde{u}\right] + \tag{3.4}$$

$$\varepsilon^2 a(x)\left[(\psi'(x,\varepsilon))^2\partial_\eta^2\tilde{u} + L_\eta\tilde{u}\right] +$$

$$\sqrt{\varepsilon}b(x)\left(\partial_x\tilde{u} + \psi'(x,\varepsilon)\partial_\eta\tilde{u}\right) -$$

$$c(x,t)\tilde{u} = f(x,t), M \in Q,$$

$$\tilde{u}|_{t=0} = \tilde{u}|_{x=\xi=\eta=0} = 0, Q = \Omega\times(0,\infty)^2.$$

Let us choose the function $\Phi_x'\left(t + \frac{1}{\sqrt{\varepsilon}}B(x)\right)$ as a solution to the equation $\varepsilon^2 a(x)\left(\Phi'\left(t + \frac{1}{\sqrt{\varepsilon}}B(x)\right)\frac{1}{\sqrt{\varepsilon}b(x)}\right)^2 = 1$ with the initial condition $\Phi(0) = 0$, then:

$$\Phi\left(t + \frac{1}{\sqrt{\varepsilon}}B(x)\right) = \frac{1}{\sqrt{\varepsilon}}\int_0^{t+\frac{1}{\sqrt{\varepsilon}}B(x)} \frac{b\left(B^{-1}(\sqrt{\varepsilon}(u-s))\right)}{\sqrt{a\left(B^{-1}(\sqrt{\varepsilon}(u-s))\right)}}du.$$

After this choice of the regularizing functions, equation (3.4) takes the form:

$$\tilde{L}_\varepsilon\tilde{u}(M,\varepsilon) \equiv -\partial_t\tilde{u} + \partial_\xi^2\tilde{u} + \partial_\eta\tilde{u} - c(x,t)\tilde{u} + \sqrt{\varepsilon}b(x)\partial_x\tilde{u} + \tag{3.5}$$

$$\varepsilon\frac{a(x)}{b^2(x)}\partial_\eta^2\tilde{u} + \varepsilon a(x)L_\xi\tilde{u} + \sqrt{\varepsilon^3}a(x)L_\eta\tilde{u} + \varepsilon^2 a(x)\partial_x^2\tilde{u} = f(x,t), M \in Q,$$

$$\tilde{u}|_{t=0} = \tilde{u}|_{x=\xi=\eta=0} = 0.$$

Equation (3.5) is regular on ε as ε tends to zero. The solution to problem (3.5) will be defined as:

$$\tilde{u}(M,\varepsilon) = \sum_{k=1}^{\infty}\varepsilon^{k/2}u_k(M).$$

For the coefficients of this series, we obtain the following iterative problems:

$$T_0u_0(M) \equiv -\partial_t u_0 + \partial_\xi^2 u_0 + \partial_\eta u_0 - c(x,t)u_0 = f(x,t), \tag{3.6}$$

$$T_0u_k = H_k(M), u_k|_{t=0} = u_k|_{x=\xi=\eta=0} = 0.$$

4 Solvability of iterative problems

We introduce the class of functions in which iterative equations will be solved:

$$U = \left\{f(M) = f_1(x,t) + f_2(x,t)erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)) : f_1, f_2 \in C^\infty(\Omega)\right\},$$

where $erfc\left(\frac{\xi}{2\sqrt{t}}\right)$ describes the parabolic boundary layer along $x = 0$, the function $\exp(-(t+\eta))$ is the boundary layer along $t = 0$.

Theorem 4.1. *Suppose that $H_k(M) \in U$, then the equation*

$$T_0u_k = H_k(M) \tag{4.1}$$

has a solution $u_k(M) \in U$.

Proof. Let $H_k(M) \in U$, namely

$$H_k(M) = h_1(x,t) + h_2(x,t) erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)), \quad \text{where} \quad h_1, h_2 \in C^\infty(\Omega).$$

Notice that functions $u_k(M)$ satisfy equation (4.1) if the functions $v_k(x,t), d_k(x,t)$ are solutions to the equations

$$\partial_t v_k(x,t) - c(x,t)v_k(x,t) = h_1(x,t), \quad \partial_t d_k(x,t) - c(x,t)d_k(x,t) = h_2(x,t). \tag{4.2}$$

By our assumptions on the functions $c(x,t), h_1(x,t), h_2(x,t)$, these equations have smooth solutions. □

Theorem 4.2. *Equation* (4.1) *under the following additional conditions:*

1. $u_k|_{t=0} = u_k|_{x=\xi=\eta=0} = 0$;
2. $L_\xi u_k = 0$

has a unique solution.

Proof. Let the function $u_k(M)$ satisfy boundary conditions 1), then:

$$v_k(x,0) = 0, d_k(x,0) = d_k^0(x), d_k(0,t) = -v_k(0,t)\exp(t), \tag{4.3}$$

where $d_k^0(x)$ is an arbitrary function. Solving equation (4.2) with respect to $d_k(x,0)$, for an arbitrary initial condition, we find:

$$d_k(x,t) = d_k^0(x)p_1(x,t) + p_2(x,t),$$

where $p_l(x,t), l = 1,2$ are known functions. Condition 2) of the theorem, taking into account the found value, is equivalent to the equation:

$$\left(d_k^0(x)\right)' + q_1(x,t)d_k^0(x) = q_2(x,t), \tag{4.4}$$

which we solve under the initial condition $d_k^0(0) = q_3(t)$, is determined from (4.3), here $q_l(.), l = 1,2,3$ are the known functions. This uniquely determines $d_k(x,t)$, also the function $v_k(x,t)$ is uniquely determined from equation (4.2) under the initial condition from (4.3). □

5 Solution of iterative problems

Consider equation (3.6) for $k = 0$, with the right-hand side $f(x,y) \in U$. By Theorem 4.1, this equation has a solution $u_0(M) \in U$, i.e.

$$u_0(M) = v_0(x,t) + d_0(x,t) erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)),$$

where the arbitrary functions $v_0(x,t)$ and $d_0(x,t)$ are defined as in Theorem 4.2. The right-hand sides of the iterative equations will have the form:

$$H_k(M) = -b(x)\left[v_{k-1}(x,t) + a(x)\partial_x^2\partial_x^2 v_{k-4}(x,t)\right] erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)) -$$

$$\left\{b(x)d_{k-1}(x,t) + a(x)\left[\partial_x^2 d_{k-4}(x,t) + \left(\frac{1}{b(x)}\right)'\right]\right\} erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)) +$$

$$a(x)\left[\frac{2}{b(x)}\partial_x d_{k-3}(x,t)+\left(\frac{1}{b(x)}\right)' d_{k-3}(x,t)\right] erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta))\in U.$$

Further, using Theorems 4.1 and 4.2, we successively determine the coefficients of the partial sum:

$$u_{\varepsilon,n}=\sum_{k=0}^{n}\varepsilon^{k/2}u(M). \tag{5.1}$$

6 Estimate of the remainder term

Considering that

$$\left(\tilde{L}_\varepsilon\tilde{u}(M,\varepsilon)\right)_{\theta=\gamma(x,t,\varepsilon)}\equiv L_\varepsilon u(x,t,\varepsilon) \tag{6.1}$$

and substituting

$$R_{\varepsilon,n}(M)=\tilde{u}(M,\varepsilon)-u_{\varepsilon,n}(M)$$

into equation (3.5), then, taking into account (3.6) and making the restriction by means of regularizing functions, based on (6.1), we obtain the following problem for the remainder:

$$L_\varepsilon R_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)=\varepsilon^{\frac{n+1}{2}}g_n(x,t,\varepsilon),$$

$$R_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)_{t=0}=R_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)_{x=0}=0.$$

Using the maximum principle [8], we establish the estimate:

$$\|R_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)\|<c\varepsilon^{\frac{n+1}{2}}. \tag{6.2}$$

Theorem 6.1. *The function* $u_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)$ *is the asymptotic solution of problem* (2.1) *and is such that in the region* $\bar{\Omega}=(0\le x\le\infty)\times(0\le t\le T)$ *estimate* (6.2) *holds, where* c *is independent of* ε.

The solution constructed above is asymptotic, namely the difference between the exact and asymptotic solutions satisfies (6.2).

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ON THE ASSOCIATED SPACES OF THE WEIGHTED ALTERED CESÀRO SPACE

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Key words: Cesàro function spaces, associated spaces.

AMS Mathematics Subject Classification: 46E30.

Abstract: We study weighted altered Cesàro space $\mathrm{Ch}_{\infty,w}(I)$, which is a non-ideal enlargement of the usual Cesàro space. We prove the connection of this space with one weighted Sobolev space of the first order on real line and give characterizations of associate spaces of this space.

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1 Introduction

Let $I := (c,d) \subset \mathbb{R}$, $p \in [1,\infty]$, $p' := \frac{p}{p-1}$, $\mathcal{L}^1$ be the Lebesgue measure on I, $\mathfrak{M}(I)$ be the vector space of all $\mathcal{L}^1$-measurable functions $f : I \to [-\infty,\infty]$, and let $L^p(I)$ be the Lebesgue space. Also we put

$$L^p_{\mathrm{loc}}(I) := \Big\{ f \in \mathfrak{M}(I) : \|\chi_{(a,b)} f\|_{L^p(I)} < \infty,\ \forall a,b \in I \Big\},$$

$$L^p_{\mathrm{loc}}([c,d)) := \Big\{ f \in \mathfrak{M}(I) : \|\chi_{(c,x)} f\|_{L^p(I)} < \infty,\ \forall x \in I \Big\},$$

$$L^p_{\mathrm{loc}}((c,d]) := \Big\{ f \in \mathfrak{M}(I) : \|\chi_{(x,d)} f\|_{L^p(I)} < \infty,\ \forall x \in I \Big\}.$$

Let

$$w \in \mathfrak{M}(I),\ w > 0\ \mathcal{L}^1\text{-almost everywhere on } I,\ w \in L^p_{\mathrm{loc}}((c,d]) \tag{1.1}$$

and (if the measure in the integral is omitted, then the integral is taken with respect to the measure $\mathcal{L}^1$)

$$\rho(f) := \begin{cases} \left(\int_I \left|w(x)\int_c^x f\right|^p dx\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \mathcal{L}^1\text{-}\operatorname{ess\,sup}_{x\in I} w(x)\left|\int_c^x f\right|, & p = \infty; \end{cases}$$

$$\mathrm{Cs}_{p,w}(I) := \Big\{ f \in L^1_{\mathrm{loc}}([c,d)) \,\Big|\, \|f\|_{\mathrm{Cs}_{p,w}(I)} < \infty \Big\},\ \|f\|_{\mathrm{Cs}_{p,w}(I)} := \rho(|f|),$$

$$\mathrm{Ch}_{p,w}(I) := \Big\{ f \in L^1_{\mathrm{loc}}([c,d)) \,\Big|\, \|f\|_{\mathrm{Ch}_{p,w}(I)} < \infty \Big\},\ \|f\|_{\mathrm{Ch}_{p,w}(I)} := \rho(f).$$

It is clear that $\mathrm{Cs}_{p,w}(I)$ is embedded in $\mathrm{Ch}_{p,w}(I)$. Since w satisfies condition (1.1) then $f \in \mathfrak{M}(I)$ with compact support belongs to the space $\mathrm{Cs}_{p,w}(I)$. The space $(\mathrm{Cs}_{p,w}(I), \|\cdot\|_{\mathrm{Cs}_{p,w}(I)})$ is called weighted Cesàro space, it has been actively studied (see [6, 3] and the survey [1]). We call the space $(\mathrm{Ch}_{p,w}(I), \|\cdot\|_{\mathrm{Ch}_{p,w}(I)})$ weighted altered Cesàro space. This space has been studied in the works [9, 10, 14].

Let $(X, \|\cdot\|)$ be the normed space of elements of $\mathfrak{M}(I)$. We define the "strong" associated space (Köthe dual space) of X by

$$X'_{\mathrm{s}} := (X, \|\cdot\|)'_{\mathrm{s}} := \left\{ g \in \mathfrak{M}(I) \;\middle|\; \|g\|_{X'_{\mathrm{s}}} := \sup_{h \in X\setminus\{0\}} \frac{\int_I |hg|}{\|h\|} < \infty \right\}$$

and the "weak" associated space of X by

$$X'_{\mathrm{w}} :=(X, \|\cdot\|)'_{\mathrm{w}} := \left\{ g \in \mathfrak{M}(I) \;\middle|\; fg \in L^1(I)\, \forall f \in X \;\&\; \|g\|_{X'_{\mathrm{w}}} := \sup_{h \in X\setminus\{0\}} \frac{|\int_I hg|}{\|h\|} < \infty \right\},$$

which is isomorphic to the subspace of the set X^* of all continuous functionals of the form $f \mapsto \int_I fg$, $f \in X$. It is clear that $X'_{\mathrm{s}} \subset X'_{\mathrm{w}}$.

The classic Cesàro space $\mathrm{Cs}_{p,w_0}(I_0)$ (where $I_0 := (0, \infty)$ and $w_0(x) := \frac{1}{x}$, $x \in I_0$) has been studied since 1970s. For $p \in (1, \infty)$ both spaces $\mathrm{Cs}_{p,w_0}(I_0)$ and $\mathrm{Ch}_{p,w_0}(I_0)$ appeared [11, 12] when solving the problem of describing the associated spaces with order one weighted Sobolev space on the real line, defined as

$$W^1_p(I_0) := \{f \in L^1_{\mathrm{loc}}(I_0) : Df \in L^1_{\mathrm{loc}}(I_0) \,\&\, \|f\|_{W^1_p(I_0)} < \infty\}, \tag{1.2}$$

where $\|f\|_{W^1_p(I_0)} := \|f\|_{L^p(I_0)} + \|\frac{1}{w_0} Df\|_{L^p(I_0)}$. As proved in [9, Theorem 3.3]

$$\begin{aligned}
&(W^1_p(I_0))'_{\mathrm{s}} = \mathrm{Cs}_{p',w_0}(I_0),\\
&(W^1_p(I_0))'_{\mathrm{w}} = (\mathrm{Cs}_{p',w_0}(I_0), \|\cdot\|_{\mathrm{Ch}_{p',w_0}(I_0)}),\\
&(X, \|\cdot\|_{W^1_p(I_0)})'_{\mathrm{w}} = \mathrm{Ch}_{p',w_0}(I_0),
\end{aligned}$$

where

$$X := \{f \in AC(I_0) \,|\, \exists f(0+),\ \exists b \in I_0 :\ \chi_{(b,\infty)} f = 0\}.$$

Note that X differs from

$$\overset{\circ\circ}{W}{}^1_p(I_0) := \{f \in AC(I_0) \,|\, \operatorname{supp} f \text{ is a compact in } I_0\}, \tag{1.3}$$

which plays an important role in the results of [11, 12]. The example in Section 2 shows that $(\overset{\circ\circ}{W}{}^1_p(I_0), \|\cdot\|_{W^1_p(I_0)})'_{\mathrm{w}} \neq \mathrm{Ch}_{p',w_0}(I_0)$. The key difference is the fact that the space $(\overset{\circ\circ}{W}{}^1_p(I_0), \|\cdot\|_{W^1_p(I_0)})'_{\mathrm{w}}$ contains functions that are not integrable at the left end of the segment I_0.

From the definition of associated spaces it follows that $(\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}} = (\mathrm{Cs}_{p,w}(I))'_{\mathrm{w}}$ and $\|g\|_{(\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}}} = \|g\|_{(\mathrm{Cs}_{p,w}(I))'_{\mathrm{w}}}$ for $g \in (\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}}$ and $p \in [1, \infty]$. For $p \in [1, \infty)$ the space $\mathrm{Cs}_{p,w}(I)$ is an order ideal and it has an absolutely continuous norm. Then for $\Lambda \in (\mathrm{Cs}_{p,w}(I))^*$ there exists $g \in (\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}}$ such that $\|\Lambda\|_{(\mathrm{Cs}_{p,w}(I))^*} = \|g\|_{(\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}}}$ and $\Lambda f = \int_I fg$, $f \in \mathrm{Cs}_{p,w}(I)$ (see [2, Chapter 1, Theorem 4.1]).

The problem of describing the associated spaces of $\mathrm{Cs}_{p,w}(I)$ was solved in [3] with the help of an essential $\int_x^d w^p$-concave majorant (see [3, Definition 2.11]), and in [15] with the help of a monotone majorant.

For $p \in [1, \infty)$ characterizations of dual spaces of weighted altered Cesàro space are given in [10]. The key step of the proof was the approximation of an element of the space $\mathrm{Ch}_{p,w}(I)$ by elements with compact support. For $p = \infty$ there is no such approximation but it is possible (see Section 3) to describe the associated spaces of $\mathrm{Ch}_{\infty,w}(I)$ with a weight satisfying the conditions

$$w(x) = \left[\int_c^x v\right]^{-1} \in (0, \infty),\ x \in I,\ v \in \mathfrak{M}(I),\quad v \in L^1_{\mathrm{loc}}([c, d)),\ \lim_{b \to d-} \int_c^b v = \infty. \tag{1.4}$$

Throughout this article, $A \lesssim B$ and $B \gtrsim A$ mean that $A \le cB$, where the constant c depends only on p and may be different in different places. If both $A \lesssim B$ and $A \gtrsim B$ hold, then we write $A \approx B$. $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ is the set of all real numbers, Df is the weak derivative of $f \in \mathfrak{M}(I)$. The space of all locally absolutely continuous functions $f : I \to \mathbb{R}$ is denoted by $AC_{\mathrm{loc}}(I)$, $AC(I)$ is the space of all absolutely continuous functions. The symbol $BPV(I)$ denotes the space of all functions $f : I \to \mathbb{R}$ that have bounded pointwise variation (see [5, §2.1]). For any Borel measure λ defined on Borel subsets of I, the symbol $\|\lambda\|$ means $|\lambda|(I)$, where $|\lambda|$ is the total variation of λ. If $f \in BPV(I)$, then λ_f denotes the unique real Borel measure such that $\lambda_f((a,b]) = f(b+) - f(a+)$ for all $a, b \in I$, with $a \le b$ (see [5, Theorem 5.13]). $C_c^1(I)$ is the space of all real-valued continuously differentiable functions with compact support in I; $C_0(I)$ is the space of all real-valued continuous functions on I that vanish at infinity (see [13, 3.16]).

2 Connection with a Sobolev space

Let $W_p^1(I_0)$ be as defined in (1.2) and $\overset{\circ\circ}{W}{}_p^1(I_0)$ be as defined in (1.3). We start with an example showing that $(\overset{\circ\circ}{W}{}_p^1(I_0), \|\cdot\|_{W_p^1(I_0)})'_{\mathrm{w}} \neq \mathrm{Ch}_{p',w_0}(I_0)$.

Example. According to [12, Remark 5.1] the following relation holds

$$g \in (\overset{\circ\circ}{W}{}_p^1(I_0), \|\cdot\|_{W_p^1(I_0)})'_{\mathrm{w}} \quad \Leftrightarrow \quad (g \in L^1_{\mathrm{loc}}(I_0) \ \& \ [\mathbb{G}(g) + \mathfrak{G}(g)] < \infty),$$

where for $g \in L^1_{\mathrm{loc}}(I_0)$

$$\mathbb{G}(g) \approx \left(\int_0^\infty \frac{1}{t^{p'}} \left| \int_{\frac{t}{2}}^t g \right|^{p'} dt \right)^{\frac{1}{p'}},$$

$$\begin{aligned} \mathfrak{G}(g) &\approx \left(\int_0^\infty \frac{1}{t^{p'(2-p')}} \left| \int_t^{2t} y^{-p'} \left[\int_{\frac{y}{2}}^t g \right] dy \right|^{p'} dt \right)^{\frac{1}{p'}} \\ &= \left(\int_0^\infty \frac{1}{t^{p'(2-p')}} \left| \int_{\frac{t}{2}}^t g(x) \left[\int_t^{2x} y^{-p'} \, dy \right] dx \right|^{p'} dt \right)^{\frac{1}{p'}}. \end{aligned}$$

Further,

$$\begin{aligned} \mathfrak{G}(g) &\approx \left(\int_0^\infty \frac{1}{t^{p'(2-p')}} \left| \int_{\frac{t}{2}}^t g(x) \left[\frac{t^{1-p'} - (2x)^{1-p'}}{p'-1} \right] dx \right|^{p'} dt \right)^{\frac{1}{p'}} \\ &= \frac{1}{p'-1} \left(\int_0^\infty \frac{1}{t^{p'}} \left| \int_{\frac{t}{2}}^t g(x) \left[1 - \left(\frac{2x}{t} \right)^{1-p'} \right] dx \right|^{p'} dt \right)^{\frac{1}{p'}} \\ &= \frac{1}{p'-1} \left(\int_0^\infty \frac{1}{t^{p'}} \left| \int_{\frac{t}{2}}^t g - \left(\frac{2}{t} \right)^{1-p'} \int_{\frac{t}{2}}^t g(x) x^{1-p'} \, dx \right|^{p'} dt \right)^{\frac{1}{p'}}. \end{aligned}$$

Hence, for $g \in L^1_{\mathrm{loc}}(I_0)$ the inequality $[\mathbb{G}(g) + \mathfrak{G}(g)] < \infty$ is equivalent to

$$\left(\int_0^\infty \frac{1}{t^{p'}} \left| \int_{\frac{t}{2}}^t g \right|^{p'} dt \right)^{\frac{1}{p'}} + \left(\int_0^\infty \frac{1}{t^{p'(2-p')}} \left| \int_{\frac{t}{2}}^t g(x) x^{1-p'} \, dx \right|^{p'} dt \right)^{\frac{1}{p'}} < \infty.$$

Now let $p = p' = 2$, $g(x) := \frac{1}{x}\sin\frac{1}{x}\chi_{(0,1]}(x)$, $x \in I_0$. Then

$$\int_0^1 |g(x)|\,dx = \int_0^1 \frac{|\sin\frac{1}{x}|}{x}\,dx = \int_1^\infty \frac{|\sin y|}{y}\,dy = \infty,$$

$$\int_1^\infty \left|\int_{\frac{t}{2}}^t \frac{g(x)}{x}\,dx\right|^2 dt = \int_1^2 \left|\int_{\frac{t}{2}}^1 \frac{\sin\frac{1}{x}}{x^2}\,dx\right|^2 dt \le 4,$$

$$\int_0^1 \left|\int_{\frac{t}{2}}^t \frac{g(x)}{x}\,dx\right|^2 dt = \int_0^1 \left|\int_{\frac{t}{2}}^t \frac{\sin\frac{1}{x}}{x^2}\,dx\right|^2 dt = \int_0^1 \left|\int_{\frac{1}{t}}^{\frac{2}{t}} \sin y\,dy\right|^2 dt$$
$$= \int_1^\infty \frac{1}{x^2}\left|\int_x^{2x} \sin y\,dy\right|^2 dx < \infty,$$

$$\int_1^\infty \frac{1}{t^2}\left|\int_{\frac{t}{2}}^t g\right|^2 dt = \int_1^2 \frac{1}{t^2}\left|\int_{\frac{t}{2}}^1 \frac{\sin\frac{1}{x}}{x}\,dx\right|^2 dt \le 1.$$

Moreover, from

$$\left|\int_y^{2y} \frac{\sin t}{t}\,dt\right| = \left|\int_y^{2y} \frac{d\cos t}{t}\right| = \left|\frac{\cos t}{t}\Big|_y^{2y} + \int_y^{2y} \frac{\cos t}{t^2}\,dt\right| \le \frac{5}{2y},$$

we have the estimates

$$\int_0^1 \frac{1}{t^2}\left|\int_{\frac{t}{2}}^t g\right|^2 dt = \int_0^1 \frac{1}{t^2}\left|\int_{\frac{t}{2}}^t \frac{\sin\frac{1}{x}}{x}\,dx\right|^2 dt = \int_0^1 \frac{1}{t^2}\left|\int_{\frac{1}{t}}^{\frac{2}{t}} \frac{\sin y}{y}\,dy\right|^2 dt$$
$$= \int_1^\infty \left|\int_x^{2x} \frac{\sin y}{y}\,dy\right|^2 dx \le \frac{25}{4}\int_1^\infty \frac{dy}{y^2} < \infty.$$

Therefore, $g \in L^1_{\mathrm{loc}}(I_0) \setminus L^1_{\mathrm{loc}}([0,\infty))$ and $[\mathbb{G}(g) + \mathfrak{G}(g)] < \infty$, that is

$$g \in (\overset{\circ\circ}{W}{}^1_2(I_0), \|\cdot\|_{W^1_2(I_0)})'_{\mathrm{w}} \setminus \mathrm{Ch}_{2,w_0}(I_0).$$

Now we show that in the case of a decreasing weight w satisfying condition (1.4) the spaces $\mathrm{Cs}_{\infty,w}(I)$ and $\mathrm{Ch}_{\infty,w}(I)$ are associated spaces of the space $W^1_1(I)$ defined in formula (2.1). In particular, the theorem contains a criterion for the embedding of $W^1_1(I)$ into the Lebesgue space $L^1_g(I)$ with arbitrary weight g and thereby complements the results obtained in [4], [7, Chapter III], [8].

Theorem 2.1. *Let w satisfy condition* (1.4), $v > 0$ $\mathcal{L}^1$*-almost everywhere on* I,

$$X := \{f \in AC(I), \,|\, \exists f(c+),\ \exists b \in I : \chi_{(b,d)}f = 0\}$$

and

$$W^1_1(I) := \{f \in L^1_{\mathrm{loc}}(I) : Df \in L^1_{\mathrm{loc}}(I)\,\&\,\|f\|_{W^1_1(I)} < \infty\}, \tag{2.1}$$

where $\|f\|_{W^1_1(I)} := \|vf\|_{L^1(I)} + \|\frac{1}{w}Df\|_{L^1(I)}$. *Then*

$$(W^1_1(I))'_{\mathrm{s}} = \mathrm{Cs}_{\infty,w}(I), \tag{2.2}$$
$$(W^1_1(I))'_{\mathrm{w}} = (\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)}), \tag{2.3}$$
$$(X, \|\cdot\|_{W^1_1(I)})'_{\mathrm{w}} = \mathrm{Ch}_{\infty,w}(I). \tag{2.4}$$

Proof. We fix an arbitrary element $f \in W_1^1(I)$. Then there exists an $AC_{\mathrm{loc}}(I)$ representative $\tilde{f}$ of f. For any $x, y \in I$ such that $x > y$ we have

$$|\tilde{f}(x) - \tilde{f}(y)| \leq \int_y^x |Df| \leq \|\chi_{(y,d)} w\|_{L^\infty(I)} \|\chi_{(y,d)} \tfrac{1}{w} Df\|_{L^1(I)}.$$

Hence, there exists the limit $\tilde{f}(d-)$. Since $\|vf\|_{L^1(I)} < \infty$ and $v \notin L^1_{\mathrm{loc}}((c,d])$ then $\tilde{f}(d-) = 0$. In addition, $w \in L^\infty_{\mathrm{loc}}((c,d])$ implies $Df \in L^1_{\mathrm{loc}}((c,d])$. Consequently, $f(x) = -\int_x^d Df$ for $\mathcal{L}^1$-almost all $x \in I$.

Further, for an arbitrary $h \in L^1(I)$ since $w \in L^\infty_{\mathrm{loc}}((c,d])$ then $wh \in L^1_{\mathrm{loc}}((c,d])$, and for $f_h(y) := \int_y^d wh$, $y \in I$, we have

$$\|vf_h\|_{L^1(I)} = \int_I \left| v(y) \int_y^d wh \right| \, dy \leq \|h\|_{L^1(I)}, \quad \|\tfrac{1}{w} Df_h\|_{L^1(I)} = \|h\|_{L^1(I)},$$

that is $f_h \in W_1^1(I)$ and $\|f_h\|_{W_1^1(I)} \leq 2\|h\|_{L^1(I)}$.

If $g \in \mathrm{Cs}_{\infty,w}(I)$ then for any $f \in W_1^1(I) \setminus \{0\}$

$$\frac{\int_I |fg|}{\|f\|_{W_1^1(I)}} \leq \frac{\int_I |(Df)(x)| \left(\int_c^x |g|\right) \, dx}{\|\frac{1}{w} Df\|_{L^1(I)}} \leq \|g\|_{\mathrm{Cs}_{\infty,w}(I)}, \tag{2.5}$$

that is $g \in (W_1^1(I))'_{\mathrm{s}}$ and $\|g\|_{(W_1^1(I))'_{\mathrm{s}}} \leq \|g\|_{\mathrm{Cs}_{\infty,w}(I)}$.

Now let $g \in (W_1^1(I))'_{\mathrm{s}}$. Since (see [2, Lemma 2.8]) for $g \in \mathfrak{M}(I)$ the equalities $\|g\|_{(L^1(I))'_{\mathrm{s}}} = \|g\|_{(L^1(I))'_{\mathrm{w}}} = \|g\|_{L^\infty(I)}$ hold, we get the estimate

$$\begin{aligned}
\|g\|_{(W_1^1(I))'_{\mathrm{s}}} &\geq \sup_{h \in L^1(I)\setminus\{0\}} \frac{\int_I |f_{|h|} g|}{\|f_{|h|}\|_{W_1^1(I)}} \geq \sup_{h \in L^1(I)\setminus\{0\}} \frac{\int_I |g(y)| \left(\int_y^d w|h|\right) \, dy}{2\|h\|_{L^1(I)}} \\
&= \sup_{h \in L^1(I)\setminus\{0\}} \frac{\int_I |h(x)| w(x) \left(\int_c^x |g|\right) \, dx}{2\|h\|_{L^1(I)}} = \frac{1}{2}\|g\|_{\mathrm{Cs}_{\infty,w}(I)},
\end{aligned} \tag{2.6}$$

and (2.2) is proved.

By [11, Theorem 2.5] the equalities $(W_1^1(I))'_{\mathrm{w}} = (W_1^1(I))'_{\mathrm{s}} = \mathrm{Cs}_{\infty,w}(I)$ hold. Besides that, for any $g \in \mathrm{Cs}_{\infty,w}(I)$, $f \in W_1^1(I)$ we have

$$\int_I fg = \int_I g(x) \left(\int_x^d Df\right) \, dx = \int_I (Df)(y) \left(\int_c^y g\right) \, dy. \tag{2.7}$$

Hence, similarly to (2.5) and (2.6) we get $\|g\|_{(W_1^1(I))'_{\mathrm{w}}} \approx \|g\|_{\mathrm{Ch}_{\infty,w}(I)}$, and (2.3) is proved.

Further, for any $a \in I$ there exists a function $f \in X$ such that $\chi_{(c,a)} f = \chi_{(c,a)}$, and this implies $(X, \|\cdot\|_{W_1^1(I)})'_{\mathrm{w}} \subset L^1_{\mathrm{loc}}([c,d))$. Therefore, for any $a \in I$, $f \in X$, $g \in L^1_{\mathrm{loc}}([c,d))$, taking into account the decrease of the function w we have

$$\begin{aligned}
\left| \int_a^d fg \right| &= \left| \int_a^d g(x) \left(\int_x^d Df\right) \, dx \right| = \left| \int_a^d (Df)(y) \left(\int_a^y g\right) \, dy \right| \\
&\leq \|f\|_{W_1^1(I)} \sup_{y \in [a,d)} \left| w(y) \left[\int_c^y g - \int_c^a g\right] \right| \leq 2\|g\|_{\mathrm{Ch}_{\infty,w}(I)} \|f\|_{W_1^1(I)}.
\end{aligned}$$

Passing to the limit as $a \to c+$, we obtain $\|g\|_{(X, \|\cdot\|_{W_1^1(I)})'_{\mathrm{w}}} \leq 2\|g\|_{\mathrm{Ch}_{\infty,w}(I)}$.

If $g \in (X, \|\cdot\|_{W_1^1(I)})'_{\mathrm{w}}$ and $h \in L^1(I)$ with $\operatorname{supp} h \subset (c, b]$ for some $b \in I$, equalities (2.7) hold with $f := f_h$. Therefore,

$$\begin{aligned}\|g\|_{(X,\|\cdot\|_{W_1^1(I)})'_{\mathrm{w}}} &\geq \sup_{b\in I} \sup_{h\in L^1(I)\setminus\{0\}, \operatorname{supp} h\subset(c,b]} \frac{\left|\int_c^b g(y) \left(\int_y^b wh\right) dy\right|}{2\|h\|_{L^1(I)}} \\ &\geq \sup_{b\in I} \sup_{h\in L^1(I)\setminus\{0\}, \operatorname{supp} h\subset(c,b]} \frac{\left|\int_c^b h(x)w(x) \left(\int_c^x g\right) dx\right|}{2\|h\|_{L^1(I)}} \\ &= \frac{1}{2} \sup_{b\in I} \sup_{x\in(c,b]} \left|w(x) \int_c^x g\right| = \frac{1}{2}\|g\|_{\mathrm{Ch}_{\infty,w}(I)},\end{aligned}$$

and (2.4) follows. □

3 Associated spaces of $\mathrm{Ch}_{\infty,w}(I)$

As in the case $p < \infty$ the "strong" associated space of $\mathrm{Ch}_{\infty,w}(I)$ is the null space. This follows from Lemma 3.1, the proof of which is similar to the proof of [10, Lemma 2.2].

Lemma 3.1. *Let w satisfy condition* (1.1), $[a, b] \subset I$ *and* $h \in L^1([a, b])$. *For any $\varepsilon > 0$ there exists $f \in \mathfrak{M}(I)$ such that* $\operatorname{supp} f \subset [a, b]$, $|f| = |h|$ *on* (a, b) *and* $\|f\|_{\mathrm{Ch}_{\infty,w}(I)} < \varepsilon$.

The next two lemmas contain the key constructions for obtaining a criterion for an element to belong to the "weak" associated space of $\mathrm{Ch}_{\infty,w}(I)$.

Lemma 3.2. *Let w satisfy condition* (1.4).

1. *If $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$ then $vg \in L^1(I)$. If $g \in (\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$ and $v > 0$ $\mathcal{L}^1$-almost everywhere on I then*

$$\|vg\|_{L^1(I)} \leq \|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}}. \tag{3.1}$$

2. *Let (a) $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$ and $A_g := \|g\|_{(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}}$, or (b) $v > 0$ $\mathcal{L}^1$-almost everywhere on I, $g \in (\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$ and $A_g := \|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}}$. Then there exists a $BPV(I)$ representative $\tilde{g}$ of $\frac{g}{w}$ and the estimate*

$$\|\lambda_{\tilde{g}}\| \leq \|vg\|_{L^1(I)} + A_g \tag{3.2}$$

holds.

Proof. 1. Since $v \in \mathrm{Ch}_{\infty,w}(I)$ then $vg \in L^1(I)$ for $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$. Now let $v > 0$ $\mathcal{L}^1$-almost everywhere on I. Then for any $f \in \mathfrak{M}(I)$

$$\|f\|_{\mathrm{Cs}_{\infty,w}(I)} \leq \|\tfrac{f}{v}\|_{L^\infty(I)},$$

and for $g \in (\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$ the relations

$$\|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}} \geq \sup_{f:\frac{f}{v}\in L^\infty(I)\setminus\{0\}} \frac{\left|\int_I \frac{f}{v} vg\right|}{\|\frac{f}{v}\|_{L^\infty(I)}} = \sup_{h\in L^\infty(I)\setminus\{0\}} \frac{\left|\int_I hvg\right|}{\|h\|_{L^\infty(I)}} = \|vg\|_{L^1(I)}$$

hold.

2. We fix an arbitrary function $\phi \in C_c^1(I)$ and put $f := D(\frac{1}{w}\phi) = v\phi + \frac{1}{w}D\phi$. Then $f \in L^1(I)$ and $\|f\|_{\mathrm{Ch}_{\infty,w}(I)} = \max_{x\in I}|\phi(x)|$. If $v > 0$ $\mathcal{L}^1$-almost everywhere on I then $f \in \mathrm{Cs}_{\infty,w}(I)$. From $vg \in L^1(I)$ we have $\int_I |\phi v g| < \infty$. Hence, $\int_I |\frac{1}{w} g D\phi| < \infty$ and

$$\frac{|\int_I \frac{1}{w} g D\phi|}{\max\limits_{x\in I}|\phi(x)|} \le \frac{|\int_I \phi v g|}{\max\limits_{x\in I}|\phi(x)|} + \frac{|\int_I fg|}{\|f\|_{\mathrm{Ch}_{\infty,w}(I)}} \le \|vg\|_{L^1(I)} + A_g.$$

For $\phi \in C_c^1(I)$ we put $\Lambda\phi := \int_I \frac{1}{w} g D\phi$. By the Hahn– Banach theorem there exists an extension $\tilde{\Lambda} \in (C_0(I))^*$ of the functional Λ for which the estimate

$$\|\tilde{\Lambda}\|_{(C_0(I))^*} \le \|vg\|_{L^1(I)} + A_g$$

holds.

By the Riesz theorem [13, 6.19] on the representation of a linear continuous functional on $C_0(I)$ there exists a unique regular real Borel measure λ such that $\|\lambda\| = \|\tilde{\Lambda}\|_{(C_0(I))^*}$ and $\tilde{\Lambda}\varphi = \int_I \varphi\, d\lambda$ for any $\varphi \in C_0(I)$.

We define $h_g(x) := \lambda(I \cap (-\infty, x])$, $x \in I$. Then $h_g \in BPV(I)$ and applying [5, Corollary 5.41], we have

$$\int_I \frac{1}{w} g D\phi = \tilde{\Lambda}\phi = \int_I \phi\, d\lambda = -\int_I h_g D\phi$$

for any $\phi \in C_c^1(I)$. Hence, $\frac{1}{w}g + h_g$ $\mathcal{L}^1$-almost everywhere on I coincides with a constant function. Therefore, there exists a $BPV(I)$ representative $\tilde{g}$ of $\frac{g}{w}$, and $\lambda_{\tilde{g}} = \lambda_{h_g} = \lambda$ are valid (see [5, Remark 5.14]). □

Lemma 3.3. *Let w satisfy condition* (1.4), *$f \in \mathrm{Ch}_{\infty,w}(I)$, $g \in L^\infty_{\mathrm{loc}}(I)$, $vg \in L^1(I)$, $\int_I |fg| < \infty$, $\frac{g}{w}$ has an $BPV(I)$ representative $\tilde{g}$. Then*

$$\left|\int_I fg\right| \le 2\left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde{g}}\|\right)\|f\|_{\mathrm{Ch}_{\infty,w}(I)}. \tag{3.3}$$

Proof. We fix $\gamma \in (0,1)$. For $n \in \mathbb{N}$ we define

$$b_n := \sup\Big\{x \in I : \frac{1}{w(x)} \le n\Big\}, \quad a_n := \inf\Big\{x \in I : \frac{1}{w(x)} \ge \frac{1}{n}\Big\}.$$

Since $\frac{1}{w}$ is a continuous function, $\lim_{x\to d-}\frac{1}{w(x)} = \infty$ and $\lim_{x\to c+}\frac{1}{w(x)} = 0$, then $\frac{1}{w(b_n)} = n$, $\frac{1}{w(a_n)} = \frac{1}{n}$. Moreover, since $\{x \in I : \frac{1}{w(x)} \le n\} \subset \{x \in I : \frac{1}{w(x)} \le n+1\}$, then $b_n \le b_{n+1}$. If $b := \lim_{n\to\infty} b_n < d$ then $v \in L^1_{\mathrm{loc}}([c,d))$ implies $\infty = \lim_{n\to\infty}\frac{1}{w(b_n)} = \frac{1}{w(b)} < \infty$ and we get a contradiction. Hence, $\lim_{n\to\infty} b_n = d$. Analogously, $a_n \downarrow c$ as $n \to \infty$.

Let $n_0 \in \mathbb{N}$ be such that $a_{n_0} < b_{n_0}$. For $n \ge n_0$ we define $\alpha_n \in [a_n, b_n]$ such that $\frac{1}{w(\alpha_n)} = \min_{x\in[a_n,b_n]}\frac{1}{w(x)}$. Then $\frac{1}{w(\alpha_n)} > 0$ and $\alpha_n < b_n$. We claim that $\lim_{n\to\infty}\alpha_n = c$. We fix an arbitrary $a > c$. Since $a_n \downarrow c$ as $n \to \infty$ there exists $n_1 > n_0$ such that $a_{n_1} < a$. Let $n_2 > n_1$ and $\frac{1}{n_2} < \frac{1}{w(\alpha_{n_1})}$. Then for $n > n_2$ we have $\alpha_n \in [a_n, a_{n_1}]$ because of $\frac{1}{w(x)} \ge n_1 > \frac{1}{w(a_{n_1})}$ for $x \ge b_{n_1}$. Hence, $\alpha_n < a$.

Since $\int_{\alpha_n}^{b_n}|f| < \infty$ then by [13, 3.14] for $n \ge n_0$ there exists a function $\bar{f}_n \in C_c((\alpha_n, b_n))$ such that $\int_{\alpha_n}^{b_n}|f - \bar{f}_n| \le \frac{1}{w(\alpha_n)n}(1 + \|g\chi_{[\alpha_n,b_n]}\|_{L^\infty(I)})^{-1}$. Now we choose $\beta_n \in (b_n, d)$, $\theta_n \in \{-1, 1\}$ so that the equality $\theta_n\int_{b_n}^{\beta_n} w^\gamma v + \int_{\alpha_n}^{b_n}\bar{f}_n = 0$ holds. For

$$f_n := \bar{f}_n\chi_{[\alpha_n,b_n]} + \theta_n w^\gamma v\chi_{[b_n,\beta_n]},\ n \ge n_0$$

we have

$$\int_c^x f_n = 0, \quad x \in (c, \alpha_n] \cup [\beta_n, d),$$

$$\sup_{x\in(\alpha_n,b_n]} w(x)\left|\int_c^x f_n\right| = \sup_{x\in(\alpha_n,b_n]} w(x)\left|\int_{\alpha_n}^x (\bar f_n - f) + \int_c^x f - \int_c^{\alpha_n} f\right|$$
$$\le \frac{1}{n} + 2\sup_{x\in[\alpha_n,b_n]} w(x)\left|\int_c^x f\right| \le 2\|f\|_{\mathrm{Ch}_{\infty,w}(I)} + \frac{1}{n}$$

and

$$\sup_{x\in(b_n,\beta_n)} w(x)\left|\int_c^x f_n\right| \le \sup_{x\in(b_n,\beta_n)} w(x)\left[\left|\int_{\alpha_n}^{b_n} (\bar f_n - f) + \int_c^{b_n} f - \int_c^{\alpha_n} f\right| + \int_{b_n}^x w^\gamma v\right]$$
$$\le 2\|f\|_{\mathrm{Ch}_{\infty,w}(I)} + \frac{1}{n} + \sup_{x\in(b_n,\beta_n)} \frac{w(x)[w(x)^{\gamma-1} - w(b_n)^{\gamma-1}]}{(1-\gamma)}$$
$$\le 2\|f\|_{\mathrm{Ch}_{\infty,w}(I)} + \frac{1}{n} + \frac{1}{(1-\gamma)n^\gamma}.$$

Moreover,

$$\left|\int_I fg - \int_I f_n g\right| \le \int_c^{\alpha_n} |fg| + \|g\chi_{[\alpha_n,b_n]}\|_{L^\infty(I)} \int_{\alpha_n}^{b_n} |f - \bar f_n| + \int_{b_n}^\infty |fg| + \left|\int_{\beta_n}^\infty w^\gamma v g\right|$$
$$\le \int_c^{\alpha_n} |fg| + \frac{1}{n^2} + \int_{b_n}^\infty |fg| + \frac{1}{n^\gamma}\|vg\|_{L^1(I)}.$$

Thus, $\lim_{n\to\infty} \int_I f_n g = \int_I fg$.

Now we put $F_n(x) := w(x)\int_c^x f_n$, $x \in I$. Then $F_n \in AC_{\mathrm{loc}}(I)$, $\operatorname{supp} F_n$ is a compact in I and $f_n = vF_n + \frac{1}{w} DF_n$ $\mathcal{L}^1$-almost everywhere on I. Using [5, Corollary 5.40], we get

$$\int_I f_n g = \int_I vgF_n + \int_I \frac{1}{w} g DF_n = \int_I vgF_n - \int_I F_n \, d\lambda_{\tilde g}.$$

Consequently,

$$\left|\int_I f_n g\right| \le \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde g}\|\right) \sup_{x\in I} w(x)\left|\int_c^x f_n\right|$$

and (3.3) follows by passing to the limit as $n \to \infty$. □

Now we can formulate the criterion of an element $g \in \mathfrak{M}(I)$ belonging to the space $(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$ and get a two-sided estimate on the norm of the element of the "weak" space in case $v > 0$ $\mathcal{L}^1$-almost everywhere on I.

Theorem 3.1. *Let w satisfy condition (1.4), $g \in \mathfrak{M}(I)$. The following statements are equivalent:*
(i) $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$;
(ii) $vg \in L^1(I)$, $g \in L^\infty(I)$, and $\chi_{(b,d)} g = 0$ for some $b \in I$, $\frac{g}{w}$ has an $BPV(I)$ representative. Moreover, if $v > 0$ $\mathcal{L}^1$-almost everywhere on I, then

$$\|g\|_{(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}} \approx \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde g}\|\right),$$

where $\tilde g$ is an $BPV(I)$ representative of $\frac{g}{w}$.

Proof. (*ii*) ⇒ (*i*). For $f \in \mathrm{Ch}_{\infty,w}(I)$ we have $f \in L^1_{\mathrm{loc}}([c,d))$ and therefore

$$\int_I |fg| \le \|g\|_{L^\infty(I)} \int_c^b |f| < \infty.$$

Using Lemma 3.3 for $\tilde{g} \in \frac{g}{w} \cap BPV(I)$ we get the estimate

$$\|g\|_{(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}} \le 2\left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde{g}}\|\right) < \infty.$$

(*i*) ⇒ (*ii*). We denote $E := \{x \in I : g(x) \ne 0\}$. Suppose that $\mathcal{L}^1((t,d) \cap E) > 0$ for any $t \in I$. Then there exists $\{[a_k,b_k]\}_1^\infty$ such that $b_k < a_{k+1}$ and $\int_{a_k}^{b_k} |g| > 0$. We choose $\theta_k \in (0,\infty)$ so that the inequality $\theta_k \int_{a_k}^{b_k} |g| \ge 1$ holds. By Lemma 3.1 there exists $f_k \in \mathfrak{M}(I)$ with the properties: $\|f_k\|_{\mathrm{Ch}_{\infty,w}(I)} < 2^{-k}$, $\operatorname{supp} f_k \subset [a_k,b_k]$ and $|f_k| = \theta_k$ on (a_k,b_k). Then for the function $f := \sum_{k=1}^\infty f_k$ we have $\|f\|_{\mathrm{Ch}_{\infty,w}(I)} \le 1$ and

$$\int_I |fg| \ge \sum_{k=1}^\infty \theta_k \int_{a_k}^{b_k} |g| \ge \sum_{k=1}^\infty 1 = \infty.$$

This contradicts $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$. Thus, there exists point $b \in I$ such that $g\chi_{(b,d)} = 0$.

Now we assume that $g \notin L^\infty(I)$. Then there exists $h \in L^1((c,b))$ such that $\int_c^b |hg| = \infty$. Let $a_1 := b$ and $a_k \downarrow c$ as $k \to \infty$. By Lemma 3.1 there exists $f_k \in \mathfrak{M}(I)$ with properties: $\operatorname{supp} f_k \subset [a_{k+1},a_k]$, $\|f_k\|_{\mathrm{Ch}_{\infty,w}(I)} < 2^{-k}$ and $|f_k| = |h|$ on (a_{k+1},a_k). Then for the function $f := \sum_{k=1}^\infty f_k$ we have $\|f\|_{\mathrm{Ch}_{\infty,w}(I)} \le 1$ and

$$\int_I |fg| \ge \int_c^b |hg| = \infty.$$

This contradicts the relation $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$, that is $g \in L^\infty(I)$.

By Lemma 3.2 we have $vg \in L^1(I)$, $\frac{g}{w} \cap BPV(I) \ne \emptyset$. If $v > 0$ $\mathcal{L}^1$-almost everywhere on I the statement 1 of Lemma 3.2 implies the estimate $3\|g\|_{(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}} \ge \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde{g}}\|\right)$ for $\tilde{g} \in \frac{g}{w} \cap BPV(I)$. □

Using the results for the weighted Cesàro space, we can also characterize the space $(\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$ in the case of $v > 0$ $\mathcal{L}^1$-almost everywhere on I.

Theorem 3.2. *Let w satisfy condition* (1.4), *$v > 0$ $\mathcal{L}^1$-almost everywhere on I and $g \in \mathfrak{M}(I)$. The following statements are equivalent:*

(*i*) $g \in (\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$;

(*ii*) $vg \in L^1(I)$, $\frac{g}{w}$ *has an $BPV(I)$ representative and* $\int_I v(t)\|g\|_{L^\infty([t,d))}\,dt < \infty$.

Moreover, $\|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}} \approx \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde{g}}\|\right)$, *where $\tilde{g}$ is an $BPV(I)$ representative of $\frac{g}{w}$.*

Proof. First, $\int_I v(t)\|g\|_{L^\infty([t,d))}\,dt < \infty$ is equivalent to $g \in (\mathrm{Cs}_{\infty,w}(I))'_{\mathrm{w}}$ by [1, Remark 4.3], [15, Theorem 4].

(*ii*) ⇒ (*i*). Since $\int_I v(t)\|g\|_{L^\infty([t,d))}\,dt < \infty$ then $g \in L^\infty_{\mathrm{loc}}(I)$. Moreover, for $f \in \mathrm{Cs}_{\infty,w}(I)$ we have $\int_I |fg| < \infty$, and the estimate

$$\|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}} \le 2\left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde{g}}\|\right) < \infty$$

follows from Lemma 3.3.

(*i*) ⇒ (*ii*). By Lemma 3.2 we have $vg \in L^1(I)$, $\frac{g}{w} \cap BPV(I) \ne \emptyset$ and the estimate $3\|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}} \ge \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde{g}}\|\right)$ holds for $\tilde{g} \in \frac{g}{w} \cap BPV(I)$. Further, since $\|g\|_{(\mathrm{Cs}_{\infty,w}(I))'_{\mathrm{w}}} \le \|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}}$, we obtain $\int_I v(t)\|g\|_{L^\infty([t,d))}\,dt < \infty$. □

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CONCINNITY OF DYNAMIC INEQUALITIES DESIGNED ON CALCULUS OF TIME SCALES

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Abstract. We present some reverse dynamic inequalities of Radon's and Bergström's type on time scales in general form. The extension of Clarkson's dynamic inequality on time scales is also given. Our further investigations explore some dynamic inequalities by using Kantorovich's and Specht's ratios. The calculus of time scales unifies and extends continuous results and their corresponding discrete and quantum analogues.

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1 Introduction

Motivated by the recent developments of the theory and applications of time scales, we will prove some results on time scales. The calculus of time scales was introduced by Stefan Hilger [9]. A time scale is an arbitrary nonempty closed subset of $\mathbb{R}$ of all real numbers. The theory of time scales calculus is applied to harmonize results in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The three popular branches of time scales calculus are delta calculus, nabla calculus and diamond-α calculus. Many dynamic inequalities (see [1, 4, 6, 12, 13, 14, 15]) have been investigated by using this hybrid theory. Basic work on dynamic inequalities is done by Agarwal, Anastassiou, Bohner, Peterson, O'Regan, Saker and several other authors.

In this paper, it is assumed that all integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of the interval $[a, b]$ the given time scale.

2 Preliminaries

We present basic concepts of delta calculus. The results of delta calculus are taken from monographs [4, 5].

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^Δ is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$, such that for all $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be delta differentiable at t, and $f^\Delta(t)$ is called the delta derivative of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous), if it is continuous at each right-dense point and there exists a finite left-sided limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [4, 5].

Definition 1. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [2, 4, 5].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$ and $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called nabla differentiable at $t \in \mathbb{T}_k$, with nabla derivative $f^\nabla(t)$, if there exists $f^\nabla(t) \in \mathbb{R}$, such that for all $\epsilon > 0$, there is a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous (ld-continuous), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [2, 4, 5].

Definition 2. A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^\nabla(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Next, we present an introduction to the diamond-α derivative, see [1, 16].

Definition 3. Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}$, the diamond-α dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t), \quad 0 \leq \alpha \leq 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem 2.1 (See [16]). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$ and we write $f^\sigma(t) = f(\sigma(t))$, $g^\sigma(t) = g(\sigma(t))$, $f^\rho(t) = f(\rho(t))$ and $g^\rho(t) = g(\rho(t))$. Then*

(i) $f \pm g : \mathbb{T} \to \mathbb{R}$ *is diamond-α differentiable at $t \in \mathbb{T}$, with*

$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t).$$

(ii) $fg : \mathbb{T} \to \mathbb{R}$ *is diamond-α differentiable at $t \in \mathbb{T}$, with*

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^\sigma(t)g^\Delta(t) + (1-\alpha)f^\rho(t)g^\nabla(t).$$

(iii) For $g(t)g^\sigma(t)g^\rho(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^\sigma(t)g^\rho(t) - \alpha f^\sigma(t)g^\rho(t)g^\Delta(t) - (1-\alpha)f^\rho(t)g^\sigma(t)g^\nabla(t)}{g(t)g^\sigma(t)g^\rho(t)}.$$

Definition 4 (See [16]). Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s, \quad 0 \leq \alpha \leq 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 2.2 (See [16]). *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$-integrable functions on $[a, b]_\mathbb{T}$. Then*

(i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;

(ii) $\int_a^t cf(s) \diamond_\alpha s = c \int_a^t f(s) \diamond_\alpha s$;

(iii) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$;

(iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;

(v) $\int_a^a f(s) \diamond_\alpha s = 0$.

We also consider *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0.$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, +\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement of Young's inequality [20] in terms of Kantorovich's ratio holds

$$K^\eta\left(\frac{a}{b}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q} \tag{2.1}$$

for $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ and $\eta = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Specht's ratio [7, 17] is defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} \qquad (h > 0, h \neq 1).$$

We present here some properties of Specht's ratio. See [7, 17, 18] for the proof and details:

(i) $S(1) = 1$ and $S(h) = S\left(\frac{1}{h}\right) > 1$ for all $h > 0$.

(ii) $S(h)$ is a monotone increasing function on $(1, +\infty)$ and monotone decreasing function on $(0, 1)$.

The following inequality is due to Furuichi [8] and provides a refinement for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{\eta}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q} \tag{2.2}$$

for $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ and $\eta = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

3 Main results

In this section, we give the following extension of reverse Radon's inequality on time scales.

Theorem 3.1. *Let $w, f, g \in C\left([a, b]_{\mathbb{T}}, \mathbb{R} \setminus \{0\}\right)$ be $\diamond_\alpha$-integrable functions. If $\beta > 0$, $\gamma \geq 1$ and $0 < m \leq \left(\frac{|f(x)|}{|g(x)|}\right)^{\beta+\gamma} \leq M < \infty$ on the set $[a, b]_{\mathbb{T}}$, then*

$$\int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x \leq \left(\frac{M}{m}\right)^{\frac{\beta+\gamma-1}{\beta+\gamma}} \frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}}. \tag{3.1}$$

Proof. Let $p = \beta + \gamma$ and $q = \frac{\beta+\gamma}{\beta+\gamma-1}$. Consider the conditions $0 < m \leq \frac{|f(x)|^p}{|g(x)|^q} \leq M$, therefore

$$|g(x)| \geq M^{-\frac{1}{q}} |f(x)|^{\frac{p}{q}} \Rightarrow |f(x)g(x)| \geq M^{-\frac{1}{q}} |f(x)|^p, \quad \forall x \in [a, b]_{\mathbb{T}}.$$

Thus, we have

$$\left(\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x\right)^{\frac{1}{p}} \geq M^{-\frac{1}{pq}} \left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}}. \tag{3.2}$$

On the other hand, we have that

$$|f(x)| \geq m^{\frac{1}{p}} |g(x)|^{\frac{q}{p}} \Rightarrow |f(x)g(x)| \geq m^{\frac{1}{p}} |g(x)|^q, \quad \forall x \in [a, b]_{\mathbb{T}}.$$

Thus,

$$\left(\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x\right)^{\frac{1}{q}} \geq m^{\frac{1}{pq}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}. \tag{3.3}$$

Multiplying (3.2) and (3.3), we obtain

$$\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x. \tag{3.4}$$

Replacing $|f(x)|$ by $\frac{|f(x)|}{|g(x)|^{\frac{1}{q}}}$ and $|g(x)|$ by $|g(x)|^{\frac{1}{q}}$ in inequality (3.4), simultaneously, we obtain

$$\left(\int_a^b \frac{|w(x)||f(x)|^p}{|g(x)|^{\frac{p}{q}}} \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int_a^b |w(x)||f(x)| \diamond_\alpha x. \tag{3.5}$$

Hence (3.5) takes the form

$$\int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta+\gamma-1}} \diamond_\alpha x \leq \left(\frac{M}{m}\right)^{\frac{\beta+\gamma-1}{\beta+\gamma}} \frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\beta+\gamma-1}}. \tag{3.6}$$

Replacing $|w(x)|$ by $|w(x)||g(x)|^{\gamma-1}$ in inequality (3.6), we obtain (3.1). □

Next, we give extended reverse Bergström's inequality on time scales.

Corollary 3.1. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R} \setminus \{0\})$ be $\diamond_\alpha$-integrable functions. If $0 < m \leq \left(\frac{|f(x)|}{|g(x)|}\right)^2 \leq M < \infty$ on the set $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b \frac{|w(x)||f(x)|^2}{|g(x)|} \diamond_\alpha x \leq \left(\frac{M}{m}\right)^{\frac{1}{2}} \frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^2}{\int_a^b |w(x)||g(x)| \diamond_\alpha x}. \tag{3.7}$$

Proof. Putting $\beta = \gamma = 1$ in Theorem 3.1, we get (3.7). □

Remark 1. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$ for $k \in \{1, 2, \ldots, n\}$. If $\gamma = 1$, then (3.1) reduces to

$$\sum_{k=1}^{n} \frac{x_k^{\beta+1}}{y_k^\beta} \leq \left(\frac{M}{m}\right)^{\frac{\beta}{\beta+1}} \frac{\left(\sum\limits_{k=1}^{n} x_k\right)^{\beta+1}}{\left(\sum\limits_{k=1}^{n} y_k\right)^{\beta}}. \tag{3.8}$$

Inequality (3.8) is just the reverse of the classical inequality

$$\frac{\left(\sum\limits_{k=1}^{n} x_k\right)^{\beta+1}}{\left(\sum\limits_{k=1}^{n} y_k\right)^{\beta}} \leq \sum_{k=1}^{n} \frac{x_k^{\beta+1}}{y_k^\beta}. \tag{3.9}$$

The inequality from (3.9) is called, in literature, Radon's inequality *[10]*.

Remark 2. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$ for $k \in \{1, 2, \ldots, n\}$. Then inequality given in (3.7) reduces to

$$\sum_{k=1}^{n} \frac{x_k^2}{y_k} \leq \left(\frac{M}{m}\right)^{\frac{1}{2}} \frac{\left(\sum\limits_{k=1}^{n} x_k\right)^2}{\sum\limits_{k=1}^{n} y_k}. \tag{3.10}$$

Inequality (3.10) is just the reverse of the classical inequality

$$\frac{\left(\sum\limits_{k=1}^{n} x_k\right)^2}{\sum\limits_{k=1}^{n} y_k} \leq \sum_{k=1}^{n} \frac{x_k^2}{y_k}. \tag{3.11}$$

Inequality (3.11) is called Bergström's or Titu Andreescu's inequality or also Engel's inequality in literature as given in *[3]* with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{x_n}{y_n}$.

The following result is an extension of dynamic Clarkson's type inequality on time scales.

Theorem 3.2. *Let $p \geq 1$, $w, f, g \in C\left([a,b]_{\mathbb{T}}, \mathbb{R}\right)$. If $1 < m \leq \frac{|f(x)|}{|g(x)|} \leq M < \infty$, $\forall x \in [a,b]_{\mathbb{T}}$, then*

$$\int_a^b |w(x)| \left(|f(x)|^p + |g(x)|^p\right) \diamond_\alpha x$$
$$\leq \Lambda \int_a^b |w(x)| \left(|f(x)| + |g(x)|\right)^p \diamond_\alpha x + \Omega \int_a^b |w(x)| \left(|f(x)| - |g(x)|\right)^p \diamond_\alpha x, \quad (3.12)$$

where $\Lambda = \frac{M^p(m+1)^p + (M+1)^p}{2(M+1)^p(m+1)^p}$ and $\Omega = \frac{1+m^p}{2(m-1)^p}$.

Proof. By using the given condition $\frac{|f(x)|}{|g(x)|} \leq M$, we have

$$(M+1)^p |f(x)|^p \leq M^p \left(|f(x)| + |g(x)|\right)^p, \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Therefore

$$\int_a^b |w(x)||f(x)|^p \diamond_\alpha x \leq \left(\frac{M}{M+1}\right)^p \int_a^b |w(x)| \left(|f(x)| + |g(x)|\right)^p \diamond_\alpha x. \quad (3.13)$$

On the other hand, we have that

$$\left(1 + \frac{1}{m}\right)^p |g(x)|^p \leq \left(\frac{1}{m}\right)^p \left(|f(x)| + |g(x)|\right)^p, \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Thus,

$$\int_a^b |w(x)||g(x)|^p \diamond_\alpha x \leq \left(\frac{1}{m+1}\right)^p \int_a^b |w(x)| \left(|f(x)| + |g(x)|\right)^p \diamond_\alpha x. \quad (3.14)$$

Adding (3.13) and (3.14), we obtain

$$\int_a^b |w(x)| \left(|f(x)|^p + |g(x)|^p\right) \diamond_\alpha x$$
$$\leq \left\{\left(\frac{M}{M+1}\right)^p + \left(\frac{1}{m+1}\right)^p\right\} \int_a^b |w(x)| \left(|f(x)| + |g(x)|\right)^p \diamond_\alpha x. \quad (3.15)$$

By given hypothesis, we have

$$m - 1 \leq \frac{|f(x)|}{|g(x)|} - 1 \Rightarrow |g(x)| \leq \frac{|f(x)| - |g(x)|}{m-1},$$

where $\forall x \in [a,b]_{\mathbb{T}}$. Thus,

$$\int_a^b |w(x)||g(x)|^p \diamond_\alpha x \leq \left(\frac{1}{m-1}\right)^p \int_a^b |w(x)| \left(|f(x)| - |g(x)|\right)^p \diamond_\alpha x. \quad (3.16)$$

On the other hand, we have that

$$1 - \frac{1}{m} \leq 1 - \frac{|g(x)|}{|f(x)|} \Rightarrow |f(x)| \leq \frac{m}{m-1} \left(|f(x)| - |g(x)|\right), \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Thus,

$$\int_a^b |w(x)||f(x)|^p \diamond_\alpha x \leq \left(\frac{m}{m-1}\right)^p \int_a^b |w(x)| \left(|f(x)| - |g(x)|\right)^p \diamond_\alpha x. \quad (3.17)$$

Adding (3.16) and (3.17), we obtain

$$\int_a^b |w(x)| \left(|f(x)|^p + |g(x)|^p\right) \diamond_\alpha x$$
$$\leq \left\{ \left(\frac{m}{m-1}\right)^p + \left(\frac{1}{m-1}\right)^p \right\} \int_a^b |w(x)| \left(|f(x)| - |g(x)|\right)^p \diamond_\alpha x. \quad (3.18)$$

Adding (3.15) and (3.18), we get the desired inequality (3.12). □

Remark 3. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$ for $k \in \{1, 2, \ldots, n\}$. Then (3.12) reduces to

$$\sum_{k=1}^{n} (x_k^p + y_k^p) \leq \Lambda \sum_{k=1}^{n} (x_k + y_k)^p + \Omega \sum_{k=1}^{n} (x_k - y_k)^p . \quad (3.19)$$

Our next result concerning extended Young's inequality with Kantorovich's ratio on time scales is investigated.

Theorem 3.3. *Let $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $f \equiv 0$ nor $g \equiv 0$. If $0 < m \leq \left|\frac{f(x)}{g(x)}\right| \leq M < \infty$, $\forall x \in [a,b]_{\mathbb{T}}$, then*

$$\int_a^b K^\eta \left(\frac{|f(x)|^p}{|g(x)|^q}\right) |w(x)||f(x)g(x)| \diamond_\alpha x \leq \Lambda \int_a^b |w(x)| \left(|f(x)|^p + |g(x)|^p\right) \diamond_\alpha x$$
$$+ \Omega \int_a^b |w(x)| \left(|f(x)|^q + |g(x)|^q\right) \diamond_\alpha x, \quad (3.20)$$

where $\Lambda = \frac{2^{p-1}M^p}{p(M+1)^p}$, $\Omega = \frac{2^{q-1}}{q(m+1)^q}$, $\eta = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $K(.)$ is Kantorovich's ratio.

Proof. By using the given hypothesis, we have that

$$\frac{|f(x)|}{|g(x)|} \leq M \Rightarrow (M+1)|f(x)| \leq M(|f(x)| + |g(x)|), \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Therefore

$$\int_a^b |w(x)||f(x)|^p \diamond_\alpha x \leq \left(\frac{M}{M+1}\right)^p \int_a^b |w(x)|(|f(x)| + |g(x)|)^p \diamond_\alpha x. \quad (3.21)$$

On the other hand, we have that

$$m \leq \frac{|f(x)|}{|g(x)|} \Rightarrow (m+1)|g(x)| \leq |f(x)| + |g(x)|, \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Thus,

$$\int_a^b |w(x)||g(x)|^q \diamond_\alpha x \leq \left(\frac{1}{m+1}\right)^q \int_a^b |w(x)|(|f(x)| + |g(x)|)^q \diamond_\alpha x. \quad (3.22)$$

Now, using Young's inequality (2.1), we have

$$K^\eta \left(\frac{|f(x)|^p}{|g(x)|^q}\right) |f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q, \quad \forall x \in [a,b]_{\mathbb{T}}. \quad (3.23)$$

Inequality (3.23) takes the form

$$\int_a^b K^\eta\left(\frac{|f(x)|^p}{|g(x)|^q}\right)|w(x)||f(x)g(x)|\diamond_\alpha x \leq \frac{1}{p}\int_a^b |w(x)||f(x)|^p\diamond_\alpha x+\frac{1}{q}\int_a^b |w(x)||g(x)|^q\diamond_\alpha x. \quad (3.24)$$

By using the results from (3.21) and (3.22), inequality (3.24) becomes

$$\int_a^b K^\eta\left(\frac{|f(x)|^p}{|g(x)|^q}\right)|w(x)||f(x)g(x)|\diamond_\alpha x \leq \frac{1}{p}\left(\frac{M}{M+1}\right)^p\int_a^b |w(x)|(|f(x)|+|g(x)|)^p\diamond_\alpha x + \frac{1}{q}\left(\frac{1}{m+1}\right)^q\int_a^b |w(x)|(|f(x)|+|g(x)|)^q\diamond_\alpha x. \quad (3.25)$$

Using the elementary inequality

$$(x+y)^\delta \leq 2^{\delta-1}(x^\delta+y^\delta), \quad \delta>1, \quad x,y\geq 0,$$

inequality (3.20) follows from inequality (3.25). □

Our next result concerning extended Young's inequality with Specht's ratio on time scales is explored.

Theorem 3.4. *Let $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $w,f,g\in C([a,b]_{\mathbb{T}},\mathbb{R})$, neither $f\equiv 0$ nor $g\equiv 0$. If $0<m\leq\left|\frac{f(x)}{g(x)}\right|\leq M<\infty$, $\forall x\in[a,b]_{\mathbb{T}}$, then*

$$\int_a^b S\left(\left(\frac{|f(x)|^p}{|g(x)|^q}\right)^\eta\right)|w(x)||f(x)g(x)|\diamond_\alpha x \leq \Lambda\int_a^b |w(x)|\left(|f(x)|^p+|g(x)|^p\right)\diamond_\alpha x + \Omega\int_a^b |w(x)|\left(|f(x)|^q+|g(x)|^q\right)\diamond_\alpha x, \quad (3.26)$$

where $\Lambda=\frac{2^{p-1}M^p}{p(M+1)^p}$, $\Omega=\frac{2^{q-1}}{q(m+1)^q}$, $\eta=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $S(.)$ is Specht's ratio.

Proof. Applying (2.2) and the rest of this proof is similar to that of Theorem 3.3. □

Remark 4. Let $\alpha=1$, $\mathbb{T}=\mathbb{Z}$, $a=1$, $b=n+1$, $w\equiv 1$, $f(k)=x_k>0$ and $g(k)=y_k>0$ for $k\in\{1,2,\ldots,n\}$. Then (3.20) reduces to

$$\sum_{k=1}^n K^\eta\left(\frac{x_k^p}{y_k^q}\right)x_ky_k\leq\Lambda\sum_{k=1}^n\left(x_k^p+y_k^p\right)+\Omega\sum_{k=1}^n\left(x_k^q+y_k^q\right) \quad (3.27)$$

and (3.26) reduces to

$$\sum_{k=1}^n S\left(\left(\frac{x_k^p}{y_k^q}\right)^\eta\right)x_ky_k\leq\Lambda\sum_{k=1}^n\left(x_k^p+y_k^p\right)+\Omega\sum_{k=1}^n\left(x_k^q+y_k^q\right). \quad (3.28)$$

4 Conclusion and future work

By using Hölder's reverse fractional integral inequality, weighted Radon's reverse integral inequality [11] was established in continuous form. Inspired by this work, we have presented an extended dynamic reverse Radon's inequality given in Theorem 3.1 on time scales in a more general form. A fractional integral Clarkson-type inequality [11] was also established in continuous form. We have presented Clarkson-type dynamic inequality in the extended form given in Theorem 3.2 on time scales. Motivated by the works of [6, 19], some dynamic inequalities in hybrid and comprehensive forms are established in this research article by using Kantorovich's ratio and Specht's ratio, respectively.

In our future research work, we will continue to find further dynamic inequalities and their reverse versions and applications in extended and generalized forms. It will be interesting to explore dynamic inequalities by using fractional calculus on time scales.

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POROSITY IN THE CONTEXT OF HYPERGROUPS

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Abstract. In this paper we show that the set of all elements $g \in L^p(\mathcal{H})$ for which $(|g|*|g|)(x) < \infty$ for a center element $x \in B$, is σ-c-lower porous, where $p > 2$, $\mathcal{H}$ is a non-compact unimodular hypergroup and B is some special symmetric compact neighborhood of the identity element. As an application, we give some new equivalent condition for the finiteness of a discrete Hermitian hypergroup. Moreover, we give some sufficient conditions for the set of all pairs (f, g) in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$ for which for a center element $x \in B$, $(|f| * |g|)(x) < \infty$, is a σ-c-lower porous, where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} < 1$. Also, we show that the complement of this set is spaceable in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$.

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1 Introduction and preliminaries

σ-porous sets, as a category of small sets, were introduced by Dolženko in 1967 in order to study of singular points of holomorphic functions. In recent decades, the relationship between this concept and many topics has been discovered, including hypercyclicity, L^p-conjecture, spaceability etc; see [4, 8, 9, 10]. We refer the reader to survey papers [23, 24] for more information on porosity on the real line, metric spaces and normed spaces. If $c \in (0, 1)$ and X is a metric space, a subset $M \subseteq X$ is called *c-lower porous* if for each $x \in M$,

$$\liminf_{r>0} \frac{\gamma(x, r, M)}{r} \geq \frac{c}{2}, \tag{1.1}$$

where

$$\gamma(x, r, M) := \sup\{s \geq 0 : \text{ for some } z \in X,\ B(z; s) \subseteq B(z; r) \setminus M\}, \tag{1.2}$$

and $B(x; r)$ is the open ball with center x and radius r. We denote also $B(x; 0) := \varnothing$. We say that M is *σ-c-lower porous* if it can be represented as a countable union of c-lower porous subsets of X. In the following result which was proved in [25] one can find some equivalent condition for σ-lower porosity of subsets of normed spaces; see also [24, Proposition 2.2].

Theorem 1.1. *Let X be a normed linear space and $c \in (0, 1]$. Then, a subset $E \subseteq X$ is σ-c-lower porous if and only if $E = \bigcup_{m=1}^{\infty} P_m$, where for each $m \in \mathbb{N}$, $x \in X$ and $r > 0$ there exists some $y \in X$ such that $B(y; cr) \subseteq B(x; r) \setminus P_m$.*

In 2010, due to study of some classes of convolution Banach function algebras, S. Głąb and F. Strobin started to investigate the σ-lower porous subsets of Lebesgue spaces in the context of locally compact groups. They proved that there is a σ-porous set such that for each (f, g) in its complement, $f * g$ does not exist on a set of positive measure [8], and also proved the following statement.

Theorem 1.2. *If G is a non-compact unimodular group and $p > 2$, then for each compact subset $F \subset G$ there exists some $c > 0$ such that the set*

$$E := \{f \in L^p(G) : \exists\, x \in F \text{ such that } |f| * |f|(x) < \infty\}$$

is a σ-c-lower porous subset of $L^p(G)$.

After that, I. Akbarbaglu and S. Maghsoudi in [1] draw a similar picture for some Orlicz spaces which are generalization of Lebesgue ones. They and J.B. Seoane-Sepúlveda in [2] studied also this topic for Lebesgue spaces on discrete semigroups.

On the other hand, locally compact hypergroups which are important generalizations of locally compact groups were introduced in [6, 11, 19]; see [12, 13, 14, 15, 16, 17] as recent works on locally compact hypergroups. There exists some convolutions among the regular measures of a locally compact hypergroup, while in contrast to the group case, the convolution of two Dirac measures of a hypergroup is not necessarily a Dirac measure. In Section 2 of this paper we initiate investigations of porosity in the context of hypergroups and as a main result we give an extension of Theorem 1.2 to hypergroups. Indeed, we prove that whenever $\mathcal{H}$ is a non-compact unimodular hypergroup and $p > 2$, then for each symmetric compact neighborhood B of the identity element of $\mathcal{H}$ if there is a constant $L > 0$ such that for each $x_1, \ldots, x_n \in \mathcal{H}$ we have

$$\frac{\sum_{k=1}^{n} \lambda(x_k * B * B)}{\sum_{k=1}^{n} \lambda(B * \check{x_k})} \leq L, \tag{1.3}$$

then the set

$$E_B := \{g \in L^p(\mathcal{H}) : \text{ for some } x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|g| * |g|)(x) < \infty\}$$

is σ-c-lower porous in $L^p(\mathcal{H})$, where $\mathrm{Ma}(\mathcal{H})$ is the center of the hypergroup and the Lebesgue space is with respect to a left invariant measure on $\mathcal{H}$. This fact directly implies that for each infinite discrete Hermitian hypergroup $\mathcal{H}$ and $p > 2$, $L^2(\mathcal{H})$ is a σ-c-lower porous subset of $L^p(\mathcal{H})$.

In Section 3 among other results we give some equivalent condition for a hypergroup to be compact. In fact, in Corollary 3.2 we prove that if $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$, $\mathcal{H}$ is a unimodular hypergroup and B is a symmetric compact neighborhood of the identity e in $\mathcal{H}$ with L-property, then $\mathcal{H}$ is non-compact if and only if the set M_B is a σ-c-lower porous subset of $L^p(\mathcal{H}) \times L^q(\mathcal{H})$ for some $c \in (0, 1)$, and this holds if and only if the set $\big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B$ is spaceable in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$, where

$$M_B := \{(f, g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H}) : \exists\, x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|f| * |g|)(x) < \infty\}.$$

We will show that in several classes of hypergroups one can find such neighborhoods B with L-property.

Next, we recall some basic information regarding hypergroups.

1.1 Hypergroups

We denote the space of all complex Radon measures on a locally compact Hausdorff space $\mathcal{X}$ by $\mathcal{M}(\mathcal{X})$. Also, the set of all non-negative measures in $\mathcal{M}(\mathcal{X})$ by $\mathcal{M}^+(\mathcal{X})$. The support of each $\mu \in \mathcal{M}(\mathcal{X})$ is denoted by $\mathrm{supp}(\mu)$. We denote a point-mass measure at $x \in \mathcal{X}$ by δ_x.

Definition 1. A locally compact Hausdorff space $\mathcal{H}$ equipped with a (*convolution*) product $*$ on $\mathcal{M}(\mathcal{H})$ and an *involution* map $x \mapsto \check{x}$ is called a *locally compact hypergroup* (or simply a *hypergroup*) if the following conditions hold.

1. $(\mathcal{M}(\mathcal{H}), +, *)$ is a Banach algebra.

2. For each $x, y \in \mathcal{H}$, $\delta_x * \delta_y$ is a compact supported probability measure.

3. The mappings $(x, y) \mapsto \delta_x * \delta_y$ from $\mathcal{H} \times \mathcal{H}$ into $\mathcal{M}^+(\mathcal{H})$ is continuous, where $\mathcal{M}^+(\mathcal{H})$ is equipped with the cone topology.

4. The mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $\mathcal{H} \times \mathcal{H}$ into the family of all nonempty compact subsets of $\mathcal{H}$, $\mathbf{C}(\mathcal{H})$, is continuous, where $\mathbf{C}(\mathcal{H})$ is equipped with the Michael topology.

5. The involution map is an involutive homeomorphism from $\mathcal{H}$ onto $\mathcal{H}$ such that for each $x, y \in \mathcal{H}$, $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$.

6. There exists an element $e \in \mathcal{H}$ (called *identity*) such that for each $x \in \mathcal{H}$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$. Moreover, for each $x, y \in \mathcal{H}$, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $y = \check{x}$.

Any locally compact group, equipped with the usual convolution and the inverse mapping as involution, is a hypergroup. Contrary to the group case, for each x, y in a hypergroup $\mathcal{H}$, the convolution $\delta_x * \delta_y$ of two Dirac measures is not necessarily a Dirac measure. We refer to the book [5] for more information and examples. $\mathcal{H}$ is called *commutative* if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in \mathcal{H}$.

A nonzero nonnegative Radon measure λ on a hypergroup $\mathcal{H}$ is called *left-invariant* if for each $x \in \mathcal{H}$, $\delta_x * \lambda$ is defined and $\delta_x * \lambda = \lambda$. For each measurable set $E \subseteq \mathcal{H}$ we have

$$\|\chi_E\, \Delta^{-1}\|_1 = \lambda(\check{E}), \tag{1.4}$$

where Δ is the modular function. By [11, Theorem 4.3C], any hypergroup $\mathcal{H}$ admits a left subinvariant measure λ with $\operatorname{supp}(\lambda) = \mathcal{H}$, while so far it has been remained as a conjecture that any hypergroup has a left-invariant measure.

In sequel, $\mathcal{H}$ is a hypergroup and λ is a left-invariant measure on $\mathcal{H}$. Also, for each $p \geq 1$, $L^p(\mathcal{H})$ is the Lebesgue space with the measure λ.

For each complex-valued Borel functions f and g on $\mathcal{H}$ and all $x, y \in \mathcal{H}$ we denote

$$f(x * y) := \int_{\mathcal{H}} f d(\delta_x * \delta_y) \qquad \text{and} \qquad (g * f)(x) := \int_{\mathcal{H}} g(y)\, f(\check{y} * x)\, d\lambda(y).$$

The convolution of two subsets $A, B \subseteq \mathcal{H}$ is defined by

$$A * B := \bigcup_{x \in A,\, y \in B} \operatorname{supp}(\delta_x * \delta_y).$$

The *center* of a hypergroup $\mathcal{H}$ is defined by

$$\operatorname{Ma}(\mathcal{H}) := \{x \in \mathcal{H} :\ \delta_x * \delta_{\check{x}} = \delta_{\check{x}} * \delta_x = \delta_e\}.$$

$\operatorname{Ma}(\mathcal{H})$ is the maximal subgroup of $\mathcal{H}$. Let $x \in \operatorname{Ma}(\mathcal{H})$ and $y \in \mathcal{H}$. Then, by [11, Section 10.4], $\delta_x * \delta_y$ is a Dirac measure; see also [18]. In this case, we denote the unique element in $\operatorname{supp}(\delta_x * \delta_y)$ by xy. Similarly, $\delta_y * \delta_x$ is the Dirac measure δ_{yx}. Note that xy and yx do not belong to the center in general. For each Borel measurable function $f : \mathcal{H} \to \mathbb{C}$, $x \in \operatorname{Ma}(\mathcal{H})$ and $y \in \mathcal{H}$ we have

$$\begin{aligned} |f(x * y)| &= \left| \int_{\mathcal{H}} f(t)\, d(\delta_x * \delta_y)(t) \right| \\ &= \left| \int_{\mathcal{H}} f(t)\, d\delta_{xy}(t) \right| \\ &= |f(xy)| = |f|(xy) \\ &= |f|(x * y). \end{aligned}$$

Example 1. Let G be a locally compact group such that the quotient space $G/Z(G)$ is compact, where $Z(G) := \{z \in G : \text{ for each } x \in G,\ zx = xz\}$. Let $I := \mathrm{Inn}(G)$ be the set of all inner automorphisms of G. Then, the orbit space $G^I := \{x^I : x \in G\}$ is a hypergroup where $x^I := \{g^{-1}xg : g \in G\}$ [11, Theorem 8.3A], and by [18] we have $\mathrm{Ma}(G^I) = \{z^I : z \in Z(G)\}$.

Remark 1. For each $A, B \subseteq \mathcal{H}$, $a \in \mathrm{Ma}(\mathcal{H})$ and $b \in \mathcal{H}$ we have:

1. $\lambda(A * \{\check{a}\}) = (\lambda * \delta_a)(A)$ and $(\lambda * \delta_b)(A) = \Delta(\check{b})\lambda(A)$.
2. $(A \cap B) * \{a\} = (A * \{a\}) \cap (B * \{a\})$ and $(A \cup B) * \{b\} = (A * \{b\}) \cup (B * \{b\})$.
3. for each $a \in \mathrm{Ma}(\mathcal{H})$ and $A, B \subseteq K$, we have
$$(A \cap B) * \{a\} = (A * \{a\}) \cap (B * \{a\}).$$

2 Porosity on $L^p(\mathcal{H})$

In this section, we study some porous subsets of Lebesgue spaces on hypergroups which helps us to give new equivalent conditions for a discrete Hermitian hypergroup to be infinite. The following lemma which was shortly proved in the proof of [21, Theorem 2.3] plays a key role in the main results of this paper.

Lemma 2.1. *Let $\mathcal{H}$ be a non-compact hypergroup and B be a compact symmetric neighborhood of the identity e in $\mathcal{H}$. Then, there exists a sequence $(a_n)_n$ in $\mathcal{H}$ with $\Delta(a_n) \leq 1$ for all $n \in \mathbb{N}$ such that for each distinct $m, n \in \mathbb{N}$,*

$$(\{a_n\} * B * B) \bigcap (\{a_m\} * B * B) = \varnothing \tag{2.1}$$

and

$$(B * \{\check{a_n}\}) \bigcap (B * \{\check{a_m}\}) = \varnothing. \tag{2.2}$$

Now, we give the main result of this paper which improves Theorem 1.2 proved by S. Głąb and F. Strobin. The method of the proof is similar to [8, Theprem 2] but its details and basics are different.

Theorem 2.1. *Let $\mathcal{H}$ be a non-compact unimodular hypergroup and $p > 2$. Let B be a symmetric compact neighborhood of the identity e in $\mathcal{H}$ and there is a constant $L > 0$ such that for each $x_1, \dots, x_n \in \mathcal{H}$,*

$$\frac{\sum_{k=1}^{n} \lambda(x_k * B * B)}{\sum_{k=1}^{n} \lambda(B * \check{x_k})} \leq L. \tag{2.3}$$

Then, there is a constant $c > 0$ such that the set

$$E_B := \{g \in L^p(\mathcal{H}) : \textit{for some } x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|g| * |g|)(x) < \infty\}$$

is σ-c-lower porous in $L^p(\mathcal{H})$.

Proof. Trivially we have $E_B = \bigcup_{m=1}^{\infty} P_m$, where

$$P_m := \{g \in L^p(\mathcal{H}) : \text{for some } x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|g| * |g|)(x) < m\}.$$

We show that the collection $\{P_m\}_m$ satisfies the equivalent condition in Theorem 1.1.

Step 1. By Lemma 2.1 one can find $a_1, a_2, \dots$ in $\mathcal{H}$ satisfying

$$(\{a_n\} * B * B) \bigcap (\{a_m\} * B * B) = \varnothing \tag{2.4}$$

and

$$(B * \{\check{a_n}\}) \bigcap (B * \{\check{a_m}\}) = \varnothing \tag{2.5}$$

for all distinct $m, n \in \mathbb{N}$.

Step 2. Fix a number k with $0 < k < \frac{1}{1+L^{\frac{1}{p}}}$. For each $0 < x < 1$ we define

$$F(x) := 2\left(\frac{x}{k\,(1-x)}\right)^p. \tag{2.6}$$

Then, F is a continuous strictly increasing function on the interval $(0,1)$, $\lim_{x\to 0^+} F(x) = 0$ and $\lim_{x\to 1^-} F(x) = \infty$. This implies that there exists a number $0 < \gamma < 1$ such that $F(\gamma) = 1$, and so for each fixed number $0 < c < \gamma$, $0 < F(c) < 1$. Define

$$G(x) := 1 - 2\left(\frac{c}{kx}\right)^p. \tag{2.7}$$

By the continuity of G on $(0,1)$, since $G(1-c) = 1 - F(c) > 0$, there are $0 < \eta < 1-c$ and $0 < \alpha < 1$ such that

$$1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p > 0.$$

Step 3. Let $r > 0$ and $f \in L^p(\mathcal{H})$. Then, by disjointness properties (2.4) and (2.5) we have

$$\sum_{n=1}^{\infty} \|\chi_{B*\check{a_n}}\, f\|_p^p \le \|f\|_p^p < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|\chi_{a_n*B*B}\, g\|_q^q \le \|g\|_q^q < \infty.$$

Hence, there is some $n_0 \in \mathbb{N}$ such that

$$\|\chi_I\, f\|_p^p = \sum_{n=n_0}^{\infty} \|\chi_{B*\check{a_n}}\, f\|_p^p < [\frac{1}{2}\,(1-c-\eta)r]^p \tag{2.8}$$

and

$$\|\chi_J\, f\|_p^p = \sum_{n=n_0}^{\infty} \|\chi_{a_n*B*B}\, f\|_p^p < [\frac{1}{2}\,(1-c-\eta)r]^p, \tag{2.9}$$

where

$$I := \bigcup_{n=n_0}^{\infty} B * \check{a_n} \quad \text{and} \quad J := \bigcup_{n=n_0}^{\infty} a_n * B * B.$$

Let $m \in \mathbb{N}$. Choose a natural number $n_1 > n_0$ such that

$$\alpha^2 \eta^2\, r^2\, k^2 (n_1 - n_0 + 1)^{1-\frac{2}{p}}\, \lambda(B)^{1-\frac{2}{p}} \left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right) > m, \tag{2.10}$$

Set

$$A := \bigcup_{n=n_0}^{n_1} B * \check{a_n} \quad \text{and} \quad D := \bigcup_{n=n_0}^{n_1} a_n * B * B.$$

Then, by (2.8) and (2.9) we have

$$\|\chi_A\, f\|_p \le \frac{1}{2}\,(1-c-\eta)r \quad \text{and} \quad \|\chi_D\, f\|_p \le \frac{1}{2}\,(1-c-\eta)r. \tag{2.11}$$

We have

$$M\,\lambda(A)^{\frac{1}{p}} + M\,\lambda(D)^{\frac{1}{p}} \le \eta r, \tag{2.12}$$

where

$$M := r\eta k\,\lambda(A)^{\frac{-1}{p}}. \tag{2.13}$$

Define $\tilde{f} := M\chi_{A\cup D} + f\chi_{(A\cup D)^c}$. Then, we have

$$\begin{aligned}
\|f-\tilde{f}\|_p &= \left\|\chi_{A\cup D}\,(f-\tilde{f}) + \chi_{(A\cup D)^c}\,(f-\tilde{f})\right\|_p \\
&= \left\|\chi_{A\cup D}\,(f-\tilde{f})\right\|_p \\
&= \left\|(\chi_A + \chi_{D-A})\,(f-\tilde{f})\right\|_p \\
&\le \|\chi_A\,f\|_p + \|\chi_A\,\tilde{f}\|_p + \|\chi_{D-A}\,f\|_p + \|\chi_{D-A}\,\tilde{f}\|_p \\
&= \|\chi_A\,f\|_p + \|\chi_A\,M\|_p + \|\chi_{D-A}\,f\|_p + \|\chi_{D-A}\,M\|_p \\
&\le \|\chi_A\,f\|_p + M\,\lambda(A)^{\frac{1}{p}} + \|\chi_D\,f\|_p + M\,\lambda(D)^{\frac{1}{p}} \\
&\le (1-c-\eta)r + \eta r = (1-c)\,r.
\end{aligned}$$

This implies that $B(\tilde{f}, cr) \subseteq B(f; r)$.

Step 4. In this step we show that $B(\tilde{f}; cr) \bigcap P_m = \varnothing$. Let $g \in B(\tilde{f}; cr)$ and $x \in B \cap \mathrm{Ma}(\mathcal{H})$. Set

$$A_1 := \{x \in A : |g(x)| \le \alpha M\} \quad \text{and} \quad D_1 := \{x \in D : |g(x)| \le \alpha M\}. \tag{2.14}$$

Then,

$$(1-\alpha)M\lambda(A_1)^{\frac{1}{p}} \le \|\chi_{A_1}\,(|\tilde{f}| - |g|)\|_p \le \|\chi_{A_1}\,(\tilde{f} - g)\|_p \le \|\tilde{f} - g\|_p < cr. \tag{2.15}$$

By a similar argument it follows that relation (2.15) holds for D_1 too. So, we can conclude that

$$\max\{\lambda(A_1), \lambda(D_1)\} \le \left(\frac{cr}{(1-\alpha)M}\right)^p = \left(\frac{c}{\eta(1-\alpha)k}\right)^p \lambda(A). \tag{2.16}$$

Put $A_2 := A \setminus A_1$ and $D_2 := D \setminus D_1$. Also, we put $F := A_2 \cap (\{x\} * \check{D}_2)$.

Then, $F \subseteq A_2$ and $\check{F} * \{x\} \subseteq D_2$. This implies that

$$\begin{aligned}
(|g| * |g|)(x) &= \int_{\mathcal{H}} |g(t)|\; |g|(\check{t} * x)\, d\lambda(t) \\
&\ge \int_F |g(t)|\; |g(\check{t} * x)|\, d\lambda(t) \\
&\ge \alpha^2 M^2 \lambda(F).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lambda(F) &= \lambda(\check{A}_2 \cap (D_2 * \{\check{x}\})) \\
&= \lambda(\check{A}_2) - \lambda(\check{A}_2 \setminus (D_2 * \{\check{x}\})) \\
&\ge \lambda(A) - \lambda(A_1) - \lambda(D_1) \\
&\ge \lambda(A)\left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right).
\end{aligned}$$

Therefore, by (2.10) we have

$$\begin{aligned}
(|g| * |g|)(x) &\geq \alpha^2 M^2 \lambda(A) \left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right) \\
&= \alpha^2 \eta^2 r^2 k^2 \lambda(A)^{1-\frac{2}{p}} \left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right) \\
&\geq \alpha^2 \eta^2 r^2 k^2 (n_1 - n_0 + 1)^{1-\frac{2}{p}} \lambda(B)^{1-\frac{2}{p}} \left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right) \\
&> m,
\end{aligned}$$

since

$$\begin{aligned}
\lambda(A) &= \sum_{n=n_0}^{n_1} \lambda(B * \{\check{a_n}\}) \\
&= \sum_{n=n_0}^{n_1} \lambda(\{a_n\} * B) \\
&\geq \sum_{n=n_0}^{n_1} \lambda(B) \\
&= (n_1 - n_0 + 1)\, \lambda(B).
\end{aligned}$$

thanks to [11, Lemma 3.3C] and the assumption that $\mathcal{H}$ is unimodular. □

Note that Theorem 2.1 is a generalization of Theorem 1.2 because if $\mathcal{H}$ is a locally compact group, then $\mathrm{Ma}(\mathcal{H}) = \mathcal{H}$ and condition (2.3) holds with $L = \frac{\lambda(B^2)}{\lambda(B)}$.

For each function $f : \mathcal{H} \to \mathbb{C}$ we define $\check{f}(x) := f(\check{x})$ for all $x \in \mathcal{H}$. We mention that in each discrete commutative hypergroup, $B := \{e\}$ is a compact symmetric neighborhood of the identity element, and in this case $B \cap \mathrm{Ma}(\mathcal{H}) = \{e\}$. Also, condition (2.3) trivially holds (with $L = 1$) for this neighborhood in the case in which the discrete hypergroup $\mathcal{H}$ is unimodular too. So, we can conclude the following result.

Corollary 2.1. *Let $\mathcal{H}$ be an infinite discrete commutative hypergroup and $p > 2$. Then, there is a constant $c > 0$ such that the set*

$$E := \{f \in L^p(\mathcal{H}) : f\check{f} \in L^1(\mathcal{H})\}$$

is σ-c-lower porous.

Recall that a hypergroup $\mathcal{H}$ is called *Hermitian* if $\check{x} = x$ for all $x \in \mathcal{H}$. Clearly, any Hermitian hypergroup is commutative.

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be equipped with the discrete topology and let p be a fixed prime number. For any $k \in \mathbb{N}_0$ and distinct $m, n \in \mathbb{N}$ define $\delta_k * \delta_0 = \delta_0 * \delta_k := \delta_k$, $\delta_m * \delta_n := \delta_{\max\{m,n\}}$ and

$$\delta_n * \delta_n := \frac{1}{p^{n-1}(p-1)}\delta_0 + \sum_{k=1}^{n-1} p^{k-1}\delta_k + \frac{p-2}{p-1}\delta_n.$$

Then, $\mathbb{N}_0$ is a Hermitian hypergroup with the left invariant measure m defined by

$$m(\{k\}) := \begin{cases} 1, & \text{if } k = 0, \\ (p-1)p^{k-1}, & \text{if } k \geq 1. \end{cases}$$

This class of discrete Hermitian hypergroups was introduced by Dunkl and Ramirez in [7]. In the final result of this paper, we give some porosity property of discrete Hermitian hypergroups. Just note that thanks to [11, Theorem 7.1A], if $\mathcal{H}$ is a discrete hypergroup, the measure λ given by

$$\lambda(\{x\}) := \frac{1}{(\delta_{\check{x}} * \delta_x)(\{e\})} \qquad (x \in \mathcal{H}) \tag{2.17}$$

is a left-invariant measure on $\mathcal{H}$. So, since the convolution of any two Dirac measures is a probability measure, we have

$$\inf\{\lambda(A) : \lambda(A) > 0\} \geq 1. \tag{2.18}$$

Hence, by [22, Theorem 1], for each $p > 2$, $L^2(\mathcal{H}, \lambda) \subseteq L^p(\mathcal{H}, \lambda)$. Now, Corollary 2.1 implies the next fact.

Corollary 2.2. *Let $\mathcal{H}$ be a discrete Hermitian hypergroup. Then, the following conditions are equivalent.*

1. *$\mathcal{H}$ is infinite.*
2. *There exists some $p > 2$ and a constant $c > 0$ such that the set $L^2(\mathcal{H})$ is a σ-c-lower porous subset of $L^p(\mathcal{H})$.*
3. *For each $p > 2$ there exists a constant $c > 0$ such that the set $L^2(\mathcal{H})$ is a σ-c-lower porous subset of $L^p(\mathcal{H})$.*

3 Porosity and spaceability on hypergroups

In this section we intend to give some equivalent conditions by porosity and spaceability for a hypergroup to be compact.

Remark 2. We say that a neighborhood B has *L-property* for some constant $L > 0$, if there exists a sequence $(a_n)_n$ satisfying the conditions of Lemma 2.1 such that

$$\sup\{\lambda(\{a_n\} * B * B) : n \in \mathbb{N}\} \leq L. \tag{3.1}$$

We will use this condition in the assumptions of some results in this paper. Next, we show that any locally compact group has this condition and also we present some classes of hypergroups which are not groups, but have L-property.

Example 2. 1. Let a hypergroup $\mathcal{H}$ have a non-compact open center. Then, there exists a compact symmetric neighborhood B of e such that $B \subset \mathrm{Ma}(\mathcal{H})$. In this case, B has L-property for some $L > 0$, because by the proof of Lemma 2.1 one can choose a sequence $(a_n)_n \subseteq \mathrm{Ma}(\mathcal{H})$ satisfying condition (3.1). In particular, if $\mathcal{H}$ is a non-compact group, then we have $\mathcal{H} = \mathrm{Ma}(\mathcal{H})$ and so any compact symmetric neighborhood of e in $\mathcal{H}$ has L-property for some $L > 0$.

2. Let G is a non-compact group with a left Haar measure $\lambda = dx$ and let H be a compact non-normal subgroup of G with normalized Haar measure dh. Let $\mathcal{H} = H\backslash G/H := \{HxH : x \in G\}$ be the double coset hypergroup with convolution $\delta_{\dot{x}} * \delta_{\dot{y}} = \int_H \delta(xhy)^{\cdot} dh$ and left Haar measure $\dot{\lambda} = \int_G \delta_{\dot{x}} dx$, where $\dot{x} := HxH$ is the image of x in $H\backslash G/H$. Let B be a compact symmetric neighborhood of the identity element HeH in $\mathcal{H}$. Then, there exists a compact subset $E \subseteq G$ such that $B = \dot{E}$ and $\dot{x} * \dot{E} = (HxHEH)^{\cdot}$. Now, thanks to Lemma 2.1, this implies that if H is connected, compact and open, or if H is finite, then there is a constant $L > 0$ such that B has L-property.

In this paper, we consider the maximum norm on the product of two Banach spaces.

Theorem 3.1. *Let $\mathcal{H}$ be a non-compact hypergroup and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$. Let B be a symmetric compact neighborhood of e in $\mathcal{H}$ with L-property. For each $m \in \mathbb{N}$, put*

$$M_{B,m} := \{(f,g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H}) : \exists x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|f| * |g|)(x) < m\}.$$

Then, there exists some $c \in (0,1)$ such that for each $(f,g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H})$ and $r > 0$ there exists an element $(\tilde{f}, \tilde{g}) \in L^p(\mathcal{H}) \times L^q(\mathcal{H})$ such that

$$B((\tilde{f}, \tilde{g}); cr) \subseteq B((f,g); r) \setminus M_{B,m}.$$

Proof. Put $S := \sup_{x \in B} \Delta(x)$. For each $0 < x < 1$ we define

$$F(x) := \left(\frac{x}{1-x}\right)^p + \left(\frac{x}{1-x}\right)^q \frac{SL}{\lambda(B)}.$$

Then, F is a continuous strictly increasing function on the interval $(0,1)$, $\lim_{x \to 0^+} F(x) = 0$ and $\lim_{x \to 1^-} F(x) = \infty$. This implies that there exists a number $0 < \gamma < 1$ such that $F(\gamma) = 1$, and so for each fixed number $0 < c < \gamma$, $0 < F(c) < 1$. Define

$$G(x) := 1 - \left(\frac{c}{x}\right)^p - \left(\frac{c}{x}\right)^q \frac{SL}{\lambda(B)}.$$

Since G is continuous on $(0,1)$, there are $0 < \eta < 1-c$ and $0 < \alpha < 1$ such that $P := G((1-\alpha)\,\eta) > 0$.

Assume that $m \in \mathbb{N}$. Let $(f,g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H})$ and $r > 0$. Assume that $(a_n)_n$ is the sequence given in Remark 2 corresponding to the symmetric compact neighborhood B. Then, by disjointness properties (2.4) and (2.5) there are some $n_0 \in \mathbb{N}$ and a natural number $n_1 > n_0$ such that

$$\alpha^2 \eta^2 r^2 \left(\frac{L}{\lambda(B)}\right)^{\frac{-1}{q}} S^{\frac{1}{p}-1}\, P\, \lambda(B)^{1-\frac{1}{p}-\frac{1}{q}} (n_1 - n_0 + 1)^{1-\frac{1}{p}-\frac{1}{q}} > m, \tag{3.2}$$

$$\sum_{n=n_0}^{n_1} \|\chi_{B*\{\check{a_n}\}}\, f\|_p^p < [(1-c-\eta)r]^p \tag{3.3}$$

and

$$\sum_{n=n_0}^{n_1} \|\chi_{\{a_n\}*B*B}\, g\|_q^q < [(1-c-\eta)r]^p. \tag{3.4}$$

Set

$$A := \bigcup_{n=n_0}^{n_1} \{a_n\} * B \quad \text{and} \quad D := \bigcup_{n=n_0}^{n_1} \{a_n\} * B * B.$$

Then, by (3.3) and (3.4) we have

$$\|\chi_{\check{A}}\, f\|_p \le (1-c-\eta)r \quad \text{and} \quad \|\chi_D\, g\|_q \le (1-c-\eta)r. \tag{3.5}$$

Also, by [11, Lemma 3.3C] and property (3.1),

$$\lambda(B)\,(n_1 - n_0 + 1) \le \lambda(A) \le L\,(n_1 - n_0 + 1) \tag{3.6}$$

and

$$\lambda(B * B)\,(n_1 - n_0 + 1) \le \lambda(D) \le L\,(n_1 - n_0 + 1). \tag{3.7}$$

Put

$$M_1 := \eta\, r \lambda(A)^{\frac{-1}{p}} \quad \text{and} \quad M_2 := \eta\, r \lambda(D)^{\frac{-1}{q}}. \tag{3.8}$$

We define $\tilde{f} := M_1 \Delta^{\frac{-1}{p}} \chi_{\check{A}} + f \chi_{(\check{A})^c}$ and $\tilde{g} := M_2\, \chi_D + g \chi_{D^c}$. Then,

$$\begin{aligned}
\|\tilde{f} - f\|_p &= \left\| \chi_{\check{A}}\, M_1\, \Delta^{\frac{-1}{p}} - \chi_{\check{A}}\, f \right\|_p \\
&\le \left\| \chi_{\check{A}}\, M_1\, \Delta^{\frac{-1}{p}} \right\|_p + \|\chi_{\check{A}}\, f\|_p \\
&\le M_1\, \|\chi_{\check{A}} \Delta^{-1}\|_1^{\frac{1}{p}} + (1 - c - \eta) r \\
&= M_1\, \lambda(A)^{\frac{1}{p}} + (1 - c - \eta) r \\
&= \eta\, r + (1 - c - \eta) r = (1 - c) r
\end{aligned}$$

thanks to (3.5), (3.8) and (1.4). Similarly, $\|\tilde{g} - g\|_q \le (1-c)r$. Therefore, $B((\tilde{f}, \tilde{g}); cr) \subseteq B((f, g); r)$. Now, let $(h, s) \in B((\tilde{f}, \tilde{g}); c\, r)$. Setting

$$A_1 := \{x \in \check{A} :\, |h(x)| \le \alpha\, \tilde{f}(x)\},$$

we have

$$\begin{aligned}
c\, r &> \|h - \tilde{f}\|_p \\
&\ge \|\, |h| - |\tilde{f}|\, \|_p \\
&\ge \|\chi_{A_1} \big(|h| - |\tilde{f}|\big)\|_p \\
&\ge \big(1 - \alpha\big)\ \|\chi_{A_1}\, \tilde{f}\|_p \\
&= \big(1 - \alpha\big)\, M_1\ \|\chi_{A_1}\, \Delta^{-1}\|_1^{\frac{1}{p}} \\
&= \big(1 - \alpha\big)\, M_1\, \lambda(\check{A}_1)^{\frac{1}{p}},
\end{aligned}$$

and so,

$$\lambda(\check{A}_1) < \left(\frac{c\, r}{(1 - \alpha)\, M_1}\right)^p = \left(\frac{c}{(1 - \alpha)\, \eta}\right)^p \lambda(A). \tag{3.9}$$

Similarly, setting

$$D_1 := \{x \in D :\, |s(x)| \le \alpha\, \tilde{g}(x)\}.$$

we have

$$\begin{aligned}
c\, r &> \|s - \tilde{g}\|_q \\
&\ge \|\, |s| - |\tilde{g}|\, \|_q \\
&\ge \|\chi_{D_1} \big(|s| - |\tilde{g}|\big)\|_q \\
&\ge \big(1 - \alpha\big)\ \|\chi_{D_1}\, \tilde{g}\|_q \\
&= \big(1 - \alpha\big)\, M_2\ \|\chi_{D_1}\|_q \\
&= \big(1 - \alpha\big)\, M_2\, \lambda(D_1)^{\frac{1}{q}},
\end{aligned}$$

and therefore by inequalities (3.6) and (3.7),

$$\lambda(D_1) < \left(\frac{c\, r}{(1 - \alpha)\, M_2}\right)^q = \left(\frac{c}{(1 - \alpha)\, \eta}\right)^q \lambda(D) \le \left(\frac{c}{(1 - \alpha)\, \eta}\right)^q \frac{L\, \lambda(A)}{\lambda(B)}. \tag{3.10}$$

Let $x \in B \cap \mathrm{Ma}(\mathcal{H})$. Put $H := \{x\} * \big[(\{\check{x}\} * A_2) \cap \check{D}_2\big]$, where $D_2 := D \setminus D_1$ and $A_2 := \check{A} \setminus A_1$. Then, since x is a center element,

$$H \subseteq \{x\} * \{\check{x}\} * A_2 = A_2 \subseteq \check{A},$$

and $\check{H} * \{x\} \subseteq D_2 \subseteq D$. In fact, for each $t \in H$, we have $\check{t}x \in D_2$, and since $\check{H} \subseteq A$, for each $t \in \check{H}$, there is some $j \in \{n_0, \ldots, n_1\}$ such that $t \in \{a_j\} * B$. This means that there exists $y \in B$ such that $t \in \{a_j\} * \{y\}$. Now, by [11, Theorem 5.3C], we have $\Delta(t) = \Delta(a_j)\Delta(y) \leq S$ because $\Delta(a_j) \leq 1$ and $y \in B$. For each $t \in \mathcal{H}$ we have $|\phi|(t * x) = |\phi(t * x)|$ for all complex-valued measurable function ϕ on $\mathcal{H}$. This implies that

$$\begin{aligned}
(|h| * |s|)(x) &= \int_{\mathcal{H}} |h(t)|\, |s|(\check{t} * x)\, d\lambda(t) \\
&\geq \int_{H} |h(t)|\, |s(\check{t}x)|\, d\lambda(t) \\
&\geq \alpha^2 \int_{H} \tilde{f}(t)\, \tilde{g}(\check{t}x)\, d\lambda(t) \\
&= \alpha^2 M_1 M_2 \int_{H} \Delta(t)^{\frac{-1}{p}}\, d\lambda(t) \\
&= \alpha^2 M_1 M_2 \int_{\check{H}} \Delta(t)^{\frac{1}{p}-1}\, d\lambda(t) \\
&\geq \alpha^2 M_1 M_2\, S^{\frac{1}{p}-1} \int_{\check{H}} d\lambda(t) \\
&= \alpha^2 M_1 M_2\, S^{\frac{1}{p}-1}\, \lambda(\check{H}),
\end{aligned}$$

thanks to [11, Theorem 5.3B]. On the other hand,

$$\begin{aligned}
\check{H} &= [(\check{A}_2 * \{x\}) \bigcap D_2] * \{\check{x}\} \\
&= \big((\check{A}_2 * \{x\} * \{\check{x}\})\big) \bigcap \big(D_2 * \{\check{x}\}\big) \\
&= \check{A}_2 \bigcap \big(D_2 * \{\check{x}\}\big) \\
&= \check{A}_2 - [\check{A}_2 - \big(D_2 * \{\check{x}\}\big)],
\end{aligned}$$

since $x \in \mathrm{Ma}(\mathcal{H})$. Also, we have

$$\check{A}_2 * \{x\} \subseteq A * \{x\} = \bigcup_{n=n_0}^{n_1} \big(\{a_n\} * B * \{x\}\big) \subseteq \bigcup_{n=n_0}^{n_1} \big(\{a_n\} * B * B\big) = D,$$

and so $\check{A}_2 \subseteq (D * \{\check{x}\})$. This implies that

$$\begin{aligned}
\lambda(\check{H}) &= \lambda(\check{A}_2) - \lambda\big(\check{A}_2 - (D_2 * \{\check{x}\})\big) \\
&\geq \lambda(\check{A}_2) - \lambda\big((D * \{\check{x}\}) - \big(D_2 * \{\check{x}\}\big)\big) \\
&= \lambda(\check{A}_2) - \lambda\big((D - D_2) * \{\check{x}\}\big) \\
&= \lambda(\check{A}_2) - \lambda\big(D_1 * \{\check{x}\}\big) \\
&= \lambda(A) - \lambda(\check{A}_1) - \lambda\big(D_1 * \{\check{x}\}\big) \\
&= \lambda(A) - \lambda(\check{A}_1) - (\lambda * \delta_x)\big(D_1\big) \\
&= \lambda(A) - \lambda(\check{A}_1) - \Delta(\check{x})\lambda\big(D_1\big).
\end{aligned}$$

Therefore, by inequalities (3.9) and (3.10)

$$\begin{aligned}\lambda(\check{H}) &\geq \lambda(A) - \left(\frac{c}{(1-\alpha)\,\eta}\right)^p \lambda(A) - \Delta(\check{x}) \left(\frac{c}{(1-\alpha)\,\eta}\right)^q \frac{L\,\lambda(A)}{\lambda(B)} \\ &\geq \left(1 - \left(\frac{c}{(1-\alpha)\,\eta}\right)^p - \left(\frac{c}{(1-\alpha)\,\eta}\right)^q \frac{SL}{\lambda(B)}\right) \lambda(A) \\ &= P\,\lambda(A).\end{aligned}$$

Therefore,

$$\begin{aligned}(|h| * |s|)(x) &\geq \alpha^2 M_1 M_2\, S^{\frac{1}{p}-1}\, P\,\lambda(A) \\ &= \alpha^2 \eta^2\, r^2 \lambda(A)^{\frac{-1}{p}}\, \lambda(D)^{\frac{-1}{q}}\, S^{\frac{1}{p}-1}\, P\,\lambda(A) \\ &\geq \alpha^2 \eta^2\, r^2 \left(\frac{L}{\lambda(B)}\right)^{\frac{-1}{q}} S^{\frac{1}{p}-1}\, P\,\lambda(A)^{1-\frac{1}{p}-\frac{1}{q}} \\ &\geq \alpha^2 \eta^2\, r^2 \left(\frac{L}{\lambda(B)}\right)^{\frac{-1}{q}} S^{\frac{1}{p}-1}\, P\,\lambda(B)^{1-\frac{1}{p}-\frac{1}{q}}\, (n_1 - n_0 + 1)^{1-\frac{1}{p}-\frac{1}{q}} \\ &> m,\end{aligned}$$

thanks to inequality (3.2). This shows that $(h, s) \notin M_{B,m}$ and the proof is complete. □

Corollary 3.1. *Let $\mathcal{H}$ be a non-compact hypergroup and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} < 1$. Let B be a symmetric compact neighborhood of e in $\mathcal{H}$ with L-property. Then, there exists some $c \in (0, 1)$ such that the set*

$$M_B := \{(f, g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H}) : \exists x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|f| * |g|)(x) < \infty\} \tag{3.11}$$

is a σ-c-lower porous.

Proof. Note that $M_B = \bigcup_{n=1}^{\infty} M_{B,n}$, and directly apply Theorem 1.1 and Theorem 3.1. □

In the sequel, we intend to give some extension of [10, Theorem 13]. In the proof of this fact, we use a recent result regarding spaceability subsets of Banach spaces from [3]. Recall that a subset S of a topological vector space $\mathcal{E}$ is called *spaceable* if $S \cup \{0\}$ contains a closed infinite-dimensional linear subspace of $\mathcal{E}$. We need the next definition given in [3] for proving our main theorem.

Definition 2. Let $\mathcal{E}$ be a topological vector space. We say that a relation $\sim$ on $\mathcal{E}$ has property (D) if the following conditions hold.

1. If (x_n) is a sequence in $\mathcal{E}$ such that $x_n \sim x_m$ for all distinct index m, n, then for each disjoint finite subsets A, B of $\mathbb{N}$ we have

$$\sum_{n \in A} \alpha_n x_n \sim \sum_{m \in B} \beta_m x_m,$$

   where α_n and β_m's are arbitrary scalars.

2. If a sequence (x_n) converges to x in $\mathcal{E}$ and for some $y \in \mathcal{E}$, $x_n \sim y$ for all $n \in \mathbb{N}$, then $x \sim y$.

We say that a subset B of a vector space is a *cone* if for each scalar c, $cB \subseteq B$.

Theorem 3.2. *Let $(\mathcal{E}, \|\cdot\|)$ be a Banach space, $\sim$ be a relation on $\mathcal{E}$ with property (D), and K be a nonempty subset of $\mathcal{E}$. Assume that:*

1. *there is a constant $k > 0$ such that $\|x + y\| \geq k\|x\|$ for all $x, y \in \mathcal{E}$ with $x \sim y$;*

2. *K is a cone;*

3. *if $x, y \in \mathcal{E}$ such that $x + y \in K$ and $x \sim y$ then $x, y \in K$;*

4. *there is an infinite sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{E} \setminus K$ such that for each distinct $m, n \in \mathbb{N}$, $x_m \sim x_n$.*

Then, $\mathcal{E} \setminus K$ is spaceable in $\mathcal{E}$.

Proof. See [3, Theorem 4.2]. □

Now, the next result which is a generalization of [10, Theorem 13] can be obtained with some different proof.

Theorem 3.3. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$. If $\mathcal{H}$ is a non-compact unimodular hypergroup and B is a fixed symmetric compact neighborhood of e in $\mathcal{H}$ with L-property, then the set $\big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B$ is spaceable in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$, where M_B is given by (3.11).*

Proof. Trivially M_B is a cone in the space $L^p(\mathcal{H}) \times L^q(\mathcal{H})$. We define the relation $\sim$ on $L^p(\mathcal{H}) \times L^q(\mathcal{H})$ by

$$(f_1, g_1) \sim (f_2, g_2) \text{ if and only if } \sigma(f_1) \cap \sigma(f_2) = \sigma(g_1) \cap \sigma(g_2) = \varnothing$$

up to a null set, for all $f_1, f_2 \in L^p(\mathcal{H})$ and $g_1, g_2 \in L^q(\mathcal{H})$, where $\sigma(f) := \{x \in G : f(x) \neq 0\}$. One can easily see that this relation satisfies condition (D) because convergence with respect to the L^p-norm implies almost everywhere subsequence convergence. This relation also satisfies conditions (1) (with $k = 1$) and (3) in Theorem 3.2. Indeed, if $(f_1, g_1) \sim (f_2, g_2)$, then we have $(|f_1| + |f_2|) * (|g_1| + |g_2|) = |f_1 + f_2| * |g_1 + g_2|$. In the sequel, we will show that condition (4) holds too. In this case the proof is complete. Assume that $(a_n)_n$ is the sequence in $\mathcal{H}$ obtained in Remark 2 regarding the neighborhood B. Define

$$f(x) := \sum_{n=1}^{\infty} n^{\frac{-q}{p+q}} \chi_{B*\{\check{a_n}\}} \qquad \text{and} \qquad g(x) := \sum_{n=1}^{\infty} n^{\frac{-p}{p+q}} \chi_{\{a_n\}*B*B} \tag{3.12}$$

for all $x \in \mathcal{H}$. Then, since $\mathcal{H}$ is unimodular we have

$$\begin{aligned}
\int_{\mathcal{H}} |f|^p \, d\lambda &= \int_{\mathcal{H}} \sum_{n=1}^{\infty} n^{\frac{-pq}{p+q}} \chi_{B*\{\check{a_n}\}} \, d\lambda \\
&= \sum_{n=1}^{\infty} n^{\frac{-pq}{p+q}} \lambda(B * \{\check{a_n}\}) \\
&= \sum_{n=1}^{\infty} n^{\frac{-pq}{p+q}} \lambda(\{a_n\} * B) \\
&\leq L \sum_{n=1}^{\infty} n^{\frac{-pq}{p+q}} < \infty,
\end{aligned}$$

because $\frac{pq}{p+q} > 1$. So $f \in L^p(\mathcal{H})$. Similarly, $g \in L^q(\mathcal{H})$. For each $N \subseteq \mathbb{N}$ we set

$$A_N := \bigcup_{n \in N} B * \{\check{a_n}\} \quad \text{and} \quad B_N := \bigcup_{n \in N} \{a_n\} * B * B.$$

Then, $f_N := \chi_{A_N} f \in L^p(\mathcal{H})$ and $g_N := \chi_{B_N} g \in L^q(\mathcal{H})$. Let $(N_k)_{k\in\mathbb{N}}$ be a partition of $\mathbb{N}$ with $\sum_{n\in N_k} \frac{1}{n} = \infty$ for all $k \in \mathbb{N}$. We denote $f_k := f_{A_{N_k}}$ and $g_k := g_{B_{N_k}}$. Then for each $k \in \mathbb{N}$ we have $(f_k, g_k) \in \big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B$ because

$$\begin{aligned}(f_k * g_k)(x) &= \int_{\mathcal{H}} f_k(y) g_k(\check{y} * x)\, d\lambda(y)\\ &= \int_{A_{N_k}} f_k(y) g_k(\check{y} * x)\, d\lambda(y)\\ &= \sum_{n\in N_k} \frac{1}{n}\, \lambda(B * \{\check{a_n}\})\\ &= \sum_{n\in N_k} \frac{1}{n}\, \lambda(\{a_n\} * B)\\ &\geq \lambda(B) \sum_{n\in N_k} \frac{1}{n} = \infty\end{aligned}$$

for all $x \in B \cap \mathrm{Ma}(\mathcal{H})$, thanks to [11, Lemma 3.3C]. Finally, it is easy to see that for each distinct numbers $k, m \in \mathbb{N}$, $(f_k, g_k) \sim (f_m, g_m)$. $\square$

Corollary 3.2. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$. Let $\mathcal{H}$ be a unimodular hypergroup and B be a symmetric compact neighborhood of e in $\mathcal{H}$ with L-property. Then, the following conditions are equivalent.*

1. *$\mathcal{H}$ is non-compact.*
2. *$\big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B \neq \varnothing$.*
3. *M_B is a σ-c-lower porous subset of $L^p(\mathcal{H}) \times L^q(\mathcal{H})$ for some $c \in (0, 1)$*
4. *The set $\big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B$ is spaceable in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$.*

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FINITE GROUPS WITH GIVEN SYSTEMS OF PROPERMUTABLE SUBGROUPS

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Abstract. Let H be a subgroup of a finite group G. Then we say that H is propermutable in G provided G has a subgroup B such that $G = N_G(H)B$ and H permutes with all subgroups of B. In this paper, we present new properties of propermutable subgroups. Also we provide new information on the structure of a group with propermutable Sylow (Hall, maximal) subgroups and a group $G = AB$ with propermutable subgroups A and B.

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1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [4]. The notation $Y \leq X$ ($Y < X$) means that Y is a subgroup (proper subgroup) of a group X.

A subgroup H of G is called *seminormal* in G if there exists a subgroup B such that $G = HB$ and HX is a subgroup of G for each subgroup X of B. The groups with given systems of seminormal subgroups were investigated in works of many authors, see, for example, the references in [13].

Following [16] a subgroup H is called *propermutable* in G if G has a subgroup B such that $G = N_G(H)B$ and H permutes with all subgroups of B. The groups with some propermutable subgroups were investigated in [1, 16, 17].

Obviously, if a subgroup H is seminormal in G, then H is propermutable in G. The opposite is not always true. For example, in the group

$$G = \langle a, b, c \mid |a| = |b| = 3, |c| = 2, ab = ba, ac = ca, b^c = b^{-1} \rangle \simeq Z_3 \times S_3$$

([6], IdGroup=[18,3]), the subgroup $A = \langle c \rangle$ is propermutable in G, since $N_G(A) = \langle ac \rangle$ and $B = \langle b \rangle$, but A is not seminormal in G.

In this paper, we present new properties of propermutable subgroups. Also we provide new information on the structure of a group with propermutable Sylow (Hall, maximal) subgroups and a group $G = AB$ with propermutable subgroups A and B.

2 Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel. A group whose chief factors have prime orders is called *supersoluble*. Recall that a *p-closed* group is a group with a normal Sylow p-subgroup and a *p-nilpotent* group is a group with a normal Hall p'-subgroup.

Denote by G', $Z(G)$, $F(G)$ and $\Phi(G)$ the derived subgroup, centre, Fitting and Frattini subgroups of G, respectively, and by $\mathrm{O}_p(G)$ the largest normal p-subgroup of G. Denote by $\pi(G)$ the set of all prime divisors of order of G. We use E_{p^t} to denote an elementary abelian group of order p^t and Z_m to denote a cyclic group of order m. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$.

The monographs [5, 10] contain the necessary information of the theory of formations.

A class group $\mathfrak{F}$ is called a formation if the following statements is true:

(1) if $G \in \mathfrak{F}$ and $N \lhd G$, then $G/N \in \mathfrak{F}$.

(2) if $G/N_1 \in \mathfrak{F}$ and $G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$.

A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. The formations of all supersoluble, nilpotent and abelian groups are denoted by $\mathfrak{U}$, $\mathfrak{N}$ and $\mathfrak{A}$, respectively. Let $\mathfrak{F}$ be a formation. Recall that the $\mathfrak{F}$-residual of G is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$ and is denoted by $G^{\mathfrak{F}}$.

Recall that a group G is said to be *siding* if every subgroup of the derived subgroup G' is normal in G, see [14, Definition 2.1]. It is clear that if G is a siding group, then G is supersoluble, every subgroup and quotient subgroup of G is a siding group. Metacyclic groups and soluble T-groups (groups in which every subnormal subgroup is normal) are siding groups. The group $G = (Z_6 \times Z_2) \rtimes Z_2$ ([6], IdGroup(G)=[24,8]) is a siding group but is neither a metacyclic nor a T-group.

Lemma 2.1. ([7, VI.9]) (1) *The class $\mathfrak{U}$ is a hereditary saturated formation.*

(2) *Every minimal normal subgroup of a supersoluble group has prime order.*

(3) *Let N be a normal subgroup of G and assume that G/N is supersoluble. If N is either cyclic or $N \leq Z(G)$, or $N \leq \Phi(G)$, then G is supersoluble.*

(4) *Each supersoluble group has an Sylow tower of supersoluble type.*

(5) *The derived subgroup of a supersoluble group is nilpotent.*

(6) *A group G is supersoluble if and only if every maximal subgroup of G has prime index.*

If H is a subgroup of G, then $H_G = \bigcap_{x \in G} H^x$ is called *the core* of H in G. If a group G contains a maximal subgroup M with trivial core, then G is said to be *primitive* and M is its *primitivator*. A simple check proves the following lemma.

Lemma 2.2. *Let $\mathfrak{F}$ be a saturated formation and G be a group. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all non-trivial normal subgroups N of G. Then G is a primitive group.*

Lemma 2.3. ([7, II.3.2]) *Let G be a soluble primitive group and M be a primitivator of G. Then the following statements hold:*

(1) $\Phi(G) = 1$*;*

(2) *$F(G) = C_G(F(G)) = O_p(G)$ and $F(G)$ is an elementary abelian subgroup of order p^n for some prime p and some positive integer n;*

(3) *G contains a unique minimal normal subgroup N and, moreover, $N = F(G)$;*

(4) *$G = F(G) \rtimes M$ and $O_p(M) = 1$;*

Lemma 2.4. ([10, Lemma 5.8, Lemma 5.11]) *Let $\mathfrak{F}$ and $\mathfrak{H}$ be non-empty formations, K be normal in G. Then:*

(1) $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$*;*

(2) $G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$*;*

(3) *if $\mathfrak{H} \subseteq \mathfrak{F}$, then $G^{\mathfrak{F}} \leq G^{\mathfrak{H}}$;*

3 Finite groups with propermutable Sylow, Hall and maximal subgroups

Recall that $A^G = \langle A^g \mid g \in G\rangle$ is the smallest normal subgroup of G containing A.

Basic properties of propermutable subgroups are given in [16]. Some of them are presented in the following lemma.

Lemma 3.1. ([16]) *Let A and B be subgroups of G and let N be a normal subgroup of G.*

(1) *If A is propermutable in G, then AN/N is propermutable in G/N.*

(2) *If $AB = BA$ and $G = N_G(A)B$, then $A^G = A(A^G \cap B)$.*

It is clear that the following lemma is true.

Lemma 3.2. *Let A be a subgroup of G. If A is propermutable in G, then A is seminormal in A^G. In particular, if $A^G = G$, then A is seminormal in G.*

Lemma 3.3. 1. *Let A be a subgroup of G. If A is propermutable in G, then A^G is soluble in each of the following cases:*

(1.1) *A is 2-nilpotent;*

(1.2) *A is soluble and $3 \notin \pi(A)$.*

2. *Let p be the smallest prime divisor of the order of G. If A is propermutable in G and p does not divide the order of A, then p does not divide the order of A^G.*

3. *Let A be propermutable in a soluble group G and let r be the largest in $\pi(G)$. If A is r-closed, then A_r is subnormal in G.*

Proof. 1. Let us prove both assertions 1 and 2 at once. By Lemma 3.2, A is seminormal in A^G. Then by [8, Lemmas 10–11], A^{A^G} is either soluble or a p'-group. Since A^{A^G} is subnormal in G, it follows that by [10, Theorem 5.31], $(A^{A^G})^G = A^G$ is either soluble or a p'-group.

3. By Lemma 3.2, A is seminormal in A^G. Then A_r is subnormal in A^G by [13, Lemma 1.8]. Hence, A_r is subnormal in G. □

The following theorem generalizes some results of the papers [8, 9, 13].

Theorem 3.1. 1. *Let H be a Hall π-subgroup of G. Suppose that H is propermutable in G. Then G is π-soluble in each of the following cases:*

(1.1) *H is 2-nilpotent;*

(1.2) *H is soluble and $3 \notin \pi$.*

2. *Let P be a Sylow p-subgroup of G. If P is propermutable in G, then G is p-soluble.*

3. *Let p be the largest prime in $\pi(G)$ and let P be a Sylow p-subgroup in G. If P is propermutable in G, then P is normal in G.*

4. *If all Sylow subgroups in G are propermutable, then G is supersoluble.*

5. *If all maximal subgroups in G are propermutable, then G is supersoluble.*

Proof. 1. By Lemma 3.3 (1), H^G is soluble. Since G/H^G is π'-group, it follows that G is π-soluble.

2. Since P is 2-nilpotent then from Step 1, G is p-soluble.

3. By Lemma 3.2, P is seminormal in P^G. Then P is normal in P^G by [13, Lemma 1.8]. Hence, P is normal in G, because P is subnormal in G.

4. Assume that the statement is not true and let G be a counterexample of minimal order. Let N be an arbitrary nontrivial normal subgroup in G and let S/N be a Sylow s-subgroup of G/N. Then $S/N = S_1N/N$, where S_1 is a Sylow s-subgroup of G. Since S_1 is propermutable in G, we have by Lemma 3.1 (1), S/N is propermutable in G/N. Thus, the condition of the lemma holds for the quotient group and by induction, G/N is supersoluble and G is primitive by Lemma 2.2.

From Step 2 it follows that G is p-soluble for every $p \in \pi(G)$. Hence, G is soluble. By Lemma 2.3, G has a unique minimal normal subgroup N, $N = F(G) = \mathrm{O}_p(G) = C_G(N)$, N is an elementary abelian subgroup of order p^n and $G = N \rtimes M$, where M is a maximal subgroup of G with trivial core. From Step 3 follows that p is the largest prime in $\pi(G)$ and $N = P$, where P is a Sylow p-subgroup of G. It is clear that M is a Hall p'-subgroup of G.

Let $N_1 \leq N = P$ such that $|N_1| = p$, and Q is a Sylow q-subgroups of M. Since Q is propermutable in G, we have $G = N_G(Q)Y$ and QX is a subgroup of G for every subgroup X of Y. By Lemma 3.1 (2), $Q^G = Q(Q^G \cap Y)$. Because $N \leq Q^G$, it follows that $N \leq Q^G \cap Y \leq Y$ and $QN_1 \leq G$ by Lemma 3.2. Since G is p-closed, $Q \leq N_G(N_1)$. Hence, $M \leq N_G(N_1)$ and N_1 is normal in $G = NM$. Then $N_1 = N$ and by Lemma 2.1 (3), G is supersoluble, a contradiction.

5. Let M be a maximal subgroup of G. By Lemma 3.2, M is seminormal in M^G. Since M is maximal in G, we have either $M^G = M$ or $M^G = G$. If $M^G = M$, then M is normal in G and $|G : M|$ is prime. If $M^G = G$, then M is seminormal in G. By [13, Lemma 1.4], $|G : M|$ is prime. By Lemma 2.1 (6), G is supersoluble. □

4 Finite factorizable groups with propermutable factors

Theorem 4.1. *Assume that A and B are propermutable subgroups of a group G and $G = AB$. Then the following statements hold.*

1. *Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. If $A, B \in \mathfrak{F}$ and the derived subgroup G' is nilpotent, then $G \in \mathfrak{F}$.*
2. *If A and B are supersoluble, then $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$.*
3. *If A and B have Sylow towers of supersoluble type, then G has a Sylow tower of supersoluble type.*
4. *If A is nilpotent and B is supersoluble, then G is supersoluble.*
5. *If A is supersoluble and B is a normal siding subgroup of G, then G is supersoluble.*

Proof. 1. Assume that the claim is false and let G be a minimal counterexample. If N is a nontrivial normal subgroup of G, then the subgroups AN/N and BN/N are propermutable in G/N by Lemma 3.1 (1) and belong to $\mathfrak{F}$. Since

$$(G/N)' = G'N/N \simeq G'/G' \cap N,$$

it follows that the derived subgroup $(G/N)'$ is nilpotent. Consequently, G/N satisfies the hypothesis of the theorem and by induction, $G/N \in \mathfrak{F}$. Then G is primitive by Lemma 2.2. Since G is soluble, therefore we apply Lemma 2.3. We save to G the notation of this lemma, in particular, $N = G'$ and G/N is abelian.

If $A^G = G$ and $B^G = G$, then by Lemma 3.2, the subgroups A and B are seminormal in G. By [15, Corollary 3.1 (2)], $G \in \mathfrak{F}$. Suppose that $A^G < G$. Since AN is normal in G, we have $A^G \leq AN$. On the other hand, $AN \leq A^G$, because N is the unique minimal normal subgroup of G. Hence, $AN = A^G$. By Lemma 3.2, A is seminormal in A^G, hence $A^G \in \mathfrak{F}$ by induction. If $B^G < G$, then by analogy, $B^G \in \mathfrak{F}$ and $G = AB = A^G B^G \in \mathfrak{F}$ by [15, Corollary 3.1 (2)].

If $B^G = G$, then by Lemma 3.2, B is seminormal in G. Then $G = AB = A^G B \in \mathfrak{F}$ by [15, Corollary 3.1 (2)].

2. Let $H = (G')^{\mathfrak{N}}$. Then the derived subgroup $(G/H)' = G'H/H = G'/H$ is nilpotent. From Step 1 it follows that G/H is supersoluble. Therefore, $G^{\mathfrak{U}} \leq H$. Because $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$, we have $G^{(\mathfrak{N}\mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{N}} = (G')^{\mathfrak{N}} = H \leq G^{\mathfrak{U}}$. Hence, $G^{\mathfrak{U}} = H$.

3. We proceed by induction on $|G|$. Since A is 2-nilpotent, it follows that by Lemma 3.3 (1), A^G is soluble and $G = A^G B$ is soluble. Let $r \in \pi(G)$ and let r be the largest. It is clear that a Sylow r-subgroup A_r is normal in A. By Lemma 3.3 (3), A_r is subnormal in G. Similarly, a Sylow r-subgroup

B_r of B is subnormal in G. Since $R = A_rB_r$ is a Sylow subgroup of G, we have G is r-closed. The subgroups $AR/R \simeq A/A \cap R$ and $BR/R \simeq B/B \cap R$ are propermutable in $G/R = (AR/R)(BR/R)$ and have Sylow towers of supersoluble type. By induction, G/R has an Sylow tower of supersoluble type, hence G has an Sylow tower of supersoluble type.

4. Assume that the claim is false and let G be a minimal counterexample. If N is a nontrivial normal subgroup of G, then the subgroups AN/N and BN/N are propermutable in G/N by Lemma 3.1 (1), $AN/N \simeq A/A \cap N$ is nilpotent and $BN/N \simeq B/B \cap N$ is supersoluble. Then by induction, $G/N = (AN/N)(BN/N)$ is supersoluble and G is primitive by Lemma 2.2. By Lemma 2.1 (4) and from Step 3, G has an Sylow tower of supersoluble type and therefore we apply Lemma 2.3. We save to G the notation of this lemma, in particular, $N = G_p$ is the Sylow p-subgroup for the largest $p \in \pi(G)$. Since $G = AB$, it follows that $N = A_pB_p$, where A_p and B_p are Sylow p-subgroups of A and B respectively, see [7, VI.4.6]. Since A is propermutable in G, $G = N_G(A)Y$ and $AX \leq G$ for all subgroups X of Y.

Suppose that $A_p = 1$. Then $N = B_p \leq B$. We choose a minimal normal subgroup N_1 of B such that $N_1 \leq N$. Since B is supersoluble, we have $|N_1| = p$ by Lemma 2.1 (2). By Lemma 3.1 (2), $A^G = A(A^G \cap Y)$. Since $A_p = 1$ and $N \leq A^G$, we have $N_1 \leq N \leq Y$ and there exists a subgroup $AN_1 = N_1 \rtimes A$ by Lemma 3.2. Hence, N_1 is normal in G. Therefore, $N_1 = N$ and by Lemma 2.1 (3), G is supersoluble, a contradiction. Thus, the assumption $A_p = 1$ is false and $A_p \neq 1$.

Assume that $B_p = 1$. Hence, $N = A_p \leq A$ and $N = A$ by Lemma 2.3 (2). Then $B \cap N = 1$ and B is maximal in G. By Lemma 3.2, B is seminormal in B^G. Since B is maximal in G, we have either $B^G = B$ or $B^G = G$. If $B^G = B$, then B is normal in G and $|G : B|$ is prime. If $B^G = G$, then B is seminormal in G. By [13, Lemma 1.4], $|G : B|$ is prime. Hence, $|N| = p$ and by Lemma 2.1 (3), G is supersoluble, a contradiction. Thus, the assumption $B_p = 1$ is false and $B_p \neq 1$.

Let Y_1 be a Hall p'-subgroup of Y. Then AY_1 is a subgroup of G and $Y_1 \leq N_G(A_p)$, because A_p is normal in AY_1. Since N is abelian, a Sylow p-subgroup Y_p of Y centralizes A_p. Because A_p is characteristic in A and A is normal in $N_G(A)$, we have A_p is normal in $N_G(A)$. Hence, A_p is normal in $G = N_G(A)Y = N_G(A)Y_pY_1$ and $A_p = N$. Because A is nilpotent and by Lemma 2.3 (2), it follows that $A = N$. Since B is supersoluble, we have B_p is normal in B. In this case, B_p is normal in $N = A$ and therefore is normal in G. Thus $B_p = N$ and $G = AB = NB = B$ is supersoluble, a contradiction.

5. If $A^G = G$, then by Lemma 3.2, A is seminormal in G. Then G is supersoluble by [13, Corollary 2.2]. Hence, $A^G < G$. By Dedekind's identity, $A^G = A(A^G \cap B)$. Since A is seminormal in A^G by Lemma 3.2 and $A^G \cap B$ is a normal siding subgroup of A^G, it follows that A^G is supersoluble by indution. Then by [13, Corollary 2.2], $G = A^GB$ is supersoluble.

□

In monograph [4, p. 149], it is presented the following definition: two subgroups A and B of a group G are said to be *mutually permutable* if $UB = BU$ and $AV = VA$ for all $U \leq A$ and $V \leq B$.

Since every normal subgroup and every subgroup of prime index are seminormal and therefore are propermutable in a group, the following corollary holds.

Corollary 4.1. *Let A and B be supersoluble subgroups of G and $G = AB$.*

1. *Suppose that A is nilpotent. Then G is supersoluble in each of the following cases:*

(1.1) *A and B are mutually permutable, see* [2, Theorem 3.2]*;*

(1.2) *A and B are seminormal in G, see* [13, Theorem 2.1]*;*

(1.3) *the indices of A and B in G are prime, see* [12, Theorem A]*;*

2. *If G' is nilpotent, then G is supersoluble in each of the following cases:*

(2.1) *A and B are normal in G, see* [3]*;*

(2.2) *A and B are mutually permutable, see* [2, Theorem 3.8]*;*

(2.3) *A and B are seminormal in G, see* [13, Theorem 2.2]*;*
(2.4) *the indices of A and B in G are prime, see* [12, Corollary 3.6].
3. *If B is normal and siding, then G is supersoluble in each of the following cases:*
(3.1) *A is normal in G and B is a soluble* T*-group, see* [11, Theorem 3]*;*
(3.2) *A is seminormal in G, see* [13, Corollary 2.2]*;*
(3.3) *the indices of A and B in G are prime, see* [12, Theorem B]*;*

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Events

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INTERNATIONAL CONFERENCE "FUNCTION SPACES, ANALYSIS AND APPROXIMATION" ASTANA, KAZAKHSTAN

Nazarbayev University (Astana) and Institute of Mathematics and Mathematical Modeling (Almaty) with the support of the International Society for Analysis, its Applications and Computation (ISAAC) organized the international conference "Function Spaces, Analysis and Approximation", dedicated to Professors Oleg Besov and Vladimir Temlyakov on their 90th and 70th birthdays.

The conference was held through February 3–6, 2024 at Nazarbayev University in Astana. It was dedicated to a number of areas in the theory of function spaces, analysis and the theory of approximation, which are some of the central trends of modern mathematics.

The conference was attended by more than 150 mathematicians. 43 talks were given by speakers from more than 10 countries. There were 22 invited speakers from such scientific centers as Nazarbayev University, University of Cambridge, Lomonosov Moscow State University, RUDN University, Beijing Normal University, University of Alberta, Johannes Kepler University of Linz, etc.

The conference "Function Spaces, Analysis and Approximation" is a satellite conference of the 15th ISAAC Congress (organized by Nazarbayev University), which is planned to take place in Astana in 2025.

Invited speakers:

Dauren Bazarkhanov (Institute of Mathematics and Mathematical Modeling)
Oleg Besov (Steklov Mathematica Institute)
Victor Burenkov (RUDN University)
Feng Dai (University of Alberta)
Mikhail Dyachenko (MSU)
Arran Fernandez (Eastern Mediterranean University)
Amiran Gogatishvili (Institute of Mathematics CAS)
Egor Kosov (Centre de Recerca Matematica)
Alexander Meskhi (Razmadze Mathematical Institute)
Erlan Nursultanov (Institute of Mathematics and Mathematical Modeling)
Lubos Pick (Charles University)
Makhmud Sadybekov (Institute of Mathematics and Mathematical Modeling)
Alexei Shadrin (University of Cambridge)
Vladimir Stepanov (Computing Center of FEB RAS)
Durvudkhan Suragan (Nazarbayev University)
Vladimir Temlyakov (University of South Carolina)
Sergey Tikhonov (Catalan Institution for Research and Advanced Studies)
Eugene Tyrtyshnikov (MSU)
Mario Ullrich (Johannes Kepler Universitat Linz)
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Participants of the conference

V.I. Burenkov, E.D. Nursultanov, D. Suragan

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THE 15th INTERNATIONAL ISAAC CONGRESS

FIRST INFORMATION LETTER

Dear Colleagues,

We are pleased to announce that Nazarbayev University in Astana, Kazakhstan, will host the 15th International ISAAC Congress from July 21-25, 2025. The International Society for Analysis, its Applications, and Computation (ISAAC) Congress is a prestigious event that continues a successful series of meetings previously held across the globe.

The congress will cover a wide range of topics, including but not limited to:

1. Application of Dynamical Systems Theory in Biology
2. Complex Analysis and Partial Differential Equations
3. Complex Variables and Potential Theory
4. Constructive Methods in Boundary Value Problems and Applications
5. Function Inequalities: New Perspectives and New Applications
6. Function Spaces and their Applications to Nonlinear Evolutional Equations
7. Fractional Calculus and Fractional Differential Equations
8. Generalized Functions and Applications
9. Harmonic Analysis and Partial Differential Equations
10. Integral Transforms and Reproducing Kernels
11. Partial Differential Equations on Curved Spacetimes
12. Pseudo Differential Operators
13. Quaternionic and Clifford Analysis
14. Recent Progress in Evolution Equations
15. Wavelet Theory and its Related Topics

The conference will feature plenary and sectional talks, as well as poster presentations. The official language of the conference is English. We plan to publish the abstracts prior to the conference's commencement.

Registration Fees:

- ISAAC Members:
  - Before April 30, 2025: 150 EUR or 73,155 KZT
  - From May 1, 2025: 200 EUR or 97,540 KZT
- Non-Members:
  - Before April 30, 2025: 200 EUR or 97,540 KZT
  - From May 1, 2025: 250 EUR or 121,925 KZT
- Students and Participants from Developing Countries:
  - Before April 30, 2025: 80 EUR or 39,016 KZT
  - From May 1, 2025: 130 EUR or 63,400 KZT

Further Information:

Details on registration procedures, abstract submission guidelines, accommodation options in Astana, and information about the Programme and Organizing Committees, as well as invited speakers, will be announced in due course.

Contact Information:

Organizing Committee, School of Science and Humanities, Nazarbayev University, Qabanbay Batyr Ave 53, Astana 010000, Kazakhstan.
Email: info@isaac2025.org
Website: `https://isaac2025.org/`

Important Dates:

- Registration and abstract submission deadline: May 1, 2025
- Arrival day: July 20, 2025
- Departure day: July 26, 2025

We encourage you to share this information with interested colleagues. We look forward to welcoming you to Astana for an engaging and fruitful congress.

Warm regards,

Prof. Durvudkhan Suragan, Nazarbayev University, Chairman of Organizing Committee
Dr. Bolys Sabitbek, Queen Mary University of London, Member of Organizing Committee

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