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A COMPACTNESS LEMMA FOR CONVEX PLANE SETS

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Abstract. Given a closed bounded convex plane set covered by a family of open sets, we show that there is a finite dissection of the convex set into convex subsets, each of which lies within some open set of the covering.

This lemma embodies an idea used in the investigation of the uniform distribution of configurations of lattice points in the plane.

Lemma 1. *Let S be a closed bounded convex set in the plane. Let $\{O_\alpha\}$ be a family of open sets that covers S. Then there is a finite partition of S into closed bounded convex sets K_i which meet only on their boundaries, such that each set K_i lies in the interior of some open set O_α of the given family.*

Proof. Each point P of S lies in some open set O_α of the family. There is an open disc $D(P)$ (in the Euclidean metric) with centre P that lies in O_α. The family of open discs $\{D(P)\}$ covers S. Since S is compact, there is a finite sub-cover of S by open discs $D(P_i)$ for some points $P_1, \ldots, P_I$. We can suppose that this cover is minimal, by considering each $D(P_i)$ in turn, and omitting $D(P_i)$ if $D(P_i)$ is a subset of $D(P_1) \cup \cdots \cup D(P_{i-1})$. Let C_i denote the boundary circle of the disc $D(P_i)$.

The power of a point Q for a circle C with centre A and radius r is defined to be

$$P(Q, C) = AQ^2 - r^2.$$

Let ℓ be any line through Q that meets the circle C, in two points, X and Y say, that need not be distinct. Then

$$QX.QY = \pm P(Q, C)$$

(Hall and Stevens [1], citing Euclid's Elements of Geometry III), with the minus sign taken when Q is inside the circle. If C_i and C_j are two circles, then the locus of points Q with $P(Q, C_i) = P(Q, C_j)$ is a straight line, the co-axis of the pair of circles. If the circles C_i and C_j intersect in two distinct points X and Y, then the co-axis is the line XY.

We define the set K_i to be

$$K_i = \{Q \in S | P(Q, C_i) = \min_j P(Q, C_j)\}.$$

Since the cover by discs D_i is minimal, each open disc D_i contains some point Q_i of S that lies in no other disc D_j of the covering. Then $P(Q_i, C_i)$ is negative, but $P(Q_i, C_j) \geq 0$ for all $j \neq i$. The point Q_i lies in the interior of K_i. The boundary of K_i consists of segments of the co-axes of C_i and certain other circles C_j, and possibly one or more arcs of the boundary of S. Let C_j be one of these circles, and let L_{ij} be the co-axis of the circles C_i and C_j. All points of K_i lie on the same side of L_{ij} as Q_i, and Q_j lies on the opposite side to Q_i, so all points of K_j lie on the far side of the line L_{ij}. The sets K_i and K_j are closed, and they meet in a segment of the co-axis L_{ij}.

Hence the set K_i can be defined as a subset of S by linear inequalities, and so K_i is a convex set. We now have a partition of S into convex sets K_i, which meet only along boundary line segments. By construction K_i is a subset of $D(P_i)$, which is a subset of some open set O_α of the given family of open sets. This establishes the Lemma. $\square$

This question arose in considering the uniform distribution over translation vectors in the unit square of the sets of integer points in a translated plane oval domain. The direct approach involves estimating double exponential sums indexed by the integer points inside the Minkowski difference set of the oval (see [2]). The region of summation has to be subdivided to consider cases separately in which some determinant of partial derivatives becomes small or vanishes, and the double sums can be renumbered using an automorphism of the integer lattice, provided that the subregion is convex.

References

[1] H.S. Hall, F.H. Stevens, *A School Geometry*. Macmillan, London, 1949.

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