Short communications

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ORDER-SHARP ESTIMATES FOR HARDY-TYPE OPERATORS ON CONES OF QUASIMONOTONE FUNCTIONS

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Abstract. The two-sided estimates are obtained for two types of generalized Hardy operators on cones of functions in weighted Lebesgue spaces with some properties of monotonicity.

1. Let β and γ be nonnegative Borel measures on $\mathbb{R}_+ = (0, \infty)$; $p, q \in \mathbb{R}_+$, Ω be a certain cone of nonnegative Borel - measurable functions on $\mathbb{R}_+$, and A be a positive operator. Let

$$H_\Omega(A) = \sup_{f \in \Omega} \left[\left(\int_0^\infty (Af)^q \, d\gamma \right)^{1/q} \left(\int_0^\infty f^p d\beta \right)^{-1/p} \right]. \tag{1}$$

Here, we consider the cones of functions that are monotone with respect to prescribed positive Borel functions k and m:

$$\Omega_k = \left\{ f \geq 0 : \frac{f(\tau)}{k(\tau)} \downarrow \right\}; \qquad \Omega^m = \left\{ f \geq 0 : \frac{f(\tau)}{m(\tau)} \uparrow \right\}. \tag{2}$$

As operator A, we consider the generalized Hardy operators $A = A_\mu$, and $A = B_\mu$ where μ is a nonnegative Borel measure on $\mathbb{R}_+$;

$$(A_\mu f)(t) = \int_0^t f d\mu; \qquad (B_\mu)(t) = \int_t^\infty f d\mu. \tag{3}$$

2. First, we formulate the result for $H_{\Omega_k}(B_\mu)$. For this purpose we need some notation:

$$\omega_p(t) = \left(\int_0^t k^p d\beta \right)^{1/p}, \quad t > 0; \qquad \Psi(t, \tau) = \int_t^\tau k d\mu, \quad t < \tau;$$

$$V_p(t) = \sup_{\tau \in [t, \infty)} \left[\Psi(t, \tau) \frac{1}{\omega_p(\tau)} \right], \quad p \in (0, 1];$$

$$V_p(t) = \left[\int_t^\infty \Psi^{p'}(t,\tau)\left(-d\left[\frac{1}{\omega_p^{p'}(\tau)}\right]\right)\right]^{1/p'}, \ \ p>1 \ \text{ and } \ \frac{1}{p}+\frac{1}{p'}=1;$$

$$W_q(\tau) = \left(\int_0^\tau d\gamma\right)^{1/q}; \qquad \xi_\alpha(\tau) = \omega_p^{-1}\left(\alpha\omega_p(\tau)\right), \ \ \tau\in\mathbb{R}_+. \tag{4}$$

Here $\alpha \in (0,\,1)$ is fixed; ω_p^{-1} is the right-continuous inverse function for the (non-decreasing) continuous function ω_p. Obviously, $\xi_\alpha(\tau) < \tau$.

The criterion of the boundedness for $H_{\Omega_k}(B_\mu)$ is determined by the following quantities:

$$\begin{aligned}
E_{pq} &= \sup_{\tau\in\mathbb{R}_+}\left[\left(\int_0^\tau \Psi^q(t,\tau)d\gamma(t)\right)^{1/q}\frac{1}{\omega_p(\tau)}\right], \ \ p\le q;\\
E_{pq} &= \left[\int_0^\infty\left(\int_{\xi_\alpha(\tau)}^\tau \Psi^q(t,\tau)d\gamma(t)\right)^{s/q}\left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)\right]^{1/s}, \ \ p>q;\\
F_{pq} &= \sup_{t\in\mathbb{R}_+}\left[V_p(t)W_q(t)\right], \ \ p\le q;\\
F_{pq} &= \left[\int_0^\infty V_p^s(t)d\left[W_q^s(t)\right]\right]^{1/s}, \ \ p>q,
\end{aligned} \tag{5}$$

where $s = pq/(p-q)$ for $p>q$. In addition, introduce the non-degeneracy condition for measure the β:

$$\beta\in N_p(k) \Leftrightarrow \int_0^1 k^p d\beta = 1, \qquad \int_1^\infty k^p d\beta = \infty.$$

Theorem 1. *Let $\beta\in N_p(k)$ and functions ω_p and W_q be positive and continuous on $\mathbb{R}_+$. Then there exists $c_0 = c_0(p,q)\in[1,\infty)$ such that*

$$c_0^{-1}\left(E_{pq}+F_{pq}\right) \le H_{\Omega_k}(B_\mu) \le c_0\left(E_{pq}+F_{pq}\right).$$

3. Now, we present the corresponding results concerning $H_{\Omega^m}(A_\mu)$ (see (1) – (3)). To this end we denote

$$\overline{\omega}_p(t) = \left(\int_t^\infty m^p d\beta\right)^{1/p}, \ \ t>0; \qquad \Phi(\tau,t) = \int_\tau^t m d\mu, \ \ \tau<t;$$

$$V_p^{(0)}(t) = \sup_{\tau\in(0,t]}\left[\Phi(\tau,t)\frac{1}{\overline{\omega}_p(\tau)}\right], \ \ p\in(0,1];$$

$$V_p^{(0)}(t) = \left[\int_0^t \Phi^{p'}(\tau,t)\left(-d\left[\frac{1}{\overline{\omega}_p^{p'}(\tau)}\right]\right)\right]^{1/p'}, \ \ p>1 \ \text{ and } \ \frac{1}{p}+\frac{1}{p'}=1;$$

$$\overline{W}_q(\tau) = \left(\int_\tau^\infty d\gamma\right)^{1/q}; \qquad \zeta_\alpha(\tau) = \overline{\omega}_p^{-1}\left(\alpha\overline{\omega}_p(\tau)\right), \ \ \tau\in\mathbb{R}_+. \tag{6}$$

Here $\alpha \in (0,\, 1)$ is fixed; $\overline{\omega}_p^{-1}$ is the right-continuous inverse function for the (decreasing) continuous function $\overline{\omega}_p$. Obviously, $\tau < \zeta_\alpha(\tau)$. Now, we introduce the following quantities:

$$\begin{aligned}
E^{(0)}_{pq} &= \sup_{\tau\in\mathbb{R}_+}\left[\left(\int_\tau^\infty \Phi^q(\tau,t)d\gamma(t)\right)^{1/q}\frac{1}{\overline{\omega}_p(\tau)}\right],\ \ p\le q;\\
E^{(0)}_{pq} &= \left[\int_0^\infty\left(\int_\tau^{\zeta_\alpha(\tau)}\Phi^q(\tau,t)d\gamma(t)\right)^{s/q}\left(-d\left[\frac{1}{\overline{\omega}_p^s(\tau)}\right]\right)\right]^{1/s},\ \ p>q;\\
F^{(0)}_{pq} &= \sup_{t\in\mathbb{R}_+}\left[V_p^{(0)}(t)\overline{W}_q(t)\right],\ \ p\le q;\\
F^{(0)}_{pq} &= \left[\int_0^\infty {V_p^{(0)}}^s(t)\left(-d\left[\overline{W}_q^s(t)\right]\right)\right]^{1/s},\ \ p>q.
\end{aligned}\tag{7}$$

Theorem 2. *Let* $\int_0^1 m^p d\beta = \infty$, $\int_1^\infty m^p d\beta = 1$ *and functions* $\overline{\omega}_p$ *and* $\overline{W}_q$ *be positive and continuous on* $\mathbb{R}_+$. *Then there exists* $c_1 = c_1(p,q) \in [1,\infty)$ *such that*

$$c_1^{-1}\left(E^{(0)}_{pq}+F^{(0)}_{pq}\right)\le H_{\Omega^m}(A_\mu)\le c_1\left(E^{(0)}_{pq}+F^{(0)}_{pq}\right).$$

Remark 1. The results concerning $H_{\Omega_k}(A_\mu)$ and $H_{\Omega^m}(B_\mu)$ were obtained in our paper [4; Theorems 1.2 and 1.4], and in some other forms in [1, 2, 3]. The detailed comparison for the corresponding results from [1, 2, 4] was made in [5].

Remark 2. It was found by A. Gogatishvili that for $p > q$ the statements of Theorems 1.1 and 1.3 in [4] were not correct (personal communication). Here we present the corrected version of these results. This is done by inserting the function ξ_α defined by (4) in (5), and by inserting the function ζ_α defined by (6) in (7). In the next paper we will present the detailed proofs for Theorems 1 and 2.

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