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# EURASIAN MATHEMATICAL JOURNAL

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## BOKAYEV NURZHAN ADILKHANOVICH

(to the 70th birthday)

January 5, 2026, marks the 70th birthday of Nurzhan Adilkhanovich Bokayev, Doctor of Physical and Mathematical Sciences (1996), Professor (2002), member of the Editorial Board of the Eurasian Mathematical Journal (2010).

Nurzhan Adilkhanovich Bokayev was born on 5 January, 1956 in the village of Urnek, Karabalyk District, Kostanay Region. He graduated in 1972, with a gold medal from the Burlin Secondary School in the district. That same year, he entered the Mathematics Department of Karaganda State University and graduated with honors in 1977. From 1978 to 1979, he served in the Soviet Army. In 1980, he completed an internship, and from 1981 to 1984, he studied in the graduate program at Lomonosov Moscow State University in the Department of Function Theory and Functional Analysis. In 1985, he defended his candidate's dissertation there under the supervision of Corresponding Member of the Academy of Sciences of the USSR D.E. Menshov and Professor V.A. Skvortsov. In 1996, he defended his doctoral dissertation, "Fourier Coefficients and Uniqueness Theorems for Series in Generalized Walsh and Haar Systems", at the Institute of Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan, speciality Mathematical Analysis (01.01.01).

After completing his postgraduate studies, he worked as a lecturer, senior lecturer, associate professor, and professor in the Department of Mathematical Analysis at E.A. Buketov Karaganda State University (1985-1999). He headed the Department of Mathematics and Mathematical Modeling (1996-1999), and was a dean of the Faculty of Mathematics at E.A. Buketov Karaganda State University (1999-2005). Since 2005, he has been a professor in the Faculty of Mechanics and Mathematics at the L.N. Gumilyov Eurasian National University. From 2009 to 2018, he was the Head of the Department of Higher Mathematics at the L.N. Gumilyov Eurasian National University, and from 2018 to the present, he has been a professor in the Department of Fundamental Mathematics.

Professor Bokaev's research focuses on problems in function theory and functional analysis, the theory of orthogonal series for generalized Walsh and Haar systems, and operator theory in various function spaces. He has proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, and constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems. He obtained conditions for functions to belong to various functional classes in terms of the Fourier coefficients of generalized Haar and Walsh systems, and embedding criteria for Nikol'skii-Besov spaces constructed on the basis of multiplicative systems. He also obtained conditions for the boundedness and compactness of the commutator of the Riesz potential in general Morrey-type spaces, and conditions for boundedness of generalizedRiesz and Bessel potentials and generalized fractional-maximal operators in rearrangement-invariant spaces.

His co-authors include Professor V.A. Skvortsov (Moscow State University, Moscow), Professors V.I. Burenkov and M.L. Goldman (Peoples' Friendship University of Russia (RUDN University). Moscow), Dr. A. Gogatishvili (Institute of Mathematics of the Czech Academy of Sciences, Prague). His doctoral students' foreign advisors include Professors W. Sickel (Friedrich-Schiller-Universitity, Jena, Germany), Massimo Lanza de Cristoforis (University of Padova, Padova, Italy), V. Ruzhansky (Ghent University, Ghent, Belgium), U. Goginava (United Arab Emirates University, Al Ain, United Arab Emirates), and E. Panakhov (Institute of Applied Mathematics at Baku State University, Baku, Azerbaijan).

Under his supervision, 15 dissertations (4 candidate's and 11 PhD) were defended. He has published over 220 scientific papers, 2 monographs and 2 textbooks.

He is a three-time recipient of the state grant “Best University Teacher” of the Republic of Kazakhstan (2006, 2010, 2024) and served as Vice President of the Mathematical Society of Turkic-Speaking Countries (2014-2023). He was awarded the “For Contribution to the Development of Science” badge (2022).

Over the last ten years, he has been and continues to be a head of more than 5 national and international funded projects.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Nurzhan Adilkhanovich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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# REDUCIBILITY TO MULTIPERIODIC LINEAR SYSTEMS WITH A DIAGONAL DIFFERENTIATION OPERATOR AND ITS APPLICATION TO CONDITIONALLY PERIODIC SYSTEMS

**Zh. Sartabanov**

Communicated by V.I. Burenkov

**Key words:** reducibility, multiperiodicity, differentiation operator, periodic characteristic, cylindrical surface, Gershgorin's circles.

**AMS Mathematics Subject Classification:** 35B10, 35F05, 35R01, 34C46.

**Abstract.** In this paper, the main theorem is proved by establishing the reducibility to an equivalent multiperiodic linear system with a differentiation operator directed along the diagonal of the independent variables space. It is shown that helical lines on a circular cylindrical surface form periodic characteristics of the operator. The reducibility of a multiperiodic system is examined near a helix starting from the initial point of the phase circle, following the classical approach used for periodic systems. A monodromy matrix is introduced, which remains constant along the first integrals of the characteristic equations and possesses the properties of smoothness and multiperiodicity. The existence of localised positive eigenvalues consistent with the properties of this matrix is demonstrated. It is assumed that at the initial point of the phase circle, the monodromy matrix attains the maximal number of distinct eigenvalues. Their localisation on the cylindrical surface is established using the Gershgorin method.

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## 1 Introduction

The reducibility of linear systems with continuously periodic coefficients to a system with constant coefficients was given for the first time in the dissertation work of A.M. Lyapunov [19]. In modern scientific literature, the issue of the reducibility of periodic systems is primarily met with a rationale associated with Floquet's theorem [10], which pertains to the representation of their fundamental solutions.

The equivalence of the reducibility problem to the representation of fundamental solutions by Floquet was shown later in [8].

Thus, nowadays, the reducibility of periodic systems of ordinary differential equations has become well known. The interest in the reducibility of conditionally periodic systems probably began in the 1930s of the last century in connection with the application of electronic tubes in technology for generating and receiving electromagnetic oscillations with different frequencies. Work [2] is considered to be beginning of a new stage in the development of nonlinear oscillation theory, and linear oscillation theory with periodic and quasi-periodic behaviour, was at the centre of interest as a basis of studying oscillations [3].

The unresolved and important issue of the reducibility of linear conditionally periodic systems is particularly highlighted in [4]. In [15], to study this issue a method based on the transition to multiperiodic systems of partial differential equations with a special differentiation operator is proposed and the sufficiency of studying it in a small neighborhood of the diagonal of the space of independent variables is noted.

In this paper, we followed the author's recommendation of the proposed method related to the transition to systems of partial differential equations by applying Bohr's theorem [6] and used some ideas from [5, 7, 11, 12, 22].

The specificity of the characteristic equation $dt/d\tau = e$ for the operator $D$ is as follows: *first*, although the basic system depends on this vector field, it does not depend on the original given system with the operator $D$, *secondly*, it can be considered an autonomous equation with a given vector field $v(t) = e$ in the phase space and, *thirdly*, the vector field $v(t) = e$ can be considered periodic with any period $p$ and, in particular, we can put $p = \theta$.

Obviously, one and only one phase curve $t = h(\tau, \xi, \eta), \tau \in \mathbb{R}$ passes through each point of the characteristic equation's phase space $(\xi, \eta)$. Hence, different phase curves of this equation do not intersect.

It is also known from the theory of autonomous systems that the solution $t = h(\tau, \xi, \eta)$ to the characteristic equation, which takes the same value $h(\tau_1, \xi, \eta) = h(\tau_2, \xi, \eta)$ at two different points $\tau = \tau_1$ and $\tau = \tau_2$, is periodic with the smallest period $\theta = \tau_2 - \tau_1 > 0$, and $\{k\theta, k \in \mathbb{Z}\}$ is the set of periods, where $\mathbb{Z}$ is a set of integers.

The characteristic equation does not have a constant solution since all components of the vector field $v(t) = e$ are positive numbers.

Note also that the initial $D$-system is given in the Euclidean space, therefore, following this premise in [15] the Euclidean space is also considered as the phase space of the characteristic equation. At the same time, the phase integral lines are straight lines that violate the basic specificity of the original system associated with its $\theta$-periodicity in $\tau$. This was a big insurmountable obstacle in the study of the system's problems related to the periodicity in $\tau$ of period $\theta$. Therefore, the author of this method and his followers [1, 21, 26, 27, 28], along with other problems about multiperiodic oscillations, limited themselves to the study of special cases of this problem. The multiperiodic and periodic solutions of various classes of ordinary differential equations and partial differential equations were studied also in [9, 13, 14, 16, 17, 18].

It turns out that if we consider the above-mentioned specifics of the characteristic equation, this difficulty can be overcome by replacing the Euclidean phase space with a cylindrical surface [20, 23, 24, 25]. Then, as the phase integral curves of the characteristic equation of the differentiation operator, we can take a helical line at an angle of elevation $\frac{\pi}{4}$. In this case, if $\tau$ represents the shift of a point along the generatrix of the cylinder, and by $t_j$ we mean the length of the arc of the phase circle, then the point $(\tau, t_j)$ changes along the helix in the three-dimensional Euclidean space, and $t_j$ changes $\theta$-periodically with respect to $\tau$. Thus, the concept of a helical line in the three-dimensional space is introduced. Then, based on the Cartesian product of three-dimensional helical lines we have multidimensional helical lines.

Following this innovation, in this paper there is investigated the reducibility of a linear homogeneous multiperiodic system with a diagonal differentiation operator of independent variables to a similar system but with a constant on the diagonal multi-periodic matrix in terms of equivalent systems introduced in [22].

To prove the existence of the logarithm of the monodromy matrix of the original system in the $\varepsilon$-neighbourhood of the diagonal of independent variables we use the method of localisation of eigenvalues according to Gershgorin [11], [12], from which follows the existence of a closed line not passing through the zero of the complex plane covering the whole spectrum of the monodromy matrix. Consequently, the required logarithm is determined by the Cauchy integral formula [7].

In conclusion, the main result of the reducibility of a conditionally periodic matrix equation with an ordinary differentiation operator to a matrix equation with a constant matrix is obtained based on the transition from an equation with a differentiation operator to an equation with a diagonal matrix according to [15].

## 2 Problem statement

We consider the following linear system

$$\begin{aligned} &\frac{d}{d\tau}X = P(\tau,t)X, \quad \frac{dt}{d\tau} = e, \\ &P(\tau+\theta, t+k\omega) = P(\tau,t) \in C^{(0,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m), \\ &\sum_{j=0}^{m} k_j\omega_j \neq 0, \ \sum_{j=0}^{m}|k_j| \neq 0, \ k_j \in \mathbb{Z}, j=\overline{1,m}, \end{aligned} \tag{2.1}$$

where $\tau \in (-\infty,+\infty) = \mathbb{R}$, $t = (t_1,\ldots,t_m) \in \mathbb{R}\times\ldots\times\mathbb{R} = \mathbb{R}^m$, $e = (1,\ldots 1)$ and $\omega = (\omega_1,\ldots,\omega_m)$ are $m$-vectors, $\omega_j \in \mathbb{R}, j = \overline{1,m}$, $\theta = \omega_0 = const \in \mathbb{R}$, $P(\tau,t)$ is a given $n$-matrix function, $X$ is the required $n$-matrix function, $C^{(0,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m)$ is a class of functions of variables $(\tau,t) \in \mathbb{R}\times\mathbb{R}^m$ of smoothness $(0,e)$.

The reducibility of system (2.1) to a system

$$\frac{d}{d\tau}Y = AY, \tag{2.2}$$

with a constant matrix $A$ is investigated, based on the linear matrix replacement

$$X = Q(\tau,t)Y, \ Q(\tau+\theta, t+\omega) = Q(\tau,t) \in C^{(1,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m), \ \det Q(\tau,t) \neq 0, \tag{2.3}$$

which is quasi-periodic along the diagonal $t = e\tau$.

The problem (2.1)–(2.3) is formulated in [4] with an indication of the characteristic equation

$$\frac{dt}{d\tau} = e, \quad e = (1,...,1), \tag{2.4}$$

which is a subsystem of system (2.1).

System (2.4) is a Cartesian product of independent scalar equations

$$\frac{dt_j}{d\tau} = 1, \ j = \overline{1,m}, \tag{2.5}$$

which are convenient for further study of vector equation (2.4).

The problem of reducibility is investigated based on the method [15], according to which, along with equation (2.1), we consider a linear multiperiodic matrix equation with a diagonal differentiation operator of the form

$$DX = P(\tau,t)X, \quad D = \frac{\partial}{\partial\tau} + \sum_{j=1}^{m}\frac{\partial}{\partial t_j}, \tag{2.6}$$

where all input data satisfies the conditions of equations (2.1) respectively.

Since [15] along the characteristics $t = \beta(\tau,\xi,\eta)$ of equation (2.4), the operator $D$ of differentiation of the function $x(\tau,t) \in C^{(1,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m)$ passes into the ordinary operator $\frac{d}{d\tau}$ of differentiation of the function $x(\tau,\beta(\tau,\xi,\eta))$ and inversely, then for $t = e\tau = \beta(\tau,0,0)$ conditionally periodic equation (2.1) and multiperiodic equation (2.6) are equivalent.

According to Bohr's theorem [6], which states the relationship between continuous periodic functions of many variables and continuous almost periodic functions of one variable, the matrix $P(\tau,t)$ given in (2.1) and (2.6) at $t = e\tau$ becomes a conditionally periodic matrix $P(\tau,e\tau)$ with a constant basis $(\omega_0^{-1},\omega_1^{-1},...,\omega_m^{-1})$ with incommensurable frequencies $\nu_0 = \omega_0^{-1}, \nu_1 = \omega_1^{-1}, ..., \nu_m = \omega_m^{-1}$.

Thus, the problem of the reducibility of (2.1)–(2.3) was reduced to the problem of reducibility of multiperiodic equation (2.6) to equation (2.2) along the diagonal of independent variables $t = e\tau$ based on substitution (2.3).

## 3 Periodic characteristics of the diagonal differentiation operator

Assuming the variable t is one-dimensional, equation (2.5) is takes the form

$$\frac{dt}{d\tau} = 1. \tag{3.1}$$

Then if we take the straight lines

$$t = \eta + \tau - \xi$$

with the initial point $(\xi, \eta) \in \mathbb{R}^2$ as phase lines, we have rectilinear motion, which is a particular case of curvilinear motion.

If we take a circle $S$ of length $2\pi r = \theta$ on the $vOw$ plane as a phase line and by $t$ we mean the length of the arc $s$ of the circle $S$, and by $\varphi = \theta^{-1}\tau$ the angle of rotation of the radius vector of the arc $t$, then $t = t(v, w)$ is determined by its coordinates as a point $(v, w)$ of the circle $S$:

$$v = rsin\left(2\pi\theta^{-1}\tau\right), \quad w = r - rcos\left(2\pi\theta^{-1}r\right) = r + rsin\left(2\pi\theta^{-1}\left(r - \frac{\theta}{4}\right)\right).$$

Obviously, $dt = d\tau$, and

$$t = \eta + \oint_{\xi}^{\tau} \sqrt{v'^2 + w'^2} d\tau \equiv \gamma(\tau, \xi, \eta)$$

represents circular motion with period $\theta$, which is a special case of general curvilinear motion. The points $t$ and $t + k\theta, k \in \mathbb{Z}$ are identical on the circle $S$.

These planar straight-line and circular motions constitute strong restrictions on the systems under consideration, namely (2.1) and (2.6).

In [15] the author limited himself to the flat rectilinear case and the problem was not investigated from a more general point of view. In the case of circular motion, systems (2.1) and (2.6) are found to be $T$-periodic, for which this problem is solved in [15] and [19].

The most general kind of motion is the rotational-reciprocating motion. Since, according to systems (2.1) and (2.6), rotation must have the property of periodicity, the phase integral curves of equation (2.4) must be helical in nature. This circumstance suggests the consideration of equation (3.1) on an infinite circular cylindrical surface of space $(u, v, w) \in \mathbb{R}^3$.

In this regard, we consider the characteristic equation of the operator $D$ on a $\theta$-circular cylindrical surface with the parametric equations

$$u = \tau, \quad v = rsin\left(\frac{t}{r}\right), \quad w = r - rcos\left(\frac{t}{r}\right) = r + rsin\left(\frac{t}{r} - \frac{\pi}{2}\right) \tag{3.2}$$

with two parameters $\tau$ and $t$, which are related to equation (3.1), where $2\pi r = \theta$.

According to equation (3.1), we have a $\beta$-line of the surface $\mathcal{M}$ of the form

$$t = \eta + \tau - \xi \equiv \beta(\tau, \xi, \eta), \quad (\tau, t) \in \mathcal{M} \tag{3.3}$$

with the initial data $(\xi, \eta) \in \mathcal{M}$.

Substituting (3.3) into (3.2), we obtain the equation of the phase integral curve of equation (3.1) on the surface $\mathcal{M}$ of the form

$$u = \tau, \quad v = rsin\left(\frac{\eta + \tau - \xi}{r}\right), \quad w = r + rsin\left(\frac{\eta + \tau - \xi}{r} - \frac{\pi}{2}\right) \tag{3.4}$$

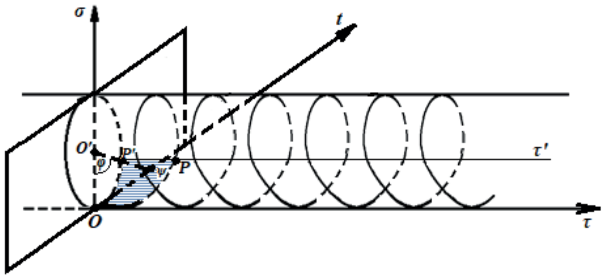


Figure 2: Projection of a circular helix onto a plane

with the parameter $\tau \in \mathbb{R}, (\xi, \eta) \in \mathcal{M}$ (see Fig. 2). The sweep function is the projection of the circular helix (see Fig. 2) on the plane $\tau Ot$, where $O(0,0,0) = O$, $O^{'}(0,0,r) = O^{'}$, $P^{'}(0,t,\sigma) = P^{'}$, $P(\tau, t, \sigma) = P$, $\psi = \tan\varphi$, $[0,\psi] \subset (Ot) = \mathbb{R}$, $\varphi = \angle OO^{'}P^{'}$.

Equations (3.4) are parametric equations of a helix, where $\tau$ is the length of displacement of a point $P(u, v, w)$ along the cylinder generatrix, passing through this point, and $t = t(v, w)$ represents the arc length $s$ of the phase circle corresponding to the polar angle $\varphi = \eta + \tau - \xi$ in radians. Obviously,

$$t = \eta + (\beta)\int_{\xi}^{\tau} \sqrt{v'^2(\tau) + w'^2(\tau)}d\tau \equiv \beta(\tau, \xi, \eta), (\tau, t) \in \mathcal{M}, \tag{3.5}$$

with the initial point $(\xi, \eta) \in \mathcal{M}$.

Thus, from the geometric interpretation of equation (3.1) in the plane $\mathbb{R}^2$ we have switched to its kinematic interpretation in the three-dimensional space $\mathbb{R}^3$. The independent variable $\tau$ in $\mathbb{R}^2$ is one of the coordinates of the point $(\tau, t)$, and in $\mathbb{R}^3$ space becomes a time parameter characterizing the motion of the point along a helical line drawn on a cylindrical surface.

The other dependent coordinate $t$ of equation (3.1) on the plane $\mathbb{R}^2$, in space $\mathbb{R}^3$ characterizes the motion of a point depending on time $\tau$ (numerically equal to the displacement along the cylinder's generatrix) along the path of a kind of helical spiral located on the cylinder, moreover, the phase trajectory $t$ is a circle $S$ obtained by projecting a helix in the direction of the $\tau$ axis.

In general, on $\mathcal{M}$, the $\beta$-characteristics of the operator $D$ have the following properties

$$\beta(\tau + \theta, \xi, \eta) = \beta(\tau, \xi + \theta, \eta) = \beta(\tau, \xi, \eta + \omega) - \omega = \beta(\tau, \xi, \eta) = \beta(\tau, \zeta, \beta(\zeta, \xi, \eta)). \tag{3.6}$$

It is important to have flat equivalents of helical lines (3.3)–(3.5) for quantitative analysis of system (2.6) (see Fig. 3).

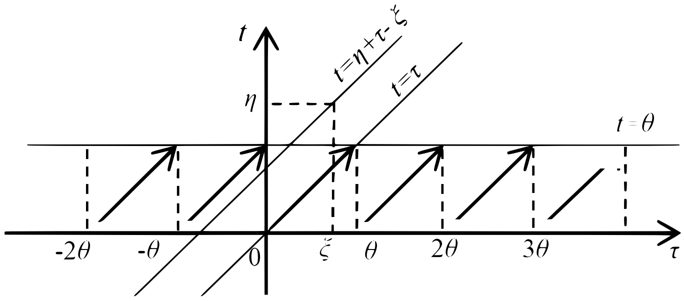


Figure 3: Cycloidal projection of helical curves

We consider the cycloidal sweep of $\chi$-mapping of a $\theta$-circular infinite cylindrical surface to get an idea of the equation of the helix $t = \beta(\tau, 0, 0) \equiv \beta^*(\tau)$ on the

$(\tau, t) \in \mathbb{R}^2$ plane. This surface is obtained by cutting the cylinder along its lower generatrix and unfolding it upward onto the plane by shifting the phase circle $S$ without sliding. Then, the cylindrical surface turns into an infinite strip $\Pi_0$ with boundaries $\tau = 0$ and $\tau = \theta$.

In the $\chi$-mapping, the areas, angles and lengths of the arcs are preserved, and the arcs of the helix are straightened and become segments. For example, the equation of the helical line $t = \beta^*(\tau)$ originating from the initial point $O$ of space $(u, v, w) \in \mathbb{R}^3$ on the plane has the form $t = \theta\{\theta^{-1}\tau\} \equiv \}(\tau)$, where $\{\tau\}$ is the fractional part of the number $\tau$. The graph of this jump function is well-known (Fig. 3).

The quantitative analysis related to the issue of integrating functions of variables $(\tau, t)$ along the helical lines on the plane $\mathbb{R}^2$ is carried out based on the functions $\}(\tau)$ and

$$](\tau) = \tau - \}(\tau) \equiv \theta[\theta^{-1}, \tau]$$

where $[\tau]$ is the integer part of the number $\tau$.

Note that the fractional partial function $\}(\tau)$ is a smooth, $\theta$-periodic on $\mathcal{M} \subset \mathbb{R}$, and the $\theta$-step function $](\tau)$ is a function identical to zero, moreover, they are summands of the expansion of the diagonal function $t = \tau \equiv ](\tau) + \}(\tau)$ on the plane $\mathbb{R}^2$. The results of the integration of functions over $\xi \in \mathbb{R}$ in the limits from $\tau^0$ to $\tau$ are presented in terms of the first characteristic integrals of the form

$$\eta = \beta(\xi, \tau, t), \quad \tau^0 \xrightarrow{\xi} \tau, \quad (\tau, t) \in \mathcal{M}, \tag{3.7}$$

which relate to the first integrals $\eta = \delta(\xi, \tau, t)$ of the operator $D$ by the relation

$$\beta(\xi, \tau, t)|_{(\tau,t)\in\mathcal{M}} = \delta(\xi, \tau, t) - ](\xi - \tau)|_{(\tau,t)\in\mathbb{R}^2}, \quad \xi \in \mathbb{R}, \tag{3.8}$$

on the plane $(\tau, t) \in \mathbb{R}^2$.

Here we tried to consider the recommendations of [22] on a reasonable definition of the phase space of a vector field based on the specifics of the statement problem. In this case, the specifics of the problem include: 1) the vector field describes the differentiation operator of functions along its direction and, 2) on the manifold where the differentiation operator was defined, this vector field must allow periodic integral curves with a given period, although it has only linear solutions in the plane.

Information about the differentiation operator can be obtained from [5] and [15]. Based on the $T$-periodic characteristics (3.3)–(3.5) of equations (3.1), the characteristics

$$t_j = \beta_j(\tau, \xi, \eta_j), \quad (\tau, t_j) \in \mathcal{M}$$

of equation (3.1) with initial data $(\xi, \eta_j) \in \mathcal{M}$, $j = \overline{1, m}$ were obtained. Then, the Cartesian product of these characteristics is a vector characteristic

$$t = (\beta_1(\tau, \xi, \eta_1), ..., \beta_m(\tau, \xi, \eta_m)) \equiv \beta(\tau, \xi, \eta), \quad (\tau, t) \in \mathcal{M}^m, \tag{3.9}$$

of system (2.4) with initial data $(\xi, \eta) \in \mathcal{M}^m$, where $\eta = (\eta_1, ..., \eta_m)$.

The $\beta$-characteristics (3.9) of system (2.4) have the properties of $(\theta, \omega)$-periodicity and a group of form (3.6).

The first characteristic integrals of system (2.4) by (3.7) and (3.9) are defined in the form

$$\eta = \beta(\xi, \tau, t), (\tau, t) \in \mathcal{M}^m, \tag{3.10}$$

with the parameters $(\xi, \eta) \in \mathcal{M}^m$.

The connection of helical characteristic integrals (3.10) with the linear Euclidean characteristic integrals

$$\eta = \delta(\xi, \tau, t), \quad (\tau, t) \in \mathbb{R} \times \mathbb{R}^m = \mathbb{R}^{1+m}$$

following (3.8) and (3.9) is determined by the relation

$$\beta(\xi,\tau,t)|_{(\tau,t)\in\mathcal{M}} = \delta(\xi,\tau,t)|_{(\tau,t)\in R^{1+m}}-](\xi-\tau), \quad \xi\in\mathbb{R}, \tag{3.11}$$

Thus, the following statement is justified.

**Theorem 3.1.** *Characteristic equation (2.4) of the diagonal differentiation operator $D$ on an infinite cylindrical surface $\mathcal{M}^m$ has $\beta$-characteristics (3.9) which have the properties of $(\theta,\omega)$-periodicity and group (3.6). Moreover, its corresponding first characteristic integrals (3.10) through $\delta$-characteristics equation (2.4) in the Euclidean space $\mathbb{R}^{1+m}$ are related to relation (3.11).*

## 4 Reducibility of a multiperiodic linear homogeneous system with a diagonal differentiation operator

We consider a multiperiodic linear homogeneous system with a diagonal differentiation operator in matrix form (2.6). The characteristic equation (2.4) of the operator $D$, its $\beta$-characteristics (3.9) with properties (3.6) and the first integrals

$$\eta=\beta(\xi,\tau,t), \quad (\xi,\eta)\in\mathcal{M}^m, \quad D\beta(\xi,\tau,t)=0 \tag{4.1}$$

are defined on the cylindrical surface $\mathcal{M}^m$, where $(\xi,\eta)$ is the initial point of the characteristic $t=\beta(\tau,\xi,\eta)$, which is smooth in all arguments, having the properties of $(\theta,\omega)$-periodicity and the group given in (4.1), and the first integrals $\eta=\beta(\xi,\tau,t)$ are defined based on the characteristics.

The *matricant* $X(\tau,t)$ of equation (2.6) is uniquely determined by the integral equation below and has the properties of $\omega$-periodicity in $t$ and smoothness in $(\tau,t)$:

$$\begin{aligned} &X(\tau,t)=E+\int_0^\tau P(\zeta,\beta(\zeta,\tau,t))X(\zeta,\beta(\zeta,\tau,t))d\zeta,\\ &X(\tau,t+\omega)=X(\tau,t)\in C^{(1,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m), \quad detX(\tau,t)\neq 0,\end{aligned} \tag{4.2}$$

where $E$ is the unit matrix.

The *monodromy matrix* $X(\theta,\beta(0,\tau,t))$ of equation (2.6) satisfies the equation

$$X(\tau+\theta,t)=X(\tau,t)X(\theta,\beta(0,\tau,t)). \tag{4.3}$$

Obviously, both parts of equality (2.4) satisfy equation (2.6) and for $\tau=0$, by virtue of (4.1), we have the same initial condition. Hence, from the unique solvability of the initial problem for equation (2.6), we have identity (4.3).

If $\Phi(\tau,t)$ is any fundamental solution with a $\omega$-periodic initial value of equation (2.6), then it can be represented in the form

$$\begin{aligned} &\Phi(\tau,t)=X(\tau,t)\Phi(0,\beta(0,\tau,t)),\\ &\Phi(0,t+\omega)=\Phi(0,t), \quad det\Phi(0,t)\neq 0,\end{aligned} \tag{4.4}$$

which has the following properties:

$$\begin{aligned} &\Phi(\tau+\theta,t)=\Phi(\tau,t)\cdot\Phi^{-1}(0,\beta(0,\tau,t))X(\theta,\beta(0,\tau,t))\Phi(0,\beta(0,\tau,t)),\\ &\Phi(\tau,t+\omega)=\Phi(\tau,t)\in C^{(1,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m),\end{aligned} \tag{4.5}$$

The matrix

$$U(\beta(0,\tau,t))=\Phi^{-1}(0,\beta(0,\tau,t))X(\theta,\beta(0,\tau,t))\Phi(0,\beta(0,\tau,t)) \tag{4.6}$$

is called the *main matrix* of the solution (4.4). Next by the fundamental solution, we mean a solution with a $\omega$-periodic in $t$ a *nonsingular* initial matrix.

Let us show that representations (4.4) and (4.5) are valid.

Indeed, both the right and left sides of these equalities satisfy equation (2.6) and, by virtue of (4.1)–(4.3), turn into identities in $\tau = 0$. Therefore, based on the unique solvability of the initial problem for system (2.6), we have the identities for $\tau \in \mathbb{R}$. This is exactly what is needed to be proved.

Thus, the following theorem is proved.

**Theorem 4.1.** *Any fundamental solution $\Phi(\tau, t)$ of system (2.6) has the property*

$$\Phi(\tau+\theta, t) = \Phi(\tau, t) U(\beta(0, \tau, t)) \tag{4.7}$$

*with its basic matrix $U(\beta(0, \tau, t))$ that is defined by relation (4.6).*

Along with system (2.6), we consider another similar multiperiodic system

$$\begin{aligned} &DY = Q(\tau, t)Y, \\ &Q(\tau+\theta, t+\omega) = Q(\tau, t) \in C_{\tau,t}^{(0,e)}(\mathbb{R}\times\mathbb{R}^m) \end{aligned} \tag{4.8}$$

with the matricant $Y(\theta, t)$ having the properties of $(\theta, \omega)$-periodicity of the form

$$\begin{aligned} &Y(\tau+\theta, t) = Y(\tau, t)\cdot Y(\theta, \beta(0, \tau, t)), \\ &Y(\tau, t+\omega) = Y(\tau, t) \in C_{\tau,t}^{(1,e)}(\mathbb{R}\times\mathbb{R}^m). \end{aligned} \tag{4.9}$$

**Definition 1.** Systems (2.6) and (4.11) are called equivalent if there exists a matrix $T(\tau, t)$ having properties

$$T(\tau+\theta, t+\omega) = T(\tau, t) \in C_{\tau,t}^{(1,e)}(\mathbb{R}\times\mathbb{R}^m), \quad detT(\tau, t) \neq 0, \tag{4.10}$$

such that linear substitution

$$X = T(\tau, t)Y \tag{4.11}$$

transforms system (2.6) to system (4.11), where $Y$ has properties (4.9).

**Theorem 4.2.** *For systems (2.6) and (4.8) to be equivalent, it is necessary and sufficient that they admit fundamental solutions of $\Phi(\tau, t)$ and $\Psi(\tau, t)$ with a common basic matrix:*

$$\begin{aligned} &U(\beta(0, \tau, t)) = V(\beta(0, \tau, t)), \\ &\Phi(\tau+\theta, t) = \Phi(\tau, t)U(\beta(0, \tau, t)), \\ &\Psi(\tau+\theta, t) = \Psi(\tau, t)V(\beta(0, \tau, t)). \end{aligned} \tag{4.12}$$

*The existence of the basic matrix $V(\beta(0, \tau, t))$ follows by Theorem 4.1 by virtue of (4.7).*

*Proof.* Let systems (2.6) and (4.8) be equivalent. Then, by virtue of (4.10) and (4.11), we have a sequence of identities of the form

$$\Phi(\tau, t) = T(\tau, t)\ \Psi(\tau, t),$$

$$\Phi(\tau+\theta, t) = \Phi(\tau, t)U(\tau, t) = T(\tau, t)\Psi(\tau+\theta, t) = T(\tau, t)\Psi(\tau, t)\cdot V(\tau, t) = \Phi(\tau, t)V(\tau, t).$$

From here we get (4.12).

Inversely, let we have (4.12). Let us prove (4.11). Dividing the second identity of (4.12) by the third one, we obtain

$$\begin{aligned} \Phi(\tau+\theta, t)\Psi^{-1}(\tau+\theta, t) &= \Phi(\tau, t)U\left(\beta(0, \tau, t)\right)V^{-1}\left(\beta(0, \tau, t)\right)\Psi^{-1}(\tau, t) \\ &= D\Phi(\tau, t)\Psi^{-1}(\tau, t). \end{aligned} \tag{4.13}$$

Further, let the following equality hold:

$$\Phi(\tau,t)\Psi^{-1}(\tau,t)=T(\tau,t). \tag{4.14}$$

As can be seen from (4.13) matrix (4.14) is $(\theta,\omega)$-periodic, and

$$\Phi(\tau,t)=T(\tau,t)\Psi(\tau,t). \tag{4.15}$$

Since $\Phi(\tau,t)=X$ and $\Psi(\tau,t)=Y$ are arbitrary fundamental solutions of equations (2.6) and (4.8), respectively, then the established relation between them (4.15) defines linear transformation (4.11). □

The $(\theta,\omega)$-periodic system (2.6) can be considered as $(2\theta,\omega)$-periodic in $(\tau,t)\in\mathbb{R}\times\mathbb{R}^m$. Then,

$$X\left(2\theta,\beta(0,\tau,t)\right)=X^2\left(\theta,\beta(0,\tau,t)\right). \tag{4.16}$$

Eigenvalues of (4.16) for fixed $(\tau,t)$ are real, and the elementary divisors corresponding to negative eigenvalues are repeated an even number of times. This is important in further studies on the logarithm of the monodromy matrix in terms of real-valued matrix functions.

The function $F\left(\beta(\xi,\tau,t)\right)$ of the first characteristic integrals $\eta=\beta(\xi,\tau,t)$ is said to be constant along the diagonal $t=e\tau$, since $\beta(\xi,\tau,e\tau)=e\xi$.

**Theorem 4.3.** *Equation (2.1) is reduced by linear substitution (4.11) with (4.10) to the equation*

$$DZ=K\left(\beta(0,\tau,t)\right)Z \tag{4.17}$$

*with a diagonal constant matrix $K\left(\beta(0,\tau,t)\right)$:*

$$K(t+\omega)=K(t)\in C_t^{(e)}\left(\mathbb{R}^m\right),\quad |t-e\tau|<\varepsilon \tag{4.18}$$

*where $\varepsilon>0$ is a sufficiently small number.*

*Proof.* Under assumptions (4.1) and (4.8), equation (4.17) is $(\theta,\omega)$-periodic in $(\tau,t)$. Therefore, it is $(2\theta,\omega)$-periodic in $(\tau,t)$. For each fixed $(\tau,t)$, we have a fixed value $\eta=\beta(0,\tau,t)$ and suppose that

$$e^{2\theta K(\eta)}=X(2\theta,\eta), \tag{4.19}$$

hence, $X(2\theta,\eta)=E+\sum_{j=1}^{\infty}\frac{(2\theta)^j}{j!}K^j(\eta)$.

Since $X(2\theta,\eta)=X^2(\theta,\eta)$ according to (4.16), then the equality

$$K(\eta)=\frac{1}{2\theta}\ln X(2\theta,\eta) \tag{4.20}$$

is valid for every fixed value of the variable $\eta$.

In the case of differentiability $K\left(\beta(0,\tau,t)\right)$, the matricant of equation (4.17) has the form

$$Z=e^{\tau K(\beta(0,\tau,t))}$$

and the monodromy matrix is

$$W\left(\beta(0,\tau,t)\right)=e^{2\theta K(\beta(0,\tau,t))}. \tag{4.21}$$

By virtue of (4.18) and (4.21) we obtain the equality

$$W\left(\beta(0,\tau,t)\right)=X\left(2\theta,\beta(0,\tau,t)\right) \tag{4.22}$$

for the monodromy matrix of equation (4.17). The coincidence of the monodromy matrices (4.22) by Theorem 4.2 proves the equivalence of equations (2.6) and (4.17).

To complete the proof, it is required to show the validity of property (4.20).

Assuming, that $t = e\tau$, we have $\eta = \beta(0, \tau, e\tau)$ and the monodromy matrix of system (2.6) can be represented as

$$\begin{aligned} &X\left(2\theta, \beta(0, \tau, t)\right) = X(2\theta, 0) + \left[X\left(2\theta, \beta(0, \tau, t)\right) - X(2\theta, 0)\right] \\ &= X(2\theta, 0) + \Omega\left(\beta(0, \tau, t)\right), \quad \Omega(\eta + \omega) = \Omega(\eta) \in C_\eta^{(e)}(\mathbb{R}^m), \end{aligned} \tag{4.23}$$

where $\Omega(\eta) = X(2\theta, \eta) - X(2\theta, 0) \to 0$ for $\eta \to 0$.

Let the constant matrix $A$ reduce the matrix $X(2\theta, 0)$ to the Jordan normal form

$$A^{-1}X(2\theta, 0)A = J = \operatorname{diag}\left[\lambda_1 \mathrm{E}_1 + \delta_1 \mathrm{I}_1, \ldots, \lambda_l \mathrm{E}_l + \delta_l \mathrm{I}_l\right], \tag{4.24}$$

where $\lambda_1, \ldots, \lambda_l$ are real non-zero eigenvalues of the matrix $X(2\theta, 0)$, $E_j$ are unit matrices of dimension $n_j \times n_j$, and $I_j$ are over diagonal unit matrices, $n_1 + \ldots + n_l = n$; $\delta_j$ are sufficiently small positive constants, j=$\overline{1, l}$.

Then, by (4.24), equality (4.23) can be written in the form

$$\begin{aligned} A^{-1}X\left(2\theta, \beta(0, \tau, t)\right)A = J + F\left(\beta(0, \tau, t)\right), \quad F(\eta) = \left[f_{ij}(\eta)\right], \\ F(\eta) = A^{-1}\Omega(\eta)A \to 0, \ \eta \to 0; \quad F(\eta + \omega) = F(\eta) \in C_\eta^{(e)}(\mathbb{R}^m). \end{aligned} \tag{4.25}$$

According to Gershgorin's theorem [11, 12], we define circles $\Gamma_j$, in which lie the eigenvalues $\mu(\beta) = \mu_k(\beta) = \lambda_k + \nu_k(\beta)$ of the matrix $J + F(\beta)$ of form (4.25),

$$\Gamma_j : |\lambda_j + f_{jj}(\beta) - \mu(\beta)| \le \delta_j + \sum_{k=1, k\ne j}^{n} |f_{jk}(\beta)|, \ j = \overline{1, n}, \ \beta = \beta(0, \tau, t), \ |t - e\tau| < \varepsilon.$$

Obviously, $\beta(0, \tau, t) \to 0$ for $t \to e\tau$ and $f_{jk}(\eta) \to 0$ for $\eta \to 0$, and the constants $\delta_1, \ldots, \delta_i$ can be arbitrarily small. Then, the radii

$$\Delta_j(\beta) = \delta_j + \sum_{k=1, k\ne j}^{n} |f_{jk}(\beta)|$$

can be made arbitrarily small, and the centers $O_j(\beta) = \lambda_j + f_{jj}(\beta)$ can be brought closer to $\lambda_j$ with any accuracy by diminishing the number $\varepsilon > 0$.

Here, inequalities $|t - e\tau| < \varepsilon$ and the concept $t \to e\tau$ should be understood in the meaning that the coordinates $t_j$ of the vector $t$ represent the length of the arc $s_j$ of the circle $S_\Theta$, and $\tau$ is the length of the shift along the generatrix of the cylinder are approximately equal to $t_j \approx \tau$ in a narrow helical strip $\mathbf{B}_\varepsilon$ of width $2\varepsilon$ with a central helix line $t_j = \tau$.

It can be shown that any eigenvalue of $\mu\left(\beta(0, \tau, t)\right) = \mu(\beta)$ and $\mu(\beta)$ satisfy the estimates

$$|\mu(\beta)| \ge \frac{1}{3}r, \quad |\mu(\beta) - \lambda_j| \le \frac{2}{3}r,$$

where $\min\limits_{1\le j\le l} |\lambda_j| = r > 0$,

$$\delta_j < \varepsilon, \quad |f_{jk}\left(\beta(0, \tau, t)\right)| \le \varepsilon$$

for $(\tau, t) \in \mathbf{B}_\varepsilon$ and $\varepsilon = \dfrac{r}{3n}$ for all $j, k = \overline{1, n}$.

Thus, all eigenvalue $\mu(\beta)$ of the matrix $J + F(\beta)$ will lie in one of the Gershgorin circles $\Gamma_j$, which do not intersect and do not cover the zero of the complex plane at a sufficiently small $\varepsilon > 0$.

Then, there is a closed contour $\gamma$, which includes all Gershgorin circles containing the eigenvalues of $\mu(\beta)$, but not covering the zero of the complex plane.

Consequently by [7], the logarithm $\ln[J+F(\beta)]$ is represented by the Cauchy integral formula

$$\ln[J+F(\beta)] = \frac{1}{2\pi i}\int\limits_{\gamma}(\lambda E - J - F(\beta))^{-1}\ln\lambda d\lambda, \tag{4.26}$$

and the eigenvalues of $\mu_j(\beta)$ inherit all the properties of the matrix related to $(\theta,\omega)$-periodicity and smoothness:

$$\mu_j\left(\beta(0,\tau+\theta,t+\omega)\right) = \mu_j\left(\beta(0,\tau,t)\right) \in C^{(1,e)}_{\tau,t}\left(\mathbb{R}\times\mathbb{R}^m\right), \quad 1\le j\le n. \tag{4.27}$$

Thus, from (4.26) and (4.27) follows (4.18), since by virtue of (4.25) the monodromy matrix $F(2\theta,\beta)$ is logarithmic. □

## 5 Reducibility of a conditionally periodic ordinary differential equation

A linear conditionally periodic matrix equation is given by (2.1). Since by [15] along the characteristics $t=\beta(\tau,\xi,\eta)$ of the equation $dt = ed\tau$, the operator $D$ of differentiation of a function $x(\tau,t)\in C^{(1,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m)$ transforms into the ordinary operator $\dfrac{d}{d\tau}$ of differentiation of the function $x\left(\tau,\beta(\tau,\xi,\eta)\right)$ and inversely, it follows that for $t = e\tau = \beta(\tau,0,0)$ systems (2.6) and (2.1) are equivalent.

According to Bohr's theorem [6] on the relation between continuous periodic functions of many variables and continuous almost periodic functions of one variable, the matrix $P(\tau,t)$ given in (2.6) and (2.1) for $t=e\tau$ becomes a quasiperiodic matrix $P(\tau,e\tau)$ with constant basis $\left(\omega_0^{-1},\omega_1^{-1},\ldots,\omega_m^{-1}\right)$ with incommensurable with each other components

$$\nu_0=\omega_0^{-1}, \nu_1=\omega_1^{-1},\ldots,\nu_m=\omega_m^{-1}.$$

Thus, along the diagonal $t=e\tau$ of the independent variables $(\tau,t)\in(\mathbb{R}\times\mathbb{R}^m)$ both equation (2.6) and (2.1) can be represented as the equation

$$\begin{aligned}&\frac{d}{d\tau}Y = P(\tau,e\tau)Y, \quad Y = X(\tau,e\tau),\\ &P(\tau+\theta,e\tau+k\omega) = P(\tau,e\tau)\in C^{(0,e)}_{\tau,e\tau}(\mathbb{R}\times\mathbb{R}^m),\end{aligned} \tag{5.1}$$

where $\sum_{j=0}^m k_j\omega_j\neq 0$, $\sum_{j=0}^m |k_j|\neq 0$, $k_j\in\mathbb{Z}$, $j=\overline{0,m}$, $\omega_0=\theta$, $\omega=(\omega_0,\omega_1,\ldots,\omega_m)$.

As a result, by Theorem 4.3 applied to equation (5.1), we have the following reducibility theorem.

**Theorem 5.1.** *Conditionally periodic equation* (5.1) *is reduced by the conditionally periodic linear substitution*

$$\begin{aligned}&Y = T(\tau,e\tau)Z, \quad detT(\tau,e\tau)\neq 0,\\ &T(\tau+\theta,e\tau+\omega) = T(\tau,e\tau)\in C^{(1,e)}_{\tau,e\tau}\left(\mathbb{R}\times\mathbb{R}^m\right)\end{aligned} \tag{5.2}$$

*to the equation*

$$\frac{d}{d\tau}Z = AZ \tag{5.3}$$

*with the constant matrix $A=K(0)$, where the transformation matrix $T(\tau,t)$ is defined by relations* (4.14), (4.11) *and* (4.10), *and the matrix $K(\eta)$ by relations* (4.19).

*Proof.* First of all, it should be noted that Theorem 5.1 is a direct corollary of Theorem 4.3. Therefore, its justification is contained in the formulation of the theorem itself, with reference to transformation (4.11), which yields (5.2), and to equation (4.17), from which (5.3) follows. □

## 6 Conclusion

The general theory of almost-periodic functions of one variable and their connection with periodic functions of many variables originates from the works of P. Bohl, G. Bohr, A. Bezikovitch, V.V. Stepanov, S. Bochner, B.M. Levitan, and others.

Applied aspects of this theory in the form of differential models of oscillatory phenomena and constructive-technical processes were first investigated in the works of A.A. Andronov, N.N. Bogolyubov and N.M. Krylov, G. Bohr and O. Neugebauer, S.L. Sobolev, and others.

The problem of reducibility of linear almost periodic systems of ordinary differential equations probably appeared in these years in connection with the needs of practice and the tasks of simplifying the qualitative study of such equations as the basis of the theory of oscillations.

Research on this problem has been particularly intensified after creation of the KAM theory in 1954-1966, the creators of which are A.N. Kolmogorov, V.I. Arnold and J. Moser.

In his work [4], V.I. Arnold raised questions about the reducibility of general conditionally periodic linear equations. Further, he noted that the problem of the reducibility of linear equations with conditionally periodic coefficients naturally appears in the study of neighborhoods of invariant tori of autonomous systems that admit conditionally periodic motions.

Although this problem was formulated in terms of ordinary differential equations, it was consonant with the formulation of the reducibility problem for linear multiperiodic systems of partial differential equations by V.Kh. Kharasakhal [15].

In this study, it was established that in the Euclidean space there exists an infinite circular cylindrical surface on which the vector field of the differentiation operator with respect to the diagonal variables admits periodic characteristics [24, 25]. Using these helical characteristics, the reducibility of multiperiodic linear systems with a diagonal differentiation operator to systems with constant coefficients along the diagonal was demonstrated in a neighbourhood of the diagonal. The logarithm of the monodromy matrix was shown to exist by means of Gershgorin's localization of eigenvalues. By subsequently passing to the diagonal of independent variables, the principal result on the reducibility of conditionally periodic linear systems of ordinary differential equations was obtained in a sufficiently general situation.

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