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# KOLMOGOROV-TYPE INEQUALITIES ON THE WHOLE LINE OR HALF LINE AND THE LAGRANGE PRINCIPLE IN THE THEORY OF EXTREMUM PROBLEMS

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**Abstract.** The paper is devoted to solving extremal problems related to Kolmogorov-type inequalities. All solutions are obtained with the help of the so-called Lagrange principle, which is a generalization of the Lagrange multiplier rule.

## 1 Preliminaries

**Extremal problems.** Let $X$ be a set, $C$ a subset of $X$, $f$ a function defined on $X$. Minimization (maximization) of $f(x)$ where $x \in C$ will be formally written as

$$f(x) \to \min\ (\max), \quad x \in C. \qquad (\mathcal{P})$$

One says in this case that $(\mathcal{P})$ is an *extremal problem with constraint* $x \in C$. If $X = C$, problem $(\mathcal{P})$ is called a *problem without constraints*. Any point of $C$ is said to be *admissible* in problem $(\mathcal{P})$. An admissible point $\widehat{x}$ is a *solution of problem* $(\mathcal{P})$ or *absolute minimum (maximum) in problem* $(\mathcal{P})$, if $f(x) \geq f(\widehat{x})$ ($f(x) \leq f(\widehat{x})$) for all $x \in C$. The maximum value of $f$ in maximization problem $(\mathcal{P})$ is called the *value of problem* $(\mathcal{P})$. We denote it by $S_{\max}(\mathcal{P})$.

The following family of extremal problems will be considered:

**Kolmogorov-type inequalities on the whole line or half line.** Let $T$ be $\mathbb{R}$ or $\mathbb{R}_+$, $n \in \mathbb{N}$, $1 \leq p, q, r \leq \infty$, $\mathcal{W}^n_{pr}(T)$ the space all functions $x(\cdot) \in L_p(T)$ with $(n-1)$th derivative locally absolutely continuous on $T$ and $x^{(n)}(\cdot) \in L_r(T)$, $k$ an integer, $0 \leq k < n$. Consider the following family of problems defined on the space $\mathcal{W}^n_{pr}(T)$:

$$f_0(x(\cdot)) = \|x^{(k)}(\cdot)\|_{L_q(T)} \to \max, f_1(x(\cdot)) = \|x(\cdot)\|_{L_p(T)} \leq 1, f_2(x(\cdot)) = \|x^{(n)}(\cdot)\|_{L_r(T)} \leq 1. \qquad (P_T(k,n,p,q,r))$$

Problem $(P_T(k,n,p,q,r))$ has an equivalent formalization as a problem without constraints:

$$f(x(\cdot)) = \frac{\|x^{(k)}(\cdot)\|_{L_q(T)}}{\|x(\cdot)\|^{\alpha}_{L_p(T)} \|x^{(n)}(\cdot)\|^{\beta}_{L_r(T)}} \to \max, \quad x(\cdot) \in \mathcal{W}^n_{pr}(T),$$

where $\alpha = \frac{(n-k-1/r+1/q)}{(n-1/r+1/p)}, \beta = 1-\alpha$. The value of problem $(P_T(k,n,p,q,r))$ we denote by $K_T(k,n,p,q,r)$ and call *Kolmogorov's constant* of $(P_T(k,n,p,q,r))$.

**Lemma.** *If $p = r$, $q = \infty$, then a solution of the problem $(P_T(k,n,p,q,r))$ can be reduced to the following problem with one constraint:*

$$f_0(x(\cdot)) = x^{(k)}(0) \to \max, \quad f_1(x(\cdot)) = \|x(\cdot)\|^p_{L_p(T)} + \|x^{(n)}(\cdot)\|^p_{L_p(T)} \le 1. \qquad (P'_T(k,n,p,q,r))$$

*Let $A_T(k,n,p)$ denote the value of the problem $(P'_T(k,n,p,q,r))$. Then the equality*

$$K_T(k,n,p,\infty,p) = A_T(k,n,p)\left(\left(\frac{np}{(n-k)p-1}\right)^{1-k/n-1/(np)}\left(\frac{np}{kp+1}\right)^{k/n+1/(np)}\right)^{1/p}$$

*holds.*

The first problem (namely the problem $(P_{\mathbb{R}_+}(1,2,\infty,\infty,\infty))$) of the series $(P_T(k,n,p,q,r))$ was solved by E. Landau in 1913. Then in 1914 J. Hadamard solved the problem $(P_{\mathbb{R}}(1,2,\infty,\infty,\infty))$. In 1938 A. Kolmogorov generalized the result of Hadamard and found all constants $K_{\mathbb{R}}(k,n,\infty,\infty,\infty)$ for $n \ge 2$, $0 < k < n$. This result remains one of the most remarkable in this area, and problems from family $(P_T(k,n,p,q,r))$ are usually called now *Kolmogorov-type inequalities* (or sometimes *Landau–Kolmogorov type inequalities*) *on the whole line or half line.*

In this survey all problems from this family will be considered from the point of view of only one general principle of the extremal theory.

**Lagrange principle.** The idea how to solve extremal problems with constraints was first expressed by Lagrange. He wrote in [1], that a procedure of solution of a smooth finite dimensional problem with equality constraints, can be reduced to the following *general principle: "If a function of several variables should attain maximum or minimum, and these variables satisfy one or several equations, then it will suffice to add to the proposed function the functions that should be zero, each multiplied by an undetermined quantity and then to look for the maximum or the minimum as if the variables were independent; the equations that one will find, combined with the given equations, will serve to determine all the unknowns".*

Let us express this text in mathematical language. Consider the problem:

$$f_0(x) \to \text{extr}, \quad f_i(x) = 0,\ 1 \le i \le m,\ x \in \mathbb{N}.$$

The function $\mathcal{L}(x,\bar\lambda) = \sum_{i=0}^{m} \lambda_i f_i(x)$ is called the *Lagrange function* of the problem, the vector $\bar\lambda = (\lambda_0, \dots, \lambda_m)$ is called *a collection of the Lagrange multipliers*. The necessary condition of a local extremum at a point $\widehat{x}$ in the problem $\mathcal{L}(x,\bar\lambda) \to \text{extr}$ («when variables are independent») by virtue of the Fermat theorem is the following *stationarity condition*

$$\mathcal{L}_x(\hat x, \bar\lambda) = 0 \Leftrightarrow \sum_{i=0}^{m} \lambda_i f'_i(\hat x) = 0$$

for $\bar\lambda \ne 0$. There are $n$ equations with $(n+m+1)$ unknowns ($x \in \mathbb{N}$, $\bar\lambda \in \mathbb{R}^{m+1}$, but we can normalize the collection of the Lagrange multipliers, for example, as follows

$\sum_{i=0}^{m} |\lambda_i| = 1$). If these equations are completed by the "given equations" $f_i(x) = 0,\ 1 \le i \le m$, we obtain $(n + m + 1)$ equations with $(n + m + 1)$ unknowns. Such systems in general case has a finite number of solutions and one can find the solution of the problem among them.

We will use below the following generalization of the Lagrange's idea for more general problems (which we call the Lagrange principle). According to this principle, when one searches necessary conditions of an extremal problem, in which smoothness is interlaced with convexity, it suffices to construct the Lagrange function of the problem and then to apply necessary conditions for minimum of the Lagrange function *as if the variables were independent.* Equations which we obtain by this procedure combined with the given equations, nonnegativity conditions and conditions of complementary slackness for the Lagrange multipliers corresponding to inequality constraints will serve to determine all unknowns.

Now we begin with formulating *necessary conditions for problems without constraints* (Propositions 1 – 4, we formulate them exactly) and then we formulating the *Lagrange principle* (**LP**) *for problems with constraints.* We shall use the LP as a rule heuristically, so we formulate them not as theorems, but *as principles.* Exact formulations and proofs the reader can find in [2].

## Necessary conditions for problems without constraints

**Proposition 1 (necessary conditions for smooth problems, [2]).** *Let $X$ be a normed space, $V$ a neighborhood of $\hat{x} \in X$, $f_0$ be a real-valued function on $V$ differentiable at $\hat{x}$. If*
*hatx is a local minimum of the problem*

$$f_0(x) \to \text{extr},$$

*then the stationarity condition*

$$f_0'(\hat{x}) = 0$$

*holds.*

**Proposition 2 (necessary conditions for problems of calculus of variations (Bolza problems), [2]).** *Let $X$ be the space $C^1([t_0, t_1], \mathbb{R}^n)$ of continuously differentiable vector-functions, $\hat{x}(\cdot) \in X$, $V_1$ be a neighborhood of the graph $\Gamma_{\hat{x}(\cdot)} = \{(t, \hat{x}(t), \dot{\hat{x}}(t)) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \mid t \in [t_0, t_1]\}$, $L : V_1 \to \mathbb{R}^{2n+1}$ be a continuously differentiable function, $V_2$ be a neighborhood of the point $(x(t_0), x(t_1)) \in \mathbb{R}^{2n}$ and $l : V_2 \to \mathbb{R}$ be a continuously differentiable function. If $\hat{x}(\cdot)$ is a local minimum in the space $X$ of the problem*

$$f_0(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt + l(x(t_0), x(t_1)) \to \min,$$

*(Bolza problem), then $\widehat{L}_{\dot{x}}(\cdot) \in C^1([t_0, t_1], \mathbb{R}^n)$ and*
*a) Euler equation $-\frac{d}{dt}\widehat{L}_{\dot{x}}(t) + \widehat{L}_x(t) = 0$,*

*b) transversality conditions* $\widehat{L}_{\dot{x}}(t_i) = (-1)^i \widehat{l}_{x(t_i)}$, $i = 0, 1$,
*hold, where*

$$\widehat{L}_{\dot{x}}(t) = L_{\dot{x}}(t, \widehat{x}(t), \dot{\widehat{x}}(t)),\ \widehat{L}_x(t) = L_x(t, \widehat{x}(t), \dot{\widehat{x}}(t)),\ \widehat{l}_{x(t_i)} = \frac{\partial l(\widehat{x}(t_0), \widehat{x}(t_1))}{\partial(x(t_i))},\ i = 0, 1.$$

**Proposition 3 (criterion of minimum for elementary optimal control problems, [2]).** *Let* $U \subset \mathbb{R}^r$, $L : \mathbb{R} \times U \to \mathbb{R}$ *be a continuous function,* $\mathcal{U}$ *be the set of all piece-wise continuous functions from* $[t_0, t_1]$ *to* $U$*. Then* $\widehat{u}(\cdot) \in \mathcal{U}$ *is a solution of the problem*

$$f_0(u(\cdot)) = \int_{t_0}^{t_1} L(t, u(t))dt \to \min,\ u(t) \in U,$$

*if and only if the minimum condition*

$$c)\ L(t, u) \geq L(t, \widehat{u}(t))\ \forall u \in U\ t \in [t_0, t_1]\ a.e.$$

*holds.*

**Proposition 4 (criterion of minimum for convex problems, [2]).** *Let* $U$ *be a convex subset of* $\mathbb{R}^n$, $f_0 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a convex function. Then* $\widehat{u}$ *is a solution of the problem*

$$f_0(u) \to \min,\ u \in U,$$

*if and only if (by definition) c) minimum condition* $f_0(u) \geq f_0(\widehat{u})\ \forall u \in U$ *holds, or if* $U = \mathbb{R}^n$*, then* $\alpha)\ 0 \in \partial f_0(\widehat{u})$.

For solving our extremal problems we shall use the following conditions of extremum: $\alpha$) stationarity conditions, $\beta$) nonnegativity conditions, $\gamma$) conditions of complementary slackness, a) Euler equations, b) transversality conditions, and c) minimum conditions.

Let us illustrate the Lagrange principle for *a finite dimensional smooth problem with equality and inequality constraints*:

$$f_0(x) \to \min, \quad f_i(x) \leq 0, \quad 1 \leq i \leq m',\ f_i(x) = 0,\ m' + 1 \leq i \leq m. \qquad (P_{Lagr})$$

The Lagrange function of this problem has the form: $\mathcal{L}(x, \bar{\lambda}) = \sum_{i=0}^{m} \lambda_i f_i(x)$ where $\bar{\lambda} = (\lambda_0, \ldots, \lambda_m)$. Application of Proposition 1 to the problem of minimization of the Lagrange function without constraints together with conditions of nonnegativity and complementary slackness leads to the following relations:

$$\alpha)\ \mathcal{L}_x(\widehat{x}, \bar{\lambda}) = 0 \ \Leftrightarrow\ \sum_{k=0}^{m} \lambda_k f_k'(\widehat{x}) = 0,\ \beta)\ \lambda_k \geq 0,\ 0 \leq k \leq m',\ \gamma)\ \lambda_k f_k(\widehat{x}) = 0,\ 1 \leq k \leq m'.$$

Now we formulate the Lagrange principle for five classes of problems.

**LP 1. Necessary conditions of extremum for a smooth problem with inequality and equality constraints.** *If an extremum to the problem*

$$f_0(x) \to \text{extr},\ f_i(x) \leq 0,\ 0 \leq i \leq m',\ f_i(x) = 0,\ m' + 1 \leq i \leq m,$$

*where $f_i$ are smooth functions defined on a normed space is attained at an element $\widehat{x}$, then the necessary conditions for this problem at the element $\widehat{x}$ correspond to the Lagrange principle.*

**LP 2. Necessary and sufficient conditions of extremum for a convex problem with inequality constraints.** *If $X$ is a locally convex vector space and an minimum to the problem*

$$f_0(x) \to \min,\ f_i(x) \le 0,\ 0 \le i \le m,\ x \in A,$$

*(where $f_i$ are convex functions defined on $X$ and $A \subset X$ is a convex set) is attained at an element $\widehat{x}$, then the necessary conditions at the element $\widehat{x}$ for this problem correspond to the Lagrange principle; the necessary conditions with $\lambda_0 \neq 0$ which correspond to Lagrange principle are sufficient.*

**LP 3. Necessary conditions of extremum for a problem of calculus of variations.** *If an extremum to the problem*

$$f_0(x(\cdot), u(\cdot)) \to \text{extr},\ f_i(x(\cdot), u(\cdot)) \le 0,\ 0 \le i \le m',$$

$$f_i(x(\cdot), u(\cdot)) = 0,\ m'+1 \le i \le m,\ \dot{x} = \varphi(t, x, u),$$

*(where $f_i(x(\cdot), u(\cdot)) = \int\limits_{t_0}^{t_1} L_i(t, x(t), u(t))dt + l_i(x(t_0), x(t_1))$ and $L_i$, $\varphi$, $l_i$ are smooth functions) is attained at a pair $(\widehat{x}(\cdot), \widehat{u}(\cdot))$, then the necessary conditions for this problem at the pair $(\widehat{x}(\cdot), \widehat{u}(\cdot)$ correspond to the Lagrange principle.*

The problem

$$f_0(u(\cdot)) \to \min, \quad f_i(u(\cdot)) \le 0,\, 1 \le i \le m', \quad f_i(u(\cdot)) = 0,\ m' + 1 \le i \le m,$$

where

$$f_i(u(\cdot)) = \int\limits_{t_0}^{t_1} L_i(t, u(t)dt,\ 0 \le i \le m,$$

is called a *Lyapunov problem.*

**LP 4. Necessary and sufficient conditions of extremum for a Lyapunov problem.** *If the absolute minimum in the problem*

$$f_0(u(\cdot)) \to \min,\ f_i(u(\cdot)) \le 0,\ 0 \le i \le m,\ u(t) \in U \quad a.e.,$$

*(where $f_i(u(\cdot)) = \int\limits_{t_0}^{t_1} L_i(t, u(t))dt$ are continuous functions defined on $[t_0, t_1] \times \mathbb{R}^r$ and $U \subset \mathbb{R}^r$) is attained at a function $\widehat{u}(\cdot) \in L_\infty([t_0, t_1], \mathbb{R}^r)$, then the necessary conditions for this problem at the function $\widehat{u}(\cdot)$ correspond to the Lagrange principle; the necessary conditions with $\lambda_0 \neq 0$ which correspond to Lagrange principle are sufficient.*

We call the problem, posed in the **LP 3** and completed by constraint $u(t) \in U \subset \mathbb{R}^r$, an *elementary problem of optimal control.*

**LP 5. Necessary conditions of extremum for a problem of optimal control.** *If an extremum to a problem of optimal control is attained at a pair $(\widehat{x}(\cdot), \widehat{u}(\cdot))$, then the* *necessary conditions for this problem at the pair* $(\widehat{x}(\cdot), \widehat{u}(\cdot)$ *correspond to the Lagrange principle.*

In the problem of optimal control which is *linear over phase coordinates* (when the differential constraints have the form

$$\dot{x} = A(t)x + F(t, u(t))$$

and the functional is

$$f(x(\cdot)) = \int_{t_0}^{t_1} (a(t)x(t) + L(t, u(t)))dt + l_0 x(t_0) + l_1 x(t_1))$$

the necessary conditions with $\lambda_0 \neq 0$ which correspond to the Lagrange principle are sufficient.

## 2 Solutions of problems from the family $(P_T(k, n, p, q, r))$

Solving of any such problem consists of the following steps:
1. Formalization (with some commentaries and writing out the Lagrange function).
2. Application of one of Propositions 1–4 to the problem of minimization of the Lagrange function as the problem without constraints.
3. Solving the equations obtained at Step 2.
4. Verification of the fact that among solutions obtained at Step 3 there exists a solution of the problem.

The first two examples have a propaedeutic character.
**Problem 1**: $(P_{\mathbb{R}_+}(0, 1, 2, \infty, 2))$

**1**. $$f_0(x(\cdot)) = x(0) \to \max,\ f_1(x(\cdot)) = \int\limits_{\mathbb{R}_+} x^2(t) \leq 1, \quad f_2(x(\cdot)) = \int\limits_{\mathbb{R}_+} \dot{x}^2(t)\, dt \leq 1. \qquad (P_1)$$

This is on one hand a smooth problem with inequality constraints and on the other hand this is a problem of calculus of variations. The Lagrange function of problem $(P_1)$ has the form (up to the constant $(-\lambda_1 - \lambda_2)$):

$$\mathcal{L}(x(\cdot), \bar{\lambda}) = -x(0) + \int\limits_{\mathbb{R}_+} (\lambda_1 x^2(t) + \lambda_2 \dot{x}^2(t))dt,\ \bar{\lambda} = (-1, \lambda_1, \lambda_2).$$

We denote a solution of problem $(P_1)$ by $\widehat{x}(\cdot)$.
**2.** Proposition 1 for minimization of the Lagrange function of smooth problem with constraints consists of the stationarity condition $\alpha)$ $\mathcal{L}_{x(\cdot)}(\widehat{x}(\cdot), \bar{\lambda}) = 0$. After differentiation of the Lagrange function we come to the identity:

$$-x(0) + \int\limits_{\mathbb{R}_+} (2\lambda_1 \widehat{x}(t)x(t) + 2\lambda_2 \dot{\widehat{x}}(t)\dot{x}(t))dt = 0 \quad \forall x(\cdot).$$

After integrating by parts and writing out $\beta), \gamma)$ conditions we obtain the following collection of relations

$$a)\ \lambda_2\ddot{\widehat{x}}(t) - \lambda_1\widehat{x}(t) = 0,\ b)\ 2\lambda_2\dot{\widehat{x}}(0) = -1,\ \beta)\ \lambda_i \geq 0,\ \gamma)\ \lambda_i(f_i(\widehat{x}(\cdot)) - 1) = 0,\ i = 1, 2. \tag{1}$$

They are a) Euler equations, b) transversality conditions, $\beta$) nonnegativity conditions and $\gamma$) conditions of complementary slackness.

**3.** It can be easily shown that system (1) has the following (unique) solution: $\widehat{x}(t) = \sqrt{2}e^{-t}$, $\lambda_i = \frac{1}{2\sqrt{2}}$, $i = 1, 2$.

The identity, which was written above, has the following explicit form:

$$x(0) = \int\limits_{\mathbb{R}_+} e^{-t}(x(t) - \dot{x}(t))dt\ \ \forall x(\cdot). \tag{1'}$$

Of course, it was possible to prove this identity directly.

**4.** Application of the Cauchy inequality to (1′) leads to the following estimate from above:

$$|x(0)| \leq 2\sqrt{\int\limits_{\mathbb{R}_+} e^{-2t}dt} = \sqrt{2}\,.$$

On the other hand, the function $\widehat{x}(t) = \sqrt{2}e^{-t}$ is admissible in our problem. Consequently,
$K_{\mathbb{R}_+}(0, 1, 2, \infty, 2) \geq \widehat{x}(0) = \sqrt{2}$. Thus $K_{\mathbb{R}_+}(0, 1, 2, \infty, 2) = \sqrt{2}$.

**Theorem 1.** $K_{\mathbb{R}_+}(0, 1, 2, \infty, 2) = \sqrt{2}$.

**Problem 2:** $(P_{\mathbb{R}}(0, 1, 2, \infty, 2))$

$$\mathbf{1}.\ \ f_0(x(\cdot)) = x(0) \to \max,\ \ f_1(x(\cdot)) = \int\limits_{\mathbb{R}} x^2(t)\,dt \leq 1,\ \ f_2(x(\cdot)) = \int\limits_{\mathbb{R}} \dot{x}^2(t)\,dt \leq 1. \tag{$P_2$}$$

**2.** Proposition 1 leads to relations similar to (1) (obtaining them is left to the reader). Solving these relations leads to the following identity (which is possible to check directly, moreover, it immediately follows from (1′)):

$$x(0) = \frac{1}{2}\int\limits_{\mathbb{R}} e^{-|t|}(x(t) - \dot{x}(t))dt \quad \forall x(\cdot). \tag{2}$$

Application of the Cauchy inequality to (2) leads to the estimate from above and the admissible function $\widehat{x}(t) = e^{-|t|}$ gives the estimate from below $K_{\mathbb{R}}(0, 1, 2, \infty, 2) \geq \widehat{x}(0) = 1$. Consequently,

**Theorem 2.** $K_{\mathbb{R}}(0, 1, 2, \infty, 2) = 1$.

There exists another way of solving of $(P_2)$ based on the Fourier transform. Let $Fx(\cdot)$ denote the Fourier transform of $x(\cdot)$. Parseval's inequality leads to the following reformulation of problem $(P_2)$:

$$\int_{\mathbb{R}} Fx(\tau)d\tau \to \max, \quad \int_{\mathbb{R}} Fx^2(\tau)d\tau \le 1, \quad \int_{\mathbb{R}} \tau^2 Fx^2(\tau)d\tau \le 1. \qquad (P_2')$$

This is a Lyapunov problem, which has a very simple solution by means of the Lagrange principle.

**Problem 3:** $(P_{\mathbb{R}_+}(1,2,\infty,\infty,\infty))$

**1**. $$y(0) \to \max,\ |x(t)| \le 1,\ \dot{x} = y,\ \dot{y} = u,\ |u(t)| \le 1 \quad \forall\, t \in \mathbb{R}_+. \qquad (P_3)$$

This is a problem of optimal control with phase constraints linear over the phase coordinates. Assume that a solution $\widehat{x}(\cdot)$ attains its minimum at $t = 0$ and its maximum at a point $T$. Then the Lagrange function has the following form

$$\mathcal{L}(x(\cdot), u(\cdot), \bar{\lambda}) = -y(0) + \mu_1 x(T) - \mu_0 x(0) + \int_0^T \big(q(t)(\dot{x}(t) - y(t)) + p(t)(\dot{y}(t) - u(t))\big)dt,$$

where $\bar{\lambda} = (\mu_0, \mu_1, p(\cdot), q(\cdot)),\ \mu_i \ge 0,\ i = 0, 1.$

**2.** Application of Proposition 1 to $\min\{\mathcal{L}(x(\cdot), \widehat{u}(\cdot), \bar{\lambda}) \mid x(\cdot)\}$ leads to the following identity

$$\dot{x}(0) = \mu_1 x(T) - \mu_0 x(0) + \int_0^T (q(t)(\dot{x}(t) - y(t)) + p(t)\dot{y}(t))dt.$$

Application of Proposition 3 to $\min\{\mathcal{L}(\widehat{x}(\cdot), u(\cdot), \bar{\lambda}) \mid |u(t)| \le 1\}$ gives the identity $\widehat{u}(t) = \operatorname{sgn} p(t)$.

After integrating by parts in the first identity one obtains the following relations: $a$) Euler equation and $b$) transversality conditions, namely

$a$) $\dot{p} = -q,\ \dot{q} = 0,$

$b$) $p(0) = -1,\ p(T) = 0,\ q(0) = -\mu_0,\ q(T) = -\mu_1 \Rightarrow p(t) = \frac{t}{T} - 1,\ \frac{1}{T} = \mu_1 = \mu_0.$

Thus, the exact form of the identity is the following:

$$\dot{x}(0) = \frac{x(T) - x(0)}{T} + \int_0^T p(t)\ddot{x}(t)dt \qquad (3)$$

(one can check this identity directly).

**3, 4.** From the identity one obtains the following estimate:

$$K_{\mathbb{R}_+}(1,2,\infty,\infty,\infty) \le \min_{T>0}\Big(\frac{2}{T} + \int_0^T \Big(1 - \frac{t}{T}\Big)dt\Big) = \min_{T>0}\Big(\frac{2}{T} + \frac{T}{2}\Big) = 2,\ (\widehat{T} = 2).$$

The equality is attained at the function $\widehat{x}(t) = 1 - \frac{(2-t)_+^2}{2}$.

**Theorem 3 (Landau, 1913, [3]).** $K_{\mathbb{R}_+}(1,2,\infty,\infty,\infty) = 2.$

**Problem 4:** $(P_{\mathbb{R}}(1,2,\infty,\infty,\infty))$

**1**. $$y(0) \to \max,\ |x(t)| \le 1,\ \dot{x} = y,\ \dot{y} = u,\ |u(t)| \le 1 \quad \forall\, t \in \mathbb{R}. \qquad (P_4)$$

**2.** Propositions 2 and 3 lead to an identity similar to (3)

$$\dot{x}(0) = \frac{x(T) - x(-T)}{2T} - \frac{1}{2}\int_{-T}^{T} (\operatorname{sgn} t - \frac{t}{T})\ \ddot{x}(t)\, dt. \qquad (4)$$

It is possible to check this identity directly.
**3, 4.** From this identity we obtain the following estimate

$$K_{\mathbb{R}}(1,2,\infty,\infty,\infty) \le \min_{T>0}(\frac{1}{T} + \frac{1}{2}\int_{-T}^{T}(1 - \frac{|t|}{T})dt) = \min_{T>0}(\frac{1}{T} + \frac{T}{2}) = \sqrt{2},\ (\widehat{T} = \sqrt{2}).$$

The equality is attained at the function $\widehat{x}(t) = (1 - \frac{(\sqrt{2}-|t|)_+^2}{2})\operatorname{sgn} t$.

**Theorem 4 (Hadamard, 1914, [4]).** $K_{\mathbb{R}}(1,2,\infty,\infty,\infty) = \sqrt{2}$.

**Problem 5:** $(P_{\mathbb{R}_+}(1,2,2,2,2))$
**1.** It is possible to formalize the problem $(P_{\mathbb{R}_+}(1,2,2,2,2))$ in the following equivalent form

$$\int_{\mathbb{R}_+} \dot{x}^2(t)dt \to \max, \quad \int_{\mathbb{R}_+} \left(x^2(t) + (\ddot{x}(t))^2\right) dt \le 1. \qquad (P_5)$$

This is a problem of calculus of variations.
**2.** Proposition 2 leads to $a)$ Euler–Poisson equation for the Lagrangian of the Lagrange function of the problem $(P_5)$, $b)$ transversality conditions, which must be complemented by the conditions of $\beta)$ nonnegativity and $\gamma)$ complimentary slackness:

$$a)\ x^{(4)} + \mu\ddot{x} + x = 0,\ b)\ x^{(3)}(0) + \mu\dot{x}(0) = 0,\ \ddot{x}(0) = 0,$$
$$\beta)\ \mu \ge 0,\ \gamma)\ \mu(\int_{\mathbb{R}_+} (x^2 + \ddot{x}^2)dt - 1) = 0. \qquad (5)$$

**3.** The characteristic polynomial $z^4 + \mu z^2 + 1$ of the Euler–Poisson equation is the product of two factors one of which $z^2 + \nu z + 1$ $(\nu = \sqrt{2-\mu})$ has a root in the left half plane. Hence equation $a)$ is satisfied if $\ddot{x} + \nu\dot{x} + x = 0$.

Differentiating this equation and substituting $t = 0$, we obtain (using (5) $b)$) the equality $x^{(3)}(0) + \dot{x}(0) = 0$, thus $\mu = 1$, and consequently $\nu = 1$. Solving now the equation $\ddot{x} + \dot{x} + x = 0$ with the boundary condition $\dot{x}(0) + x(0) = 0$ we obtain a family of solutions $x(t) = Ae^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{6}\right)$, where $A$ can be calculated by using the isoperimetric condition.
**4.** It is easy to check that the following identity takes place:

$$\int_{\mathbb{R}_+} \left(\ddot{x}^2(t) - \dot{x}^2(t) + x^2(t)\right) dt =$$

$$= \int_{\mathbb{R}_+} (\ddot{x}(t) + \dot{x}(t) + x(t))^2 \, dt + (x(0) + \dot{x}(0))^2 \quad \forall x(\cdot) \in \mathcal{W}_2^2(\mathbb{R}_+).$$

(It is nothing else but the Weierstrass formula in calculus of variations). From this identity it follows that the value of the problem $(P_5)$ is equal to 1. Application of Lemma from Section 1 proves the inequality.

**Theorem 5 (Hardy–Littlwood–Polya I, 1934, [5]).** $K_{\mathbb{R}_+}(1, 2, 2, 2, 2) = 2$.

**Problem 6:** $(P_{\mathbb{R}}(k, n, 2, 2, 2))$

**1.** $$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \to \max, \; \|x(\cdot)\|_{L_2(\mathbb{R})} \leq 1, \; \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})} \leq 1. \qquad (P_6)$$

Application of the Fourier transform leads to the following reformulation:

$$\int\limits_{\mathbb{R}} t^{2k} d\mu(t) \to \max, \; \int\limits_{\mathbb{R}} d\mu(t) \leq 1, \; \int\limits_{\mathbb{R}} t^{2n} d\mu(t) \leq 1, \; d\mu \geq 0, \; d\mu(t) = F^2 x(t) dt. \quad (P_6')$$

This is a Lyapunov problem. The Lagrange function of this problem is the function

$$\mathcal{L}(\mu(\cdot), \bar{\lambda}) = \int\limits_{\mathbb{R}} L(t, \bar{\lambda}) \, d\mu, \quad \text{where } L(t, \bar{\lambda}) = -t^{2k} + \lambda_0 + \lambda_1 t^{2n}.$$

**2.** Proposition 4 leads to the minimum condition:

$$c) \; \min_{\mu} \int\limits_{\mathbb{R}} L(t, \bar{\lambda}) \, d\mu = \int\limits_{\mathbb{R}} L(t, \bar{\lambda}) \, d\widehat{\mu}. \qquad (6)$$

**3, 4.** From (6) it follows that $d\widehat{\mu}(t) = \widehat{C}\delta(t - \widehat{\tau})dt$, where constants $\widehat{C}$ and $\widehat{\tau}$ can be found from the conditions of complementary slackness.

But the following direct proof is much more simple:

$$\|x^{(k)}(\cdot)\|^2_{L_2(\mathbb{R})} = \int\limits_{\mathbb{R}} (x^{(k)}(t))^2 dt$$

$$\overset{Parseval}{=} (2\pi) \int\limits_{\mathbb{R}} t^{2k} (F(x(t)))^2 dt = (2\pi) \int\limits_{\mathbb{R}} t^{2k} (F(x(t)))^{\frac{2k}{n}} (F(x(t)))^{2 - \frac{2k}{n}} \, dt$$

$$\leq (2\pi) (\int\limits_{\mathbb{R}} t^{2n} (F(x(t)))^2 dt)^{\frac{k}{n}} (\int\limits_{\mathbb{R}} (F(x(t)))^2 dt)^{1 - \frac{k}{n}} = \|x(\cdot)\|_{L_2(\mathbb{R})})^{1 - \frac{k}{n}} \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})})^{\frac{k}{n}}.$$

Sharpness of this inequality is trivial.

**Theorem 6 (Hardy–Littlwood–Polya II, 1934, [5]).** $K_{\mathbb{R}}(k, n, 2, 2, 2) = 1$.

**Problem 7:** $(P_{\mathbb{R}}(k, n, \infty, \infty, \infty))$

**1.** $$x^{(k)}(0) \to \max, \quad \|x(\cdot)\|_{C(\mathbb{R})} \leq 1, \quad |x^{(n)}(t)| \leq 1 \text{ a. e. on } \mathbb{R}. \qquad (P_7)$$

Consider the problem $(P_{\mathbb{R}}(k,n,\infty,\infty,\infty))$ where $n=4l$ and $k=1$ (the case $n\in\mathbb{N}$, $0\le k\le n-1$ is considered analogously). Denote by

$$K_n=\frac{4}{\pi}\sum_{j\in\mathbb{N}}\frac{(-1)^{(j+1)(n+1)}}{(2j-1)^{n+1}}$$

the Favard constant and consider the problem

$$\dot{x}(0)\to\max,\quad \|x(\cdot)\|_{C(\mathbb{R})}\le K_n,\quad |x^{(n)}(t)|\le 1 \text{ a. e. on } \mathbb{R}. \tag{$P_7'$}$$

This is a problem of optimal control with phase constraints. It is natural to suppose that the solution of $(P_7')$ is a $2\pi$-periodic Euler spline $\widehat{x}(\cdot)$ for which

$$\int_{-\pi}^{\pi}\widehat{x}(t)\,dt=0,\quad \text{and } \widehat{x}^{(n)}(t)=\operatorname{sgn}\sin t\quad (\text{then } \|\widehat{x}(\cdot)\|_{C(\mathbb{R})}=K_n).$$

**2.** If the Lagrange principle is valid, then the following identity has to be fulfilled:

$$\dot{x}(0)=\sum_{j\in\mathbb{Z}}\mu_j x(j\pi+\pi/2)+\int_{\mathbb{R}}p(t)x^{(n)}(t)\,dt\quad \forall x(\cdot) \tag{7}$$

where $\operatorname{sgn}p(t)=\operatorname{sgn}\sin t$.

**3, 4.** Substituting $x(\cdot)=\exp(i\sigma\cdot)$, $\sigma\in\mathbb{R}$, in (7) we obtain that

$$Fp(\sigma)=\int_{\mathbb{R}}p(t)e^{i\sigma t}dt=\frac{1}{(i\sigma)^{n-1}}-\frac{F\nu(\sigma)}{(i\sigma)^n},$$

where

$$\nu(\cdot)=\sum_{j\in\mathbb{Z}}\mu_j\,\delta(\cdot-j\pi-\pi/2),\quad F\nu(\sigma)=\sum_{j\in\mathbb{Z}}\mu_j\,\exp(i\sigma(j\pi+\pi/2)).$$

Let $\tilde{p}(\sigma)=\sum_{s\in\mathbb{Z}}Fp(\sigma+2s)$ be 2-periodization of $Fp(\cdot)$. Then (as is easy to check) the Fourier coefficients of $\tilde{p}(\cdot)$ are equal to zero and hence $\tilde{p}(\cdot)=0$. Thus,

$$0=\tilde{p}(\sigma)=\sum_{s\in\mathbb{Z}}\frac{i}{(\sigma+2s)^{n-1}}-\sum_{s\in\mathbb{Z}}\frac{(-1)^sF\nu(\sigma)}{(\sigma+2s)^n},$$

and consequently

$$F\nu(\sigma)=2\pi i(n-1)\frac{(\cot\frac{\pi\sigma}{2})^{(n-2)}}{(\csc\frac{\pi\sigma}{2})^{(n-1)}}.$$

These formulae are obtained by Buslaev. It follows from this that

$$K_{\mathbb{R}}(1,n,\infty,\infty,\infty)=\frac{K_{n-1}}{(K_n)^{(n-1)/n}}.$$

**Theorem 7 (Kolmogorov, 1938,[6]).** $K_{\mathbb{R}}(k,n,\infty,\infty,\infty)=K_{n-k}/K_n^{\frac{n-k}{n}}$.

**Problem 8:** $(P_{\mathbb{R}_+}(0,1,p,\infty,r))$

**1.** $$x(0) \to \max, \quad \|x\|_{L_p(\mathbb{R}_+)} \le 1, \quad \|\dot{x}\|_{L_r(\mathbb{R}_+)} \le 1. \qquad (P_8)$$

This is a problem of calculus of variations. The Lagrange function has the following form:

$$\mathcal{L}(x(\cdot),\bar{\lambda}) = -x(0) + \lambda_1 \int_{\mathbb{R}_+} |x|^p\,dt + \lambda_2 \int_{\mathbb{R}_+} |\dot{x}|^r\,dt, \quad \bar{\lambda} = (-1,\lambda_1,\lambda_2).$$

**2.** Proposition 2 leads to a) Euler equation and b) transversality conditions:

$$a)\ \frac{d}{dt} r\lambda_2(\dot{\widehat{x}})_r = p\lambda_1(\widehat{x})_p, \quad b)\ r\lambda_2(\widehat{x}(0))_r = -1. \tag{8}$$

**3.** The integrand in the problem does not depend on $t$, thus the Euler equation has the energy integral: $(r-1)\lambda_2|\dot{\widehat{x}}|^r - p\lambda_1|\widehat{x}|^p = 0$. After integrating these equations (selecting solutions which tends to zero when $t \to \infty$) we find the following solution of the Euler equations:

$$\widehat{x}(t) = \begin{cases} p^{1/p}\cdot e^{-t} & ,\ \text{if } p = r; \\ \left(\frac{p+r'}{r'}\right)^{\frac{r'}{p+r'}} \left(1 + \frac{p-r}{pr-p+r}at\right)_+^{\frac{r}{r-p}} & ,\ \text{if } p \ne r, \end{cases}$$

where $a = (1-s)^{-r's}$ and $s = (1+r'/p)^{-1}$. After finding the Lagrange multiples and substituting them in the Lagrange function we obtain the following identity:

$$x(0) = p(1-s)^s/r' \int_{\mathbb{R}_+} (\widehat{x})_p \cdot x\,dt + (1-s)^s \int_{\mathbb{R}_+} (\dot{\widehat{x}})_r \cdot \dot{x}.\,dt$$

(This identity can be checked directly).

**4.** Application of the Hölder inequality gives the value of the problem.

**Theorem 8 (S. Nagy, 1941, [7]).** $K_{\mathbb{R}_+}(0,1,p,\infty,r,) = (1-s)^{s-1},\ s = (1+r'/p)^{-1}$.

**Problem 9:** $(P_{\mathbb{R}}(k,n,1,1,1))$

This is a unique case when we do not know how to solve the problem by means of general principles of the theory. We prove the appropriate inequality by reducing the problem to the Kolmogorov inequality. Denote

$$\Phi(t) := \int_{\mathbb{R}} x(t+\tau)\,\mathrm{sgn}\,x^{(k)}(\tau)\,d\tau.$$

Then the following relations are evidently satisfied:

$$\|\Phi(\cdot)\|_{C(\mathbb{R})} \le \|x(\cdot)\|_{L_1(\mathbb{R})},\ \|\Phi^{(k)}(\cdot)\|_{C(\mathbb{R})} \ge \Phi^{(k)}(0) = \|x^{(k)}(\cdot)\|_{L_1(\mathbb{R})},$$

$$\|\Phi^{(n)}(\cdot)\|_{L_\infty(\mathbb{R})} \le \|x^{(n)}(\cdot)\|_{L_1(\mathbb{R})} \tag{9}$$

Using (9) and the Kolmogorov inequality for the problem $(P_{\mathbb{R}}(k,n,\infty,\infty,\infty))$ we obtain

$$\|x^{(k)}(\cdot)\|_{L_1(\mathbb{R})} \le \|\Phi^{(k)}(\cdot)\|_{C(\mathbb{R})} \le$$

$$\leq \frac{K_{n-k}}{K_n^{\frac{n-k}{n}}} \|\Phi(\cdot)\|_{C(\mathbb{R})}^{\frac{n-k}{n}} \|\Phi^{(n)}(\cdot)\|_{L_\infty(\mathbb{R})}^{\frac{k}{n}} \leq \frac{K_{n-k}}{K_n^{\frac{n-k}{n}}} \|x(\cdot)\|_{L_1(\mathbb{R})}^{\frac{n-k}{n}} \|x^{(n)}(\cdot)\|_{L_1(\mathbb{R})}^{\frac{k}{n}},$$

where $C_{kn} = \frac{K_{n-k}}{K_n^{\frac{n-k}{n}}}$. To complete the proof it suffices to prove the sharpness of this inequality.

**Theorem 9 (Stein, 1957, [8]).** $K_{\mathbb{R}}(k, n, 1, 1, 1) = \frac{K_{n-k}}{K_n^{\frac{n-k}{n}}}$.

**Problem 10:** $(P_{\mathbb{R}_+}(0, 2, 2, \infty, \infty))$

**1**. $x_1(0) \to \max, \quad \dot{x}_1 = x_2,\ \dot{x}_2 = u, \quad \int_{\mathbb{R}_+} |x_1(t)|^2\, dt \leq 1, \quad |u(t)| \leq 1$ a. e. $\qquad (P_{10})$

This is a problem of optimal control. The Lagrange function has the following form:

$$\mathcal{L}((x_1(\cdot), x_2(\cdot), u(\cdot)), \bar{\lambda}) = \int_{\mathbb{R}_+} \left( \frac{x_1^2}{2} + p_1(\dot{x}_1 - x_2) + p_2(\dot{x}_2 - u) \right) dt - \lambda x_1(0),$$

$$\bar{\lambda} = (p_1(\cdot), p_2(\cdot), \lambda).$$

**2.** Proposition 2 applied to the problem $\mathcal{L}((x_1(\cdot), x_2(\cdot), \widehat{u}(\cdot)), \bar{\lambda}) \to \min$ leads to *a*) Euler equation and *b*) transversality conditions; Proposition 3 applied to the problem $\mathcal{L}((\widehat{x}_1(\cdot), \widehat{x}_2(\cdot), u(\cdot)), \bar{\lambda}) \to \min,\ |u| \leq 1$, leads to *c*) minimum condition:

$$a)\ -\dot{p}_1 + x_1 = 0,\ -\dot{p}_2 - p_1 = 0, \quad b)\ p_2(0) = 0,\ c) \quad u = \operatorname{sgn} p_2.$$

If we denote $(p_2 = y)$, then we obtain the following equations which are satisfied only by the function expected to be a solution of the problem:

$$\ddot{x} = \operatorname{sgn} y,\ \ddot{y} = -x,\ y(0) = 0. \tag{10}$$

We will solve equations (10) with the normalization $x(0) = 1$.
**3.** Solutions of (10) have an «energy integral»

$$\dot{y} \cdot \dot{x} + x^2/2 = |y|, \quad (\Rightarrow\ \dot{y}(0)\dot{x}(0) = -x^2(0)/2)$$

(one can check it by differentiation). When $t$ is small we have:

$$x(t) = \frac{t^2}{2} - \alpha t + 1, \quad y(t) = \frac{t}{2\alpha} - \frac{t^2}{2} + \frac{\alpha t^3}{6} - \frac{t^4}{24}.$$

Next by using the invariance of our equations with respect to the following transformations:
$\lambda \mapsto (x_\lambda(\cdot), y_\lambda(\cdot)),\ x_\lambda(t) = \lambda^2 x(t/\lambda),\ y_\lambda(t) = \lambda^4 y(t/\lambda)$, it is possible to find the solution

$$x(t) = x(\tau)^{-1} x(|x(\tau)|^{1/2} t + \tau), \quad y(t) = -x(\tau)^{-2} y(|x(\tau)|^{1/2} t + \tau),$$

$$\left( \Rightarrow\ \dot{x}(0) = \dot{x}(\tau) \frac{\sqrt{|x(\tau)|}}{x(\tau)} \right),$$

where $\tau > 0$ is the point where $y(\cdot)$ attains zero for the first time. Continuing this process we shall construct an optimal process with countable number of switchings on a finite segment of time. (This phenomenon is called «chattering regime».)

$$\frac{\tau^2}{2} - \alpha\tau + 1 + (\frac{\tau}{\alpha} - 1)^2 = 0, \quad \tau - 2\alpha(\frac{\tau^2}{2} - \frac{\alpha\tau^3}{6} + \frac{\tau^4}{24}) = 0.$$

Excluding $\tau$ from these equations we obtain that

$$(\alpha^2-2)(2\alpha^8-3\alpha^4-36) = 0, \quad \alpha = \left(\frac{3(1+\sqrt{33})}{4}\right)^{1/4}, \quad \tau = \frac{\alpha}{4}\Big(7+\sqrt{33}-\sqrt{26+6\sqrt{33}}\Big).$$

This leads to the following result (which in this form was obtained by Kochurov):

**Theorem 10 (Fuller, 1960, [9]).** $K_{\mathbb{R}_+}(0,2,2,\infty,\infty) = 5^{2/5}2^{-3/5}(3\sqrt{33}+3)^{1/10}$.

**Problem 11:** $(P_{\mathbb{R}}(k,n,2,\infty,2))$

**1.** $$\|x^{(k)}(\cdot)\|_{C^b(\mathbb{R})} \to \max, \|x(\cdot)\|_{L_2(\mathbb{R})} \le 1, \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})} \le 1. \qquad (P_{11})$$

Let $Fx(\cdot)$ be the Fourier transform of $x(\cdot)$ and $u(\tau)$ be $2\pi Fx(\tau)$ (for $x(\cdot) \in L_2(\mathbb{R})$). Then by the Parseval equality $\|x(\cdot)\|^2_{L_2(\mathbb{R})} = 2\pi\|Fx(\cdot)\|^2_{L_2(\mathbb{R})}$, we obtain the following reformulation of the problem:

$$\int_{\mathbb{R}} t^{2k}u(t)dt \to \max, \int_{\mathbb{R}} u^2(t) \le 1, \int_{\mathbb{R}} t^{2n}u^2(t) \le 1. \qquad (P'_{11})$$

This is a Lyapunov problem. The Lagrange function of this problem is the function

$$\mathcal{L}(u(\cdot),\bar\lambda) = \int_{\mathbb{R}} L(t,u(t),\bar\lambda)dt,$$

where $L(t,u,\bar\lambda) = -t^{2k}u + (\lambda_0 + \lambda_1 t^{2n})u^2$.

**2.** Proposition 3 applied to the problem of minimization of the Lagrange function leads to the minimum condition:

*c)* $\min\limits_{u\in\mathbb{R}} L(t,u,\bar\lambda) = L(t,\widehat{u}(t),\bar\lambda). \qquad (11)$

**3, 4.** From (11) it follows that $\widehat{u}(t) = \frac{t^{2k}}{2(\lambda_0+\lambda_1 t^{2n})}$.

Integrals $\int_{\mathbb{R}} \frac{t^{2k}\,dt}{\lambda_0+\lambda_1 t^{2n}}$ are expressed via trigonometric functions. It gives possibility to find the Lagrange multipliers $\lambda_0$ and $\lambda_1$ and then calculate $\int_{\mathbb{R}} \widehat{u}(t)dt$ and thus the value of the problem.

The Cauchy inequality gives the solution very quickly:

$$\big(x^{(k)}(0)\big)^2 = \Big|\int_{\mathbb{R}} \tau^k Fx(\tau)d\tau\Big|^2 = \Big|\int_{\mathbb{R}} \frac{\tau^k}{\sqrt{1+\tau^{2n}}} Fx(\tau)\sqrt{1+\tau^{2n}}d\tau\Big|^2$$

$$\overset{Cauchy}{\le} \frac{A_{kn}}{2\pi}\int_{\mathbb{R}} (x^2(\tau) + (x^{(n)})^2(\tau))\,d\tau \le \frac{A_{kn}}{2\pi},$$

where $A_{kn} = \pi\big(n\sin\frac{(2k+1)\pi}{2n}\big)^{-1}$. This is a sharp estimate. Lemma from Section 1 immediately gives the expression for $K_{\mathbb{R}}(k,n,2,\infty,2)$.

**Theorem 11 (Taikov, 1968, [10]).**

$$K_{\mathbb{R}}(k,n,2,\infty,2)=\sqrt{\frac{1}{2n\sin\frac{(2k+1)\pi}{2n}}}\left(\frac{2n}{2(n-k)-1}\right)^{\frac{2(n-k)-1}{4n}}\left(\frac{2n}{2k+1}\right)^{\frac{2k+1}{4n}}.$$

**Problem 12:** $(P_{\mathbb{R}_+}(1,2,\infty,\infty,r))$

**1.** $$\dot{x}(0)\to\max,\quad |x(t)|\le 1,\ t\in\mathbb{R}_+,\quad \int\limits_{\mathbb{R}_+}|\ddot{x}(t)|^r\le 1,\quad r\ge 1. \qquad (P_{12})$$

This is a problem of optimal control with phase constraints. Consider the case $r>1$, the case $r=1$ will be obtained by passage to the limit.
**2.** The Lagrange principle leads to the following identity

$$\dot{x}(0)=\frac{x(T)-x(0)}{T}-\int\limits_{\mathbb{R}_+}(1-\frac{t}{T})_+\ddot{x}(t)dt \qquad (13)$$

which was obtained when the Landau problem was solved. As a result we come to the differential equation $\ddot{x}(t)=a(1-\frac{t}{T})_+^{r'-1}$ with boundary conditions

$$x(0)=-1,\quad x(T)=1,\quad \dot{x}(T)=0,\quad \int_0^T|\ddot{x}(t)|^r dt=1.$$

**3, 4.** Solving these equations we obtain that

$$\widehat{x}(t)=1-2(1-\frac{t}{T})_+^{r'+1},\ T=(2r')^{\frac{r'}{r'+1}}(r'+1)^{\frac{1}{r'+1}}.$$

For an estimate from below it suffices to calculate $\dot{\widehat{x}}(0)$. For the estimate from above it suffices to apply the Hölder inequality to the general identity.

**Theorem 12 (Arestov, 1972, [11]).** $K_{\mathbb{R}_+}(1,2,\infty,\infty,r)=2^{1/(r'+1)}\left(\frac{r'+1}{r'}\right)^{r'/(r'+1)}$.

**Problem 13:** $(P_{\mathbb{R}_+}(k,n,2,\infty,2))$

**1.** $$\int_{\mathbb{R}_+}(x^2+(x^{(n)})^2)dt\to\min,\ x^{(k)}(0)=1. \qquad (P_{13})$$

This is one of possible formalizations of the problem. Problem $(P_{13})$ belongs to the class of convex problems of the calculus of variations (on an infinite interval).

The Lagrange function of the problem has the form:

$$\mathcal{L}(x(\cdot),\bar{\lambda})=\int_{\mathbb{R}_+}(x^2(t)+(x^{(n)}(t))^2)\,dt+\lambda x^{(k)}(0),$$

where $\bar{\lambda}=(1,\lambda)$.
**2.** The Lagrange principle gives the *a*) Euler–Lagrange equation and *b*) transversality conditions:

$$a)\ (-1)^n\widehat{x}^{(2n)}+\widehat{x}=0,\quad b)\ \widehat{x}^{(n+l-1)}(0)=(-1)^{n-k}\delta_{l,n-k}.$$

**3, 4.** Admissible solutions of the Euler–Lagrange equation are represented in the form: $\widehat{x}(t)=\sum_{j=1}^{n} c_j e^{\lambda_j t}$, where $\lambda_j$ are the roots of $(2n)$ th degree of $+1$ if $n$ is even and of $-1$ if $n$ is odd. Then we have to satisfy the transversality conditions. Thus it is necessary to solve the system:

$$\sum\nolimits_{j=1}^{n} c_j\lambda_j^{n+l-1}=(-1)^{n-k}\delta_{l,n-k} \tag{13}$$

and to calculate

$$\widehat{x}^{(k)}(0)=\sum_{j=1}^{n} c_j\lambda_j^k.$$

The matrix of system (13) is the Vandermonde matrix $A=(\mu_{lm})$ up to multiplication by a constant with absolute value 1, where $0\le l\le n-1$, $n\le m\le 2n-1$. The solution of system (13) has the form $C\cdot\frac{\det A_k}{\det A}$, where the matrix $A_k$ is obtained from $A$ by replacing the $(n-k)$ row with the row $(1,\mu_k,\dots,\mu_k^{n-1})$. The modulus of the Vandermonde determinant generated by numbers $w_1,\dots,w_n$ is equal to the product of numbers $|w_j-w_k|$. In our case $w_k$ are roots of $+1$ or $-1$ of degree $(2n)$. Thus, $|w_j-w_k|=\left|\sin\frac{l\pi}{n}\right|$ or $\left|\sin\frac{l\pi+\pi/2}{n}\right|$. After simplification of the expression $\frac{\det A_k}{\det A}$ we obtain

$$A_{kn}=\left(\sin\frac{\pi(2k+1)}{2n}\right)^{-1/2}\prod_{j=1}^{k}\cot\frac{\pi j}{2n}\,.$$

Applying now Lemma from Section I we conclude the proof.

**Theorem 13 (Gabushin, 1969, [12], Kalyabin, 2002, [14]).**

$$K_{\mathbb{R}_+}(k,n,2,\infty,2)=A_{kn}\cdot\Big(\frac{2n}{2n-2k-1}\Big)^{\frac{2n-2k-1}{4n}}\Big(\frac{2n}{2k+1}\Big)^{\frac{2k+1}{2n}}.$$

**Problem 14**: $(P_{\mathbb{R}_+}(0,2,p,\infty,1))$

After some reformulation of the problem (the inequality $\operatorname{Var}\dot{x}(\cdot)\le 1$ instead of $\int\limits_{\mathbb{R}_+}|\ddot{x}(t)|dt\le 1$) we obtain the following formalization:

**1.**

$$x(0)\to\max,\ \int\limits_{\mathbb{R}_+}|x(t)|^p dt\le 1,\ \dot{x}=u,\ \operatorname{Var}u(\cdot)\le 1. \tag{$P_{14}$}$$

This is a problem of optimal control with a nonstandard constraint of the control parameter. The Lagrange function of the problem $(P_{14})$ has the form:

$$-x(0)+\int\limits_{\mathbb{R}_+}(\lambda|x(t)|^p+q(t)(\dot{x}(t)-u(t))\,dt.$$

**2.** The Lagrange principle leads to the identity

$$x(0)=\int_{\mathbb{R}_+}(\lambda p(\widehat{x}(t))_p\,x(t)+q(t)\dot{x}(t))\,dt, \tag{14}$$

$a)$ the Euler equation $-\dot{q}(t)+\lambda p(\widehat{x}(t))_p = 0$, $b)$ the transversality condition $q(0)=-1$ and $c)$ the minimum condition:

$$\min_{\mathrm{Var}\ u(\cdot)\le 1} -\int_{\mathbb{R}_+} q(t)u(t)dt = -\int_{\mathbb{R}_+} q(t)\widehat{u}(t)dt.$$

**3, 4**. From relations $\dot{x}=u$, $x(\cdot)\in L_p(\mathbb{R}_+)$ it follows that $u(t)$ tends to zero when $t\to\infty$. If we assume that $(-\widehat{u}(t))$ is the characteristic function of $[0,T]$, where $T$ is to be found then our assumptions give possibility to define all unknowns:
$\widehat{x}(t) = -\int\limits_t^T \widehat{u}(t)\,dt = (T-t)_+, \quad \dot{q}(t)=\lambda p(\widehat{x}(t))_p \ \Rightarrow q(t) = -\lambda (T-t)_+^p, \quad q(0)=-1$
$\Rightarrow \lambda = \frac{1}{T^p}$, $\|\widehat{x}(\cdot)\|_{L_p(\mathbb{R}_+)} = 1 \Rightarrow T = (p+1)^{\frac{1}{p+1}}$.
Thus $K_{\mathbb{R}_+}(0,2,p,\infty,1) = (p+1)^{\frac{1}{p+1}}$. Returning to (14), we obtain the following identity:

$$x(0) = \int_{\mathbb{R}_+} \Big(\frac{p}{T^p}(T-t)_+^{p-1}x(t) - (1-\frac{t}{T})_+^p\,\dot{x}(t)\Big)dt.$$

It is possible to check this identity directly (so we could start our investigation from this identity). Using the admissible function $t\mapsto (T-t)_+$, we obtain the estimate from below
$K_{\mathbb{R}_+}(0,2,p,\infty,1) \ge (p+1)^{\frac{1}{p+1}}$.
If we change in the identity (14) the expression

$$-\int_0^\infty (1-\frac{t}{T})_+^p\,\dot{x}(t)\,dt \quad \text{by} \quad \int_0^\infty \frac{T}{p+1}\big(1-(1-\frac{t}{T})_+^{p+1}\big)\,d\dot{x}(t),$$

then from Hölder inequality and inequalities $\|x(\cdot)|_{L_p(\mathbb{R}_+)} \le 1$ and $\mathrm{Var}\,x(\cdot)\le 1$, we obtain the estimate from above $K_{\mathbb{R}_+}(0,2,p,\infty,1) \le (p+1)^{\frac{1}{p+1}}$. Thus we proved the following

**Theorem 14 (Magaril-Il'yaev, 1983, [13]).** $K_{\mathbb{R}_+}(0,2,p,\infty,1) = (p+1)^{\frac{1}{p+1}}$.

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