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BOKAYEV NURZHAN ADILKHANOVICH

(to the 70th birthday)

January 5, 2026, marks the 70th birthday of Nurzhan Adilkhanovich Bokayev, Doctor of Physical and Mathematical Sciences (1996), Professor (2002), member of the Editorial Board of the Eurasian Mathematical Journal (2010).

Nurzhan Adilkhanovich Bokayev was born on 5 January, 1956 in the village of Urnek, Karabalyk District, Kostanay Region. He graduated in 1972, with a gold medal from the Burlin Secondary School in the district. That same year, he entered the Mathematics Department of Karaganda State University and graduated with honors in 1977. From 1978 to 1979, he served in the Soviet Army. In 1980, he completed an internship, and from 1981 to 1984, he studied in the graduate program at Lomonosov Moscow State University in the Department of Function Theory and Functional Analysis. In 1985, he defended his candidate's dissertation there under the supervision of Corresponding Member of the Academy of Sciences of the USSR D.E. Menshov and Professor V.A. Skvortsov. In 1996, he defended his doctoral dissertation, "Fourier Coefficients and Uniqueness Theorems for Series in Generalized Walsh and Haar Systems", at the Institute of Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan, speciality Mathematical Analysis (01.01.01).

After completing his postgraduate studies, he worked as a lecturer, senior lecturer, associate professor, and professor in the Department of Mathematical Analysis at E.A. Buketov Karaganda State University (1985-1999). He headed the Department of Mathematics and Mathematical Modeling (1996-1999), and was a dean of the Faculty of Mathematics at E.A. Buketov Karaganda State University (1999-2005). Since 2005, he has been a professor in the Faculty of Mechanics and Mathematics at the L.N. Gumilyov Eurasian National University. From 2009 to 2018, he was the Head of the Department of Higher Mathematics at the L.N. Gumilyov Eurasian National University, and from 2018 to the present, he has been a professor in the Department of Fundamental Mathematics.

Professor Bokaev's research focuses on problems in function theory and functional analysis, the theory of orthogonal series for generalized Walsh and Haar systems, and operator theory in various function spaces. He has proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, and constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems. He obtained conditions for functions to belong to various functional classes in terms of the Fourier coefficients of generalized Haar and Walsh systems, and embedding criteria for Nikol'skii-Besov spaces constructed on the basis of multiplicative systems. He also obtained conditions for the boundedness and compactness of the commutator of the Riesz potential in general Morrey-type spaces, and conditions for boundedness of generalizedRiesz and Bessel potentials and generalized fractional-maximal operators in rearrangement-invariant spaces.

His co-authors include Professor V.A. Skvortsov (Moscow State University, Moscow), Professors V.I. Burenkov and M.L. Goldman (Peoples' Friendship University of Russia (RUDN University). Moscow), Dr. A. Gogatishvili (Institute of Mathematics of the Czech Academy of Sciences, Prague). His doctoral students' foreign advisors include Professors W. Sickel (Friedrich-Schiller-Universitity, Jena, Germany), Massimo Lanza de Cristoforis (University of Padova, Padova, Italy), V. Ruzhansky (Ghent University, Ghent, Belgium), U. Goginava (United Arab Emirates University, Al Ain, United Arab Emirates), and E. Panakhov (Institute of Applied Mathematics at Baku State University, Baku, Azerbaijan).

Under his supervision, 15 dissertations (4 candidate's and 11 PhD) were defended. He has published over 220 scientific papers, 2 monographs and 2 textbooks.

He is a three-time recipient of the state grant "Best University Teacher" of the Republic of Kazakhstan (2006, 2010, 2024) and served as Vice President of the Mathematical Society of Turkic-Speaking Countries (2014-2023). He was awarded the "For Contribution to the Development of Science" badge (2022).

Over the last ten years, he has been and continues to be a head of more than 5 national and international funded projects.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Nurzhan Adilkhanovich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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GENERALIZATIONS OF HARDY-TYPE INTEGRAL INEQUALITIES FOR QUASIMONOTONE FUNCTIONS IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

M. Sofrani, A. Senouci

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Abstract. In 1992 V.I. Burenkov proved some Hardy's inequalities with sharp constants in Lebesgue spaces for monotone functions for $0 < p < 1$. Later R.A. Bandaliev established analogous estimates in weighted variable exponent Lebesgue spaces for monotone functions for $0 < p(x) \leq q(x) < 1$. In 2020 A. Senouci and A. Zanou generalized the results of R.A. Bandaliev for quasi-monotone functions. The aim of this paper is to obtain some generalizations of the previous results cited above for weighted Hardy operators by introducing a parameter $\alpha \in \mathbb{R}$. Moreover, by using the quasi-norms $\|f\|^{BT}_{L_{p(x)}(\Omega)}$ introduced by V.I. Burenkov and T.V. Tararykova, we obtain an improvement of constants in our previous estimates.

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1 Introduction

For the first time the variable exponent Lebesgue space appeared in the literature already in the thirties of the last century, being introduced by W. Orlicz. At the beginning these spaces had theoretical interest. Later, at the end of the last century, their first use beyond the function spaces theory itself, was in variational problems and studies of $p(x)$-Laplacian (see Zhikov [11], [12]) which in its turn gave an essential impulse for the development of this theory. The extensive investigation of these spaces was also widely stimulated by applications to various problems of applied mathematics, e.g., in modelling of electrorheological fluids [8].

The variable exponent Lebesgue spaces $L^{p(x)}$ for $p(x) \geq 1$ appeared in the literature for the first time in [7]. Further developement of this theory was connected with the theory of modular functions.

Many investigations are devoted to the problem of boundedness of the Hardy operator in the Lebesgue spaces $L^{p(x)}$ for $p(x) \geq 1$ (see for example [1]). However, investigations of the Hardy inequality in the variable exponent Lebesgue spaces $L^{p(x)}$ for $0 < p(x) < 1$ are much less known.

It is well known that for L_p-spaces with $0 < p < 1$, the Hardy inequalities are not satisfied for arbitrary non-negative measurable functions, but are satisfied for non-negative monotone functions (for more details see [5]). The aim of this work is to obtain weighted inequalities for the Hardy operators acting from one weighted variable exponent Lebesgue space $L_{p(x),w_1(x)}(0,\infty)$ to another weighted variable exponent Lebesgue space $L_{q(x),w_2(x)}(0,\infty)$ for $0 < p(x) \leq q(x) < 1$, for functions defined on $(0,\infty)$ and satisfying conditions of the quasi-monotonicity. Some results obtained in [10] are generalized. Moreover, by using the quasi-norms $\|f\|^{BT}_{L_{p(x),\omega(x)}}$ introduced by V.I. Burenkov and T.V. Tararykova (for more details see [4]) and a new parameter α, we establish some weighted inequalities for the same operators with improved constants.

2 Preliminaries

In this section, we state definitions, lemmas, corollaries and theorems that are useful in the proofs of main results. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, x_2, \ldots, x_n)$, Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. Suppose that p is a Lebesgue measurable function on Ω such that $0 < p(x) \leq \infty \, \forall \, x \in \Omega$, and ω is a weight function, that is a positive Lebesgue measurable function on Ω.

Definition 1. Let p be a Lebesgue measurable function, $0 < p(x) < \infty$ for all $x \in \Omega$. By $L_{p(x),\omega(x)}(\Omega)$ we denote the set of all Lebesgue measurable functions f on Ω such that

$$\rho_{p(x),\omega(x)}(f) = \int_\Omega (|f(x)|\omega(x))^{p(x)} dx < \infty. \tag{2.1}$$

Note that the expression

$$\|f\|_{L_{p(x),\omega(x)}(\Omega)} = \inf\{\lambda > 0; \int_\Omega \Big(\frac{|f(x)|\omega(x)}{\lambda}\Big)^{p(x)} dx \leq 1\} \tag{2.2}$$

defines a quasi-norm on $L_{p(x),\omega(x)}(\Omega)$. $L_{p(x),\omega(x)}(\Omega)$ is a quasi-Banach space equipped with this quasi-norm (see [9]).

In Definition 1, the case $p(x) = \infty$ is not considered. For $\omega(x) = 1$ definitions, including this case, were considered by O. Kovachik, J. Rakosnik (quasi-norm $\|f\|^{KR}_{L_{p(x)}(\Omega)}$) and V.I. Burenkov, T.V. Tararykova (quasi-norm $\|f\|^{BT}_{L_{p(x)}(\Omega)}$) (see [6] and [4], respectively). We shall use the quasi-norm based on the definition given in [4].

Definition 2. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$, $p(x) : \Omega \to (0, \infty]$, $f : \Omega \to \mathbb{C}$, Lebesgue measurable functions on Ω. Following [4], we say that $f \in L^{BT}_{p(x),\omega(x)}(\Omega)$ if

$$\|f\|^{BT}_{L_{p(x),\omega(x)}(\Omega)} = \inf\left\{\lambda > 0; \int_\Omega \left(\frac{|f(x)|\omega(x)}{\lambda}\right)^{p(x)} dx \leq 1\right\} < \infty. \tag{2.3}$$

If $\frac{|f(x)|\omega(x)}{\lambda} < 1$ and $p(x) = \infty$, then it is assumed that $\left(\frac{|f(x)|\omega(x)}{\lambda}\right)^{p(x)} = 0$.

If $\frac{|f(x)|\omega(x)}{\lambda} > 1$ and $p(x) = \infty$, then it is assumed that $\left(\frac{|f(x)|\omega(x)}{\lambda}\right)^{p(x)} = \infty$.

Remark 1. Note that $L^{BT}_{p(x)}(\Omega)$ is a quasi-normed space with the quasi-norm $\|f\|^{BT}_{L_{p(x),\omega(x)}(\Omega)}$ (norm if $p(x) \geq 1$).

Clearly, if $p(x) < \infty \; \forall x \in \Omega$, then $\|f\|^{BT}_{L_{p(x),\omega(x)}(\Omega)} = \|f\|_{L_{p(x),\omega(x)}(\Omega)}$.

The following statement is known (see [1]).

Lemma 2.1. *Let $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ be measurable sets, p be a Lebesgue measurable function on Ω_1 and q be a Lebesgue measurable function on Ω_2, $1 \leq \underline{p} \leq p(x) \leq q(y) \leq \overline{q} < \infty$ for all $x \in \Omega_1 \subset \mathbb{R}^n$ and $y \in \Omega_2 \subset \mathbb{R}^m$. If $p \in C(\Omega_1)$, $q \in C(\Omega_2)$, then the inequality*

$$\Big\|\|f\|_{L_{p(x)}(\Omega_1)}\Big\|_{L_{q(x)}(\Omega_2)} \leq C_{p,q}\Big\|\|f\|_{L_{q(x)}(\Omega_2)}\Big\|_{L_{p(x)}(\Omega_1)} \tag{2.4}$$

is valid, where

$$C_{p,q} = \Big(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + \frac{\overline{p}}{\underline{q}} + \frac{\underline{p}}{\overline{\overline{q}}}\Big)(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty), \tag{2.5}$$

$$\underline{p} = \operatorname{ess}\inf_{\Omega_1} q(x),\ \overline{p} = \operatorname{ess}\sup_{\Omega_1} q(x),\ \underline{q} = \operatorname{ess}\inf_{\Omega_2} q(x),\ \overline{q} = \operatorname{ess}\sup_{\Omega_2} q(x),$$

$$\Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2;\ p(x) = q(y)\}, \qquad \Delta_2 = (\Omega_1 \times \Omega_2)\backslash\Delta_1,$$

$C(\Omega_1)$, $C(\Omega_2)$ are the spaces of continuous functions in Ω_1, Ω_2 and $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is any Lebesgue measurable function such that $\Big\| \|f\|_{L_{q(x)}(\Omega_2)} \Big\|_{L_{p(x)}(\Omega_1)} < \infty$.

The following definition was introduced in [3].

Definition 3. We say that a non-negative function f is quasimonotone on $]0,\infty[$, if for some real number α, $x^\alpha f(x)$ is a decreasing or an increasing function of x. More precisely, given $\beta \in \mathbb{R}$, we say that $f \in Q_\beta$ if only if $x^{-\beta} f(x)$ is non-increasing and $f \in Q^\beta$ if only if $x^{-\beta} f(x)$ is non-decreasing.

The following theorems were proved in [10].

Theorem 2.1. *Let p, q be Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta > -1$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

1) If $f \in Q_\beta$, then the inequality

$$\|Hf\|_{L_{q(x),\omega_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\overline{p}'}} C_{p,q} d_p \left\| \frac{t^{\frac{1}{\overline{p}'}} \| \frac{\omega_2(x)}{x} \|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{2.6}$$

holds.

2) $f \in Q^\beta$, then the inequality

$$\|Hf\|_{L_{q(x),\omega_2}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\overline{p}'}} C_{p,q} d_p \left\| \frac{\| [y^{-\beta}(x^{\beta+1} - y^{\beta+1})]^{\frac{1}{\overline{p}'}} \frac{\omega_2(x)}{x} \|_{L_{q(x)}(y,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{2.7}$$

holds, where

$$(Hf)(x) = \frac{1}{x}\int_0^x f(y)dy,$$

$$C_{p,q} = \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \left(\frac{\overline{p}}{\underline{q}} + \frac{\underline{p}}{\overline{q}} \right) \right) \left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)} \right),$$

$S_1 = \{x \in (0,\infty) :\ p(x) = \underline{p}\}$, $S_2 = (0,\infty)\backslash S_1$ and

$$d_p = \left(1 + \frac{\overline{p} - \underline{p}}{\overline{p}} + \|\chi_{S_1}\|_{L_\infty(0,\infty)} \right)^{\frac{1}{\underline{p}}}.$$

Theorem 2.2. *Let p, q Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$ and $\beta = -1$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

1) If $f \in Q_{-1}$, then the inequality

$$\|H^* f\|_{L_{q(x),\omega_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}} C_{p,q} d_p \left\| \frac{t^{\frac{1}{\overline{p}'}} (\ln\frac{t}{x})^{\frac{1}{\overline{p}'}} \| \frac{\omega_2(x)}{x} \|_{L_{q(x)}(0,t)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{2.8}$$

holds.

2) *If* $f \in Q^{-1}$, *then the inequality*

$$\|Hf\|_{L_{q(x),\omega_2(x)}(0,\infty)} \leq \underline{p}^{\frac{1}{p}} C_{p,q} d_p \Bigg\| \frac{t^{\frac{1}{\overline{p}'}} (\ln\frac{t}{x})^{\frac{1}{\overline{p}'}} \|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \Bigg\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{2.9}$$

holds.

In (2.8) - (2.9)

$$(Hf)(x) = \frac{1}{x}\int_0^x f(t)dt, \ (H^*f)(x) = \frac{1}{x}\int_x^\infty f(t)dt$$

and $C_{p,q}$, d_p *are the constants in Theorem* 2.1.

The following proposition was proved in [3].

Proposition 2.1. (*a*) *Let* $-\infty < \beta < \infty$, $f \in Q_\beta$, $0 \leq a < b < \infty$ *for* $\beta > -1$ *and* $0 < a < b \leq \infty$ *for* $\beta \leq -1$.

If $0 < p \leq 1$ *and* $\beta \neq -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \leq p|\beta+1|^{1-p} \int_a^b \left(\frac{|y^{\beta+1} - a^{\beta+1}|}{y^\beta}\right)^{p-1} f^p(y)dy. \tag{2.10}$$

If $0 < p \leq 1$ *and* $\beta = -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \leq p \int_a^b \left(y \ \ln\frac{y}{a}\right)^{p-1} f^p(y)dy. \tag{2.11}$$

The inequalities hold in the reversed direction if $1 \leq p < \infty$.

(*b*) *Let* $-\infty < \beta < \infty$, $f \in Q^\beta$ *and* $0 < a < b \leq \infty$ *for* $\beta < -1$ *and* $0 \leq a < b < \infty$ *for* $\beta \geq -1$.

If $0 < p \leq 1$ *and* $\beta \neq -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \leq p|\beta+1|^{1-p} \int_a^b \left(\frac{|y^{\beta+1} - b^{\beta+1}|}{y^\beta}\right)^{p-1} f^p(y)dy. \tag{2.12}$$

If $0 < p \leq 1$ *and* $\beta = -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \leq p \int_a^b \left(y \ \ln\frac{b}{y}\right)^{p-1} f^p(y)dy. \tag{2.13}$$

The inequalities hold in the reversed direction if $1 \leq p < \infty$.

(*c*) *The constants in these inequalities are the best possible in all cases.*

We note the following special cases of Proposition 2.1 which are useful in the proofs of main results. If we set $a = 0$ and $b = x$ in (2.10) and assume that f is nonnegative and $\omega \in Q^\alpha$, then we get for $0 < x < \infty$

$$\left(\int_0^x f(y)y^{-\alpha}\omega(y)dy\right)^p \leq \left(x^{-\alpha}\omega(x)\right)^p \left(\int_0^x f(y)dy\right)^p$$
$$\leq p(\beta+1)^{1-p}\omega^p(x)x^{-\alpha p} \int_0^x y^{p-1} f^p(y)dy.$$

Corollary 2.1. *Let* $0 < p \leq 1$, $0 < x < \infty$.

(*a*) *If* $\beta > -1$, $f \in Q_\beta$, $\omega \in Q^\alpha$, $\alpha \in \mathbb{R}$, *then*

$$\left(\int_0^x f(y)y^{-\alpha}\omega(y)dy\right)^p \leq p(\beta+1)^{1-p}\omega^p(x)x^{-\alpha p} \int_0^x y^{p-1} f^p(y)dy. \tag{2.14}$$

(*b*) *If* $\beta < -1$, $f \in Q^\beta$, $\omega \in Q_\alpha$, $\alpha \in \mathbb{R}$, *then*

$$\left(\int_x^\infty f(y)y^{-\alpha}\omega(y)dy\right)^p \le p|\beta+1|^{1-p}\omega^p(x)x^{-\alpha p}\int_x^\infty y^{p-1}f^p(y)dy. \tag{2.15}$$

(*c*) *If* $\beta > -1$, $f \in Q^\beta$, $\omega \in Q^\alpha$, $\alpha \in \mathbb{R}$, *then*

$$\left(\int_0^x f(y)y^{-\alpha}\omega(y)dy\right)^p \le p(\beta+1)^{1-p}\omega^p(x)x^{-\alpha p}\int_0^x \left[y^{-\beta}\left(x^{\beta+1}-y^{\beta+1}\right)\right]^{p-1} f^p(y)dy. \tag{2.16}$$

The estimates (2.15) and (2.16) are proved similarly by putting $a = x$, $b = \infty$ and $a = 0$, $b = x$ in (2.12).

If we take $a = x$, $b = \infty$ and $a = 0$, $b = x$ in (2.11) and (2.13) respectively, we obtain the following corollary.

Corollary 2.2. *Let* $0 < p \le 1$, $\beta = -1$, $\alpha \in \mathbb{R}$, $0 < x < \infty$.

1) *If* $\omega \in Q_\alpha$, *then*

$$\left(\int_x^\infty f(y)y^{-\alpha}\omega(y)dy\right)^p \le p\omega^p(x)x^{-\alpha p}\int_x^\infty \left(y \ln \frac{y}{x}\right)^{p-1} f^p(y)dy. \tag{2.17}$$

2) *If* $\omega \in Q^\alpha$, *then*

$$\left(\int_0^x f(y)y^{-\alpha}\omega(y)dy\right)^p \le p\omega^p(x)x^{-\alpha p}\int_0^x \left(y \ln \frac{x}{y}\right)^{p-1} f^p(y)dy. \tag{2.18}$$

The following theorem was proved in [4].

Theorem 2.3. *Let* $\Omega \subset \mathbb{R}^n$ *be a Lebesgue measurable set;* $p, q : \Omega \to (0,\infty]$, $f : \Omega \to \mathbb{C}$, *Lebesgue measurable functions, such that*

1) *for all* $x \in \Omega$ $0 < p(x) \le q(x) \le \infty$;

2) $r(x) = \frac{p(x)q(x)}{q(x)-p(x)}$, *if* $p(x) < q(x) < \infty$, $r(x) = p(x)$, *if* $p(x) < q(x) = \infty$, *and* $r((x) = \infty$, *if* $p(x) = q(x)$;

3) $m = \operatorname*{ess\,inf}_{x\in\Omega} \frac{p(x)}{q(x)}$, $M = \operatorname*{ess\,sup}_{x\in\Omega} \frac{p(x)}{q(x)}$, $\underline{p} = \operatorname*{ess\,inf}_{x\in\Omega} p(x)$.

If $\underline{p} > 0$, *then*

$$\|fg\|^{BT}_{L_{p(x)}(\Omega)} \le (1+M-m)^{\frac{1}{\underline{p}}}\|f\|^{BT}_{L_{q(x)}(\Omega)}\|g\|^{BT}_{L_{r(x)}(\Omega)} \tag{2.19}$$

for all $f \in L^{BT}_{q(x)}(\Omega)$ *and* $g \in L^{BT}_{r(x)}(\Omega)$.

Remark 2. The constant $(1+M-m)^{\frac{1}{\underline{p}}}$ in inequality (2.19) is an improvement of the constant in Corollary 2.1 of [2], with $A = M, B = 1-m$.

If the functions p, q satisfy the conditions of Theorem 2.3, f is replaced by $f\omega_2$ and $g = \frac{\omega_1}{\omega_2}$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.3. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is a Lebesgue measurable set, then the inequality (2.19) takes the form*

$$\|f\|^{BT}_{L_{p(x),\omega_1(x)}(\Omega)} \le (1+M-m)^{\frac{1}{\underline{p}}}\left\|\frac{\omega_1(x)}{\omega_2(x)}\right\|^{BT}_{L_{r(x)}(\Omega)}\|f\|^{BT}_{L_{q(x),\omega_2(x)}(\Omega)}, \tag{2.20}$$

for every $f \in L^{BT}_{q(x),\omega_2(x)}(\Omega)$ *and* $\frac{\omega_1}{\omega_2} \in L^{BT}_{r(x)}(\Omega)$.

3 Main results

Let ω be a weight function on $(0,\infty)$ and $\alpha \in \mathbb{R}$.

Consider the weighted Hardy operators

$$H_{\omega,\alpha} = \frac{1}{W(x)} \int_0^x f(y) y^{-\alpha} \omega(y) dy, \ \alpha \in \mathbb{R},$$

$$(H^*_{\omega,\alpha} f)(x) = \frac{1}{W(x)} \int_x^\infty f(y) y^{-\alpha} \omega(y) dy,$$

where $W(x) = \int_0^x y^{-\alpha}\omega(y)dy, \ y > 0$ and f is a non-negative Lebesgue measurable function on $(0,\infty)$.

Note that for $\omega(y) \equiv 1$, and $\alpha = 0$, the $H_{\omega,\alpha}$ and $H^*_{\omega,\alpha}$ are the usual Hardy operators H and H^*.

Theorem 3.1. *Let p, q be Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta > -1$, $f \in Q_\beta$, $\alpha \in \mathbb{R}$, and $\omega \in Q^\alpha$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then, the inequality

$$\|H_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\underline{p}'}} C_{p,q} (1+M-m)^{\frac{1}{\underline{p}}} \Big\| \frac{y^{\frac{1}{\underline{p}'}} \|\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\|_{L_{q(x)}(y,\infty)}}{\omega_1(x)} \Big\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.1}$$

holds, where $C_{p,q}$, M and m are the constants in Theorems 2.1, 2.3 *respectively.*

Proof. Taking in account Remark 2.1 and by applying Corollary 2.1 and inequality (2.14) with $p = \underline{p}$, we get

$$\|H_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)} = \|\omega_2(x) H_{\omega,\alpha} f\|_{L_{q(x)}(0,\infty)} = \Big\| \frac{\omega_2(x)}{W(x)} \int_0^x f(y) y^{-\alpha} w dy \Big\|_{L_{q(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\underline{p}'}} \Big\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big(\int_0^x f^{\underline{p}}(y) y^{\underline{p}-1} dy \Big)^{\frac{1}{\underline{p}}} \Big\|_{L_{q(x)}(0,\infty)}.$$

Let

$$K_1 = \Big\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big(\int_0^x f^{\underline{p}}(y) y^{\underline{p}-1} dy \Big)^{\frac{1}{\underline{p}}} \Big\|_{L_{q(x)}(0,\infty)},$$

then,

$$K_1 = \Big\| \Big(\int_0^\infty f^{\underline{p}}(y) \chi_{(0,x)}(y) \Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big]^{\underline{p}} y^{\underline{p}-1} dy \Big)^{\frac{1}{\underline{p}}} \Big\|_{L_{q(x)}(0,\infty)}$$

$$= \Big\| \Big(\int_0^\infty f^{\underline{p}}(y) \chi_{(\omega,x)}(y) \Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big]^{\underline{p}} y^{\underline{p}-1} dy \Big) \Big\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}$$

$$= \Big\| \Big\| f^{\underline{p}}(y) \chi_{(0,x)}(y) \Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big]^{\underline{p}} y^{\underline{p}-1} \Big\|_{L_1(0,\infty)} \Big\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}.$$

Now, one can use Lemma 2.1 and get that

$$K_1 \le C_{p,q} \Big(\int_0^\infty \Big\| [f^{\underline{p}}(y)] \chi_{(0,x)}(y) \Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \Big]^{\underline{p}} y^{\underline{p}-1} \Big\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dy \Big)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q}\Big(\int_0^\infty f^{\underline{p}}(y)y^{\underline{p}-1}\Big\|\chi_{(0,x)}(y)\Big[\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\Big]^{\underline{p}}\Big\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}dy\Big)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q}\Big(\int_0^\infty f^{\underline{p}}(y)y^{\underline{p}-1}\Big\|\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\Big\|^{\underline{p}}_{L_{q(x)}(y,\infty)}dy\Big)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q}\Big\|f(y)y^{\frac{1}{\underline{p}'}}\Big\|\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\Big\|_{L_{q(x)}(y,\infty)}\Big\|_{L_{\underline{p}}(0,\infty)}.$$

Let

$$K_2 = \Big\|f(y)y^{\frac{1}{\underline{p}'}}\Big\|\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\Big\|_{L_{q(x)}(y,\infty)}\Big\|_{L_{\underline{p}}(0,\infty)}.$$

Finally, applying Corollary 2.3, we have

$$K_2 \le (1+M-m)^{\frac{1}{\underline{p}}}\Big\|\frac{y^{\frac{1}{\underline{p}'}}\|\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\|_{L_{q(x)}(y,\infty)}}{\omega_1(x)}\Big\|^{BT}_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),\omega_1(x)}(0,\infty)},$$

consequently,

$$\|H_{\omega,\alpha}f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}}C_{p,q}(1+M-m)^{\frac{1}{\underline{p}}}\Big\|\frac{y^{\frac{1}{\underline{p}'}}\|\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\|_{L_{q(x)}(y,\infty)}}{\omega_1(x)}\Big\|^{BT}_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),\omega_1(x)}(0,\infty)}.$$

□

Remark 3. Note that $(1+M-m)^{\frac{1}{\underline{p}}} < d_p$. Since, under the assumptions of Theorem 3.1, $\|f\|^{BT}_{L_{p(x)}(\Omega)} = \|f\|_{L_{p(x)}(\Omega)}$, if we take $\alpha = 0$, $\omega(x) = 1$ in inequality (3.1), we get inequality (2.6) of Theorem 2.1 with an improved constant. Moreover, by putting $\beta = 0$, we get Theorem 3.1 of [2].

By applying Corollary 2.1, inequality (2.15), the following theorem is proved in a similar way.

Theorem 3.2. *Let p, q be Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta < -1$, $f \in Q^\beta$, $\alpha \in \mathbb{R}$, and $\omega \in Q_\alpha$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$. Then, the inequality*

$$\|H^*_{\omega,\alpha}f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}}|\beta+1|^{-\frac{1}{\underline{p}'}}C_{p,q}(1+M-m)^{\frac{1}{\underline{p}}}\Big\|\frac{y^{\frac{1}{\underline{p}'}}\|\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\|_{L_{q(x)}(0,y)}}{\omega_1(x)}\Big\|^{BT}_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.2}$$

holds, where $C_{p,q}$ and M, m are the constants in Theorem 3.1.

By using Corollary 2.1, inequality (2.16), the following theorem is proved similarly.

Theorem 3.3. *Let p, q Lebesgue be measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta > -1$, $f \in Q^\beta$, $\alpha \in \mathbb{R}$, and $\omega \in Q^\alpha$. Suppose that ω_1, ω_2 are weight functions defined on $(0,\infty)$.*

Then, the inequality

$$\|H_{\omega,\alpha}f\|_{L_{q(x),\omega_2}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}}C_{p,q}(1+M-m)^{\frac{1}{\underline{p}}}$$

$$\times\Big\|\frac{\|[y^{-\beta}(x^{\beta+1}-y^{\beta+1})]^{\frac{1}{\underline{p}'}}\frac{\omega_2(x)\omega(x)}{x^\alpha W(x)}\|_{L_{q(x)}(y,\infty)}}{\omega_1(x)}\Big\|^{BT}_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.3}$$

holds, where $C_{p,q}$ and M, m are the constants in Theorem 3.1.

Remark 4. Since, under the assumptions of Theorem 3.3, $\|f\|^{BT}_{L_{p(x)}(\Omega)} = \|f\|_{L_{p(x)}(\Omega)}$, if we take $\alpha = 0$, $\omega(x) = 1$ in inequality (3.3), we get inequality (2.7) of Theorem 2.1, with an improved constant. Moreover, by putting $\beta = 0$, we get Theorem 3.2 of [2].

Remark 5. For constant $p(x) = q(x) = p$ and $\omega_1(x) = \omega_2(x) = x^\alpha$ and $\omega(x) = 1$, inequalities (3.1) and (3.3) with sharp constants, were proved in [3] and if $\beta = 0$ earlier in [5].

Now we consider the case $\beta = -1$.

Theorem 3.4. *Let p, q be Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ for $x \in (0,\infty)$, $\beta = -1$, and $\alpha \in \mathbb{R}$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

1) If $f \in Q_{-1}$ and $\omega \in Q_\alpha$, then the inequality

$$\|H^*_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} C_{p,q} (1 + M - m)^{\frac{1}{\underline{p}}} \left\| \frac{t^{\frac{1}{\overline{p}'}} (\ln \frac{t}{x})^{\frac{1}{\overline{p}'}} \| \frac{\omega_2(x)}{x} \|_{L_{q(x)}(0,t)}}{\omega_1(x)} \right\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.4}$$

holds.

2) If $f \in Q^{-1}$ and $\omega \in Q^\alpha$, then the inequality

$$\|H_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} C_{p,q} (1 + M - m)^{\frac{1}{\underline{p}}} \left\| \frac{t^{\frac{1}{\overline{p}'}} (\ln \frac{x}{t})^{\frac{1}{\overline{p}'}} \| \frac{\omega_2(x)}{x} \|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|^{BT}_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.5}$$

holds.

Proof. 1. By using inequality (2.17) with $p = \underline{p}$, we obtain

$$\|H^*_{\omega,\alpha} f\|_{L_{q(x),\omega_2(x)}(0,\infty)} = \|\omega_2(x) H^*_{\omega,\alpha} f\|_{L_{q(x)}(0,\infty)} = \left\| \frac{\omega_2(x)}{W(x)} \int_x^\infty f(y) y^{-\alpha} \omega dy \right\|_{L_{q(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}} \left\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \left(\int_x^\infty \left(y \ln \frac{y}{x} \right)^{p-1} f^p dy \right)^{\frac{1}{p}} \right\|_{L_{q(x)}(0,\infty)}.$$

We put

$$J_1 = \left\| \frac{\omega_2(x)\omega(x)}{x^\alpha W(x)} \left(\int_x^\infty \left(y \ln \frac{y}{x} \right)^{p-1} f^p dy \right)^{\frac{1}{p}} \right\|_{L_{q(x)}(0,\infty)}.$$

The rest is similar to the proof of Theorem 3.1.

2. We apply inequality (2.18) with $p = \underline{p}$ and the rest of the proof is similar to that of Theorem 3.1. □

Remark 6. Since, under the assumptions of Theorem 3.4, $\|f\|^{BT}_{L_{p(x)}(\Omega)} = \|f\|_{L_{p(x)}(\Omega)}$, if we set $\alpha = 0$, $\omega(x) = 1$ in inequalities (3.4) and (3.5), we obtain inequalities (2.8) and (2.9), respectively, of Theorem 2.2 with improved constants.

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References

[1] R.A. Bandaliev, *On an inequality in Lebesgue space with mixed norm and with variable summability.* Matem. Zametki 84 (2008), no. 3, 323–333 (in Russian). English translation in Math. Notes, 84 (2008), no. 3, 303–313.

[2] R.A. Bandaliev, *On Hardy-type inequalities in weighted variable exponent Lebesgue spaces for 0<p<1.* Eurasian Math. Journal 4 (2013), no. 4, 5–16.

[3] J. Bergh, V.I. Burenkov, L.-E. Persson, *Best constants in reversed Hardy's inequalities for quasimonotone functions.* Acta Sci. Math. (Szeged) 59 (1994), 223–241.

[4] V.I. Burenkov, T.V. Tararykova, *About Hölder's inequality in Lebesgue spaces with variable summability.* Contemporary Mathematics. Fundamental Directions 67 (2021), no. 3, 472-482 (in Russian).

[5] V.I. Burenkov, *On the best constants in Hardy's inequalities for 0<p<1 for monotone functions.* Trudy Matem. Inst. Steklov 194 (1992), 58–62 (in Russian). English translation in Proc. Steklov Inst. Math. 194 (1993), no. 4, 59–63.

[6] O. Kovachik, J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$.* Czechoslovak Math. J. 41 (1991), no. 4, 592–618.

[7] W. Orlicz, *Über konjugierte Exponentenfolgen.* Stud. Math. 3 (1931), 200–212.

[8] M. Rúžička, *Electrorheological fluids : modeling and mathematical theory.* Lecture Notes in Mathematics. 1748, Springer, Berlin (2000).

[9] S.G. Samko, *Differentiation and integration of variable order and the variable exponent Lebesgue spaces*, Proc. Inter. Conf. "Operator theory for complex and hypercomplex analysis", Mexico, 1994, Contemp. Math. 212 (1998), 203-219.

[10] A. Senouci, A. Zanou, *Some integral inequalities for quasi-monotone functions in weighted variable exponent Lebesgue spaces with* $0 < p(x) < 1$. Eurasian Math. Journal 4 (2020), no. 4, 58–65.

[11] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory.* Izv. Akad. Nauk SSSR Ser. Matem. 50 (1986), 675–710, (in Russian). English translation in Math. USSR, Izv. 29 (1987), 33–66.

[12] V.V. Zhikov, *On some variational problems.* Russian J. Math. Phys. 5 (1997), no. 1, 105–116.

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