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BOKAYEV NURZHAN ADILKHANOVICH

(to the 70th birthday)

January 5, 2026, marks the 70th birthday of Nurzhan Adilkhanovich Bokayev, Doctor of Physical and Mathematical Sciences (1996), Professor (2002), member of the Editorial Board of the Eurasian Mathematical Journal (2010).

Nurzhan Adilkhanovich Bokayev was born on 5 January, 1956 in the village of Urnek, Karabalyk District, Kostanay Region. He graduated in 1972, with a gold medal from the Burlin Secondary School in the district. That same year, he entered the Mathematics Department of Karaganda State University and graduated with honors in 1977. From 1978 to 1979, he served in the Soviet Army. In 1980, he completed an internship, and from 1981 to 1984, he studied in the graduate program at Lomonosov Moscow State University in the Department of Function Theory and Functional Analysis. In 1985, he defended his candidate's dissertation there under the supervision of Corresponding Member of the Academy of Sciences of the USSR D.E. Menshov and Professor V.A. Skvortsov. In 1996, he defended his doctoral dissertation, “Fourier Coefficients and Uniqueness Theorems for Series in Generalized Walsh and Haar Systems”, at the Institute of Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan, speciality Mathematical Analysis (01.01.01).

After completing his postgraduate studies, he worked as a lecturer, senior lecturer, associate professor, and professor in the Department of Mathematical Analysis at E.A. Buketov Karaganda State University (1985-1999). He headed the Department of Mathematics and Mathematical Modeling (1996-1999), and was a dean of the Faculty of Mathematics at E.A. Buketov Karaganda State University (1999-2005). Since 2005, he has been a professor in the Faculty of Mechanics and Mathematics at the L.N. Gumilyov Eurasian National University. From 2009 to 2018, he was the Head of the Department of Higher Mathematics at the L.N. Gumilyov Eurasian National University, and from 2018 to the present, he has been a professor in the Department of Fundamental Mathematics.

Professor Bokaev's research focuses on problems in function theory and functional analysis, the theory of orthogonal series for generalized Walsh and Haar systems, and operator theory in various function spaces. He has proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, and constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems. He obtained conditions for functions to belong to various functional classes in terms of the Fourier coefficients of generalized Haar and Walsh systems, and embedding criteria for Nikol'skii-Besov spaces constructed on the basis of multiplicative systems. He also obtained conditions for the boundedness and compactness of the commutator of the Riesz potential in general Morrey-type spaces, and conditions for boundedness of generalizedRiesz and Bessel potentials and generalized fractional-maximal operators in rearrangement-invariant spaces.

His co-authors include Professor V.A. Skvortsov (Moscow State University, Moscow), Professors V.I. Burenkov and M.L. Goldman (Peoples' Friendship University of Russia (RUDN University). Moscow), Dr. A. Gogatishvili (Institute of Mathematics of the Czech Academy of Sciences, Prague). His doctoral students' foreign advisors include Professors W. Sickel (Friedrich-Schiller-Universitity, Jena, Germany), Massimo Lanza de Cristoforis (University of Padova, Padova, Italy), V. Ruzhansky (Ghent University, Ghent, Belgium), U. Goginava (United Arab Emirates University, Al Ain, United Arab Emirates), and E. Panakhov (Institute of Applied Mathematics at Baku State University, Baku, Azerbaijan).

Under his supervision, 15 dissertations (4 candidate's and 11 PhD) were defended. He has published over 220 scientific papers, 2 monographs and 2 textbooks.

He is a three-time recipient of the state grant “Best University Teacher” of the Republic of Kazakhstan (2006, 2010, 2024) and served as Vice President of the Mathematical Society of Turkic-Speaking Countries (2014-2023). He was awarded the “For Contribution to the Development of Science” badge (2022).

Over the last ten years, he has been and continues to be a head of more than 5 national and international funded projects.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Nurzhan Adilkhanovich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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ANISOTROPIC MORREY-TYPE SPACES AND THEIR INTERPOLATION PROPERTIES

J.G. Jumabayeva, E.D. Nursultanov

Communicated by R. Oinarov

Key words: anisotropic local Morrey-type spaces, anisotropic generalized Morrey-type spaces, embedding properties of anisotropic Morrey-type spaces, interpolation properties of anisotropic Morrey-type spaces.

AMS Mathematics Subject Classification: 46E30, 46B70, 46B20.

Abstract In this paper, there are defined the anisotropic local Morrey-type spaces $LM_{\bar{p},\bar{q}}^{\bar{\lambda}}$ and the anisotropic generalized Morrey-type spaces $M_{\bar{p},\bar{q}}^{\bar{\lambda}}$, where $\bar{p}$, $\bar{q}$, and $\bar{\lambda}$ are vectors. The spaces $LM_{\bar{p},\bar{q}}^{\bar{\lambda}}$ allow relaxation of the conditions on the parameter $\bar{\lambda}$, namely, the components of the given vector can take any real value, i.e., $-\infty < \lambda_i < \infty$, $i = \overline{1,d}$, in contrast to previously studied spaces. The embedding properties of the defined spaces are investigated. Additionally, an anisotropic interpolation method is considered, which allows the study of the interpolation properties of these spaces.

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1 Introduction

To study specific operators of analysis, such as the Riesz operator, the O'Neil operator, the convolution operator, and others, it is important to correctly choose the spaces in which various properties of these operators can be described. In recent decades, the Morrey spaces M_p^λ and their various generalizations [17, 7, 8, 1] have played an important role in analysis. At the same time, there are only few studies dedicated to the anisotropic Morrey-type spaces and local Morrey-type spaces, in which functions have different characteristics for each variable. This is because traditional methods are not always effective for such spaces. For example, the real interpolation method is not applicable to the anisotropic local Morrey spaces.

The classical Morrey space was introduced in the work of Morrey [17] in 1938 in connection with the study of the properties solutions to quasilinear elliptic differential equations.

Let $0 \le \lambda \le \frac{d}{p}$ and $0 < p \le \infty$. The Morrey space $M_p^\lambda(\mathbb{R}^d)$ is the set of all Lebesgue measurable functions $f \in L_p^{loc}(\mathbb{R}^d)$ for which the following quantity is finite:

$$\|f\|_{M_p^\lambda} \equiv \|f\|_{M_p^\lambda(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B_r(x))}.$$

Here, $B_r(x)$ is the ball centered at point x with radius $r > 0$. Note that if $\lambda = 0$, then $M_p^0(\mathbb{R}^d) = L_p(\mathbb{R}^d)$; if $\lambda = \frac{d}{p}$ and $0 < p < \infty$, then $M_p^{\frac{d}{p}}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$; and if $\lambda < 0$ or $\lambda > \frac{d}{p}$, then $M_p^\lambda = \Theta$, where Θ is the set of all functions equivalent to zero on $\mathbb{R}^d$.

The question of interpolation in classical Morrey spaces was addressed in the work of Stampacchia [21] in 1964, Campanato and Murthy [16] in 1965, as well as in the work of Peetre [20] in 1969.

Peetre's studies led to the conclusion that $\left(M_p^{\lambda_0}, M_p^{\lambda_1}\right)_{\theta,\infty} \hookrightarrow M_p^{\lambda}$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, $0 < \theta < 1$. Note that this embedding is strict. From the work of Blasco, Ruiz, and Vega [6], it follows that $\left(M_{p_0}^{\lambda_0}, M_{p_1}^{\lambda_1}\right)_{\theta,\infty} \neq M_p^{\lambda}$.

In the work of Burenkov V.I. and Guliyev H.V. [10], the local Morrey-type spaces were introduced

$$LM_{p,q,x}^{\lambda} = \left\{ f : \left(\int_0^{\infty} \left(t^{-\lambda} \|f\|_{L_p(B_t(x))}\right)^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

where $\lambda > 0$. The interpolation properties were studied by Burenkov V.I. and Nursultanov E.D. in works [9]-[14]. In particular, it was established in [9], [12] that the scale of the local Morrey-type spaces $LM_{p,\theta}^{\lambda} \equiv LM_{p,\theta,0}^{\lambda}$ is closed under interpolation with respect to the upper parameter, that is, the following equality holds:

$$\left(LM_{p,q}^{\lambda_0}, LM_{p,q}^{\lambda_1}\right)_{\theta,\infty} = LM_{p,q}^{\lambda},$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, $0 < \theta < 1$.

In the present work, the anisotropic local Morrey-type spaces and anisotropic generalized Morrey-type spaces are defined and their properties are investigated. The apparatus for studying these spaces, namely, the anisotropic interpolation method was developed in the works of Nursultanov E.D. and Bekmaganbetov K.A. [18] and [4].

2 Anisotropic local Morrey-type spaces $LM_{\bar{p},\bar{q}}^{\bar{\lambda}}(\mathbb{T})$

In the work of Nursultanov E.D. and Suragan D. [19], the interpolation properties of the following local Morrey-type spaces were studied.

Let $k \in \mathbb{Z}$, and let G_k denote the set of all cubes in $\mathbb{R}^d$ of the form

$$[0, 2^k)^d + 2^k m, \quad m \in \mathbb{Z}^d. \tag{2.1}$$

It is obvious that

$$\mathbb{R}^d = \bigsqcup_{Q \in G_k} Q, \tag{2.2}$$

where $\bigsqcup Q$ denotes the union of mutually disjoint sets.

The set $\mathbb{G} = \bigcup\limits_{k\in\mathbb{Z}} G_k$ is called the family of dyadic cubes in $\mathbb{R}^d$. Note that each cube $Q \in G_k$ is subdivided into 2^d cubes from G_{k-1}.

Let μ be the d-dimensional Lebesgue measure in $\mathbb{R}^d$. The family of mutually disjoint cubes $\mathbb{T} = \{Q\} \subset \mathbb{G}$ is called a local decomposition of the space $\mathbb{R}^d$ if:

1. $\mu\left(\mathbb{R}^d \setminus \bigsqcup_{Q\in G_k} Q\right) = 0$;
2. $|\mathbb{T} \cap G_k| < \infty$.

Here and in the sequel, $|A|$ denotes the number of elements in the set A.

Let $\lambda \in \mathbb{R}$, $0 < p, q \leq \infty$, and $\mathbb{T}$ be a local decomposition of $\mathbb{R}^d$. The local Morrey-type space $LM_{p,q}^{\lambda}(\mathbb{T})$ is defined as the set of all measurable functions f, for which

$$\|f\|_{LM_{p,q}^{\lambda}(\mathbb{T})} = \left(\sum_{k\in\mathbb{Z}} \left(2^{-k\lambda} \sum_{Q \in \mathbb{T}_k = \mathbb{T}\cap G_k} \|f\|_{L_p(Q)} \right)^q \right)^{\frac{1}{q}} < \infty.$$

Now we define the anisotropic local Morrey-type spaces.

Let $\overline{n} = (n_1, \dots, n_d)$ where $n_i \in \mathbb{N}$, $|\overline{n}| = n_1 + \dots + n_d$, and let $\overline{k} = (k_1, \dots, k_d)$ where $k_i \in \mathbb{Z}$. Denote $G_{\overline{k}} = \{Q = Q_1 \times \dots \times Q_d : Q_i \subset G_{k_i}, i = 1, \dots, d\}$. The family of all mutually non-intersecting cubes $\mathbb{T}_i = \{Q_i\} \subset G_{k_i}$ is called a local decomposition of the space $\mathbb{R}^{n_i}$, and the families $\mathbb{T}_1, \dots, \mathbb{T}_d$ are local decompositions of the spaces $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_d}$, respectively. The family of all mutually disjoint parallelepipeds $\mathbb{T} = \mathbb{T}_1 \times \dots \times \mathbb{T}_d = \{Q = Q_1 \times \dots \times Q_d : Q_i \subset \mathbb{T}_i, i = 1, \dots, d\}$ is called, respectively, a local decomposition of the space $\mathbb{R}^{|\overline{n}|}$.

Let $\overline{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$, vectors $\overline{p} = (p_1, \dots, p_d)$ and $\overline{q} = (q_1, \dots, q_d)$ be such that $0 < p_i, q_i \leq \infty$, $i = \overline{1, d}$, and $\mathbb{T}_{\overline{k}} = \mathbb{T} \cap G_{\overline{k}}$.

For arbitrary vectors $\overline{a} = (a_1, \dots, a_d)$ and $\overline{b} = (b_1, \dots, b_d)$, let $\langle \overline{a}, \overline{b} \rangle$ denote $\langle \overline{a}, \overline{b} \rangle = a_1 b_1 + \dots + a_d b_d$.

We define the anisotropic local Morrey-type space $LM^{\overline{\lambda}}_{\overline{p},\overline{q}}(\mathbb{T})$ as the set of all measurable functions f, for which

$$\|f\|_{LM^{\overline{\lambda}}_{\overline{p},\overline{q}}(\mathbb{T})} = \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\langle \overline{k}, \overline{\lambda} \rangle} \sum_{Q \in \mathbb{T}_{\overline{k}}} \|f\|_{L_{\overline{p}}(Q)} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}} < \infty. \tag{2.3}$$

In particular, in $q_i = \infty$ for $i = \overline{1, d}$, the expressions

$$\left(\int_{\Omega_d} \cdots \left(\int_{\Omega_1} |\varphi(\overline{t})|^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \cdots \frac{dt_d}{t_d} \right)^{\frac{1}{q_d}} \quad \text{and} \quad \left(\sum_{k_d \in \Omega_d} \cdots \left(\sum_{k_1 \in \Omega_1} |a_{\overline{k}}|^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}}$$

are understood as $\sup\limits_{\overline{t} \in \Omega} |\varphi(\overline{t})|$ and $\sup\limits_{\overline{k} \in \Omega} |a_{\overline{k}}|$, respectively, where $\Omega = \Omega_1 \times \dots \times \Omega_d$.

For the anisotropic local Morrey-type spaces the following lemma holds.

Lemma 2.1. *(i) For vectors* $\overline{n} = (n_1, \dots, n_d)$, $\overline{p}_0 = (p^0_1, \dots, p^0_d)$, $\overline{p}_1 = (p^1_1, \dots, p^1_d)$, *and* $\overline{q} = (q_1, \dots, q_d)$ *such that* $0 < p^0_i < p^1_i < \infty$, $0 < q_i \leq \infty$ *for* $i = 1, \dots, d$, *we have*

$$LM^{\overline{\alpha}}_{\overline{p}_1,\overline{q}}(\mathbb{T}) \hookrightarrow LM^{\overline{\beta}}_{\overline{p}_0,\overline{q}}(\mathbb{T}),$$

where $\overline{\alpha} = (\alpha_1, \dots, \alpha_d)$ *and* $\overline{\beta} = (\beta_1, \dots, \beta_d)$ *such that* $\beta_i = \alpha_i - \frac{n_i}{p^1_i} + \frac{n_i}{p^0_i}$ *for* $i = 1, \dots, d$, *and* "$\hookrightarrow$" *denotes the continuous embedding.*

(ii) For vectors $\overline{p} = (p_1, \dots, p_d)$, $\overline{q}_0 = (q^0_1, \dots, q^0_d)$, $\overline{q}_1 = (q^1_1, \dots, q^1_d)$ *such that* $0 < q^0_i < q^1_i \leq \infty$ *for* $i = 1, \dots, d$, *we have*

$$LM^{\overline{\lambda}}_{\overline{p},\overline{q}_0}(\mathbb{T}) \hookrightarrow LM^{\overline{\lambda}}_{\overline{p},\overline{q}_1}(\mathbb{T}).$$

Proof. Let us prove (i). Let $f \in LM^{\bar{\alpha}}_{\bar{p}_1,\bar{q}}$. Applying Hölder's inequality and taking into account that $|Q_i| = 2^{k_i n_i}$, we obtain

$$\|f\|_{LM^{\bar{\beta}}_{\bar{p}_0,\bar{q}}} \leq \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\langle \bar{k}, \bar{\beta} \rangle} \sum_{Q \in \mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}_1}(Q)} \prod_{i=1}^{d} |Q_i|^{\frac{1}{p^0_i} - \frac{1}{p^1_i}} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}}$$

$$\leq \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\sum\limits_{i=1}^{d} k_i(\beta_i - (\frac{n_i}{p^1_i} - \frac{n_i}{p^0_i}))} \sum_{Q \in \mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}_1}(Q)} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \right)^{\frac{1}{q_d}} = \|f\|_{LM^{\bar{\alpha}}_{\bar{p}_1,\bar{q}}}.$$

Let us prove (ii). Let $f \in LM^{\bar{\lambda}}_{\bar{p},\bar{q}_0}$. Applying Jensen's inequality, we obtain

$$\|f\|_{LM^{\bar{\lambda}}_{\bar{p},\bar{q}_1}} = \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\langle \bar{k}, \bar{\lambda} \rangle} \sum_{Q \in \mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}}(Q)} \right)^{q_1^1} \right)^{\frac{q_2^1}{q_1^1}} \cdots \right)^{\frac{1}{q_d^1} \frac{q_d^0}{q_d^0}}$$

$$\leq \left(\sum_{k_d \in \mathbb{Z}} \left(\sum_{k_{d-1} \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\langle \bar{k}, \bar{\lambda} \rangle} \sum_{Q \in \mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}}(Q)} \right)^{q_1^1} \right)^{\frac{q_2^1}{q_1^1}} \cdots \right)^{\frac{q_{d-1}^0}{q_{d-1}^1} \frac{q_d^0}{q_{d-1}^0}} \right)^{\frac{1}{q_d^0}}$$

$$\cdots$$

$$\leq \left(\sum_{k_d \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} \left(2^{-\langle \bar{k}, \bar{\lambda} \rangle} \sum_{Q \in \mathbb{T}_{\bar{k}}} \|f\|_{L_{\bar{p}}(Q)} \right)^{q_1^0} \right)^{\frac{q_2^0}{q_1^0}} \cdots \right)^{\frac{1}{q_d^0}} = \|f\|_{LM^{\bar{\lambda}}_{\bar{p},\bar{q}_0}}.$$

□

Note that in [11] periodic Morrey spaces were studied, while in [15] local Morrey-type spaces with mixed quasi-norms were considered.

Lemma 2.2. [Hardy's inequality] *For* $\alpha > 0$, $0 < q, h \leq \infty$, *and* $d > 1$, *the following inequalities hold:*

$$\left(\sum_{k=0}^{\infty} \left(d^{-\alpha k} \left(\sum_{r=0}^{k} |b_r|^h \right)^{\frac{1}{h}} \right)^q \right)^{\frac{1}{q}} \leq c_{\alpha,q} \left(\sum_{k=0}^{\infty} \left(d^{-\alpha k} |b_k| \right)^q \right)^{\frac{1}{q}},$$

$$\left(\sum_{k=0}^{\infty} \left(d^{\alpha k} \left(\sum_{r=k}^{\infty} |b_r|^h \right)^{\frac{1}{h}} \right)^q \right)^{\frac{1}{q}} \leq c_{\alpha,q} \left(\sum_{k=0}^{\infty} \left(d^{\alpha k} |b_k| \right)^q \right)^{\frac{1}{q}}$$

for some $c_{\alpha,q} > 0$, *depending only on* α *and* q.

3 Interpolation of anisotropic local Morrey-type spaces $LM^{\overline{\lambda}}_{\overline{p},\overline{q}}(\mathbb{T})$

Let $\overline{A}_0 = (A_1^0, ..., A_d^0)$, $\overline{A}_1 = (A_1^1, ..., A_d^1)$ be a pair of anisotropic spaces. $E = \{\varepsilon = (\varepsilon_1, ..., \varepsilon_d) : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1,\ i = 1, ..., d\}$ be the vertices of the d-dimensional unit cube. For an arbitrary $\varepsilon \in E$, consider the space $A_\varepsilon = (A_1^{\varepsilon_1}, ..., A_d^{\varepsilon_d})$ with the norm

$$\|a_\varepsilon\|_{A_\varepsilon} = \| \ldots \|a\|_{A_1^\varepsilon} \ldots \|_{A_d^{\varepsilon_d}}. \tag{3.1}$$

Let us define the method of anisotropic interpolation. Let $A = (\overline{A}_0, \overline{A}_1)$ be a compatible pair of Banach spaces [5]. For an arbitrary vector $\overline{k} \in \mathbb{Z}^d$ we denote $2^{\overline{k}} = (2^{k_1}, ..., 2^{k_d})$. Let us consider the K–functional

$$K(2^{\bar{k}}, a; \bar{A}_0, \bar{A}_1) = \inf\left\{\sum_{\varepsilon\in E} 2^{\langle \bar{k},\varepsilon\rangle}\|a_\varepsilon\|_{A_\varepsilon} :\ a = \sum_{\varepsilon\in E} a_\varepsilon,\ a_\varepsilon \in A_\varepsilon\right\}.$$

If vectors $\overline{q} = (q_1, ..., q_d)$, $\overline{\theta} = (\theta_1, ..., \theta_d)$ are such that $0 < q_i < \infty$, $0 < \theta_i < 1$, then

$$A_{\overline{\theta},\overline{q}} = (\overline{A}_0, \overline{A}_1)_{\overline{\theta},\overline{q}} = \left\{ a = \sum_{\varepsilon\in E} a_\varepsilon,\ a_\varepsilon \in A_\varepsilon : \right.$$

$$\left. \|a\|_{A_{\overline{\theta},\overline{q}}} = \left(\sum_{k_d\in\mathbb{Z}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \left(2^{-\langle\overline{\theta},\overline{k}\rangle} K(2^{\overline{k}}, a)\right)^{q_1}\right)^{\frac{q_2}{q_1}} \cdots\right)^{\frac{1}{q_d}} < \infty \right\}$$

and for $\overline{q} = \overline{\infty}$

$$A_{\overline{\theta},\overline{\infty}} = (\overline{A}_0, \overline{A}_1)_{\overline{\theta},\overline{\infty}} = \left\{ a = \sum_{\varepsilon\in E} a_\varepsilon,\ a_\varepsilon \in A_\varepsilon :\ \|a\|_{A_{\overline{\theta},\overline{\infty}}} = \sup_{\overline{k}\in\mathbb{Z}^d} 2^{-\langle\overline{\theta},\overline{k}\rangle} K(2^{\overline{k}}, a) < \infty \right\}.$$

For some vector $\overline{b} = (b_1, ..., b_d)$, $b_i > 1$, $i = \overline{1, d}$, the K-functional takes the form

$$K(b^{\bar{k}}, a; \bar{A}_0, \bar{A}_1) = \inf\left\{\sum_{\varepsilon\in E} \prod_{i=1}^{d} b_i^{k_i\theta_i}\|a_\varepsilon\|_{A_\varepsilon} :\ a = \sum_{\varepsilon\in E} a_\varepsilon,\ a_\varepsilon \in A_\varepsilon\right\}$$

and

$$\|a\|_{A_{\overline{\theta},\overline{q}}} = \left(\sum_{k_d\in\mathbb{Z}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \left(\prod_{i=1}^{d} b_i^{-k_i\theta_i} K(\overline{b}^{\overline{k}}, a)\right)^{q_1}\right)^{\frac{q_2}{q_1}} \cdots\right)^{\frac{1}{q_d}}.$$

Lemma 3.1. *(see [18]) Let $\{A_\varepsilon\}_{\varepsilon\in E}$ and $\{B_\varepsilon\}_{\varepsilon\in E}$ be Banach spaces that are subspaces of some linear space. Let vectors $\overline{\theta} = (\theta_1, ..., \theta_d)$ and $\overline{r} = (r_1, ..., r_d)$ be such that $0 < \theta_i < 1$, $0 < r_i \le \infty$. If T is a linear operator such that $T: \quad A_\varepsilon \to B_\varepsilon$ with the norm M_ε for any $\varepsilon \in E$ (here the spaces A_ε and B_ε are defined by norm (3.1)), then*

$$T: \quad A_{\overline{\theta},\overline{r}} \to B_{\overline{\theta},\overline{r}}$$

with the norm $\|T\| \le \max_{\varepsilon\in E} M_\varepsilon$.

Here we may mention the papers [2] and [3].

Theorem 3.1. *Let vectors $\overline{\lambda}_0 = (\lambda_1^0, ..., \lambda_d^0)$, $\overline{\lambda}_1 = (\lambda_1^1, ..., \lambda_d^1)$, $\overline{\theta} = (\theta_1, ..., \theta_d)$, $\overline{p} = (p_1, ..., p_d)$, $\overline{q}_0 = (q_1^0, ..., q_d^0)$, $\overline{q}_1 = (q_1^1, ..., q_d^1)$, $\overline{q} = (q_1, ..., q_d)$, $\overline{n} = (n_1, ..., n_d)$ be such that $0 < q_i^0, q_i^1, q_i \le \infty$, $0 < p_i \le \infty$, $-\infty < \lambda_i^0 < \lambda_i^1 < +\infty$, $\theta_i \in (0, 1)$, $n_i \in \mathbb{N}$, $i = \overline{1, d}$ and $\mathbb{T}$ is a local partition of $\mathbb{R}^{|\overline{n}|}$. Then*

$$\left(LM^{\overline{\lambda}_0}_{\overline{p},\overline{q}_0}(\mathbb{T}), LM^{\overline{\lambda}_1}_{\overline{p},\overline{q}_1}(\mathbb{T})\right)_{\overline{\theta},\overline{q}} = LM^{\overline{\lambda}}_{\overline{p},\overline{q}}(\mathbb{T}),$$

where the vector $\overline{\lambda} = (\lambda_1, ..., \lambda_d)$ is such that $\lambda_i = (1 - \theta_i)\lambda_i^0 + \theta_i\lambda_i^1$, $i = \overline{1, d}$.

Proof. Let $r = \min\limits_{1\le i\le d}\{q_i, q_i^0, q_i^1\}$, $\overline{r} = (r, ..., r)$. Since for any $\overline{\lambda}$ with coordinates $\lambda_i \in \mathbb{R}$, $i = \overline{1,d}$

$$LM_{\overline{p},\overline{r}}^{\overline{\lambda}}(\mathbb{T}) \hookrightarrow LM_{\overline{p},\overline{q}}^{\overline{\lambda}}(\mathbb{T}) \hookrightarrow LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}}(\mathbb{T}),$$

it suffices to prove

$$\left(LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}(\mathbb{T}), LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}(\mathbb{T})\right)_{\overline{\theta},\overline{q}} \hookrightarrow LM_{\overline{p},\overline{q}}^{\overline{\lambda}}(\mathbb{T}) \hookrightarrow \left(LM_{\overline{p},\overline{r}}^{\overline{\lambda}_0}(\mathbb{T}), LM_{\overline{p},\overline{r}}^{\overline{\lambda}_1}(\mathbb{T})\right)_{\overline{\theta},\overline{q}}. \tag{3.2}$$

Let us first prove the left relation in (3.2). Let $f \in \left(LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}(\mathbb{T}), LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}(\mathbb{T})\right)_{\overline{\theta},\overline{q}}$, $\overline{m} \in \mathbb{Z}^d$, $\varepsilon = (\varepsilon_1, ..., \varepsilon_d) \in E$. Let us denote $\overline{\lambda}_\varepsilon = (\lambda_1^{\varepsilon_1}, ..., \lambda_d^{\varepsilon_d})$, where $\lambda_i^{\varepsilon_i} = \begin{cases}\lambda_i^0, & \varepsilon_i = 0,\\ \lambda_i^1, & \varepsilon_i = 1.\end{cases}$ For an arbitrary representation $f = \sum\limits_{\varepsilon\in E} f_\varepsilon$, where $f_\varepsilon \in LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_\varepsilon}(\mathbb{T})$ we obtain

$$\sum_{Q\in\mathbb{T}_{\overline{m}}} \|f\|_{L_{\overline{p}}(Q)} \le \sum_{Q\in\mathbb{T}_{\overline{m}}} \sum_{\varepsilon\in E} \|f_\varepsilon\|_{L_{\overline{p}}(Q)}$$

$$= \sum_{Q\in\mathbb{T}_{\overline{m}}} \left(2^{\langle\overline{m},\overline{\lambda}_0\rangle} \sum_{\varepsilon\in E} 2^{\langle\overline{m},\overline{\lambda}_\varepsilon-\overline{\lambda}_0\rangle} 2^{-\langle\overline{m},\overline{\lambda}_\varepsilon\rangle} \|f_\varepsilon\|_{L_{\overline{p}}(Q)}\right)$$

$$\le 2^{\langle\overline{m},\overline{\lambda}_0\rangle} \left(\sum_{\varepsilon\in E} 2^{\langle\overline{m},\overline{\lambda}_\varepsilon-\overline{\lambda}_0\rangle} \|f_\varepsilon\|_{LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_\varepsilon}(\mathbb{T})}\right).$$

Taking into account the arbitrariness of the representation $f = \sum\limits_{\varepsilon\in E} f_\varepsilon$ and the definition of the K-functional, we obtain

$$\sum_{Q\in\mathbb{T}_{\overline{m}}} \|f\|_{L_{\overline{p}}(Q)} \le 2^{\langle\overline{m},\overline{\lambda}_0\rangle} K(2^{\langle\overline{m},\overline{\lambda}_\varepsilon\rangle}, f; LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_\varepsilon}).$$

Let $b_i = 2^{\lambda_i^1-\lambda_i^0}$, $i = \overline{1,d}$, then from the last inequality we obtain

$$\|f\|_{LM_{\overline{p},\overline{q}}^{\overline{\lambda}}(\mathbb{T})} = \left(\sum_{k_d\in\mathbb{Z}} \sum_{b_d^{k_d}\le 2^{m_d}<b_d^{k_d+1}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \sum_{b_1^{k_1}\le 2^{m_1}<b_1^{k_1+1}} \left(2^{-\langle\overline{m},\overline{\lambda}\rangle} \sum_{Q\in\mathbb{T}_{\overline{m}}} \|f\|_{L_{\overline{p}}(Q)}\right)^{q_1}\right)^{\frac{q_2}{q_1}} \cdots\right)^{\frac{1}{q_d}}$$

$$\le \left(\sum_{k_d\in\mathbb{Z}} \sum_{b_d^{k_d}\le 2^{m_d}<b_d^{k_d+1}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \sum_{b_1^{k_1}\le 2^{m_1}<b_1^{k_1+1}} \left(2^{-\langle\overline{m},\overline{\lambda}-\overline{\lambda}_0\rangle} K(2^{\langle\overline{m},\overline{\lambda}_\varepsilon\rangle}, f)\right)^{q_1}\right)^{\frac{q_2}{q_1}} \cdots\right)^{\frac{1}{q_d}}$$

$$\le c\left(\sum_{k_d\in\mathbb{Z}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \left(\prod_{i=1}^d b_i^{-\theta_i k_i} K(b^{\overline{k}}, f)\right)^{q_1}\right)^{\frac{q_2}{q_1}} \cdots\right)^{\frac{1}{q_d}}$$

$$\le c\|f\|_{\left(LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}(\mathbb{T}), LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}(\mathbb{T})\right)_{\overline{\theta},\overline{q}}},$$

here $c > 0$ depends only on the parameters $\overline{\lambda_0}$, $\overline{\lambda_1}$. Therefore,

$$\left(LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}, LM_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}\right)_{\overline{\theta},\overline{q}} \hookrightarrow LM_{\overline{p},\overline{q}}^{\overline{\lambda}}.$$

Let us prove the second embedding in (3.2). Let $\overline{k}=(k_1,...,k_d):\ k_i\in\mathbb{Z},\ b_i=2^{\lambda_i^1-\lambda_i^0}>1,\ i=\overline{1,d}$, $f\in LM^{\overline{\lambda}}_{\overline{p},\overline{q}}(\mathbb{T})$. For an arbitrary $\varepsilon\in E$ we will consider the following intervals:

$$\Delta_{\varepsilon_i}=\begin{cases}(-\infty;k_i]\cap\mathbb{Z}, & \text{for } \varepsilon_i=0,\\ [k_i+1;+\infty)\cap\mathbb{Z}, & \text{for } \varepsilon_i=1\end{cases}$$

and $\Delta_\varepsilon=\Delta_{\varepsilon_1}\times\cdots\times\Delta_{\varepsilon_d}$. We define the functions f_ε as follows:

$$f_\varepsilon(\overline{x})=\begin{cases} f(\overline{x}), & \text{for } \overline{x}\in\bigsqcup\limits_{\overline{m}\in\Delta\varepsilon}\ \bigsqcup\limits_{Q\in\mathbb{T}_{\overline{m}}}Q,\\ 0, & \text{for } \overline{x}\notin\bigsqcup\limits_{\overline{m}\in\Delta\varepsilon}\ \bigsqcup\limits_{Q\in\mathbb{T}_{\overline{m}}}Q.\end{cases}$$

Then, since

$$\|f_\varepsilon\|_{LM^{\overline{\lambda}_\varepsilon}_{\overline{p},\overline{r}}}=\left(\sum_{m_d\in\Delta_{\varepsilon_d}}\cdots\left(\sum_{m_1\in\Delta_{\varepsilon_1}}\left(2^{-\langle\overline{m},\overline{\lambda}_\varepsilon\rangle}\sum_{Q\in\mathbb{T}_{\overline{m}}}\|f\|_{L_{\overline{p}}(Q)}\right)^{r_1}\right)^{\frac{r_2}{r_1}}\cdots\right)^{\frac{1}{r_d}},$$

we get

$$\|f\|_{(LM^{\overline{\lambda}_0}_{\overline{p},\overline{r}},LM^{\overline{\lambda}_1}_{\overline{p},\overline{r}})_{\overline{\theta}q}}\le c\left(\sum_{k_d\in\mathbb{Z}}\cdots\left(\sum_{k_1\in\mathbb{Z}}\left(\prod_{i=1}^d b_i^{-k_i\theta_i}K(\overline{b}^{\overline{k}},f)\right)^{q_1}\right)^{\frac{q_2}{q_1}}\cdots\right)^{\frac{1}{q_d}}$$

$$\le c\left(\sum_{k_d\in\mathbb{Z}}\cdots\left(\sum_{k_1\in\mathbb{Z}}\left(\prod_{i=1}^d b_i^{-k_i\theta_i}\sum_{\varepsilon\in E}\|f_\varepsilon\|_{LM^{\overline{\lambda}_\varepsilon}_{\overline{p},\overline{r}}}\right)^{q_1}\right)^{\frac{q_2}{q_1}}\cdots\right)^{\frac{1}{q_d}}$$

$$\le c\sum_{\varepsilon\in E}\left(\sum_{k_d\in\mathbb{Z}}\cdots\left(\sum_{k_1\in\mathbb{Z}}\left(\prod_{i=1}^d b_i^{(\varepsilon_i-\theta_i)k_i}\right.\right.\right.$$

$$\left.\left.\left.\times\left(\sum_{m_d\in\Delta_{\varepsilon_d}}\cdots\left(\sum_{m_1\in\Delta_{\varepsilon_1}}\left(2^{-\langle\overline{m},\overline{\lambda}_\varepsilon\rangle}\sum_{Q\in\mathbb{T}_{\overline{m}}}\|f\|_{L_{\overline{p}}(Q)}\right)^{r_1}\right)^{\frac{r_2}{r_1}}\cdots\right)^{\frac{1}{r_d}}\right)^{q_1}\right)^{\frac{q_2}{q_1}}\cdots\right)^{\frac{1}{q_d}}.$$

Taking into account, that $b_i=2^{\lambda_i^1-\lambda_i^0}$, $\lambda_i=(1-\theta_i)\lambda_i^0+\theta_i\lambda_i^1$ $i=\overline{1,d}$, and applying for each parameter r_i, $i=\overline{1,d}$ the generalized Minkowski inequality and Hardy's inequality (Lemma 2.2), we obtain

$$\|f\|_{\left(LM^{\overline{\lambda}_0}_{\overline{p},\overline{r}},LM^{\overline{\lambda}_1}_{\overline{p},\overline{r}}\right)_{\overline{\theta},\overline{q}}}\le c\|f\|_{LM^{\overline{\lambda}}_{\overline{p},\overline{q}}}.$$

Here, $c>0$ depends only on the parameters $\overline{\lambda_0}$, $\overline{\lambda_1}$, $\overline{p}$, $\overline{q}$. □

4 Anisotropic generalized Morrey-type spaces $M^{\overline{\lambda}}_{\overline{p},\overline{q}}$

Let $\overline{p}=(p_1,...,p_d)$, $\overline{q}=(q_1,...,q_d)$, $\overline{\lambda}=(\lambda_1,...,\lambda_d)$ be such that $0<p_i\le\infty$, $0<q_i\le\infty$, $0<\lambda_i<\infty$. We define the anisotropic generalized Morrey-type spaces $M^{\overline{\lambda}}_{\overline{p},\overline{q}}$ as the set of all Lebesgue measurable functions $f\in L^{loc}_{\overline{p}}(\mathbb{R}^{|\overline{n}|})$, for which the following norm is finite

$$\|f\|_{M^{\overline{\lambda}}_{\overline{p},\overline{q}}}=\left(\sum_{k_d\in\mathbb{Z}}\cdots\left(\sum_{k_1\in\mathbb{Z}}\left(2^{-\langle\overline{k},\overline{\lambda}\rangle}\sup_{Q\in G_{\overline{k}}}\|f\|_{L_{\overline{p}}(Q)}\right)^{q_1}\right)^{\frac{q_2}{q_1}}\cdots\right)^{\frac{1}{q_d}}<\infty,$$

where $G_{\overline{k}} = \{Q = Q_1 \times \cdots \times Q_d :\ Q_i \subset G_{k_i},\ i = \overline{1,d}\}$.

By the anisotropic Morrey space $M_{\overline{p}}^{\overline{\lambda}}$ we mean the set of all Lebesgue measurable functions $f \in L_{\overline{p}}^{loc}(\mathbb{R}^{|\overline{n}|})$, for which

$$\|f\|_{M_{\overline{p}}^{\overline{\lambda}}} = \sup_{\overline{k}\in\mathbb{Z}^d}\left(2^{-\langle\overline{k\lambda}\rangle} \sup_{Q\in G_{\overline{k}}} \|f\|_{L_{\overline{p}}(Q)}\right) < \infty.$$

Note, that for $\overline{q} = \overline{\infty}$

$$M_{\overline{p},\overline{\infty}}^{\overline{\lambda}} = M_{\overline{p}}^{\overline{\lambda}}. \tag{4.1}$$

Let us present some properties of the introduced spaces.

Lemma 4.1. *(i) Let vectors* $\overline{n} = (n_1, ..., n_d)$, $\overline{p}_0 = (p_1^0, \dots, p_d^0)$, $\overline{p}_1 = (p_1^1, \dots, p_d^1)$ *and* $\overline{q} = (q_1, \dots, q_d)$ *be such that* $0 < p_i^0 < p_i^1 < \infty, 0 < q_i \le \infty$, $i = \overline{1,d}$. *Then*

$$M_{\overline{p}_1,\overline{q}}^{\overline{\lambda}_1} \hookrightarrow M_{\overline{p}_0,\overline{q}}^{\overline{\lambda}_0},$$

where $\overline{\lambda}_0 = (\lambda_1^0, \dots \lambda_d^0)$, $\overline{\lambda}_1 = (\lambda_1^1, \dots \lambda_d^1)$ *are such that* $\lambda_i^0 = \lambda_i^1 - \frac{n_i}{p_i^1} + \frac{n_i}{p_i^0}$, $i = \overline{1,d}$.

(ii) Let vectors $\overline{p} = (p_1, \dots, p_d)$, $\overline{q}_0 = (q_1^0, \dots, q_d^0)$, $\overline{q}_1 = (q_1^1, \dots, q_d^1)$ *such that* $0 < q_i^0 < q_i^1 \le \infty$, $i = \overline{1,d}$. *Then*

$$M_{\overline{p},\overline{q}_0}^{\overline{\lambda}} \hookrightarrow M_{\overline{p},\overline{q}_1}^{\overline{\lambda}}.$$

The proof is similar to the proof of Lemma 2.1.

Theorem 4.1. *Let vectors* $\overline{p} = (p_1, \dots, p_d)$, $\overline{q} = (q_1, \dots, q_d)$, $\overline{q}_0 = (q_1^0, \dots, q_d^0)$, $\overline{q}_1 = (q_1^1, \dots, q_d^1)$, $\overline{\lambda}_0 = (\lambda_1^0, \dots, \lambda_d^0)$, $\overline{\lambda}_1 = (\lambda_1^1, \dots, \lambda_d^1)$, $\overline{\theta} = (\theta_1, \dots, \theta_d)$ *be such that* $\lambda_i^0 \ne \lambda_i^1$, $0 < p_i \le \infty$, $0 < q_i, q_i^0, q_i^1 \le \infty$, $\theta_i \in (0,1)$. *Then*

$$\left(M_{\overline{p},\overline{q}_0}^{\overline{\lambda}_0}, M_{\overline{p},\overline{q}_1}^{\overline{\lambda}_1}\right)_{\overline{\theta},\overline{q}} \hookrightarrow M_{\overline{p},\overline{q}}^{\overline{\lambda}}, \tag{4.2}$$

where $\overline{\lambda} = (\lambda_1, \dots, \lambda_d) :\ \lambda_i = (1-\theta_i)\lambda_i^0 + \theta_i\lambda_i^1$, $i = \overline{1,d}$.

Proof. Let $f \in \left(M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}, M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}\right)_{\overline{\theta},\overline{q}}$.

For vectors $\overline{k} = (k_1, \dots, k_d)$, $\overline{\lambda} = (\lambda_1, \dots, \lambda_d)$ and $\varepsilon \in E$ we denote $\overline{\lambda}_\varepsilon = (\lambda_1^{\varepsilon_1}, \dots, \lambda_d^{\varepsilon_d}) :\ \lambda_i^{\varepsilon_i} = \begin{cases} \lambda_i^0, & \text{for } \varepsilon_i = 0, \\ \lambda_i^1, & \text{for } \varepsilon_i = 1. \end{cases}$

Let $f = \sum\limits_{\varepsilon\in E} f_\varepsilon$ be an arbitrary representation of the function f, where $f_\varepsilon \in M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_\varepsilon}$. Then we have

$$2^{-\langle\overline{k},\overline{\lambda}\rangle} \sup_{Q\in G_{\overline{k}}} \|f\|_{L_{\overline{p}}(Q)} \le 2^{-\langle\overline{k},(\overline{\lambda}-\overline{\lambda}_0)\rangle} \sum_{\varepsilon\in E}\left(2^{\langle\overline{k},\overline{\lambda}_\varepsilon\rangle} \sup_{Q\in G_{\overline{k}}} \|f_\varepsilon\|_{L_{\overline{p}}(Q)}\right).$$

Taking into account the arbitrariness of the representation $f = \sum\limits_{\varepsilon\in E} f_\varepsilon$, we obtain

$$2^{-\langle\overline{k},\overline{\lambda}\rangle} \sup_{Q\in G_{\overline{k}}} \|f\|_{L_{\overline{p}}(Q)} \le 2^{-\langle\overline{k},(\overline{\lambda}-\overline{\lambda}_0)\rangle} K(2^{\langle\overline{k},\overline{\lambda}_\varepsilon\rangle}, f; M_{\overline{p}\overline{\infty}}^{\overline{\lambda}_\varepsilon}).$$

Hence,

$$\|f\|_{M_{\overline{p}\overline{q}}^{\overline{\lambda}}} \le \left(\sum_{k_d\in\mathbb{Z}} \cdots \left(\sum_{k_1\in\mathbb{Z}} \left(2^{-\langle\overline{k},(\overline{\lambda}-\overline{\lambda}_0)\rangle} K(2^{\overline{k}}, f)\right)^{q_1}\right)^{\frac{q_2}{q_1}} \cdots\right)^{\frac{1}{q_d}} \asymp \|f\|_{\left(M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_0}, M_{\overline{p},\overline{\infty}}^{\overline{\lambda}_1}\right)_{\overline{\theta},\overline{q}}}.$$

Thus, we have

$$\left(M^{\overline{\lambda}_0}_{\overline{p},\overline{q}_0}, M^{\overline{\lambda}_1}_{\overline{p},\overline{q}_1}\right)_{\overline{\theta},\overline{q}} \hookrightarrow \left(M^{\overline{\lambda}_0}_{\overline{p},\overline{\infty}}, M^{\overline{\lambda}_1}_{\overline{p},\overline{\infty}}\right)_{\overline{\theta},\overline{q}} \hookrightarrow M^{\overline{\lambda}}_{\overline{p},\overline{q}}.$$

□

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