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**BOKAYEV NURZHAN ADILKHANOVICH**

(to the 70th birthday)

January 5, 2026, marks the 70th birthday of Nurzhan Adilkhanovich Bokayev, Doctor of Physical and Mathematical Sciences (1996), Professor (2002), member of the Editorial Board of the Eurasian Mathematical Journal (2010).

Nurzhan Adilkhanovich Bokayev was born on 5 January, 1956 in the village of Urnek, Karabalyk District, Kostanay Region. He graduated in 1972, with a gold medal from the Burlin Secondary School in the district. That same year, he entered the Mathematics Department of Karaganda State University and graduated with honors in 1977. From 1978 to 1979, he served in the Soviet Army. In 1980, he completed an internship, and from 1981 to 1984, he studied in the graduate program at Lomonosov Moscow State University in the Department of Function Theory and Functional Analysis. In 1985, he defended his candidate's dissertation there under the supervision of Corresponding Member of the Academy of Sciences of the USSR D.E. Menshov and Professor V.A. Skvortsov. In 1996, he defended his doctoral dissertation, "Fourier Coefficients and Uniqueness Theorems for Series in Generalized Walsh and Haar Systems", at the Institute of Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan, speciality Mathematical Analysis (01.01.01).

After completing his postgraduate studies, he worked as a lecturer, senior lecturer, associate professor, and professor in the Department of Mathematical Analysis at E.A. Buketov Karaganda State University (1985-1999). He headed the Department of Mathematics and Mathematical Modeling (1996-1999), and was a dean of the Faculty of Mathematics at E.A. Buketov Karaganda State University (1999-2005). Since 2005, he has been a professor in the Faculty of Mechanics and Mathematics at the L.N. Gumilyov Eurasian National University. From 2009 to 2018, he was the Head of the Department of Higher Mathematics at the L.N. Gumilyov Eurasian National University, and from 2018 to the present, he has been a professor in the Department of Fundamental Mathematics.

Professor Bokaev's research focuses on problems in function theory and functional analysis, the theory of orthogonal series for generalized Walsh and Haar systems, and operator theory in various function spaces. He has proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, and constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems. He obtained conditions for functions to belong to various functional classes in terms of the Fourier coefficients of generalized Haar and Walsh systems, and embedding criteria for Nikol'skii-Besov spaces constructed on the basis of multiplicative systems. He also obtained conditions for the boundedness and compactness of the commutator of the Riesz potential in general Morrey-type spaces, and conditions for boundedness of generalizedRiesz and Bessel potentials and generalized fractional-maximal operators in rearrangement-invariant spaces.

His co-authors include Professor V.A. Skvortsov (Moscow State University, Moscow), Professors V.I. Burenkov and M.L. Goldman (Peoples' Friendship University of Russia (RUDN University). Moscow), Dr. A. Gogatishvili (Institute of Mathematics of the Czech Academy of Sciences, Prague). His doctoral students' foreign advisors include Professors W. Sickel (Friedrich-Schiller-Universitity, Jena, Germany), Massimo Lanza de Cristoforis (University of Padova, Padova, Italy), V. Ruzhansky (Ghent University, Ghent, Belgium), U. Goginava (United Arab Emirates University, Al Ain, United Arab Emirates), and E. Panakhov (Institute of Applied Mathematics at Baku State University, Baku, Azerbaijan).

Under his supervision, 15 dissertations (4 candidate's and 11 PhD) were defended. He has published over 220 scientific papers, 2 monographs and 2 textbooks.

He is a three-time recipient of the state grant "Best University Teacher" of the Republic of Kazakhstan (2006, 2010, 2024) and served as Vice President of the Mathematical Society of Turkic-Speaking Countries (2014-2023). He was awarded the "For Contribution to the Development of Science" badge (2022).

Over the last ten years, he has been and continues to be a head of more than 5 national and international funded projects.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Nurzhan Adilkhanovich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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# ADAMS THEOREM FOR THE $B$-RIESZ POTENTIAL IN THE TOTAL $B$-MORREY SPACES

**V.S. Guliyev, A. Akbulut, M.N. Omarova, A. Serbetci**

Communicated by E.D. Nursultanov

**Key words:** $B$-maximal operator, $B$-Riesz potential, total $B$-Morrey space, Adams theorem for $B$-Riesz potential.

**AMS Mathematics Subject Classification:** 42B20, 42B25, 42B35.

**Abstract.** We prove Adams theorem for the Riesz potential $I_\gamma^\alpha$ ($B$-Riesz potential) in the total Morrey spaces $L_{p,(\lambda,\mu),\gamma}$ (total $B$-Morrey spaces), associated with the Laplace-Bessel differential operator $\Delta_B$. More precisely, we obtain necessary and sufficient conditions for the operator $I_\gamma^\alpha$ to be bounded from the total $B$-Morrey space $L_{p,(\lambda,\mu),\gamma}$ to the total $B$-Morrey space $L_{q,(\lambda,\mu),\gamma}$ and from the total $B$-Morrey space $L_{1,(\lambda,\mu),\gamma}$ to the weak total $B$-Morrey space $WL_{q,(\lambda,\mu),\gamma}$.

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## 1 Introduction

The classical Morrey spaces were introduced by Morrey [4] for the study of solutions of some quasi-linear elliptic partial differential equations. For more applications of the Morrey spaces to partial differential equations, the reader is referred to [4, 15, 29]. In [16] the first author introduced a variant of the Morrey spaces called the total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. The total Morrey spaces generalize the classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ so that $L_{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L_{p,\lambda}(\mathbb{R}^n)$ and the modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ so that $L_{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$. See also [1, 7, 8, 14, 17, 18, 23, 24, 27, 28].

The Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \dots, \gamma_k > 0, 1 \le k \le n$$

is known as an important operator in the Fourier-Bessel harmonic analysis and applications. The maximal operator, potentials and related topics associated with the Laplace-Bessel differential operator $\Delta_B$ have been investigated by many researchers. See B. Muckenhoupt and E. Stein [26], I. Kipriyanov [20], K. Trimeche [33], L. Lyakhov [22], K. Stempak [32], A.D. Gadjiev and I.A. Aliev [9], I.A. Aliev and S. Bayrakci [5], V.S. Guliyev [10, 11], V.S. Guliyev and J.J. Hasanov [12], S. Bayrakci [6], V.S. Guliyev, A. Serbetci and I. Ekincioglu [13], A. Akbulut, M. Dziri and I. Ekincioglu [3], A. Serbetci and I. Ekincioglu [30], E.L. Shishkina [31] and others.

In this paper, we consider the generalized shift operator generated by the Laplace-Bessel differential operator $\Delta_B$ in terms of which the $B$-maximal operator and the $B$-Riesz potential are investigated in the total $B$-Morrey space. We prove Adams theorem for the $B$-Riesz potential $I_\gamma^\alpha$ in the total $B$-Morrey spaces $L_{p,(\lambda,\mu),\gamma}$, namely, we will obtain necessary and sufficient conditions for the operator $I_\gamma^\alpha$ to be bounded from one total $B$-Morrey space $L_{p,(\lambda,\mu),\gamma}$ to another one $L_{q,(\lambda,\mu),\gamma}$ and from the total $B$-Morrey space $L_{1,(\lambda,\mu),\gamma}$ to the weak total $B$-Morrey space $WL_{q,(\lambda,\mu),\gamma}$.

The paper is organized as follows. In Section 2 we present some definitions and auxiliary results. In Section 3 we study some embeddings for the total $B$-Morrey spaces. In Section 4 the boundedness of the $B$-maximal operator $M_\gamma$ on the total $B$-Morrey spaces $L_{p,(\lambda,\mu),\gamma}$ is proved. The main result of the paper is Adams theorem for the $B$-Riesz potential $I_\gamma^\alpha$ in the total $B$-Morrey space $L_{p,(\lambda,\mu),\gamma}$, established in Section 5.

Finally, we make some conventions on notation. Throughout this paper, we assume that the letter $C$ denotes a positive constant that may vary at each occurrence but is independent of the essential variables. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant $C$ depending only on the numerical parameters.

## 2 Notations and preliminaries

Suppose that $n$ and $k$ are positive integers with $1 \le k \le n$ and $\mathbb{R}^n$ is $n$-dimensional Euclidean space. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$, $x = (x', x'') \in \mathbb{R}^n$, $n \ge 2$. Let

$$\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n;\, x_1 > 0, \ldots, x_k > 0\}$$

and

$$E(x, r) = \{y \in \mathbb{R}^n_{k,+}\ ;\ |x - y| < r\}, \quad E_r = E(0, r).$$

For a measurable set $E$, let $|E|_\gamma = \int_E (x')^\gamma dx$, where $\gamma = (\gamma_1, \ldots, \gamma_k)$ are positive numbers and $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$, $|E_r|_\gamma = \omega(n, k, \gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n, k, \gamma) = \int_{E_1} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i + 1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Define the generalized shift operator ($B$-shift operator) by

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \ldots \int_0^\pi f\left((x', y')_\beta, x'' - y''\right)\, d\nu(\beta),$$

where $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x', y')_\beta = ((x_1, y_1)_{\beta_1}, \ldots, (x_k, y_k)_{\beta_k})$, $d\nu(\beta) = \prod_{i=1}^k \sin^{\gamma_i - 1}\beta_i \, d\beta_1 \ldots d\beta_k$, $1 \le k \le n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i + 1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)} = \frac{2^k}{\pi^k} \omega(2k, k, \gamma).$$

We remark that the generalized shift operator $T^y$ is closely connected with the Bessel differential operator $B$ (if $n = k = 1$ see [21] for details, if $n > 1$, $k = 1$ see [20], if $n, k > 1$ see [22]).

Let $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ be the space of all measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\gamma}} \equiv \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = \left( \int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} |f(x)|.$$

The $B$-maximal function (see [10, 11]) is defined by

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int\limits_{E_r} T^y |f(x)| (y')^\gamma dy$$

and the $B$-Riesz potential (see [10, 11]) is defined by

$$I_\gamma^\alpha f(x) = \int\limits_{\mathbb{R}^n_{k,+}} T^y \big(|x|^{\alpha-Q}\big)\, f(y)\, (y')^\gamma dy, \ 0<\alpha<Q,$$

where $T^y$ is the generalized shift operator generated by the Laplace-Bessel differential operator $\Delta_B$.

The operator $M_\gamma$ was introduced by Guliyev in [10]. Moreover, the strong-$(L_{p,\gamma}, L_{p,\gamma})$, $1<p\le\infty$ and weak-$(L_{1,\gamma}, L_{1,\gamma})$ boundedness of $M_\gamma$ was proved in [10] (see also [11]). Also, the strong-$(L_{p,\gamma}, L_{q,\gamma})$, $1<p<q<\infty$, $1/p-1/q=\alpha/Q$ and weak-$(L_{1,\gamma}, L_{q,\gamma})$, $1<q<\infty$, $1-1/q=\alpha/Q$ boundedness of $I_\gamma^\alpha$ was proved in [10] (see also [11]).

**Theorem 2.1.** [10, 11] *1. If $f\in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, then $M_\gamma f\in WL_{1,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{WL_{1,\gamma}} \le C_{1,\gamma}\|f\|_{L_{1,\gamma}},$$

*for all $f\in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, where $C_{1,\gamma}>0$ depends only on $\gamma$,$k$ and $n$.*

*2. If $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1<p\le\infty$, then $M_\gamma f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{L_{p,\gamma}} \le C_{p,\gamma}\,\|f\|_{L_{p,\gamma}},$$

*for all $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, where $C_{p,\gamma}>0$ depends only on $p$, $\gamma$, $k$ and $n$.*

**Theorem 2.2.** [10, 11] *Let $0<\alpha<Q$ and $1\le p<\frac{Q}{\alpha}$.*

*1) If $1<p<\frac{Q}{\alpha}$, then the condition $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}$ is necessary and sufficient for the boundedness $I_\gamma^\alpha$ from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.*

*2) If $p=1$, then the condition $1-\frac{1}{q}=\frac{\alpha}{Q}$ is necessary and sufficient for the boundedness $I_\gamma^\alpha$ from $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.*

## 3 Some embeddings for the total $B$-Morrey spaces

In this section we define the total $B$-Morrey spaces $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, give auxiliary results and some embeddings for these spaces.

**Definition 1.** Let $1\le p<\infty$, $\lambda\in\mathbb{R}$, $\mu\in\mathbb{R}$, $[t]_1=\min\{1,t\}$, $t>0$. We denote by $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the classical $B$-Morrey spaces [12], by $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the modified $B$-Morrey spaces [14], and by $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ the total $B$-Morrey spaces: the sets of all locally integrable functions $f$ with the finite norms

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left(t^{-\lambda}\int_{E_t} T^x|f(y)|^p\,(y')^\gamma dy\right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left([t]_1^{-\lambda}\int_{E_t} T^x|f(y)|^p\,(y')^\gamma dy\right)^{1/p},$$

$$\|f\|_{L_{p,(\lambda,\mu),\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left([t]_1^{-\lambda}\,[1/t]_1^{-\mu}\int_{E_t} T^x|f(y)|^p\,(y')^\gamma dy\right)^{1/p},$$

respectively.

**Definition 2.** Let $1 \le p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey spaces $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ [12], the weak modified Morrey spaces $W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ [14] and the weak total Morrey spaces $WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ as the sets of all locally integrable functions $f$ with the finite norms

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \Big(t^{-\lambda} \int_{\{y\in E_t : T^x|f(y)|^p\}} (y')^\gamma dy\Big)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \Big([t]_1^{-\lambda} \int_{\{y\in E_t : T^x|f(y)|^p\}} (y')^\gamma dy\Big)^{1/p},$$

$$\|f\|_{WL_{p,(\lambda,\mu),\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \Big([t]_1^{-\lambda}\,[1/t]_1^{-\mu} \int_{\{y\in E_t : T^x|f(y)|^p\}} (y')^\gamma dy\Big)^{1/p},$$

respectively.

Note that

$$L_{p,(0,0),\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+}),$$
$$WL_{p,(0,0),\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\gamma}(\mathbb{R}^n_{k,+}),$$
$$L_{p,(\lambda,\lambda),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}), \quad L_{p,(\lambda,0),\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}).$$

We have $\|f\|_{WL_{p,(\lambda,\mu),\gamma}} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}}$, therefore, the following continuous embedding holds

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}).$$

Furthermore,

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}),\ \mu \le \lambda \quad \text{and} \quad \|f\|_{L_{p,\lambda,\gamma}} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}}, \tag{3.1}$$

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+}),\ \mu \le \lambda \quad \text{and} \quad \|f\|_{L_{p,\mu,\gamma}} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}} \tag{3.2}$$

and

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\gamma}(\mathbb{R}^n_{k,+}) \quad \text{and} \quad \|f\|_{L_{p,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

If $\lambda < 0$ or $\lambda > Q$, then

$$L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta,$$

where $\Theta \equiv \Theta(\mathbb{R}^n_{k,+})$ is the set of all functions equivalent to 0 on $\mathbb{R}^n_{k,+}$.

**Lemma 3.1.** *If $1 \le p < \infty$, $0 \le \mu \le \lambda \le Q$, then*

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$$

*and*

$$\|f\|_{L_{p,(\lambda,\mu),\gamma}} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\}.$$

*Proof.* Let $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Then from (3.1) and (3.2) we have that $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$ and $\max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}}$.

Now, let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$. Then

$$\|f\|_{L_{p,(\lambda,\mu),\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+}, t>0} \left([t]_1^{-\lambda}\ [1/t]_1^{\mu} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\right)^{1/p}$$
$$= \max\Big\{ \sup_{x\in\mathbb{R}^n_{k,+}, 0<t\le 1} \Big(t^{-\lambda} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\Big)^{1/p},$$
$$\sup_{x\in\mathbb{R}^n_{k,+}, t>1} \Big(\ t^{-\mu} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy\Big)^{1/p}\Big\} \le \max\Big\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\Big\}.$$

Therefore, $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and the embedding $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+}) \subset\succ L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ is valid.

Thus, $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$ and $\max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\} = \|f\|_{L_{p,(\lambda,\mu),\gamma}}$. □

**Corollary 3.1.** [19, Lemma 5] *If $1 \le p < \infty$, $0 \le \lambda \le Q$, then*

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+})$$

*and*

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\gamma}}\right\}.$$

**Lemma 3.2.** *If $1 \le p < \infty$, $0 \le \mu \le \lambda \le Q$, then*

$$WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap WL_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$$

*and*

$$\|f\|_{WL_{p,(\lambda,\mu),\gamma}} = \max\left\{\|f\|_{WL_{p,\lambda,\gamma}}, \|f\|_{WL_{p,\mu,\gamma}}\right\}.$$

**Remark 1.** If $1 \le p < \infty$, and $\mu < 0$ or $\lambda > Q$, then

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = \Theta(\mathbb{R}^n_{k,+}).$$

**Lemma 3.3.** *If $1 \le p < \infty$, $0 \le \lambda \le Q$ and $0 \le \mu \le Q$, then*

$$L_{p,(Q,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\succ L_{\infty,\gamma}(\mathbb{R}^n_{k,+}) \subset\succ L_{p,(\lambda,Q),\gamma}(\mathbb{R}^n_{k,+})$$

*and*

$$\|f\|_{L_{p,(\lambda,Q),\gamma}} \le \omega(n,k,\gamma)^{1/p}\|f\|_{L_{\infty,\gamma}} \le \|f\|_{L_{p,(Q,\mu),\gamma}}.$$

*Proof.* Let $f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$. Then for all $x \in \mathbb{R}^n_{k,+}$ and $0 < t \le 1$

$$\left(t^{-\lambda}\int_{E_t} T^x|f(y)|^p(y')^\gamma dy\right)^{1/p} \le \omega(n,k,\gamma)^{1/p}\,\|f\|_{L_\infty}, \quad 0 \le \lambda \le Q$$

and for all $x \in \mathbb{R}^n_{k,+}$ and $t \ge 1$

$$\left(t^{-Q}\int_{E_t} T^x|f(y)|^p(y')^\gamma dy\right)^{1/p} \le \omega(n,k,\gamma)^{1/p}\,\|f\|_{L_{\infty,\gamma}}.$$

Therefore, $f \in L_{p,(\lambda,Q),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{p,(\lambda,n),\gamma}} \le \omega(n,k,\gamma)^{1/p}\|f\|_{L_{\infty,\gamma}}.$$

Let $f \in L_{p,(Q,\mu),\gamma}(\mathbb{R}^n_{k,+})$. By the Lebesgue's Theorem we have (see [12])

$$\lim_{t\to 0}|E_t|^{-1}\int_{E_t} T^x|f(y)|^p(y')^\gamma dy = |f(x)|^p$$

for almost all $x \in \mathbb{R}^n_{k,+}$.

Then for almost all $x \in \mathbb{R}^n_{k,+}$

$$\begin{aligned}|f(x)| &= \left(\lim_{t\to 0}|E_t|^{-1}\int_{E_t} T^x|f(y)|^p(y')^\gamma dy\right)^{1/p}\\ &\le \omega(n,k,\gamma)^{-1/p}\sup_{x\in\mathbb{R}^n_{k,+},\ 0<t\le 1}\left(t^{-Q}\int_{E_t} T^x|f(y)|^p(y')^\gamma dy\right)^{1/p}\\ &\le \omega(n,k,\gamma)^{-1/p}\|f\|_{L_{p,(Q,\mu)}}.\end{aligned}$$

Therefore, $f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{\infty,\gamma}} \le \omega(n,k,\gamma)^{-1/p} \|f\|_{L_{p,(Q,\mu),\gamma}}.$$

□

**Corollary 3.2.** *If $1 \le p < \infty$, then*

$$L_{p,Q,\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,Q,\gamma}(\mathbb{R}^n_{k,+}) = L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$$

*and*

$$\|f\|_{L_{p,Q,\gamma}} = \|f\|_{\widetilde{L}_{p,Q,\gamma}} = \omega(n,k,\gamma)^{1/p} \|f\|_{L_{\infty,\gamma}}.$$

**Lemma 3.4.** *If $0 \le \lambda < Q$, $0 \le \mu < Q$, $0 \le \alpha < Q-\lambda$ and $0 \le \beta < Q-\mu$, then for $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}$*

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{1,(Q-\alpha,Q-\beta),\gamma}(\mathbb{R}^n_{k,+})$$

*and for all $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ the following inequality*

$$\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,(\lambda,\mu),\gamma}}$$

*is valid.*

*Proof.* Let $0 < \alpha < Q$, $0 \le \lambda < Q$, $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}$. By Hölder's inequality we have

$$\begin{aligned}
\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} &= \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} [t]_1^{\alpha-Q}\ [1/t]_1^{Q-\beta} \int_{E_t} T^x|f(y)|(y')^\gamma dy \\
&\le \omega(n,k,\gamma)^{1/p'} \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left([t]_1\, t^{-1}\right)^{-Q/p'} [t]_1^{\alpha-\frac{Q-\lambda}{p}} \\
&\quad \times [1/t]_1^{Q-\beta-\frac{\mu}{p}} \left( [t]_1^{-\lambda}\ [1/t]_1^{\mu} \int_{E_t} T^x|f(y)|^p (y')^\gamma dy \right)^{1/p} \\
&\le \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,(\lambda,\mu),\gamma}} \sup_{t>0} \left([t]_1\, t^{-1}\right)^{\frac{Q-\mu}{p}-\beta}\ [t]_1^{\alpha-\frac{Q-\lambda}{p}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sup_{t>0} \left([t]_1\, t^{-1}\right)^{\frac{Q-\mu}{p}-\beta} [t]_1^{\alpha-\frac{Q-\lambda}{p}} &= \max\left\{ \sup_{0<t\le 1} t^{\alpha-\frac{Q-\lambda}{p}}, \sup_{t>1} t^{\beta-\frac{Q-\mu}{p}} \right\} < \infty \\
&\Longleftrightarrow \frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}.
\end{aligned}$$

Therefore, $f \in L_{1,(Q-\alpha,Q-\beta),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,(\lambda,\mu),\gamma}}.$$

□

**Corollary 3.3.** [19, *Lemma* 6] *If $0 \le \lambda < Q$ and $0 \le \alpha < Q-\lambda$, then for $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$*

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ \widetilde{L}_{1,Q-\alpha,\gamma}(\mathbb{R}^n_{k,+})$$

*and for all $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the following inequality*

$$\|f\|_{\widetilde{L}_{1,Q-\alpha,\gamma}} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$

*is valid.*

## 4 $L_{p,(\lambda,\mu),\gamma}$-boundedness of the $B$-maximal operator

In this section we will prove that the $B$-maximal operator $M_\gamma$ is bounded on the total $B$-Morrey spaces $L_{p,(\lambda,\mu),\gamma}$. Let us begin by recalling that $B$-maximal operator $M_\gamma$ is bounded on the $B$-Morrey spaces $L_{p,\lambda,\gamma}$.

**Theorem 4.1.** [12]
*1) If $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$, $0 \le \lambda < Q$, then $M_\gamma f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{L_{p,\lambda,\gamma}} \le C_{p,\lambda,\gamma}\|f\|_{L_{p,\lambda,\gamma}},$$

*for some $C_{p,\lambda,\gamma} > 0$ depending only on $p$, $\lambda$, $\gamma$, $k$ and $n$.*
*2) If $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $0 \le \lambda < Q$, then $M_\gamma f \in WL_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{WL_{1,\lambda,\gamma}} \le C_{1,\lambda,\gamma}\|f\|_{L_{1,\lambda,\gamma}},$$

*for some $C_{1,\lambda,\gamma} > 0$ depending only on $\lambda$, $\gamma$, $k$ and $n$.*

Applying Lemma 3.1 and Theorem 4.1, we obtain the following $L_{p,(\lambda,\mu),\gamma}$-boundedness of the $B$-maximal operator $M_\gamma$ in the total $B$-Morrey spaces.

**Theorem 4.2.** *1) If $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$, $0 \le \mu \le \lambda < Q$, then $M_\gamma f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{L_{p,(\lambda,\mu),\gamma}} \le C_{p,\lambda,\gamma}\|f\|_{L_{p,(\lambda,\mu),\gamma}},$$

*for some $C_{p,\lambda,\gamma} > 0$ depending only on $p$, $\lambda$, $\mu$, $\gamma$, $k$ and $n$.*
*2) If $f \in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $0 \le \mu \le \lambda < Q$, then $M_\gamma f \in WL_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{WL_{1,(\lambda,\mu),\gamma}} \le C_{1,\lambda,\gamma}\|f\|_{L_{1,(\lambda,\mu),\gamma}},$$

*for some $C_{1,\lambda,\gamma} > 0$ depending only on $\lambda$, $\mu$, $\gamma$, $k$ and $n$.*

*Proof.* 1) Suppose that $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$ and $0 \le \mu \le \lambda < Q$. It is obvious that (see Lemma 3.1)

$$\|M_\gamma f\|_{L_{p,(\lambda,\mu),\gamma}} = \max\left\{\|M_\gamma f\|_{L_{p,\lambda,\gamma}}, \|M_\gamma f\|_{L_{p,\mu,\gamma}}\right\}.$$

Then, by the boundedness of $M_\gamma$ on $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$ and $0 \le \mu \le \lambda < Q$ (see Theorem 4.1 and Lemma 3.1), we get

$$\|M_\gamma f\|_{L_{p,(\lambda,\mu),\gamma}} = \max\left\{C_{p,\lambda}, C_{p,\mu}\right\}\|f\|_{L_{p,\lambda,\gamma}}.$$

2) Suppose that $f \in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $0 \le \mu \le \lambda < Q$. It is obvious that (see Lemma 3.1)

$$\|M_\gamma f\|_{WL_{1,(\lambda,\mu),\gamma}} = \max\left\{\|M_\gamma f\|_{WL_{1,\lambda,\gamma}}, \|M_\gamma f\|_{WL_{1,\mu,\gamma}}\right\}.$$

Then, by the weak boundedness of $M_\gamma$ on $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and $0 \le \mu \le \lambda < Q$ (see Theorem 4.1 and Lemma 3.1), we get

$$\|M_\gamma f\|_{WL_{1,(\lambda,\mu),\gamma}} = \max\left\{C_{1,\lambda}, C_{1,\mu}\right\}\|f\|_{L_{1,\lambda,\gamma}}.$$

□

## 5 Adams theorem for the $B$-Riesz potential in the total $B$-Morrey spaces

In this section we prove Adams theorem for the $B$-Riesz potential $I_\gamma^\alpha$ in the total $B$-Morrey spaces $L_{p,(\lambda,\mu),\gamma}$. Namely, we will obtain necessary and sufficient conditions on the numerical parameters for the $B$-Riesz potential $I_\gamma^\alpha$ to be bounded from one total $B$-Morrey space $L_{p,(\lambda,\mu),\gamma}$ to another one $L_{q,(\lambda,\mu),\gamma}$ and from the total $B$-Morrey space $L_{1,(\lambda,\mu),\gamma}$ to the weak total $B$-Morrey space $WL_{q,(\lambda,\mu),\gamma}$. These statements are the main results of our article.

**Theorem 5.1.** *Let $1 \le p < \infty$, $0 \le \mu \le \lambda < Q$, $0 < \alpha < \frac{Q-\lambda}{p}$.*

*1) If $1 < p < \frac{Q-\lambda}{\alpha}$, then the condition $\frac{\alpha}{Q-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of the operator $I_\gamma^\alpha$ from $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}_{k,+}^n)$ to $L_{q,(\lambda,\mu),\gamma}(\mathbb{R}_{k,+}^n)$.*

*2) If $p = 1 < \frac{Q-\lambda}{\alpha}$, then the condition $\frac{\alpha}{Q-\mu} \le 1 - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of the operator $I_\gamma^\alpha$ from $L_{1,(\lambda,\mu),\gamma}(\mathbb{R}_{k,+}^n)$ to $WL_{q,(\lambda,\mu),\gamma}(\mathbb{R}_{k,+}^n)$.*

*Proof.* 1) *Sufficiency.* Let $1 < p < \frac{Q-\lambda}{\alpha}$, $\frac{\alpha}{Q-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ and $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}_{k,+}^n)$. Then

$$I_\gamma^\alpha f(x) = \Big( \int_{E_t} + \int_{\mathbb{R}_{k,+}^n \setminus E_t} \Big) T^y f(x) |y|^{\alpha-Q} (y')^\gamma dy \equiv A(x,t) + C(x,t). \tag{5.1}$$

For $A(x,t)$ we have

$$|A(x,t)| \le \int_{E_t} T^y \, |f(x)| \, |y|^{\alpha-Q} (y')^\gamma dy \le \sum_{j=-\infty}^{-1} \big(2^j t\big)^{\alpha-Q} \int_{E_{2^{j+1}t} \setminus E_{2^j t}} T^y \, |f(x)| \, (y')^\gamma dy$$

$$\le \sum_{j=-\infty}^{-1} \big(2^j t\big)^{\alpha-Q} \, |E_{2^{j+1}t}|_\gamma \, M_\gamma f(x) = t^\alpha \, \omega(n,k,\gamma) \, 2^Q \, M_\gamma f(x) \sum_{j=1}^{\infty} 2^{-j\alpha}.$$

Hence,

$$|A(x,t)| \le C_3 t^\alpha M_\gamma f(x) \qquad \text{with} \qquad C_3 = \frac{\omega(n,k,\gamma) 2^Q}{2^\alpha - 1}. \tag{5.2}$$

For $C(x,t)$ by Hölder's inequality we have

$$|C(x,t)| \le \Big( \int\limits_{\mathbb{R}_{k,+}^n \setminus E_t} |y|^{-\beta} T^y |f(x)|^p (y')^\gamma dy \Big)^{1/p} \Big( \int\limits_{\mathbb{R}_{k,+}^n \setminus E_t} |y|^{\left(\frac{\beta}{p}+\alpha-Q\right)p'} (y')^\gamma dy \Big)^{1/p'}$$

$$= J_1 \cdot J_2.$$

Let $\lambda < \beta < Q - \alpha p$. For $J_1$ we get

$$J_1 = \Big( \sum_{j=0}^{\infty} \int_{E_{2^{j+1}t} \setminus E_{2^j t}} T^y |f(x)|^p \; |y|^{-\beta} \; (y')^\gamma dy \Big)^{1/p}$$

$$\le 2^{\frac{\lambda}{p}} \, [t]_1^{\frac{\lambda-\beta}{p}} \; [1/t]_1^{-\frac{\mu-\beta}{p}} \; \|f\|_{L_{p,(\lambda,\mu),\gamma}} \Big( \sum_{j=0}^{\infty} 2^{(\lambda-\beta)j} \Big)^{1/p}$$

$$= C_4 \, [t]_1^{\frac{\lambda-\beta}{p}} \; [1/t]_1^{-\frac{\mu-\beta}{p}} \; \|f\|_{L_{p,(\lambda,\mu),\gamma}},$$

where $C_4 = \left( \frac{2^\beta}{2^{\beta-\lambda}-1} \right)^{1/p}$.

For $J_2$ we obtain

$$J_2 = \left( \int_{\mathbb{S}^{n-1}_{k,+}} (\xi')^\gamma d\xi \int_t^\infty r^{Q-1+\left(\frac{\beta}{p}+\alpha-Q\right)p'} dr \right)^{\frac{1}{p'}} = C_5 t^{\frac{\beta}{p}+\alpha-\frac{Q}{p}},$$

where $C_5 = \left( \omega(n,k,\gamma) \left( Q + \left( \frac{\beta}{p} + \alpha - Q \right) p' \right)^{-1} \right)^{1/p'}$. Then

$$|C(x,t)| \le C_6 \, [t]_1^{\frac{\lambda-Q}{p}+\alpha} \, [1/t]_1^{-\frac{\mu-Q}{p}-\alpha} \, \|f\|_{L_{p,(\lambda,\mu),\gamma}}, \tag{5.3}$$

where $C_6 = C_4 \cdot C_5$. Thus, from (5.2) and (5.3) we have

$$\left| I_\gamma^\alpha f(x) \right| \lesssim t^\alpha M_\gamma f(x) + [t]_1^{\alpha-\frac{Q-\lambda}{p}} \, [1/t]_1^{-\alpha+\frac{Q-\mu}{p}} \, \|f\|_{L_{p,(\lambda,\mu),\gamma}}$$

for all $t > 0$. Taking

$$t = \left( \frac{\|f\|_{L_{p,(\lambda,\mu),\gamma}}}{M_\gamma f(x)} \right)^{\frac{p}{Q-\mu}} \quad \text{and} \quad t = \left( \frac{\|f\|_{L_{p,(\lambda,\mu),\gamma}}}{M_\gamma f(x)} \right)^{\frac{p}{Q-\lambda}}$$

we have

$$\begin{aligned} \left| I_\gamma^\alpha f(x) \right| &\lesssim \min \left\{ \left( M_\gamma f(x) \right)^{1-\frac{\alpha p}{Q-\mu}} \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{\frac{\alpha p}{Q-\mu}}, \left( M_\gamma f(x) \right)^{1-\frac{\alpha p}{Q-\lambda}} \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{\frac{\alpha p}{Q-\lambda}} \right\} \\ &= \min \left\{ \left( \frac{M_\gamma f(x)}{\|f\|_{L_{p,(\lambda,\mu),\gamma}}} \right)^{1-\frac{\alpha p}{Q-\mu}}, \left( \frac{M_\gamma f(x)}{\|f\|_{L_{p,(\lambda,\mu),\gamma}}} \right)^{1-\frac{\alpha p}{Q-\lambda}} \right\} \|f\|_{L_{p,(\lambda,\mu),\gamma}}, \end{aligned}$$

then

$$\left| I_\gamma^\alpha f(x) \right| \lesssim \left( \frac{M_\gamma f(x)}{\|f\|_{L_{p,(\lambda,\mu),\gamma}}} \right)^{\frac{p}{q}} \|f\|_{L_{p,(\lambda,\mu),\gamma}} = \left( M_\gamma f(x) \right)^{\frac{p}{q}} \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{p}{q}}. \tag{5.4}$$

Hence, by Theorem 4.2 and inequality (5.4), we get

$$\begin{aligned} \|I_\gamma^\alpha f\|_{L_{q,(\lambda,\mu),\gamma}} &= \sup_{x \in \mathbb{R}^n_{k,+}, t>0} \left( [t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{E_t} T^x |I_\gamma^\alpha f(y)|^q (y')^\gamma dy \right)^{1/q} \\ &\lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{p}{q}} \sup_{x \in \mathbb{R}^n_{k,+}, t>0} \left( [t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{E_t} T^x \left( \left( M_\gamma f(y) \right)^p \right) (y')^\gamma dy \right)^{\frac{1}{q}} \\ &= \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{p}{q}} \|\left( M_\gamma f \right)^{\frac{p}{q}}\|_{L_{q,(\lambda,\mu),\gamma}} \\ &\lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{p}{q}} \|M_\gamma f\|_{L_{p,(\lambda,\mu),\gamma}}^{\frac{p}{q}} \\ &\lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}} \end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned} \|I_\gamma^\alpha f\|_{WL_{q,(\lambda,\mu),\gamma}} &= \sup_{r>0} \sup_{x \in \mathbb{R}^n_{k,+}, t>0} \left( [t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{\{y \in E_t : T^x |I_\gamma^\alpha f(y)|^q > r\}} (y')^\gamma dy \right)^{1/q} \\ &\lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}}^{1-\frac{1}{q}} \sup_{r>0} \sup_{x \in \mathbb{R}^n_{k,+}, t>0} \left( [t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{\{y \in E_t : T^x \left( M_\gamma f(y) \right) > r\}} (y')^\gamma dy \right)^{\frac{1}{q}} \\ &= \|f\|_{L_{1,(\lambda,\mu),\gamma}}^{1-\frac{1}{q}} \|\left( M_\gamma f \right)^{\frac{1}{q}}\|_{WL_{q,(\lambda,\mu),\gamma}} \\ &\lesssim \|f\|_{L_{1,(\lambda,\mu),\gamma}}^{1-\frac{1}{q}} \|M_\gamma f\|_{WL_{1,(\lambda,\mu),\gamma}}^{\frac{1}{q}} \\ &\lesssim \|f\|_{L_{1,(\lambda,\mu),\gamma}} \end{aligned}$$

if $p=1<q<\infty$.

Therefore, for $1<p<q<\infty$ we have $I_\gamma^\alpha f \in L_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}} \lesssim \|f\|_{L_{p,(\lambda,\mu),\gamma}},$$

also for $p=1<q<\infty$ we have $I_\gamma^\alpha f \in WL_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\left\|I_\gamma^\alpha f\right\|_{WL_{q,(\lambda,\mu),\gamma}} \lesssim \|f\|_{L_{1,(\lambda,\mu),\gamma}}.$$

*Necessity.* Let $1<p<\frac{Q-\lambda}{\alpha}$, $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and $I_\gamma^\alpha$ be bounded from $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Define $f_t(x) =: f(tx)$. Then we have

$$\begin{aligned}\left\|f_t\right\|_{L_{p,(\lambda,\mu),\gamma}} &= t^{-\frac{Q}{p}} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda}\,[1/r]_1^{\mu} \int_{E_{tr}} T^y|f(tx)|^p\,(y')^\gamma dy\right)^{1/p} \\ &= [t]_1^{-\frac{Q-\lambda}{p}}\,[1/t]_1^{\frac{Q-\mu}{p}}\,\|f\|_{L_{p,(\lambda,\mu),\gamma}}\end{aligned}$$

and

$$I_\gamma^\alpha f_t(x) = t^{-\alpha} I_\gamma^\alpha f(tx),$$

$$\begin{aligned}\left\|I_\gamma^\alpha f_t\right\|_{L_{q,(\lambda,\mu),\gamma}} &= t^{-\alpha} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda}\,[1/r]_1^{\mu} \int_{E_r} T^{ty}\left|I_\gamma^\alpha f(tx)\right|^q (y')^\gamma dy\right)^{1/q} \\ &= [t]_1^{-\alpha-\frac{Q}{q}}\,[1/t]_1^{\alpha+\frac{Q}{q}} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda}\,[1/r]_1^{\mu} \int_{E_{tr}} T^y\left|I_\gamma^\alpha f(x)\right|^q (y')^\gamma dy\right)^{1/q} \\ &= [t]_1^{-\alpha-\frac{Q-\lambda}{q}}\,[1/t]_1^{\alpha+\frac{Q-\mu}{q}} \left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}}.\end{aligned}$$

By the boundedness $I_\gamma^\alpha$ from $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ we get

$$\left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}} \le C_{p,q,\lambda,\gamma}\,[t]_1^{\alpha+\frac{Q-\lambda}{q}-\frac{Q-\lambda}{p}}\,[1/t]_1^{-\alpha-\frac{Q-\mu}{q}+\frac{Q-\mu}{p}}\,\|f\|_{L_{p,(\lambda,\mu),\gamma}},$$

where $C_{p,q,\lambda,\gamma}>0$ depends only on $p$,$q$,$\lambda$,$\gamma$,$k$ and $n$.

If $\frac{1}{p}<\frac{1}{q}+\frac{\alpha}{Q-\lambda}$, then by letting $t\to 0$ we have $\left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}}=0$ for all $f\in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Similarly, if $\frac{1}{p}>\frac{1}{q}+\frac{\alpha}{Q-\mu}$, then by letting $t\to\infty$ we obtain $\left\|I_\gamma^\alpha f\right\|_{L_{q,(\lambda,\mu),\gamma}}=0$ for all $f\in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Therefore, we have $\frac{\alpha}{Q-\mu}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$.

Let $p=1<\frac{Q-\lambda}{\alpha}$, $f\in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and $I_\gamma^\alpha$ be bounded from $L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. We have

$$\begin{aligned}\left\|f_t\right\|_{L_{1,(\lambda,\mu),\gamma}} &= t^{-Q} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda}\,[1/r]_1^{\mu} \int_{E_{tr}} T^y|f(tx)|\,(y')^\gamma dy\right) \\ &= [t]_1^{-Q+\lambda}\,[1/t]_1^{Q-\mu}\,\|f\|_{L_{1,(\lambda,\mu),\gamma}}\end{aligned}$$

and

$$\left\|I_\gamma^\alpha f_t\right\|_{WL_{q,(\lambda,\mu),\gamma}} = \sup_{r>0} r \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left([\tau]_1^{-\lambda}\,[1/\tau]_1^{\mu} \int_{\{y\in E_\tau\,:\,T^y|I_\gamma^\alpha f_t(x)|>r\}} (y')^\gamma dy\right)^{1/q}$$

$$= t^{-\alpha} \sup_{r>0} rt^{\alpha} \sup_{x\in\mathbb{R}^n_{k,+},\tau>0} \left([\tau]_1^{-\lambda}\,[1/\tau]_1^{\mu} \int_{\{y\in E_\tau\,:\,T^{ty}|I_\gamma^\alpha f(tx)|>rt^\alpha\}} (y')^\gamma dy\right)^{1/q}$$

$$= [t]_1^{-\alpha-\frac{Q}{q}}\,[1/t]_1^{\alpha+\frac{Q}{q}} \sup_{r>0} rt^{\alpha} \sup_{x\in\mathbb{R}^n_{k,+},\tau>0} \left([\tau]_1^{-\lambda}\,[1/\tau]_1^{\mu} \int_{\{y\in E_{t\tau}\,:\,T^{y}|I_\gamma^\alpha f(x)|>rt^\alpha\}} (y')^\gamma dy\right)^{1/q}$$

$$= [t]_1^{-\alpha-\frac{Q-\lambda}{q}}\,[1/t]_1^{\alpha+\frac{Q-\mu}{q}} \left\|I_\gamma^\alpha f\right\|_{WL_{q,(\lambda,\mu),\gamma}}.$$

By the boundedness $I_\gamma^\alpha$ from $L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ it follows that

$$\left\|I_\gamma^\alpha f\right\|_{WL_{q,(\lambda,\mu),\gamma}} \le C_{1,q,\lambda,\gamma}\,[t]_1^{\alpha+\frac{Q-\lambda}{q}-(Q-\lambda)}\,[1/t]_1^{-\alpha-\frac{Q-\mu}{q}+Q-\mu}\,\|f\|_{L_{1,(\lambda,\mu),\gamma}},$$

where $C_{1,q,\lambda,\gamma}>0$ depends only on $q$,$\lambda$,$\gamma$,$k$ and $n$.

If $1<\frac{1}{q}+\frac{\alpha}{Q-\mu}$, then by passing to the limit as $t\to 0$ we have $\left\|I_\gamma^\alpha f\right\|_{WL_{q,(\lambda,\mu),\gamma}}=0$ for all $f\in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$.

Similarly, if $1>\frac{1}{q}+\frac{\alpha}{Q-\lambda}$, then by passing to the limit as $t\to\infty$ we obtain $\left\|I_\gamma^\alpha f\right\|_{WL_{q,(\lambda,\mu),\gamma}}=0$ for all $f\in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. Therefore, we have $\frac{\alpha}{Q-\mu}\le 1-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$. □

From Theorem 5.1 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

**Corollary 5.1.** [12] *Let* $0<\alpha<Q$, $0\le\lambda<Q-\alpha$ *and* $1\le p<\frac{Q-\lambda}{\alpha}$.

*1) If* $1<p<\frac{Q-\lambda}{\alpha}$, *then the condition* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q-\lambda}$ *is necessary and sufficient for the boundedness* $I_\gamma^\alpha$ *from* $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

*2) If* $p=1$, *then the condition* $1-\frac{1}{q}=\frac{\alpha}{Q-\lambda}$ *is necessary and sufficient for the boundedness* $I_\gamma^\alpha$ *from* $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

**Corollary 5.2.** [19] *Let* $0<\alpha<Q$, $0\le\lambda<Q-\alpha$ *and* $1\le p<\frac{Q-\lambda}{\alpha}$.

*1) If* $1<p<\frac{Q-\lambda}{\alpha}$, *then the condition* $\frac{\alpha}{Q}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$ *is necessary and sufficient for the boundedness* $I_\gamma^\alpha$ *from* $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *to* $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

*2) If* $p=1<\frac{Q-\lambda}{\alpha}$, *then the condition* $\frac{\alpha}{Q}\le 1-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$ *is necessary and sufficient for the boundedness* $I_\gamma^\alpha$ *from* $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *to* $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

## 6 Conclusion

In this paper, necessary and sufficient conditions on the numerical parameters are found ensuring that the $B$-Riesz potential $I_\gamma^\alpha$ is bounded from the total $B$-Morrey space $L_{p,(\lambda,\mu),\gamma}$ to the total $B$-Morrey space $L_{q,(\lambda,\mu),\gamma}$ and from the total $B$-Morrey space $L_{1,(\lambda,\mu),\gamma}$ to the weak total $B$-Morrey space $WL_{q,(\lambda,\mu),\gamma}$.


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