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CONTENTS

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BOKAYEV NURZHAN ADILKHANOVICH

(to the 70th birthday)

January 5, 2026, marks the 70th birthday of Nurzhan Adilkhanovich Bokayev, Doctor of Physical and Mathematical Sciences (1996), Professor (2002), member of the Editorial Board of the Eurasian Mathematical Journal (2010).

Nurzhan Adilkhanovich Bokayev was born on 5 January, 1956 in the village of Urnek, Karabalyk District, Kostanay Region. He graduated in 1972, with a gold medal from the Burlin Secondary School in the district. That same year, he entered the Mathematics Department of Karaganda State University and graduated with honors in 1977. From 1978 to 1979, he served in the Soviet Army. In 1980, he completed an internship, and from 1981 to 1984, he studied in the graduate program at Lomonosov Moscow State University in the Department of Function Theory and Functional Analysis. In 1985, he defended his candidate's dissertation there under the supervision of Corresponding Member of the Academy of Sciences of the USSR D.E. Menshov and Professor V.A. Skvortsov. In 1996, he defended his doctoral dissertation, "Fourier Coefficients and Uniqueness Theorems for Series in Generalized Walsh and Haar Systems", at the Institute of Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan, speciality Mathematical Analysis (01.01.01).

After completing his postgraduate studies, he worked as a lecturer, senior lecturer, associate professor, and professor in the Department of Mathematical Analysis at E.A. Buketov Karaganda State University (1985-1999). He headed the Department of Mathematics and Mathematical Modeling (1996-1999), and was a dean of the Faculty of Mathematics at E.A. Buketov Karaganda State University (1999-2005). Since 2005, he has been a professor in the Faculty of Mechanics and Mathematics at the L.N. Gumilyov Eurasian National University. From 2009 to 2018, he was the Head of the Department of Higher Mathematics at the L.N. Gumilyov Eurasian National University, and from 2018 to the present, he has been a professor in the Department of Fundamental Mathematics.

Professor Bokaev's research focuses on problems in function theory and functional analysis, the theory of orthogonal series for generalized Walsh and Haar systems, and operator theory in various function spaces. He has proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, and constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems. He obtained conditions for functions to belong to various functional classes in terms of the Fourier coefficients of generalized Haar and Walsh systems, and embedding criteria for Nikol'skii-Besov spaces constructed on the basis of multiplicative systems. He also obtained conditions for the boundedness and compactness of the commutator of the Riesz potential in general Morrey-type spaces, and conditions for boundedness of generalizedRiesz and Bessel potentials and generalized fractional-maximal operators in rearrangement-invariant spaces.

His co-authors include Professor V.A. Skvortsov (Moscow State University, Moscow), Professors V.I. Burenkov and M.L. Goldman (Peoples' Friendship University of Russia (RUDN University). Moscow), Dr. A. Gogatishvili (Institute of Mathematics of the Czech Academy of Sciences, Prague). His doctoral students' foreign advisors include Professors W. Sickel (Friedrich-Schiller-Universitity, Jena, Germany), Massimo Lanza de Cristoforis (University of Padova, Padova, Italy), V. Ruzhansky (Ghent University, Ghent, Belgium), U. Goginava (United Arab Emirates University, Al Ain, United Arab Emirates), and E. Panakhov (Institute of Applied Mathematics at Baku State University, Baku, Azerbaijan).

Under his supervision, 15 dissertations (4 candidate's and 11 PhD) were defended. He has published over 220 scientific papers, 2 monographs and 2 textbooks.

He is a three-time recipient of the state grant “Best University Teacher” of the Republic of Kazakhstan (2006, 2010, 2024) and served as Vice President of the Mathematical Society of Turkic-Speaking Countries (2014-2023). He was awarded the “For Contribution to the Development of Science” badge (2022).

Over the last ten years, he has been and continues to be a head of more than 5 national and international funded projects.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Nurzhan Adilkhanovich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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p-NUMERICAL RADIUS INEQUALITIES FOR THE TENSOR PRODUCT OF OPERATORS

A. Frakis, F. Kittaneh, S. Soltani

Communicated by T. Bekjan

Key words: Schatten p-norm, p-numerical radius, tensor product, inequality.

AMS Mathematics Subject Classification: 15A60, 15A69, 47A12, 47A30, 47A80, 47B10.

Abstract. In this paper, we give several inequalities for the tensor product of two operators involving the p-numerical radius and the Schatten p-norms.

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the $\mathcal{C}^*$-algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, let $T = \Re(T) + i\Im(T)$ be the Cartesian decomposition of T, where $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$.

Let $\mathcal{K}(\mathcal{H})$ denote the class of all compact operators in $\mathcal{B}(\mathcal{H})$. For a compact operator T, the Schatten p-norm of T is defined by $\|T\|_p = (tr|T|^p)^{\frac{1}{p}}$, where $1 \le p \le \infty$ and $|T| = (T^*T)^{\frac{1}{2}}$. For $0 < p < 1$, $\|\cdot\|_p$ is a quasi-norm. The Schatten p-class in $\mathcal{B}(\mathcal{H})$, denoted by $\mathcal{B}_p(\mathcal{H})$ is defined by

$$\mathcal{B}_p(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \|T\|_p < \infty\}.$$

Note that when $p = \infty$, the Schatten p-norm of T is the operator norm $\|T\|_\infty = \|T\| = \sup_{\|x\|=1} \|Tx\|$, and when $p = 2$, it is the Hilbert–Schmidt norm $\|T\|_2 = (tr\, T^*T)^{\frac{1}{2}}$. For $1 \le p \le q \le \infty$, the Schatten p-norm of T satisfies the monotonicity property

$$\|T\|_\infty \le \|T\|_q \le \|T\|_p \le \|T\|_1.$$

For $0 < p \le \infty$, we have the following relations:

$$\|T\|_{sp}^s = \||T|^s\|_p = \||T^*|^s\|_p \quad \text{for } s > 0 \tag{1.1}$$

If $T, S \in \mathcal{B}_p(\mathcal{H})$, where $0 < p \le \infty$, then

$$\left\|\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}\right\|_p = \left\|\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right\|_p = \left(\|T\|_p^p + \|S\|_p^p\right)^{\frac{1}{p}}. \tag{1.2}$$

Moreover, if $T \in \mathcal{B}_p(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, then

$$\|TS\|_p \le \|T\|_p\|S\| \quad \text{and} \quad \|ST\|_p \le \|S\|\|T\|_p. \tag{1.3}$$

Let $1 \le p \le \infty$. The p-numerical radius of T is defined by

$$w_p(T) = \sup_{\theta\in\mathbb{R}} \left\|\Re(e^{i\theta}T)\right\|_p = \sup_{\theta\in\mathbb{R}} \left\|\Im(e^{i\theta}T)\right\|_p.$$

It is well known [1] that the p-numerical radius of T is equivalent to the Schatten p-norm. In fact, we have

$$\frac{1}{2}\|T\|_p \leq w_p(T) \leq \|T\|_p. \tag{1.4}$$

There are many papers dealing with estimates of the p-numerical radius of a bounded linear operator on a separable Hilbert space. We refer the readers to [3], [5], [6], [7] and the references therein.

Next, we focus our attention on the tensor product of operators, which has a wide range of applications in various fields. We mention here image processing, quantum computing, semidefinite programming, operator equations, operator differential equations and other disciplines.

The tensor product of operators has many interesting properties. For example the mixed-product property, which states that for any operators $T, S, A, B \in \mathcal{B}(\mathcal{H})$, we have

$$(T \otimes S)(A \otimes B) = (TA \otimes SB).$$

Other useful properties of the tensor product, that will be used later to state our results, are as follows:

$$|T \otimes S| = |T| \otimes |S|$$

and

$$(T \otimes S)^* = T^* \otimes S^*.$$

If $T, S \geq 0$ and $r > 0$, then

$$(T \otimes S)^r = T^r \otimes S^r.$$

Another useful property of the tensor product is the Schatten p-norm equality, which says that if $T, S \in \mathcal{B}_p(\mathcal{H})$, then

$$\|T \otimes S\|_p = \|T\|_p \|S\|_p.$$

In the case in which $\dim \mathcal{H} = n \in \mathbb{N}$, we identify $\mathcal{B}(\mathcal{H})$ with the matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. The tensor product (or the Kronecker product) of $T = [t_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $S = [s_{ij}] \in \mathcal{M}_n(\mathbb{C})$ is defined to be the block matrix

$$T \otimes S = [t_{ij}S] := \begin{bmatrix} t_{11}S & \dots & t_{1n}S \\ \vdots & \ddots & \vdots \\ t_{n1}S & \dots & t_{nn}S \end{bmatrix} \in \mathcal{M}_{n^2}(\mathbb{C}).$$

Many works involving the tensor product have been published. The reader may consult [2], [4], [8], [10], [11] and references therein.

In this paper, we give several p-numerical radius and Schatten p-norm inequalities for the tensor product of operators.

2 Main results

For $X \in \mathcal{M}_n(\mathbb{C})$, the Schatten p-norm distance $\Delta_p(X)$ of the matrix X from the scalar matrices is defined as

$$\Delta_p(X) = \inf_{z \in \mathbb{C}} \|X - zI\|_p.$$

By a compactness argument, one can deduce that there exists $z_0 \in \mathbb{C}$ such that $\|X - z_0 I\|_p = \Delta_p(X)$. It can be easily verified that $\Delta_p(\cdot)$ is a seminorm on $\mathcal{M}_n(\mathbb{C})$, and it is obvious that $\Delta_p(X) \leq \|X\|_p$.

In the following two theorems, we give lower bounds for the p-numerical radius of the tensor product of matrices T and S using merely the entries of these matrices.

Let $X[\mathcal{I}|\mathcal{J}]$ denote the submatrix of X consisting of the entries which belong to the rows $i \in \mathcal{I}$ and the columns $j \in \mathcal{J}$. Let $|\mathcal{I}|$ and $|\mathcal{J}|$ denote the cardinality of $\mathcal{I}$ and $\mathcal{J}$, respectively. It is easy to see that if $\mathcal{I} \cap \mathcal{J} = \emptyset$, then $(X - zI)[\mathcal{I}|\mathcal{J}] = X[\mathcal{I}|\mathcal{J}]$ for any $z \in \mathbb{C}$.

Note that the Schatten p-norm of an $m \times n$ matrix can be defined in a natural way. It is known that $\|X_0\|_p \leq \|X\|_p$, where X_0 is any submatrix of X. It should be mentioned here that some ideas of the proofs of the first two results are inspired by [9].

Theorem 2.1. *Let* $T = [t_{ij}], S = [s_{ij}] \in \mathcal{M}_n(\mathbb{C})$, *and let* $p \geq 1$. *Then*

$$\frac{1}{2n}\left(\sum_{i,j=1}^{n}|t_{ij}|^2\sum_{i,j=1}^{n}|s_{ij}|^2 - \frac{1}{n}\Big(\sum_{i=1}^{n}t_{ii}\sum_{i=1}^{n}s_{ii}\Big)^2\right)^{\frac{1}{2}} \leq w_p(T \otimes S).$$

Proof. It is well known that for any matrix $X \in \mathcal{M}_n(\mathbb{C})$, we have

$$\|X\|_p \geq \|X\| \geq \frac{1}{\sqrt{n}}\|X\|_2.$$

Now,

$$\begin{aligned}
w_p(T\otimes S) &\geq \frac{1}{2}\|T\otimes S\|_p\\
&\geq \frac{1}{2}\inf_{z\in\mathbb{C}}\|T\otimes S - zI\|_p\\
&\geq \frac{1}{2n}\inf_{z\in\mathbb{C}}\|T\otimes S - zI\|_2\\
&= \frac{1}{2n}\sqrt{\|T\otimes S\|_2^2 - \frac{1}{n}tr^2(T\otimes S)}\\
&= \frac{1}{2n}\sqrt{\|T\|_2^2\|S\|_2^2 - \frac{1}{n}tr^2(T)tr^2(S)},
\end{aligned}$$

as required. □

Theorem 2.2. *Let* $T = [t_{ij}], S = [s_{ij}] \in \mathcal{M}_n(\mathbb{C})$, *and let* $p \geq 1$. *Then*

$$\max_{\mathcal{I}}\frac{\min_{i\in\mathcal{I},j\notin\mathcal{I}}|t_{ij}|}{2\sqrt{|\mathcal{I}|}}\sqrt{\sum_{i\in\mathcal{I},j\notin\mathcal{I}}|s_{ij}|^2} \leq w_p(T\otimes S),$$

where $\emptyset \neq \mathcal{I} \subset \{1,2,\ldots,n\}$ *and* $|\mathcal{I}| \leq \frac{n}{2}$.

Proof. Let $\mathcal{I}, \mathcal{J}$ be as described above with $\mathcal{I} \cap \mathcal{J} = \emptyset$. Then

$$\begin{aligned}
w_p(T\otimes S) &\geq \frac{1}{2}\|T\otimes S\|_p\\
&\geq \frac{1}{2}\inf_{z\in\mathbb{C}}\|T\otimes S - zI\|_p\\
&\geq \frac{1}{2}\inf_{z\in\mathbb{C}}\|(T\otimes S - zI)[\mathcal{I}|\mathcal{J}]\|_p\\
&= \frac{1}{2}\|(T\otimes S)[\mathcal{I}|\mathcal{J}]\|_p\\
&\geq \frac{1}{2\sqrt{\min(|\mathcal{I}|,|\mathcal{J}|)}}\|(T\otimes S)[\mathcal{I}|\mathcal{J}]\|_2\\
&= \frac{\min_{i\in\mathcal{I},j\in\mathcal{J}}|t_{ij}|}{2\sqrt{\min(|\mathcal{I}|,|\mathcal{J}|)}}\sqrt{\sum_{i\in\mathcal{I},j\in\mathcal{J}}|s_{ij}|^2}.
\end{aligned}$$

Letting $\mathcal{J} = \{1, 2, \dots, n\} \setminus \mathcal{I}$ and restricting $|\mathcal{I}| \le \frac{n}{2}$, gives the desired result.

□

The following theorem refines the inequality $\frac{\|T\|_p\|S\|_p}{2} \le w_p(T \otimes S)$, which can be extracted from inequality (1.4).

Theorem 2.3. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \ge 1$. Then*

$$\frac{1}{2}\left(\|T\|_p\|S\|_p + \left|\, \|\Re(T\otimes S)\|_p - \|\Im(T\otimes S)\|_p \,\right|\right) \le w_p(T \otimes S).$$

Proof. We have

$$\begin{aligned} w_p(T\otimes S) &\ge \max\left\{\|\Re(T\otimes S)\|_p, \|\Im(T\otimes S)\|_p\right\} \\ &= \frac{\|\Re(T\otimes S)\|_p + \|\Im(T\otimes S)\|_p}{2} + \frac{\left|\,\|\Re(T\otimes S)\|_p - \|\Im(T\otimes S)\|_p\,\right|}{2} \\ &\ge \frac{\|\Re(T\otimes S) + i\Im(T\otimes S)\|_p}{2} + \frac{\left|\|\Re(T\otimes S)\|_p - \|\Im(T\otimes S)\|_p\right|}{2} \\ &= \frac{\|T\|_p\|S\|_p}{2} + \frac{\left|\,\|\Re(T\otimes S)\|_p - \|\Im(T\otimes S)\|_p\,\right|}{2}, \end{aligned}$$

as required. □

In the following corollary, we give a necessary condition for the equality

$$w_p(T\otimes S) = \frac{\|T\|_p\|S\|_p}{2}.$$

Corollary 2.1. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \ge 1$. If $w_p(T\otimes S) = \frac{\|T\|_p\|S\|_p}{2}$, then*

$$\|T\otimes S + T^*\otimes S^*\|_p = \|T\otimes S - T^*\otimes S^*\|_p = \|T\|_p\|S\|_p.$$

Proof. Let $w_p(T\otimes S) = \frac{\|T\|_p\|S\|_p}{2}$. Then by Theorem 2.3, we have $\|\Re(T\otimes S)\|_p = \|\Im(T\otimes S)\|_p$. On the other hand, we have

$$\begin{aligned} \|\Re(T\otimes S)\|_p &\le w_p(T\otimes S) \\ &= \frac{\|T\|_p\|S\|_p}{2} \\ &\le \frac{\|\Re(T\otimes S)\|_p + \|\Im(T\otimes S)\|_p}{2} \\ &= \|\Re(T\otimes S)\|_p. \end{aligned}$$

Hence, we obtain the required equalities. □

In the next theorem, we provide another lower bound for $w_p(T\otimes S)$.

Theorem 2.4. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \ge 2$. Then*

$$\frac{\left\||T|^2\otimes|S|^2 + |T^*|^2\otimes|S^*|^2\right\|_{p/2}}{4} + \frac{\left|\,\|\Re(T\otimes S)\|_p^2 - \|\Im(T\otimes S)\|_p^2\,\right|}{2} \le w_p^2(T\otimes S).$$

Proof. We have

$$\begin{aligned}
w_p^2(T\otimes S) &\geq \max\left\{\|\Re(T\otimes S)\|_p^2, \|\Im(T\otimes S)\|_p^2\right\}\\
&= \frac{\|\Re(T\otimes S)\|_p^2+\|\Im(T\otimes S)\|_p^2}{2}+\frac{\left|\|\Re(T\otimes S)\|_p^2-\|\Im(T\otimes S)\|_p^2\right|}{2}\\
&= \frac{\left\|\Re^2(T\otimes S)\right\|_{p/2}+\left\|\Im^2(T\otimes S)\right\|_{p/2}}{2}+\frac{\left|\|\Re(T\otimes S)\|_p^2-\|\Im(T\otimes S)\|_p^2\right|}{2}\\
&\geq \frac{\left\|\Re^2(T\otimes S)+\Im^2(T\otimes S)\right\|_{p/2}}{2}+\frac{\left|\|\Re(T\otimes S)\|_p^2-\|\Im(T\otimes S)\|_p^2\right|}{2}\\
&= \frac{\left\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\right\|_{p/2}}{4}+\frac{\left|\|\Re(T\otimes S)\|_p^2-\|\Im(T\otimes S)\|_p^2\right|}{2},
\end{aligned}$$

as required. ☐

Corollary 2.2. *Let $T,S\in\mathcal{B}_p(\mathcal{H})$, and let $p\geq 2$. If*

$$w_p^2(T\otimes S)=\frac{\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}}{4},$$

then

$$\|e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*\|_p^2=\|e^{i\theta}T\otimes S-e^{-i\theta}T^*\otimes S^*\|_p^2=\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}$$

for all $\theta\in\mathbb{R}$.

Proof. Assume that

$$w_p^2(T\otimes S)=\frac{\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}}{4}.$$

Then, from the fact that $w_p(T\otimes S)=w_p(e^{i\theta}T\otimes S)$, where $\theta\in\mathbb{R}$, and by Theorem 2.4, we get

$$\left\|\Re(e^{i\theta}T\otimes S)\right\|_p=\left\|\Im(e^{i\theta}T\otimes S)\right\|_p. \tag{2.1}$$

Now,

$$\begin{aligned}
4w_p^2(T\otimes S) &= \frac{\|(e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*)^2-(e^{i\theta}T\otimes S-e^{-i\theta}T^*\otimes S^*)^2\|_{p/2}}{2}\\
&\leq \frac{\|(e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*)^2\|_{p/2}+\|(e^{i\theta}T\otimes S-e^{-i\theta}T^*\otimes S^*)^2\|_{p/2}}{2}\\
&= \frac{\|e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*\|_p^2+\|e^{i\theta}T\otimes S-e^{-i\theta}T^*\otimes S^*\|_p^2}{2}\\
&= \|e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*\|_p^2 \ \text{(by equality (2.1))}\\
&\leq 4w_p^2(T\otimes S)\\
&= \||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}.
\end{aligned}$$

Hence, we get the desired result. ☐

Our next lower bound for $w_p(T\otimes S)$ reads as follows.

Theorem 2.5. *Let $T,S\in\mathcal{B}_p(\mathcal{H})$, and let $p\geq 1$. Then*

$$\begin{aligned}
w_p(T\otimes S) &\geq \frac{\sqrt{2}}{4}\|T\|_p\|S\|_p+\frac{\left|\|T\otimes S+T^*\otimes S^*\|_p-\|T\otimes S-T^*\otimes S^*\|_p\right|}{4}\\
&\quad+\frac{\sqrt{2}}{8}\left|\|T\otimes S+iT^*\otimes S^*\|_p-\|T\otimes S-iT^*\otimes S^*\|_p\right|.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
w_p(T\otimes S) &\geq \max\left\{\left\|\Re(T\otimes S)\right\|_p, \left\|\Im(T\otimes S)\right\|_p\right\}\\
&= \frac{\left\|\Re(T\otimes S)\right\|_p+\left\|\Im(T\otimes S)\right\|_p}{2}+\frac{\left|\left\|\Re(T\otimes S)\right\|_p-\left\|\Im(T\otimes S)\right\|_p\right|}{2}\\
&\geq \frac{1}{2}\max\left\{\left\|\Re(T\otimes S)+\Im(T\otimes S)\right\|_p, \left\|\Re(T\otimes S)-\Im(T\otimes S)\right\|_p\right\}\\
&+\frac{\left|\left\|\Re(T\otimes S)\right\|_p-\left\|\Im(T\otimes S)\right\|_p\right|}{2}\\
&= \frac{\left\|\Re(T\otimes S)+\Im(T\otimes S)\right\|_p+\left\|\Re(T\otimes S)-\Im(T\otimes S)\right\|_p}{4}\\
&+\frac{\left|\left\|\Re(T\otimes S)+\Im(T\otimes S)\right\|_p-\left\|\Re(T\otimes S)-\Im(T\otimes S)\right\|_p\right|}{4}\\
&+\frac{\left|\left\|\Re(T\otimes S)\right\|_p-\left\|\Im(T\otimes S)\right\|_p\right|}{2}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{\left\|\left(\Re(T\otimes S)+\Im(T\otimes S)\right)+i\left(\Re(T\otimes S)-\Im(T\otimes S)\right)\right\|_p}{4}\\
&+\frac{\left|\left\|\Re(T\otimes S)+\Im(T\otimes S)\right\|_p-\left\|\Re(T\otimes S)-\Im(T\otimes S)\right\|_p\right|}{4}\\
&+\frac{\left|\left\|\Re(T\otimes S)\right\|_p-\left\|\Im(T\otimes S)\right\|_p\right|}{2}\\
&= \frac{\|(1+i)T^*\otimes S^*\|_p}{4}+\frac{\left|\left\|\Re(T\otimes S)\right\|_p-\left\|\Im(T\otimes S)\right\|_p\right|}{2}\\
&+\frac{\left|\left\|\Re(T\otimes S)+\Im(T\otimes S)\right\|_p-\left\|\Re(T\otimes S)-\Im(T\otimes S)\right\|_p\right|}{4}\\
&= \frac{\sqrt{2}}{4}\|T\|_p\|S\|_p+\frac{|\|T\otimes S+T^*\otimes S^*\|_p-\|T\otimes S-T^*\otimes S^*\|_p|}{4}\\
&+\frac{\sqrt{2}}{8}\left|\|T\otimes S+iT^*\otimes S^*\|_p-\|T\otimes S-iT^*\otimes S^*\|_p\right|.
\end{aligned}$$

This completes the proof. □

In the following two theorems, we give some upper bounds for $w_p(T\otimes S)$.

Theorem 2.6. *Let $T,S\in\mathcal{B}_p(\mathcal{H})$, and let $p\geq 1$. Then*

$$\begin{aligned}
w_p(T\otimes S) &\leq \|T\|_p w_p(S)\\
&+\min\{w_p\left(\Re T\otimes S^*\right), w_p\left(\Im T\otimes S^*\right), w_p\left(\Re T\otimes S\right), w_p\left(\Im T\otimes S\right)\}.
\end{aligned}$$

Proof. Let $\theta \in \mathbb{R}$. Then

$$\begin{aligned}\|\Re(e^{i\theta}T\otimes S)\|_p &= w_p\big(\Re(e^{i\theta}T\otimes S)\big)\\ &= w_p\Big(\frac{e^{i\theta}T\otimes S}{2}+\frac{e^{-i\theta}T^*\otimes S^*}{2}\Big)\\ &= w_p\Big(T\otimes\frac{e^{i\theta}S+e^{-i\theta}S^*}{2}+e^{-i\theta}\frac{T^*\otimes S^*-T\otimes S^*}{2}\Big)\\ &\le w_p\big(T\otimes\Re(e^{i\theta}S)\big)+w_p(\Im T\otimes S^*)\\ &\le \|T\|_p\|\Re(e^{i\theta}S)\|_p+w_p\left(\Im T\otimes S^*\right).\end{aligned}$$

By taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality, we get

$$w_p(T\otimes S) \;\le\; \|T\|_p w_p(S)+w_p\left(\Im T\otimes S^*\right). \tag{2.2}$$

Replacing T by iT in inequality (2.2), yields

$$\begin{aligned} w_p(T\otimes S) &= w_p(iT\otimes S)\\ &\le \|T\|_p w_p(S)+w_p\left(\Re T\otimes S^*\right). \end{aligned}\tag{2.3}$$

From inequalities (2.2) and (2.3), we obtain

$$w_p(T\otimes S)\le \|T\|_p w_p(S)+\min\big\{w_p\left(\Re T\otimes S^*\right), w_p\left(\Im T\otimes S^*\right)\big\}.$$

The required result follows from the fact that $w_p(T\otimes S)=w_p(T^*\otimes S^*)$. □

In the next theorem, we improve the inequality $w_p(T\otimes S)\le \|T\|_p\|S\|_p$, which can be deduced from inequality (1.4).

Theorem 2.7. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \ge 1$. Then*

$$w_p^2(T\otimes S)\le \frac{1}{2}\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\|_{p/2}.$$

Proof. Let $\theta \in \mathbb{R}$. Then

$$\begin{aligned}\|\Re(e^{i\theta}T\otimes S)\|_p^2 &= \frac{1}{4}\|e^{i\theta}T\otimes S+e^{-i\theta}T^*\otimes S^*\|_p^2\\ &= \frac{1}{4}\left\|\begin{bmatrix} e^{i\theta}I & e^{-i\theta}I\\ 0 & 0\end{bmatrix}\begin{bmatrix} T\otimes S & 0\\ T^*\otimes S^* & 0\end{bmatrix}\right\|_p^2\\ &\le \frac{1}{4}\left\|\begin{bmatrix} e^{i\theta}I & e^{-i\theta}I\\ 0 & 0\end{bmatrix}\right\|^2\left\|\begin{bmatrix} T\otimes S & 0\\ T^*\otimes S^* & 0\end{bmatrix}\right\|_p^2\\ &\quad \text{(by inequality (1.3))}\\ &\le \frac{1}{2}\left\|\begin{bmatrix} T^*\otimes S^* & T\otimes S\\ 0 & 0\end{bmatrix}\begin{bmatrix} T\otimes S & 0\\ T^*\otimes S^* & 0\end{bmatrix}\right\|_{p/2}\\ &\quad \text{(by inequality (1.1))}\\ &= \frac{1}{2}\left\||T|^2\otimes|S|^2+|T^*|^2\otimes|S^*|^2\right\|_{p/2}\\ &\quad \text{(by inequality (1.2)).}\end{aligned}$$

The result follows by taking the supremum in the above inequality over all $\theta \in \mathbb{R}$. □

In the rest of this work, we present certain inequalities involving the Schatten p-norms of operators.

Theorem 2.8. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$, and let $p \geq 2$. Then*

$$\begin{aligned}\max\left\{\|T\otimes S+S\otimes T\|_p^2, \|T\otimes S-S\otimes T\|_p^2\right\} &\\ \geq \max\left\{\left\|T^2\otimes S^2+S^2\otimes T^2\right\|_p, \left\||T|^2\otimes |S|^2+|S|^2\otimes |T|^2\right\|_{p/2}\right\}&\\ +\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}.&\end{aligned}$$

Proof. Let $\mathcal{M}=\max\left\{\|T\otimes S+S\otimes T\|_p^2, \|T\otimes S-S\otimes T\|_p^2\right\}$. Then

$$\begin{aligned}\mathcal{M} &= \frac{\|T\otimes S+S\otimes T\|_p^2+\|T\otimes S-S\otimes T\|_p^2}{2}\\ &\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}\\ &\geq \frac{\|(T\otimes S+S\otimes T)^2\|_p+\|(T\otimes S-S\otimes T)^2\|_p}{2}\\ &\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}\\ &\quad\text{(by the inequality } \|X\|_p^2\geq \|X^2\|_p \text{ for any } X\in\mathcal{B}_p(\mathcal{H}))\\ &\geq \frac{\|(T\otimes S+S\otimes T)^2+(T\otimes S-S\otimes T)^2\|_p}{2}\\ &\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}\\ &= \left\|T^2\otimes S^2+S^2\otimes T^2\right\|_p\\ &\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathcal{M} &= \frac{\|T\otimes S+S\otimes T\|_p^2+\|T\otimes S-S\otimes T\|_p^2}{2}\\ &\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}\\ &= \frac{\|\,|T\otimes S+S\otimes T|^2\,\|_{p/2}+\|\,|T\otimes S-S\otimes T|^2\,\|_{p/2}}{2}\\ &\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}\\ &\geq \frac{\|\,|T\otimes S+S\otimes T|^2+|T\otimes S-S\otimes T|^2\,\|_{p/2}}{2}\\ &\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}\\ &= \left\||T|^2\otimes |S|^2+|S|^2\otimes |T|^2\right\|_{p/2}\\ &\quad+\frac{\left|\|T\otimes S+S\otimes T\|_p^2-\|T\otimes S-S\otimes T\|_p^2\right|}{2}.\end{aligned}$$

Hence, we obtain the desired result. □

Theorem 2.9. *Let* $T, S \in \mathcal{B}_p(\mathcal{H})$*, and let* $p \geq 2$*. Then*

$$\max\left\{\|T\otimes S+S\otimes T\|_p^2, \|T\otimes S-S\otimes T\|_p^2\right\} \leq \min\left\{\alpha,\beta\right\},$$

where

$$\alpha = \|\,|T|^2\otimes|S|^2+|S|^2\otimes|T|^2\|_{p/2}+\|T^*S\otimes S^*T+S^*T\otimes T^*S\|_{p/2}$$

and

$$\beta = \|\,|T^*|^2\otimes|S^*|^2+|S^*|^2\otimes|T^*|^2\|_{p/2}+\|TS^*\otimes ST^*+ST^*\otimes TS^*\|_{p/2}.$$

Proof. We have $\|T\otimes S\pm S\otimes T\|_p^2$

$$\begin{aligned}
&= \|\,|T\otimes S\pm S\otimes T|^2\,\|_{p/2}\\
&= \|\,(T\otimes S\pm S\otimes T)^*(T\otimes S\pm S\otimes T)\,\|_{p/2}\\
&= \|\,|T|^2\otimes|S|^2+|S|^2\otimes|T|^2\pm(T^*S\otimes S^*T+S^*T\otimes T^*S)\|_{p/2}\\
&\leq \|\,|T|^2\otimes|S|^2+|S|^2\otimes|T|^2\|_{p/2}+\|T^*S\otimes S^*T+S^*T\otimes T^*S\|_{p/2}.
\end{aligned}$$

Using the fact that $\|T\|_p=\|T^*\|_p$, gives

$$\|T\otimes S\pm S\otimes T\|_p^2 \leq \|\,|T^*|^2\otimes|S^*|^2+|S^*|^2\otimes|T^*|^2\|_{p/2}+\|TS^*\otimes ST^*+ST^*\otimes TS^*\|_{p/2}.$$

Hence, we get the desired inequality. □

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