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BOKAYEV NURZHAN ADILKHANOVICH

(to the 70th birthday)

January 5, 2026, marks the 70th birthday of Nurzhan Adilkhanovich Bokayev, Doctor of Physical and Mathematical Sciences (1996), Professor (2002), member of the Editorial Board of the Eurasian Mathematical Journal (2010).

Nurzhan Adilkhanovich Bokayev was born on 5 January, 1956 in the village of Urnek, Karabalyk District, Kostanay Region. He graduated in 1972, with a gold medal from the Burlin Secondary School in the district. That same year, he entered the Mathematics Department of Karaganda State University and graduated with honors in 1977. From 1978 to 1979, he served in the Soviet Army. In 1980, he completed an internship, and from 1981 to 1984, he studied in the graduate program at Lomonosov Moscow State University in the Department of Function Theory and Functional Analysis. In 1985, he defended his candidate's dissertation there under the supervision of Corresponding Member of the Academy of Sciences of the USSR D.E. Menshov and Professor V.A. Skvortsov. In 1996, he defended his doctoral dissertation, “Fourier Coefficients and Uniqueness Theorems for Series in Generalized Walsh and Haar Systems”, at the Institute of Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan, speciality Mathematical Analysis (01.01.01).

After completing his postgraduate studies, he worked as a lecturer, senior lecturer, associate professor, and professor in the Department of Mathematical Analysis at E.A. Buketov Karaganda State University (1985-1999). He headed the Department of Mathematics and Mathematical Modeling (1996-1999), and was a dean of the Faculty of Mathematics at E.A. Buketov Karaganda State University (1999-2005). Since 2005, he has been a professor in the Faculty of Mechanics and Mathematics at the L.N. Gumilyov Eurasian National University. From 2009 to 2018, he was the Head of the Department of Higher Mathematics at the L.N. Gumilyov Eurasian National University, and from 2018 to the present, he has been a professor in the Department of Fundamental Mathematics.

Professor Bokaev's research focuses on problems in function theory and functional analysis, the theory of orthogonal series for generalized Walsh and Haar systems, and operator theory in various function spaces. He has proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, and constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems. He obtained conditions for functions to belong to various functional classes in terms of the Fourier coefficients of generalized Haar and Walsh systems, and embedding criteria for Nikol'skii-Besov spaces constructed on the basis of multiplicative systems. He also obtained conditions for the boundedness and compactness of the commutator of the Riesz potential in general Morrey-type spaces, and conditions for boundedness of generalizedRiesz and Bessel potentials and generalized fractional-maximal operators in rearrangement-invariant spaces.

His co-authors include Professor V.A. Skvortsov (Moscow State University, Moscow), Professors V.I. Burenkov and M.L. Goldman (Peoples' Friendship University of Russia (RUDN University). Moscow), Dr. A. Gogatishvili (Institute of Mathematics of the Czech Academy of Sciences, Prague). His doctoral students' foreign advisors include Professors W. Sickel (Friedrich-Schiller-Universitity, Jena, Germany), Massimo Lanza de Cristoforis (University of Padova, Padova, Italy), V. Ruzhansky (Ghent University, Ghent, Belgium), U. Goginava (United Arab Emirates University, Al Ain, United Arab Emirates), and E. Panakhov (Institute of Applied Mathematics at Baku State University, Baku, Azerbaijan).

Under his supervision, 15 dissertations (4 candidate's and 11 PhD) were defended. He has published over 220 scientific papers, 2 monographs and 2 textbooks.

He is a three-time recipient of the state grant "Best University Teacher" of the Republic of Kazakhstan (2006, 2010, 2024) and served as Vice President of the Mathematical Society of Turkic-Speaking Countries (2014-2023). He was awarded the "For Contribution to the Development of Science" badge (2022).

Over the last ten years, he has been and continues to be a head of more than 5 national and international funded projects.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Nurzhan Adilkhanovich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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HÖLDER INEQUALITY ON THE SPACE OF UPPER SEMICONTINUOUS FUNCTIONS

Sh.A. Ayupov, M.R. Eshimbetov, A.A. Zaitov

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Key words: idempotent measure; max-plus linear functional; Borel sets; upper semicontinuous functions.

AMS Mathematics Subject Classification: Primary 47B65; Secondary 28C05.

Abstract. For a compact Hausdorff space X, we consider the space $I_{\mathfrak{B}}(X)$ of all idempotent probability measures on X, which are defined as set-functions on the σ-algebra of all Borel subsets of X, and also the space $I_{USC}(X)$ of all normalized max-plus linear functionals on the linear space of all upper semicontinuous functions on X, equipped with idempotent operations. In the main result it is established that a max-plus version of the Hölder inequality holds on the space of upper semicontinuous functions.

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1 Introduction

Idempotent mathematics is a branch of mathematical sciences, rapidly developing and gaining popularity over the last four decades. An important stage of development of the subject was presented in the book "Idempotency" [7] edited by J. Gunawardena. This book arose out of the well-known international workshop that was held in Bristol, England, in October 1994. Idempotent mathematics is based on replacing the usual arithmetic operations with a new set of basic operations, i.e., on replacing numerical fields by idempotent semirings and semifields. Typical example is the so-called max-plus algebra $\mathbb{R}_{\max}$ [7], [11].

In [15] M. Zarichnyi considered categorical properties of the space of idempotent probability measures on compact Hausdorff spaces. Later the space of idempotent probability measures was investigated on the class of compact metric spaces [16]. The study of spaces of idempotent probability measures leads to similar problems for topological spaces which are wider than the class of compact Hausdorff (or metrizable) spaces, in particular, for the case of Tychonoff spaces. In this case it is natural to apply the methods proposed in works [1], [2], [6] or [13].

The results obtained in [14], [19], [17] show that in order to establish "good" properties of the space of idempotent probability measures, methods are required which are different from classical methods (i.e., from methods suitable for probability measures which have been applied in [6], [13] and others). Note that idempotent probability measures in general are not linear and for a given compact Hausdorff space X they are defined as max-plus functionals on $C(X)$ [14], while the usual probability measures on X are a positive, linear, normalized functional on $C(X)$ [3], [4].

Unlike the above mentioned papers in the present work for a compact Hausdorff space X we introduce the notion of idempotent measure as a set-function on the σ-algebra $\mathfrak{B}(X)$ of all Borel subsets of X. For a compact Hausdorff space X, we denote by $I_{\mathfrak{B}}(X)$ the space of all idempotent probability measures on X, which define as set-functions $\mu\colon \mathfrak{B}(X) \to \overline{\mathbb{R}}_{+}$, and by $I_{USC}(X)$ the space of all normalized max-plus linear functionals $\nu\colon USC(X) \to \mathbb{R}_{\max}$ on the space $USC(X)$ of all upper semicontinuous functions on X. Then, we present a max-plus version of the well-known Riesz Representation Theorem. Further, we obtain the main result, which states that the spaces $I_{\mathfrak{B}}(X)$ and $I_{USC}(X)$ are homeomorphic. Finally, the Hölder inequality on the space of all upper semicontinuous functions is proved.

2 Preliminaries

Let X be a compact Hausdorff space and let $\mathfrak{B}(X)$ denote the family of all Borel subsets of X. We denote $\overline{\mathbb{R}}_+ = [0, +\infty) \cup \{+\infty\} = [0, +\infty]$. The symbol $\mathfrak{A}$ denotes a directed set. Following [10], we introduce the following notion.

Definition 1. A set function $\mu\colon \mathfrak{B}(X) \to \overline{\mathbb{R}}_+$ is said to be an idempotent measure on X if it satisfies the following conditions:

1) $\mu(\varnothing) = 0$;

2) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for any $A, B \in \mathfrak{B}(X)$;

3) $\mu\left(\bigcup\limits_{\alpha\in\mathfrak{A}} A_\alpha\right) = \sup\limits_{\alpha\in\mathfrak{A}}\{\mu(A_\alpha)\}$ for every increasing net $\{A_\alpha\colon \alpha \in \mathfrak{A}\} \subset \mathfrak{B}(X)$ such that $\bigcup\limits_{\alpha\in\mathfrak{A}} A_\alpha \in \mathfrak{B}(X)$.

Remark 1. Every idempotent measure μ is increasing, i.e., for $A, B \in \mathfrak{B}(X)$ if $A \subset B$ then $\mu(A) \le \mu(B)$.

The set of all idempotent measure on X we denote by $IM(X)$. Let $\mathcal{B}$ be a base of the topology on X. For an idempotent measure $\mu \in IM(X)$ a system of sets

$$\langle \mu;\, U_1, \ldots, U_n;\, \varepsilon\rangle = \{\nu \in IM(X)\colon\ |\nu(U_i) - \mu(U_i)| < \varepsilon,\ i = 1, \ldots, n\} \tag{2.1}$$

forms [8] a base of a topology on $IM(X)$ at μ. Here $U_i \in \mathcal{B}$, $i = 1, \ldots, n$, and $\varepsilon > 0$.

If $\mu(X) = 1$, the idempotent measure μ is called an idempotent probability measure on X. We denote

$$I_{\mathfrak{B}}(X) = \{\mu \in IM(X)\colon\ \mu(X) = 1\}.$$

Let (X, μ) be an idempotent measure space such that $\mu(X) < \infty$. We adopt the convention that $\infty \cdot 0 = 0$. For $A \subset X$ its characteristic function χ_A is defined as $\chi_A(x) = 1$ at $x \in A$, and $\chi_A(x) = 0$ at $x \in X \setminus A$.

Definition 2. [10] For a function $f\colon X \to \overline{\mathbb{R}}_+$ we define the idempotent integral of f with respect to μ by

$$\int\limits_X^{\oplus} f d\mu = \sup_{t\in\overline{\mathbb{R}}_+}\{t \cdot \mu\{x \in X\colon f(x) \ge t\}\}.$$

For $A \subset X$, we let $\int\limits_A^{\oplus} f d\mu = \int\limits_X^{\oplus} f\chi_A d\mu$.

Lemma 2.1. [18] *For every couple A and B of Borel subsets of a compact Hausdorff space X the following equality holds*

$$\int\limits_{A\cup B}^{\oplus} f\, d\mu = \int\limits_A^{\oplus} f\, d\mu \oplus \int\limits_B^{\oplus} f\, d\mu.$$

The following two statements will be applied to establish Lemma 2.3

Lemma 2.2. [9] *For a function* $f\colon X \to \overline{\mathbb{R}}_+$ *we have*

$$\int\limits_X^{\oplus} f d\mu = \sup_{x\in X}\{f(x)\cdot \mu(f^{-1}(f(x)))\}.$$

Theorem 2.1. [10] *For every* $c \in \overline{\mathbb{R}}_+$, *any* $\overline{\mathbb{R}}_+$*-valued functions* f, g *and a net* $\{f_j\}_{j\in J}$ *of* $\overline{\mathbb{R}}_+$*-valued functions on* X *the following properties hold:*

1) $\int\limits_X^{\oplus} 0\, d\mu = 0$;

2) $\int\limits_X^{\oplus} 1\, d\mu = \mu(X)$;

3) $\int\limits_X^{\oplus} f\, d\mu \leq \int\limits_X^{\oplus} g\, d\mu$ if $f \leq g$;

4) $\int\limits_X^{\oplus} (c\cdot f)\, d\mu = c\cdot \int\limits_X^{\oplus} f\, d\mu$;

5) $\int\limits_X^{\oplus} (f\oplus g)\, d\mu = \int\limits_X^{\oplus} f\, d\mu \oplus \int\limits_X^{\oplus} g\, d\mu$;

6) $\int\limits_X^{\oplus} (f+g)\, d\mu \leq \int\limits_X^{\oplus} f\, d\mu + \int\limits_X^{\oplus} g\, d\mu$;

7) $\left|\int\limits_X^{\oplus} f\, d\mu - \int\limits_X^{\oplus} g\, d\mu\right| \leq \int\limits_X^{\oplus} |f-g|\, d\mu$ provided the left-hand side is well defined;

8) $\int\limits_X^{\oplus} \sup_{j\in J} f_j\, d\mu = \sup_{j\in J} \int\limits_X^{\oplus} f_j\, d\mu$.

Now, for $\varphi\colon X \to \overline{\mathbb{R}}_+$ and $p > 0$ we define

$$||\varphi||_p = \left(\int\limits_X^{\oplus} \varphi^p(x) d\mu\right)^{\frac{1}{p}} \quad \text{and} \quad ||\varphi||_\infty = \sup_{\{x\in X\colon\, \mu(\{x\})>0\}}\{\varphi(x)\}.$$

Lemma 2.3. *Let* φ, $\psi\colon X \to \overline{\mathbb{R}}_+$.

1) *Let* $p \in [1, +\infty]$ *and* $q \in [1, +\infty]$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then,*

$$\int\limits_X^{\oplus} \varphi(x)\psi(x) d\mu \leq ||\varphi||_p \cdot ||\psi||_q;$$

2) *If* $\mu(X) = 1$, *then* $||\varphi||_p \leq ||\varphi||_q$, *where* $0 < p < q$.

Proof. 1) From [3], [12], we have the following inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{ where } a,\, b \geq 0,\ p \geq 1,\, q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

Now, let $I_1 = \int\limits_X^{\oplus} \varphi^p(x)d\mu \neq 0$ and $I_2 = \int\limits_X^{\oplus} \psi^q(x)d\mu \neq 0$. Then,

$$\frac{\varphi}{I_1^{\frac{1}{p}}} \cdot \frac{\psi}{I_2^{\frac{1}{q}}} \leq \frac{\varphi^p}{pI_1} + \frac{\psi^q}{qI_2} \Rightarrow$$

$$\int\limits_X^{\oplus} \frac{\varphi(x)\psi(x)}{I_1^{\frac{1}{p}} I_2^{\frac{1}{q}}} d\mu \leq \int\limits_X^{\oplus} \left(\frac{\varphi^p(x)}{pI_1} + \frac{\psi^q(x)}{qI_2} \right) d\mu \leq$$

$$\leq \int\limits_X^{\oplus} \frac{\varphi^p(x)}{pI_1} d\mu + \int\limits_X^{\oplus} \frac{\psi^q(x)}{qI_2} d\mu = \frac{1}{pI_1} \int\limits_X^{\oplus} \varphi^p(x)d\mu + \frac{1}{qI_2} \int\limits_X^{\oplus} \psi^q(x)d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

Consequently, we obtain

$$\int\limits_X^{\oplus} \varphi(x)\psi(x)d\mu \leq I_1^{\frac{1}{p}} \cdot I_2^{\frac{1}{q}} = ||\varphi||_p \cdot ||\psi||_q.$$

If $I_1 = 0$ or $I_2 = 0$, then $0 \leq \int\limits_X^{\oplus} \varphi(x)\psi(x)d\mu \leq I_1^{\frac{1}{p}} \cdot I_2^{\frac{1}{q}} = 0$.

Hence, $\int\limits_X^{\oplus} \varphi(x)\psi(x)d\mu = 0$.

2) If $0 < p < q$, then $r = \frac{q}{p} > 1$. According to the first part of Lemma 2.3 and $\mu(X) = 1$, we have

$$\int\limits_X^{\oplus} (\varphi^p(x) \cdot 1)d\mu \leq \left(\int\limits_X^{\oplus} \varphi^{p \cdot r}(x)d\mu \right)^{\frac{1}{r}} \cdot \left(\int\limits_X^{\oplus} 1^q d\mu \right)^{\frac{1}{q}} =$$

$$= \left(\int\limits_X^{\oplus} \varphi^q(x)d\mu \right)^{\frac{p}{q}} \cdot (\mu(X))^{\frac{1}{q}} = \left(\int\limits_X^{\oplus} \varphi^q(x)d\mu \right)^{\frac{p}{q}} \Rightarrow$$

$$\Rightarrow \left(\int\limits_X^{\oplus} \varphi^p(x)d\mu \right)^{\frac{1}{p}} \leq \left(\int\limits_X^{\oplus} \varphi^q(x)d\mu \right)^{\frac{1}{q}}.$$

Thus, $||\varphi||_p \leq ||\varphi||_q$.

□

To obtain the main results we need the following notions and facts.

Definition 3. [10] We say that a function $f\colon X \to \overline{\mathbb{R}}_+$ is maximable (or μ-maximable), if $\int\limits_X^{\oplus} f(x)d\mu < \infty$ and, moreover, $\int\limits_X^{\oplus} f(x)\chi_{\{x\in X\colon f(x)>t\}}d\mu \to 0$ as $t \to \infty$.

Theorem 2.2. [10] *A function $f\colon X \to \overline{\mathbb{R}}_+$ is maximable if and only if there exists a monotonically increasing function $F\colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that $\frac{F(x)}{x} \to \infty$ as $x \to \infty$ and $\int\limits_X^{\oplus} (F(x) \circ f(x))d\mu < \infty$.*

Definition 4. [10] A net $\{f_\alpha\colon \alpha \in \mathfrak{A}\}$ of $\overline{\mathbb{R}}_+$-valued functions on X is said to be uniformly maximable (or μ-uniformly maximable) if

$$\limsup_{\alpha\in\mathfrak{A}} \overset{\oplus}{\int_X} f_\alpha(x)\chi_{\{x\in X\colon f_\alpha(x)>t\}}d\mu \to 0 \quad \text{as} \quad t\to\infty.$$

Theorem 2.3. [10] *A net $\{f_\alpha\colon \alpha \in \mathfrak{A}\}$ is uniformly maximable if and only if the following conditions hold:*

1) $\limsup\limits_{\alpha\in\mathfrak{A}} \overset{\oplus}{\int_X} f_\alpha(x)d\mu < \infty$;

2) *for every $\varepsilon > 0$ there exists $\eta > 0$ such that $\limsup\limits_{\alpha\in\mathfrak{A}} \overset{\oplus}{\int_{A_\alpha}} f_\alpha(x)d\mu < \varepsilon$ for every net of sets $\{A_\alpha\colon \alpha\in\mathfrak{A}\}$ such that $\limsup\limits_{\alpha\in\mathfrak{A}}\{\mu(A_\alpha)\} < \eta$.*

Theorem 2.4. [10] *Let $\{f_\alpha\colon \alpha \in \mathfrak{A}\}$ be a net of $\overline{\mathbb{R}}_+$-valued functions of X and f be an $\overline{\mathbb{R}}_+$-valued function on X.*

1) *If $\liminf\limits_{\alpha\in\mathfrak{A}}\{f_\alpha\} \overset{\mu-a.e.}{\geq} f$, then $\liminf\limits_{\alpha\in\mathfrak{A}} \overset{\oplus}{\int_X} f_\alpha(x)d\mu \geq \overset{\oplus}{\int_X} f(x)d\mu$;*

2) *If $f_\alpha \overset{\mu}{\to} f$ and net $\{f_\alpha\}$ is uniformly maximable, then $\lim\limits_{\alpha\in\mathfrak{A}} \overset{\oplus}{\int_X} f_\alpha(x)d\mu = \overset{\oplus}{\int_X} f(x)d\mu$;*

3) *If $f_\alpha \overset{\mu-a.e.}{\uparrow} f$, then $\lim\limits_{\alpha\in\mathfrak{A}} \overset{\oplus}{\int_X} f_\alpha(x)d\mu = \overset{\oplus}{\int_X} f(x)d\mu$.*

Definition 5. [5] Let X be a Tychonoff space. A function $\varphi\colon X \to \mathbb{R}_{\max}$ is said to be upper semicontinuous if for every $a \in \mathbb{R}$ the set $\{x \in X\colon \varphi(x) < a\}$ is open.

3 Max-plus version of the Riesz representation theorem

We denote by $USC(X)$ the linear space of all upper semicontinuous functions defined on X which is endowed with the sup-norm $\|\varphi\| = \sup\{|\varphi(x)|\colon x\in X\}$ and denote by $USC_b(X)$ the space of all real-valued upper semicontinuous functions on X bounded above. Note that $(-\infty)_X \in USC(X)$ and $\|(-\infty)_X\| = +\infty$.

Now we present the version of the convergence theorem in the Lebesgue sense of idempotent integrals.

Theorem 3.1. *Let for an arbitrary sequence of functions $\varphi_n(x) \in USC_b(X)$, the sequence $\{e^{\varphi_n(x)}\colon n\in\mathbb{N}\}$ be uniformly maximable. If $\varphi_n \overset{\mu}{\to} \varphi$, then*

$$\lim_{n\to\infty} \overset{\oplus}{\int_X} e^{\varphi_n(x)}d\mu = \overset{\oplus}{\int_X} e^{\lim\limits_{n\to\infty}\varphi_n(x)}d\mu = \overset{\oplus}{\int_X} e^{\varphi(x)}d\mu. \tag{3.1}$$

Proof. First, let us prove the first part of the theorem. We introduce the following notation $A_{n,t} = \{x\in X\colon e^{\varphi_n(x)} > t\}$. Clearly, $A_{n,t}\to\varnothing$ as $t\to\infty$. (Note the symbol $A_{n,t}\to\varnothing$ means the following: $A_{n,t}\supset A_{n,t'}$ for $t<t'$ and $\bigcap\limits_t A_{n,t} = \varnothing$ for each n).

Then, according to the Definitions 2 and 4

$$\sup_{n\in\mathbb{N}} \int\limits_X^{\oplus} e^{\varphi_n(x)} \chi_{A_{n,\,t}} d\mu = \sup_{n\in\mathbb{N}} \int\limits_{A_{n,\,t}}^{\oplus} e^{\varphi_n(x)} d\mu =$$
$$= \sup_{n\in\mathbb{N}}\{ \sup_{t\in[0,\,\infty]} \{t\cdot\mu\{x\in A_{n,t}\colon\ e^{\varphi_n(x)}\geq t\}\} =$$
$$= \sup_{n\in\mathbb{N}}\{ \sup_{t\in[0,\,\infty]} \{t\cdot\mu(A_{n,t})\} \to \sup_{n\in\mathbb{N}}\{\infty\cdot\mu(\varnothing)\} = 0 \quad \text{as} \quad t\to\infty.$$

So, the set $\{e^{\varphi_n(x)}\colon\ n\in\mathbb{N}\}$ is uniformly maximable.

Now we will prove equality (3.1). By the assumption of the theorem $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$. Then, according to the property of continuous and measurable functions

$$\lim_{n\to\infty} e^{\varphi_n(x)} = e^{\lim\limits_{n\to\infty}\varphi_n(x)} = e^{\varphi(x)}.$$

Hence and by part 2) of Theorem 2.4, we have

$$\lim_{n\to\infty}\int\limits_X^{\oplus} e^{\varphi_n(x)} d\mu = \int\limits_X^{\oplus} e^{\lim\limits_{n\to\infty}\varphi_n(x)} d\mu = \int\limits_X^{\oplus} e^{\varphi(x)} d\mu.$$

□

Now, we introduce the notion of a max-plus-linear functional on $USC(X)$.

For each $c\in\mathbb{R}_{\max}$ we denote by c_X the constant function in $USC(X)$ defined by the formula $c_X(x)=c$ for each $x\in X$. Define on the set $USC(X)$ operations $\oplus$ and $\odot$ by $\varphi\oplus\psi=\max\{\varphi,\,\psi\}$ and $\varphi\odot\psi=\varphi+\psi$, where $\varphi,\,\psi\in USC(X)$.

Definition 6. We say that a functional $\nu\colon\ USC(X)\to\mathbb{R}_{\max}$ is max-plus-linear, if it has the following properties:

1) $\nu(\varphi\oplus\psi)=\nu(\varphi)\oplus\nu(\psi)$ for any $\varphi,\,\psi\in USC(X)$;

2) $\nu(c\odot\varphi)=c\odot\nu(\varphi)$ for every $c\in\mathbb{R}_{\max}$ and $\varphi\in USC(X)$.

The set of all max-plus linear functionals on $USC(X)$ we denote by $USC(X)^{\oplus}$. For a max-plus linear functional $\nu\in USC(X)^{\oplus}$ a system of sets

$$\langle\nu;\,\varphi_1,\,\ldots,\,\varphi_n;\,\varepsilon\rangle=\{\nu^{'}\in USC(X)^{\oplus}\colon\ |\nu^{'}(\varphi_i)-\nu(\varphi_i)|<\varepsilon,\,i=1,\,\ldots,\,n\}$$

forms a base of $USC(X)^{\oplus}$ at ν. Here, $\varphi_i\in USC(X)$, $i=1,\,\ldots,\,n$, and $\varepsilon>0$. The proof of the last statement can be obtained in the same way as in [15], if we accept the convention $(-\infty)-(-\infty)=0$.

In order to give the following definition we note that $\nu(0_X)=0$ if and only if $\nu(c_X)=c$ for every $c\in\mathbb{R}_{\max}$. Indeed, for every $c\in\mathbb{R}_{\max}$ we have $\nu(c_X)=\nu(c\odot 0_X)=c\odot\nu(0_X)$. Hence, we get $\nu(c_X)-c=\nu(0_X)$.

Definition 7. A max-plus linear functional $\nu\colon\ USC(X)\to\mathbb{R}_{\max}$ is said to be normalized, if

3) $\nu(c_X)=c$ for each $c\in\mathbb{R}_{\max}$.

Put

$$I_{USC}(X)=\{\nu\in USC(X)^{\oplus}\colon\ \nu(0_X)=0\}.$$

Theorem 3.2. *For each normalized max-plus linear functional* $\nu\colon USC(X)\to\mathbb{R}_{\max}$ *a set-function* $\mu_\nu\colon \mathfrak{B}(X)\to\mathbb{R}$ *defined by the rule*

$$\mu_\nu(A)=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\, \varphi\geq\chi_A\},\qquad A\in\mathfrak{B}(X),$$

is an idempotent probability measure on X.

Proof. Clearly, $\mu_\nu(\varnothing)=0$ and $\mu_\nu(X)=1$. Equality 2) in Definition 1 holds. Indeed,

$$\begin{aligned}\mu_\nu(A\cup B)&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\, \varphi\geq\chi_{A\cup B}\}=\\&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\, \varphi\geq\chi_A\oplus\chi_B\}=\\&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\, (\varphi\geq\chi_A)\wedge(\varphi\geq\chi_B)\}=\\&=\inf\{\nu(\varphi_1\oplus\varphi_2)\colon \varphi_1,\,\varphi_2\in USC(X),\, (\varphi_1\geq\chi_A)\wedge(\varphi_2\geq\chi_B)\}=\\&=\inf\{\nu(\varphi_1)\oplus\nu(\varphi_2)\colon \varphi_1,\,\varphi_2\in USC(X),\, (\varphi_1\geq\chi_A)\wedge(\varphi_2\geq\chi_B)\}=\\&=\max\{\inf\{\nu(\varphi_1)\colon \varphi_1\in USC(X),\, \varphi_1\geq\chi_A\},\\&\qquad\quad \inf\{\nu(\varphi_2)\colon \varphi_2\in USC(X),\, \varphi_2\geq\chi_B\}\}=\\&=\mu_\nu(A)\oplus\mu_\nu(B).\end{aligned}$$

We have to show that equality 3) in Definition 1 is also true.
Let $\{A_\alpha\colon \alpha\in\mathfrak{A}\}\subset\mathfrak{B}(X)$ be an increasing net such that $\bigcup\limits_{\alpha\in\mathfrak{A}}A_\alpha\in\mathfrak{B}(X)$. Then, we have

$$\begin{aligned}\mu_\nu(\bigcup_{\alpha\in\mathfrak{A}}A_\alpha)&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\, \varphi\geq\chi_{\bigcup\limits_{\alpha\in\mathfrak{A}}A_\alpha}\}=\\&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\, \varphi\geq\sup_{\alpha\in\mathfrak{A}}\chi_{A_\alpha}\}=\\&=\inf\{\nu(\varphi)\colon \varphi\in USC(X),\, \varphi\geq\chi_{A_\alpha},\, \alpha\in\mathfrak{A}\}=\\&=\inf\{\nu(\bigoplus_{\alpha\in\mathfrak{A}}\varphi_\alpha)\colon \varphi_\alpha\in USC(X),\, \varphi_\alpha\geq\chi_{A_\alpha},\, \alpha\in\mathfrak{A}\}=\\&=\inf\{\sup_{\alpha\in\mathfrak{A}}\nu(\varphi_\alpha)\colon \varphi_\alpha\in USC(X),\, \varphi_\alpha\geq\chi_{A_\alpha},\, \alpha\in\mathfrak{A}\}=\\&=\sup_{\alpha\in\mathfrak{A}}\{\inf\{\nu(\varphi_\alpha)\colon \varphi_\alpha\in USC(X),\, \varphi_\alpha\geq\chi_{A_\alpha}\}\}=\\&=\sup_{\alpha\in\mathfrak{A}}\{\mu_\nu(A_\alpha)\}.\end{aligned}$$

□

Theorem 3.3. *For a compact Hausdorff space* X *and for any normalized max-plus functional* $\nu\colon USC(X)\to\mathbb{R}_{\max}$ *there exists a unique idempotent probability measure* μ_ν *on* $\mathfrak{B}(X)$ *such that*

$$\nu(\varphi)=\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}e^{\varphi(x)}d\mu_\nu\right),\qquad \varphi\in USC(X). \tag{3.2}$$

Proof. We need to verify that the functional ν defined by (3.2) satisfies the conditions in Definition 1. Indeed,

1) according to Lemma 2.1 for every pair of $\varphi,\,\psi\in USC(X)$ one has

$$\begin{aligned}\nu(\varphi\oplus\psi)&=\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}e^{\varphi\oplus\psi}d\mu_\nu\right)=\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}\left(e^{\varphi}\oplus e^{\psi}\right)d\mu_\nu\right)=\\&=\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}e^{\varphi}d\mu_\nu\right)\oplus\ln\left(\frac{1}{\mu_\nu(X)}\int_X^{\oplus}e^{\psi}d\mu_\nu\right)=\nu(\varphi)\oplus\nu(\psi);\end{aligned}$$

2) for every $c \in \mathbb{R}_{\max}$ and $\varphi \in USC(X)$

$$\nu(c \odot \varphi) = \ln\left(\frac{1}{\mu_\nu(X)}\int\limits_X^{\oplus} e^{c\odot\varphi} d\mu_\nu\right) = \ln\left(\frac{1}{\mu_\nu(X)}\int\limits_X^{\oplus} e^{c+\varphi} d\mu_\nu\right) =$$
$$= \ln\left(\frac{1}{\mu_\nu(X)}\int\limits_X^{\oplus} (e^c \cdot e^\varphi)\, d\mu_\nu\right) = \ln\left(e^c \cdot \frac{1}{\mu_\nu(X)}\int\limits_X^{\oplus} e^\varphi d\mu_\nu\right) =$$
$$= c + \ln\left(\frac{1}{\mu_\nu(X)}\int\limits_X^{\oplus} e^\varphi d\mu_\nu\right) = c + \nu(\varphi) = c \odot \nu(\varphi).$$

Finally, $\nu(c_X) = \ln\left(\frac{1}{\mu_\nu(X)}\int\limits_X^{\oplus} e^c d\mu_\nu\right) = \ln\left(\frac{1}{\mu_\nu(X)} \cdot e^c \cdot \mu_\nu(X)\right) = c$.

Uniqueness. Suppose, that for some ν there exist two different measures μ_1 and μ_2 satisfying (3.2). Then, according to Theorem 3.2, there exists a set $A \in \mathfrak{B}(X)$ such that $\mu_1(A) \neq \mu_2(A)$. Recall that one has

$$|\mu_1(A) - \mu_2(A)| =$$
$$= |\inf\{\nu(\varphi)\colon \varphi \in USC(X),\, \varphi \geq \chi_A\} - \inf\{\nu(\varphi)\colon \varphi \in USC(X),\, \varphi \geq \chi_A\}| = 0.$$

This contradiction shows that $\mu_1 = \mu_2$.

□

Clearly, Theorem 3.3 is a max-plus variant of the well-known Riesz Representation Theorem.

Considering $I_{\mathfrak{B}}(X)$ with the topology generated by the sets of form (2.1) and $I_{USC}(X)$ equipped with the pointwise convergence topology, by Theorems 3.2 and 3.3 we get

Corollary 3.1. *For every compact Hausdorff space X the topological spaces $I_{\mathfrak{B}}(X)$ and $I_{USC}(X)$ are homeomorphic.*

Now, let us consider a more general case. Let X be a Tychonoff space. Denote by βX the Stone-Čech compactification of X. We define the following set:

$$I_\tau(X) = \{\mu \in I(\beta X)\colon\ \mu(F) = 0 \text{ for every } F \in \mathfrak{B}(\beta X), F \subset \beta X \setminus X\}.$$

Elements of $I_\tau(X)$ are called *τ-smooth idempotent probability measures* (see [8]).

For each $\mu \in I_\tau(X)$ we define the set function $\tilde{\mu}\colon \mathfrak{B}(X) \to \mathbb{R}$ on the family $\mathfrak{B}(X)$ of all Borel subsets of X by the formula

$$\tilde{\mu}(A) = \inf\{\mu(B)\colon\ B \in \mathfrak{B}(\beta X),\, B \supset A\}, \qquad A \in \mathfrak{B}(X). \tag{3.3}$$

Lemma 3.1. [8] *$\tilde{\mu}$ is an idempotent probability measure on X.*

Now, we shall extend the assertions of Theorems 3.2 and 3.3 to a wider class of topological spaces.

Theorem 3.4. *Let X be a Tychonoff space. If $\tilde{\mu}$ is τ-smooth idempotent probability measure on $\mathfrak{B}(X)$, then integration*

$$\tilde{\varphi} \mapsto \ln\left(\frac{1}{\tilde{\mu}(X)}\int\limits_X^{\oplus} e^{\tilde{\varphi}(x)} d\tilde{\mu}\right)$$

is a normalized max-plus linear functional on the linear space $USC_b(X)$. Conversely, for any normalized max-plus linear functional $\tilde{\nu}\colon USC_b(X) \to \mathbb{R}_{\max}$ there exists a unique τ-smooth idempotent probability measure $\tilde{\mu}$ on $\mathfrak{B}(X)$ such that

$$\tilde{\nu}(\tilde{\varphi}) = \ln\left(\frac{1}{\tilde{\mu}(X)}\int_X^{\oplus} e^{\tilde{\varphi}(x)}d\tilde{\mu}\right), \qquad \tilde{\varphi} \in USC_b(X).$$

Proof. We define a max-plus linear functional ν on $USC(\beta X)$ by $\nu(\varphi) = \tilde{\nu}(\tilde{\varphi})$, where $\tilde{\varphi}$ denotes the restriction of $\varphi \in USC(\beta X)$ to X. It is obvious that ν satisfies the conditions of Definition 6. According to Theorem 3.3 there exists a unique idempotent probability measure μ on $\mathfrak{B}(\beta X)$ such that

$$\nu(\varphi) = \ln\left(\frac{1}{\mu(\beta X)}\int_{\beta X}^{\oplus} e^{\varphi(x)}d\mu\right), \qquad \varphi \in USC(\beta X).$$

Now, we prove the converse part of Theorem 3.4. By Theorem 3.2, we can write

$$\mu(B) = \inf\{\nu(\varphi)\colon \varphi \in USC(\beta X),\, \varphi \geq \chi_B\}, \qquad B \in \mathfrak{B}(\beta X).$$

Then, applying (3.3) for each $\mu \in I_\tau(X)$ we have

$$\tilde{\mu}(A) = \inf\{\mu(B)\colon\ B \in \mathfrak{B}(\beta X),\, B \supset A\}, \qquad A \in \mathfrak{B}(X).$$

According to Lemma 3.1 $\tilde{\mu}$ is an idempotent probability measure on X.

Uniqueness. Suppose, for some μ there exist two different measures $\tilde{\mu_1}$ and $\tilde{\mu_2}$. Then, there exists a set $A \in \mathfrak{B}(X)$ such that

$$\begin{aligned}0 \neq &|\tilde{\mu_1}(A) - \tilde{\mu_2}(A)| = \\ &= |\inf\{\mu(B)\colon B \in \mathfrak{B}(\beta X),\, B \supset A\} - \inf\{\mu(B)\colon B \in \mathfrak{B}(\beta X),\, B \supset A\}| = 0.\end{aligned}$$

This contradiction implies that $\tilde{\mu_1} = \tilde{\mu_2}$.

□

4 Max-plus version of the Hölder inequality

Now, we introduce a notion of an inner product in the space $USC_b(X)$.

Theorem 4.1. *The following equality defines an inner product on the linear space $USC_b(X)$:*

$$(\varphi,\, \psi) = \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{\varphi(x)\odot\psi(x)}d\mu\right) \tag{4.1}$$

for $\varphi,\, \psi \in USC_b(X)$.

Proof. By the definition of an inner product, we have to verify the following properties:

1) for every $\varphi,\, \psi \in USC_b(X)$ one has $(\varphi,\, \psi) = (\psi,\, \varphi)$ (it is obvious);

2) for every $\varphi,\, \psi,\, \chi \in USC_b(X)$ we have

$$(\varphi \oplus \psi,\, \chi) = \ln\left(\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{(\varphi(x)\oplus\psi(x))\odot\chi(x)}d\mu\right) =$$

$$= \ln\left(\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{(\varphi(x)\odot\chi(x))\oplus(\psi(x)\odot\chi(x))}d\mu\right) =$$

$$= \ln\left(\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{\varphi(x)\odot\chi(x)}d\mu\right) \oplus \ln\left(\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{\psi(x)\odot\chi(x)}d\mu\right) =$$

$$= (\varphi,\, \chi) \oplus (\psi,\, \chi);$$

3) for each $\varphi,\, \psi \in USC_b(X)$ and $c \in \mathbb{R}_{\max}$ we have

$$(c \odot \varphi,\, \psi) = \ln\left(\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{(c\odot\varphi(x))\odot\psi(x)}d\mu\right) = \ln\left(\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{c\odot(\varphi(x)\odot\psi(x))}d\mu\right) =$$

$$= \ln\left(\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{c}\cdot e^{\varphi(x)\odot\psi(x)}d\mu\right) = \ln\left(e^{c}\cdot\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{\varphi(x)\odot\psi(x)}d\mu\right) =$$

$$= c + \ln\left(\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{\varphi(x)\odot\psi(x)}d\mu\right) = c + (\varphi,\, \psi) = c \odot (\varphi,\, \psi);$$

4) for any $\varphi \in USC_b(X)$ we have $e^{\varphi(x)} \geq 0$. According to the monotonicity of the idempotent integration $\overset{\oplus}{\int_X} e^{\varphi(x)}d\mu \geq 0$. Then, $(\varphi,\, \varphi) \geq \mathbf{0}$.

□

Lemma 4.1. *Inner product (4.1) is continuous.*

Proof. Let $\varphi_n \to \varphi$, $\psi_n \to \psi$ and $\lambda_n \to \lambda$ for $\varphi_n,\, \psi_n \in USC_b(X)$ and $\lambda_n \in \mathbb{R}$. Then, $\varphi_n \oplus \psi_n \to \varphi \oplus \psi$ and $\lambda_n \odot \varphi_n \to \lambda \odot \varphi$. By Theorem 3.1 $(\varphi_n,\, \psi_n) \to (\varphi,\, \psi)$. □

For $\varphi \in USC_b(X)$, we define a $\oplus$-norm of φ by $||\varphi||_{\oplus 2} = \sqrt{(|\varphi|,\, |\varphi|)}$. For $p > 2$ we put $||\varphi||_{\oplus p} = \ln\left(\frac{1}{\mu(X)}\overset{\oplus}{\int_X} e^{p\cdot|\varphi|(x)}d\mu\right)^{\frac{1}{p}}$.

Theorem 4.2. *(**Hölder inequality**)* 1) *Let $p \in [1,\, +\infty]$ and $q \in [1,\, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have the following inequality*

$$|(\varphi,\, \psi)| \leq ||\varphi||_{\oplus p} \odot ||\psi||_{\oplus q}, \quad \textit{for every} \quad \varphi,\, \psi \in USC_b(X);$$

2) *If $\mu(X) = 1$, then $||\varphi||_{\oplus p} \leq ||\varphi||_{\oplus q}$, where $0 < p < q$.*

Proof. By the first part of Lemma 2.3 we have

$$\int_X^{\oplus} e^{\varphi(x)} \cdot e^{\psi(x)} d\mu \leq \left(\int_X^{\oplus} e^{p\cdot\varphi(x)} d\mu \right)^{\frac{1}{p}} \cdot \left(\int_X^{\oplus} e^{q\cdot\psi(x)} d\mu \right)^{\frac{1}{q}} \Rightarrow$$

$$\frac{1}{\mu(X)} \int_X^{\oplus} e^{\varphi(x)} \cdot e^{\psi(x)} d\mu \leq \left(\frac{1}{\mu(X)} \int_X^{\oplus} e^{p\cdot\varphi(x)} d\mu \right)^{\frac{1}{p}} \cdot \left(\frac{1}{\mu(X)} \int_X^{\oplus} e^{q\cdot\psi(x)} d\mu \right)^{\frac{1}{q}} \Rightarrow$$

$$\left| \ln \left(\frac{1}{\mu(X)} \int_X^{\oplus} e^{\varphi(x)} \cdot e^{\psi(x)} d\mu \right) \right| \leq \ln \left(\frac{1}{\mu(X)} \int_X^{\oplus} e^{p\cdot|\varphi|(x)} d\mu \right)^{\frac{1}{p}} \odot \ln \left(\frac{1}{\mu(X)} \int_X^{\oplus} e^{q\cdot|\psi|(x)} d\mu \right)^{\frac{1}{q}}.$$

Hence, $|(\varphi, \psi)| \leq ||\varphi||_{\oplus p} \odot ||\psi||_{\oplus q}$.

According to the second part of Lemma 2.3 and from a property of logarithmic function one has $||\varphi||_{\oplus p} \leq ||\varphi||_{\oplus q}$. □

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References

[1] S. Albeverio, Sh.A. Ayupov, A.A. Zaitov, *On certain properties of the spaces of order-preserving functionals.* Topology and its Applications 155 (2008), no. 16, 1792-1799. https://doi.org/10.1016/j.topol.2008.05.019

[2] Sh.A. Ayupov, A.A. Zaitov, *Functor of weakly additive τ-smooth functionals and mappings.* Ukrainian Mathematical Journal 61 (2009), 1380-1386. https://doi.org/10.1007/s11253-010-0283-0

[3] V.I. Bogachev, *Measure theory, Vo I.* Springer, 2007.

[4] V.I. Bogachev, *Measure theory, Vo II.* Springer, 2007.

[5] R. Engelking, *General topology.* – Revised and completed ed. – Berlin: Heldermann, 1989.

[6] V.V. Fedorchuk, Yu.V. Sadovnichii, *On some topological and categorical properties of measures with alternating signs.* Fundam. Prikl. Mat. 5 (1999), no. 2, 597-618.

[7] J. Gunawardena, *Idempotency.* Publ. of the Newton Institute, 1998, Vol. 11, Cambridge University Press, Cambridge, (Online publication date: May 2010).

[8] L.D.R. Kocinac, A.A. Zaitov, M.R. Eshimbetov, *On the Čech-completeness of the space of τ-smooth idempotent measures.* Axioms, 13 (2024), no. 8, 569-582.

[9] B. Makarov, A. Podkorytov, *Real analysis: measures, integrals and applications.* Springer-Verlag, London 2013.

[10] A. Puhalskii, *Large deviations and idempotent probability.* Chapman & Hall/CRC, 2001.

[11] T. Radul, *Idempotent measures: absolute retracts and soft maps.* Topological Methods in Nonlinear Analysis [online] 56 (2020), no. 1, 161-172.

[12] A. Sheldon, *Measure, integration and real analysis.* Springer Open, 2010.

[13] V.S. Varadarajan, *Measures on topological spaces.* Mat.Sb. (N.S.) 55 (1961), no. 1, 35-100.

[14] A.A. Zaitov, *On a metric on the space of idempotent probability measures.* Applied General Topology 16 (2020), no. 1, 35-51.

[15] M. Zarichnyi, *Spaces and mappings of idempotent measures.* Izvestiya: Math. 74(2010), no. 3, 45-64.

[16] M. Zarichnyi, L. Bazylevych, D. Repovs, *Spaces of idempotent measures of compact metric spaces.* Topology and its Applications 157 (2010), no. 1, 136-144.

[17] A.A. Zaitov, Kh.F. Kholturayev, *On some properties of infinite iterations of the functor of idempotent probobility measures.* Lobachevskii Journal of Mathematics 43 (2022), no. 8, 2341-2348.

[18] A.A. Zaitov, M.R. Eshimbetov, *On a max-plus variant of the Riesz representation theorem.* Bulletin of the Institute of Mathematics 6 (2023), no. 5, 73-79.

[19] A.A. Zaitov, A.Ya. Ishmetov, *Homotopy properties of the space $I_f(X)$ of idempotent probability measures.* Math. Notes 106 (2019), no. 3-4, 562-571.

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