EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 2, Number 3 (2011), 98 – 124

METHODS OF TRIGONOMETRIC APPROXIMATION AND GENERALIZED SMOOTHNESS. I

K. Runovski, H.-J. Schmeisser

Communicated by E.D. Nursultanov

Key words: trigonometric approximation, Fourier means, families of linear polynomial operators, L_p - convergence ($0 < p \leq +\infty$), classical kernels and summability methods, stochastic approximation.

AMS Mathematics Subject Classification: 42A10, 42A15, 42A24, 42B08, 42B35.

Abstract. We give a unified approach to trigonometric approximation and study its interrelation with smoothness properties of functions. In the first part our concern lies on convergence of the Fourier means, interpolation means and families of linear trigonometric polynomial operators in the scale of the L_p-spaces with $0 < p \leq +\infty$. We establish a general convergence theorem which allows to determine the ranges of convergence for approximation methods generated by classical kernels.

The second part will deal with the equivalence of the approximation errors for families of linear polynomial operators generated by classical kernels in terms of K-functionals generated by homogeneous functions and general moduli of smoothness. It will also be shown that the results of the classical approximation theory on the Fourier means and interpolation means in the case $1 \leq p \leq +\infty$, classical differential operators and moduli of smoothness are direct consequences of our general approach.

1 Introduction

Problems related to various constructive methods of trigonometric approximation, their convergence in the scale of L_p-spaces of 2π-periodic functions of one or several variables with $1 \leq p \leq +\infty$ and the description of the decay of their approximation errors in terms of smoothness quantities of a given function form an essential part of both classical and modern approximation theory. For detailed investigations, precise definitions, key results we refer, for instance, to [6], [8], [21], [42], [43], [44], [39], [50], [51]. It is well known (see, e. g., [23]) that in the case $0 < p < 1$ non-trivial linear polynomial operators do not exist. For this reason constructive methods of trigonometric approximation were not found in this case and some important problems as, for instance, the direct Jackson type estimate and the inverse Bernstein type estimate in L_p with $0 < p < 1$ were first proved by using some non-constructive special methods ([17], [45]-[47]). This gap has been closed only by introducing the method of approximation by families of linear polynomial operators in the papers [26] and [27].

It is the aim of this survey to discuss the unified approach to trigonometric approximation and smoothness in the scale of the L_p-spaces for all $0 < p \le +\infty$ described in detail in our joint papers [24], [25], [28]-[38]. Within this approach we show, in particular, that many both already solved and new problems of approximation theory can be reduced to the study of three universal objects - families of linear polynomial operators, generalized K-functionals and moduli of smoothness. We are able to determine the sharp ranges of convergence of families in terms of the Fourier transforms of their generators. Moreover, it turns out that their approximation errors are equivalent to the associated K-functionals and smoothness moduli if their generators satisfy some natural conditions. As can be seen in the above mentioned papers our approach is mainly based on the following crucial ideas:

- interpretation of a family of linear polynomial operators as an operator mapping into a space of functions of doubled number of variables;
- representation of the objects listed above in (the Fourier) multiplier form and their classification in dependence of multiplier properties;
- generation of generalized smoothness by homogeneous multipliers.

Our paper consists of two parts. The first part presented here is mainly concerned with the convergence of approximation processes and the universality of our approach. Recall that the classical methods of trigonometric approximation - the Fourier means and interpolation means - are defined as follows. Let us consider the matrix of complex numbers

$$\Lambda = \{a_{n,k}\}_{(n,k)\in\mathbb{N}_0\times\mathbb{Z}^d}\,, \tag{1.1}$$

where

$$\begin{cases} a_{n,\,0} = 1 & \text{if}\,,\ n\in\mathbb{N}_0 \\ a_{n,\,-k} = \overline{a_{n,\,k}} \in \mathbb{C}\,, & \text{if}\ |k| \le r(\Lambda)n,\ \ n\in\mathbb{N}_0 \\ a_{n,k} = 0 & \text{if}\,,\ |k| > r(\Lambda)n,\ \ n\in\mathbb{N}_0 \end{cases}$$

and where $r(\Lambda)$ is a real positive number. We put

$$W_n(\Lambda)(h) = \sum_{|\,k\,|\,\le\, r(\Lambda)\,n} a_{n,\,k}\, e^{ikh}\,,\ \ n\in\mathbb{N}_0\,. \tag{1.2}$$

As special cases of (1.2) we mention, for example, the classical kernels of Dirichlet, Fejér, de la Vallée-Poussin, Rogosinski, Jackson, Cesaro, Riesz, Bochner-Riesz, Zygmund, and Favard. If $f \in L_p(\mathbb{T}^d)$, $1 \le p \le +\infty$, ($\mathbb{T}^d$ stands for the d-dimensional torus; the case $p = +\infty$ corresponds to the space $C(\mathbb{T}^d)$ of continuous functions), then the Fourier means are defined by the convolution integrals

$$\mathcal{F}_n^{(\Lambda)}(f;x) = (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(h)\, W_n(\Lambda)(x-h)\, dh\,,\ n\in\mathbb{N}_0\,, x\in\mathbb{T}^d\,. \tag{1.3}$$

If $f \in C(\mathbb{T}^d)$, then the interpolation means (or sampling operators) are defined as

$$\mathcal{I}_n^{(\Lambda)}(f;x) = (2N+1)^{-d}\cdot \sum_{\nu=0}^{2N} f\left(t_N^\nu\right)\cdot W_n(\Lambda)\left(x - t_N^\nu\right)\,,\ n\in\mathbb{N}_0\,, x\in\mathbb{T}^d\,, \tag{1.4}$$

where

$$N = [rn]\,,\ r \geq r(\Lambda)\,;\ t_N^\nu = \frac{2\pi\nu}{2N+1}\,,\quad \nu \in \mathbb{Z}^d\,;\quad \sum_{\nu=0}^{2N} \equiv \sum_{\nu_1=0}^{2N} \dots \sum_{\nu_d=0}^{2N}\,. \tag{1.5}$$

From the historical point of view the convergence of these methods in $L_p(\mathbb{T}^d)$ and $C(\mathbb{T}^d)$, respectively, in the sense that

$$\lim_{n\to+\infty} \|\, f - L_n^{(\Lambda)}(f)\,\|_p \;=\; 0 \tag{1.6}$$

for all $f \in L_p(\mathbb{T}^d)\,(C(\mathbb{T}^d))$, where L is $\mathcal{F}$ or $\mathcal{I}$, has been established first for special kernels which are nowadays called classical ones (see, e. g., [6], [8], [21], [42]-[44]). In the general case the convergence of $\mathcal{F}_n^{(\Lambda)}$ in L_1, C or in L_p for all $1 \leq p \leq +\infty$ and the convergence of $\mathcal{I}_n^{(\Lambda)}$ in C is equivalent to the boundedness of the sequence of the L_1-norms of the kernels in view of the Banach-Steinhaus principle (see, e. g., [8], [21]). Moreover, applying the approach elaborated in [5] we could prove that the approximation errors of these methods are equivalent to each other in C (see e.g. [31]). If

$$\Lambda \equiv \Lambda(\varphi) = \left\{\, a_{n,k} : \quad a_{0,0} = 1;\ a_{n,k} = \varphi\left(\frac{k}{\sigma(n)}\right)\,,\ k \in \mathbb{Z}^d\,,\ n \in \mathbb{N}\,\right\}, \tag{1.7}$$

where $(\sigma(n))_n$ is a certain strictly increasing sequence of positive real numbers satisfying $\sigma(n) \asymp n$ and φ is a complex-valued continuous function on $\mathbb{R}^d$ with compact support satisfying $\varphi(0) = 1$ and $\varphi(-\xi) = \overline{\varphi(\xi)}$ for each $\xi \in \mathbb{R}^d$, then the condition of the uniform boundedness of the L_1-norms of the kernels in the convergence criterion is equivalent to the condition that the Fourier transform $\widehat{\varphi}$ of the generator φ belongs to the space $L_1(\mathbb{R}^d)$ (see, e. g., [31] or [15]).

In contrast to the classical methods of trigonometric approximation the method of approximation by families of linear (trigonometric) polynomial operators which are given by ($\lambda \in \mathbb{R}^d$ is a parameter and $x \in \mathbb{T}^d$)

$$\mathcal{L}_{n;\lambda}^{(\Lambda)}(f;x) = (2N+1)^{-d} \cdot \sum_{\nu=0}^{2N} f\left(t_N^\nu + \lambda\right) \cdot W_n(\Lambda)\left(x - t_N^\nu - \lambda\right)\,,\ n \in \mathbb{N}_0\,, \tag{1.8}$$

is comparatively new (see [26], [27]). Its systematical study (also in the non-periodic case of approximation by band-limited functions) has been developed in [2]-[4], [14], [24], [25], [28], [32], [33], [34] and further papers. In particular, it has been shown that this method is universal in the sense that it is relevant for both $p \geq 1$ and $0 < p < 1$, where the range of admissible parameters p depends on the properties of Λ. Moreover, the averaged approximation error with respect to the parameter λ, that is, the quantity

$$\|\, f - \mathcal{L}_{n;\lambda}^{(\Lambda)}(f)\,\|_{\overline{p}} \;=\; (2\pi)^{-d/p}\left(\int_{\mathbb{T}^d} \|\, f(\cdot) - \mathcal{L}_{n;\lambda}^{(\Lambda)}(f;\cdot)\,\|_p^p\, d\lambda\right)^{1/p} \tag{1.9}$$

(in the case $p = +\infty$ the average is replaced by the maximum over all λ) is equivalent to the approximation error of the corresponding Fourier means and the corresponding interpolation means in the cases $1 \le p \le +\infty$ and $p = +\infty$, respectively. Let us also mention that under the assumption $\widehat{\varphi} \in L_1(\mathbb{R}^d)$ the family $\{\mathcal{L}_{n;\lambda}^{(\varphi)}\}$ generated by the matrix $\Lambda(\varphi)$ of type (1.7) converges in L_p, which means that the averaged approximation error defined in (1.9) converges to 0, if and only if $\widehat{\varphi} \in L_p(\mathbb{R}^d)$ ([24]). This result enables us to find the sharp convergence ranges for families generated by classical kernels as, for instance, the kernels of Fejér, de la Vallée-Poussin, Bochner-Riesz, Riesz and others ([24], [32], [34]). The problem of convergence of families of linear polynomial operators generated by general matrices of multipliers and their special cases corresponding to the classical kernels of Cesaro, Jackson and Fejér-Korovkin has been considered in [33]. For applications of the method, in particular, for the algorithm of stochastic approximation we refer to [28].

Let us give an outlook to the topics considered in Part II which will be published in one of forthcoming volumes. First we shall introduce a concept of generalized smoothness (ψ - smoothness) associated with a symbol generated by a homogeneous function. We define corresponding function spaces and investigate their interrelations with periodic Sobolev (Triebel-Lizorkin) and Besov spaces. This will be important for the study of generalized K - functionals. We briefly sketch the setting. Let ψ be a complex-valued function which is continuous on $\mathbb{R}^d$, infinitely differentiable on $\mathbb{R}^d \setminus \{0\}$, homogeneous of order $s > 0$ and satisfies $\psi(\xi) \neq 0$ for $\xi \neq 0$ and $\psi(-\xi) = \overline{\psi(\xi)}$ for each $\xi \in \mathbb{R}^d$. The ψ-derivative is formally defined as

$$\mathcal{D}(\psi)g(x) = \sum_{k\in\mathbb{Z}^d} \psi(k)g^\wedge(k)e^{ikx} \,. \tag{1.10}$$

If $1 \le p \le \infty$ then we consider the space

$$X_p(\psi) = \left\{ \; g \in L_p(\mathbb{T}^d) \; : \; \mathcal{D}(\psi)g \in L_p(\mathbb{T}^d) \; \right\} \tag{1.11}$$

equipped with the norm

$$\|g\|_{X_p(\psi)} = \|g\|_p + \|\mathcal{D}(\psi)g\|_p. \tag{1.12}$$

(Recall that $L_\infty(\mathbb{T}^d) = C(\mathbb{T}^d)$ by our convention.) Here we follow our paper [35]. On the one hand, homogeneity of ψ seems to be a rather general assumption. Various differential operators as, for instance, the classical derivatives, Weyl and Riesz derivatives, mixed derivatives, the Laplace-operator and its (fractional) powers are generated by homogeneous multipliers (symbols). On the other hand, taking into account that the Fourier transform (in the sense of distributions) of a homogeneous function of order s is also a homogeneous function of order $-(d+s)$ (see e.g. [19], Theorem 7.1.6) one can derive substantial statements concerning the corresponding operators and related function spaces. It turns out that the spaces $X_p(\psi)$ coincide with the periodic fractional Sobolev spaces $H_p^s(\mathbb{T}^d)$ for all ψ as above if $1 < p < \infty$. However, we obtain new spaces in the limiting cases $p = 1$ and $p = \infty$ which will be of peculiar interest later on.

The problem of describing the quality of approximation by the Fourier means and interpolation means in terms of smoothness quantities of a given function has a long history. In the classical theory (one-dimensional case) one usually deals with moduli of smoothness of order $k \in \mathbb{N}$ given by $(f \in L_p(\mathbb{T})\,,\ \delta \ge 0)$

$$\omega_k(f,\delta)_p \;=\; \sup_{0\le h\le\delta} \left\| \sum_{\nu=0}^{k} \frac{(-1)^{k-\nu}k(k-1)\dots(k-\nu+1)}{\nu\,!}\, f(x+\nu h) \right\|_p \tag{1.13}$$

and K-functionals due to J. Peetre which are defined as

$$K_k(f,\delta)_p \;=\; \inf_{g\in W_p^k} \left\{ \,\|\, f-g \,\|_p \,+\, \delta^k \,\|\, g^{(k)} \,\|_p \right\}\,, \tag{1.14}$$

where the symbol $(\cdot)^{(k)}$ stands for the usual derivative of order k and where $W_p^k = W_p^k(\mathbb{T})$ is the classical Sobolev space. It is well known (see, e. g., [8], Theorem 2.4, p. 177) that these quantities are equivalent in L_p with $1 \le p \le +\infty$. It holds

$$c_1\,\omega_k(f,\delta)_p \;\le\; K_k(f,\delta)_p \;\le\; c_2\,\omega_k(f,\delta)_p\,, \tag{1.15}$$

where the positive constants c_1 and c_2 do not depend on f and δ. In the case $0 < p < 1$ quantity (1.14) is equal to 0 (see [20]). More precisely we have

$$\inf_{g\in\mathcal{T}} \left\{ \,\|\, f-g \,\|_p \,+\, \delta^k \,\|\, g^{(k)} \,\|_p \right\} = 0$$

for all $f \in L_p(\mathbb{T})$. For this reason a functional $\mathcal{K}_k(f,\delta)_p$ originally called "realization of K-functional" was introduced, where the infimum over all functions $g \in W_p^k$ in (1.14) was replaced by the infimum over the space $\mathcal{T}_{1/\delta}$ of all real-valued trigonometric polynomials of order at most $1/\delta$ ([12], [20]). It was proved in [20] that such a modification allows to extend equivalence (1.15) to the case of all admissible parameters $0 < p \le +\infty$. In what follows we shall use the notation "polynomial K-functional" in place of "realization of K-functional". In the literature one can also find K-functionals related to Riesz or Weyl derivatives ([10], [11], [48]) and a modulus of smoothness related to the Weyl derivative which is obtained by replacing the natural parameter k in (1.13) by an arbitrary positive real number $\alpha > 0$ and by replacing the finite sum by the sum over $\nu \in \mathbb{N}_0$ (see [48]). Further modifications are, for example, the K-functional related to the Laplace-operator ([7], [11], [13], [14], [32]), the mixed moduli of smoothness [22] and the discrete modulus of smoothness related to the Laplace-operator ([9], [14]) in the multivariate case. These smoothness quantities have been used to characterize the approximation error of some concrete Fourier means. In particular, it could be shown that the approximation error of the Fejér means is equivalent to the K-functional related to the Riesz derivative ([11]), that the approximation error of the Rogosinski means is equivalent to the modulus of smoothness of second order ([49]). Moreover, the approximation error of the Bochner-Riesz means with index α with respect to the norm in $L_p(\mathbb{T}^d)$ is equivalent to the K-functional related to the Laplace-operator provided that $\alpha > (d-1)/2$ (Bochner's critical index) ([10], [32]).

In contrast to the just mentioned results concerning special cases, concrete methods and smoothness quantities restricted to L_p with $p \ge 1$, our approach tackles the problem of quality of approximation in a general form. We deal with families of linear polynomial operators generated by arbitrary kernels as universal approximation methods in the scale of the spaces L_p for all $0 < p \le +\infty$ and we intend to raise the concepts of smoothness moduli and (polynomial) K-functionals at the same level. Here we shall follow our recent papers ([25], [35], [36], [37], [38]), The K-functional and the polynomial K-functional generated by a homogeneous function ψ of order s are given by $(f \in L_p\,,\ \delta \ge 0)$

$$K_\psi(f,\delta)_p \;=\; \inf_{g \in X_p(\psi)} \{\, \| f - g \|_p \,+\, \delta^s \,\| \mathcal{D}(\psi) g \,\|_p \,\}\;, \tag{1.16}$$

$$K_\psi^{(\mathcal{P})}(f,\delta)_p \;=\; \inf_{T \in \mathcal{T}_{1/\delta}} \{\, \| f - T \|_p \,+\, \delta^s \,\| \mathcal{D}(\psi) T \,\|_p \,\}\;, \tag{1.17}$$

respectively, where the space $X_p(\psi)$ had the meaning of (1.11) and $\mathcal{D}(\psi)g$ stands for ψ-derivative of g defined in (1.10). For a systematic study we refer [35], [36], and [37]. In particular, it was shown that functionals (1.16) and (1.17) are equivalent in the case $1 \le p \le +\infty$.

As was shown in [25], the approximation error of the family of linear polynomial operators generated by the function $\varphi \in \mathcal{K}$ is equivalent to the polynomial K-functional generated by $\psi \in \mathcal{H}_s$ in L_p, if p belongs to the range of convergence of the family, if the generator φ is sufficiently smooth and if the functions $1 - \varphi(\cdot)$ and $\psi(\cdot)$ are close to each other in a certain sense in a neighbourhood of 0.

Let $f \in L_p\,,\ n \in \mathbb{N}_0$. The main result in [25] reads as

$$c_1\, K_\psi^{(\mathcal{P})}(f,(n+1)^{-1})_p \;\le\; \| f - \mathcal{L}_{n;\,\lambda}^{(\varphi)}(f) \,\|_{\overline{p}} \;\le\; c_2\, K_\psi^{(\mathcal{P})}(f,(n+1)^{-1})_p\,, \tag{1.18}$$

where the positive constants c_1 and c_2 do not depend on f and n. This general result enables us, in particular, to characterize the approximation by families generated by classical kernels. As a by-product we obtain some new results even in the classical case of the Fourier means and $1 \le p \le +\infty$. For instance, in view of the above mentioned equivalences of families and the corresponding Fourier means as well as of K-functionals and polynomial K-functionals in the case $1 \le p \le +\infty$ we conclude that the approximation error of the Riesz means (generated by $\varphi(\xi) = (1 - |\,\xi\,|^\beta)_+^\alpha$) in the multivariate case is equivalent to the K-functional related to the (fractional) power of the Laplace operator $(-\Delta)^{\beta/2}$ $(\psi(\xi) = |\,\xi\,|^\beta)$ ([34]).

The extension of the concept of a smoothness modulus (see [38] for a model case) is based on the observation that an operator generated by a periodic multiplier (symbol) admits a representation as an (infinite) linear combination of shift operators. More precisely, in the one-dimensional case one has

$$\sum_{\nu \in \mathbb{Z}} \theta(\nu h)\, f^\wedge(\nu)\, e^{i\nu x} \;=\; \sum_{\nu \in \mathbb{Z}} \theta^\wedge(\nu)\, f(x + \nu h) \tag{1.19}$$

for an appropriate periodic function θ. Assume that θ is complex-valued, 2π-periodic and continuous satisfying $\theta(0) = 0$, $\theta^\wedge(0) = -1$ and $\theta(-\xi) = \overline{\theta(\xi)}$ for each $\xi \in \mathbb{R}$. The modulus of smoothness generated by θ (θ-modulus) in L_p is now defined as

$$\omega_\theta(f,\delta)_p \;=\; \sup_{0 \le h \le \delta} \Big\| \sum_{\nu \in \mathbb{Z}} \theta^\wedge(\nu)\, f(x + \nu h) \Big\|_p\,,\ \ \delta \ge 0\,. \tag{1.20}$$

Clearly, if the sequence of the Fourier coefficints $\{\theta^\wedge(k)\}_{k\in\mathbb{Z}^d}$ belongs to $l_{\widetilde{p}}$, where $\widetilde{p} = \min(1,p)$, then the θ-modulus is well defined in L_p in the sense that the series on the right-hand side of (1.20) converges in L_p and

$$\omega_\theta(f,\delta)_p \le \left(\| \{\theta^\wedge(k)\}_{k\in\mathbb{Z}} \|_{l_{\widetilde{p}}} \right) \| f \|_p \,, \;\; f \in L_p \,, \;\; \delta \ge 0 \,. \tag{1.21}$$

The θ-modulus (1.20) can be studied by the scheme elaborated in [38], where the case $\theta(\xi) = -2/\pi|\,\xi\,|$ for $\xi \in [-\pi,\pi]$ corresponding to the Riesz derivative has been considered and the equivalence to the approximation error for the Fejér means has been proved. We refer to Part II for details. Let us mention also that some work has to be done in future research. In particular, the task is to prove the equivalence of the θ-modulus and the polynomial K-functional generated by an appropriate ψ in the multivariate case for all admissible values of the parameters p. In view of (1.18) this result would enable us to characterize the approximation error in the case of general methods also in terms of moduli of smoothness.

2 Preliminaries

2.1 Notations

The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}^d$, $\mathbb{R}^d$ denote the sets of natural, non-negative integer, integer, real numbers and d-dimensional vectors with integer and real components, respectively. The symbol $\mathbb{T}^d$ is reserved for the d-dimensional torus $[0,2\pi)^d$. We will also use the notation

$$xy \;=\; x_1y_1 + \ldots + x_dy_d \,, \;\; |\,x\,|_q = \begin{cases} (x_1^q + \ldots + x_d^q)^{1/q} & , \; 0 < q < +\infty \\ \max(|\,x_1\,|,\ldots,|\,x_d\,|) \,, & q = +\infty \,, \end{cases} ,$$

for the scalar product and the l_q-norm of vectors. For brevity we put $|\,x\,| \equiv |\,x\,|_2$. Furthermore,

$$B_r \;=\; \{\, x \in \mathbb{R}^d : |\,x\,| < r \,\} \,, \;\; \overline{B}_r \;=\; \{\, x \in \mathbb{R}^d : |\,x\,| \le r \,\}$$

stand for open and closed balls in $\mathbb{R}^d$, respectively.

Unimportant positive constants, denoted by c (with subscripts and superscripts) may have different values in different formulas (but not in the same formula).

By $A \lesssim B$ we denote the relation $A(j) \le cB(j)$, where c is a positive constant independent of the parameter j belonging to a certain index set J. The symbol $\asymp$ indicates equivalence. It means that $A \lesssim B$ and $B \lesssim A$ simultaneously.

2.2 L_p-spaces and spaces of trigonometric polynomials

Let $L_p \equiv L_p(\mathbb{T}^d)$, where $0 < p < +\infty$, be the space of measurable real-valued functions f which are 2π-periodic with respect to each variable and satisfy

$$\|\, f \,\|_p \;=\; \left(\int\limits_{\mathbb{T}^d} |f(x)|^p \; dx \right)^{1/p} \;<\; +\infty \,.$$

As usual in the approximation theory we understand $L_\infty(\mathbb{T}^d)$ as the space $C \equiv C(\mathbb{T}^d)$ of real-valued 2π-periodic continuous functions equipped with the Chebyshev norm

$$\| f \|_C \;=\; \max_{x \in \mathbb{T}^d} |f(x)| \,.$$

For L_p-spaces of non-periodic functions defined on a measurable set $\Omega \subseteq \mathbb{R}^d$ we shall use the notation $L_p(\Omega)$.

In what follows we often deal with functions in $L_p(\mathbb{T}^{2d})$ depending on the "main" variable $x \in \mathbb{R}^d$ and on the parameter $\lambda \in \mathbb{R}^d$. The L_p-norm with respect to x is denoted by $\| \cdot \|_p$ or $\| \cdot \|_{p;\,x}$. We use the symbol $\| \cdot \|_{p;\,\lambda}$ for the L_p-norm with respect to the parameter λ. For brevity the space $L_p(\mathbb{T}^{2d})$ equipped with the norm

$$\| \cdot \|_{\overline{p}} \;=\; (2\pi)^{-d/p} \, \| \quad \| \cdot \|_{p;\,x} \, \|_{p;\,\lambda}, \tag{2.1}$$

is denoted by the symbol $L_{\overline{p}}$. Clearly, L_p can be considered as a subspace of $L_{\overline{p}}$. Obviously, we have

$$\| f \|_{\overline{p}} \;=\; \| f \|_p \,, \; f \in L_p \,.$$

The functional $\| \cdot \|_{\overline{p}}$ is a norm if and only if $1 \le p \le +\infty$. For $0 < p < 1$ it is a quasi-norm and the "triangle" inequality is valid for its pth power. If we put $\widetilde{p} = \min(1, p)$ then the inequality

$$\| f + g \|_{\overline{p}}^{\widetilde{p}} \;\le\; \| f \|_{\overline{p}}^{\widetilde{p}} + \| g \|_{\overline{p}}^{\widetilde{p}}, \;\; f\,, \;\; g \in L_{\overline{p}}\,, \tag{2.2}$$

is valid for all $0 < p \le +\infty$. Such a form of the "triangle" inequality is convenient, because both cases can be treated uniformly. Moreover, for the sake of simplicity we shall use the notation "norm" also in the case $0 < p < 1$.

Let σ be a real non-negative number. Let us denote by $\mathcal{T}_\sigma$ the space of all real-valued trigonometric polynomials of (spherical) order of at most σ. It means

$$\mathcal{T}_\sigma \;=\; \Big\{ T(x) = \sum_{k \in \mathbb{Z}^d :\, |\, k \,| \le \sigma} c_k \, e^{ikx} \,:\; c_{-k} = \overline{c_k} \Big\}\,, \tag{2.3}$$

where $\bar{c}$ is a complex conjugate to c. Further, $\mathcal{T}$ stands for the space of all real-valued trigonometric polynomials of arbitrary order.

The space $\mathcal{T}_\sigma$ equipped with the L_p-norm, where $0 < p \le +\infty$, is denoted by $\mathcal{T}_{\sigma,\,p}$. Moreover, $\mathcal{T}_{\sigma,\,\overline{p}}$ stands for the subspace of $L_{\overline{p}}$ which consists of all functions $g(x, \lambda)$ such that $g(x, \lambda)$ belongs to $\mathcal{T}_\sigma$ for almost all λ if it is considered as a function of x. Clearly, $\mathcal{T}_{\sigma,\,p}$ can be understood as a subspace of $\mathcal{T}_{\sigma,\,\overline{p}}$ with coincidence of associated norms. Thus, in our notation the line over the index p indicates that we are dealing with functions of $2d$ variables.

Finally, we define the best approximation of f in L_p by trigonometric polynomials of order at most σ as

$$E_\sigma(f)_p \;=\; \inf_{t \in \mathcal{T}_\sigma} \| f - T \|_p \,, \;\; \sigma \ge 0\,. \tag{2.4}$$

Here, $0 < p \le +\infty$.

2.3 Fourier coefficients and Fourier transform

The Fourier coefficients of $f \in L_1$ are defined by

$$f^{\wedge}(k) \;=\; (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(x)\, e^{-ikx}\, dx\,, \;\; k \in \mathbb{Z}^d\,. \tag{2.5}$$

The Fourier transform and its inverse are given by

$$\widehat{f}(\xi) = \int\limits_{\mathbb{R}^d} f(x)\, e^{-ix\xi}\, dx\;, \quad f^{\vee}(x) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} f(\xi)\, e^{ix\xi}\, d\xi \tag{2.6}$$

for $f \in L_1(\mathbb{R}^d)$. For the sake of clarity we shall sometimes use also the notations $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ in place of $\widehat{f}$ and $f^{\vee}$, respectively.

Suppose g is defined everywhere on $\mathbb{R}^d$ and belongs to $L_1(\mathbb{R}^d)$. The equation

$$\sum_{k\,\in\,\mathbb{Z}^d} g(k) \;=\; \sum_{\nu\,\in\,\mathbb{Z}^d} \widehat{g}(2\pi\nu) \tag{2.7}$$

is called the Poisson summation formula. By elementary properties of the Fourier transform it can be rewritten as

$$\sum_{k\,\in\,\mathbb{Z}^d} g(k)\, e^{ikx} \;=\; \sum_{\nu\,\in\,\mathbb{Z}^d} \widehat{g}(x+2\pi\nu)\,, \;\; x \,\in\, \mathbb{R}^d\,. \tag{2.8}$$

As well known, formulas (2.7), (2.8) are valid pointwise if g satisfies some additional conditions. In particular, if the function g is continuous and if

$$|\, g(\xi)\,| \;\le\; c_1(\,1+|\,\xi\,|\,)^{-d-\delta}\,, \;|\,\widehat{g}(x)\,| \;\le\; c_2(\,1+|\,x\,|\,)^{-d-\delta}\,, \;\xi,\, x \,\in\, \mathbb{R}^d\,, \tag{2.9}$$

for some $\delta > 0$, then (2.8) holds for each $x \in \mathbb{R}^d$ ([42], p. 252).

It was shown in [2] (Lemma 2.2, p. 681) that formula (2.8) holds almost everywhere on $\mathbb{R}^d$ if g is continuous, compactly supported and if its Fourier transform belongs to $L_p(\mathbb{R}^d)$ for some $0 < p \le 1$. Moreover, in this case

$$\Big\|\sum_{k\,\in\,\mathbb{Z}^d} g(k)\, e^{ikx}\Big\|_p \;\le\; \|\,\widehat{g}\,\|_{L_p(\mathbb{R}^d)}\;. \tag{2.10}$$

2.4 Polynomial operators in (quasi-)Banach spaces

As we shall see in Section 2 and already sketched in the Introduction (cf. formulae (1.3), (1.4) and (1.8)) all approximation methods we deal with can be understood as linear operators of type

$$\mathcal{L}_\sigma \,:\, L_p \;\longrightarrow\; \mathcal{T}_{r\sigma,\overline{p}} \,\subset\, L_{\overline{p}}\,, \;\sigma \,\in\, \Omega\,, \tag{2.11}$$

where $0 < p \le +\infty$, $r > 0$ and the set Ω is $\{\sigma \ge 0\}$ or $\{\sigma_n,\, n \in \mathbb{N}_0 : \lim_{n\to+\infty} \sigma_n = +\infty\}$. If $p = +\infty$ then the operators are defined on C. Since L_p is a subspace of $L_{\overline{p}}$ with coincidence of norms, the well-known linear polynomial operators of classical approximation theory are special cases of (2.11). It was already mentioned in the Introduction that such operators do not make sense if $0 < p < 1$. For this reason $\mathcal{T}_{r\sigma,\overline{p}}$ can not be replaced by $\mathcal{T}_{r\sigma}$ if $0 < p < 1$.

Our unified and universal approach to treat the norm convergence of constructive methods of trigonometric approximation is based on the application of standard concepts and principles of classical functional analysis to linear and bounded operators mapping from L_p into $L_{\overline{p}}$. Recall that a linear operator $\mathcal{L}_\sigma : L_p \to L_{\overline{p}}$ is bounded if its norm

$$\| \mathcal{L}_\sigma \|_{(p)} \;=\; \sup_{\| f \|_p \le 1} \| \mathcal{L}_\sigma(f) \|_{\overline{p}} \tag{2.12}$$

is finite. A family $(\mathcal{L}_\sigma)_\sigma$ is called bounded in L_p if their norms are uniformly bounded, i. e. if

$$\sup_{\sigma \in \Omega} \| \mathcal{L}_\sigma \|_{(p)} \;<\; +\infty \;. \tag{2.13}$$

A family $(\mathcal{L}_\sigma)_\sigma$ is said to be convergent in L_p if

$$\lim_{\sigma \to +\infty} \| f - \mathcal{L}_\sigma(f) \|_{\overline{p}} \;=\; 0 \tag{2.14}$$

for each $f \in L_p$. Obviously, for operators mapping into L_p this concept coincides with L_p-convergence in the usual sense.

It turns out that some properties of approximation methods do not depend on their specific structure. In particular, as direct consequences of classical theorems of functional analysis (as, for example collected in [16], Appendix G) and interpolation theory (see [1] or [16], Section 1.3) for quasi-normed spaces, the generalized Lebesgue theorem for operators of de la Vallée-Poussin type (see, e. g., [2], [28]) and the comparison ideas described in [5] we obtain the following general statements for families of type (2.11):

(i) (***Banach-Steinhaus convergence principle***) *Let $0 < p \le \infty$. A family of linear bounded operators of type (1.25) converges in L_p if and only if the following conditions are satisfied:*

1) $\lim_{\sigma\to+\infty} \| e^{ik\cdot} - \mathcal{L}_\sigma(e^{ik\cdot}) \|_{\overline{p}} \;=\; 0$ *for all* $k \in \mathbb{Z}^d$ *;*

2) $(\mathcal{L}_\sigma)_\sigma$ *is bounded in L_p (i.e. (2.13) holds).*

(ii) (***Riesz-Thorin interpolation theorem***) *Suppose that the family of linear bounded operators of type (1.25) is bounded in L_{p_0} and in L_{p_1}, where $1 \le p_0 < p_1 \le +\infty$. Then it is bounded in L_p for all $p_0 \le p \le p_1$ and*

$$\| \mathcal{L}_\sigma \|_{(p)} \;\le\; \| \mathcal{L}_\sigma \|_{(p_0)}^{1-\theta} \cdot \| \mathcal{L}_\sigma \|_{(p_1)}^{\theta} \,,\; \sigma \in \Omega \,,\; 1/p = (1-\theta)/p_0 + \theta/p_1.$$

(iii) (***Comparison principle***) *Let $0 < p \le \infty$ and let $(\mathcal{L}_\sigma^{(j)})_\sigma$, $j = 1, 2$, be families of linear bounded operators of type (1.25). If they are bounded in L_p and if $\mathcal{L}_\sigma^{(1)}(T) = \mathcal{L}_\sigma^{(2)}(T)$ for all $T \in \mathcal{T}_{r\sigma}$, then*

$$\| f - \mathcal{L}_\sigma^{(1)}(f) \|_{\overline{p}} \;\asymp\; \| f - \mathcal{L}_\sigma^{(2)}(f) \|_{\overline{p}} \,,\; f \in L_p \,,\; \sigma \in \Omega \;.$$

(iv) (***Generalized Lebesgue estimate***) *Let a family of linear bounded operators of type (1.25) satisfy* $\mathcal{L}_\sigma(T) = T$ *for all* $T \in \mathcal{T}_{\rho\sigma}$*, where* $0 < \rho < r$*. Let* $0 < p \le \infty$ *and let* $\widetilde{p} = \min(1, p)$*. Then*

$$\| f - \mathcal{L}_\sigma(f) \|_{\overline{p}} \le \left(1 + \| \mathcal{L}_\sigma \|_{(p)}^{\widetilde{p}}\right)^{1/\widetilde{p}} E_{\rho\sigma}(f)_p\,, \;\; f \in L_p\,, \;\; \sigma \in \Omega\,.$$

Note that statement **(ii)** remains true for $0 < p_0 < p_1 \le \infty$ with a multiplicative constant c, depending only on p_0, p_1, θ, in the right-hand side (the Marcinkiewicz interpolation theorem).

Let us also mention that the general comparison principle established in **(iii)** has been proved in [24]. Forerunners of comparing the rates of convergence of convolution integrals and generalized sampling series generated by the same kernel in the space of uniformly continuous and bounded functions on $\mathbb{R}$ can be found in the paper [5]. For a trigonometric version we refer to [31], where the equivalence of the rates of convergence of Fourier means (1.3) and interpolation means (1.4) generated by the same kernel has been proved.

3 Approximation methods and their convergence

3.1 Generators and kernels

Generators of approximation methods. By definition the class $\mathcal{K}$ consists of functions φ satisfying

1) $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is centrally symmetric ($\varphi(-\xi) = \overline{\varphi(\xi)}$ for each $\xi \in \mathbb{R}^d$);

2) φ is continuous on $\mathbb{R}^d$;

3) φ has a compact support;

4) $\varphi(0) = 1$.

Important characteristics of the generator φ are the radius of its support

$$r(\varphi) = \sup\{\,|\,\xi\,| : \varphi(\xi) \ne 0\,\} < +\infty \tag{3.1}$$

and the set

$$\mathcal{P}_\varphi = \{\, p \in (0, +\infty] : \widehat{\varphi} \in L_p(\mathbb{R}^d)\,\}\,. \tag{3.2}$$

Since $\lim_{|\,x\,| \to +\infty} |\,\widehat{\varphi}(x)\,| = 0$, we have $\widehat{\varphi} \in L_q(\mathbb{R}^d)$ for $p \le q \le +\infty$ if $\widehat{\varphi} \in L_p(\mathbb{R}^d)$. Hence, $\mathcal{P}_\varphi$ is either $(p_0, +\infty]$ or $[p_0, +\infty]$, where $p_0 = \inf \mathcal{P}_\varphi$.

Kernels. Let

$$\Lambda = \{a_{n,\,k}\}_{n \in \mathbb{N}_0,\, k \in \mathbb{Z}} \tag{3.3}$$

be a matrix of complex numbers satisfying $a_{n,-k} = \overline{a_{n,k}}$ if $|k| \le r(\Lambda)n$ and $a_{n,k} = 0$ if $|k| > r(\Lambda)n$ for some positve real number $r(\Lambda)$ and for all $n \in \mathbb{N}_0$. Note, that in contrast to (1.1) we do not assume that $a_{n,0} = 1$ for all $n \in \mathbb{N}_0$. It generates the real-valued trigonometric polynomials (kernels)

$$W_n(\Lambda)(h) = \sum_{|k| \leq r(\Lambda) n} a_{n,k} \, e^{ikh}, \quad n \in \mathbb{N}_0 , \tag{3.4}$$

of order at most $r(\Lambda)\, n$ (see also (1.2)). Special cases are, in particular, the classical kernels of Dirichlet, Fejér, de la Vallée-Poussin, Rogosinski, Jackson, Cesaro, Riesz, Bochner-Riesz, Zygmund and Favard. For precise definitions see Subsection 2.5. For basic properties and historical remarks we refer, for instance, to [6], [8], [42]-[44].

In classical approximation theory one traditionally deals with real-valued multipliers (matrices) and even kernels. However, such a restriction does not enable us to describe the smoothness of odd order (in particular, the classical first derivative and the modulus of continuity) via approximation methods. Details will be discussed in Part II based on our paper [25]. For this reason we have extended the set of generators admitting complex-valued matrices (multipliers).

Following [33] we give a classification of kernels containing at least all classical ones.

Type (G). If Λ is generated by some function $\varphi \in \mathcal{K}$ in the sense that

$$\Lambda \equiv \Lambda(\varphi) = \{ a_{n,k} : \quad a_{0,0} = 1; \; a_{n,k} = \varphi\left(\frac{k}{\sigma(n)}\right), \; k \in \mathbb{Z}^d, \; n \in \mathbb{N} \}, \tag{3.5}$$

where $\sigma(n)$ is a certain strictly increasing sequence of positive real numbers satisfying $\sigma(n) \asymp n$, then the kernels $W_n(\Lambda)$ are said to be of type (G). In this situation we shall replace the discrete parameter $\sigma(n)$ by a continuous one and the notation $W_n(\Lambda)$ by the symbol $W_\sigma(\varphi)$. Thus, kernels of type (G) generated by $\varphi \in \mathcal{K}$ are given by

$$W_0(h) \equiv 1, \; W_\sigma(h) = \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{\sigma}\right) e^{ikh}, \; \sigma > 0 . \tag{3.6}$$

It is well known that the kernels of Fejér, de la Vallée-Poussin, Rogosinski, Riesz, Bochner-Riesz, Zygmung, Favard belong are of this type. Their generators can be found in the General Convergence Table in Subsection 2.5.

Type (GR). We say that the kernels $W_n(\Lambda)$ are of type (GR) if the corresponding matrix can be represented in the form

$$\Lambda = \Lambda(\varphi) + R, \; R = \{ r_{n,k} \}; \; \lim_{n \to +\infty} r_{n,k} = 0, \; k \in \mathbb{Z}^d, \tag{3.7}$$

where $\Lambda(\varphi)$ is of type (3.5) and representation (3.7) is unique. The kernels of Fejér-Korovkin and Cesaro are of such a type (see [33] for further details).

The main idea to deal with kernels of type (GR) and corresponding approximation methods in the spaces L_p with $0 < p < 1$ consists in the replacement of the norms of the kernels $W_n(R)$ generated by the remainder-matrix of R by their L_2-norms. This has been pointed out in [33]. More precisely, if $0 < q < +\infty$ then we put

$$M_{q;\Lambda}(n) = (n+1)^{d(1/q-1)} \| W_n(\Lambda) \|_q , \; n \in \mathbb{N}_0 ; \quad M_{q;\Lambda} = \sup_n M_{q;\Lambda}(n) ; \tag{3.8}$$

$$\mathcal{M}_{q;\Lambda}(n) = (n+1)^{d(1/q-1)} \,\|\, W_n(\Lambda) \,\|_2,\ n \in \mathbb{N}_0\,;\ \mathcal{M}_{q;\Lambda} = \sup_n \mathcal{M}_{q;\Lambda}(n)\,. \tag{3.9}$$

Clearly,

$$\mathcal{M}_{q;R}(n) \ = \ (2\pi)^{d/2}(n+1)^{d(1/q-1)} \left(\sum_{|\,k\,| \le r(\Lambda)n} |\, r_{n,k}|^2 \right)^{1/2},\ n \in \mathbb{N}_0\,, \tag{3.10}$$

by Parseval's equality.

Let now $0 < q < 1$. A kernel $W_n(\Lambda)$ is said to be of type $(\mathbf{GR}^{\boldsymbol{q}})$ and of type $(\mathbf{GR}_{\boldsymbol{q}})$ if its generating matrix is of type (3.7) with $M_{q;R} < +\infty$, $\mathcal{M}_{q;R} < +\infty$ respectively. Hölder's inequality implies

$$M_{q;R}(n) \ \le \ (2\pi)^{d(1/q-1/2)} \mathcal{M}_{q;R}(n)\,,\ n \in \mathbb{N}_0\,. \tag{3.11}$$

Hence, each kernel of type (GR_q) is also of type (GR^q) and we have

$$\mathrm{GR}_q \ \subset \ \mathrm{GR}^q\,. \tag{3.12}$$

Notation (3.8) turn out to be also convenient for kernels of type (G). In this case we shall use the symbols $M_{q;\varphi}(\sigma)$ and $M_{q;\varphi}$ in place of $M_{q;\Lambda}(n)$, $M_{q;\Lambda}$ respectively.

Type $(\mathbf{G}^{\boldsymbol{q}})$. Let $q \in \mathbb{N}$. By definition a kernel belongs to the class (G^q) if it can be represented as

$$W_n(\varphi, q) = (\gamma_n(\varphi, q))^{-1} (W_n(\varphi))^q(h)\,, \tag{3.13}$$

where

$$\gamma_n(\varphi, q) = (2\pi)^{-d} \int_{\mathbb{T}^d} (W_n(\varphi)(h))^q \, dh \tag{3.14}$$

is a normalizing factor. It is shown in [33] that $\gamma_n(\varphi, q) \neq 0$ if $n \ge n_0$, where $n_0 \equiv n_0(\varphi, q)$ is a certain integer. Therefore, the functions $W_n(\varphi, q)(h)$ are well defined for $n \ge n_0$ and belong to $\mathcal{T}_{qr(\varphi)\sigma}$. A typical example of a kernel of type (G^q) is the generalized Jackson kernel introduced by S. Stechkin (see, e.g., [40]). We also mention that kernels of type (G^q) are also of type (GR). They are generated by the q-th convolution power of the function φ (see [33] for further details).

In contrast to the case of kernels of type (GR), where finding appropriate estimates for $\mathcal{M}_{q;R}(n)$ can be a rather complicated problem, the L_p-norms of kernels of type (G) can be exactly calculated applying a well-known scheme adapted to the case $0 < p < 1$. It is based on the Poisson summation formula as well as on the interpretation of the kernels as integral sums of Riemann type for the function $\widehat{\varphi}$. For the classical case $p = 1$ we refer to [6], [18], [41], and [42]. Following [24] we present here the corresponding general result.

Theorem 3.1. *Let $\varphi \in \mathcal{K}$ and $0 < p \le 1$. The set*

$$\{\, \sigma^{d(1/p-1)} \,\|\, W_\sigma(\varphi) \,\|_p \ : \ \sigma \ge 0 \,\}$$

is bounded if and only if $\widehat{\varphi} \in L_p(\mathbb{R}^d)$. Moreover, in this case

$$\lim_{\sigma \to +\infty} \sigma^{d(1/p-1)} \,\|\, W_\sigma(\varphi) \,\|_p \ = \ \sup_{\sigma \ge 0} \sigma^{d(1/p-1)} \,\|\, W_\sigma(\varphi) \,\|_p \ = \ \|\, \widehat{\varphi} \,\|_{L_p(\mathbb{R}^d)}\,. \tag{3.15}$$

Proof. Step 1. Applying (2.10) to $\varphi(\cdot/\sigma)$ and using the equality $\widehat{\varphi(\sigma^{-1}\cdot)}(x) = \sigma^d \widehat{\varphi}(\sigma x)$ we get

$$\sup_{\sigma \geq 0} \sigma^{d(1/p-1)} \| W_\sigma(\varphi) \|_p \leq \| \widehat{\varphi} \|_{L_p(\mathbb{R}^d)} . \tag{3.16}$$

Step 2. Now we prove the following: if there exists a sequence $(\nu_n)_{n \in \mathbb{N}}$ of strictly increasing natural numbers such that the sequence $\nu_n^{d(1/p-1)} \|W_{\nu_n}(\varphi)\|_p$ is bounded, then $\widehat{\varphi} \in L_p(\mathbb{R}^d)$ and

$$\| \widehat{\varphi} \|_{L_p(\mathbb{R}^d)} \leq \sup_{n \in \mathbb{N}} \nu_n^{d(1/p-1)} \| W_{\nu_n}(\varphi) \|_p . \tag{3.17}$$

Consider the sequence of functions $F_n(x)$, $n \in \mathbb{N}$, given by

$$F_n(x) = \begin{cases} n^{-dp} \left| W_{\nu_n}(\varphi) \left(\dfrac{x}{\nu_n} \right) \right|^p & , \quad x \in [-\pi\nu_n, \pi\nu_n]^d \\ 0 & , \quad \text{otherwise} \end{cases} . \tag{3.18}$$

Clearly, the functions $F_n(x)$, $n \in \mathbb{N}$, are non-negative and measurable. Let $x_0 \in \mathbb{R}^d$. Then there exists $n_0 \in \mathbb{N}$ such that $x_0 \in [-\pi\nu_n, \pi\nu_n]^d$ for $n \geq n_0$. The function $\varphi(\cdot)e^{ix_0\cdot}$ is Riemann integrable on a cube $Q \subset \mathbb{R}^d$ containing its support. By definition of the Riemann integral we get

$$\lim_{n \to +\infty} \nu_n^{-d} \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{\nu_n}\right) \cdot e^{(ikx_0)/\nu_n} = \int_Q \varphi(\xi) \cdot e^{i\xi x_0} d\xi = \widehat{\varphi}(-x_0) .$$

Hence,

$$\lim_{n \to +\infty} F_n(x_0) = | \widehat{\varphi}(-x_0) |^p . \tag{3.19}$$

By the definition of F_n in (3.18) it follows

$$\begin{aligned} \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} F_n(x) dx &= \sup_{n \in \mathbb{N}} \nu_n^{d(1-p)} \int_{[-\pi\nu_n, \pi\nu_n]^d} \left| W_{\nu_n}(\varphi) \left(\frac{x}{\nu_n} \right) \right|^p dx \\ &= \sup_{n \in \mathbb{N}} \nu_n^{d(1-p)} \| W_{\nu_n}(\varphi) \|_p^p < +\infty . \end{aligned} \tag{3.20}$$

Thus, we have proved that the sequence $(F_n(x))_{n \in \mathbb{N}}$ satisfies all conditions of Fatou's lemma. Its combination with (3.19) and (3.20) yields $\widehat{\varphi} \in L_p(\mathbb{R}^d)$ and (3.17).

Step 3. Now, the criterion and the second relation in (3.15) follow immediately from the statements above. Suppose that the set $\{ \sigma^{d(1/p-1)} \| W_\sigma(\varphi) \|_p \}$ is bounded and let a be one of its accumulation points. By (3.16) a does not exceed the norm of $\widehat{\varphi}$ in $L_p(\mathbb{R}^d)$. The inverse estimate follows from (3.17). As a consequence we get (3.15). □

3.2 Fourier means and interpolation means

Recall that the Fourier means generated by the matrix Λ as given in (1.1) are defined as

$$\mathcal{F}_n^{(\Lambda)}(f;x) = (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(h)\, W_n(\Lambda)(x-h)\, dh\,,\ n \in \mathbb{N}_0 \tag{3.21}$$

for functions $f \in L_1$. If the kernel $W_n(\Lambda)$ is of type (G) then we use the notation $\mathcal{F}_\sigma^{(\varphi)}$, $\sigma \geq 0$. In this case the function $W_n(\Lambda)$ is replaced by $W_\sigma(\varphi)$ given in (3.6). The Fourier means are well defined only in L_p for $1 \leq p \leq +\infty$. Their properties are well known and can be found in the literature (see, e. g., [6], [8], [15], [31], [42], [50], etc.). For the sake of completeness and better understanding we collect those, which are of interest here, in the following theorem.

Theorem 3.2. *Let $1 \leq p \leq \infty$, $n \in \mathbb{N}_0$, and let Λ be a matrix given in* (1.1). *Then the following statements hold true.*

(i) $\mathcal{F}_n^{(\Lambda)}$ *is a linear polynomial operator mapping L_p into $\mathcal{T}_{r(\Lambda)\,n}$. Moreover,*

$$\mathcal{F}_n^{(\Lambda)}(f;x) \;=\; \sum_{|\,k\,| \leq r(\Lambda)\,n} a_{n,k}\, f^\wedge(k)\, e^{ikx}\,,\ f \in L_p\,. \tag{3.22}$$

(ii) $\mathcal{F}_n^{(\Lambda)}$ *is a bounded operator in L_p and*

$$\|\,\mathcal{F}_n^{(\Lambda)}\,\|_{(p)} \;=\; \sup_{\|\,f\,\|_p \leq 1} \|\,\mathcal{F}_n^{(\Lambda)}(f)\,\|_p \;\leq\; (2\pi)^{-d}\,\|\,W_n(\Lambda)\,\|_1\,. \tag{3.23}$$

If $p=1$ or $p=+\infty$ then

$$\|\,\mathcal{F}_n^{(\Lambda)}\,\|_{(p)} \;=\; (2\pi)^{-d}\,\|\,W_n(\Lambda)\,\|_1\,. \tag{3.24}$$

(iii) *The means $\mathcal{F}_n^{(\Lambda)}$ converge in L_p, that is,*

$$\lim_{n\to+\infty} \|\,f - \mathcal{F}_n^{(\Lambda)}(f)\,\|_p \;=\; 0\,,\ f \in L_p\,,$$

for $p=1$, $p=+\infty$ or for all $1 \leq p \leq +\infty$ if and only if $M_{1;\Lambda} < +\infty$ (where the numbers $M_{q;\Lambda}$ are defined in (3.8)).

(iv) *If $a_{n,k}=1$ for $|\,k\,| \leq \rho n$, where $0<\rho<r(\Lambda)$, then*

$$\|\,f - \mathcal{F}_n^{(\Lambda)}(f)\,\|_p \;\leq\; \big(1 + \|\,\mathcal{F}_n^{(\Lambda)}\,\|_{(p)}\big)\, E_{\rho n}(f)_p\,,\ f \in L_p\,. \tag{3.25}$$

Note that part (i) follows directly from (2.5), (1.2) and (3.21). Inequality (3.23) can be proved applying the generalized Minkowski inequality. For the proof of equality (3.24) in the one-dimensional case we refer to [8]. In the multivariate case the proof is similar. Part (iii) is a consequence of (i), (ii) and the Banach-Steinhaus convergence principle. Combining (i) and the generalized Lebesgue estimate (iv) in Subsection 1.4 we obtain (3.25).

For the Fourier means $\mathcal{F}_\sigma^{(\varphi)}$ generated by $\varphi \in \mathcal{K}$ our Theorem 2.2 can be reformulated in terms of the continuous parameter $\sigma \geq 0$ and the generator φ with obvious modifications. In particular, the condition in part (iv) should be rewritten in the form: $\varphi(\xi) = 1$ for $\xi \in \overline{B}_\rho$, where $0 < \rho < r$. Let us also emphasize that the convergence criterion for the Fourier means related to kernels of type (G) takes a very simple and clear form (see, for instance, [31] or [42]). As a consequence of part (iii) of Theorem 2.2 and Theorem 2.1 we find (see also [15]).

Theorem 3.3. *Let $\varphi \in \mathcal{K}$. Then the Fourier means $\mathcal{F}_\sigma^{(\varphi)}$ converge in L_p for $p = 1$, $p = +\infty$ or for all $1 \leq p \leq +\infty$ if and only if $1 \in \mathcal{P}_\varphi$.*

This theorem enables us to give very simple proofs of the convergence of the means generated by the kernels of Fejér, Vallée-Pousssin, Rogosinski and other classical kernels which are based on direct calculation or estimates of the Fourier transforms of their generators (see Subsection 2.5 for further details). In this respect we want to remind that the original proofs of the convergence of classical means are based on the specific properties of their kernels. Sometimes these proofs are very complicated (see [8], [43], [44], [51], etc.). Let us also mention that Theorem 2.3 is a direct consequence of the general result on the convergence of families of linear polynomial operators as will be discussed in the next subsection.
Recall that the interpolation means generated by the matrix Λ read as (cf. (1.4))

$$\mathcal{I}_n^{(\Lambda)}(f;x) = (2N+1)^{-d} \cdot \sum_{\nu=0}^{2N} f\left(t_N^\nu\right) \cdot W_n(\Lambda)\left(x - t_N^\nu\right), \quad n \in \mathbb{N}_0, \tag{3.26}$$

where

$$N = [rn], \quad r \geq r(\Lambda); \quad t_N^\nu = \frac{2\pi\nu}{2N+1}, \quad \nu \in \mathbb{Z}^d; \quad \sum_{\nu=0}^{2N} \equiv \sum_{\nu_1=0}^{2N} \cdots \sum_{\nu_d=0}^{2N}. \tag{3.27}$$

Again we use the notation $\mathcal{I}_\sigma^{(\varphi)}$, $\sigma \geq 0$ if the kernel is of type (G). In this case the function $W_n(\Lambda)$ is replaced by $W_\sigma(\varphi)$ given in (3.6) and the number $r(\Lambda)$ is replaced by $r(\varphi)$.The function f in the right-hand side of (3.26) should be defined at all points of the uniform grids, that is, on a set of measure 0. For this reason within the scale of spaces L_p, $0 < p \leq +\infty$ the interpolation means are well defined only in the space $C(\mathbb{T}^d)$ of 2π-periodic continuous functions. Their convergence properties are completely studied in the literature (see, e.g., [31]). The main results read as follows.

Theorem 3.4. *Let $n \in \mathbb{N}_0$, and Λ be a matrix of multipliers given in* (1.1)*. Then the following statements hold true.*

(i) $\mathcal{I}_n^{(\Lambda)}$ *is a linear polynomial operator mapping C to $\mathcal{T}_{r(\Lambda)\,n}$.*

(ii) *For each polynomial $T \in \mathcal{T}_{rn}$ ($r \geq r(\Lambda)$ is fixed, $N = [rn]$) one has*

$$\mathcal{I}_n^{(\Lambda)}(T;x) = \sum_{|k| \leq r(\Lambda)\,n} a_{n,k}\, T^\wedge(k)\, e^{ikx}. \tag{3.28}$$

(iii) $\mathcal{I}_n^{(\Lambda)}$ *is a bounded operator in* C *and*

$$\| \mathcal{I}_n^{(\Lambda)} \|_{(C)} = \sup_{\| f \|_C \le 1} \| \mathcal{I}_n^{(\Lambda)}(f) \|_C \asymp \| W_n(\Lambda) \|_1 . \tag{3.29}$$

(iv) *The means* $\mathcal{I}_n^{(\Lambda)}$ *converge in* C*, that is, it holds*

$$\lim_{n \to +\infty} \| f - \mathcal{I}_n^{(\Lambda)}(f) \|_C = 0 , \; f \in C , \tag{3.30}$$

if and only if $M_{1;\Lambda} < +\infty$.

(v) *If* $a_{n,k} = 1$ *for* $|k| \le \rho n$*, where* $0 < \rho < r(\Lambda)$*, then*

$$\| f - \mathcal{I}_n^{(\Lambda)}(f) \|_C \le \big(1 + \| \mathcal{I}_n^{(\Lambda)} \|_{(C)} \big) E_{\rho n}(f)_C , \; f \in C . \tag{3.31}$$

(vi) *If* $M_{1;\Lambda} < +\infty$ *then*

$$\| f - \mathcal{I}_n^{(\Lambda)}(f) \|_C \asymp \| f - \mathcal{F}_n^{(\Lambda)}(f) \|_C , \; f \in C , \; n \in \mathbb{N}_0 . \tag{3.32}$$

Note that part (i) follows directly from the definition. The proof of (ii) is straightforward. The asymptotic formula for norms (3.29) follows from the classical estimate of the discrete norm of a trigonometric polynomial by its continuous one (see, e.g. [51], Vol. 2). For the complete proofs of (ii) and (iii) we refer to [31]. Part (iv) is a consequence of (ii), (iii) and the Banach-Steinhaus convergence principle. Part (v) is similar to part (iv) of Theorem 2.2. The equivalence of the approximation errors of $\mathcal{I}_n^{(\Lambda)}(f)$ and $\mathcal{F}_n^{(\Lambda)}(f)$ in the space C follows immediately from the comparison principle in combination with part (i) of Theorem 2.2 and part (ii) of Theorem 2.4.

In analogy to the case of the Fourier means Theorem 2.4 can be reformulated for the interpolation means $\mathcal{I}_\sigma^{(\varphi)}$ generated by $\varphi \in \mathcal{K}$ in terms of the continuous parameter $\sigma \ge 0$ and the generator φ with obvious modifications. In particular, in view of Theorem 2.1 the convergence principle and the comparison principle take the following form (see also [31]).

Theorem 3.5. *Let* $\varphi \in \mathcal{K}$*. Then the interpolation means* $\mathcal{I}_\sigma^{(\varphi)}$ *converge in* C *if and only if* $1 \in \mathcal{P}_\varphi$*. Moreover, in this case we have*

$$\| f - \mathcal{I}_\sigma^{(\varphi)}(f) \|_C \asymp \| f - \mathcal{F}_\sigma^{(\varphi)}(f) \|_C , \; f \in C , \; \sigma \ge 0. \tag{3.33}$$

As in the case of the Fourier means we immediately obtain the convergence of the interpolation means generated by the Fejér, de la Vallée-Poussin, Rogosinski and other classical kernels of type (G) from this result.

3.3 Families of linear polynomial operators

Recall that the families of linear polynomial operators generated by a matrix Λ as given in (1.1) are defined by

$$\mathcal{L}_{n;\lambda}^{(\Lambda)}(f;x) = (2N+1)^{-d} \cdot \sum_{\nu=0}^{2N} f\left(t_N^\nu + \lambda\right) \cdot W_n(\Lambda)\left(x - t_N^\nu - \lambda\right) , \tag{3.34}$$

where the number N and the points $t_N^\nu \in \mathbb{T}^d$, $\nu \in \mathbb{Z}^d$, are described in (3.27). This has to be understood in the sense that $\mathcal{L}_n^{(\Lambda)}(f) = \mathcal{L}_{n;\lambda}^{(\Lambda)}(f;x)$ is considered as a function of x and λ on $\mathbb{T}^d \times \mathbb{T}^d$, or on $\mathbb{R}^d \times \mathbb{R}^d$ respectively. Here $n \in \mathbb{N}_0$ is fixed. If the kernel is of type (G) then we use the notation $\{\mathcal{L}_\sigma^{(\varphi)}\}_{\sigma\geq 0}$. In this case the function $W_n(\Lambda)$ is replaced by $W_\sigma(\varphi)$ given in (3.6) and the number $r(\Lambda)$ is replaced by $r(\varphi)$.

In contrast to the classical methods of trigonometric approximation the method of approximation by families is comparatively new. It has been introduced in [26] and [27] as a constructive method for trigonometric approximation in L_p where $0 < p < 1$. Its systematical study was continued in [2]-[4], [14], [24], [25], [28], [32], [33], [34], [37] and other papers. The main properties of families are given in the theorem below.

Theorem 3.6. *Let $0 < p \leq +\infty$, $n \in \mathbb{N}_0$, and Λ be a matrix as given in* (1.1). *Then the following statements hold.*

(i) *If $f \in L_p(\mathbb{T}^d)$ then for almost all λ the values $f(t_N^\nu + \lambda)$, $\nu \in \mathbb{Z}^d$, are well defined and $\mathcal{L}_{n;\lambda}^{(\Lambda)}(f;x)$ belongs to $\mathcal{T}_{r(\Lambda)\,n}$ as a function of x.*

(ii) *We have*

$$\mathcal{L}_{n\,;\lambda}^{(\Lambda)}(T;x) \;=\; \sum_{|\,k\,| \,\leq\, r(\Lambda)\,n} a_{n,\,k}\, T^\wedge(k)\, e^{ikx} \tag{3.35}$$

for each polynomial $T \in \mathcal{T}_{rn}$ ($r \geq r(\Lambda)$ is fixed, $N = [rn]$) and for each $\lambda \in \mathbb{R}^d$.

(iii) *The operator $\mathcal{L}_n^{(\Lambda)}$ is a linear bounded mapping from L_p to $\mathcal{T}_{r(\Lambda)\,n,\overline{p}} \subset L_{\overline{p}}$ for each number $n \in \mathbb{N}_0$. Moreover, it holds*

$$M_{\widehat{p};\,\Lambda}(n) \;\lesssim\; \|\, \mathcal{L}_n^{(\Lambda)} \,\|_{(p)} \;\lesssim\; M_{\widetilde{p};\,\Lambda}(n)\,,\;\; n \in \mathbb{N}_0\,, \tag{3.36}$$

for its norm

$$\|\, \mathcal{L}_n^{(\Lambda)} \,\|_{(p)} \;=\; \sup_{\|\,f\,\|_p \,\leq\, 1} \|\, \mathcal{L}_n^{(\Lambda)}(f) \,\|_{\overline{p}}\,, \tag{3.37}$$

where $M_{q;\,\Lambda}$ is defined by (3.8), *$\widetilde{p} = \min(1,p)$, $\widehat{p} = p$ for $0 < p < +\infty$ and $\widehat{p} = 1$ for $p = +\infty$.*

(iv) *If $M_{1;\,\Lambda} < +\infty$, then the family $\{\mathcal{L}_n^{(\Lambda)}\}_n$ converges in L_p, that is,*

$$\lim_{n\,\to\,+\infty} \|\, f - \,\mathcal{L}_n^{(\Lambda)}(f) \,\|_{\overline{p}} \;=\; 0\,,\;\; f \in L_p\,, \tag{3.38}$$

if and only if $M_{p;\,\Lambda} < +\infty$.

(v) *If* $a_{n,k} = 1$ *for* $|k| \le \rho n$, *where* $0 < \rho < r(\Lambda)$, *then*

$$\| f - \mathcal{L}_n^{(\Lambda)}(f) \|_{\overline{p}} \le \left(1 + \| \mathcal{L}_n^{(\Lambda)} \|_{(p)}^{\widetilde{p}}\right)^{1/\widetilde{p}} E_{\rho n}(f)_p \,, \ f \in L_p \,. \tag{3.39}$$

(vi) *If* $1 \le p \le +\infty$ *and if* $M_{1;\Lambda} < +\infty$, *then*

$$\| f - \mathcal{L}_n^{(\Lambda)}(f) \|_{\overline{p}} \asymp \| f - \mathcal{F}_n^{(\Lambda)}(f) \|_p \,, \ f \in L_p \,, \ n \in \mathbb{N}_0 \,. \tag{3.40}$$

If $p = +\infty$ *then* (3.40) *holds with* $\mathcal{I}_n^{(\Lambda)}$ *in place of* $\mathcal{F}_n^{(\Lambda)}$ *for all* $f \in C$ *and* $n \in \mathbb{N}_0$.

Theorem 2.6 shows that the method of approximation by families of linear polynomial operators is universal in the sense that it is relevant for both $1 \le p \le +\infty$ and $0 < p < 1$ where the range of admissible parameters depends on the properties of the generating matrix Λ and, moreover, their approximation errors are equivalent to the approximation errors of the corresponding Fourier means in case of L_p, $1 \le p \le +\infty$ and to the approximation errors of the corresponding interpolation means in case of the space C.

Complete proofs of statements (i)-(vi) can be found in [24] and [33]. The proof of the "key"-statement in the theory of families polynomial operators on operator norms (part (iii)) is also given below. Note that part (i) follows from the Fubini theorem and the definition. Part (ii) is a consequence of part (ii) of Theorem 2.4 and the operator equality

$$\mathcal{L}_{n;\lambda}^{(\Lambda)} = S_{-\lambda} \circ \mathcal{I}_\sigma^{(\Lambda)} \circ S_\lambda \,, \tag{3.41}$$

where $S_t f(\cdot) = f(\cdot + t)$ is the translation operator, which is valid on $\mathcal{T}_{rn}$. Part (iv) follows from (ii), (iii) and the Banach-Steinhaus convergence principle. Part (v) is a consequence of the generalized Lebesgue estimate (iv) in Susection 1.4. Part (vi) follows immediately from the comparison principle (iii) in Subsection 1.4 combined with part (i) of Theorem 2.2, part (ii) of Theorem 2.6 and part (iv) of Theorem 2.4.

Proof of part (iii). Step 1. First we prove the upper estimate for $0 < p \le 1$. By (3.34) and (2.1)-(2.2) we get the estimates

$$\begin{aligned}
&(2\pi)^{d/p} \| \mathcal{L}_{n;\lambda}^{(\Lambda)}(f;x) \|_{\overline{p}}^p \\
&\quad \le (2N+1)^{-dp} \sum_{\nu=0}^{2N} \| f(t_N^k + \lambda) \| W_n(\Lambda)(x - -t_N^k - \lambda) \|_{p;x} \|_{p;\lambda}^p \\
&\quad \le (2N+1)^{-dp} \| W_n(\Lambda) \|_p^p \cdot \sum_{\nu=0}^{2N} \| f(t_N^k + \lambda) \|_p^p \\
&\quad \le (2N+1)^{d(1-p)} \| W_n(\Lambda) \|_p^p \, \| f \|_p^p
\end{aligned}$$

for each $f \in L_p$. This implies

$$\| \mathcal{L}_n^{(\Lambda)} \|_{(p)} \le (2\pi)^{-d/p} (2N+1)^{d(1/p-1)} \| W_n(\Lambda) \|_p \,, \ 0 < p \le 1 \,. \tag{3.42}$$

Step 2. In order to prove the lower estimate in (3.36) for $0 < p < +\infty$ we consider the 2π-periodic function f_* which is defined on $[-\pi, \pi)^d$ by (μ denotes the d-dimensional

Lebesgue measure)

$$f_*(h) \; = \; \begin{cases} (\mu(B_{\tau/2}(0)))^{-1/p} & , \;\; h \in B_{\tau/2}(0) \\ 0 & , \;\; \text{otherwise} \end{cases} \;\; , \quad \left(\tau = \frac{2\pi}{2N+1} \right) \, .$$

For each $\lambda \in B_{\tau/2}(0)$ and for each vector $k \in \mathbb{Z}^d \setminus \{0\}$ with components $0 \le k_j \le 2N$, $j = 1, \ldots, d$, we have

$$|\, t_N^k + \lambda \,| \;\ge\; |\, t_N^k \,| \,-\, |\, \lambda \,| \;\ge\; \tau \,-\, \tau/2 \;=\; \tau/2 \, .$$

Hence, by the definition of f_* we get

$$f_*(t_N^k + \lambda) \;=\; 0 \;\text{ for }\; \lambda \in B_{\tau/2}(0) \; , \; k \in \mathbb{Z}^d, \; k \ne 0 \, , \; 0 \le k_j \le 2N \, , \; j = 1, \ldots, d \, .$$

Therefore, in view of (3.34)

$$\mathcal{L}_{n;\lambda}^{(\Lambda)}(f_*; x) = (2N+1)^{-d} \, f_*(\lambda) \, W_n(\Lambda)(x - \lambda) \; , \quad x \in \mathbb{T}^d , \; \lambda \in B_{\tau/2}(0) \, . \tag{3.43}$$

Since $\mathcal{L}_{n;\lambda}^{(\Lambda)}(f_*; x)$ is τ-periodic with respect to each λ_j, $j = 1, \ldots, d$ as a function of λ and since $\|f_*\|_p = 1$, we obtain

$$\begin{aligned} \| \, \mathcal{L}_n^{(\Lambda)} \, \|_{(p)}^p &\ge \| \, \mathcal{L}_{n;\lambda}^{(\Lambda)}(f_*; x) \, \|_{\overline{p}}^p = \tau^{-d} \int\limits_{[-\tau/2,\tau/2]^d} \left(\int\limits_{\mathbb{T}^d} |\mathcal{L}_{n;\lambda}^{(\Lambda)}(f_*; x)|^p \, dx \right) d\lambda \\ &\ge \tau^{-d} \int\limits_{B_{\tau/2}(0)} \left(\int\limits_{\mathbb{T}^d} |\, \mathcal{L}_{n;\lambda}^{(\Lambda)}(f_*; x)|^p \, dx \right) d\lambda \\ &= (2\pi)^{-d} \, (2N+1)^{d(1-p)} \int\limits_{B_{\tau/2}(0)} |\, f_*(\lambda)|^p \left(\int\limits_{\mathbb{T}^d} |\, W_n(\Lambda)(x - \lambda)|^p dx \right) d\lambda \\ &= (2\pi)^{-d} \, (2N+1)^{d(1-p)} \, \| \, W_n(\Lambda) \, \|_p^p \, . \end{aligned}$$

from (3.43). Thus, we have

$$\| \, \mathcal{L}_n^{(\Lambda)} \|_{(p)} \;\ge\; (2\pi)^{-d/p} \, (2N+1)^{d(1/p-1)} \, \| \, W_n(\Lambda) \, \|_p \tag{3.44}$$

for $0 < p < +\infty$.

Step 3. Now, let $p = +\infty$. Using (3.41) we obtain

$$\begin{aligned} \| \, \mathcal{I}_n^{(\Lambda)} \, \|_{(C)} \;\; &\le \;\; \| \mathcal{L}_n^{(\Lambda)} \|_{(C)} = \sup_{\| \, f \, \|_C \le 1} \max_{\lambda} \| \, S_{-\lambda} \circ \mathcal{I}_n^{(\Lambda)} \circ S_\lambda(f) \, \|_C \\ &= \;\; \sup_{\| \, f \, \|_C \le 1} \max_{\lambda} \| \, \mathcal{I}_n^{(\Lambda)} \circ S_\lambda(f) \, \|_C \\ &\le \;\; \sup_{\| \, f \, \|_C \le 1} \max_{\lambda} \| \, \mathcal{I}_n^{(\Lambda)} \, \|_{(C)} \cdot \| \, S_\lambda(f) \, \|_C = \| \, \mathcal{I}_n^{(\Lambda)} \, \|_{(C)} \, . \end{aligned}$$

Hence,

$$\|\mathcal{L}_n^{(\Lambda)}\|_{(C)} \;=\; \|\, \mathcal{I}_n^{(\Lambda)}\,\|_{(C)}\ .$$

Applying the estimate for the norm of $\mathcal{I}_n^{(\Lambda)}$ (see part (iii) of Theorem 2.4) we obtain (3.36) for $p = +\infty$.

Step 4. By the Riesz-Thorin interpolation theorem (ii) in Subsection 1.4 we get

$$\|\, \mathcal{L}_n^{(\Lambda)}\,\|_{(p)} \;\le\; \|\, \mathcal{L}_n^{(\Lambda)}\|_{(1)}^{1/p} \cdot \|\, \mathcal{L}_n^{(\Lambda)}\|_{(C)}^{1-1/p}\ .$$

for $1 < p < +\infty$. This leads to the upper estimate in (3.36) for $1 < p < +\infty$ by Step 1 - Step 3. The proof is complete. □

Similarly to the case of the Fourier means and interpolation means Theorem 2.6 can be reformulated for the families $\{\mathcal{L}_\sigma^{(\varphi)}\}_\sigma$ generated by $\varphi \in \mathcal{K}$ in terms of the continuous parameter $\sigma \ge 0$ and the generator φ with obvious modifications. In particular, in view of Theorem 2.1 the convergence principle and the comparison principle read as follows (see [24] for the complete proof and further details).

Theorem 3.7. *Let $\varphi \in \mathcal{K}$ and assume $1 \in \mathcal{P}_\varphi$. Then the family $\{\mathcal{L}_\sigma^{(\varphi)}\}_\sigma$ converges in L_p, $0 < p \le +\infty$, if and only if $p \in \mathcal{P}_\varphi$.*

This theorem has many applications. In particular, it enables us to find the sharp ranges of convergence (range of convergence is by definition the set of $p \in (0, +\infty]$ for which the method converges in L_p) for families related to classical kernels of type (G) (see [24] and also Subsection 2.5 for details).

Combining part (iv) of Theorem 2.6, the Riesz-Thorin interpolation theorem and the ideas described in Subsection 2.1 one can obtain some efficient criteria for the convergence of families of linear polynomial operators related to kernels of type (GR). Following [33] we give precise statements.

Theorem 3.8. *Let Λ be of type (GR^q) for some $0 < q < 1$ and $\varphi \in \mathcal{K}$. Assume $1 \in \mathcal{P}_\varphi$ and $q \notin \mathcal{P}_\varphi$. Then the family $\{\mathcal{L}_n^{(\Lambda)}\}_n$ converges in L_p if and only if $p \in \mathcal{P}_\varphi$.*

Theorem 3.9. *Let $\Lambda = \Lambda(\varphi) + R' + R''$ with $\varphi \in \mathcal{K}$. Suppose $R' = \{r'_{n,k}\}$ satisfies $\mathcal{M}_{q;R'} < +\infty$ for some $0 < q < 1$ and let $R'' = \{r''_{n,k}\}$ be a matrix satisfying*

$$r''_{n,k} \;=\; \lambda_n \psi\left(\frac{k}{n}\right)\ ,\ n \in \mathbb{N}\,,$$

where $\psi \in \mathcal{K}$, $\widehat{\psi}(x) = O(|\,x\,|^{-\delta})$ for $x \to +\infty$, $\lambda_n = O(n^{\delta - d/q})$ and $d < \delta < d/q$. Assume $1 \in \mathcal{P}_\varphi$ and $q \notin \mathcal{P}_\varphi$. Then the family $\{\mathcal{L}_n^{(\Lambda)}\}_n$ converges in L_p if and only if $p \in \mathcal{P}_\varphi$.

Theorems 2.8 and 2.9 show that under some conditions the "remainder matrices" do not affect the range of convergence determined by the first item $\Lambda(\varphi)$ in (3.7). It turns out that the classical kernels of type (GR), in particular, the Cesaro kernels, satisfy the conditions of Theorem 2.9. This enables us to find their sharp ranges of convergence given in Subsection 2.5. For further details and for complete proofs of the statements above we refer to [33].

To complete the description of convergence results for various types of kernels we now deal with families generated by kernels of type (G^q) described in (3.13)-(3.14) (see also [33]).

Theorem 3.10. *Let $q \in \mathbb{N}$ and let $\varphi \in \mathcal{K}$ such that*

$$\int\limits_{\mathbb{R}^d} (\,\widehat{\varphi}(x))^q\, dx \,\neq\, 0\,.$$

Assume $1 \in \mathcal{P}_\varphi$. Then the family associated with the kernels $W_n(\varphi, q)$ defined in (3.13) *converges in L_p if and only if $p \in q^{-1}\,\mathcal{P}_\varphi$.*

3.4 Stochastic approximation

As we have seen in Theorem 2.1 and parts (iii), (v) of Theorem 2.6, the averaged approximation error of the linear polynomial operators $\{\mathcal{L}_\sigma^{(\varphi)}\}$ in L_p, $0 < p < +\infty$, with respect to the parameter λ can be estimated up to a constant multiple by the best approximation of order $\asymp n$ provided that the generator $\varphi \in \mathcal{K}$ satisfies the following additional conditions: $\varphi(\xi) = 1$ in a neighborhood of 0 and the Fourier transform $\widehat{\varphi}$ belongs to $L_{\widetilde{p}}(\mathbb{R}^d)$, where $\widetilde{p} = \min(1, p)$ (see Theorem 2.11 below). It was shown in [28] that the same quality of approximation can be achieved, if the parameter λ is randomly chosen. This result served as a theoretical background for the algorithm of stochastic approximation which reduces the problem of trigonometric approximation for all admissible parameters $0 < p \leq +\infty$ to the problem of interpolation (sampling) with randomly shifted nodes (see Theorem 2.12 below). For the description of all advantages of this method, for the corresponding computational procedure as well as for examples and complete proofs of the theoretical results we refer to [28]. In this subsection we formulate the main statements, the algorithm of stochastic approximation is based on, following [28]. Henceforth, the probability of an event A is denoted by $P(A)$.

Theorem 3.11. *Let $0 < p \leq +\infty$, $0 < \rho < 1$. Let $\varphi \in \mathcal{K}$ be real-valued with $r(\varphi) \leq 1$ and let the parameter N in* (3.27) *be equal to $[\sigma]$. Assume that $\widehat{\varphi}$ belongs to $L_{\widetilde{p}}(\mathbb{R}^d)$, where $\widetilde{p} = \min(1, p)$, and $\varphi(\xi) = 1$ if $|\xi| \leq \rho$. Then*

$$\|\, f - \,\mathcal{L}_\sigma^{(\varphi)}(f)\,\|_{\overline{p}} \,\leq\, c(d, p, \varphi)\, E_{\rho\sigma}(f)_p\,,\;\; f \in L_p\,,\;\; \sigma \geq 0\,,$$

where

$$c(d,\, p,\, \varphi) \;=\; \begin{cases} \;1 \,+\, (2\pi)^{-d}\, 3^{d(1-1/p)}\, \|\,\widehat{\varphi}\,\|_{L_1(\mathbb{R}^d)} & ,\;\; 1 \leq p \leq +\infty \\ \Big(1 \,+\, (2\pi)^{-d}\, 3^{d(1-p)}\, \|\,\widehat{\varphi}\,\|^p_{L_p(\mathbb{R}^d)}\Big)^{1/p} & ,\;\; 0 < p < 1 \end{cases}.$$

Theorem 3.12. *Let the conditions of Theorem 2.11 be satisfied. Let also $\gamma > 1$, $m \in \mathbb{N}$ and let η_j, $j = 1, \ldots, m$, be independent random vectors uniformly distributed on the unit cube $[0, 1]^d$. Then for $f \in L_p$ and $\sigma \geq 0$*

$$P\left\{ \min_{j=1,\ldots,m} \|\, f - \mathcal{L}^{\varphi}_{\sigma;\,\theta_j}(f)\,\|_p \;\leq\; \gamma\, c(d,\, p,\, \varphi)\, E_{\rho\sigma}(f)_p \right\} \;\geq\; 1 \,-\, \gamma^{-pm}\;,$$

where $\theta_j = \tau\eta_j$, $j = 1, \ldots, m$, and $\tau = 2\pi/(2[\sigma] + 1)$.

3.5 Methods generated by classical kernels

As already mentioned above, Theorems 2.7-2.10 show that in some cases the problem of convergence can be reduced to the problem of finding the set $\mathcal{P}_\varphi$ for the generator φ. It turns out that the classical kernels satisfy these conditions and the sets $\mathcal{P}_\varphi$ can be determined exactly. Hence, we are able to find the sharp ranges of convergence for the associated families of linear polynomial operators. The corresponding results will be established in form of the General Convergence Table (GCT) presented below.

Type	***d***	***Generator*** φ	***Range*** $\mathcal{P}_\varphi$
Fejér (G)	1	$(1-\lvert\xi\rvert)_+$	$(1/2,+\infty]$
Jackson (GR)	1	$3/2\,(1-\lvert\xi\rvert)_+ * (1-\lvert\xi\rvert)_+$	$(1/4,+\infty]$
Korovkin (GR)	1	$(1-\lvert\xi\rvert)\cos\pi\xi + (1/\pi)\sin\pi\lvert\xi\rvert$, $(\lvert\xi\rvert\le 1)$	$(1/4,+\infty]$
de la Vallée-Poussin (G)	1	$\begin{cases} 1, & \lvert\xi\rvert\le 1 \\ 2-\lvert\xi\rvert, & 1<\lvert\xi\rvert\le 2 \\ 0, & \lvert\xi\rvert>2 \end{cases}$	$(1/2,+\infty]$
Rogosinski (G)	1	$\cos\frac{\pi\xi}{2}$, $(\lvert\xi\rvert\le 1)$	$(1/2,+\infty]$
Bochner-Riesz (G)	≥ 1	$(1-\lvert\xi\rvert^2)^\alpha_+$, $(\alpha\ge 0)$	$\left(\frac{2d}{d+2\alpha+1},+\infty\right]$
Riesz (G)	≥ 1	$(1-\lvert\xi\rvert^\beta)^\alpha_+$, $\alpha>0$, $\beta=2k$ or $0<\alpha<\beta+(d-1)/2$, $\beta\ne 2k$	$\left(\frac{2d}{d+2\alpha+1},+\infty\right]$
Riesz (G)	≥ 1	$\alpha\ge\beta+(d-1)/2$, $\beta\ne 2k$	$\left(\frac{d}{d+\beta},+\infty\right]$
Cesaro (GR)	1	$(1-\lvert\xi\rvert)^\alpha_+$, $(\alpha>0)$	$(1/\min(2,\alpha+1),+\infty]$
Zygmund (G)	1	$(1-\lvert\xi\rvert^\beta)_+$, $(\beta>0)$	$(1/(1+\min(1,\beta)),+\infty]$

Table 1. General Convergence Table (GCT)

In this table a_+ denotes $\max(a,0)$. The symbol $f*g$ stands for the convolution of functions f and g defined on $\mathbb{R}^d$. The ranges of convergence of the families of linear polynomial operators generated by Fejér, de la Vallée-Poussin, Rogosinski and Bochner-Riesz kernels were obtained in [24], [32] applying Theorem 2.7 in combination with direct formulas for the Fourier transforms of the corresponding generators. Upper and lower estimates for the Fourier transform of the generators of the Riesz kernels, which enable us to find $\mathcal{P}_\varphi$ in this case, can be found in [34]. For representations of the Korovkin and Cesaro kernels in form (3.7) and for the results on the ranges can be derived from Theorems 2.9 and 2.10, respectively, we refer to [33]. Let us also mention that in view of parts (vi) of Theorems 2.4 and 2.6 all convergence results for families automatically imply the classical results on convergence of the corresponding Fourier means and the corresponding interpolation means in the spaces L_p, $1 \le p \le +\infty$, and in the space C ($p = +\infty$), respectively.

Acknowledgments

This research was supported by the DFG-project 436 UKR 113/103/0-1.

References

[1] J. Bergh, J. Löfström, *Interpolation spaces.* Springer, Berlin – Heidelberg, 1976.

[2] Z. Burinska, K. Runovski, H.-J. Schmeisser, *On the method of approximation by families of linear polynomial operators.* Z. Anal. Anw., 19, no. 3 (2000), 677 – 693.

[3] Z. Burinska, K. Runovski, H.-J. Schmeisser, *On the approximation by generalized sampling series in* L_p*-metrics.* Sampling Theory in Signal and Image Proc. (STSIP), 5, no. 1 (2006), 59 – 87.

[4] Z. Burinska, K. Runovski, H.-J. Schmeisser, *On quality of approximation by families of generalized sampling series.* Sampling Theory in Signal and Image Proc. (STSIP), 8, no. 2 (2009), 105 – 126.

[5] P. Butzer,W. Splettstösser, R. Stens, *The sampling theorem and linear prediction in signal analysis.* Jahresberichte der Dt. Math-Verein, 90 (1988), 1 – 70.

[6] P. Butzer, R. Nessel, *Fourier analysis and approximation.* Vol. 1. Academic Press, New York – London: 1971.

[7] W. Chen, Z. Ditzian, *Best approximation and* K*-functionals.* Acta Math. Hungar., 75, no. 3 (1997), 165 – 208.

[8] R. DeVore, G. Lorentz, *Constructive approximation.* Springer-Verlag, Berlin – Heidelberg, 1993.

[9] Z. Ditzian, *Measure of smoothness related to the Laplacian.* Trans. AMS, 326 (1991), 407 – 422.

[10] Z. Ditzian, *On Fejér and Bochner-Riesz means.* J. Fourier Analysis and Appl., 11 (2005), 489 – 496.

[11] Z. Ditzian, K. Ivanov, *Strong converse inequalities.* J. d'Analyse Math., 61 (1993), 61 – 111.

[12] Z. Ditzian, V. Hristov, K. Ivanov, *Moduli of smoothness and* K*-functionals in* L_p, $0 < p < 1$. Constr. Approx., 11 (1995), 67 – 83.

[13] Z. Ditzian, K. Runovski, *Averages and* K*-functionals related to the Laplacian.* J. Approx. Theory, 97 (1999), 113 – 139.

[14] Z. Ditzian, K. Runovski, *Realization and smoothness related to the Laplacian.* Acta Math. Hungar., 93 (2001), 189 – 223.

[15] H.G. Feichtinger, F. Weisz, *The Segal algebra* $S_0(\mathbb{R}^d)$ *and norm summability of Fourier series and Fourier transforms.* Monatsh. Math., 148 (2006), 333 – 349.

[16] L. Grafakos, *Classical Fourier analysis.* Springer, New York, 2008.

[17] V. Ivanov, *Direct and inverse theorems of approximation theory in the metrics* L_p *for* $0 < p < 1$. Mat. Zametki, 18, no. 5 (1975), 641 – 658 (in Russian).

[18] V. Ivanov, V. Yudin, *On trigonometric system in* L_p, $0 < p < 1$. Mat. Zametki, 28 (1980), 859 – 868 (in Russian).

[19] L. Hörmander, *The analysis of linear partial differential operators I.* Springer, Berlin, 1990.

[20] V. Hristov, K. Ivanov, *Realization of* K*-functionals on subsets and constrained approximation.* Math. Balkanica (New Series), 4 (1990), 236 – 257.

[21] P. Korovkin, *Linear operators and approximation theory.* Fizmatgiz, Moscow, 1959 (in Russian).

[22] M. Potapov, *Approximation "by angle"and embedding theorems*. Math. Balkanica, 2 (1972), 183 – 188.

[23] W. Rudin, *Functional analysis*. McGraw-Hill Inc., New York, 1991.

[24] V. Rukasov, K. Runovski, H.-J. Schmeisser, *On convergence of families of linear polynomial operators*. Functiones et Approx., 41 (2009), 41 – 54.

[25] V. Rukasov, K. Runovski, H.-J. Schmeisser, *Approximation by families of linear trigonometric polynomial operators and smoothness properties of functions*. Math. Nachr. (2011) (*to appear*).

[26] K. Runovski, *On families of linear polynomial operators in L_p-spaces,* $0 < p < 1$. Russian Acad. Sci. Sb. Math., 78 (1994), 165 – 173 (Translated from Ross. Akad. Nauk Matem. Sbornik, 184 (1993), 33 – 42).

[27] K. Runovski, *On approximation by families of linear polynomial operators in L_p – spaces,* $0 < p < 1$. Russian Acad. Sci. Sb. Math., 82 (1995), 441 -459 (Translated from Ross. Akad. Sci. Matem. Sbornik, 185 (1994), 81 – 102).

[28] K. Runovski, I. Rystsov, H.-J. Schmeisser: *Computational aspects of a method of stochastic approximation.* Z. Anal. Anw., 25 (2006), 367 – 383.

[29] K. Runovski, H.-J. Schmeisser, *On some extensions of Bernstein inequalities for trigonometric polynomials*. Functiones et Approx., 29 (2001), 125 – 142.

[30] K. Runovski, H.-J. Schmeisser, *Inequalities of Calderon-Zygmund type for trigonometric polynomials*. Georgian Math. J., 8, no. 1 (2001), 165 – 179.

[31] K. Runovski, H.-J. Schmeisser, *On the convergence of Fourier means and interpolation means.* Journal of Comp. Anal. and Appl., 6, no. 3 (2004), 211 – 220.

[32] K. Runovski, H.-J. Schmeisser, *On approximation methods generated by Bochner-Riesz kernels.* J. Fourier Anal. and Appl., 14 (2008), 16 – 38.

[33] K. Runovski, H.-J. Schmeisser, *On convergence of families of linear polynomial operators generated by matrices of multipliers.* Eurasian Math. J., 1, no. 3 (2010), 112 – 133.

[34] K. Runovski, H.-J. Schmeisser, *On families of linear polynomial operators generated by Riesz kernels*. Eurasian Math. J., 1, no. 4 (2010), 124 – 139.

[35] K. Runovski, H.-J. Schmeisser, *Smoothness and function spaces generated by homogeneous multipliers*. J. Function Spaces and Appl. (2011) (to appear).

[36] K. Runovski, H.-J. Schmeisser, *On K-Functionals generated by homogeneous functions.* Preprint, 2010.

[37] K. Runovski, H.-J. Schmeisser, *On polynomial K-functionals generated by homogeneous functions.* Preprint, 2011.

[38] K. Runovski, H.-J. Schmeisser, *On modulus of continuity related to Riesz derivative.* Jenaer Schriften zur Mathematik und Informatik, Math/Inf/01/11 (2011).

[39] H.-J. Schmeisser, H. Triebel, *Topics in Fourier analysis and function spaces*. John Wiley & Sons, Chichester, 1987.

[40] S.B. Stechkin, *On the order of approximation of continuous functions*. Izv. Akad. Nauk USSR, 15 (1951), 219 – 242.

[41] E.M. Stein, *Singular integrals and differentiability properties of functions.* Princeton Univ. Press, Princeton, 1970.

[42] E.M. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces.* Princeton Univ. Press, Princeton, 1971.

[43] A.I. Stepanets, *Methods of approximation theory I.* Proc. of Inst. of Math., Kiev, 2002.

[44] A.I. Stepanets, *Methods of approximation theory II.* Proc. of Inst. of Math., Kiev, 2002.

[45] E.A. Storozhenko, V.G. Krotov, P. Oswald, *Direct and inverse theorems of Jackson type in the spaces L_p,* $0 < p < 1$. Mat. Sbornik, 98, no. 3 (1975), 395 – 415 (in Russian).

[46] E.A. Storozhenko, P. Oswald, *Moduli of smoothness and best approximation in the spaces L_p,* $0 < p < 1$. Analysis Math., 3, no. 2 (1977), 141 – 150.

[47] E.A. Storozhenko, P. Oswald, *Jackson theorems in the spaces $L_p(\mathbb{R}^n)$,* $0 < p < 1$. Sib. Mat J., 19, no. 4 (1978), 888 – 901 (in Russian).

[48] S. Tikhonov, *Moduli of smoothness and the interrelation of some classes of functions.* In: Function Spaces, Interpolation Theory and Related Topics. Proc. of Int. Conf. in Honour of J. Peetre on his 65-th birthday, Lund, Sweden, August 17-22, 2000, 413 – 424. W. de Gruyter, Berlin – New York, 2002.

[49] R. Trigub, *Absolute convergence of Fourier integrals, summability of Fourier series and approximation by polynomial functions on the torus.* Izv. Akad. Nauk SSSR, Ser. Mat., 44 (1980), 1378 – 1409 (in Russian).

[50] R.M. Trigub, E.S. Belinsky, *Fourier analysis and approximation of functions.* Kluwer, Dordrecht, 2004.

[51] A. Zygmund, *Trigonometric series*, Vol. 1 & 2. Cambridge Univ. Press, Cambridge, 1990.

Konstantin Runovski
V.I. Vernadskiy Taurida National University
Vernadskogo Str., 4
95007 Simferopol, Ukraine
E-mail: k_runov@mail.ru

Hans-Jurgen Schmeisser
Mathematisches Institut
Friedrich-Schiller University
D-07737 Jena, Germany
E-mail: mhj@uni-jena.de

Received: 23.06.2011