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VANISHING OF THE BOCHNER CURVATURE TENSOR OF INDEFINITE ALMOST HERMITIAN MANIFOLDS

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Abstract. The aim of this paper is to discuss indefinite almost Hermitian manifold with the vanishing Bochner curvature tensor. Relations between the anti-holomorphic sectional curvature, the holomorphic sectional curvature and the Bochner curvature tensor have also been established.

1 Introduction

Bochner [3] obtained a modified version of Weyl's conformal curvature tensor for a Kaehler manifold presently known as the Bochner curvature tensor. Tachibana [6] also obtained analogous expression for the Bochner curvature tensor. Many geometers have studied vanishing of the Bochner curvature tensor and established necessary and sufficient conditions relating the sectional curvature, the anti-holomorphic sectional curvature and the holomorphic sectional curvature. The present paper extends these relations to the indefinite metric with the same condition of vanishing of Bochner curvature tensor.

2 Preliminaries

Let (M^n, g, J) be an indefinite almost Hermitian manifold of dimension $n(= 2m)$ with an almost complex structure J and an indefinite metric g such that

$$g(JX, JY) = g(X, Y),$$

where $X, Y \in \chi(M)$ and $\chi(M)$ is a set of all smooth vector fields on M. The metric g is known as degenerate if there exists a non-zero vector $X \in \chi(M)$ such that $g(X, Y) = 0$ for all Y and a vector field X is called a space-like, time-like or null if $g(X, X) > 0$, $g(X, X) < 0$, or $g(X, X) = 0$ respectively for $X \neq 0$. If ∇ is the Riemannian connection then the Riemannian curvature tensor $R(X, Y)Z$ is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and the sectional curvature $K(X,Y)$ for a 2-plane spanned by X and Y is defined as

$$K(X,Y) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where $R(X,Y,Z,W)$ is the Riemannian curvature tensor. Then the holomorphic sectional curvature $H(X)$ for a unit vector X is the sectional curvature $K(X,JX)$.

In [4], for an almost Hermitian manifold if Q, respectively S, denote the Ricci operator, respectively the scalar curvature, then the Bochner curvature tensor B of type (1, 3) on $X,Y,Z,W \in \chi(M)$ is given by

$$B(X,Y,Z,W) = R(X,Y,Z,W) + \frac{1}{n+4}U(X,Y,Z,W), \tag{1}$$

where

$$\begin{aligned} U(X,Y,Z,W) = {} & g(QX,Z)g(Y,W) - g(QY,Z)g(X,W) + g(QY,W)g(X,Z) \\ & - g(QX,W)g(Y,Z) + g(QJX,Z)g(JY,W) - g(QJY,Z)g(JX,W) \\ & + g(QJY,W)g(JX,Z) - g(QJX,W)g(JY,Z) + 2g(QJX,Y)g(JZ,W) \\ & + 2g(QJZ,W)g(JX,Y) - \tfrac{S}{n+2}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W) \\ & + g(JX,Z)g(JY,W) - g(JY,Z)g(JX,W) + 2g(JX,Y)g(JZ,W)\}. \end{aligned}$$

This gives

$$U(X,Y,X,Y) = -\{g(QX,X) + g(QY,Y) - \frac{S}{n+2}\}. \tag{2}$$

Moreover (1) is equivalent to

$$\begin{aligned} B(X,Y,Z,W) = {} & R(X,Y,Z,W) + L(X,W)g(Y,Z) - L(X,Z)g(Y,W) \\ & + L(Y,Z)g(X,W) - L(Y,W)g(X,Z) + L(JX,W)g(JY,Z) \\ & - L(JX,Z)g(JY,W) + L(JY,Z)g(JX,W) - L(JY,W)g(JX,Z) \\ & - 2L(JX,Y)g(JZ,W) - 2L(JZ,W)g(JX,Y), \end{aligned} \tag{3}$$

where

$$L(X,Y) = -\frac{1}{n+4}g(QX,Y) + \frac{S}{2(n+2)(n+4)}g(X,Y).$$

Vanhecke and Yano [7], generalized results in [4] and [5]. The aim of this paper is to generalize these results for an indefinite metric. Before we proceed further, we remark the following statement [2].

Lemma. *Let P be a semi curvature like tensor, that is, a tensor field of type (1, 3) such that*

(i) P(X, Y, Z, W) = - P(Y, X, Z, W).

(ii) P(X, Y, Z, W) = P(Z, W, X, Y).

(iii) Bianchi's first identity is satisfied.

Then $P = 0$ *if and only if* $P(X,Y,X,Y) = 0$ *for every base.*

3 Vanishing of the Bochner curvature tensor

Theorem. *Let (M^n, g, J) be an indefinite almost Hermitian manifold of dimension $n(=2m)$ satisfying the curvature identity, $R(X,Y,Z,W) = R(X,Y,JZ,JW)$. Then the following statements are equivalent:*

(i) $B = 0$.

(ii) $H(X) + H(Y) = \epsilon\ 8K(X,\ Y)$, $\epsilon = 1$, *(respectively - 1), when metric is definite, (respectively indefinite) and X and Y form an arbitrary antiholomorphic orthonormal pair.*

(iii) $K(X,\ Y) = K(X,\ JY)$, *where X and Y are as in (ii).*

(iv) $R(X,\ Y,\ Z,\ W) = 0$, *for any antiholomorphic 4-plane spanned by the orthogonal X, Y, Z and W.*

(v) For every orthonormal X, Y, Z, W spanning an antiholomorphic 4-plane $K(X,\ Y) + K\ (Z,W) = K(X,\ W) + K(Y,\ Z)$.

(vi) For each holomorphic 8-plane, $K(X,\ Y) + K\ (Z,\ W)$ *is independent of the orthonormal basis* $\{\ X,\ Y,\ Z,\ W,\ JX,\ JY,\ JZ,\ JW\ \}$.

Proof. We shall consider two different cases:

Case I: When $g(X,X) = g(Y,Y)$ and the proof of this case follows from [7].

Case II: When $g(X,X) = -g(Y,Y)$.
(i)⇒ (ii)

If $B = 0$, then (3) implies

$$\begin{aligned} R(X,Y,Z,W) = -[&L(X,W)g(Y,Z) - L(X,Z)g(Y,W) + L(Y,Z)g(X,W) \\ &- L(Y,W)g(X,Z) + L(JX,W)g(JY,Z) - L(JX,Z)g(JY,W) \\ &+ L(JY,Z)g(JX,W) - L(JY,W)g(JX,Z) - 2L(JX,Y)g(JZ,W) \\ &- 2L(JZ,W)g(JX,Y)]. \end{aligned} \tag{4}$$

Then the above yields

$$R(X,Y,X,Y) = -L(X,X) + L(Y,Y). \tag{5}$$

Since $L(X,X) = L(JX,JX)$, therefore (4) also yields

$$R(X,JX,X,JX) = H(X) = 8L(X,X), \tag{6}$$

and

$$R(Y,JY,Y,JY) = H(Y) = -8L(Y,Y). \tag{7}$$

Thus from (5), (6) and (7), we have

$$H(X) + H(Y) = -8K(X,Y). \tag{8}$$

(ii) ⇒ (i)

Using (8), for a local orthonormal frame field $\{E_i, JE_i\}_i^m$, we have

$$\sum_{j\neq i=1}^{m}[H(E_i)+H(E_j)] = -8\sum_{j}^{m} K(E_i, E_j).$$

This gives

$$H(E_i) = \frac{1}{m+2}[-4g(QE_i, E_i) - \sum_{j=1}^{m} H(E_j)]. \tag{9}$$

Taking summation over $i = 1, 2, ..., m$, we get

$$S = -(m+1)\sum_{j=1}^{m} H(E_j). \tag{10}$$

From (9) and (10), we get

$$H(E_i) = \frac{1}{n+4}[-8g(QE_i, E_i) + \frac{4S}{n+2}]$$

where $n = 2m$. Now using (8) in above, we obtain

$$R(E_i, E_j, E_i, E_j) = \frac{1}{n+4}\{g(QE_i, E_i) + g(QE_j, E_j) - \frac{S}{n+2}\}.$$

Thus using (1), (2) in above, we get

$$B(X, Y, X, Y) = 0$$

So, by Lemma in Section 2, the result follows.

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (ii)

For an arbitrary antiholomorphic orthonormal pair X and Y, we have

$$K(X, Y) = K(X, JY). \tag{11}$$

It is obvious that $(X+iY)/\sqrt{2}$ and $(iJX+JY)/\sqrt{2}$ span an antiholomorphic orthonormal pair, consequently from (11), we have

$$H(X) + H(Y) = -8K(X, Y).$$

(i) ⇒ (iv) is trivial.

(iv) ⇒ (i)

Suppose $R(X, Z, Y, W) = 0$ for X, Y, Z and W spanning an antiholomorphic 4-plane then replacing Z (respectively W) by $aZ + bW$ (respectively $bZ + aW$) such that $a^2 - b^2 = 1$ and $ab \neq 1$, we get

$$R(Z, X, Z, Y) = -R(W, X, W, Y).$$

Replacing Z by JZ in above, we get

$$R(Z, X, Z, Y) = R(JZ, X, JZ, Y).$$

Replacing X (respectively Y) by $aX + bY$ (respectively $bX + aY$) in above, such that $a^2 - b^2 = 1$ and $ab \neq 1$, we get

$$R(Z, X, Z, X) + R(Z, Y, Z, Y) = R(JZ, X, JZ, X) + R(JZ, Y, JZ, Y),$$

this gives

$$R(Z, Y, Z, Y) = R(JZ, Y, JZ, Y).$$

Again, replacing Z (respectively Y) by $aZ + bY$ (respectively $bZ + aY$) in above, such that $a^2 - b^2 = 1$ and $ab \neq 1$, we get

$$H(Z) + H(Y) = -8K(Z, Y)$$

then by using (ii) $\Rightarrow$ (i), we get the result.

(i) $\Rightarrow$ (v)

Since $B = 0$ implies

$$H(X) + H(Y) = -8K(X, Y),$$

then result follows immediately.

(v) $\Rightarrow$ (i)

Replacing Y (respectively Z) by $aY + ibZ$ (respectively $-ibY + aZ$) in

$$K(X, Y) + K(Z, W) = K(X, W) + K(Y, Z),$$

we obtain

$$R(X, Y, Y, W) = R(X, Z, Z, W).$$

Replacing Y (respectively Z) by $aY + ibZ$ (respectively $-ibY + aZ$) in above, we get

$$R(X, Z, Y, W) + R(X, Y, Z, W) = 0.$$

Using Bianchi's identity, we get $R(X, Y, Z, W) = 0$, then (iv)$\Rightarrow$(i), gives $B = 0$.

(i)$\Rightarrow$(vi)

Let $B = 0$, then for a J-basis, we have

$$R(X,Y,X,Y) = \frac{1}{n+4}\{g(QX,X) + g(QY,Y) - \frac{S}{n+2}\}.$$

Let $W_1 = \{E_1, E_2, E_3, E_4, \ldots, E_m, JE_1, JE_2, JE_3, JE_4, \ldots, JE_m\}$ and $W_2 = \{E'_1, E'_2, E'_3, E'_4, E_5, \ldots, E_m, JE'_1, JE'_2, JE'_3, JE'_4, JE_5, \ldots, JE_m\}$, be two basis of tangent space, then we have

$$K(E_1, E_2) + K(E_3, E_4) = \frac{1}{n+4}\sum_{i=1}^{4}[g(QE_i, E_i) - \frac{2S}{n+2}]. \tag{12}$$

Let $g(QE_i, E_i)$ and $g'(QE_i, E_i)$ be the components of the Ricci tensor with respect to the bases W_1 and W_2. Also

$$S = \sum g(QE_i, E_i) = \sum g'(QE_i, E_i),$$

and $g(QE_i, E_i) = g'(QE_i, E_i)$ for $i > 4$, thus we have

$$\sum_{i=1}^{4} g(QE_i, E_i) = \sum_{i=1}^{4} g'(QE_i, E_i),$$

then using (12), it is clear that $K(X,Y) + K(Z,W)$ is independent of an orthonormal basis.

(vi)⇒ (i)

In this case (vi)⇒ (v) is trivial. Thus the result follows from (v) ⇒ (i). □

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