EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 2, Number 3 (2011), 67 – 81

ON CONTINUITYOF THE SPECTRUM OF A SINGULAR QUASI-DIFFERENTIAL OPERATOR WITH RESPECT TO A PARAMETER

Kh.K. Ishkin

Communicated by E.D. Nursultanov

Key words: quadratic form of a quasi-differential operators, continuity of eigenvalues with respect to parameters.

AMS Mathematics Subject Classification: 47E05, 34L15, 34L16, 34L40, 34B40.

Abstract. We obtain sufficient conditions for continuity of the eigenvalues of semi-bounded quasi-differential operators of order $2n$ on the half-axis with respect to the parameters that appear in the corresponding differential expression. In addition we obtain a generalization of the well-known result of M.G. Krein [9] concerning description of the quadratic form of a regular quasi-differential operator in the singular case, when the deficiency indices of the minimal operator are equal to (n, n).

1 Introduction

Let Ω be an open set in $\mathbb{R}^k$ or $\mathbb{C}^k$, $L(\omega), \quad \omega \in \Omega,$ – a family of closed operators. It is well known (see, for example, [6, p. 213]) that if for a certain $z_0 \in \mathbb{C}$ the resolvent $(L(\omega) - z_0)^{-1}$ exists and is continuous in a neighborhood of $\omega_0 \in \Omega$, then the spectrum of the operator $L(\omega)$ is continuous near ω_0 in the following sense: if λ_0 is an isolated eigenvalue of multiplicity m of the operator $L(\omega_0)$, then there exist m functions $\lambda_1(\omega), \dots, \lambda_m(\omega)$ continuous at ω_0 such that these functions are the only points of the spectrum of the operator $L(\omega)$ near ω_0. There are known sufficient conditions for generalized resolvent continuity [13, p. 287] of a family of operators. However, in a given situation these conditions might happen to be either inefficient or hard to verify. At the same time, the issue of continuity and differentiability of eigenvalues and eigenfunctions gains special relevance due to emergence of various software packages [1, 12, 5, 2] for their approximate computation. This question has been resolved in the most comprehensive form in [7] for a regular[1] Sturm – Liouville operator. Later in [8] these results were extended to regular ordinary differential operators of any order. A similar problem for operators with partial derivatives (also in the regular case) has been studied in the paper [4]. In the paper [10], a result concerning smooth dependence of a simple isolated eigenvalue of an arbitrary Fredholm operator on a Banach space was obtained. There were also given many applications of the derived results for various operators.

[1]In accordance with [11], a differential operator will be called regular, if the corresponding differential expression is defined in a bounded interval, and its coefficients are summable on this interval.

In the paper we study one natural class of differential operators, where the standard technique of [10] is not applicable (for more details see Example 2).

Consider the quasidifferential expression depending on k parameters:

$$\mathcal{L}(y) = (-1)^n y^{(2n)} + (-1)^{n-1} \left(p_1 y^{(n-1)}\right)^{(n-1)} + \ldots + p_n y, \quad x > 0, \tag{1.1}$$

$p_i = p_i(x,\omega), \quad \omega \in \Omega, \quad \Omega$ is an open set in $\mathbb{R}^k$.

We assume that for all $\omega \in \Omega$ the following holds:

1) the functions $p_i(x,\omega), \quad i = 1,\ldots,n$, are real-valued and locally summable in $[0,\infty)$;

2) the deficiency indices of the minimal operator $L_0(\omega)$ generated in $L^2(0,\infty)$ by the expression $\mathcal{L}(\omega)(y)$ are equal to (n,n).

Then (for details see [11, p. 213]) any self-adjoint extension of the operator $L_0(\omega)$ can be defined in terms of boundary conditions:

$$\begin{aligned} D(L(A(\omega),\omega)) &= \{y \in L^2(0,\infty) : \quad y^{[k]} \quad (k = \overline{0, 2n-1}) \text{ are absolutely continuous}, \\ &\qquad \mathcal{L}(\omega)(y) \in L^2(0,\infty), \quad A(\omega) Y_{2n}(0) = 0\}, & (1.2) \\ L(A(\omega),\omega) y &= \mathcal{L}(\omega)(y), \quad y \in D(L(A(\omega),\omega)), & (1.3) \end{aligned}$$

where $Y_{2n} = (y, y^{[1]}, \ldots, y^{[2n-1]})^T, \quad y^{[k]}$ is a k-th quasi-derivative (see [11, p. 182]) defined by the formula

$$y^{[k]} = \begin{cases} \dfrac{d^k y}{dx^k}, & k = 1,\ldots,n; \\ p_{k-n} \dfrac{d^{2n-k} y}{dx^{2n-k}} - \dfrac{d}{dx}\left(y^{[k-1]}\right), & k = n+1,\ldots,2n. \end{cases} \tag{1.4}$$

Here the matrix $A(\omega)$ satisfies the requirements for boundary conditions to be self-adjoint:

$$A = (A_1,\ A_2), \tag{1.5}$$

where $A_1 = A_1(\omega),\ A_2 = A_2(\omega)$ are square matrices of order n such that $\operatorname{rank} A = n$,

$$A_1 J_n A_2^* = A_2 J_n A_1^*, \tag{1.6}$$

$$J_n = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \ldots & 0 & 0 \end{pmatrix}, \qquad n \geq 2,\ J_1 = 1. \tag{1.7}$$

We denote by $L(\omega)$ the operator $L(A,\omega)$, where A is a fixed matrix *independent* of $\omega \in \Omega$. Let $\lambda_1(\omega), \lambda_2(\omega), \ldots$ be the eigenvalues of $L(\omega)$ numbered in non-decreasing order of magnitude and repeated as many times as their multiplicities.[2]

The aim of this paper is to obtain sufficient conditions for continuity of any finite collection of eigenvalues of $L(\omega)$. Our main results are follows.

---

[2]It follows from condition 4) of Theorem 1 that $L(\omega)$ is semi-bounded from below and for some $a > 0$ and all $\omega \in \Omega$ the resolvent $(L(\omega) + a)^{-1}$ is compact so that the spectrum has a unique limit point $+\infty$.

Theorem 1. *Let $\omega_0 \in \Omega$ and, in addition to the conditions 1) – 2), the following conditions hold:*

3) for any $\omega \in \Omega$ the functions $p_1(x, \omega), \ldots, p_n(x, \omega)$ are bounded from below on $[a, \infty)$ for some $a > 0$;

4) for all $0 < b < \infty$

$$\int_0^b \mid p_i(x, \omega) - p_i(x, \omega_0) \mid \, dx \to 0, \quad \omega \to \omega_0, \quad i = \overline{1, n};$$

5) the operator $L(\delta, \omega)$, obtained from $L(\omega)$ by replacing the coefficients $p_i(x, \omega)$ by

$$p_i(x, \delta, \omega) \equiv \inf_{|\omega' - \omega| < \delta} p_i(x, \omega'), \quad i = \overline{1, n},$$

with some $\delta > 0$, has compact resolvent.

Then the eigenvalues $\lambda_k(\omega)$ are continuous at ω_0.

Theorem 2. *Let the eigenvalue λ_0 of the operator $L(\omega_0)$ have multiplicity k and let $\lambda_i(\omega)(i = \overline{1, k})$ be the eigenvalues of $L(\omega)$ such that $\lambda_i(\omega) \to \lambda_0$, $\omega \to \omega_0$, and $\varphi_i(x, \omega)$ be corresponding normalized eigenfunctions. Furthermore, let P_0 be the projector onto the eigenspace corresponding to λ_0.*

Then, if the conditions of Theorem 1 are satisfied, we have

$$\|\varphi_i(\cdot, \omega) - (P_0 \varphi_i)(\cdot, \omega)\| \to 0, \quad \omega \to \omega_0 \tag{1.8}$$

for any $i = \overline{1, k}$, where $\| \cdot \|$ denotes the norm in $L^2(0, \infty)$.

If λ_0 is a simple eigenvalue, then $\varphi(x, \omega)$ can be chosen in such a way that

$$\|\varphi(\cdot, \omega) - \varphi(\cdot, \omega_0)\|_{n-1} \to 0, \quad \omega \to \omega_0,$$

where

$$\|\varphi\|_s = \sup_{x \geq 0} \left(\sum_{\nu=0}^{s} \left| \varphi^{(\nu)}(x) \right| \right). \tag{1.9}$$

The proofs of these theorems are based on application of the well-known min-max principle (see, for instance, [14, p. 78]), for which there is an explicit description of the quadratic form of the operator $L(\omega)$ (Lemmas 2, 4). It turns out that the domain of the quadratic form of the operator $L(\omega)$ is defined by the maximal set of linear independent boundary conditions $AY_{2n}(0) = 0$, which do not contain derivatives of orders higher than $n - 1$. These conditions, according to the regular case [9], will be called *main*. The other boundary conditions (more precisely: the matrix consisting of the coefficients of these conditions) appear in the formula for the quadratic form. Section 1 is devoted to proofs of these statements. In Sections 2 and 3 we give proofs of Theorems 1, 2. In Section 4 we give several examples and build a counterexample showing that condition 5) is essential.

2 Sesquilinear forms and associated operators

It is known from [6, p. 308] that there is a one-to-one correspondence between closed sectorial sesquilinear forms and m-sectorial operators. In this section we will give an explicit description of this correspondence when the form is generated by the differential expression (1.1). We assume till the end of the proof of Lemma 7 that a condition more restrictive than condition 3) is satisfied, namely:

3′) for all $\omega\in\Omega$ the functions $p_1(x,\omega),\dots,p_n(x,\omega)$ are nonnegative on $[a,\infty)$ for some $a>0$.

We denote by $D_l(\omega)$ the set of all functions $y\in L^2(0,\infty)$ satisfying the following conditions:

a) $y,y',\dots,y^{(n-1)}$ are absolutely continuous on $(0,\infty)$;

b) $y^{(n)}\in L^2(0,\infty),\quad p_1^{1/2}(\cdot,\omega)y^{(n-1)},\dots,p_n^{1/2}(\cdot,\omega)y\in L^2(a,\infty)$,

c) $CY_n(0)=0,$ where C is a fixed matrix of rank $0\le r\le n$ independent of ω and $Y_n(0)=(y(0),\dots,y^{(n-1)}(0))^T$.

We define the following sesquilinear form on $D_l(\omega)\times D_l(\omega)$:

$$\begin{aligned} l(\omega)[y,z] &= \int_0^\infty \big(y^{(n)}(x)\overline{z}^{(n)}(x)+p_1(x,\omega)y^{(n-1)}(x)\overline{z}^{(n-1)}(x)+\dots+ \\ &+ p_n(x,\omega)y(x)\overline{z}(x)\big)\,dx-\langle A_0Y_n(0),Z_n(0)\rangle, \end{aligned}$$

where A_0 is a self-adjoint matrix of order n independent of ω, $\langle\cdot,\cdot\rangle$ denotes the scalar product in the n-dimensional Euclidean space. Next, for brevity, we denote the quadratic form $l(\omega)[y,y]$ by $l(\omega)[y]$ or $l(\omega)$.

Lemma 1. *If conditions 1) and 3′) are satisfied, then for any $\omega\in\Omega$ the quadratic form $l(\omega)$ is bounded from below and closed in $D_l(\omega)$.*

Proof. According to Sobolev's inequalities (see, for example,[3, pp. 129, 142]), we have

$$\sup_{x\in[0,a]}|u(x)| \le M\varepsilon^{-1}\|u\|_{L^2(0,a)}+\varepsilon\|u'\|_{L^2(0,a)},\ u\in W_2^{(1)}(0,a), \tag{2.1}$$

$$\|u^{(k)}\|_{L^2(0,a)} \le C_k(\delta)\|u\|_{L^2(0,a)}+\delta\|u^{(n)}\|_{L^2(0,a)},\ u\in W_2^{(n)}(0,a),\ k=\overline{1,n-1}, \tag{2.2}$$

where $W_2^{(n)}(0,a)$ is the Sobolev space, constants ε,δ can be arbitrarily small, constants $M,C_k(\delta)$ are independent of u. Combining (2.1) for $u=y^{(k)}$ with (2.2) for $u=y$ and $\delta=\varepsilon^2$, we obtain

$$\sup_{x\in[0,a]}|y^{(k)}(x)|\le M_k(\varepsilon)\|y\|_{L^2(0,a)}+N_k(\varepsilon)\varepsilon\|y^{(n)}\|_{L^2(0,a)},\quad y\in W_2^{(n)}(0,a),\quad k=\overline{1,n-1}, \tag{2.3}$$

where

$$M_k(\varepsilon)=\begin{cases} MC_k(\varepsilon^2)\varepsilon^{-1}, & k=\overline{1,n-2},\\ MC_{n-1}(\varepsilon^2), & k=n-1;\end{cases}\quad N_k(\varepsilon)=\begin{cases} M+\varepsilon, & k=\overline{1,n-2},\\ 1, & k=n-1.\end{cases}$$

Consider the quadratic form

$$l_0(\omega)[y]=\int_0^a\left(\sum_1^{n-1}p_k(x,\omega)\mid y^{(n-k)}\mid^2\right)dx-\langle A_0Y_n(0).Y_n(0)\rangle,\quad y\in W_2^{(n)}(0,a),$$

From inequality (2.3) we obtain the inequality

$$|l_0(\omega)[y]| \leq K_1(\omega,\varepsilon)\|y\|^2_{L^2(0,a)} + K_2(\omega)\varepsilon^2\|y^{(n)}\|^2_{L^2(0,a)}, \quad y \in W_2^{(n)}(0,a), \quad \omega \in \Omega, \quad (2.4)$$

which holds for all sufficiently small positive ε, where $K_1(\omega,\varepsilon), K_2(\omega)$ are independent of y. It follows from this and condition 3′) that

$$l(\omega)[y] \geq (1 - K_2(\omega)\varepsilon^2)\|y^{(n)}\|^2 - K_1(\omega,\varepsilon)\|y\|^2, \quad y \in D_l(\omega). \quad (2.5)$$

Therefore, the form $l(\omega)$ is semi-bounded.

Now we prove that for all $\omega \in \Omega$ the form $l(\omega)$ is closed. Let $\omega \in \Omega$ and $\{y_k\}$ be a sequence in $D_l(\omega)$ such that $||y_k - y_m|| \to 0,\ l(\omega)[y_k - y_m] \to 0, \quad k,m \to \infty$. Using (2.5), we get

$$\|y^{(n)}\|^2 \leq (1 - K_2(\omega)\varepsilon^2)^{-1}\left(l(\omega)[y] + K_1(\omega,\varepsilon)\|y\|^2\right), \quad y \in W_2^{(n)}(0,\infty),$$

where ε is a positive constant such that $\varepsilon < K_2^{-1/2}(\omega)$, therefore, the sequence $\{y_k\}$ converges in $W_2^{(n)}(0,\infty)$ to a certain function $y \in W_2^{(n)}(0,\infty)$. Hence, the limit function y satisfies conditions a), c) and $y \in L^2(0,\infty)$. So we only need to show that

$$p_1^{1/2}(\cdot,\omega)y^{(n-1)}, \ldots, p_n^{1/2}(\cdot,\omega)y \in L^2(a,\infty). \quad (2.6)$$

Since the sequence $\{l(\omega)[y_k]\}$ is convergent, it is bounded and, therefore,

$$\int_a^\infty \left(\sum_{i=1}^n p_i(x,\omega)\left|y_k^{(n-i)}(x)\right|^2\right) dx \leq C(\omega),$$

where $C(\omega) > 0$ is independent of k. Passing to the limit in this inequality as $k \to \infty$, we obtain (2.6). □

Remark 1. Lemma 1 allows us to assume without loss of generality that for any $\omega \in \Omega$ the quadratic form $l(\omega)$ is positive.

Remark 2. Suppose that in addition to the conditions of Lemma 1, condition 4) of Theorem 1 holds. Then it follows from the proof of inequality (2.4) that for any compact $\Delta \subset \Omega$ and any $\varepsilon > 0$ there are some constants $K_1(\Delta,\varepsilon) > 0$ and $K_2(\Delta) > 0$ such that for all $\omega \in \Delta$ the following inequality holds:

$$\|y^{(n)}\|^2 \leq (1 - K_2(\Delta)\varepsilon)^{-1}\left(l(\omega)[y] + K_1(\Delta,\varepsilon)\|y\|^2\right), \quad y \in W_2^{(n)}(0,\infty). \quad (2.7)$$

Let $L(\omega)$ be the self-adjoint operator associated with the form $l(\omega)$ [6, p. 308].

Lemma 2. *If conditions 1) – 2) and 3') are satisfied, then* $\widetilde{L}(\omega) = L(\omega)$, *where*

$$\mathcal{A} = \begin{pmatrix} C & 0 \\ B^*A_0 & B^*J_n \end{pmatrix}, \quad (2.8)$$

B is a $n \times (n-r)$ matrix, the columns of which form a basis in the space of solutions of the system $CX = 0$, and J_n is defined by (1.7).

If $y \in D(\widetilde{L}(\omega))$, then $\widetilde{L}(\omega)y = \mathcal{L}(\omega)(y)$.

Proof. Let $y \in D(\widetilde{L}(\omega))$ and $\widetilde{L}(\omega)y = g$. We will show that y is absolutely continuous on $[0,\infty)$ together with quasi-derivatives of orders up to and including $2n-1$, and that

$$\mathcal{A}Y_{2n}(0) = 0. \tag{2.9}$$

We will also show that $\mathcal{L}(\omega)(y) \in L^2(0,\infty)$ and $g = \mathcal{L}(\omega)(y)$.

Taking into account the representation theorem in [6, p. 322], we obtain

$$(g, z) = (\widetilde{L}(\omega)y, z) = l(\omega)[y, z], \qquad \forall z \in D_l(\omega). \tag{2.10}$$

Let $D_l'(\omega) = D_l(\omega) \bigcap C_0[0,\infty)$ and $z \in D_l'(\omega)$. Then the equalities in (2.10) show that, assuming $p_0 \equiv 1$,

$$\int_0^\infty g\,\overline{z}\,dt = \int_0^\infty \sum_0^n p_k\, y^{(n-k)}\,\overline{z}^{(n-k)}\,dt - \langle A_0 Y_n(0),\ Z_n(0)\rangle. \tag{2.11}$$

We denote by u_1 an antiderivative of $-g + p_n y$ so that

$$u_1' = p_n y - g. \tag{2.12}$$

Therefore,

$$\int_0^\infty (g - p_n\, y)\,\overline{z}\,dt = -\int_0^b u_1'\,\overline{z}\,dt = u_1(0)\,\overline{z}(0) + \int_0^b u_1\,\overline{z}\,'\,dt.$$

Here we integrate over the interval $[0,b]$ such that the support of the function $z \in D_l'$ is contained in $[0,b]$. We obtain from (2.11) that

$$\int_0^\infty \left(\sum_0^{n-2} p_k\, y^{(n-k)}\,\overline{z}^{\,(n-k)} + (p_{n-1}\, y\,' - u_1)\,\overline{z}\,' \right) dt - \langle A_0\, Y_n(0),\ Z_n(0)\rangle - u_1(0)\,\overline{z}(0) = 0. \tag{2.13}$$

Next, we denote by u_k antiderivatives of $p_{n-k+1}\, y^{(k-1)} - u_{k-1}$ so that

$$u_k' = p_{n-k+1}\, y^{(k-1)} - u_{k-1}, \qquad\qquad k = \overline{2, n}. \tag{2.14}$$

Then, iterating the previous equality, we obtain

$$\int_0^\infty (y^{(n)} - u_n)\,\overline{z}^{\,(n)}\,dt - \langle A_0\, Y_n(0),\ Z_n(0)\rangle - \sum_{k=1}^n u_k(0)\,\overline{z}^{\,(k-1)}(0) = 0. \tag{2.15}$$

We fix $M > 0$ and consider the set $\triangle_M = \{\, y : y, y\,', \ldots, y^{\,(n-1)}$ are absolutely continuous on $[0,\infty)$, $\quad y^{(n)} \in L^2(0,\infty)$, $\quad y(0) = y\,'(0) = \ldots = y^{(n-1)}(0) = 0$, $\quad y(x) = 0, \forall x \geq M\}$. It is obvious that equality (2.15) holds $\forall z \in \triangle_M$. Moreover, for these z we have

$$\int_0^M (y^{(n)} - u_n)\,\overline{z}^{\,(n)}\,dt = 0,$$

so that the function $y^{(n)} - u_n$ is orthogonal to $\overline{z}^{\,(n)}$ in $L^2(0,M)$. Therefore, the function $y^{(n)} - u_n$ as an element of $L^2(0,M)$ is orthogonal to all functions that are orthogonal to polynomials of degree $\le n-1$. Thus $y^{(n)} - u_n$ coincides with a polynomial of degree not higher than $n-1$ almost everywhere on $[0, M]$:

$$y^{(n)} - u_n = \sum_{0}^{n-1} \frac{c_k\, t^k}{k!}. \tag{2.16}$$

Since M is arbitrary then equality (2.16) holds on the entire interval $[0, \infty)$. It follows from this and equalities (2.12) and (2.14) that $y^{(n)}, y^{[n+1]}, \dots, y^{[2n-1]}$ are absolutely continuous on $[0, \infty)$, $\mathcal{L}(\omega) y = g$, and

$$u_{n-k}(0) = y^{[n+k]}(0) + (-1)^{k+1} c_k, \qquad k = \overline{0, n-1}. \tag{2.17}$$

Then from (2.15) and taking into account (2.16) and (2.17) we obtain

$$\langle (A_0,\ J_n)\, Y_{2n}(0),\ Z_n(0)\rangle = 0 \tag{2.18}$$

for all $z \in D_l'(\omega)$. However, z satisfies condition c) and thus $Z_n(0) = B\, C_0$, where $C_0 = (c_1, c_2, \dots, c_n)^T$. Then using (2.18), we get

$$\langle B^*(A_0,\ J_n) Y_{2n}(0),\ C_0\rangle = 0.$$

As z varies in $D_l'(\omega)$, C_0 takes all possible values in $\mathbb{C}^n$. This proves (2.9). Thus we have shown that $\widetilde{L}(\omega) \subset L(\omega)$.

Conversely, if $y \in D(L(\omega))$, then $y \in D(\widetilde{L}(\omega))$ and $\widetilde{L}(\omega) y = \mathcal{L}(\omega)(y)$. Indeed, integration by parts shows that $l[y, z] = (\mathcal{L} y,\ z)$ for all $z \in D_l'(\omega)$. However, $D_l'(\omega)$ is the essential domain of the quadratic form $l(\omega)$, and, therefore, the statement follows directly from the representation theorem. □

Lemma 3. *Let* $A = (A_1, A_2)$, *where* A_1, A_2 *are square matrices of order* n *such that* rank $A = n$, $A_1 J_n A_2^* = A_2 J_n A_1^*$. *Then the system of equations* $AX = 0$ *is equivalent to the system* $\mathcal{A} X = 0$, *where* $\mathcal{A}$ *has the form* (2.8).

Proof. Let $\operatorname{rank} A_2 = n - r$. Then we can assume that

$$A = \begin{pmatrix} C & 0 \\ B_1 & B_2 J_n \end{pmatrix},$$

where C is a $r \times n$ matrix, $\operatorname{rank} C = r$, $\operatorname{rank} B_2 = n - r$. It follows from self-adjointness of $A_1 J_n A_2^*$ that

$$CB_2^* = 0, \tag{2.19}$$
$$B_1 B_2^* = B_2 B_1^*. \tag{2.20}$$

We can see from the first equation that the columns of matrix B_2^* solve the system $CX = 0$. Because $\operatorname{rank} B_2^* = n - r$ then, first of all, the columns of matrix B_2^* form a basis in the space of solutions of the given system, and, secondly, there is a matrix $\widetilde{A}$ such that $B_1 = B_2 \widetilde{A}$. Then from (2.20) we will obtain that $B_2(\widetilde{A} - \widetilde{A}^*) B_2^* = 0$. Combining this with (2.19) we get that there exists a matrix M such that $B_2(\widetilde{A} - \widetilde{A}^*) = MC$. Therefore, $B_2 \widetilde{A} - \frac{1}{2} MC = B_2 A_0$, where $A_0 = \frac{1}{2}(\widetilde{A} + \widetilde{A}^*)$. □

Remark 3. The proven lemma asserts that by doing algebraic transformations we can single out those from self-adjoint boundary conditions that do not contain derivatives of orders higher than $(n-1)$. They appear right in the domain of the quadratic form associated with the operator $L(\omega)$. Following [9], we will call them the **main** boundary conditions.

Lemma 4. *If $\omega \in \Omega$ and the conditions 1) – 2), 3') are satisfied, then for all $y \in D(L(\omega))$:*

(i) $y^{(n)},\, p_1^{1/2}(\cdot,\omega)y^{(n-1)},\ldots,p_n^{1/2}(\cdot,\omega)y \in L^2(0,\infty)$;

(ii) *for all z from the domain of the quadratic form of the operator $L(\omega)$*

$$[y,z] \equiv \lim_{x\to\infty}\sum_0^n y^{[2n-k]}(x)\,\overline{z}^{[k-1]}(x) = 0.$$

Proof. Let $\omega \in \Omega$ and $y \in D(L(\omega))$. It follows by Lemma 3 that the system $AY_{2n}(0) = 0$ satisfying condition (1.6) can be transformed to an equivalent system with a matrix $\mathcal{A}$ having the form (2.8). Then it follows from Lemma 2 that $L(\omega) = \widetilde{L}(\omega)$. Combining this with the inclusion $D(L(\omega)) \subset D_l(\omega)$, we get (i).

Suppose that $y \in D(L(\omega)),\ z \in D_l(\omega)$. Then

$$\begin{aligned}(L(\omega)y,z) &= \lim_{b\to\infty}\int_0^b \mathcal{L}(\omega)y\,\overline{z}\,dt = \lim_{b\to\infty}\Bigg\{\int_0^b\left(\sum_0^n p_k\,y^{(n-k)}\overline{z}^{\,(n-k)}\right)dt - \\ &- \langle A_0Y_n(0),\ Z_n(0)\rangle + \sum_{k=1}^n y^{[2n-k]}(b)\overline{z}^{(k-1)}(b)\Bigg\}.\end{aligned}$$

On the other hand, taking into account the representation theorem, we have

$$\begin{aligned}(L(\omega)y,z) &= l(\omega)[y,z] = \int_0^\infty\left(\sum_0^n p_k\,y^{(n-k)}\overline{z}^{\,(n-k)}\right)dt - \\ &- \langle A_0Y_n(0),\ Z_n(0)\rangle, \quad y\in D(L(\omega)),\ z\in D_l(\omega).\end{aligned}$$

This implies (ii). □

Lemma 5. *Let $D'(L(\omega))$ be the set of all functions from $D(L(\omega))$ which have compact support in $[0,\infty)$. Then $D'(L(\omega))$ is the core for $l(\omega)$, i.e. for all $v \in D_l(\omega)$ and $\varepsilon > 0$ there exists a $y \in D(L(\omega))$ such that $\|v-y\| + l(\omega)[v-y] < \varepsilon$.*

Proof. We set (see Remark 1)

$$(u,\,v)_{+1}(\omega) = l(\omega)[u,\,v] + (u,\,v), \quad \|u\|_{+1}(\omega) = \sqrt{(u,u)_{+1}(\omega)}, \quad u,v\in D_l(\omega). \quad (2.21)$$

It is known [13, p. 278] that $(\cdot,\cdot)_{+1}(\omega)$ and $\|\cdot\|_{+1}(\omega)$ are an inner product and a norm in $D_l(\omega)$ respectively. Then the statement to prove means that $D'_L(\omega)$ is dense in $D_l(\omega)$ in the norm $\|\cdot\|_{+1}(\omega)$.

Let $y \in D_l(\omega)$ and $(y, \varphi)_{+1}(\omega) = 0$ for all $\varphi \in D'_L(\omega)$. We will prove that $y = 0$. Fix $b > 0$ and introduce the operator $L_{0b}(\omega)$ generated in $L^2(0, b)$ by the differential expression $\mathcal{L}(\omega)(y)$ and the boundary conditions $\mathcal{A}Y_{2n}(0) = 0$, $Y_{2n}(b) = 0$. It is easy to verify that the operator $L_{0b}(\omega)$ is symmetric and $L^*_{0b}(\omega)$ is the operator generated in $L^2(0, b)$ by the expression $\mathcal{L}(\omega)(y)$ and the boundary conditions $\mathcal{A}Y_{2n}(0) = 0$. The set $D(L_{0b}(\omega))$ can be embedded in $D'_L(\omega)$ assuming that the function $\varphi \in D(L_{0b}(\omega))$ vanishes outside $[0, b]$. Therefore, $(y, \varphi)_{+1}(\omega) = 0$ for all $\varphi \in D(L_{0b}(\omega))$. Yet for these φ we have

$$(y, \varphi)_{+1}(\omega) = (y, L_{0b}(\omega)\varphi) + (y, \varphi) = (y, (L_{0b}(\omega) + 1)\varphi).$$

Hence, $y \perp \mathrm{Ran}(L_{0b}(\omega) + 1)$, which is equivalent to $y \in \mathrm{Ker}(L^*_{0b}(\omega) + 1)$. So

$$\mathcal{L}(\omega)(y) = -y, \quad 0 \le x \le b;$$

$$AY_{2n}(0) = 0.$$

Suppose that $y \neq 0$. Since $b > 0$ is arbitrary it follows that y is an eigenfunction of the operator $L(\omega)$ corresponding to the eigenvalue -1, which is impossible (see Remark 1). □

3 Proof of Theorem 1

We can now proceed to a direct proof of Theorem 1. Assume that conditions 1), 2), 3′) are satisfied. Then for all $\omega \in \Omega$ the minimal operator $L_0(\omega)$ is bounded from below and has deficiency indices equal to (n, n). Denote by $d(\omega)$ the bottom of the essential spectrum of $L(\omega) = L(\mathcal{A}, \omega)$, where $\mathcal{A}$ is of the form (2.8). (If spectrum $L(\omega)$ is discrete, then we set $d(\omega) = +\infty$.)

Lemma 6. *Suppose that conditions 1), 2), 3'), 4) are satisfied and the operator $L(\omega_0)$ has k $(1 \le k \le \infty)$ eigenvalues located to the left from*

$$d_-(\omega_0) \equiv \varliminf_{\omega \to \omega_0} d(\omega).$$

Then there exists a neighborhood of ω_0 $V_k \subset \Omega$ such that for $\omega \in V_k$ the operator $L(\omega)$ has at least k eigenvalues below $d(w)$ that satisfy the inequalities

$$\varlimsup_{\omega \to \omega_0} \lambda_j(\omega) \le \lambda_j(\omega_0), \quad j = 1, \dots, k. \tag{3.1}$$

Proof. Let $\sigma_k = (d_-(\omega_0) - \lambda_k(\omega_0))/3$ and $U_k = \{\omega \in \Omega : \quad d(\omega) > d_-(\omega_0) - \sigma\}$. Furthermore, let $f_1(\omega_0), \dots, f_k(\omega_0)$ be normalized eigenfunctions of the operator $L(\omega_0)$ corresponding to the eigenvalues $\lambda_1(\omega_0), \dots, \lambda_k(\omega_0)$. It follows from Lemma 5 that for any $\varepsilon > 0$ there exist functions $\varphi_1(x, \varepsilon), \dots, \varphi_k(x, \varepsilon)$ in $D'(L(\omega_0))$ such that $\|\varphi_j\| = 1$ and

$$\begin{aligned} \|\varphi_j(\cdot, \varepsilon) - f_j(\omega_0)\| &< \varepsilon, \\ l(\omega_0)[\varphi_j(\cdot, \varepsilon)] &\le \lambda_j(\omega_0) + \varepsilon. \end{aligned} \tag{3.2}$$

Since $\varphi_j(\cdot,\varepsilon) \in D'(L(\omega_0))$, then $\varphi_j(x,\varepsilon) \equiv 0, \quad x \geq b, \quad j = \overline{1,k}$, for some $b > 0$. Therefore, according to (3.2) and condition 4) of Theorem 1, there exists a constant $\delta(\varepsilon) > 0$ such that for all $\omega \in \Omega \cap \{|\omega - \omega_0| < \delta(\varepsilon)\}$ we have

$$l(\omega)[\varphi_j(\cdot,\varepsilon)] < \lambda_j(\omega_0) + 2\varepsilon, \quad j = \overline{1,k}. \tag{3.3}$$

By definition $(f_m(\omega_0), f_n(\omega_0)) = \delta_{mn}, \quad m,n = \overline{1,k}$. Thus there exists a constant $\varepsilon_k > 0$ such that for all $0 < \varepsilon \leq \varepsilon_k$ the functions $\varphi_1(\cdot,\varepsilon),\ldots,\varphi_k(\cdot,\varepsilon)$ are linearly independent. Let $\varepsilon^{(k)} = \min\{\varepsilon_k, \sigma_k\}$ and $V_k = \Omega \cap \{|\omega - \omega_0| < \delta(\varepsilon^{(k)})\}$. Then for any $\omega \in V_k$ we see that

$$l(\omega)[\varphi_j(\cdot,\varepsilon^{(k)})] < d(\omega), \quad j = \overline{1,k}.$$

Therefore, according to the min-max principle, for all $\omega \in V_k$ the operator $L(\omega_0)$ has at least k eigenvalues $\lambda_1(\omega),\ldots,\lambda_k(\omega)$ located to the left from $d(\omega)$. Furthermore, by the inequalities (3.3) we have

$$\lambda_j(\omega) < \lambda_j(\omega_0) + 2\varepsilon, \quad \varepsilon < \varepsilon^{(k)}, \quad \omega \in \{|\omega - \omega_0| < \delta(\varepsilon)\} \cap \Omega, \quad j = \overline{1,k}.$$

This proves (3.1). □

Remark 4. If, in addition to the conditions of Lemma 6, condition 5) of Theorem 1 is also satisfied, then by the min-max principle the spectrum of $L(\omega)$ is discrete for any $\omega \in \Omega$. Therefore, Lemma 6 will be true for <u>all</u> eigenvalues.

Lemma 7. *If conditions 1), 2), 3'), 4), 5) are satisfied for any $\varepsilon > 0$ and any finite k, then there exists a neighborhood $V = V(k,\varepsilon) \subset \Omega$ of ω_0, where the inequalities $\lambda_k(\omega) \geq \lambda_k(\omega_0) - \varepsilon$ hold.*

Proof. Assume to the contrary, that is $\lambda_k(\omega_m) < \lambda_k(\omega_0) - \sigma$ for some sequence $\omega_m \to \omega_0$ and $\sigma > 0$. Let $\varphi_j(x,\omega_m) \equiv f_{jm}(x)$ be the normalized eigenfunctions of $L(\omega_m)$ that correspond to the eigenvalues $\lambda_j(\omega_m)$. Then

$$l(\omega_m)[f_{jm}] = (L(\omega_m)f_{jm}, f_{jm}) \leq \lambda_k(\omega_0) - \sigma. \tag{3.4}$$

According to condition 5) of Theorem 1, for any $0 < \tau < \delta$ there exists $M(\tau) \in \mathbb{N}$ such that for all $j = \overline{1,k}$ and $m \geq M(\tau)$ the functions f_{jm} belong to the compact set $S = \{\psi \in D(l_\tau) : \|\psi\| \leq 1, \; l_\tau[\psi] \leq \lambda_k(\omega_0) - \sigma\}$, where l_τ is the quadratic form of the operator $L(\tau,\omega_0)$ occurring in condition 5). Without loss of generality we can assume that for every $j = \overline{1,k}$ $\varphi_j(\omega_m)$ converges to some function $f_j \in S$, then $l_\tau[f_j] \leq \lambda_k(\omega_0) - \sigma$. Moreover, since $\forall\, m \in \mathbb{N}$ $(f_{jm}, f_{km}) = \delta_{jk}$ we see that

$$(f_i, f_j) = \delta_{ij}, \; i,j = \overline{1,k}. \tag{3.5}$$

So $\forall\, b > 0$ and $0 < \tau < \delta$,

$$\int_0^b \left(|f_j^{(n)}(x)|^2 + \sum_1^n p_i(x,\tau,\omega_0)|f_j^{(n-i)}(x)|^2 \right) dx < \lambda_k(\omega_0) - \sigma, \quad j = \overline{1,k}.$$

Passing to the limit in this inequality as $\tau \to 0+$ and taking into account condition 4) of Theorem 1, we obtain

$$\int_0^b \left(|f_j^{(n)}(x)|^2 + \sum_1^n p_i(x,\omega_0)|f_j^{(n-i)}(x)|^2 \right) dx \le \lambda_k(\omega_0) - \sigma, \quad j = \overline{1,k}, \ b > 0.$$

Since $b > 0$ is arbitrary, we have

$$l(\omega_0)[f_j] \le \lambda_k(\omega_0) - \sigma, \ j = \overline{1,k}.$$

Hence, applying the min-max principle together with (3.5), we arrive at the inequality $\lambda_k(\omega_0) \le \lambda_k(\omega_0) - \sigma$. The derived contradiction proves the lemma. □

Conclusion of the proof of Theorem 1. If $p_1, \dots, p_n$ are nonnegative for $x \ge a$, then Theorem 1 follows from Lemmas 6, 7 and the min-max principle.

It is clear from proofs of Lemmas 1, 2, 4 – 7 that nonnegativity of $p_1, \dots, p_n$ for $x \ge a$ is necessary only to guarantee boundedness from below of the quadratic form $l(\omega)$. Therefore, it suffices to verify that the quadratic form $l(\omega)$ is also semi-bounded if we replace condition 3') by condition 3).

Suppose $p_i(x,\omega) \ge c_i(\omega)$, $x \ge a$, where $c_i(\omega)$ is independent of x. Then $l(\omega)[y] = l_1(\omega)[y] + l_2(\omega)[y]$, where $l_2[y] = \sum\limits_0^{n-1} c_i(\omega)\|y^{(n-i)}\|^2$, $l_1(\omega)$ satisfies (2.5) for any sufficient by small $\varepsilon > 0$. According to the well-known inequality (see, for example, [3, p. 129]), we have

$$\|y^{(k)}\| \le C(\varepsilon)\|y\| + \varepsilon\|y^{(n)}\|, \quad y \in W_2^{(n)}(0,\infty), \quad k = \overline{1, n-1}, \tag{3.6}$$

where a constant ε can be chosen to be arbitrarily small and $C(\varepsilon) > 0$ is independent of y. From this and (2.5) it follows that the form $l(\omega)$ is bounded from below. □

4 Proof of Theorem 2

First, we prove (1.8). Arguing similarly to the proof of Lemma 7, from any sequence $\{\omega_m\}$ converging to ω_0 we can choose a subsequence $\{\omega_m\}$, which we will also denote by $\{\omega_m\}$, such that there exists the limit

$$\lim_{m\to\infty} \varphi_i(\cdot,\omega_m) \equiv f_i,$$

where $f_i \in D_l(\omega_0)$, $\|f_i\| = 1$ and $l(\omega_0)[f_i] \le \lambda_0$. We will show that $l(\omega_0)[f_i] = \lambda_0$, whence (1.8) follows.

Suppose that there is a $1 \le i \le k$ such that

$$l(\omega_0)[f_i] < \lambda_0. \tag{4.1}$$

Let the operator $L(\omega_0)$ have exactly s eigenvalues to the left from λ_0. We repeat the same argument as above and find s functions $g_1, \dots, g_s$ in $D_l(\omega_0)$ satisfying (3.5), (4.1), and $(f_i, g_j) = 0, \ j = 1, \dots, k$. Whence using the min-max principle, we conclude that the operator $L(\omega_0)$ has at least $s+1$ eigenvalues to the left from λ_0. This contradiction proves (1.8).

Let us now prove (1.9). We first note that estimates (2.1) and (2.2) hold for $a=\infty$ (see [3, p. 142] and (3.6)). Therefore, inequality (2.3) is also satisfied for $a=\infty$. From this and (2.7) it follows that for any compact $\Delta\subset\Omega$ there exists a constant $K(\Delta)>0$ such that for all $\omega\in\Delta$ and $y\in D_l(\omega)$

$$\|y\|_{n-1}\le K(\Delta)\|y\|_{+1}(\omega), \tag{4.2}$$

where the norm $\|\cdot\|_{+1}$ is defined by (2.21).

Let λ_0 be a simple eigenvalue of the operator $L(\omega_0)$, $\varphi_0(x):=\varphi(x,\omega_0)$. Then using (1.8), we get $|(\varphi(\cdot,\omega),\varphi_0)|\to 1,\ \omega\to\omega_0$. We choose a normalizer $\varphi(x,\omega)$ such that for all ω sufficiently close to ω_0, $\quad (\varphi(\cdot,\omega),\varphi(\cdot,\omega_0))>0$. Then

$$\|\varphi(\cdot,\omega)-\varphi_0)\|\to 0,\quad \omega\to\omega_0. \tag{4.3}$$

Taking into account Lemma 5 we see that there exists a sequence $\{\varphi_m\}_1^\infty$ of functions from $D'(L(\omega_0))$ such that

$$\|\varphi_0-\varphi_m\|_{+1}(\omega_0)\to 0,\quad m\to\infty. \tag{4.4}$$

We choose $\delta>0$ such that $\Delta=\{|\omega-\omega_0|\le\delta\}\subset\Omega$. Then using (4.2) for $\varphi(\cdot,\omega)-\varphi_m$ and $\varphi_m-\varphi_0$ with $\omega\in\Delta,\ m\in\mathbb{N}$, we obtain

$$\begin{aligned}\|\varphi(\cdot,\omega)-\varphi_0)\|_{n-1} &\le K(\Delta)\left\{\|\varphi(\cdot,\omega)-\varphi_m\|_{+1}(\omega)+\|\varphi_m-\varphi_0\|_{+1}(\omega_0)\right\}\le\\ &\le K(\Delta)\left\{l(\omega)[\varphi(\cdot,\omega)-\varphi_m]+\|\varphi(\cdot,\omega)-\varphi_0\|+2\|\varphi_m-\varphi_0\|_{+1}(\omega_0)\right\}.\end{aligned}$$

Hence,

$$\|\varphi(\cdot,\omega)-\varphi_0)\|_{n-1}\le K(\Delta)r_m(\omega), \tag{4.5}$$

where

$$r_m(\omega)=\lambda(\omega)\{1-2\mathrm{Re}(\varphi(\cdot,\omega),\varphi_m)\}+l(\omega)[\varphi_m]+\|\varphi(\cdot,\omega)-\varphi_0\|+2\|\varphi_m-\varphi_0\|_{+1}(\omega_0).$$

By (4.3) and condition 4) it follows that for each $m\in\mathbb{N}$ $r_m(\omega)\to r_m,\ \omega\to\omega_0$, where $r_m=\lambda_0\{1-2\mathrm{Re}(\varphi_0,\varphi_m)\}+l(\omega_0)[\varphi_m]+2\|\varphi_m-\varphi_0\|_{+1}(\omega_0)$. Moreover, (4.4) implies that $r_m\to 0,\ m\to\infty$. Combining this with (4.5), we get (1.9). This concludes the proof.

5 Examples

In this section we consider several examples and one counterexample showing the importance of assumption 5).

Example 1. Let $\mathcal{L}(y)=(-1)^n y^{(2n)}+qy$, where

$$q=\sum_{k=1}^m a_k x^{\alpha_k}. \tag{5.1}$$

Then there exist two “reasonable" choices of the parameters:

a) For $a_1 > 0$ it is natural to consider the collection $(\alpha_1, \alpha_2, \dots, \alpha_m)$ as a parameter ω and choose for Ω any domain whose closure belongs to the following set: $\mathbf{\Omega_a} = \{\omega \in \mathbb{R}^m : \ \alpha_1 > 0, \ \ -1 < \alpha_k < \alpha_1, \ k = \overline{2, m}\}$;

b) If $\alpha_1 > 0$ and $-1 < \alpha_k < \alpha_1, \ k = \overline{2, m}$, then we set $\omega = (\alpha_1, \alpha_2, \dots, \alpha_m)$, and Ω is a domain in $\mathbb{R}^m$ such that $\overline{\Omega} \subset \mathbf{\Omega_b} = \{\omega \in \mathbb{R}^m : \alpha_1 > 0\}$.

In both cases conditions 1), 3) — 5) of Theorem 1 can be easily verified. Concerning condition 2) we can apply Naimark's theorem ([11, p. 336]) which shows that the deficiency indices of the minimal operator with potential (5.1) are equal to (n, n) for all ω from $\mathbf{\Omega_a}$ or $\mathbf{\Omega_b}$.

Example 2. Let $\mathcal{L}_\omega(y) = -y'' + q(\omega, x)y$, where

$$q(\omega, x) = q_0(x) + \omega q_1(x), \tag{5.2}$$

$\omega \in \mathbb{R}$, functions q_0 and q_1 are locally summable on $[0, \infty)$, bounded from below on $[a, \infty)$, $a > 0$, and q_0 satisfies Molchanov's criteria ([11, p. 393]). Let $\Omega = [0, \infty)$. Then all conditions of Theorem 1 are satisfied. Hence all eigenvalues of any self-adjoint operator generated in $L^2(0, \infty)$ by the differential expression $\mathcal{L}_\omega$ are continuous on $[0, \infty)$.

We point out that there are no restrictions on the growth rate of q_1. In particular, the first term of the perturbation theory $C_k = (q_1\varphi_k(\cdot, 0), \varphi_k(\cdot, 0))$ might be infinite. However, if the integral C_k converges, then it can be proved under several additional assumptions that $\lambda_k(\omega)$ is not only continuous (from the right) at 0 but also has a finite right derivative at 0. In fact, multiplying scalarly both sides of the equation

$$-\varphi_k''(x, \omega) + (q_0 + \omega q_1)\varphi_k(x, \omega) = \lambda_k(\omega)\varphi_k(x, \omega)$$

by $\varphi_k(x, 0)$ and taking into account the boundary condition $y'(0) - hy(0) = 0$, we obtain

$$\frac{\lambda_k(\omega) - \lambda_k(0)}{\omega} = \frac{(q_1\varphi_k(\cdot, \omega), \varphi(\cdot, 0))}{(\varphi_k(\cdot, \omega), \varphi(\cdot, 0))}. \tag{5.3}$$

As the eigenvalues of $l(\omega)$ are simple, it then follows by Theorem 2 that $(\varphi_k(\cdot, \omega), \varphi(\cdot, 0)) \to 1, \ \omega \to 0$. We require that $(q_1\varphi_k(\cdot, \omega), \varphi(\cdot, 0)) \to C_k, \ \omega \to 0$. Then from (5.3) follows the well-known formula $\lambda_k'(0) = C_k$ (see, for example, [10]).

Example 3. We will show that if condition 5) does not hold, then the statement of Theorem 1 in general is not correct.

The first trivial example is as follows: in Example 2 we set $q_0 = 0$ and choose for q_1 any function satisfying, in addition the conditions of Example 2, Molchanov's criteria. Then all conditions except for 5) are satisfied, yet for $\omega_0 = 0$ the operator $L(\omega_0)$ does not have any eigenvalues.

Then consider an example where the limit operator does have an eigenvalue that lies below the essential spectrum but does not possess continuity. We will again proceed from Example 2. We choose for q_0 a function which finite range such that the discrete spectrum of the operator $L(0)$ is not empty. For example, the function

$$q_0(x) = \begin{cases} -m^2 & , \quad 0 \le x \le 1, \\ 0 & , \quad x > 1, \end{cases}$$

where $m > \pi$, satisfies this condition if $L(0)$ is the extension corresponding to the Dirichlet condition. This is easy to verify if we apply the min-max principle to the sample function

$$\varphi(x) = \begin{cases} \sin(\pi x) & , \quad 0 \le x \le 1. \\ 0 & , \quad x > 1. \end{cases}$$

Although this function does not belong to the domain of the operator $L(0)$, it does belong to the domain of the quadratic form $l(0)$ of the operator and $l(0)[\varphi] = -(m^2 - \pi^2)\|\varphi\|^2$. Then, since the essential spectrum of the operator $L(0)$ is $[0, \infty)$, we conclude that discrete spectrum $L(0)$ is not empty and the least one eigenvalue is not greater than $-(m^2 - \pi^2)$.

The second term in (5.2) can be defined as follows:

$$q_1(\omega, x) = \begin{cases} 0 & , \quad \omega = 0, \\ p(x - 1/\omega) & , \quad \omega > 0, \end{cases}$$

where

$$p(x) = \begin{cases} 0 & , \quad x < 0, \\ -n^2 & , \quad 0 \le x \le 1, \\ \widetilde{p}(x) & , \quad x > 1, \end{cases}$$

$n > m$, $\widetilde{p}(x)$ is any function satisfying conditions of Example 2. The function $q(\omega, x)$ satisfies all conditions of Theorem 1 except 5). On the other hand, arguing as in the analysis of the operator $L(0)$, we obtain that for $0 < \omega < 1$ the least eigenvalue $\lambda_1(\omega)$ of the operator $L(\omega)$ is not greater than $-(n^2 - \pi^2)$. Choosing sufficiently large n, we can claim that $\lambda_1(\omega) \nrightarrow \lambda_1(0)$ for $\omega \to 0$.

Acknowledgments

This research was supported by the Russian Foundation for Basic Research, grants no. 09-01-00440-a, no. 08-01-97020.

References

[1] P.B. Bailey, M.K. Gordon, L.F. Shampine, *Automatic solution of the Sturm – Lioville problem.* ACM Trans. Math. Software, 4 (1978), 193 – 208.

[2] P.B. Bailey, W.N. Everitt, A. Zettl, *SLEIGN2: An eigenfunction – eigenvalue code for singular Sturm – Lioville problems.* ACM Trans. Math. Software, 21 (2001), 142 – 196.

[3] V.I. Burenkov, *Sobolev spaces on domains.* Texts in Mathematics, vol.137, B.G. Teubner, Stuttgart – Leipzig, 1998.

[4] M. Dauge, B. Helffer, *Eigenvalues variation. II. Multidimensional problems.* J. Diff. Equations, 104 (1993), 263 – 297.

[5] C.T. Fulton, S. Pruess, *Mathematical software for Sturm – Lioville problems.* ACM Trans. Math. Software, 19 (1993), 360 – 376.

[6] T. Kato, *Perturbation theory for linear operators*, Springer Verlag, Berlin – Heidelberg – New York, 1966.

[7] Q. Kong, A. Zettl, *Eigenvalues of regular Sturm-Liouville problems.* J. Diff. Equations, 131 (1996), 1 – 19.

[8] Q. Kong, H. Wu, A. Zettl, *Dependence of eigenvalues on the problem.* Math. Nachr., 188 (1997), 173 – 201.

[9] M.G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian operators and its applications, II.* Mat. sb, 21 (1947), 365 – 404 (in Russian).

[10] M. Möller, A. Zettl, *Differentiable dependence of simple eigenvalues of operators in Banach spaces.* J. Operator Theory, 36 (1996), 335 – 355.

[11] M.A. Naimark, *Linear differential operators II.* Ungar, New York, 1968.

[12] J.D. Pryce, *The NAG Sturm – Lioville codes and some applications.* NAG Newsl., 3 (1986), 4 – 26.

[13] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis.* Academic Press, New York – London, 1972.

[14] M. Reed, B. Simon, *Methods of modern mathematical physics IV: Analysis of operators.* Academic Press, New York – London, 1979.

Khabir Ishkin
Department of Mechanics and Mathematics
Bashkir State University
32 Zaki Validi st.
450074 Ufa, Bashkortostan
E-mail: Ishkin62@mail.ru

Received: 25.01.2011