EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 2, Number 3 (2011), 5 – 19

ON THE NULL-CONTROLLABILITY OF THE HEAT EXCHANGE PROCESS

Sh. Alimov

Communicated by M. Otelbaev

Key words: parabolic type operators, heat conduction equation, control of distributed systems.

AMS Mathematics Subject Classification: 93C20, 35K05, 35K20.

Abstract. A mathematical model of the heat exchange process, where the temperature inside some domain is controlled by m convectors acting on the boundary, is considered. The control parameter is a vector-function, whose components are equal to the magnitude of the output of hot or cold air produced by each convector. The necessary and sufficient conditions, which initial temperature must satisfy for achieving the zero value by the projection of the temperature into some m-dimensional subspace, are studied.

1 Introduction

1. Consider in a semi-infinite cylinder $Q = \Omega \times (0, +\infty) \subset \mathbb{R}^{n+1}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise smooth boundary $\partial\Omega$, the following equation

$$u_t(x,t) \; = \; \Delta u(x,t) \; - \; p(x)u(x,t), \quad p(x) \geq 0, \quad x \in \Omega, \quad t > 0. \tag{1.1}$$

Let $\Gamma_1, \Gamma_2, ..., \Gamma_m$ be m disjoint subsets of $\partial\Omega$, and set

$$\Gamma = \bigcup_{i=1}^{m} \Gamma_i.$$

We suppose that each Γ_i (heater or air conditioner) has piecewise smooth boundary $\partial\Gamma_i$ and $mes_{n-1}\Gamma_i > 0$ (we denote by $mes_{n-1}\Gamma$ the surface measure of Γ, distinct from Lebesgue measure $|\Omega|$).

Assume that non-negative piecewise smooth functions $a_i(x)$ defined on the boundary $\partial\Omega$ vanish outside the Γ_i . We say that a vector-function $q : [0, +\infty) \to \mathbb{R}^m$ is an admissible control if all components $q_i(t)$ are measurable real-valued functions and satisfy the condition

$$|q_i(t)| \leq 1, \quad t \geq 0, \quad i = 1, 2, ..., m.$$

Consider the following boundary conditions:

$$\frac{\partial u(x,t)}{\partial n} \; = \; q(t) \cdot a(x), \quad x \in \Gamma, \quad t > 0, \tag{1.2}$$

where

$$q(t)\cdot a(x) \;=\; q_1(t)a_1(x)+q_2(t)a_2(x)+...+q_m(t)a_m(x),$$

and

$$\frac{\partial u(x,t)}{\partial n} \;+\; h(x)u(x,t) \;=\; 0, \quad x\in\partial\Omega\setminus\Gamma, \quad t>0. \tag{1.3}$$

Condition (1.2) means that there is a blast of hot (or cold) air from Γ_i with the magnitude of the output given by the function $q_i(t)$, and condition (1.3) means that on the surface $\partial\Omega\setminus\Gamma$ a heat exchange takes place according to Newton's law (see, e.g. [14], Sect. III.1.4).

We suppose that $h(x)$ (thermal conductivity of the walls) and $a_i(x)$ (the density of the power of the i-th heater or air conditioner) are given piecewise smooth non-negative functions, which are not identically zeros.

We may extend the function $h(x)$ to the whole boundary $\partial\Omega$ by setting $h(x)=0$ for $x\in\Gamma$. In this case we may rewrite conditions (1.2) and (1.3) in the following form

$$\frac{\partial u(x,t)}{\partial n} \;+\; h(x)u(x,t) \;=\; q(t)\cdot a(x), \quad x\in\partial\Omega, \quad t>0. \tag{1.4}$$

Finally, we add the initial condition

$$u(x,0) \;=\; \psi(x). \tag{1.5}$$

We use the standard definition of a generalized solution to the initial-boundary value problem for equation (1.1) with conditions (1.4) and (1.5) (see [9], III.5, formula (5.5) and Theorem 5.1).

Namely, a generalized solution of this problem is a function $u(x,t)$ such that for any $T>0$ and any $\eta\in W_2^{1,1}(\Omega\times[0,T])$ for $0<t\le T$ the following equality is valid

$$\int_0^t ds\int_\Omega [\nabla u(x,s)\,\nabla\eta(x,s)+p(x)u(x,s)\eta(x,s)]\,dx \;-\; \int_0^t ds\int_\Omega u(x,s)\frac{\partial\eta(x,s)}{\partial s}dx \;+$$

$$+\;\int_\Omega u(x,t)\eta(x,t)dx \;-\; \int_\Omega \psi(x)\eta(x,0)dx \;=$$

$$=\;\int_0^t ds\int_{\partial\Omega}[q(s)\cdot a(x)]\eta(x,s)d\sigma(x) \;-\; \int_0^t ds\int_{\partial\Omega} h(x)u(x,s)\eta(x,s)d\sigma(x). \tag{1.6}$$

2. For the formulation of the problem studied in this paper consider the following eigenvalue problem

$$-\Delta v(x) \;+\; p(x)v(x) \;=\; \lambda v(x), \quad x\in\Omega, \tag{1.7}$$

with the boundary condition

$$\frac{\partial v(x)}{\partial n}+h(x)v(x) \;=\; 0, \quad x\in\partial\Omega. \tag{1.8}$$

We define a generalized solution of problem (1.7) – (1.8) as a function $v(x)$ in the Sobolev space $W_2^1(\Omega)$, which satisfies the equality

$$\int\limits_{\Omega} [\nabla v(x)\, \nabla \eta(x) + p(x) v(x) \eta(x)]\, dx \;=\; \lambda \int\limits_{\Omega} v(x)\eta(x)dx - \int\limits_{\partial\Omega} h(x) v(x) \eta(x) d\sigma(x), \quad (1.9)$$

for an arbitrary function $\eta \in W_2^1(\Omega)$ (see [8], Sec. III.6, formula (6.3)).

We consider this problem in the real Hilbert space $L^2(\Omega)$ with the scalar product

$$(u, v) \;=\; \int\limits_{\Omega} u(x) v(x) dx$$

and the norm $\|u\| = \sqrt{(u, u)}$.

It is well known that under the assumptions made above this problem is self-adjoint in $L^2(\Omega)$ and there exists a sequence of positive eigenvalues $\{\lambda_i\}$ such that

$$0 \;<\; \lambda_1 \;\le\; \lambda_2 \;\le\; ... \;\le\; \lambda_i \;\to \infty, \qquad i \to \infty$$

(see, e. g. [8], Sec. III.6).

The corresponding eigenfunctions form a complete orthonormal system $\{v_i\}_{i \in \mathbf{N}}$ in $L^2(\Omega)$.

Let H_m be the m-dimensional subspace formed by the eigenfunctions $v_1, v_2, ..., v_m$ and let S_m be the orthogonal projector onto H_m, i. e. :

$$S_m u(x) \;=\; \sum_{i=1}^{m} (u, v_i) v_i(x). \quad (1.10)$$

3. In the present work we consider the following problem.

NC Problem. *For a given vector-function $\psi \in L_2(\Omega)$ NC problem consists in finding the admissible control $q(t)$ such that the solution $u(x,t)$ of the initial-boundary value problem (1.4)-(1.5) for equation (1.1) exists, is unique and for some $T > 0$ satisfies the equality*

$$S_m u(x, t) \;=\; 0, \quad x \in \Omega, \quad t \ge T. \quad (1.11)$$

We may note that the detailed information on the problem of optimal control for distributed parameter systems is given in the monographs [5], [6] and [10]. More recent results concerned with the heat control problem for partial differential equations of parabolic type were established in [1]-[4], [7], [11]-[13], [15].

In the case in which $m = 1$ and P_1 is the projector to the one-dimensional subspace generated by the function $u_1(x) \equiv 1$, according to [1], the null-controllability takes place for any initial function $\psi \in L_2(\Omega)$.

In the case $m > 1$ the situation changes (see case $m = 2$ in [2]). In what follows we assume that $m \ge 2$.

4. Denote the points of the spectrum of the boundary value problem (1.7)-(1.8) by $\{\mu_k\}$, where

$$0 \;<\; \mu_1 \;<\; \mu_2 \;<\; ...$$

Each μ_k is an eigenvalue with multiplicity ν_k such that $1 \leq \nu_k < \infty$. It is well known that $\nu_1 = 1$ (see, e. g. [16]).

Set $N_0 = 0$ and

$$N_k \;=\; \nu_1 + \nu_2 + ... + \nu_k, \quad k = 1, 2, ...$$

Define the number $l = l(m)$ such that

$$N_{l-1} \;<\; m \;\leq\; N_l. \tag{1.12}$$

Inasmuch as $m \geq 2$, then $l > 1$. For a positive integer $k \leq l - 1$ set

$$E_k \;=\; \{u \in L_2(\Omega) \;:\; u(x) = \sum_{\lambda_i = \mu_k} \alpha_i v_i(x), \quad \alpha_i \in \mathbb{R}\}.$$

It is clear that for these k

$$E_k \;=\; \{u \in L_2(\Omega) \;:\; u(x) = \sum_{i=N_{k-1}+1}^{N_k} \alpha_i v_i(x), \quad \alpha_i \in \mathbb{R}\}, \tag{1.13}$$

and hence

$$\dim E_k \;=\; N_k - N_{k-1} \;=\; \nu_k, \quad 1 \leq k \leq l - 1.$$

Further, define for $k = l = l(m)$

$$E_l \;=\; \{u \in L_2(\Omega) \;:\; u(x) = \sum_{i=N_{l-1}+1}^{m} \alpha_i v_i(x), \quad \alpha_i \in \mathbb{R}\} \tag{1.14}$$

and note that

$$0 \;<\; \dim E_l \;=\; m \;-\; N_{l-1} \;\leq\; \nu_l.$$

Let P_k be the orthogonal projector onto E_k, $1 \leq k \leq l$, i. e. :

$$P_k u(x) \;=\; \sum_{\lambda_i = \mu_k} (u, v_i) v_i(x), \quad 1 \leq k \leq l - 1,$$

and

$$P_l u(x) \;=\; \sum_{i=N_{l-1}+1}^{m} (u, v_i) v_i(x).$$

Then, obviously, for projector (1.10) we get

$$S_m \;=\; \sum_{k=1}^{l} P_k.$$

5. The solution of the null-controllability problem is connected with the following boundary value problem for the equation

$$\Delta w_j(x) \;-\; p(x) w_j(x) \;=\; 0, \quad x \in \Omega, \tag{1.15}$$

and the boundary condition

$$\frac{\partial w_j(x)}{\partial n} + h(x)w_j(x) = a_j(x), \quad x \in \partial\Omega. \tag{1.16}$$

The physical meaning of the function $w_j(x)$ is clear: this is the temperature of the volume Ω in the case in which only the jth convector works and it produces heat or cold with maximal capacity (output).

Consider the following m vectors in $\mathbb{R}^m$:

$$\Theta_i \;=\; (\theta_{i1},\; \theta_{i2},\; ...,\; \theta_{im}), \quad i = 1, 2, ..., m, \tag{1.17}$$

where

$$\theta_{ij} \;=\; \int\limits_{\Omega} v_i(x)\, w_j(x)\, dx. \tag{1.18}$$

Set

$$L_k \;=\; \{\xi \in \mathbb{R}^m \;:\; \xi = \sum_{i=N_{k-1}+1}^{N_k} c_i\Theta_i, \quad c_i \in \mathbb{R}\}, \quad 1 \le k < l, \tag{1.19}$$

and

$$L_l \;=\; \{\xi \in \mathbb{R}^m \;:\; \xi = \sum_{i=N_{l-1}+1}^{m} c_i\Theta_i, \quad c_i \in \mathbb{R}\}. \tag{1.20}$$

It is clear that $\dim L_k \le \nu_k$ for $k = 1, 2, ..., l$.

Definition. *We say that convectors $\{\Gamma_i, a_i\}_{i=1}^m$ are properly arranged if for every $k, 1 \le k \le l$, the following equality*

$$\dim L_k \;=\; \dim E_k \tag{1.21}$$

is valid.

Remark. Equalities (1.21) mean that none of the vectors Θ_i is equal to zero and for every $k, 1 \le k \le l$, the vectors $\Theta_i \in L_k$ are linearly independent.

The main result is the following.

Theorem 1. *If convectors $\{\Gamma_i, a_i\}_{i=1}^m$ are properly arranged then for any function $\psi \in L_2(\Omega)$ the problem of the null-controllability has positive solution.*

The next theorem shows that this result is precise.

Theorem 2. *If convectors $\{\Gamma_i, a_i\}_{i=1}^m$ are not properly arranged then there exists a function $\psi \in L_2(\Omega)$ such that the null-controllability does not take place.*

Remark. It is clear that $v_1(x) \ge 0$ and $w_j(x) \ge 0$ for all $j = 1, 2, ..., m$, and we may state that the vector Θ_1 has positive components. Therefore, in the case $m = 1$ obviously $l(m) = 1$ and

$$\dim L_1 \;=\; \dim E_1 \;=\; 1.$$

Hence, if there is only one air conditioner then it is always properly arranged. This conforms with the results of [1].

2 Representation of the solution

Consider the $(m \times m)$-matrix $\widehat{\Theta} = \|\theta_{ij}\|$ defined by (1.18).

Lemma 2.1. *The elements of the matrix $\widehat{\Theta}$ have the following form:*

$$\theta_{ij} \;=\; \frac{1}{\lambda_i}\int\limits_{\partial\Omega} v_i(x) a_j(x)\, d\sigma(x). \tag{2.1}$$

Proof. According to the definition of a generalized solution to problem (1.15) – (1.16), the following equation

$$\int\limits_{\Omega} [\nabla w_j(x)\, \nabla \eta(x) + p(x) w_j(x) \eta(x)]\, dx \;=\; \int\limits_{\partial\Omega} [a_j(x) - h(x) w_j(x)] \eta(x)\, d\sigma(x)$$

is valid for an arbitrary function $\eta \in W_2^1(\Omega)$. In particular, for $\eta(x) = v_i(x)$ we get

$$\int\limits_{\Omega} [\nabla w_j(x)\, \nabla v_i(x) + p(x) w_j(x) v_i(x)]\, dx \;=$$

$$= \int\limits_{\partial\Omega} a_j(x) v_i(x)\, d\sigma(x) \;-\; \int\limits_{\partial\Omega} h(x) w_j(x) v_i(x)\, d\sigma(x). \tag{2.2}$$

Now we use identity (1.9). Set in this identity $\eta(x) = w_j(x)$. Then we get

$$\int\limits_{\Omega} [\nabla v_i(x)\, \nabla w_j(x) + p(x) v_i(x) w_j(x)]\, dx \;=$$

$$= \lambda_i \int\limits_{\Omega} v_i(x) w_j(x)\, dx \;-\; \int\limits_{\partial\Omega} h(x) v_i(x) w_j(x)\, d\sigma(x). \tag{2.3}$$

Comparing (2.2) and (2.3) we get

$$\int\limits_{\partial\Omega} a_j(x) v_i(x)\, d\sigma(x) \;=\; \lambda_i \int\limits_{\Omega} v_i(x) w_j(x)\, dx.$$

Since the value of the right-hand side is $\lambda_i \theta_{ij}$, the required representation (2.1) is valid. □

Set

$$b_i(t) \;=\; \int\limits_{\partial\Omega} [q(t) \cdot a(x)]\, v_i(x)\, d\sigma(x). \tag{2.4}$$

Lemma 2.2. *Let $u(x,t)$ be the solution of the initial-boundary value problem (1.4)-(1.5) for the equation (1.1). Then the eigenfunction expansion of this solution has the form*

$$u(x,t) \;=\; \sum_{i=1}^{\infty} \left[\int\limits_0^t b_i(s)\, e^{\lambda_i s}\, ds \;+\; (\psi, v_i) \right] e^{-\lambda_i t}\, v_i(x), \quad t \ge 0, \quad x \in \Omega\,. \tag{2.5}$$

Proof. The function $u(x,t)$ for almost every $t \geq 0$ belongs to $L_2(\Omega)$ and that is why the Fourier coefficients exist and are equal to

$$u_i(t) \;=\; \int\limits_{\Omega} u(x,t)\, v_i(x)\, dx \;.$$

According to the definition of a generalized solution, for any $\eta \in W_2^{1,1}(\Omega \times [0,T])$ identity (1.6) is valid.

In the case in which $\eta(x,t) \equiv v_i(x)$ we have $\dfrac{\partial v_i(x)}{\partial t} \equiv 0$ and hence

$$\int\limits_0^t ds \int\limits_{\Omega} [\nabla u(x,s)\, \nabla v_i(x) + p(x) u(x,s) v_i(x)] dx \;+$$

$$+ \int\limits_{\Omega} u(x,t) v_i(x) dx \;-\; \int\limits_{\Omega} \psi(x) v_i(x) dx \;=$$

$$= \int\limits_0^t ds \int\limits_{\partial\Omega} [q(s) \cdot a(x)]\, v_i(x)\, d\sigma(x) \;-\; \int\limits_0^t ds \int\limits_{\partial\Omega} h(x) u(x,s) v_i(x) d\sigma(x).$$

Using (2.4), we may rewrite this equality as follows

$$\int\limits_0^t ds \left(\int\limits_{\Omega} [\nabla u(x,s)\, \nabla v_i(x) + p(x) u(x,s) v_i(x)]\, dx + \int\limits_{\partial\Omega} h(x) u(x,s) v_i(x)\, d\sigma(x) \right) \;=$$

$$= -\int\limits_{\Omega} u(x,t) v_i(x) dx \;+\; \int\limits_0^t ds \int\limits_{\partial\Omega} [q(s) \cdot a(x)]\, v_i(x)\, d\sigma(x) \;+\; \int\limits_{\Omega} \psi(x) v_i(x)\, dx \;=$$

$$= \; - \, u_i(t) \;+\; \int\limits_0^t b_i(s)\, ds \;+\; (\psi, v_i).$$

Further, it follows from (1.9) for $\eta(x) = u(x,s)$ that

$$\int\limits_{\Omega} [\nabla u(x,s) \nabla v_i(x) + p(x) u(x,s) v_i(x)]\, dx \;+\; \int\limits_{\partial\Omega} h(x) u(x,s) v_i(x) d\sigma(x) \;=$$

$$= \; \lambda_i \int\limits_{\Omega} u(x,s) v_i(x)\, dx \;=\; \lambda_i u_i(s).$$

Consequently,

$$\lambda_i \int\limits_0^t u_i(s)\, ds \;=\; - \, u_i(t) \;+\; \int\limits_0^t b_i(s)\, ds \;+\; (\psi, v_i) \;. \tag{2.6}$$

Hence, $u_i(t)$ is an absolutely continuous function and

$$u_i(0) \;=\; (\psi, v_i).$$

After differentiation equation (2.6) we get

$$\lambda_i u_i(t) \;+\; u_i'(t) \;=\; b_i(t)$$

or

$$\left[e^{\lambda_i t} u_i(t)\right]' \;=\; e^{\lambda_i t} b_i(t).$$

Hence,

$$e^{\lambda_i t} u_i(t) \;=\; (\psi, v_i) \;+\; \int_0^t e^{\lambda_i s} b_i(s)\, ds.$$

This means that Fourier expansion of solution has form (2.5). □

Corollary. *For $t \geq 0$ and $x \in \Omega$ projector (1.10) has the form*

$$S_m u(x,t) \;=\; \sum_{i=1}^{m} \left[\int_0^t b_i(s)\, e^{\lambda_i s}\, ds \;+\; (\psi, v_i)\right] e^{-\lambda_i t}\, v_i(x), \quad t \geq 0, \quad x \in \Omega\,. \tag{2.7}$$

Define the operator $\Lambda_m : L_2(\Omega) \to \mathbb{R}^m$ by the equality

$$\Lambda_m \psi \;=\; ((\psi, v_1), (\psi, v_2), ..., (\psi, v_m)). \tag{2.8}$$

We introduce the matrix

$$\widehat{E}(t) \;=\; \begin{Vmatrix} e^{\lambda_1 t} & 0 & ... & 0 \\ 0 & e^{\lambda_2 t} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & e^{\lambda_m t} \end{Vmatrix}$$

If a vector in $\mathbb{R}^m$ is on the right of some $m \times m$ matrix we always assume that it is a column-vector.

Lemma 2.3. *Condition (1.11) may be rewritten in the following form:*

$$\int_0^t \widehat{E}\,'(s) \widehat{\Theta}\, q(s)\, ds \;=\; -\,\Lambda_m \psi. \tag{2.9}$$

Proof. We use representation (2.7). According to Lemma 2.1, we may write for the value $b_i(t)$, which is defined by (2.4),

$$b_i(t) \;=\; \sum_{j=1}^{m} q_j(t) \int_{\partial\Omega} a_j(x) v_i(x)\, d\sigma(x) \;=\; \lambda_i \sum_{j=1}^{m} \theta_{ij}\, q_j(t). \tag{2.10}$$

Note that

$$\widehat{E}'(t) \;=\; \begin{Vmatrix} \lambda_1 e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & \lambda_2 e^{\lambda_2 t} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_m e^{\lambda_m t} \end{Vmatrix}$$

Hence, taking into account (2.10), we get for the vector

$$B(t) \;=\; \big(b_1(t)e^{\lambda_1 t},\; b_2(t)e^{\lambda_2 t},\; \ldots,\; b_m(t)e^{\lambda_m t}\big)$$

the following representation

$$B(t) \;=\; \widehat{E}\,'(t)\widehat{\Theta}\, q(t). \tag{2.11}$$

According to (2.7), condition (1.11) means that the following equalities

$$\int\limits_0^t b_i(s)\, e^{\lambda_i s}\, ds \;+\; (\psi, v_i) \;=\; 0, \quad i = 1, 2, \ldots, m,$$

must be fulfilled. We may rewrite them as

$$\int\limits_0^t B_i(s)\, ds \;=\; -(\psi, v_i), \quad i = 1, 2, \ldots, m,$$

or, due to (2.8),

$$\int\limits_0^t B(s)\, ds \;=\; -\Lambda_m \psi.$$

Hence, taking into account (2.11), we get (2.9). □

Remark. We may also rewrite condition (2.9) in the following form:

$$\int\limits_0^t \lambda_i e^{\lambda_i s}\, (\Theta_i, q(s))\, ds \;=\; -\;(\psi, v_i), \quad i = 1, 2, \ldots, m. \tag{2.12}$$

3 Proof of the Theorem 1

We suppose in this section that convectors $\{\Gamma_i, a_i\}_{i=1}^m$ are properly arranged.

Lemma 3.1. *Let*

$$0 \;<\; \mu_1 \;<\; \mu_2 \;<\; \cdots \;<\; \mu_l\;. \tag{3.1}$$

There exist real numbers $p_l > p_{l-1} > ... > p_1 > 0$ such that for the determinant of the matrix

$$D_l \ = \ \left\| \begin{matrix} 1 & 1 & 1 & ... & 1 \\ 1 & e^{\mu_1 p_1} & e^{\mu_1 p_2} & ... & e^{\mu_1 p_l} \\ 1 & e^{\mu_2 p_1} & e^{\mu_2 p_2} & ... & e^{\mu_2 p_l} \\ ... & ... & ... & ... & ... \\ 1 & e^{\mu_l p_1} & e^{\mu_l p_2} & ... & e^{\mu_l p_l} \end{matrix} \right\|$$

the inequality

$$|\det D_l| \ \geq \ 1 \tag{3.2}$$

is valid.

Proof. Is clear by induction. Indeed, for sufficiently large p_l we have expanding by last column

$$\det D_l \ = \ (-1)^l e^{\mu_l p_l} \, \det D_{l-1} \ + \ O(e^{\mu_{l-1} p_l})$$

and

$$|\det D_l| \ = \ e^{\mu_l p_l} \, |\det D_{l-1}| \left[1 + \ O(e^{-(\mu_l - \mu_{l-1}) p_l})\right].$$

□

Lemma 3.2. *Let s be a fixed integer, $1 \leq s \leq l$, and let condition (3.1) be fulfilled. Then for any $\tau > 0$ and $\alpha \in \mathbb{R}$ there exist a vector $\beta \in \mathbb{R}^{l+1}$ and numbers $t_1, t_2, ..., t_{l+1}$ such that*

$$t_{l+1} \ > \ t_l \ > \ ... \ > \ t_1 \ \geq \ \tau \tag{3.3}$$

and the following relations are valid:

$$\sum_{j=1}^{l+1} \beta_j \ = \ 0, \tag{3.4}$$

$$\sum_{j=1}^{l+1} \beta_j \, e^{\mu_k t_j} \ = \ \alpha \cdot \delta_{sk} \, , \quad k = 1, 2, ..., l, \tag{3.5}$$

and

$$|\beta_j| \ \leq \ C|\alpha| e^{-\mu_s \tau}, \quad j = 1, 2, ..., l+1, \tag{3.6}$$

where the constant C does not depend on the numbers τ and α.

Proof. Obviously, it suffices to prove this lemma in the case $\alpha = 1$. Let $p_l > p_{l-1} > ... > p_1 > 0$ be the numbers in Lemma 3.1. Set $t_1 = \tau$, $t_2 = \tau + p_1$, ..., $t_{l+1} = \tau + p_l$.

Consider the system of (l+1) equations for (l+1) unknowns β_j:

$$\beta_1 \ + \ \beta_2 \ + \ ... + \ \beta_{l+1} \ = \ 0 \, ,$$

$$\beta_1 \;+\; \beta_2 e^{\mu_k p_1} \;+\; ...+\; \beta_{l+1} e^{\mu_k p_l} \;=\; e^{-\mu_k \tau}\delta_{sk}, \quad k=1,2,...,l.$$

According to Cramer's rule, the solution exists, is unique and has the form

$$\beta_j \;=\; \frac{\Delta_j}{\Delta}.$$

It is clear that the matrix Δ of this system does not depend on τ and coincides with D_l in Lemma 3.1. Hence, it satisfies inequality (3.2), i. e. $|\Delta| \geq 1$. It is clear also that

$$\Delta_j \;=\; O(e^{-\mu_s \tau}).$$

Inequality (3.6) follows from this estimate. □

Set $\mathbf{e}_1 = (1,0,...,0)$, $\mathbf{e}_2 = (0,1,0,...,0), ..., \mathbf{e}_m = (0,0,...,0,1)$.

Lemma 3.3. *Let r be a fixed integer, $1 \leq r \leq m$. There exists a vector $h \in \mathbb{R}^m$ such that for any $\tau > 0$ there exist a vector $\beta \in \mathbb{R}^{l+1}$ and numbers $t_1, t_2, ..., t_{l+1}$ with following properties:*

1) conditions (3.3) and (3.4), and the equality

$$\sum_{j=1}^{l+1} \beta_j \widehat{E}(t_j)\widehat{\Theta} h \;=\; \mathbf{e}_r \tag{3.7}$$

are fulfilled;

2) the estimate

$$|\beta_j| \;\leq\; Ce^{-\lambda_1 \tau}, \quad j=1,2,...,l+1, \tag{3.8}$$

where the constant C does not depend on the number τ, is valid.

Proof. Choose the integer s, $1 \leq s \leq l$, such that

$$N_{s-1} \;<\; r \;\leq\; N_s.$$

According to the assumptions, the vectors $\Theta_i \in L_s$ are linearly independent. Denote by h any nonzero vector in L_s, which is orthogonal to all $\Theta_i \in L_s$ except Θ_r. It means that $(\Theta_r, h) \neq 0$.

Consequently,

$$(\widehat{\Theta} h)_i \;=\; (\Theta_i, h) \;=\; 0, \quad \Theta_i \in L_s, \quad i \neq r, \tag{3.9}$$

and

$$(\widehat{\Theta} h)_r \;=\; (\Theta_r, h) \;\neq\; 0. \tag{3.10}$$

Consider equation (3.7), which we rewrite in the following form:

$$\left(\sum_{j=1}^{l+1} \;\beta_j\, e^{\lambda_i t_j}\right)(\widehat{\Theta} h)_i \;=\; \delta_{ri}\,, \quad i=1,2,...,m\;. \tag{3.11}$$

To find the solution $\beta \in \mathbb{R}^{n+1}$ of the system (3.11) and (3.4), we apply Lemma 3.2, which guarantees the existence of the required $\beta \in \mathbb{R}^{n+1}$ and t_j for

$$\alpha \;=\; \frac{1}{(\widehat{\Theta}h)_r} \;=\; \frac{1}{(\Theta_r, h)}. \tag{3.12}$$

1) Let $N_{k-1} < i \le N_{k-1}$ and $k \ne s$. Then, according to (3.5),

$$\left(\sum_{j=1}^{l+1} \beta_j \, e^{\lambda_i t_j}\right) (\widehat{\Theta}h)_i \;=\; \left(\sum_{j=1}^{l+1} \beta_j \, e^{\mu_k t_j}\right) (\widehat{\Theta}h)_i \;=\; 0 \cdot (\widehat{\Theta}h)_i \;=\; 0. \tag{3.13}$$

2) If $N_{s-1} < i \le N_s$ and $i \ne r$ then, according to (3.9), $(\widehat{\Theta}h)_i = 0$, and

$$\left(\sum_{j=1}^{l+1} \beta_j \, e^{\lambda_i t_j}\right) (\widehat{\Theta}h)_i \;=\; 0. \tag{3.14}$$

3) If $N_{s-1} < i \le N_s$ and $i = r$ then, according to (3.5), (3.10) and (3.12),

$$\left(\sum_{j=1}^{l+1} \beta_j \, e^{\lambda_r t_j}\right) (\widehat{\Theta}h)_r \;=\; \left(\sum_{j=1}^{l+1} \beta_j \, e^{\mu_s t_j}\right) (\widehat{\Theta}h)_r \;=\; \alpha \cdot (\widehat{\Theta}h)_r \;=\; 1. \tag{3.15}$$

Equalities (3.13) – (3.15) show that this vector $\beta \in \mathbb{R}^{n+1}$ satisfies equations (3.11). Finally, we note that estimate (3.8) follows from (3.6). □

Lemma 3.4. *For any integer r, $1 \le r \le m$, and for any $\alpha \in \mathbb{R}$ there exist a number $T_0 > 0$ and an admissible control $q(t)$ such that for $T \ge T_0$*

$$\int_0^T \widehat{E}'(t)\widehat{\Theta}q(t)\, dt \;=\; \alpha \mathbf{e}_r. \tag{3.16}$$

Proof. Let β_j and t_j and vector $h \in \mathbb{R}^m$ be as in Lemma 3.3.

Choose $\tau > 0$ such that

$$|\beta_j| \;\le\; \frac{1}{(l+1)|\alpha| \cdot |h|}. \tag{3.17}$$

We introduce the following $l+1$ control functions

$$q^{[j]}(t) \;=\; \begin{cases} h, & \text{for} \quad 0 \le t \le t_j, \\ 0, & \text{for} \qquad t > t_j, \end{cases} \tag{3.18}$$

where $j = 1, 2, ..., l+1$.

Then for $T > t_j$ we have

$$\int_0^T \widehat{E}'(t)\widehat{\Theta}q^{[j]}(t)\, dt \;=\; \int_0^{t_j} \widehat{E}'(t)\widehat{\Theta}h\, dt \;=\; \widehat{E}(t_j)\widehat{\Theta}h - \widehat{E}(0)\widehat{\Theta}h. \tag{3.19}$$

Set

$$q(t) \;=\; \alpha \sum_{j=1}^{l+1} \beta_j q^{[j]}(t). \tag{3.20}$$

According to (3.17) the control $q(t)$ is admissible.

It is clear that if T is greater than all these numbers t_j then, according to (3.19), (3.4) and (3.7),

$$\int_0^T \widehat{E}'(t)\widehat{\Theta} q(t)\,dt \;=\; \alpha \sum_{j=1}^{l+1} \beta_j \widehat{E}(t_j)\widehat{\Theta} h \;-\; \alpha \sum_{j=1}^{l+1} \beta_j \widehat{E}(0)\widehat{\Theta} h \;=\; \alpha \mathbf{e}_r.$$

□

Proof of Theorem 1. Let $\psi \in L_2(\Omega)$. Set

$$g \;=\; -\Lambda_m \psi,$$

where the operator $\Lambda_m : L_2(\Omega) \to \mathbb{R}^m$ is defined by (2.8). Hence,

$$g \;=\; \sum_{r=1}^{m} c_r \mathbf{e}_r.$$

We apply Lemma 3.4 for $\alpha = mc_r$. Then for each r, $1 \le r \le m$, we get an admissible control $q^{[r]}(t)$ such that

$$\int_0^T \widehat{E}'(t)\widehat{\Theta} q^{[r]}(t)\,dt \;=\; mc_r \mathbf{e}_r, \quad T \ge T_0.$$

Set

$$q(t) \;=\; \frac{1}{m} \sum_{r=1}^{m} q^{[r]}(t).$$

It is clear that $q(t)$ is an admissible control. Then

$$\int_0^T \widehat{E}'(t)\widehat{\Theta} q(t)\,dt \;=\; \frac{1}{m} \sum_{r=1}^{m} \int_0^T \widehat{E}'(t)\widehat{\Theta} q^{[r]}(t)\,dt \;=\; \frac{1}{m} \sum_{r=1}^{m} mc_r \mathbf{e}_r \;=\; g.$$

Hence, the control function $q(t)$ satisfies equation (2.9) and, according to Lemma 2.3, condition (1.11) is fulfilled. □

4 Proof of Theorem 2

Assume that convectors $\{\Gamma_i, a_i\}_{i=1}^m$ are not properly arranged. This means that some of the vectors Θ_i, defined by (1.17) and (1.18), are equal to zero, or for some k, $1 \le k \le l$, vectors $\Theta_i \in L_k$ are linearly dependent.

1) First we assume that $\Theta_r = 0$ for some r, $1 \le r \le m$, i. e. the following equalities are valid

$$\theta_{r1} \;=\; \theta_{r2} \;=\; \cdots \;=\; \theta_{rm} =\; 0. \tag{4.1}$$

Set $\psi(x) = v_r(x)$, then $\Lambda_m\psi = \mathbf{e}_r$. Let us suppose that there exists an admissible control $q(t) = (q_1(t), q_2(t), ..., q_m(t))$ such that

$$\int\limits_0^T \widehat{E}'(t)\widehat{\Theta}q(t)\,dt \;=\; -\Lambda_m\psi \;=\; -\,\mathbf{e}_r. \tag{4.2}$$

Then

$$\int\limits_0^T \sum_{j=1}^m \lambda_r e^{\lambda_r t}\theta_{rj}q_j(t)\,dt \;=\; -\,1. \tag{4.3}$$

This equality contradicts condition (4.1) and this contradiction proves Theorem 2.

2) Now we assume that for some k, $1 \le k < l$, the vectors $\Theta_i \in L_k$ are linearly dependent:

$$\sum_{k=N_{k-1}+1}^{N_k} c_i\Theta_i \;=\; 0, \qquad \sum_{k=N_{k-1}+1}^{N_k} |c_i|^2 \;>\; 0$$

(recall that in the case $k < l$ the subspaces L_k are defined by (1.19)).

Hence, if an admissible control exists then, according to (2.12),

$$\sum_{i=N_{k-1}+1}^{N_k} c_i(\psi, v_i) \;=\; -\int\limits_0^t \mu_k e^{\mu_k s}\left(\sum_{i=N_{k-1}+1}^{N_k} c_i\Theta_i,\; q(s)\right) ds \;=\; 0.$$

This means that the initial function ψ cannot be an arbitrary function but have to satisfy the additional condition

$$\sum_{i=N_{k-1}+1}^{N_k} c_i(\psi, v_i) \;=\; 0.$$

3) Finally, in the case $k = l$ the subspace L_l is defined by (1.20) and the assumption about null controllability under the same considerations leads to the condition

$$\sum_{i=N_{l-1}+1}^{m} c_i(\psi, v_i) \;=\; 0, \qquad \sum_{i=N_{l-1}+1}^{m} |c_i|^2 \;>\; 0.$$

Hence, for

$$\psi(x) \;=\; \sum_{i=N_{l-1}+1}^{m} c_i v_i(x)$$

the null controllability does not take place. □

Acknowledgments

This work was supported by the Fund of Fundamental Investigations of Republic of Uzbekistan.

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Shavkat Alimov
Tashkent Branch of M.V. Lomonosov Moscow State University
22 Movaraunnahr St
100060 Tashkent, Uzbekistan
E-mail: sh_alimov@mail.ru

Received: 20.06.2011