EURASIAN MATHEMATICAL JOURNAL
volume 16 , number 4, 2025

EURASIAN NATIONAL UNIVERSITY

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

CONTENTS

EURASIAN MATHEMATICAL JOURNAL

ISSN 2077-9879
9 772077 987003

VOLUME 16, NUMBER 4 2025

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2025, Volume 16, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;
- compliance of the title of the paper to its content;
- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);
- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);
- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);
- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Republic of Kazakhstan

The Moscow Editorial Office
The Patrice Lumumba Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russian Federation

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 4 (2025), 76 – 94

SONIN'S INVENTORY MODEL WITH A LONG-RUN AVERAGE COST FUNCTIONAL

E.L. Presman

Communicated by K.T. Mynbayev

Key words: inventory model, Markov chain, optimality equation, canonical triplet.

AMS Mathematics Subject Classification: 93E20, 60J28, 90B05, 90C40, 91B70.

Abstract. We present an inventory model where a manufacturer (firm) uses for "production" a "commodity" (resource), which is consumed with the unit intensity. The price of the commodity follows a stochastic process, modelled by a continuous time Markov chain with a finite number of states and known transition rates. The firm can buy this commodity at the current price or use "stored" one. The storage cost is proportional to the storage level. The goal of the firm is to minimize the long-run average cost functional. We prove the existence of a canonical triple with an optimal threshold strategy, present an algorithm for constructing optimal thresholds and the optimal value of the functional, and discuss issues of uniqueness.

DOI: https://doi.org/10.32523/2077-9879-2025-16-4-76-94

1 Introduction

There are various mathematical inventory models (see, for example, Arrow et al., 1951, [1]; Bather, 1966, [2]; Rubalskiy, 1972, [15]; Browne, Zipkin 1991, [4]; Bayer et al., 2010, [3]; Bulinskaya, Sokolova, 2015, [5]. These models allow us to find the optimal strategy for purchasing a product that minimizes the total cost of purchasing, registration and delivery of the order, storage of goods, as well as losses from its shortage.

In this paper, we will consider the following inventory problem.

There is a manufacturer (firm) who needs to consume an intermediate product (commodity) with the unit intensity for production. If the price of a commodity is constant, then it is possible to purchase the commodity with the same intensity, and thereby production will be ensured. If the price changes over time, then at a low price it is reasonable to buy a one-time quantity in order not to overpay at a high price. But if you make a one-time purchase, then the purchased product must be stored, and you have to pay for this. An example of such a commodity is cotton, which, on one hand, is still one of the most important world products, and on the other hand, is characterized by significant price fluctuations, see for example Darekar and Reddy, 2017, [6].

I.M. Sonin proposed to consider the situation in which the price depends on the value of a Markov chain with continuous time, a finite number of n states and known transition intensities. In this case, it is reasonable to create a warehouse and make purchases and create a stock in accordance with the price.

It is assumed that purchases at the current price can be made both in large quantities and by continuously increasing (without decreasing) the quantity of the purchased goods.

It is assumed that the storage fee is proportional to the amount of goods in stock, the cost of ordering does not depend on the size of the order, the order is processed instantly.

The question is how to organize the work of the warehouse in order to minimize the expected costs of storage and purchase of goods, while the costs can be considered both discounted and marginal averages per unit of time.

This problem was studied first in the thesis of Hill, 2004, [8], and then in Hill, Sonin, 2006,[9], Katehakis, Sonin, 2013, [10] for the case $n = 2$ and special cases for $n = 3$. In these works, the marginal average expected costs per unit of time were considered and it was assumed that, in the general case, the strategy for the optimal organization of the work of the warehouse is of a threshold nature, i. e. :

for each state i of a Markov chain, there is a threshold a_i such that

if the inventory level x is less than or equal to a_i, then it is necessary to make a one-time purchase up to the level of this threshold, i.e. buy goods in the quantity $a_i - x$, and then, until the next jump of the Markov process (the moment of price change), it is necessary to purchase goods with the unit intensity so that the inventory level is equal to the threshold value a_i,

if $x > a_i$, then purchases should not be carried out until the next jump of the Markov process or the moment when the value of the inventory level, decreasing with the unit intensity becomes equal to a_i.

In Presman, Sonin, 2023, [14], using the methods developed in Presman, Sonin, 1982, [11], Presman, Sethi and Zhang, 1995, [12], Presman, Sethi, 2006, [13], a complete solution of this problem was given for the case of discounted costs. The existence of an optimal threshold strategy was proved, an algorithm for constructing optimal thresholds and the optimal value of the functional was formulated, the problem of the uniqueness of the optimal control was investigated.

We first describe the main ideas and features of the results and proofs of [14]. First, it turned out that in problems with Markov chains with continuous time it is convenient to write the optimality equation (the Bellman equation) in the form of choosing the optimal control until the moment of the first jump, i.e., consider an imbedded Markov chain. After that, from considerations of the convexity of the value function, it is easy to show that there is an optimal threshold control. Secondly, it turned out that in such problems it is convenient to pass from studying the value function to studying its derivative. Thirdly, it turned out that instead of the smooth gluing condition (continuity of the derivative of the value function), which arises in problems with diffusion processes, the condition of twice smooth gluing appears (continuity of the second derivative). Fourth, as a rule, the optimal control is unique, but, for some relations between the parameters, in which, in addition, the corresponding transition intensities are equal to zero, for some states the optimal control is unique, and for others, any optimal control is of quasi-threshold character, i.e., there is a whole interval of optimal thresholds (from zero to maximum), and if in the corresponding state the inventory level does not exceed the maximal optimal threshold, then any control that does not go beyond the interval of optimal thresholds is optimal.

In this paper, we study the case with the long-run average cost functional under the assumption that the chain is regular, i.e., from any state it is possible with a positive probability to get to any other (not necessarily after the first jump). A passage to the limit is carried out with the discounting parameter tending to zero. As a result, an analogue of the canonical triple, known in the theory of controlled Markov chains with discrete time and a finite number of states, arises. It is shown that, as in the case of discounting, there is an optimal threshold strategy, an algorithm for constructing optimal thresholds is given, and issues of uniqueness are considered.

2 Problem formulation and results for discounted costs

Let a *right-continuous* Markov process $\{m(t)\}_{0\le t<\infty}$, $m(0)=i$, with a finite number of states n (a Markov chain in continuous time) and an intensity matrix (infinitesimal operator)

$$\Lambda=(\lambda_{i,j})\,,\quad \text{where}\ \ \lambda_{i,j}\ge 0,\ \lambda_{i,i}=-\sum_{j\ne i}\lambda_{i,j}=-\lambda_i<0.\qquad i,j\in N=\{1,\cdots,n\}\,,$$

are given, i.e., if at some point in time the process is in state i, then in a short time $\triangle$ with probability $\lambda_{i,j}\triangle+o(\triangle)$ it goes to state j, and with probability $1-\lambda_i\triangle+o(\triangle)$ it remains in state i.

Let $\mathcal{F}=\{\mathcal{F}_t\}_{0\le t<\infty}$ be the filtration generated by it, i.e., $\mathcal{F}_t$ contains all information about the Markov process up to and including the time t. Control u is

an $\mathcal{F}$-adapted **left continuous** nondecreasing function $u(t), u(0)=0$, whose value at time t corresponds to the total purchases of goods up to time t inclusive.

In other words, the total purchases up to time t can only depend on the behavior of the Markov process up to time t and cannot take into account whether the Markov process jumped at time t.

The moments of jumps and the sizes of jumps at these moments correspond to the moments and sizes of one-time purchases.

Since the consumption of a good occurs at the unit intensity, the equation

$$x^u(t)=x-t+u(t) \tag{2.1}$$

determines the inventory level at time t with an initial level of $x\ge 0$.

Controls for which $x^u(t)\ge 0$ for all $t\ge 0$ and the values of the functionals are finite will be called admissible. Denote by $\mathcal{U}(x)$ the set of all admissible controls for the initial point x.

For $u\in\mathcal{U}(x)$, we consider the functionals

$$V_i^{\rho,u}(x)=E_{x,i}\left\{\int_0^\infty e^{-\rho t}cx^u(t)dt+\int_0^\infty e^{-\rho t}P_{m(t)}du(t)\right\}\ \text{for}\ \rho>0, \tag{2.2}$$

$$V_i^{0,u}(x)=\limsup_{T\to\infty}\frac{1}{T}E_{x,i}\left\{\int_0^T cx^u(t)dt+\int_0^T P_{m(t)}du(t)\right\}\,, \tag{2.3}$$

where c is the cost of storage of a unit of goods per unit of time, P_i is the price of a product under condition that the Markov process is in the state $i\in N$, $E_{x,i}\{\cdot\}$ – mathematical expectation when the initial state of the Markov process is equal to i and the initial inventory level is x. Without loss of generality, it is assumed that $P_i>P_{i+1}$ for $1\le i\le n-1$.

The goal is to find for $\rho\ge 0$ the value

$$V_i^\rho(x)=\inf_{u\in\mathcal{U}(x)}V_i^{\rho,u}(x),\ i\in N, \tag{2.4}$$

and determine the optimal control. The function $V_i^\rho(x)$ is called the value function. The name of the value function is also used for the column vector $V^\rho(x)$ with coordinates $V_i^\rho(x),\ i\in N$.

We will say that a control is *a_i-threshold in state* $i\in N$ if it satisfies the following conditions.

Let $m(t_0)=i,\ x(t_0)=x$, be satisfied at some time $t_0\ge 0$ and let $t_0+\tau$ be the moment of the first jump after t_0 of the process $m(t)$. To simplify the notation, we assume that $t_0=0$.

If $x \le a_i$, then a one-time purchase of $a_i - x$ is immediately made to bring the inventory level to the value of a_i, and after that, up to the moment τ, purchases are made with the unit intensity to keep the inventory at the level a_i, and hence $x^u(t) = a_i$ for $0 < t \le \tau$, i, e. $u(t) = u(0) + a_i - x + t$ for $0 < t \le \tau$;

if $x > a_i$, then at first no purchases are made (in this case, the inventory level decreases with the unit intensity), and this happens until the minimum from the moment τ and the moment when the inventory level becomes equal to a_i, and then, if $\tau > a_i - x$, until τ purchases are made with unit intensity to keep inventory level at a_i, so $u(t) = u(0) + \max[0, t - (x - a_i)]$ for $0 < t \le \tau$, which means $x^u(t) = x - t > a_i$ for $0 < t \le \min[x - a_i, \tau]$, and $x^u(t) = a_i$ for $x - a_i < t \le \tau$.

Consider a vector a with coordinates $a_1, \dots, a_n$. We call a threshold strategy a control that is a_i-threshold for every $i \in N$.

As already mentioned, [14] gave a complete solution of this problem for the case $\rho > 0$. Let us first present the results obtained there.

Let P be an n-dimensional column vector with coordinates P_i, $i \in N$; I be an n-dimensional column vector with coordinates equal to 1; E be an $n \times n$ diagonal matrix such that all diagonal element are equal to 1; $V^\rho(x)$ be the column vector with coordinates $V_i^\rho(x)$, $i \in N$; $\mathbf{0}$ be the N-dimensional column vector with zero coordinates.

We define the vector $b^\rho = (\Lambda - \rho E)(-P) + cI$, so that

$$b_i^\rho = c + (\lambda_i + \rho)P_i - \sum_{j \ne i} \lambda_{i,j} P_j = c + \rho P_i + \sum_{j \ne i} \lambda_{i,j}(P_i - P_j), \quad i \in N. \tag{2.5}$$

It turned out that for any state $i \in N$ there exists an optimal threshold control, and the equality of the optimal threshold to zero or its positiveness depends on the sign of b_i^ρ.

Let us put

$$N_+^\rho = \{i : \ b_i > 0\}, \quad N_0^\rho = \{i : \ b_i = 0\}, \quad N_-^\rho = \{i : \ b_i < 0\}. \tag{2.6}$$

Remark 1. The set N_+^ρ is non-empty because $P_1 > P_i$ for all $j \ne i$, and from the second equality in (2.5) we get that $b_1 > 0$.

Below, if it is clear which $\rho > 0$ we are talking about, we will omit the superscript ρ for all considered quantities.

The following theorem was proved in [14].

Theorem 2.1. 1) *There is a vector a^* such that the a^*-threshold strategy is optimal in the class of all admissible controls in the problem of minimization of functional (2.2).*

2) *The vector function $V(x)$ is convex and its derivative $U(x) = \dfrac{dV(x)}{dx}$ is a unique* **continuously differentiable** *solution of the equation*

$$\begin{aligned} \frac{dU(x)}{dx} = \ & \max[b(x), \mathbf{0}], \ U(0) = -P, \quad \text{where} \\ & b(x) = (\Lambda - \rho E)\, U(x) + cI = (\Lambda - \rho E)\left(U(x) - \frac{c}{\rho} I\right), \end{aligned} \tag{2.7}$$

(hereinafter for any vector d the notation $\max[d, \mathbf{0}]$ *denotes taking the coordinate-wise maximum), moreover*

$$b(0) = b, \quad b_i(x) < 0 \text{ for } 0 \le x < a_i^*, \quad b_i(x) \ge 0 \text{ for } x \ge a_i^*, \quad i \in N, \tag{2.8}$$

which means

$$a_i^* > 0 \text{ if and only if } i \in N_-. \tag{2.9}$$

Let us put $a^m = \max_i\{a_i^*\}$. It follows from (2.7) and (2.8) that for $x \geq a^m$

$$U(x)-\frac{c}{\rho}I=e^{(\Lambda-\rho E)(x-a^m)}\left(U(a^m)-\frac{c}{\rho}I\right)=e^{(\Lambda-\rho E)(x-a^m)}U(a^m)-\frac{c}{\rho}e^{-\rho(x-a^m)}I\,. \tag{2.10}$$

The function $V(x)$ itself is determined from the relation

$$V(x)=\begin{cases} c\dfrac{\rho x-1}{\rho^2}I-\displaystyle\int\limits_x^\infty\left(U(y)-\frac{c}{\rho}I\right)dy \\ \qquad =c\dfrac{e^{-\rho(x-a^m)}+\rho x-1}{\rho^2}I+e^{(\Lambda-\rho E)(x-a^m)}(\Lambda-\rho E)^{-1}U(a^m) \text{ for } x\geq a^m, \\ -\displaystyle\int\limits_x^{a^m}U(y)dy+V(a^m)=-\int\limits_x^{a^m}U(y)dy+c\frac{a^m}{\rho}I+(\Lambda-\rho E)^{-1}U(a^m) \text{ for } 0\leq x<a^m. \end{cases} \tag{2.11}$$

For $a^m > 0$, in [14] an algorithm was formulated for successive construction of the vector-function $U(x)$ and the vector a^* on successive intervals between thresholds, starting from the interval $[a^{(1)}, a^{(2)}]$, where $a^{(1)} = 0$ and $a^{(2)}$ is the minimal positive threshold. On this interval, the vector-function $U(x)$ was first constructed, and then the threshold $a^{(2)}$ and the set $I^{(2)} = \{i : a_i^* = a^{(2)}\}$. Then, if $a^m > a^{(2)}$, then the same is done for the interval $[a^{(2)}, a^{(3)}]$, where $a^{(3)}$ is the minimal threshold of those thresholds that are greater than $a^{(2)}$, etc., up to $a^{(r)}$, where the number r was determined from the condition $a^{(r)} = a^m$.

For a more precise formulation, we need some notation. For any set of thresholds a, which is convenient for us to consider as a column vector with coordinates a_i, consider:

the number r corresponding to the number of different threshold values;

increasing numbers $a^{(1)}(a), \ldots, a^{(r)}(a)$ corresponding to different threshold values;

$$\begin{aligned} &\text{sets } I^{(l)}(a),\ 1 \leq l \leq r(a), \text{ where } I^{(l)}(a)=\{i:\ a_i=a^{(l)}(a)\}; \\ &\text{sets } N_+^{(l)}(a) = \{i:\ a_i \leq a^{(l)}(a)\} = \bigcup_{k=1}^{l} I^{(k)}(a), \text{ while } N_+^{(r)}(a)=N; \\ &\text{sets } N_-^{(l)}(a) = N\setminus N_+^{(l)}(a) = \{i:\ a_i > a^{(l)}(a)\}. \end{aligned} \tag{2.12}$$

For $a = a^*$ we will omit the dependence on a and simply write r; $a^{(l)}, I^{(l)}, N_+^{(l)}, N_-^{(l)}$, $1 \leq l \leq r$. In addition, in further notation, if it is clear which l we are talking about, we will not write the superscript "(l)".

Let us introduce the following notation:

$$\Lambda_+ = (\lambda_{i,j})_{i,j\in N_+^{(l)}}\,, \quad \Lambda_- = (\lambda_{i,j})_{i,j\in N_-^{(l)}}\,, \quad \Lambda_\pm = (\lambda_{i,j})_{i\in N_+^{(l)},\, j\in N_-^{(l)}}\,,$$

$$\Lambda_\mp = (\lambda_{i,j})_{i\in N_-^{(l)},\, j\in N_+^{(l)}}\,, \quad A=\Lambda-\rho E\,, \quad A_+=\Lambda_+-\rho E_+\,, \quad A_-=\Lambda_--\rho E_-\,,$$

for any vector $d = (d_i)_{i\in N}$ we set $d_+ = (d_i)_{i\in N_+^{(l)}}$, $d_- = (d_i)_{i\in N_-^{(l)}}$. Here the subscript "+" (respectively "−") defines a vector with coordinates from $N_+^{(l)}$ (respectively $N_-^{(l)}$) and transitions from $N_+^{(l)}$ to $N_+^{(l)}$, (respectively, from $N_-^{(l)}$ to $N_-^{(l)}$). The subscript "±" (respectively "∓") defines transitions from $N_+^{(l)}$ to $N_-^{(l)}$ (respectively from $N_-^{(l)}$ to $N_+^{(l)}$).

Remark 2. From the fact that A_+ corresponds to a Markov chain with the state set $N_+^{(l)}$ and the killing rate in state i equal to $\rho_i = \rho + \sum_{j\in N_-^{(l)}} \lambda_{1,j}$, the existence of the inverse matrix $(A_+)^{-1}$ follows.

Given l, $1 \leq l < r$, for $0 \leq x < \infty$ we define the column vector-function $F(x)$ as follows: $F(x) = U(x)$ for $0 \leq x \leq a^{(l)}$, and for $y \geq 0$

$$\begin{aligned} F_-(a^{(l)} + y) = & \ -P_- \\ F_+(a^{(l)} + y) = & \ -(A_+)^{-1}\left[cI_+ - \Lambda_{\pm}P_-\right] + (A_+)^{-1} e^{A_+ x} b_+(a^{(l)}). \end{aligned} \tag{2.13}$$

Consider also the vector-function

$$f(x) = cI + (\Lambda - \rho E)\, F(x), \;\; 0 \leq x < \infty, \tag{2.14}$$

so that $f(x) = b(x)$ for $0 \leq x \leq a^{(l)}$.

It follows from relation (2.8) that $f_i(a^{(l)}) \geq 0$ for $i \in N_+^{(l)}$, $f_i(a^{(l)}) < 0$ for $i \in N_-^{(l)}$. Consider now the function

$$f^{\max}(x) = \max_{i \in N_-^{(l)}} f_i(x), \qquad x \geq a^{(l)}. \tag{2.15}$$

Proposition 2.1. Algorithm for constructing the vector-function $U(x)$ and the vector a^*. *Assume that for some $l < r$ we have constructed $a^{(i)}$ and $I^{(i)}$ for $1 \leq i \leq l$, and also $U(x)$ for $0 \leq x \leq a^{(l)}$, satisfying (2.7). If $l < r$, then*

$$U(x) = F(x) \;\; \text{for} \;\; a^{(l)} \leq x \leq a^{(l+1)}, \tag{2.16}$$

$$a^{(l+1)} = \inf\{x: \; f^{\max}(x) > 0\}, \quad I^{(l+1)} = \{i: \; i \in N_-^{(l)}, \;\; f_i(a^{(l+1)}) = 0\}, \tag{2.17}$$

where $F(x)$, $f(x)$ and $f^{\max}(x)$ are defined in (2.13), (2.14), (2.15).

Let us pass to the study of the uniqueness of the optimal control. Let $\bar{a}_i = \inf\{x: \, b_i(x) > 0\}$. From (2.8) it follows that $\bar{a}_i \geq a_i^*$. It turns out that if $\bar{a}_i = a_i^*$, then the optimal control in state i is unique, and if $\bar{a}_i > a_i^*$, then $a_i^* = 0$, and in the state i any threshold control with a threshold not exceeding $\bar{a}_i$ is optimal. Moreover, in this state, the control is optimal if and only if, in this state, it prescribes not to make purchases at an inventory level greater than $\bar{a}_i$, and if the inventory level does not exceed $\bar{a}_i$, then it prescribes to make such purchases so that the trajectory of the inventory level does not go beyond the segment $[0, \bar{a}_i]$.

We say that a control in state $i \in J$ is a_i-quasi-threshold if it satisfies the following conditions.

Let $m(t_0) = i$, $x(t_0) = x$, be satisfied at some time $t_0 \geq 0$ and let $t_0 + \tau$ be the moment of the first jump after t_0 of the process $m(t)$.

If $x > a_i$ and $t_0 < t < t_0 + \min(x - a_i, \tau)$, then, as for threshold control, $u(t) = u(t_0)$. If $x > a_i$ and $x - a_i < t < t_0 + \tau$, or $x \leq a_i$ and $t_0 < t < t_0 + \tau$, then the control u is such that $0 \leq x^u(t) \leq a_i$.

Let $\tilde{N} \subset N$. The set of $(a, \tilde{N})$-quasi-threshold strategies is the set of all admissible controls that satisfy the following properties: in states $i \notin \tilde{N}$ they are a_i-threshold, and in states $i \in \tilde{N}$ they are a_i-quasi-threshold.

Denote by N_0^- the set of those states from N_0 from which one can get to states belonging to N_+ only by visiting states from N_-. The following theorem was proved in [14].

Theorem 2.2. *If $N_0^- = \emptyset$, then $\bar{a}_i = a_i^*$ for all $i \in N$ and in the optimization problem (2.2) the optimal control is unique and is given by a^*-threshold strategy, and if $N_0^- \neq \emptyset$, then from $i \notin N_0^-$ it follows that $\bar{a}_i = a_i^*$, from $i \in N_0^-$ it follows that $\bar{a}_i > a_i^* = 0$, $b_i(x) = 0$ for $0 \leq x \leq \bar{a}_i$ and the control is optimal if and only if it belongs to the set of $(\bar{a}, N_0^-)$-quasi-threshold strategies. Wherein:*

a) *$U_i(x) = -P_i$ for $0 \leq x \leq \bar{a}_i$ and $b_i(x) > 0$ for $x > \bar{a}_i$ (and hence, by virtue of the first equality in (2.7), $U_i(x)$ strictly increases for $x > \bar{a}_i$), $i \in N$.*

b) *if $i \in N_0^{(1)}$, then there exists l, $2 \leq l \leq r$ such that $\bar{a}_i = a^{(l)} > 0$, and if $\bar{a}_i > a^{(l)}$, then $\lambda_{i,j} = 0$ for any such j, that $\bar{a}_j \leq a^{(l)}$.*

For $1 \le l \le r$ we set:

$$I_0^{(l)} = \{i : i \in N_0,\, \bar{a}_i = a^{(l)}\},\ N_0^{(l-1)} = \{i : i \in N_0,\, \bar{a}_i \ge a^{(l)}\}, \quad N_+^{(l)} = \{i : \bar{a}_i \le a^{(l)}\}.$$

It follows from here that $N_0^{(0)} = N_0 = \sum_{l=1}^{r} I_0^{(l)}$, $N_0^{(l)} = N_0^{(l-1)} \setminus I_0^{(l)}$. It follows from Theorem 2.2 that $N_0^{(1)} = N_0^-$.

In paper [14] the following algorithm for the sequential construction of sets $I_0^{(l)}$ was given. This algorithm is related to the structure of the zero elements of the matrix Λ.

At first it was shown how the set $I_0^{(1)}$ is constructed. First, one includes in it all those elements $i \in N_0$ for which there exists $j \in N_+$ such that $\lambda_{i,j} > 0$, then all those elements $i \in N_0$ for which there exists $\lambda_{i,j} > 0$ for j included in the previous step, and so on.

Proposition 2.2. Algorithm for constructing sets $I_0^{(l)}$. *Let the sets $I_0^{(i)}, 1 \le i \le l$ be constructed for some $l < r$ (this was done above for $l = 1$). From statement* b) *of* Theorem 3.2, *it follows that for any $i \in N_0^{(l)}$ it is true that: $\lambda_{i,j} = 0$ for any $j \in N_+^{(l)}$. To construct $I_0^{(l+1)}$, we first include in it those elements $i \in N_0^{(l)}$ for which there exists $j \in I^{(l+1)}$, such that $\lambda_{i,j} > 0$. If this set is empty, then the set $I_0^{(l+1)}$ is also empty. Otherwise, to those included in $I_0^{(l+1)}$ at the first stage, we add those elements i from the elements remaining in $N_0^{(l)}$ for which there exists j included in the first step such that $\lambda_{i,j} > 0$. If this set is empty, then the construction of the set $I_0^{(l+1)}$ is complete. If not, then we repeat the procedure, and so on. As a result, the set $I_0^{(l+1)}$ will be constructed.*

Remark 3. It is worth paying attention to the fact that, although in the formula (2.13) the vector $F_+(x)$ is written as an exponent, nevertheless, if $\bar{a}_i > a^{(l)}$, then $i \in N_+^{(l)}$ and $F_i(x) \equiv P_i$. It is possible to reformulate Proposition 2.2 in such a way that all coordinates of the new vector $F_+(x)$ are represented as constants minus decreasing functions, which are linear combinations of decreasing exponentials, perhaps multiplied by sines or cosines (in the case of complex eigenvalues of the new matrix A_+), and, perhaps, multiplied by polynomials (in the case of multiples of eigenvalues). To do this, $N_+^{(l)}$ should be determined not according to (2.12), but according to the formula: $N_+^{(1)} = N_+ \cup I_0^{(1)}$, $N_+^{(l)} = N_+^{(l-1)} \cup I_-^{(l)} \cup I_0^{(l)}$ for $1 < l < r-1$. In the sequel, we will talk about the algorithm in this formulation.

The aim of this paper is to consider the case $\rho = 0$, i.e., studying functional (2.3). This is carried out by passing to the limit as the discount coefficient tends to zero. Therefore, in what follows we return to the use of the superscript ρ.

3 Main results

3.1 Main theorems

If the chain $m(t)$ is regular, then it is natural to assume that in the problem of minimization of functional (2.3) (a long-run average cost) the optimal value of the functional $V_i^0(x)$ does not depend on i and x and is equal to some number, which we will denote V^*. In this case, it is natural to consider the problem of finding

$$G_i(x) = \inf_{u \in \mathcal{U}(x)} E_{x,i} \left\{ \int_0^\infty \left[(cx^u(t) - V^*)dt + P_{m(t)} du(t) \right] \right\}, \ i \in N. \tag{3.1}$$

If such a function exists, then, by virtue of the Bellman optimality principle, for any stopping time $\mathcal{T}$ (with respect to the process $m(t)$) with a finite mathematical expectation, the following relation holds:

$$G_i(x)=\inf_{u\in\mathcal{U}(x)} E_{x,i}\left\{\int\limits_0^{\mathcal{T}} \left[(cx^u(t)-V^*)dt+P_{m(t)}du(t)\right]+G_{m(\mathcal{T})}(x^u(\mathcal{T}))\right\},\ i\in N. \tag{3.2}$$

A vector-function $G(x)$ that for any Markov moment with a finite mathematical expectation satisfies relation (3.2) is naturally called an invariant function for the problem of minimization of functional (2.3).

Let there exist a strategy u^* (i.e. a set of controls that assigns to each $i \in N$ and $x \geq 0$ a control $u^*_{x,i}(t) \in \mathcal{U}(x)$) on which the minimum of functional (3.1) is achieved. Let $x^*(t)$ be the trajectory corresponding to the control $u^*_{x,i}(t)$. If $E_{x,i}G_{m(T)}(x^*(T))$ is bounded with respect to T for each x and i, then

$$G_i(x) = E_{x,i}\left\{\int\limits_0^{\mathcal{T}} \big((cx^*(t)-V^*)dt + P_{m(t)}du^*_{x,i}(t)\big) + G_{m(\mathcal{T})}(x^*(\mathcal{T}))\right\}. \tag{3.3}$$

In this case $(V^*;\ u^*;\ G(x))$ is called the canonical triplet for the problem of minimization of functional (2.4) by analogy with the corresponding concept in control problems for discrete-time Markov chains (see [7], Chapter 7).

Theorem 3.1. *Let a chain be regular and $c > 0$. Then:*

a) *there exist a vector $a^{0,*}$, a number V^*, and a vector function $W(x)$ such that*

$$W(x) = \lim_{\rho\to 0} W^\rho(x),\ \text{where}\ \ W^\rho(x) = V^\rho(x) - \frac{V^*}{\rho}I, \qquad a^{0,*} = \lim_{\rho\to 0} a^{\rho,*}, \tag{3.4}$$

and V^*, $a^{0,*}$-threshold strategy and $W(x)$ form a canonical triplet for minimization problem (2.3),

b) *the function $U^0(x) = \dfrac{dW(x)}{dx}$ for $0 \leq x \leq a^{0,m} = \max_i\{a_i^{0,*}\}$ is a unique continuously differentiable solution of the equation*

$$\frac{dU^0(x)}{dx} = \max\left(\Lambda U^0(x) + cI, \mathbf{0}\right), \quad U^0(0) = -P,\ \text{for}\ 0 \leq x \leq a^{0,m}, \tag{3.5}$$

here $V^* = ca^{0,m} - \bar{\Lambda}U^0(a^{0,m})$, where the row vector $\bar{\Lambda}$ denotes the invariant distribution of the Markov chain.

c) *$a_i^{0,*} > 0$ if $i \in N^0_-$ and $a_i^{0,*} = 0$ otherwise,*

d) *the solution of equation (3.5) and the numbers $a_i^{0,*}$ are constructed in accordance with the algorithm from Proposition* 2.1.

Remark 4. To find V^* in case $c > 0$ it is not necessary to find $W(x)$ for $x > a^{0,m}$, but it suffices to solve equation (3.5). Nevertheless, below, in (4.12) we give an explicit expression for $W(x)$ for $x > a^{0,m}$.

For $c = 0$, the situation differs from the case $c > 0$. We show that if $c = 0$ and $\rho \to 0$, then $V^{\rho,*} \to P_n$ and the solution of problem (3.2) is obtained by passing to the limit as $\rho \to 0$. It is clear that for $c = 0$ and $V^* = P_n$ to solve problem (3.2), it is necessary for $i \neq n$ to minimize the expected costs until the hitting time of state n. We show that when solving the latter problem, the optimal control is a threshold strategy for which the thresholds coincide with $a_i^{0,*} = \lim\limits_{\rho\to 0} a_i^{\rho,*}, i \neq n$. At the same time, it turns out that if for $i = n$ one uses the y-threshold control and after the first jump uses the obtained optimal strategy, then the larger y the smaller the value of functional is (it is due to the fact, that $a_n^{\rho,*} \to \infty$ as $\rho \to 0$). Thus, when solving problem (3.2), there exists an invariant function, but for $i = n$ there is no optimal control, while in problem (3.1) for any control, the value of the functional is infinity.

Before we formulate and prove these facts, we introduce the following notation. Denote $N_{+n} = N \setminus \{n\}$. By analogy with the previous ones, the subscript "+n" of the vector means that we are considering a vector with coordinates from N_{+n}.

Theorem 3.2. *Let a chain be regular and $c = 0$. Denote $\hat{a}_n^\rho = \max_{i \neq n} a_i^{\rho,*}$. Then the following statements hold*

a) *There exist $a_{+n}^{0,*} = \lim_{\rho\to 0} a_{+n}^{\rho,*}$; numbers $g > 0$, $\mu > 0$, and an integer number $m_0 \geq 0$ such that $\lim_{\rho\to 0} \dfrac{a_n^{\rho,*} - \hat{a}_n^\rho}{a(\rho)} = \dfrac{1}{\mu}$, where $a(\rho) > m_0$ is the root of the equation $[a(\rho)]^{m_0} e^{-a(\rho)} = g\rho$; for $0 \leq x < \infty$ there exists $U^0(x) = \lim_{\rho\to 0} U^\rho(x)$ such that $U_n^0(x) = -P_n$ and the function $U_{+n}^0(x)$ for $0 \leq x < \infty$ is a unique continuously differentiable solution of the equation*

$$\frac{dU_{+n}^0(x)}{dx} = \max\left(\Lambda_{+n} U_{+n}^0(x) - \Lambda^{(n)} P_n, \mathbf{0}_{+n}\right), \quad U_{+n}^0(0) = -P_{+n}, \tag{3.6}$$

where $\Lambda^{(n)}$ is the column-vector with elements $\lambda_{i,n}$, $1 \leq i \leq n-1$. The numbers $a_i^{0,*}$, $i \neq n$, and $U_{+n}^0(x)$ are constructed in accordance with the algorithm from Proposition 2.1.

b) *In the problem of minimization of functional (2.3) $V^* = P_n$ and there exists $W(x) = \lim_{\rho\to 0}\left[V^\rho(x) - \left(a_n^\rho + \dfrac{1}{\rho}\right) P_n I\right]$, which is an invariant function that for $i \neq n$ is equal to the minimal expected costs until the hitting time $\mathcal{T}^{(n)}$ of state n minus $P_n E_i\{\mathcal{T}^{(n)}\}$. Herewith*

$$W_n(x) = -xP_n, \qquad W_{+n}(x) = -xP_n I_{+n} - \int_x^\infty \left[U_{+n}^0(v) + P_n I_{+n}\right] dv, \tag{3.7}$$

where $U_{+n}^0(v) + P_n I_{+n}$ exponentially converges to $\mathbf{0_{+n}}$ as $v \to \infty$.

c) *For any $i \in N_{+n}$ the $a_i^{0,*}$-threshold control is optimal in the problem of minimization of the right hand side of (3.2) (where $c = 0$, $V^* = P_n$ and $G(x)) = W(x)$). There is no optimal control for $i = n$. For any $\varepsilon > 0$, there exists such $y(\varepsilon)$ that for $y > y(\varepsilon)$ the y-threshold control is ε-optimal for the state $i = n$. In the problem of minimization of functional (3.1) for any control, the value of the functional is infinity.*

3.2 Uniqueness

When studying functional (2.3), it makes no sense to talk about the uniqueness of the optimal control, since an arbitrary control on any fixed time interval with subsequent use of the optimal control gives the same value of the functional. But it makes sense to talk about the uniqueness of the optimal control for solving problem (3.1) with $G(x) = W(x)$, which, as mentioned, also gives a solution to problem (2.3).

After constructing $W(x)$ you can define numbers $\bar{a}_i^{0,*}$ and sets $I_0^{0,(l)}$, $N_0^{0,(l-1)}$, $1 \leq l \leq r$, just as it was done for $\rho > 0$.

Theorem 3.3. *The statements of Theorem 2.2 and Proposition 2.2 are also true in the optimization problem (3.2) with $G(x) = W(x)$ for $i \in N$ in the case $c > 0$ and for $i \in N_{+n}$ in the case $c = 0$.*

Remark 5. In [14] it was noted that, as a rule, optimal control is unique. If we fix all parameters except ρ, then non-uniqueness is possible only for a finite set of values of ρ at which some of b_i (see (2.5)) vanish, and even then, provided that the corresponding $\lambda_{i,j}$ equals zero. It follows directly from (2.5) that for $\rho = 0$ it may turn out that $N_0^{0,\{1\}} \neq \emptyset$ and therefore optimal control is not unique, while for sufficiently small ρ it is always true that $N_0^{\rho,\{1\}} = \emptyset$ and optimal control is unique.

4 Proofs

4.1 Proof of Theorem 3.1

In this section, we consider a regular Markov chain, for which with positive probability it is possible to go from any state to any other one, possibly after some jumps.

4.1.1. Let us first show how (3.4) yields 3.2) and statement b) of Theorem 3.1. Let us write down the obvious identity:

$$\begin{aligned} V_i^\rho(x) = &\inf_{u\in\mathcal{U}(x)} E_{x,i}\left\{\int_0^{\mathcal{T}} e^{-\rho t}\left(cx^u(t)dt + P_{m(t)}du(t)\right) + e^{-\rho\mathcal{T}}V^\rho_{m(\mathcal{T})}(x^u(\mathcal{T}))\right\} \\ = &E_{x,i}\left\{\int_0^{\mathcal{T}} e^{-\rho t}\left(cx^{a^{\rho,*}}(t)dt + P_{m(t)}du^{\rho,*}(t)\right) + e^{-\rho\mathcal{T}}V^\rho_{m(\mathcal{T})}(x^{a^{\rho,*}}(\mathcal{T}))\right\}. \end{aligned} \tag{4.1}$$

Using the equality $\int_0^{\mathcal{T}} e^{-\rho t}dt = \frac{1-e^{-\rho\mathcal{T}}}{\rho}$, relation (4.1) can be rewritten as:

$$\begin{aligned} W_i^\rho(x) = &\inf_{u\in\mathcal{U}(x)} E_{x,i}\left\{\int_0^{\mathcal{T}} e^{-\rho t}\left((cx^u(t) - V^*)dt + P_{m(t)}du(t)\right) + e^{-\rho\mathcal{T}}W^\rho_{m(\mathcal{T})}(x^u(\mathcal{T}))\right\} \\ = &E_{x,i}\left\{\int_0^{\mathcal{T}} e^{-\rho t}\left((cx^{a^{\rho,*}}(t) - V^*)dt + P_{m(t)}du^{\rho,*}(t)\right) + e^{-\rho\mathcal{T}}W^\rho_{m(\mathcal{T})}(x^{a^{\rho,*}}(\mathcal{T}))\right\}. \end{aligned} \tag{4.2}$$

We know the structure of the functions $W_i^\rho(x)$, and therefore we can take a limit as $\rho \to 0$. So we get (3.2) and assertion b) of Theorem 3.1.

4.1.2. Let us move on to the proof of the existence of the limit vector function $W(x)$.

First, we show that using the algorithm from Proposition 2.1, one can construct a vector $a^{0,*}$ such that $a^{0,*} = \lim_{\rho\to 0} a^{\rho,*}$, and a vector-function $U^0(x)$ defined for $0 \le x \le a^{0,m} = \max_{i\in N} a_i^{0,*}$, such that $U^0(x) = \lim_{\rho\to 0} U^\rho(x)$ for $0 \le x \le a^{0,m}$, $U^0(x)$ is the only continuously differentiable solution of equation (3.5).

It is easy to see that for this it suffices to prove the existence of a finite limit $\lim_{\rho\to 0} a^{\rho,*}$, because then relation (3.5) and assertion d) of Theorems 3.1 follow from the continuous dependence on the parameter of the solution of the system of linear differential equations. To prove the existence of a finite limit $\lim_{\rho\to 0} a^{\rho,*}$, in turn, it suffices to prove that for any $1 \le l < r^0$, there exists $i^{(l)} \in N_-^{(l)}$ such that $\lim_{x\to\infty} f^{\rho,(l)}_{i^{(l)}}(x) > 0$ for all $\rho \ge 0$ because, according to the algorithm, $f_i^{\rho,(l)}(a^{\rho,(l)}) < 0$ for any $i \in N_-^{(l)}$, and the positiveness of the corresponding limit guarantees the existence of $a^{\rho,(l+1)}$.

In [14] the proof of the existence of $a^{\rho,*}$ was based on the study of the optimality equation and made essential use of the property $\rho > 0$. Here we give an independent proof of the existence of $a^{\rho,*}$, which is valid for both $\rho > 0$ and $\rho = 0$. Therefore, later in this section we will omit index ρ.

For $f_-(x)$ formula (2.14) can be written as

$$f_-(x) = cI_- + A_{\mp}F_+(x) - A_-P_- \tag{4.3}$$

where, according to (2.13), for $x \geq a^{(l)}$

$$F_+(a^{(l)} + x) = (-A_+)^{-1}\left[cI_+ - \Lambda_\pm P_-\right] + (-A_+)^{-1} e^{A_+(x-a^{(l)})} b_+(a^{(l)}). \tag{4.4}$$

In Remark 2, it was said that since A_+ corresponds to a Markov chain with state set $N_+^{(l)}$ and the killing rate in state i is equal to $\rho_i = \rho + \sum_{j\in N_-^{(l)}} \lambda_{,j}$, the existence of the inverse matrix $(A_+)^{-1}$ follows. For $\rho = 0$, for some $i \in N_+^{(l)}$ the corresponding sum may turn out to be zero. However, the following lemma will be proved in Appendix **A1**.

Lemma 4.1. *If $\rho > 0$ or $\rho = 0$ and a chain $m(t)$ is regular, then*
a) *there exists $(A_+)^{-1}$, and all elements of this matrix are nonpositive,*
b) *the eigenvalue of the matrix A_+ with the maximal real part is negative, and therefore $e^{A_+x} \to 0$ for $x \to \infty$, and the convergence is exponential.*

From this lemma and from (4.4) we get

$$F_+ =: \lim_{x\to\infty} F_+(x) = (-A_+)^{-1}\left[cI_+ - A_\pm P_-\right], \tag{4.5}$$

and from this and from (4.3) it follows that

$$f_- =: \lim_{x\to\infty} f_-(x) = c[I_- + A_\mp (-A_+)^{-1} I_+] - BP_-, \tag{4.6}$$

where

$$B = A_- + A_\mp (-A_+)^{-1} A_\pm, \tag{4.7}$$

In Appendix **A2** an interpretation of the elements of the matrices $(A_+)^{-1}$ and B will be given and the following lemma will be proved.

Lemma 4.2. *If $\rho > 0$, or $\rho = 0$ and the chain $m(t)$ is regular, then*
a) *all off-diagonal elements of the matrix B are non-negative, and the sum of the off-diagonal elements over each row is positive,*
b)

$$BI_- + \rho\left(I_- + A_\mp (-A_+)^{-1} I_+\right) = \mathbf{0}_-. \tag{4.8}$$

Define $i^{(l)}$ from the condition $P_{i^{(l)}} = \max_{j\in N_-^{(l)}} P_j$. Multiplying (4.8) by $P_{i^{(l)}}$ and adding with (4.6) we get

$$f_- = (c + P_{i^{(l)}}\rho)[I_- + A_\mp (-A_+)^{-1} I_+] + B(P_{i^{(l)}} I_- - P_-). \tag{4.9}$$

All elements of the matrix $(-A_+)^{-1}$ are nonnegative, consequently all elements in the first square brackets are positive. All elements of the matrices $A_\mp$, and $A_\pm$ are also non-negative. Therefore, all coordinates of the vector in square brackets on the right side of (4.9) are non-negative. As for the last term on the right side of (4.9), its component corresponding to $i^{(l)}$ is obtained by multiplying the row corresponding to $i^{(l)}$ by the vector $P_{i^{(l)}} I_- - P_-$, whose coordinate corresponding to $i^{(l)}$ is equal to zero, and all other coordinates are strictly positive. Therefore, it follows from assertion a) of Lemma 4.2 that $f_{i^{(l)}} > 0$. This proves the existence of $a^{(l+1)}$, and hence the existence of the limit $a^{0,*}$ and the limit function $U^0(x)$ on the interval $[0, a^{0,m}]$, while $U^0(x)$ is the only continuously differentiable solution of equation (3.5). Thus, we have proved also both statements c) and d) of Theorem 3.1. To complete the proof of Theorem 3.1, it remains to consider the interval $[a^{\rho,m}, \infty)$.

4.1.3. Note now that on the interval $[a^{\rho,m}, \infty)$ the situation becomes more complicated for $\rho = 0$. On this interval, for $\rho > 0$, the function $V(x)$ is given by formula (2.11), which contains the matrix $(\Lambda-\rho E)^{-1}e^{(\Lambda-\rho E)(x-a^m)}$. Therefore, to carry out the passage to the limit as $\rho\to 0$, we need to consider the structure of the matrix $(\Lambda-\rho E)^{-1}e^{(\Lambda-\rho E)x}$.

Let a Markov chain $m(t)$ be regular, and let $\mu_i,\ i=1,\dots,N$ be the eigenvalues of the matrix Λ. Then the eigenvalue with the maximal real part (we will assume that this is μ_1) is single and equal to zero. It corresponds to the right column eigenvector I, consisting of ones, and the left row eigenvector $\bar{\Lambda}$, which defines the unique invariant distribution of the chain. All other $\mu_i, i\neq 1$, have a negative real part. For the case $N=3$ in [14] it is shown that among them there can be both complex conjugate and coinciding.

Let us first consider the situation in which all roots are different and real, and hence $\mu_i<0$ for $i\neq 1$. In such a case the following representation is valid:

$$\begin{aligned} e^{(\Lambda-\rho E)x}(\Lambda-\rho E)^{-1} &= X\operatorname{diag}\left(\frac{1}{\mu_i-\rho}e^{(\mu_i-\rho)x}\right)X^{-1}\\ &= -\frac{1}{\rho}e^{-\rho x}X_1Y_1+\sum_{i=2}^{N}\frac{1}{\mu_i-\rho}e^{(\mu_i-\rho)x}X_iY_i, \end{aligned}$$

where $\operatorname{diag}(f_i)$ is a diagonal matrix with diagonal elements f_i; X_i is the i-th column vector of the matrix X, which is the right eigenvector of the matrix Λ corresponding to the eigenvalue μ_i (with $X_1=I$); Y_i is the i-th row vector of the matrix $Y=X^{-1}$, which is the left eigenvector of the matrix Λ corresponding to the eigenvalue μ_i (where $Y_1=\bar{\Lambda}$).

In the general case, instead of the diagonal matrix, there are the corresponding Jordan cells, in which there are decreasing exponentials (with sines and cosines in the case of complex eigenvalues), multiplied by polynomials in the case of multiple eigenvalues.

In the general case, the following representation takes place

$$e^{(\Lambda-\rho E)x}(\Lambda-\rho E)^{-1}=-\frac{e^{-\rho x}}{\rho}I\bar{\Lambda}+B^\rho(x),\quad B^\rho(x)I=\mathbf{0}. \tag{4.10}$$

where the matrix $B^\rho(x)$ has a limit as $\rho\to 0$, while the elements of the limit matrix $B^0(x)$ are combinations of decreasing exponents, possibly multiplied by sines, cosines and polynomials, and the last equality follows from the fact that $Y=X^{-1}$.

Hence, using (2.11), (2.10), and (4.10) we obtain that for $x\geq a^{\rho,m}$

$$\begin{aligned} V^\rho(x) &= c\frac{\left(e^{-\rho(x-a^{\rho,m})}+\rho x-1\right)}{\rho^2}I-\left[\frac{e^{-\rho(x-a^{\rho,m})}}{\rho}I\bar{\Lambda}-B^\rho\left(x-a^{\rho,m}\right)\right]U^\rho\left(a^{\rho,m}\right)\\ &= c\frac{\left(e^{-\rho(x-a^{\rho,m})}+\rho x-1\right)}{\rho^2}I-\frac{e^{-\rho(x-a^{\rho,m})}}{\rho}I\bar{\Lambda}U^\rho(a^{\rho,m})+B^\rho\left(x-a^{\rho,m}\right)U^\rho\left(a^{\rho,m}\right) \qquad (4.11)\\ &= \frac{V^*}{\rho}I+\frac{c}{2}\left(x-a^{0,m}\right)^2I-\left(x-a^{0,m}\right)I\bar{\Lambda}U^0\left(a^{0,m}\right)+B^0\left(x-a^{0,m}\right)U^0\left(a^{0,m}\right)+o(\rho)I, \end{aligned}$$

where $V^*=ca^{0,m}-\bar{\Lambda}U^0\left(a^{0,m}\right)$. Thus, for $x\geq a^{\rho,m}$

$$W(x)=\frac{c}{2}\left(x-a^{0,m}\right)^2\ I-\left(x-a^{0,m}\right)I\bar{\Lambda}U^0\left(a^{0,m}\right)+\ B^0\left(x-a^{0,m}\right)U^0\left(a^{0,m}\right). \tag{4.12}$$

If $0\leq x\leq a^{0,m}$, then

$$W(x)=-\int_x^{a^{0,m}}U^0(y)dy+W\left(a^{0,m}\right),$$

where the vector-function $U^0(x)$ was constructed in **Section 4.1.2**. □

4.2 Proof of Theorem 3.2

.

First we will prove statement a) of the theorem.

For $c = 0$, the proof that there exists $a^{0,(l)} = \lim_{\rho\to 0} a^{\rho,(l)}$ for $1 \le l < r$, and for $0 \le x \le a^{0,(r-1)} = \hat{a}_n^\rho$ there exists a limit $U^0(x) = \lim_{\rho\to 0} U^\rho(x)$ is not different from the corresponding proof for $c > 0$.

In order to prove that $\lim_{\rho\to 0} a_n^\rho = \infty$ we first prove by induction that $b_n^\rho(a^{\rho,(l)}) < 0$ for sufficiently small ρ for any $1 \le l < r$. For $l = 1$, $a^{\rho,(1)} = 0$ holds, and, according to (2.5), $b_n^\rho(0) = \rho P_n + \sum_{j\neq n} \lambda_{n,j}(P_n - P_j)$. Since $P_n < P_j$ for any $j \neq n$, then for sufficiently small ρ this expression is negative. Thus, for $l = 1$ the induction hypothesis is satisfied. If $r = 2$, then everything is proved. Let $r > 2$ and we have proven that $b_n^\rho(a^{\rho,(l)}) < 0$ for some $1 \le l < r - 1$ for sufficiently small ρ.

Let us rewrite (4.6) for $c = 0$ and (4.8) in the form

$$f_- = -BP_-, \tag{4.13}$$

$$\mathbf{0}_- = BI_- + \rho\left(I_- + A_\mp\left(-A_+\right)^{-1} I_+\right). \tag{4.14}$$

Multiplying (4.14) by P_n and adding with (4.13) we get:

$$f_- = \rho[I_- + A_\mp\left(-A_+\right)^{-1} I_+]P_n + B(P_n I_- - P_-). \tag{4.15}$$

We will show that the last coordinate of the vector $B(P_nI_- - P_-)$, which corresponds to the state of the chain with the number n, is negative. In fact, this last coordinate is equal to the product of the last row of the matrix B by a vector whose last element is zero, and all the others are negative. But, according to Lemma 4.2, for $\rho \ge 0$ all coordinates other than the last one of the last row of the matrix B are non-negative, and at least one is positive. This proves the negativity of the last coordinate of the vector $B(P_nI_- - P_-)$. From here and from (4.14) it follows that for sufficiently small ρ, the last coordinate of the vector f_- is also negative. Thus, in the interval $a^{0,(l)} \le x < \infty$, the function $f_n(x)$ increases from the value $b_n^\rho(a^{\rho,(l)}) < 0$ to a negative value set by formula (4.14), remaining negative. According to the algorithm, on the interval $a^{\rho,(l)} \le x \le a^{\rho,(l+1)}$ the equality $b_n^\rho(x) = f_n(x)$ holds, and therefore $b_n^\rho(a^{\rho,(l+1)}) < 0$, which completes the proof of the induction assumption.

Thus, for $x > \hat{a}_n^\rho$ and a sufficiently small ρ, the set $N_-^{(r-1)}$ consists of one element n, and due to Lemma 4.1 for such ρ the matrix $(A_{+n}^\rho)^{-1}$ exists, is continuous at $\rho = 0$ and all its elements are nonpositive.

We define the matrix $A(\rho)$, column vectors $G(\rho)$ and $H(\rho)$, and row vector $\Lambda_{(n)}$ as follows:

$$\begin{aligned} A(\rho) = (A_{+n}^\rho)^{-1}, \quad G(\rho) = A(\rho)\Lambda^{(n)}, \quad \Lambda_{(n)} = \left(\lambda_{n,1}, \dots, \lambda_{n,n-1}\right), \\ H(\rho) = U_{+n}(\hat{a}_n^\rho) - P_n G(\rho) = A(\rho)\left[A_{+n}^\rho U(\hat{a}_n^\rho) - P_n\Lambda^{(n)}\right] = A(\rho) b_{+n}^\rho(\hat{a}_n^\rho) \le \mathbf{0}. \end{aligned} \tag{4.16}$$

It is evident, that $\Lambda_{(n)} I_{+n} = \lambda_n$, $G(\rho) = -I_{+n} - \rho A(\rho) I_{+n}$ (the last equality is obtained from the elementary equality $\Lambda^{(n)} = -\left[A_{+n}^\rho + \rho E_{+n}\right] I_{+n}$ by multiplying from the left by $A(\rho)$).

Using these notations and relations, we can rewrite (4.4) and (4.3) for $c = 0$ in the following form: for $x \ge 0$

$$\begin{aligned} F_{+n}^\rho(\hat{a}_n^\rho + x) &= P_n G(\rho) + e^{A_{+n}^\rho x} H(\rho) \\ &= -P_n I_{+n} + \rho P_n A(\rho) I_{+n} + A(\rho) e^{A_{+n}^\rho x} b_{+n}^\rho(\hat{a}_n^\rho) \end{aligned} \tag{4.17}$$

$$\begin{aligned} f_n^\rho(\hat{a}_n^\rho + x) &= P_n(\lambda_n + \rho) + \Lambda_{(n)} F_{+n}^\rho(\hat{a}_n^\rho + x) \\ &= \rho P_n\left(1 - \Lambda_{(n)} A(\rho) I_{+n}\right) + \Lambda_n A(\rho) e^{A_{+n}^\rho x} b_{+n}^\rho(\hat{a}_n^\rho). \end{aligned} \tag{4.18}$$

Note also that according to Remark 3, all coordinates of the vector $F^{\rho}_{+n}(\hat{a}^{\rho}_n + x)$ are strictly increasing.

The function $f^{\rho}_n(\hat{a}^{\rho}_n + x)$ is strictly increasing and according to Proposition 2.2 $f^{\rho}_n(\hat{a}^{\rho}_n) = b^{\rho}_n(\hat{a}^{\rho}_n) < 0$ and, according to (4.18) and due to Lemma 4.1 $f^{\rho}_{-} = \lim_{x\to\infty} f^{\rho}_n(x) > 0$. The threshold $a^{\rho,*}_n$ is determined from the condition $f^{\rho}_n(a^{\rho,*}_n) = 0$, which, according to (4.18), can be written as

$$\rho P_n \left(1 - \Lambda_n A(\rho) I_{+n}\right) = -\Lambda_n e^{A^{\rho}_{+n}(a^{\rho,*}_n - \hat{a}^{\rho}_n)} H(\rho). \tag{4.19}$$

It follows from here that $a^{\rho,*}_n \to \infty$ as $\rho \to 0$, therefore we got that for $x \geq a^{\rho,(r-1)}$ there exists a finite limit $F^0_{+n}(x) = \lim\limits_{\rho\to 0} F^{\rho}_{+n}(x)$. According to Proposition 2.2 for $a^{\rho,(r-1)} \leq x \leq a^{\rho}_n$ we have $F^{\rho}_{+n}(x) = U^{\rho}_{+n}(x),\ U^{\rho}_n(x) = -P_n$.

Thus, we proved that for $0 \leq x < \infty$ there exists a limit $U^0_{+n}(\hat{a}^0_n + x) = \lim\limits_{\rho\to 0} U^{\rho}_{+n}(\hat{a}^{\rho}_n + x)$, where

$$U^0_{+n}(x) = P_n G(0) + e^{\Lambda_{+n}(x)}(U^0_{+n}(\hat{a}^0_n) - P_n G(0)) = P_n I_+ - \rho A(\rho) I_+ + A(\rho) e^{\Lambda_{+n}(x)} b^{\rho}_{+n}(\hat{a}^{\rho}_n),$$

while $U^0_n(x) = -P_n$.

To complete the proof of statement a) of Theorem 3.2 it remains to study the limiting behavior of a^{ρ}_n as $\rho \to 0$. In Appendix **A3**, the following lemma will be proved, which completes the proof of statement a) of Theorem 3.2.

Lemma 4.3. a) *There exist numbers $g > 0$, $\mu > 0$, and integer number $m_0 \geq 0$ such that $\lim\limits_{\rho\to 0} \dfrac{a^{\rho,*}_n - \hat{a}^{\rho}_n}{a(\rho)} = \dfrac{1}{\mu}$, where $a(\rho)$ is the root of the equation $[a(\rho)]^{m_0} e^{-a(\rho)} = g\rho$.*

b) *There exists a finite limit $\lim\limits_{\rho\to 0} \dfrac{1}{\rho} e^{A^{\rho}_{+} a(\rho)}$.*

Let us proceed to the proof of statement b) of Theorem 3.2. For $c = 0$ and $0 \leq x < a^{\rho,m}$, expression (2.11) can be rewritten as

$$V^{\rho}(x) = -\int_x^{a^{\rho,*}_n} (U^{\rho}(y) + P_n I)dy + V^{\rho}(a^{\rho,*}_n) + (a^{\rho,*}_n - x))P_n I \text{ for } 0 \leq x < a^{\rho,*}_n.$$

It follows from this expression that to prove the existence of $W(x)$, it suffices to prove the existence and finiteness of two limits

$$\lim_{\rho\to 0} \int_x^{a^{\rho,*}_n} (U^{\rho}(y) + P_n I)dy = \int_x^{\infty} (U^0(y) + P_n I)dy < \infty, \tag{4.20}$$

$$\lim_{\rho\to 0} \left[V^{\rho}(a^{\rho}_n) - \frac{1}{\rho} P_n I\right] < \infty\,. \tag{4.21}$$

For $i = n$, the integrands in both parts of (4.20) are zero, since $U^{\rho}_n(x) = -P_n$ for $x < a^{\rho}_n,\ \rho \geq 0$. For $i \neq n$, convergence and finiteness in (4.20) follows from the proven statement a) of Theorems 3.2, from the second equality in (4.17), and from the exponential decay of integrands which in turn follows from Lemma 4.1.

Let us move on to the proof of relation (4.21). From the second equality in (4.11) it follows that for $c = 0$

$$V^{\rho}(a^{\rho}_n) - \frac{P_n}{\rho} I = -I\bar{\Lambda} \frac{U^{\rho}(a^{\rho}_n) + P_n I}{\rho} + B^{\rho}(0) U^{\rho}(a^{\rho}_n).$$

Since $V_n^\rho(a_n^\rho) = P_n$ it follows from here that to prove (4.20), it suffices to prove the existence and finiteness of the limit

$$\lim_{\rho\to 0} \frac{U_{+n}^\rho(a_n^\rho) + P_n I_{+n}}{\rho} < \infty. \tag{4.22}$$

This follows from the second equality in (4.17), and from statement b) of Lemma 4.3.

To complete the proof of statement b) of Theorem 3.2, it remains to verify that $W(x)$ is an invariant function. The proofs of this fact and that the $a_i^{0,*}$-threshold control in state $i \neq n$ is optimal in the problem of minimization of the right hand side of (3.2) is similar to the proof of the corresponding facts for the case $\rho > 0$, which is given in Section 4.1.1. One only needs to substitute $V_n + \rho a_n^{\rho,*}$ instead of V^* in (4.2) and take into account that $\rho a_n^{\rho,*} \to 0$ as $\rho \to 0$.

Let us show that the y-threshold control is ε-optimal for the state $i = n$ in the problem (3.2). Indeed, let us first consider the case in which $\mathcal{T}$ is the moment of the first jump of the process $m(t)$. Then the application of y-threshold control in the state (x, n) with $y > x$ is reduced to a one-time purchase of $y - x$ and a subsequent purchase with intensity P_n up to the moment τ. Therefore, the difference between the value of the functional in (3.2), corresponding to the y-threshold control, and the optimal value (equal to $-xP_n$), is

$$\Delta_y = E_{x,n}\left[yP_n + G_{m(\tau)}(y)\right] = \sum_{j=1}^{n-1} \frac{\lambda_{n,j}}{\lambda_n}\left[yP_n + G_j(y)\right] = -\sum_{j=1}^{n-1} \frac{\lambda_{n,j}}{\lambda_n} \int_y^\infty [sP_n + U_j^0(s)ds, \tag{4.23}$$

where the last equality follows from (3.7). Now ε-optimality for sufficiently large y follows from the exponential convergence to zero of the integrands in (4.23), proved in statement b) of Theorem 3.2.

For an arbitrary τ with a finite mathematical expectation, it is necessary to split the interval from zero to τ into a random number of intervals between successive hits of the chain in state n. On each such interval, except for the last one, the increment of the functional value will be equal to Δ_y, and on the last one it will be only less. □

4.3 Proof of Theorem 3.3

.

According to Remark 4, the value of V^* in (3.1) is uniquely determined. At the same time, if the coordinates of the vector-function $G(x)$ satisfy (3.1), then for any constant C the vector $G(x) + CI$ also satisfies relation (3.1). Therefore, without loss of generality, we will assume that $G(0)$ is the same as the value obtained from (3.4).

Let us write relation (3.1) for the moment τ of the first jump of the process $m(t)$

$$G_i(x) = \inf_{z\in\mathcal{Z}(x)} E_{x,i}\left\{\int_0^\tau cx^z(t)dt - V^*\tau + P_i z(\tau) + G_{m(\tau)}(x^z(\tau))\right\}, \tag{4.24}$$

where $\mathcal{Z}(x)$ is the set of admissible controls between jumps, i.e., the set of nondecreasing left continuous deterministic functions $z =: z(t)$ such that $z(0) = 0$ and $x^z(t) =: x - t + z(t) \geq 0$ for any $t > 0$.

Substituting here the expression $x^z(\tau) + \tau - x$ instead of $z(\tau)$ and taking into account that the moment τ of the first jump of the process $m(t)$ with the initial value i has exponential distribution with expectation equal to $\frac{1}{\lambda_i}$, we get

$$\begin{aligned} G_i(x) = \frac{P_i - V^*}{\lambda_i} - xP_i + \inf_{y\in\mathcal{A}(x)} \Bigg[&\int_0^\infty \lambda_i e^{-\lambda_i t}\left(\int_0^t cy(s)ds\right)dt + P_i\int_0^\infty \lambda_i e^{-\lambda_i t} y(t)dt \\ &+ \int_0^\infty e^{-\lambda_i t}\left(\sum_{j\neq i}\lambda_{i,j}G_j(y(t))\right)dt\Bigg], \end{aligned} \tag{4.25}$$

where $\mathcal{A}(x)$ is the set of all admissible trajectories between jumps, i.e. the set of deterministic functions $y =: y(t)$ such that $y(0) = x$, $y(t) \geq 0$ for any $t > 0$ and $z(t) =: y(t) - x + t$ is a non-decreasing left continuous function. Changing the order of all integration in the first term under the inf sign, we obtain

$$G_i(x) = \frac{P_i - V^*}{\lambda_i} - xP_i + \inf_{y\in\mathcal{A}(x)} \int_0^\infty e^{-\lambda_i t} \left(y(t)(c + \lambda_i P_i) + \sum_{j\neq i} \lambda_{i,j} G_j(y(t)) \right) dt\,. \tag{4.26}$$

Let $H_i(y) = y(c+\lambda_i P_i)+\sum_{j\neq i} \lambda_{i,j} W_j(y)$. The functions $W_j(y), i \in N$, are convex as limits of convex functions, and from (4.12) it follows that as $y \to \infty$ they tend to $+\infty$ (quadratically for $c > 0$ and linearly for $c = 0$). For each $i \in N$ the function $H_i(y)$ is convex, finite at zero, and tends to plus infinity at infinity, and hence reaches a minimum. It follows that if the function reaches a minimum at one point a_i^*, then the optimal control until the moment of the first jump of the Markov process is unique and is a threshold control with a threshold a_i^*, because any other admissible trajectory gives a greater value of the functional. If the minimum is reached on the interval $[a_i^*, \bar{a}_i]$, then the subset of quasi-threshold strategies for which on the interval $[a_i^*, \bar{a}_i]$ you can use any admissible control that does not go beyond this interval, and on the interval $[0, a_i^*)$ you need to make a one-time purchase in order to jump to any point in the interval $[a_i^*, \bar{a}_i]$.

It remains to prove that $a_i^* \neq \bar{a}_i$ implies that $a_i^* = 0$, $i \in N_0^{0,-}$, and investigate the properties of the solution to equation (3.5). This is done in exactly the same way as done in [14] and we will not dwell on it. □

5 Conclusion

In the paper there is considered the inventory problem, in which a manufacturer who needs to consume an intermediate product (goods) with a constant intensity for production buys this product at a price that depends on the value of a Markov process with continuous time, a finite number of states and known transition intensities. The case of discounted integral costs, consisting of purchase and storage costs, was considered in [14]. In this paper, we study the case with the long-run average cost functional. A passage to the limit is carried out with the discounting parameter tending to zero. As a result, an analogue of the canonical triple, known in the theory of controlled Markov chains with discrete time and a finite number of states, arises. It is shown that, as in the case of discounting, there is an optimal threshold strategy and an algorithm for constructing optimal thresholds is given.

Appendix

A1. Proof of Lemma 4.1. The matrix A_+ is the intensity matrix for a Markov chain with the killing (getting into a fictitious absorbing state) at the time of the first exit of the chain $\{m(t)\}_{0\leq t<\infty}$ from the set $N_+^{(l)}$, and the exit moment is associated both with the entry of the original chain into a fictitious state (for $\rho > 0$) and with the entry of the original chain into states from $N_-^{(l)}$.

Let us show that the matrix $(-A_+)^{-1}$ exists and all its elements are non-negative. To do this, we represent the matrix $-A_+$ as $-A_+ = D\left(E_+ - \bar{A}_+\right)$, where D is a diagonal matrix with entries $d_{i,i} = \rho + \lambda_i = -a_{i,i},\ i \in N_+^{(l)}$, and the matrix $\bar{A}_+$ has zeros on the diagonal, and for the remaining elements

$$\bar{a}_{i,j} = \frac{a_{i,j}}{(\rho + \lambda_i)} = \frac{\lambda_{i,j}}{\rho + \sum_{k\in N} \lambda_{i,k}},\ i, j \in N_+^{(l)},\ i \neq j.$$

The matrix $\bar{A}_+$ is the transition matrix for an embedded Markov chain (with discrete time) with respect to the chain with continuous time corresponding to A_+.

If $\rho > 0$, then for each row of the matrix $\bar{A}_+$ the sum of the elements is less than one, so there exists $(E_+ - \bar{A}_+)^{-1}$. If $\rho = 0$, then for some $i \in N_+^{(l)}$ it may turn out that the sum of the elements is equal to one. However, if the chain $m(t)$ is regular, then for each row the sum of the elements of the matrix $(\bar{A}_+)^n$ is less than one, because for a regular chain the probability of exiting the set $N_+^{(l)}$ after the nth jump is positive for any initial state $i \in N_+^{(l)}$. Therefore, in both cases, we have the representation

$$(-A_+)^{-1} = \left(E_+ - \bar{A}_+\right)^{-1} D^{-1} = \left(\sum_{k=0}^{\infty} \left(\bar{A}_+\right)^k\right) D^{-1}, \tag{A.1}$$

where the series converges. This representation implies the existence of the matrix $(-A_+)^{-1}$ and the non-negativity of all its elements, i.e., statement a) of Lemma 4.1.

If the intensity matrix of a killable Markov chain is such that the corresponding transition matrix of the imbedded chain has the property that for some power of this transition matrix the sum of the elements for all rows is less than one, then the real part of any eigenvalue of the initial intensity matrix is negative. □

A2. Proof of Lemma 4.2. $\Lambda I = 0$ implies $AI = -\rho I$. This ratio can be written as:

$$A_+ I_+ + A_\pm I_- = -\rho I_+ \tag{A.2}$$

$$A_- I_- + A_\mp I_+ = -\rho I_- \tag{A.3}$$

From (A.3) and (4.7) it follows that

$$\begin{aligned} BI_- &= A_- I_- + A_\mp \left(A_+\right)^{-1} A_\pm I_- = -\rho I_- - A_\mp I_+ + A_\mp \left(A_+\right)^{-1} A_\pm I_- \\ &= -\rho I_- + A_\mp \left(-A_+\right)^{-1} A_+ I_+ + A_\mp \left(A_+\right)^{-1} A_\pm I_- \\ &= -\rho I_- + A_\mp \left(-A_+\right)^{-1} \left(A_+ I_+ + A_\pm I_-\right). \end{aligned} \tag{A.4}$$

The matrix B is the transition matrix of the state-set $N_-^{(l)}$ Markov chain, which is obtained from the original chain (with killing for $\rho > 0$) by discarding time intervals when the original Markov chain is in states from $N_+^{(l)}$. The intensities of transitions in this chain increased compared to the intensities of the original chain, due to the exclusion of time intervals when the original circuit was in states from $N_-^{(l)}$. The first term in (A.2) corresponds to direct transitions inside $N_-^{(l)}$, and the second term corresponds to transitions after entering, staying, and leaving the set $N_+^{(l)}$, while the first factor of the second term corresponds to the transition from $N_-^{(l)}$ to $N_+^{(l)}$, the second factor corresponds to staying in $N_+^{(l)}$, and the third factor corresponds to returning from $N_+^{(l)}$ to $N_-^{(l)}$.

Both terms have off-diagonal elements that are non-negative. If for some $i \in N_-^{(l)}$ all elements of the string were equal to zero, then this would mean that from this state to go to other states from $N_-^{(l)}$ is impossible either directly or after visiting $N_+^{(l)}$, and this contradicts the regularity of the chain. □

A3. Limit properties of a_n^ρ and $e^{A(\rho)(a_n^\rho - \hat{a}_n^\rho)}$. It follows from Lemma 4.1 and (4.18) that

$$f_n^\rho(\hat{a}_n^\rho + x) = \rho P_n \left(1 - \Lambda_{(n)} A(\rho)\right) I_{+n} - q_\rho (\mu x)^{m_0} Q\left(\frac{1}{x}\right) e^{-(\mu+\rho)x} + e^{-(\mu+\rho+\bar{\mu})x} g(x), \tag{A.5}$$

where $-\mu$ is an eigenvalue of the matrix Λ_+ with the maximal real part (it is known that $\mu > 0$); m_0 is the maximal size among the Jordan blocks corresponding to $-\mu$; $Q(y)$ is a polynomial of degree m_0, where $Q(0) = 1$, $q_\rho > 0$ for $\rho \geq 0$; $\mu + \bar{\mu} < \mu_1$, where $-\mu_1$ is the real part of the second eigenvalue in absolute value; $g(x)$ is a bounded function. Recall that the right-hand side in (A.5) is a function that increases from some negative value to a positive value, since all elements of the matrix $A(\rho)$ are non-positive, and vanishes at the point $x = (a_n^\rho - \hat{a}_n^\rho)$.

Let $a(\rho) > m_0$ be the solution to the equation

$$[a(\rho)]^{m_0} e^{-a(\rho)} = \frac{\rho}{q_0} P_n \left(1 - \Lambda_{(n)} A(0)\right) I_{+n}. \tag{A.6}$$

From (A.5), (4.20) and the fact that $f_n^\rho(a_n^\rho) = 0$, statement a) of Lemma 4.3 follows. Statement b) follows from the Jordan representation of the matrix $\frac{1}{\rho} e^{A_+^\rho a(\rho)}$ in which in the limit all elements vanish, with the exception of the elements corresponding to $[a(\rho)]^{m_0} e^{-a(\rho)}$.

Acknowledgments

This work was supported by RSF grant "Econometric and probabilistic methods for the analysis of complex financial markets", project no. 20-68-47030.

References

[1] K.J. Arrow, T.E. Harris, J. Marschak, *Optimal inventory policy*, Econometrica, XIX (1951), 250-272.

[2] J. Bather, *A continuous time inventory model*, Journal of Applied Probability 3 (1966), 538-549.

[3] D. Beyer, Feng Cheng, S.P. Sethi, M. Taksar, *Markovian demand inventory models*, Springer, 2010.

[4] S. Browne, P. Zipkin, *Inventory models with continuous stochastic demands*, The Annals of Applied Probability 1 (1991), no. 3, 419-435.

[5] E.V. Bulinskaya, A.I. Sokolova, *Asymptotic behavior of some stochastic storage systems*, In: "Modern problems of mathematics and mechanics" 10 (2015), no. 3, 37–62 (in Russian).

[6] A. Darekar, A. Reddy, *Cotton price forecasting in major producing states*, Economic Affairs 62 (2017), no. 3, 1-6, New Delhi Publishers.

[7] E.B. Dynkin, A.A. Yushkevich, *Controlled Markov processes and their applications*, "Nauka", Moscow, 1975 (in Russian), English translation: Springer-Verlag, Berlin, 1978.

[8] J. Hill, *A Markov-modulated acquisition strategy*, PhD thesis. 2004.

[9] J. Hill, I. Sonin, *An inventory optimization model with Markov modulated commodity prices*, abstract, Intern. Conf. on Management Sciences, Univ. of Texas at Dallas, 2006.

[10] M. Katehakis, I. Sonin, *A Markov chain modulated inventory model*, abstract, INFORMS, 2013.

[11] E.L. Presman, I.M. Sonin, *Sequential control with incomplete information: The Bayesian approach to many-armed bandit problems*, "Nauka", Moscow, 1982 (in Russian), 256 p., English version: Academic Press, 1990, 266 pp.

[12] E.L. Presman, S.P. Sethi, Q.Zhang, (1995), *Optimal feedback production planning in a stochastic N -machine flowshop*, Automatica, 31 (1995), no. 9, 1325-1332.

[13] E.L. Presman, S.P. Sethi, *Stochastic inventory models with continuous and Poisson demands and discounted and Average Costs*, Production and Operations Management 15 (2006), no. 2, 279-293.

[14] E.L. Presman, I.M. Sonin, *An inventory model where commodity prices depend on a continuous time Markov chain.* Journal of the New Economic Association 2 (59) (2023), 12–34 (in Russian).

[15] G. Rubalskiy, *Calculation of optimum parameters in an inventory control problem*, Eng. Cybern. 10 (1972), 182-187.

Ernst L'vovich Presman
Central Economics & Mathematics Instiute
Russian Academy of Scienses
47 Nakhimovsky prospekt, 117418 Moscow, Russian Federation
E-mail: presman@cemi.rssi.ru

Received: 30.12.2024