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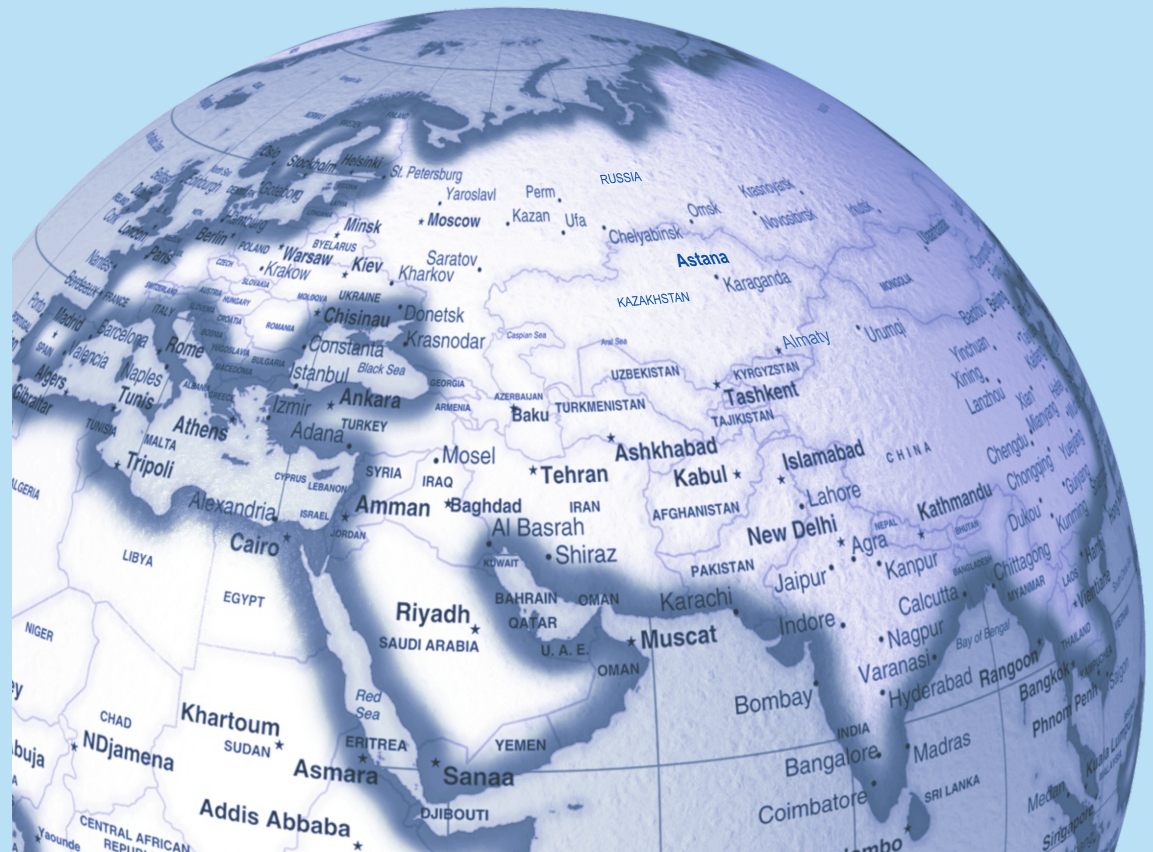

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CONTINUOUS EXTENSION TO THE BOUNDARY OF A DOMAIN OF THE LOGARITHMIC DOUBLE LAYER POTENTIAL

S.A. Plaksa

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Key words: logarithmic double layer potential, Cauchy-type integral, Ahlfors-regular curve, Král curve, Radon curve, Lyapunov curve.

AMS Mathematics Subject Classification: 30E20, 31A10.

Abstract. For the real part of the Cauchy-type integral that is known to be the logarithmic potential of the double layer, a necessary and sufficient condition for the continuous extension to the Ahlfors-regular boundary is established. Sufficient conditions involving subclasses of Ahlfors-regular curves are also considered. Illustrative examples are presented.

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1 Introduction

Let γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$, and let D^+ and D^- be the interior and exterior domains bounded by γ, respectively.

The classical theory of the logarithmic double layer potential

$$\frac{1}{2\pi}\int\limits_{\gamma} g(t)\frac{\partial}{\partial \mathbf{n}_t}\left(\ln\frac{1}{|t-z|}\right) ds_t \qquad \forall\, z\in D^{\pm} \tag{1.1}$$

is developed in the case in which the integration curve γ is a Lyapunov curve (see, for example, J. Plemelj [23]). Here $\mathbf{n}_t$ and s_t denote the unit vector of the outward normal to the curve γ at a point $t\in\gamma$ and an arc coordinate of this point, respectively, and the integral density $g:\gamma\to\mathbb{R}$ takes values in the set of real numbers $\mathbb{R}$.

J. Radon [26] established the continuous extension of the logarithmic double layer potential from the domains D^+ and D^- to the boundary γ in the case in which γ is a curve of bounded rotation, i.e., a curve for which the angle between the tangent to the curve and a fixed direction is a function of bounded variation. It is known that the class of Lyapunov curves and the class of Radon curves of bounded rotation are different, i.e., each of them contains curves that do not belong to the other class (see, for example, I.I. Danilyuk [4, p. 26]).

J. Král [19] established a necessary and sufficient condition for the curve γ, under which the logarithmic double layer potential is continuously extended from the domains D^+ and D^- to the boundary γ for all continuous functions $g:\gamma\to\mathbb{R}$.

The logarithmic double layer potential (1.1) is the real part of the Cauchy-type integral (see, for example, F.D. Gakhov [8], N.I. Muskhelishvili [20])

$$\widetilde{g}(z) := \frac{1}{2\pi i}\int\limits_{\gamma}\frac{g(t)}{t-z}\,dt \qquad \forall\, z\in D^{\pm}\,. \tag{1.2}$$

The theory of the boundary properties of the integral in (1.2) is presented in the monographs by F.D. Gakhov [8] and N.I. Muskhelishvili [20] under the classical assumptions about the smoothness of the integration curve and the Hölder density of the integral. In the papers of N.A. Davydov [6], V.V. Salaev [27], T.S. Salimov [28], E.M. Dyn'kin [7], O.F. Gerus [9, 11], the theory of the Cauchy-type integral and the Cauchy singular integral is developed on an arbitrary rectifiable Jordan curve in classes that are more general than the Hölder class of the integral density, which are defined, as a rule, in terms of the modulus of continuity of the function g.

In the paper of O.F. Gerus and M. Shapiro [12], an analog of the Davydov theorem [6] is proved for an appropriate Cauchy-type integral along an arbitrary rectifiable Jordan curve in $\mathbb{R}^2$, which takes values in the algebra of quaternions. This result is applied in the paper of O.F. Gerus and M. Shapiro [13] to establish sufficient conditions for the continuous extension to the boundary of a domain of metaharmonic potentials, a partial case of which is logarithmic double layer potential (1.1).

At the same time, the results mentioned above about the continuous extension of the logarithmic double layer potential to the boundary of a domain, which are contained in papers [23, 26, 19], are valid for arbitrary continuous functions g. It is linked to the fact that the real part of the Schwartz integral, i.e. the Poisson integral, is continuously extended to the boundary of the unit disk for an arbitrary continuous integral density, while the continuous extension of the imaginary part of the Schwartz integral requires additional assumptions about the integral density.

The purpose of this paper is to establish general results about the continuous extension of the real part of the Cauchy-type integral with a real-valued integral density, which are usable for the cases in which the classical results of papers [23, 26, 19] as well as the corresponding result of paper [13] are not applicable, generally speaking.

2 Preliminary information

In what follows, a closed rectifiable Jordan curve γ satisfies the condition (see V.V. Salaev [27])

$$\theta(\varepsilon) := \sup_{\xi\in\gamma} \theta_\xi(\varepsilon) = O(\varepsilon), \quad \varepsilon \to 0\,, \tag{2.1}$$

where $\theta_\xi(\varepsilon) := \operatorname{meas}\gamma_\varepsilon(\xi)$, $\gamma_\varepsilon(\xi) := \{t \in \gamma : |t-\xi| \le \varepsilon\}$ and meas denotes the linear Lebesgue measure on γ. Curves satisfying condition (2.1) are important in solving various problems (see, for example, V.V. Salaev [27], L. Ahlfors [1], G. David [5], C. Pommerenke [24], A. Böttcher and Y.I. Karlovich [2]). Such curves are often called *regular* (see, for example, [5]) or *Ahlfors-regular* (see, for example, [24]), or *Carleson curves* (see, for example, [2]).

It is well known that a closed rectifiable Jordan curve γ has a tangent at almost all points $t \in \gamma$. For such a point $t \in \gamma$ we denote by ϑ_t the angle between the tangent to the curve γ at this point and the direction of the real axis. J. Radon [26] called γ a *curve of bounded rotation* if the angle ϑ_t is a function of bounded variation on γ.

This implies that for a curve γ of bounded rotation, the angle ϑ_t can have at most a countable set of discontinuity points, and there are one-sided tangents at each point of the curve γ. Moreover, a curve of bounded rotation can have only a finite set of cusp points and at most a countable set of corner points. At the same time, every curve of bounded rotation satisfies condition (2.1).

This follows, for example, from the fact that the Cauchy singular integral operator is bounded in Lebesgue spaces on any curve γ of bounded rotation (see I.I. Danilyuk [4], I.I. Daniljuk and V.Yu. Šelepov [3], È. G. Gordadze [15]), and a necessary condition for this is condition (2.1) on the curve (see V.A. Paatashvili and G.A. Khuskivadze [21]).

A curve γ is called a *Lyapunov curve* if the angle ϑ_t satisfies the Hölder condition:

$$|\vartheta_{t_1} - \vartheta_{t_2}| \le c\,|t_1 - t_2|^\alpha \qquad \forall\, t_1, t_2 \in \gamma,$$

where $\alpha \in (0, 1]$ and the constant c does not depend on t_1 and t_2. It is clear that the Lyapunov curve is a smooth curve and also satisfies condition (2.1). There are Lyapunov curves that are not the Radon curves of bounded rotation (see, for example, I.I. Danilyuk [4, p. 26]).

J. Král [19] proved that logarithmic double layer potential (1.1) is extended continuously from the domains D^+ and D^- to the boundary γ for all continuous functions $g : \gamma \to \mathbb{R}$ if and only if the curve γ satisfies the condition

$$\sup_{\xi\in\gamma} \int_0^{2\pi} \mu_\gamma(\xi, \phi)\, d\phi < \infty, \tag{2.2}$$

where $\mu_\gamma(\xi, \phi)$ is the number of intersection points of the curve γ with the ray $\{z = \xi + re^{i\phi} : r > 0\}$. It will be shown below that each curve γ, which satisfies condition (2.2), also satisfies condition (2.1), i.e, it is an Ahlfors-regular curve.

Note that not every smooth curve satisfies condition (2.2). In particular, an example of such a curve will be given below.

For a function $f\colon E \to \mathbb{C}$ continuous on a set $E \subset \mathbb{C}$, we shall use its modulus of continuity

$$\omega_E(f, \varepsilon) := \sup_{t_1,t_2\in E\,:\,|t_1-t_2|\le\varepsilon} |f(t_1) - f(t_2)|\,.$$

Quite often, the conditions for a given domain are formulated in terms of a mapping of the unit disk onto this domain (see, for example, I.I. Priwalow [25, §§ 12–17 of Chapter III]).

Consider a conformal mapping $\sigma : U \to D^+$ of the unit disk U onto the domain D^+. It is well known that the mapping σ is continuously extended to the boundary ∂U and defines a homeomorphism between the unit circle ∂U and the curve γ.

In paper [16], when solving boundary value problems for monogenic hypercomplex functions associated with a biharmonic equation, the following result on the continuous extension of logarithmic double layer potential (1.1) has actually been established, although it was not formulated as a theorem.

Theorem 2.1. *Let $\sigma : U \to D^+$ be a conformal mapping of the unit disk U onto the domain D^+, and let the continuous extension of σ to the circle ∂U have the nonvanishing continuous contour derivative σ' on ∂U, and let its modulus of continuity satisfy the Dini condition*

$$\int_0^1 \frac{\omega_{\partial U}(\sigma', \eta)}{\eta}\, d\eta < \infty. \tag{2.3}$$

Then, for each continuous function $g\colon \gamma \to \mathbb{R}$, integral (1.1) has a continuous extension from the domain D^+ to the boundary γ.

Theorem 2.1 generalizes the corresponding result of the classical theory of the logarithmic double layer potential on the Lyapunov curves (see, for example, J. Plemelj [23]), because in the case in which γ is a Lyapunov curve, condition (2.3) is satisfied owing to the Kellogg theorem (see, for example, G.M. Goluzin [14]). Condition (2.3) is also satisfied in the more general case in which the modulus of continuity of the angle ϑ_t satisfies the condition

$$\int_0^1 \frac{\omega_\gamma(\vartheta_t, \eta)}{\eta}\; \ln\frac{2}{\eta}\, d\eta < \infty\,. \tag{2.4}$$

It follows from the estimate of the modulus of continuity of the function σ' presented in Theorem 2 in the paper of J.L. Heronimus [17] (see also S.E. Warschawski [30]).

If the modulus of continuity of the function $g:\gamma\to\mathbb{R}$ satisfies the Dini condition

$$\int_0^1 \frac{\omega_\gamma(g,\eta)}{\eta}\,d\eta<\infty, \tag{2.5}$$

then the reduced singular Cauchy integral

$$\int_\gamma \frac{g(t)-g(\xi)}{t-\xi}\,dt := \lim_{\delta\to0^+}\int_{\gamma\setminus\gamma_\delta(\xi)} \frac{g(t)-g(\xi)}{t-\xi}\,dt \qquad \forall\,\xi\in\gamma \tag{2.6}$$

exists (see O.F. Gerus [9], and also V.V. Salaev [27], where the Dini condition of form (2.5) is given in terms of the regularized modulus of continuity using the Stechkin construction).

From this and from the result of N.A. Davydov [6], it follows that Cauchy-type integral (1.2) has the limiting values $\widetilde{g}^{\,\pm}(\xi)$ at every point $\xi\in\gamma$ from the domains $D^\pm$, which are expressed by the Sokhotski–Plemelj formulas:

$$\widetilde{g}^{\,+}(\xi)=g(\xi)+\frac{1}{2\pi i}\int_\gamma \frac{g(t)-g(\xi)}{t-\xi}\,dt\,, \tag{2.7}$$

$$\widetilde{g}^{\,-}(\xi)=\frac{1}{2\pi i}\int_\gamma \frac{g(t)-g(\xi)}{t-\xi}\,dt\,. \tag{2.8}$$

Let $d:=\max\limits_{t_1,t_2\in\gamma}|t_1-t_2|$ be the diameter of the curve γ.

To single out the real part of integral (2.6), we define a branch $\arg(z-\xi)$ continuous on $\gamma\setminus\{\xi\}$ of the multivalued function $\operatorname{Arg}(z-\xi)$ in the following way. For each positive $\delta<d/2$, we select that connected component $\gamma_{\xi,\delta}$ of the set $\gamma_\delta(\xi)$ which contains the point ξ, and we take such a point $\xi_1\in\gamma_{\xi,\delta}$ at which there is a tangent to γ and which does not precede the point ξ under the given orientation of the curve γ. It is obvious that in the case in which there is a tangent to γ at the point ξ, we can set $\xi_1=\xi$. Let us cut the complex plane along the curve $\Gamma_{\xi,\delta}:=\gamma[\xi,\xi_1]\cup\Gamma[\xi_1,\infty]$, where $\gamma[\xi,\xi_1]$ is the arc of γ with the initial point ξ and the end point ξ_1, and $\Gamma[\xi_1,\infty]$ is a smooth curve that connects the points ξ_1 and ∞ and lies completely (except for its ends ξ_1 and ∞) in the domain D^-. Now, let us single out a branch $\arg_\delta(z-\xi)$ of the multivalued function $\operatorname{Arg}(z-\xi)$, which is continuous outside the cut $\Gamma_{\xi,\delta}$ with the normalization condition $\arg_\delta(z_0-\xi)=\phi_0$, where $z_0\in D^+$ and ϕ_0 is one of the values of the function $\operatorname{Arg}(z-\xi)$ at $z=z_0$. We shall use the fixed values z_0 and ϕ_0 for all positive $\delta<d/2$. As a result, we have the obvious equality

$$\arg_{\delta_1}(t-\xi)=\arg_\delta(t-\xi) \qquad \forall\,\delta_1,\delta: 0<\delta_1<\delta<d/2 \quad \forall\,t\in\gamma\setminus\gamma_\delta(\xi)$$

that implies the existence of the following limit:

$$\arg(t-\xi):=\lim_{\delta\to0^+}\arg_\delta(t-\xi) \qquad \forall\,t\in\gamma\setminus\{\xi\}\,.$$

Thus, under the assumption that the function $g:\gamma\to\mathbb{R}$ satisfies the condition (2.5), from equality (2.6) we get the equality

$$\operatorname{Re}\left(\frac{1}{2\pi i}\int_\gamma \frac{g(t)-g(\xi)}{t-\xi}\,dt\right)=\frac{1}{2\pi}\lim_{\delta\to0^+}\int_{\gamma\setminus\gamma_\delta(\xi)}\big(g(t)-g(\xi)\big)\,d\arg(t-\xi) \qquad \forall\,\xi\in\gamma\,,$$

where in the right-hand side of the equality the integral is the Stieltjes integral.

Let us accept by definition

$$\int\limits_{\gamma} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi) := \lim_{\delta\to 0^+} \int\limits_{\gamma\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi) \qquad \forall\,\xi\in\gamma\,. \tag{2.9}$$

Finally, denoting

$$\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi) := \lim_{z\to\xi,\, z\in D^{\pm}} \mathrm{Re}\,\widetilde{g}(z) \qquad \forall\,\xi\in\gamma,$$

as a corollary of formulas (2.7) and (2.8), for all $\xi\in\gamma$, we obtain the equalities

$$\big(\mathrm{Re}\,\widetilde{g}\big)^{+}(\xi) = g(\xi) + \frac{1}{2\pi}\int\limits_{\gamma} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\,, \tag{2.10}$$

$$\big(\mathrm{Re}\,\widetilde{g}\big)^{-}(\xi) = \frac{1}{2\pi}\int\limits_{\gamma} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\,. \tag{2.11}$$

Below, we investigate the fulfillment of equalities (2.10) and (2.11), not assuming, generally speaking, neither the fulfillment of condition (2.5) for the function $g:\gamma\to\mathbb{R}$ nor the fulfillment of condition (2.2) for the curve γ.

3 A necessary and sufficient condition for the continuous extension of the real part of the Cauchy-type integral to the boundary of the domain bounded by an Ahlfors-regular curve

The following statement is true.

Theorem 3.1. *Let a closed Jordan curve γ be Ahlfors-regular and let a function $g\colon\gamma\to\mathbb{R}$ be continuous on γ. The function $\mathrm{Re}\,\widetilde{g}(z)$ has a continuous extension to the boundary γ from the domain D^+ or D^- if and only if the following condition is satisfied:*

$$\sup_{\xi\in\gamma}\ \sup_{\delta\in(0,\varepsilon)} \Bigg| \int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi) \Bigg| \to 0\,, \qquad \varepsilon\to 0\,. \tag{3.1}$$

In the case in which condition (3.1) is satisfied, the limiting values $\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi)$ are represented by formulas (2.10) and (2.11) for all $\xi\in\gamma$.

Proof. Sufficiency. Obviously, if condition (3.1) is satisfied, the limit exists in equality (2.9).

Let us prove equality (2.10). Let $\xi\in\gamma$, $z\in D^+$ and $\varepsilon:=|z-\xi|<d/8$. Denote $\varepsilon_1 := \min\limits_{t\in\gamma}|t-z|$. Let us choose the point $\xi_z\in\gamma$ closest to the point z.

We use the following representation of the difference:

$$\mathrm{Re}\,\widetilde{g}(z) - g(\xi) - \frac{1}{2\pi}\int\limits_{\gamma} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)$$

$$= \mathrm{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma} \frac{g(t)-g(\xi_z)}{t-z}\,dt\right) - \frac{1}{2\pi}\int\limits_{\gamma} \big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z) + g(\xi_z) - g(\xi)$$

$$+ \frac{1}{2\pi}\int\limits_{\gamma} \big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z) - \frac{1}{2\pi}\int\limits_{\gamma} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\,.$$

Consider the difference

$$\operatorname{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma}\frac{g(t)-g(\xi_z)}{t-z}\,dt\right)-\frac{1}{2\pi}\int\limits_{\gamma}\big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z)$$
$$=\operatorname{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma_{2\varepsilon_1}(\xi_z)}\frac{g(t)-g(\xi_z)}{t-z}\,dt\right)-\frac{1}{2\pi}\int\limits_{2\varepsilon_1(\xi_z)}\big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z)$$
$$+\operatorname{Re}\left(\frac{z-\xi_z}{2\pi i}\int\limits_{\gamma\setminus\gamma_{2\varepsilon_1}(\xi_z)}\frac{g(t)-g(\xi_z)}{(t-z)(t-\xi_z)}\,dt\right)=:I_1-I_2+I_3\,.$$

Taking into account condition (2.1), we obtain the relation

$$|I_1|\le\frac{1}{2\pi}\int\limits_{\gamma_{2\varepsilon_1}(\xi_z)}\frac{|g(t)-g(\xi_z)|}{|t-z|}\,|dt|\le\frac{\omega_\gamma(g,2\varepsilon_1)}{2\pi\varepsilon_1}\,\theta_{\xi_z}(2\varepsilon_1)\le c\,\omega_\gamma(g,2\varepsilon_1)\to0,\qquad\varepsilon_1\to0,$$

where the constant c depends only on the curve γ.

Condition (3.1) implies the relation

$$|I_2|\to0,\qquad\varepsilon_1\to0.$$

To estimate the integral I_3, we use Proposition 7.2 in [22] (see also the proof of Theorem 1 in the paper of O.F. Gerus [10]) and condition (2.1) so that we have

$$|I_3|\le\frac{|z-\xi_z|}{\pi}\int\limits_{\gamma\setminus\gamma_{2\varepsilon_1}(\xi_z)}\frac{|g(t)-g(\xi_z)|}{|t-\xi_z|^2}\,|dt|\le\frac{\varepsilon_1}{\pi}\int\limits_{[2\varepsilon_1,d]}\frac{\omega_\gamma(g,\eta)}{\eta^2}\,d\theta_{\xi_z}(\eta)$$
$$\le\frac{2\varepsilon_1}{3\pi}\int\limits_{\varepsilon_1}^{d}\frac{\theta_{\xi_z}(2\eta)\omega_\gamma(g,2\eta)}{\eta^3}\,d\eta\le c\,\varepsilon_1\int\limits_{\varepsilon_1}^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}\,d\eta\to0,\qquad\varepsilon_1\to0,$$

where the constant c depends only on the curve γ.

Now, consider the difference

$$\frac{1}{2\pi}\int\limits_{\gamma}\big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z)-\frac{1}{2\pi}\int\limits_{\gamma}\big(g(t)-g(\xi)\big)\,d\arg(t-\xi)$$
$$=\frac{1}{2\pi}\int\limits_{\gamma_\varepsilon(\xi_z)}\big(g(t)-g(\xi_z)\big)\,d\arg(t-\xi_z)+\operatorname{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma_{4\varepsilon}(\xi)\setminus\gamma_\varepsilon(\xi_z)}\frac{g(t)-g(\xi_z)}{t-\xi_z}\,dt\right)$$
$$-\frac{1}{2\pi}\int\limits_{\gamma_{4\varepsilon}(\xi)}\big(g(t)-g(\xi)\big)\,d\arg(t-\xi)+\operatorname{Re}\left(\frac{\xi_z-\xi}{2\pi i}\int\limits_{\gamma\setminus\gamma_{4\varepsilon}(\xi)}\frac{g(t)-g(\xi)}{(t-\xi)(t-\xi_z)}\,dt\right)$$
$$+\operatorname{Re}\left(\frac{g(\xi)-g(\xi_z)}{2\pi i}\int\limits_{\gamma\setminus\gamma_{4\varepsilon}(\xi)}\frac{dt}{t-\xi_z}\right)=:J_1+J_2-J_3+J_4+J_5\,.$$

Condition (3.1) implies the relations

$$|J_1|\to0\quad\text{and}\quad|J_3|\to0,\qquad\varepsilon\to0.$$

The integrals J_2 and J_4 are estimated similarly to the integrals I_1 and I_3, respectively. As a result, we have the relations

$$|J_2| \to 0 \quad \text{and} \quad |J_4| \to 0, \qquad \varepsilon \to 0.$$

In addition, the following relations are satisfied (see the proof of Theorem 1 in the paper of V.V. Salaev [27]):

$$|J_5| \le \frac{|g(\xi) - g(\xi_z)|}{2\pi} \left| \int\limits_{\gamma \setminus \gamma_{4\varepsilon}(\xi)} \frac{dt}{t - \xi_z} \right| \le 2\,\omega_\gamma(g, 2\varepsilon) \to 0, \qquad \varepsilon \to 0.$$

An obvious corollary of the given relations is equality (2.10). Equality (2.11) is similarly established.

Necessity. Since the curve γ satisfies condition (2.1), the singular Cauchy integral operator is bounded in the Lebesgue spaces L_p for $p > 1$ on γ (see Theorem 1 in the paper of G. David [5]). At the same time, Cauchy-type integral (1.2) belongs to the Smirnov classes E_p (see, for example, I.I. Priwalow [25]) for $p > 1$ in the domains D^+ and D^-. In addition, its angular boundary values $\widetilde{g}^{\,\pm}_{\text{ang}}(\xi)$ from $D^\pm$ exist for almost all points $\xi \in \gamma$, and the following equality holds almost everywhere on γ:

$$g(\xi) = \widetilde{g}^{\,+}_{\text{ang}}(\xi) - \widetilde{g}^{\,-}_{\text{ang}}(\xi).$$

We denote by $\widetilde{g}^{\,\pm}$ the function that is defined by equality (1.2) in the domain $D^\pm$ and is extended almost everywhere on γ by means of the values $\widetilde{g}^{\,\pm}_{\text{ang}}$. We denote also the real part of this function by $\operatorname{Re}\widetilde{g}^{\,\pm}$.

Note that the values of the functions $\operatorname{Re}\widetilde{g}^{\,+}$ and $\operatorname{Re}\widetilde{g}^{\,-}$ are expressed by equalities of form (2.10) and (2.11) for almost all points $\xi \in \gamma$. It obviously follows that in the case in which the function $\operatorname{Re}\widetilde{g}(z)$ is continuously extended to the boundary γ from one of the domains D^+ or D^-, this function is also continuously extended to γ from the other domain.

For $\xi \in \gamma$ and $0 < \delta < \varepsilon < d$, consider the open sets $D^\pm_{\delta,\varepsilon}(\xi) := \{z \in D^\pm : \delta < |z - \xi| < \varepsilon\}$ and their boundaries $\partial D^\pm_{\delta,\varepsilon}(\xi)$, the orientation of which is induced by the orientation of γ. Denote $\Gamma^\pm_\delta := \{t \in \partial D^\pm_{\delta,\varepsilon}(\xi) \setminus \gamma : |z - \xi| = \delta\}$, $\Gamma^\pm_\varepsilon := \{t \in \partial D^\pm_{\delta,\varepsilon}(\xi) \setminus \gamma : |z - \xi| = \varepsilon\}$.

We have the equalities:

$$\int\limits_{\gamma_\varepsilon(\xi) \setminus \gamma_\delta(\xi)} \big(g(t) - g(\xi)\big)\, d\arg(t - \xi) = \operatorname{Im}\Bigg(\int\limits_{\gamma_\varepsilon(\xi) \setminus \gamma_\delta(\xi)} \frac{g(t) - g(\xi)}{t - \xi}\, dt \Bigg)$$

$$= \operatorname{Im}\Bigg(\int\limits_{\gamma_\varepsilon(\xi) \setminus \gamma_\delta(\xi)} \frac{\widetilde{g}^{\,+}(t) - \widetilde{g}^{\,-}(t) - \operatorname{Re}\widetilde{g}^{\,+}(\xi) + \operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t - \xi}\, dt \Bigg)$$

$$= \operatorname{Im}\Bigg(\int\limits_{\partial D^+_{\delta,\varepsilon}(\xi)} \frac{\widetilde{g}^{\,+}(t) - \operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t - \xi}\, dt - \int\limits_{\Gamma^+_\delta} \frac{\widetilde{g}^{\,+}(t) - \operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t - \xi}\, dt - \int\limits_{\Gamma^+_\varepsilon} \frac{\widetilde{g}^{\,+}(t) - \operatorname{Re}\widetilde{g}^{\,+}(\xi)}{t - \xi}\, dt \Bigg)$$

$$- \operatorname{Im}\Bigg(\int\limits_{\partial D^-_{\delta,\varepsilon}(\xi)} \frac{\widetilde{g}^{\,-}(t) - \operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t - \xi}\, dt - \int\limits_{\Gamma^-_\delta} \frac{\widetilde{g}^{\,-}(t) - \operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t - \xi}\, dt - \int\limits_{\Gamma^-_\varepsilon} \frac{\widetilde{g}^{\,-}(t) - \operatorname{Re}\widetilde{g}^{\,-}(\xi)}{t - \xi}\, dt \Bigg).$$

Further, taking into account that the integrals of functions from the Smirnov classes along the closed curves $\partial D^{\pm}_{\delta,\varepsilon}(\xi)$ are equal to zero, we have

$$\begin{aligned}
\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} &\big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \\
&= -\mathrm{Im}\bigg(\int\limits_{\Gamma^+_\delta} \frac{\widetilde{g}^{\,+}(t)-\mathrm{Re}\,\widetilde{g}^{\,+}(\xi)}{t-\xi}\,dt + \int\limits_{\Gamma^+_\varepsilon} \frac{\widetilde{g}^{\,+}(t)-\mathrm{Re}\,\widetilde{g}^{\,+}(\xi)}{t-\xi}\,dt\bigg) \\
&\quad + \mathrm{Im}\bigg(\int\limits_{\Gamma^-_\delta} \frac{\widetilde{g}^{\,-}(t)-\mathrm{Re}\,\widetilde{g}^{\,-}(\xi)}{t-\xi}\,dt + \int\limits_{\Gamma^-_\varepsilon} \frac{\widetilde{g}^{\,-}(t)-\mathrm{Re}\,\widetilde{g}^{\,-}(\xi)}{t-\xi}\,dt\bigg) \\
&= -\mathrm{Im}\bigg(\int\limits_{\Gamma^+_\delta} \frac{\mathrm{Re}\,\widetilde{g}^{\,+}(t)-\mathrm{Re}\,\widetilde{g}^{\,+}(\xi)}{t-\xi}\,dt + \int\limits_{\Gamma^+_\varepsilon} \frac{\mathrm{Re}\,\widetilde{g}^{\,+}(t)-\mathrm{Re}\,\widetilde{g}^{\,+}(\xi)}{t-\xi}\,dt\bigg) \\
&\quad + \mathrm{Im}\bigg(\int\limits_{\Gamma^-_\delta} \frac{\mathrm{Re}\,\widetilde{g}^{\,-}(t)-\mathrm{Re}\,\widetilde{g}^{\,-}(\xi)}{t-\xi}\,dt + \int\limits_{\Gamma^-_\varepsilon} \frac{\mathrm{Re}\,\widetilde{g}^{\,-}(t)-\mathrm{Re}\,\widetilde{g}^{\,-}(\xi)}{t-\xi}\,dt\bigg).
\end{aligned}$$

Since the function $\mathrm{Re}\,\widetilde{g}(z)$ is continuously extended to the boundary γ from $D^{\pm}$ and vanishes at infinity, the function $\mathrm{Re}\,\widetilde{g}^{\,\pm}$ is uniformly continuous in the closure $\overline{D^{\pm}}$ of the domain $D^{\pm}$. Therefore, we obtain the following estimates:

$$\begin{aligned}
\bigg|\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} &\big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\bigg| \\
&\le \int\limits_{\Gamma^+_\delta} \frac{\big|\mathrm{Re}\,\widetilde{g}^{\,+}(t)-\mathrm{Re}\,\widetilde{g}^{\,+}(\xi)\big|}{|t-\xi|}\,|dt| + \int\limits_{\Gamma^+_\varepsilon} \frac{\big|\mathrm{Re}\,\widetilde{g}^{\,+}(t)-\mathrm{Re}\,\widetilde{g}^{\,+}(\xi)\big|}{|t-\xi|}\,|dt| \\
&\quad + \int\limits_{\Gamma^-_\delta} \frac{\big|\mathrm{Re}\,\widetilde{g}^{\,-}(t)-\mathrm{Re}\,\widetilde{g}^{\,-}(\xi)\big|}{|t-\xi|}\,|dt| + \int\limits_{\Gamma^-_\varepsilon} \frac{\big|\mathrm{Re}\,\widetilde{g}^{\,-}(t)-\mathrm{Re}\,\widetilde{g}^{\,-}(\xi)\big|}{|t-\xi|}\,|dt| \\
&\le 4\pi\,\omega_{\overline{D^+}}(\mathrm{Re}\,\widetilde{g}^{\,+},\varepsilon) + 4\pi\,\omega_{\overline{D^-}}(\mathrm{Re}\,\widetilde{g}^{\,-},\varepsilon),
\end{aligned}$$

which imply condition (3.1). □

Theorem 3.1 is similar in a certain sense to the corresponding theorem for the Cauchy-type integral, which is proved by A.O. Tokov [29].

Let us note that in the case in which condition (3.1) is satisfied for a function $g\colon \gamma \to \mathbb{R}$ given on an Ahlfors-regular curve γ, a similar condition with the Stieltjes integral

$$\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \frac{g(t)-g(\xi)}{|t-\xi|}\, d|t-\xi|$$

may not be satisfied if the function g does not satisfy Dini condition (2.5). In this case, the function $\mathrm{Im}\,\widetilde{g}(z)$ has no continuous extension to the boundary γ from the domains D^+ and D^-. Indeed, the necessary condition established by A.O. Tokov [29] for the continuous extension of the Cauchy-type integral to the boundary γ is not satisfied.

4 Some properties of Ahlfors-regular curves

Note that for each $\xi \in \gamma$ and each $\delta > 0$, the function $\arg(t-\xi)$ has a bounded variation on the set $\gamma \setminus \gamma_\delta(\xi)$. However, in general, the function $\arg(t-\xi)$ can be a function of unbounded variation on γ, because, in particular, it can be unbounded in a neighborhood of the point ξ.

Consider the class of curves γ, for which the function $\arg(t-\xi)$ has a bounded variation $V_\gamma[\arg(t-\xi)]$ on $\gamma \setminus \{\xi\}$ for all $\xi \in \gamma$ and, moreover, satisfies the condition

$$\sup_{\xi\in\gamma} V_\gamma[\arg(t-\xi)] < \infty\,. \tag{4.1}$$

It is obvious that a curve satisfying the condition (4.1) has one-sided tangents at each point $\xi \in \gamma$.

Note that condition (4.1) is equivalent to condition (2.2), which follows from the Banach indicatrix theorem (see J. Král [19, Lemma 1.2]). Thus, the class of curves satisfying condition (4.1) includes curves from the corresponding classical results of J. Plemelj [23] and J. Radon [26] and from Theorem 2.1.

Curves satisfying the condition (4.1) will be called the *Král curves.*

Proposition 4.1. Every Král curve is an Ahlfors-regular curve.

Proof. Let γ be a Král curve and $\xi \in \gamma$. Let us first show that for an arbitrary $\varepsilon > 0$, the variation of the function $|t-\xi|$ on the set $\gamma_\varepsilon(\xi)$ satisfies the inequality

$$V_{\gamma_\varepsilon(\xi)}[|t-\xi|] \le c\,\varepsilon, \tag{4.2}$$

where the constant c does not depend on ξ and ε.

In the case in which we consider a fixed point $z \in \mathbb{C} \setminus \gamma$ and a variable $t \in \gamma$, we understand $\arg(t-z)$ as an arbitrary branch of the multivalued function $\mathrm{Arg}\,(t-z)$.

We use the representation $\gamma_\varepsilon(\xi) = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

$$\gamma_1 := \gamma_\varepsilon(\xi) \cap \big\{t = \xi + r\,e^{i\phi} : r > 0,\ \phi \in (-\pi/4, \pi/4) \cup (3\pi/4, 5\pi/4)\big\},$$

$$\gamma_2 := \gamma_\varepsilon(\xi) \cap \big\{t = \xi + r\,e^{i\phi} : r > 0,\ \phi \in (-3\pi/4, -\pi/4) \cup (\pi/4, 3\pi/4)\big\},$$

$$\gamma_3 := \gamma_\varepsilon(\xi) \cap \big\{t = \xi + r\,e^{i\phi} : r \ge 0,\ \phi \in \{-\pi/4, \pi/4, -3\pi/4, 3\pi/4\}\big\}.$$

For the variation of the function $|t-\xi|$ on the set γ_1, the following inequality holds (see J. Král [18, Theorem 2.10]):

$$\frac{V_{\gamma_1}[|t-\xi|]}{\varepsilon} \le c_0\Big(V_{\gamma_1}[\arg(t-\xi)] + V_{\gamma_1}[\arg(t-\xi-\varepsilon)]\Big),$$

where $c_0 = 6/\sin^2(\pi/4) = 12$. Moreover, since the curve γ satisfies condition (4.1), by virtue of Theorem 1.11 from the paper J. Král [19], the following condition is also satisfied:

$$\sup_{z\in\mathbb{C}} V_\gamma[\arg(t-z)] < \infty\,.$$

As a result, under condition (4.1) for the curve γ, we obtain the inequality

$$V_{\gamma_1}[|t-\xi|] \le c_1\,\varepsilon, \tag{4.3}$$

where the constant c_1 does not depend on ξ and ε.

In a similar way, for the variation of the function $|t-\xi|$ on the set γ_2, we obtain the inequalities

$$V_{\gamma_2}[|t-\xi|] \le c_0\Big(V_{\gamma_2}[\arg(t-\xi)] + V_{\gamma_2}[\arg(t-\xi-i\varepsilon)]\Big)\,\varepsilon \le c_1\,\varepsilon. \tag{4.4}$$

In addition, it is obvious that

$$V_{\gamma_3}[|t-\xi|] \le 4\,\varepsilon. \tag{4.5}$$

Inequalities (4.3) – (4.5) imply inequality (4.2), where $c = 2c_1 + 4$.

Now, taking into account condition (4.1) and inequality (4.2), for arbitrary $\xi \in \gamma$ and $\varepsilon > 0$, we obtain the relations

$$\theta_\xi(\varepsilon) = \int\limits_{\gamma_\varepsilon(\xi)} |dt| \le \int\limits_{\gamma_\varepsilon(\xi)} \big|d|t-\xi|\big| + \int\limits_{\gamma_\varepsilon(\xi)} |t-\xi||d\arg(t-\xi)|$$
$$\le V_{\gamma_\varepsilon(\xi)}[|t-\xi|] + \varepsilon\, V_\gamma[\arg(t-\xi)] \le c\,\varepsilon,$$

where the constant c does not depend on ξ and ε. Thus, the curve γ satisfies condition (2.1), i.e., it is an Ahlfors-regular curve. □

Among the Král curves there are curves that are not the Radon curves of bounded rotation, as the following example shows:

Example 1. Consider the curve

$$\gamma = \Big\{z = e^{i\phi} : \phi \in [0,\pi]\Big\} \cup [-1,0] \cup \bigcup_{n=1}^{\infty} [2^{-2n+1}, 2^{-2n+2}]$$
$$\cup \bigcup_{n=1}^{\infty} \Big\{z = 2^{-n} e^{i\phi} : \phi \in [0, 2^{-n}]\Big\} \cup \bigcup_{n=1}^{\infty} \Big\{z = re^{ir} : r \in [2^{-2n}, 2^{-2n+1}]\Big\}.$$

It is clear that $V_\gamma[\vartheta_t] = \infty$, but at the same time, the following relations are fulfilled:

$$V_\gamma[\arg t] \le \pi + \sum_{n=1}^{\infty} 2^{-n} + \frac{1}{2} = \pi + \frac{3}{2}\,,$$

$$V_\gamma[\arg(t-\xi)] = V_{\gamma_{|\xi|/2}(0)}[\arg(t-\xi)] + V_{\gamma_{|\xi|}(\xi)\setminus\gamma_{|\xi|/2}(0)}[\arg(t-\xi)] + V_{\gamma\setminus\gamma_{|\xi|}(\xi)\setminus\gamma_{|\xi|/2}(0)}[\arg(t-\xi)]$$
$$\le V_{\gamma_{|\xi|/2}(0)}[\arg t] + 2\pi + 2V_{\gamma\setminus\gamma_{|\xi|}(\xi)\setminus\gamma_{|\xi|/2}(0)}[\arg t] \le 2V_\gamma[\arg t] + 2\pi \qquad \forall\, \xi \in \gamma : 0 < |\xi| < 1,$$

$$V_\gamma[\arg(t-\xi)] \le V_{\gamma_{1/2}(\xi)}[\arg(t-\xi)] + \int\limits_{\gamma\setminus\gamma_{1/2}(\xi)} \frac{|dt|}{|t-\xi|} \le \pi + 2\,\mathrm{mes}\,\gamma \qquad \forall\, \xi \in \gamma : |\xi| = 1.$$

Thus, γ is a Král curve that is not the Radon curve of bounded rotation.

For points $\xi_1, \xi_2 \in \gamma$, we denote by $\gamma[\xi_1, \xi_2]$ the arc of the curve γ with the initial point ξ_1 and the end point ξ_2 at the orientation of this arc, which is induced by the orientation of the curve γ.

The following statement defines a class of smooth curves satisfying condition (4.1).

Proposition 4.2. If for a closed smooth Jordan curve γ the angle ϑ_t satisfies the condition

$$\int_0^1 \frac{\omega_\gamma(\vartheta_t, \eta)}{\eta}\, d\eta < \infty, \tag{4.6}$$

then γ is a Král curve.

Proof. Let $\xi \in \gamma$. It is known (see, for example, N.I. Muskhelishvili [20]) that there exists $r_0 > 0$, which does not depend on ξ, such that each circle of radius $r \le r_0$ centered at the point ξ intersects γ in only two points.

We denote by t_- and t_+ the points of intersection of the circle $\{z \in \mathbb{C} : |z-\xi| = r_0\}$ and the curve γ, and with the given orientation γ the point t_- precedes the point ξ and the point t_+ follows it. Considering one of the arcs either $\gamma[t_-,\xi]$ or $\gamma[\xi,t_+]$, we will denote it $\widetilde{\gamma}$.

The arc $\widetilde{\gamma}$ allows the parameterization $t = \xi + r\,e^{i(\phi(r)+\phi_0)}$, $r \in [0,r_0]$, where ϕ_0 is a real constant and $\phi(r) \to 0$ as $r \to 0$. Denote $\widetilde{x}(r) := r\cos\phi(r)$, $\widetilde{y}(r) := r\sin\phi(r)$.

For all $t \in \widetilde{\gamma} \setminus \{\xi\}$, the following equalities hold:

$$d\arg(t-\xi) = d\phi(r) = d\,\mathrm{arctg}\,\frac{\widetilde{y}(r)}{\widetilde{x}(r)} = \frac{1}{1+\left(\frac{\widetilde{y}(r)}{\widetilde{x}(r)}\right)^2}\left(\frac{\widetilde{y}\,'(r)}{\widetilde{x}(r)} - \frac{\widetilde{y}(r)\widetilde{x}\,'(r)}{\big(\widetilde{x}(r)\big)^2}\right)dr$$
$$= \widetilde{x}\,'(r)\,\cos^2\phi(r)\,\frac{\frac{\widetilde{y}\,'(r)}{\widetilde{x}\,'(r)} - \mathrm{tg}\,\phi(r)}{\widetilde{x}(r)}\,dr = \widetilde{x}\,'(r)\,\cos\phi(r)\,\frac{\mathrm{tg}\,(\vartheta_t-\vartheta_\xi) - \mathrm{tg}\,\phi(r)}{r}\,dr\,.$$

Note that for a smooth arc $\widetilde{\gamma}$, for each $r \in (0,r_0]$ there exists $r_* \in [0,r]$ such that for $t_* = \xi + r_*\,e^{i(\phi(r_*)+\phi_0)}$ the following relations are fulfilled:

$$|\phi(r)| = |\vartheta_{t_*} - \vartheta_\xi| \le \omega_\gamma(\vartheta_t, r). \tag{4.7}$$

Without loss of generality, we assume r_0 to be small enough to satisfy the inequality $\omega_\gamma(\vartheta_t,r_0) < 1$. Then we get the estimate

$$\int\limits_{\widetilde{\gamma}} |d\arg(t-\xi)| \le c\int\limits_0^{r_0}\frac{\omega_\gamma(\vartheta_t,r)}{r}\,dr < \infty,$$

where the constant c depends on r_0, but does not depend on ξ.

Finally, using the obtained estimate, we estimate the variation

$$V_\gamma[\arg(t-\xi)] \le \int\limits_{\gamma[t_1,\xi]} |d\arg(t-\xi)| + \int\limits_{\gamma[\xi,t_2]} |d\arg(t-\xi)| + \int\limits_{\gamma\setminus\gamma_{r_0}(\xi)} \frac{|dt|}{|t-\xi|}$$
$$\le 2c\int\limits_0^{r_0}\frac{\omega_\gamma(\vartheta_t,r)}{r}\,dr + \frac{\mathrm{mes}\,\gamma}{r_0}\,,$$

which yields the fulfillment of condition (4.1) for the curve γ. □

It is obvious that condition (4.6) is a weaker constraint on the curve γ compared to condition (2.4).

Relation (4.7) implies the estimate

$$\omega_{\widetilde{\gamma}}\big(\arg(t-\xi),\eta\big) \le \omega_\gamma(\vartheta_t,\eta) \qquad \forall\,\eta \in [0,r_0],$$

where the arc $\widetilde{\gamma}$ is defined in the proof of Proposition 4.2. Therefore, if the modulus of continuity of the angle ϑ_t satisfies Dini condition (4.6), then the condition of the same form is also satisfied for the modulus of continuity $\omega_{\widetilde{\gamma}}\big(\arg(t-\xi),\eta\big)$ at all points $\xi \in \gamma$:

$$\int\limits_0^1 \frac{\omega_{\widetilde{\gamma}}\big(\arg(t-\xi),\eta\big)}{\eta}\,d\eta < \infty. \tag{4.8}$$

Let us show that the class of smooth Král curves differs from the class of smooth curves γ that satisfy the conditions of form (4.8) at all points $\xi \in \gamma$. First, we give an example of a smooth curve γ that is a Král curve, but condition (4.8) is not satisfied at a point $\xi \in \gamma$.

Example 2. Consider the smooth arc

$$\widetilde{\gamma} = \Big\{ t(r) = r \exp\Big(-i\,\frac{1}{\ln r}\Big) : r \in (0, r_0] \Big\},$$

where r_0 is the smallest positive root of the equation $\operatorname{Re} t'(r) = 0$. It is obvious that the one-sided tangent to the arc $\widetilde{\gamma}$ at the beginning point $t_0 = 0$ is the positive semi-axis of the real axis. At the end point $t(r_0)$, the arc $\widetilde{\gamma}$ has the one-sided tangent parallel to the imaginary axis of the complex plane.

Let Γ be such an arc of the ellipse that includes the points $z = x + iy$ satisfying the equation

$$\frac{x^2}{(\operatorname{Re} t(r_0))^2} + \frac{(y - \operatorname{Im} t(r_0))^2}{(\operatorname{Im} t(r_0))^2} = 1,$$

which is smoothly glued to the arc $\widetilde{\gamma}$ at the points 0 and $t(r_0)$. Then $\gamma = \widetilde{\gamma} \cup \Gamma$ is a closed smooth Jordan curve.

It is obvious that the curve γ satisfies condition (4.1) because $V_\gamma[\arg(t-\xi)] = \pi$ for all $\xi \in \gamma$. At the same time, condition (4.8) is not satisfied at the point $\xi = 0$ owing to the fact that

$$\int_0^1 \frac{\omega_{\widetilde{\gamma}}\big(\arg t, \eta\big)}{\eta}\, d\eta \geq -\int_0^{r_0} \frac{1}{\eta \ln \eta}\, d\eta = \infty.$$

Now we give an example of a smooth curve γ for which the conditions of form (4.8) are satisfied at all points $\xi \in \gamma$, but it is not a Král curve.

Example 3. Consider the smooth arc

$$\Gamma_1 = \Big\{ t(r) = r \exp\Big(-i\,\frac{r}{\ln r}\,\cos\frac{\pi}{r}\Big) : r \in (0, 1/2] \Big\}.$$

Let Γ_2 be such an arc of the ellipse that includes the points $z = x + iy$ satisfying the equation

$$\frac{x^2}{a^2} + \frac{(y-b)^2}{b^2} = 1$$

with fully defined positive a and b, which is smoothly glued to the arc Γ_1 at the points 0 and $t(1/2)$. Then $\gamma = \Gamma_1 \cup \Gamma_2$ is a closed smooth Jordan curve.

For each point $\xi \in \gamma$, consider the arcs $\gamma[t_-, \xi]$ and $\gamma[\xi, t_+]$ defined in the proof of Proposition 4.2, and denote them by γ_ξ^- and γ_ξ^+, respectively. Considering the function $\arg(t-\xi)$ on the arc $\gamma_\xi^\pm$, we redefine it at the point $t = \xi$ by the limiting value

$$\lim_{t \to \xi, t \in \gamma_\xi^\pm} \arg(t - \xi).$$

As a result, for each $\xi \in \gamma$, the function $\arg(t-\xi)$ satisfies the Hölder condition on each of the arcs γ_ξ^- and γ_ξ^+:

$$|\arg(t_1 - \xi) - \arg(t_2 - \xi)| \leq c\,|t_1 - t_2|^\alpha \qquad \forall\, t_1, t_2 \in \gamma_\xi^\pm$$

for all $\alpha \in (0, 1/2]$, where the constant c does not depend on t_1 and t_2. Therefore, the conditions of form (4.8) are satisfied at all points of $\xi \in \gamma$.

At the same time,

$$V_\gamma[\arg t] \geq V_{\Gamma_1}[\arg t] \geq \sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty,$$

i.e., condition (4.1) is not satisfied for the curve γ.

Example 3 also shows that condition (4.6) on the angle ϑ_t in Proposition 4.2 can not be replaced by a similar condition of form (4.8) on the function $\arg(t-\xi)$.

5 Sufficient conditions for the continuous extension of the real part of the Cauchy-type integral to the boundary of domain with unbounded variation of the function $\arg(t-\xi)$

We shall now consider curves for which condition (4.1) is not satisfied, generally speaking.

In what follows, we use the following characteristic of the function $f\colon E \to \mathbb{C}$ continuous on the set $E \subset \mathbb{C}$ (see O.F. Gerus [11]):

$$\Omega_{E,f}(a,b) := \sup_{a\le\eta\le b} \frac{\omega_E(f,\eta)}{\eta} \qquad \text{for} \quad 0<a\le b.$$

The function $\Omega_{E,f}(a,b)$ does not increase monotonically with respect to the variable a and does not decrease monotonically with respect to the variable b. In addition, the function $a\,\Omega_{E,f}(a,b)$ does not decrease monotonically with respect to the variable a.

Denote $E^{R,\psi_1,\psi_2}(\xi) := \{z=\xi+re^{i\phi} : R/2<r<R, \psi_1<\phi<\psi_2\}$.

Let us describe a certain finite set of Jordan arcs placed in the closure of domain $E^{R,0,\psi}(0)$. For this purpose, we consider two sets of points $\{\tau_j\}_{j=1}^n$ and $\{\eta_j\}_{j=1}^n$ located on the rectilinear parts of the boundary of domain $E^{R,0,\psi}(0)$ such that

$$R\ge\tau_1\ge\tau_2\ge\cdots\ge\tau_n\ge R/2,$$

$$\eta_j=|\eta_j|\,e^{i\psi} \quad \text{and} \quad R\ge|\eta_1|\ge|\eta_2|\ge\cdots\ge|\eta_n|\ge R/2.$$

Let $\Gamma := \bigcup_{j=1}^n \Gamma_j$, where Γ_j is a Jordan arc with ends at the points τ_j and η_j. Moreover, the arcs Γ_j, $j=1,2,\ldots,n$, excluding the ends, lie in the domain $E^{R,0,\psi}(0)$ and pairwise do not intersect within this domain. In addition, if the arc Γ_j is oriented from the point τ_j to the point η_j, then the next arc Γ_{j+1} is oriented in the opposite direction from the point η_{j+1} to the point τ_{j+1}, and vice versa, if the arc Γ_j is oriented from the point η_j to the point τ_j, then the arc Γ_{j+1} is oriented from the point τ_{j+1} to the point η_{j+1}.

Consider the auxiliary statements.

Lemma 5.1. *If a function $f:\Gamma\to\mathbb{R}$ is continuous on Γ, then*

$$\left|\int_\Gamma f(t)\,d\arg t\right| \le \left(R\,\Omega_{\Gamma,f}\Big(\frac{R}{n},R\Big)+\max_{t\in\Gamma}|f(t)|\right)\psi+\frac{2\,\omega_\Gamma(f,\lambda)\,\mathrm{mes}\,\Gamma}{R},$$

where $\lambda := \max\limits_j \mathrm{mes}\,\Gamma_j$ and $\arg t$ is any branch of the multivalued function $\mathrm{Arg}\,z$, which is continuous on Γ.

Proof. As in the paper T.S. Salimov [28], we use the representation

$$\int_\Gamma f(t)\,d\arg t=\sum_{j=1}^n\int_{\Gamma_j}\big(f(t)-f(\tau_j)\big)\,d\arg t+\sum_{j=1}^n f(\tau_j)\int_{\Gamma_j}d\arg t$$

and the estimates

$$\left|\sum_{j=1}^n\int_{\Gamma_j}\big(f(t)-f(\tau_j)\big)\,d\arg t\right|\le\sum_{j=1}^n\left|\mathrm{Im}\int_{\Gamma_j}\frac{f(t)-f(\tau_j)}{t}\,dt\right|\le$$

$$\sum_{j=1}^n\int_{\Gamma_j}\frac{|f(t)-f(\tau_j)|}{|t|}\,|dt|\le\sum_{j=1}^n\frac{2\,\omega_{\Gamma_j}(f,\mathrm{mes}\,\Gamma_j)\,\mathrm{mes}\,\Gamma_j}{R}\le\frac{2\,\omega_\Gamma(f,\lambda)\,\mathrm{mes}\,\Gamma}{R},$$

$$\left|\sum_{j=1}^{n} f(\tau_j) \int\limits_{\Gamma_j} d\arg t\right| \leq \left(\sum_{j=1}^{n_0} |f(\tau_{2j-1}) - f(\tau_{2j})| + 2q_0|f(\tau_n)|\right)\psi,$$

where n_0 is the integer part of the number $n/2$ and q_0 is the fractional part of the number $n/2$.

Next, taking into account the estimates (see Lemma 1 in the paper O.F. Gerus [11])

$$\sum_{j=1}^{n_0} |f(\tau_{2j-1}) - f(\tau_{2j})| \leq \sum_{j=1}^{n_0} \omega_\gamma(f, |\tau_{2j-1} - \tau_{2j}|) \leq R\,\Omega_{\Gamma,f}\Big(\frac{R}{2n_0}, R\Big)$$

and the inequality $2n_0 \leq n$, we have

$$\left|\sum_{j=1}^{n} f(\tau_j) \int\limits_{\Gamma_j} d\arg t\right| \leq \left(R\,\Omega_{\Gamma,f}\Big(\frac{R}{n}, R\Big) + \max_{t\in\Gamma}|f(t)|\right)\psi.$$

The given estimates imply the statement of the lemma. □

For a given closed rectifiable Jordan curve γ, we shall consider its intersections with the domains $E^{R,\psi_1,\psi_2}(\xi)$, where $\xi \in \gamma$. We shall denote these intersections by $\gamma_{R,\psi_1,\psi_2}(\xi)$. By $n_\gamma(\xi, R, \psi_1, \psi_2)$ we denote the number of connected components of the set $\gamma_{R,\psi_1,\psi_2}(\xi)$, the ends of which lie on the different segments $\{z = \xi + re^{i\psi_1} : R/2 \leq r \leq R\}$ and $\{z = \xi + re^{i\psi_2} : R/2 \leq r \leq R\}$. We can say that the number $n_\gamma(\xi, R, \psi_1, \psi_2)$ expresses the number of complete oscillations of the function $\arg(t - \xi)$ in the domain $E^{R,\psi_1,\psi_2}(\xi)$. Since the curve γ is rectifiable, the number $n_\gamma(\xi, R, \psi_1, \psi_2)$ is finite, but with fixed ξ and R it can tend to infinity when $\psi_2 \to \psi_1$.

Consider the case in which there exist $\xi \in \gamma$ and $R \in (0, d]$ such that

$$k_\gamma(\xi, R) := \max\Big\{1, \sup_{0\leq\psi_1<\psi_2<2\pi} n_\gamma(\xi, R, \psi_1, \psi_2)\Big\} < \infty. \tag{5.1}$$

Denote by $\varphi_\gamma(\xi, R)$ the Lebesgue measure (given on the segment $[0, 2\pi]$) of the set of those $\phi \in [0, 2\pi]$ for which the rays $\{z = \xi + re^{i\phi} : r > 0\}$ have a nonempty intersection with the set $\gamma_R(\xi) \setminus \gamma_{R/2}(\xi)$.

Lemma 5.2. *Let a closed Jordan curve γ be Ahlfors-regular and satisfy condition (5.1) and let a function $g\colon \gamma \to \mathbb{R}$ be continuous on γ. Then the following estimate holds:*

$$\left|\int\limits_{\gamma_R(\xi)\setminus\gamma_{R/2}(\xi)} \big(g(t) - g(\xi)\big)\, d\arg(t - \xi)\right| \leq 6\,R\,\varphi_\gamma(\xi, R)\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi, R)}\,,\, R\Big). \tag{5.2}$$

Proof. For the proof, we apply the method developed by T.S. Salimov [28] for estimating the modulus of continuity of the Cauchy singular integral on an arbitrary closed rectifiable Jordan curve and adapted by O.F. Gerus [11] for the purpose of using the modulus of continuity of the integral density instead of the regularized (by means of the Stechkin construction) modulus of continuity, which is used in the paper [28].

The set $\breve\gamma := \gamma_R(\xi) \setminus \gamma_{R/2}(\xi)$ is the union of no more than a countable collection of connected components of the set $\breve\gamma \setminus \{z \in \gamma : |z - \xi| = R\}$, which is open in the topology of the curve γ, and the closed set $\{z \in \gamma : |z - \xi| = R\}$ for which

$$\mathrm{mes}\,\{z \in \gamma : |z - \xi| = R\} \leq \varphi_\gamma(\xi, R)R.$$

Therefore, there exists a finite union $\widehat\gamma$ of the specified connected components such that

$$\mathrm{mes}\,(\breve\gamma \setminus \widehat\gamma) \leq 2\,\varphi_\gamma(\xi, R)R.$$

We have the equality

$$\int\limits_{\gamma_R(\xi)\setminus\gamma_{R/2}(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)$$

$$= \int\limits_{\widehat{\gamma}} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi) + \int\limits_{\breve{\gamma}\setminus\widehat{\gamma}} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi) \quad (5.3)$$

and the estimate

$$\Bigg|\int\limits_{\breve{\gamma}\setminus\widehat{\gamma}} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\Bigg| = \Bigg|\mathrm{Im}\int\limits_{\breve{\gamma}\setminus\widehat{\gamma}} \frac{g(t)-g(\xi)}{t-\xi}\,dt\Bigg|$$

$$\le \int\limits_{\breve{\gamma}\setminus\widehat{\gamma}} \frac{\big|g(t)-g(\xi)\big|}{|t-\xi|}\,|dt| \le \frac{\omega_\gamma(g,R)\,\mathrm{mes}\,(\breve{\gamma}\setminus\widehat{\gamma})}{R/2}$$

$$\le 4\,\varphi_\gamma(\xi,R)\,\omega_\gamma(g,R) \le 4\,R\,\varphi_\gamma(\xi,R)\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)}\,,\,R\Big). \quad (5.4)$$

To estimate the first integral in the right-hand side of equality (5.3), we partition the ring $\{z\in\mathbb{C}:R/2<|z-\xi|<R\}$ by the rays $\{z=\xi+re^{i\phi_m}:r>0\}$, $m=1,2,\ldots,k$, for $0=\phi_0<\phi_1<\cdots<\phi_k=2\pi$, so that the ends of all connected components of all nonempty sets $\widehat{\gamma}_{R,\phi_{m-1},\phi_m}(\xi):=\widehat{\gamma}\cap E^{R,\phi_{m-1},\phi_m}(\xi)$, $m=1,2,\ldots,k$, lay on the indicated rays. It is always possible due to the finite number of connected components of the set $\widehat{\gamma}$. In this case, there is the representation $\widehat{\gamma}_{R,\phi_{m-1},\phi_m}(\xi)=\widehat{\gamma}_{m,1}\cup\widehat{\gamma}_{m,2}$, where $\widehat{\gamma}_{m,1}:=\bigcup_j\widehat{\gamma}_{m,1,j}$ is the union of connected components $\widehat{\gamma}_{m,1,j}$ of the set $\widehat{\gamma}_{R,\phi_{m-1},\phi_m}(\xi)$ with the ends on one of the specified rays, and $\widehat{\gamma}_{m,2}:=\bigcup_j\widehat{\gamma}_{m,2,j}$ is the union of a finite number of connected components $\widehat{\gamma}_{m,2,j}$ of the set $\widehat{\gamma}_{R,\phi_{m-1},\phi_m}(\xi)$ with the ends on different rays.

We have the following equality for these sets:

$$\widehat{\gamma}=\bigcup_{m=1}^{k}\widehat{\gamma}_{m,1}\cup\bigcup_{m=1}^{k}\widehat{\gamma}_{m,2}\cup\bigcup_{m=1}^{k}\widehat{\gamma}_{m,3},$$

where $\widehat{\gamma}_{m,3}:=\widehat{\gamma}\cap\{z=\xi+re^{i\phi_m}:R/2\le r\le R\}$, which implies the following equality for the corresponding integrals:

$$\int\limits_{\widehat{\gamma}} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi) = \sum_{m=1}^{k}\Bigg(\int\limits_{\widehat{\gamma}_{m,1}}+\int\limits_{\widehat{\gamma}_{m,2}}+\int\limits_{\widehat{\gamma}_{m,3}}\Bigg)\big(g(t)-g(\xi)\big)\,d\arg(t-\xi)$$

$$= \sum_{m=1}^{k}\Bigg(\int\limits_{\widehat{\gamma}_{m,1}}+\int\limits_{\widehat{\gamma}_{m,2}}\Bigg)\big(g(t)-g(\xi)\big)\,d\arg(t-\xi), \quad (5.5)$$

because the integrals over the sets $\widehat{\gamma}_{m,3}$ are equal to zero.

Denote $\lambda:=\max\limits_{l=1,2}\max\limits_{m,j}\mathrm{mes}\,\widehat{\gamma}_{m,l,j}$.

Estimating the integrals over the nonempty sets $\widehat{\gamma}_{m,1}$, we denote one of the ends of the arc $\widehat{\gamma}_{m,1,j}$

by $\tau_{m,j}$, and as a result, we get

$$\Big|\int\limits_{\widehat{\gamma}_{m,1}} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\Big| = \Big|\sum_j \int\limits_{\widehat{\gamma}_{m,1,j}} \big(g(t)-g(\tau_{m,j})\big)\, d\arg(t-\xi)\Big|$$
$$= \Big|\sum_j \operatorname{Im} \int\limits_{\widehat{\gamma}_{m,1,j}} \frac{g(t)-g(\tau_{m,j})}{t-\xi}\, dt\Big| \le \sum_j \int\limits_{\widehat{\gamma}_{m,1,j}} \frac{\big|g(t)-g(\tau_{m,j})\big|}{|t-\xi|}\, |dt|$$
$$\le \sum_j \frac{\omega_\gamma(g,\lambda)\operatorname{mes}\widehat{\gamma}_{m,1,j}}{R/2} \le \frac{2\,\omega_\gamma(g,\lambda)\operatorname{mes}\widehat{\gamma}_{m,1}}{R}. \qquad (5.6)$$

The integrals over the nonempty sets $\widehat{\gamma}_{m,2}$ are estimated by applying Lemma 5.1, for the application of which it is necessary to establish an obvious correspondence between the parameters of the sets Γ and $\widehat{\gamma}_{m,2}$. As a result, we have

$$\Big|\int\limits_{\widehat{\gamma}_{m,2}} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\Big|$$
$$\le (\phi_m-\phi_{m-1})\Big(R\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)},R\Big)+\omega_\gamma(g,R)\Big)+\frac{2\,\omega_\gamma(g,\lambda)\operatorname{mes}\widehat{\gamma}_{m,2}}{R}$$
$$\le 2\,(\phi_m-\phi_{m-1})\,R\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)},R\Big)+\frac{2\,\omega_\gamma(g,\lambda)\operatorname{mes}\widehat{\gamma}_{m,2}}{R}. \qquad (5.7)$$

Taking into account equalities (5.3), (5.5) and estimates (5.4), (5.6) and (5.7), as well as condition (2.1) on the curve γ, we obtain the estimate

$$\Big|\int\limits_{\gamma_R(\xi)\setminus\gamma_{R/2}(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\Big| \le 4\,R\,\varphi_\gamma(\xi,R)\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)}\,,\,R\Big)$$
$$+\,2\sum_{m=1}^{k}\Big((\phi_m-\phi_{m-1})\,R\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)},R\Big)+\frac{\omega_\gamma(g,\lambda)\,\big(\operatorname{mes}\widehat{\gamma}_{m,1}+\operatorname{mes}\widehat{\gamma}_{m,2}\big)}{R}\Big)$$
$$\le 6\,R\,\varphi_\gamma(\xi,R)\,\Omega_{\gamma,g}\Big(\frac{R}{k_\gamma(\xi,R)}\,,\,R\Big)+c\,\omega_\gamma(g,\lambda),$$

where the constant c does not depend on ξ and R.

Now, as a result of a refinement of the partition of the ring $\{z\in\mathbb{C}:\,R/2<|z-\xi|<R\}$, if $k\to\infty$, hence $\lambda\to 0$, from the last estimate we obtain estimate (5.2). □

Lemma 5.3. *Let a closed Jordan curve γ be Ahlfors-regular and let a function $g\colon\gamma\to\mathbb{R}$ be continuous on γ. Let for $\xi\in\gamma$ condition (5.1) be satisfied for all $R\in[\delta,2\varepsilon]$, where $0<\delta<\varepsilon\le d/2$. Then the following estimate holds:*

$$\Big|\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\Big|$$
$$\le c\Big(\int\limits_{\delta}^{2\varepsilon} \widehat{\varphi}_\gamma(\xi,\eta)\,\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)}\,,\,\eta\Big)\,d\eta+\omega_\gamma(g,\varepsilon)\Big), \qquad (5.8)$$

where $\widehat{\varphi}_\gamma(\xi,R):=\sup\limits_{r\in[R/2,\,R]}\varphi_\gamma(\xi,r)$, $\widehat{k}_\gamma(\xi,R):=\sup\limits_{r\in[R/2,\,R]}k_\gamma(\xi,r)$, *and the constant $c>0$ does not depend on ξ, δ and ε.*

Proof. Let $\delta \in [\varepsilon/2^n, \varepsilon/2^{n-1})$ for some natural n. Using estimate (5.2), we obtain

$$\Big|\int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\Big| \le \sum_{m=0}^{n-2}\Big|\int\limits_{\gamma_{\varepsilon/2^m}(\xi)\setminus\gamma_{\varepsilon/2^{m+1}}(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\Big|$$
$$+\Big|\int\limits_{\gamma_{\varepsilon/2^{n-1}}(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\Big|$$
$$\le 6\sum_{m=0}^{n-2}\frac{\varepsilon}{2^m}\,\varphi_\gamma(\xi,\varepsilon/2^m)\,\Omega_{\gamma,g}\Big(\frac{\varepsilon}{2^m\,k_\gamma(\xi,\varepsilon/2^m)}\,,\,\frac{\varepsilon}{2^m}\Big)$$
$$+\Big|\operatorname{Im}\int\limits_{\gamma_{\varepsilon/2^{n-1}}(\xi)\setminus\gamma_\delta(\xi)} \frac{g(t)-g(\xi)}{t-\xi}\,dt\Big| =: I(\xi,\delta,\varepsilon)\,.$$

Next, we continue the estimation, using the monotonicity properties of the function $\Omega_{\gamma,g}$ and performing the transition to the functions $\widehat{\varphi}_\gamma$ and $\widehat{k}_\gamma$:

$$I(\xi,\delta,\varepsilon) \le \frac{6}{\ln 2}\sum_{m=0}^{n-2}\frac{\varepsilon}{2^m}\,\varphi_\gamma(\xi,\varepsilon/2^m)\,\Omega_{\gamma,g}\Big(\frac{\varepsilon}{2^m\,k_\gamma(\xi,\varepsilon/2^m)}\,,\,\frac{\varepsilon}{2^m}\Big)\int\limits_{\varepsilon/2^m}^{\varepsilon/2^{m-1}}\frac{d\eta}{\eta}$$
$$+\int\limits_{\gamma_{\varepsilon/2^{n-1}}(\xi)\setminus\gamma_\delta(\xi)}\frac{\big|g(t)-g(\xi)\big|}{|t-\xi|}\,|dt|$$
$$\le \frac{6}{\ln 2}\sum_{m=0}^{n-2}\int\limits_{\varepsilon/2^m}^{\varepsilon/2^{m-1}}\widehat{\varphi}_\gamma(\xi,\eta)\,\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)}\,,\,\eta\Big)\,d\eta + \frac{\omega_\gamma(g,\varepsilon/2^{n-1})\,\theta_\xi(\varepsilon/2^{n-1})}{\varepsilon/2^n}\,.$$

As a result, taking into account condition (2.1) on the curve γ, we get estimate (5.8). □

The following statement is true.

Theorem 5.1. *Let a closed Jordan curve γ be Ahlfors-regular and let a function $g\colon \gamma\to\mathbb{R}$ be continuous on γ. Consider a partition $\gamma=\gamma^1\cup\gamma^2$, for which there exists $R_0\in(0,d]$ such that:*

(a) for all $\xi\in\gamma^1$ and all $R\in(0,R_0]$, the curve γ satisfies condition (5.1) and, in addition, the following condition is satisfied:

$$\sup_{\xi\in\gamma^1}\int\limits_0^{R_0}\widehat{\varphi}_\gamma(\xi,\eta)\,\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)}\,,\,\eta\Big)\,d\eta<\infty\,; \tag{5.9}$$

(b) for each $\xi\in\gamma^2$ there exists $r(\xi)\in(0,R_0]$ such that the curve γ satisfies condition (5.1) for all $R\in(r(\xi),R_0]$, the function $\arg(t-\xi)$ has bounded variation on the set $\gamma_{r(\xi)}(\xi)\setminus\{\xi\}$ and, in addition, the following condition is satisfied:

$$\sup_{\xi\in\gamma^2}\Big(V_{\gamma_{r(\xi)}(\xi)}[\arg(t-\xi)]+\int\limits_{r(\xi)}^{R_0}\widehat{\varphi}_\gamma(\xi,\eta)\,\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)}\,,\,\eta\Big)\,d\eta\Big)<\infty\,. \tag{5.10}$$

Then the function $\operatorname{Re}\widetilde{g}(z)$ has a continuous extension to the boundary γ from the domains D^+ and D^-, and the limiting values $\big(\operatorname{Re}\widetilde{g}\big)^\pm(\xi)$ are represented by formulas (2.10) and (2.11) for all $\xi\in\gamma$.

Proof. Let us show that under the assumptions of the theorem, condition (3.1) is also satisfied.

Let $\varepsilon \in (0, R_0/2]$ and $\delta \in (0, \varepsilon)$. Then estimate (5.8) holds for all $\xi \in \gamma^1$, and for each $\xi \in \gamma^2$, taking into account Lemma 5.3, we obtain the estimate

$$\Big| \int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \Big|$$

$$\le \Big| \int\limits_{\gamma_{r(\xi)}(\xi)\setminus\gamma_\delta(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \Big| + \Big| \int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_{r(\xi)}(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \Big|$$

$$\le \omega_\gamma(g,\varepsilon)\, V_{\gamma_{r(\xi)}(\xi)}[\arg(t-\xi)] + c\Bigg(\int\limits_{r(\xi)}^{2\varepsilon} \widehat{\varphi}_\gamma(\xi,\eta)\, \Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)}\,,\, \eta\Big)\, d\eta + \omega_\gamma(g,\varepsilon)\Bigg),$$

where the constant $c>0$ does not depend on ξ, δ and ε.

Under conditions (5.9) and (5.10), the given estimates yield condition (3.1).

Now, to complete the proof, it remains to apply Theorem 3.1. □

Corollary 5.1. *The statement of Theorem 5.1 remains valid if conditions (5.9), (5.10) are replaced by the conditions*

$$\sup_{\xi\in\gamma^1} \int\limits_0^{R_0} \frac{\widehat{\varphi}_\gamma(\xi,\eta)\, \widehat{k}_\gamma(\xi,\eta)\, \omega_\gamma(g,\eta)}{\eta}\, d\eta < \infty\,, \tag{5.11}$$

$$\sup_{\xi\in\gamma^2} \Bigg(V_{\gamma_{r(\xi)}(\xi)}[\arg(t-\xi)] + \int\limits_{r(\xi)}^{R_0} \frac{\widehat{\varphi}_\gamma(\xi,\eta)\, \widehat{k}_\gamma(\xi,\eta)\, \omega_\gamma(g,\eta)}{\eta}\, d\eta \Bigg) < \infty\,, \tag{5.12}$$

respectively.

It is clear that Corollary 5.1 follows from Theorem 5.1 and the inequality

$$\Omega_{\gamma,g}\Big(\frac{\eta}{\widehat{k}_\gamma(\xi,\eta)}\,,\,\eta\Big) \le \frac{\widehat{k}_\gamma(\xi,\eta)\, \omega_\gamma(g,\eta)}{\eta} \qquad \forall\, \eta \in (0, R_0].$$

Note that in the papers of T.S. Salimov [28], E.M. Dyn'kin [7] and O.F. Gerus [11], the singular Cauchy integral (2.6) on an arbitrary closed rectifiable Jordan curve is considered under certain conditions on the integral density, which are reduced to a condition of form (2.5) in the case of a curve satisfying condition (2.1). Under such conditions on the curve and the integral density, Cauchy-type integral (1.2) is continuously extended to the boundary γ from the domains D^+ and D^-, and the limiting values $\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi)$ are expressed by formulas (2.10) and (2.11) for all $\xi \in \gamma$.

At the same time, under additional assumptions of type (5.1) about the curve γ, Theorem 5.1 and Corollary 5.1 allow to construct examples (see the next example) of the curves γ that do not satisfy condition (4.1) and the functions g that do not satisfy condition (2.5) and a similar condition (see N.A. Davydov [6]) used in Theorem 2 in the paper of O.F. Gerus and M. Shapiro [13], but the limiting values $\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi)$ of logarithmic double layer potential (1.1) exist at all points $\xi \in \gamma$ and are expressed by formulas (2.10) and (2.11).

Example 4. Consider the curve

$$\gamma = \Big\{ z = e^{i\phi} : \phi \in [0,\pi] \Big\} \cup [-1,0] \cup \bigcup_{n=1}^{\infty} [2^{-2n+1}, 2^{-2n+2}]$$

$$\cup \bigcup_{n=1}^{\infty} \Big\{ z = 2^{-n} e^{i\phi} : \phi \in [0, 1/n] \Big\} \cup \bigcup_{n=1}^{\infty} \Big\{ z = r e^{-i\frac{\ln 2}{\ln r}} : r \in [2^{-2n}, 2^{-2n+1}] \Big\}$$

and the function

$$g(t) = \begin{cases} -1/\big(\ln|t| - 1\big) & \text{for} \quad t \in \gamma \setminus \{0\}, \\ 0 & \text{for} \quad t = 0 \end{cases}$$

that does not satisfy Dini condition (2.5) as well as a similar condition used in Theorem 2 in paper [13] because

$$\min\left\{ \int_0^1 \frac{\omega_\gamma(g,\eta)}{\eta}\, d\eta,\ \int_\gamma \frac{|g(t) - g(0)|}{|t - 0|}\, |dt| \right\} \geq \int_{-1}^0 \frac{|g(t) - g(0)|}{|t - 0|}\, dt$$

$$= -\int_{-1}^0 \frac{dt}{|t|\,(\ln|t| - 1)} = \infty\,.$$

For $0 < \varepsilon \leq 1/2$, denoting by n_0 the smallest natural number n that satisfies the inequality $2^{-n} \leq \varepsilon$, we obtain the estimate

$$\theta(\varepsilon) = \theta_0(\varepsilon) \leq 2\varepsilon + \sum_{n=n_0}^\infty \frac{1}{n\, 2^n} + \int_0^\varepsilon \sqrt{1 + \frac{\ln^2 2}{\ln^4 r}}\, dr \leq 2\varepsilon + 2\varepsilon + \frac{\sqrt{1 + \ln^2 2}}{\ln 2}\, \varepsilon$$

$$\leq \left(4 + \frac{\sqrt{1 + \ln^2 2}}{\ln 2} \right) \varepsilon\,, \quad (5.13)$$

which proves the validity of condition (2.1) for the curve γ. Thus, γ is an Ahlfors-regular curve.

At the same time,

$$V_\gamma[\arg t] \geq \sum_{n=1}^\infty \frac{1}{n} = \infty$$

and condition (4.1) is not satisfied for the curve γ, i.e., γ is not a Král curve.

Let us show that the curve γ and the function g satisfy the conditions of Corolary 5.1. There is the partition $\gamma = \gamma^1 \cup \gamma^2$, where $\gamma^1 = \{0\}$ and $\gamma^2 = \gamma \setminus \{0\}$. Let $R_0 = 1/2$ and $r(\xi) = \min\{2|\xi|, 1/2\}$ for all $\xi \in \gamma^2$. Then for all $\xi \in \gamma^2$, we have the estimate

$$V_{\gamma_{r(\xi)}(\xi)}[\arg(t - \xi)] \leq V_{\gamma_{r(\xi)/4}(\xi)}[\arg(t - \xi)] + \int\limits_{\gamma_{r(\xi)}(\xi) \setminus \gamma_{r(\xi)/4}(\xi)} \frac{|dt|}{|t - \xi|} \leq \pi + \frac{\theta\big(r(\xi)\big)}{r(\xi)/4} \leq \pi + 4c,$$

where $c = 4 + (\ln 2)^{-1} \sqrt{1 + \ln^2 2}$ as it is follows from estimate (5.13).

It is obvious that $\widehat{k}_\gamma(0,\eta) \leq 2$ and $\widehat{\varphi}_\gamma(0,\eta) < -1/\ln\eta$ for all $\eta \in (0, R_0]$.

In addition, $\widehat{k}_\gamma(\xi,\eta) \leq 2$ for all $\xi \in \gamma^2$ and all $\eta \in (r(\xi), R_0]$.

Finally, taking into account the relations $|t| \leq |t - \xi| + |\xi| \leq 3\eta/2$, which hold for all $\xi \in \gamma^2$ and all $t \in \gamma$ such that $|t - \xi| = \eta \in (r(\xi), R_0]$, for the specified ξ and η, we obtain the inequality $\widehat{\varphi}_\gamma(\xi,\eta) < -3/\ln(3\eta/2)$.

Now, the validity of conditions (5.11) and (5.12) follows from the estimates

$$\sup_{\xi\in\gamma} \int_{\beta(\xi)}^{1/2} \frac{\widehat{\varphi}_\gamma(\xi,\eta)\, \widehat{k}_\gamma(\xi,\eta)\, \omega_\gamma(g,\eta)}{\eta}\, d\eta < 6 \int_0^{1/2} \frac{d\eta}{\eta\, \ln(3\eta/2)(\ln\eta - 1)} < \infty\,,$$

where $\beta(\xi) = 0$ for $\xi = 0 \in \gamma^1$ and $\beta(\xi) = r(\xi)$ for $\xi \in \gamma^2$.

Thus, all conditions of Corollary 5.1 are satisfied for the given curve γ and the given function g.

As a result, we can state that the limiting values $\big(\mathrm{Re}\,\widetilde{g}\big)^{\pm}(\xi)$ of logarithmic double layer potential (1.1) exist at all points $\xi \in \gamma$ and are expressed by formulas (2.10) and (2.11).

The results of this paper have been announced at a preprint of Arxiv (arXiv:2405.01482v1 [math.CV]).

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