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PROPAGATION OF NONSMOOTH WAVES ALONG A STAR GRAPH WITH FIXED BOUNDARY VERTICES

B. Kanguzhin

Communicated by V.I. Burenkov

Key words: star graph, d'Alembert formula, eigenvalues problem, matching conditions

AMS Mathematics Subject Classification: 34A55, 34B05, 58C40.

Abstract. The paper studies the spread of waves along the star graph. The continuation of the initial data from the graph edges for the entire numerical axis allows to represent an analogue of the d'Alembert formula for waves on the star graph. At the same time, the continuation of the initial data is closely related to the continuation of the system of its eigenfunctions of the Sturm-Liouville problem originally defined on the star graph. The continuation of the eigenfunctions defined on the star graph is based on the continuation of the initial data of the mixed problem for the wave equation. The indicated continuation of the initial data of the mixed problem was proposed by B.M. Levitan.

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1 Statement of the main result

Let $\Gamma = \{V, E\}$ be a star graph. Here V is a set of vertices and E is a set of edges. The vertices are numerated by integer numbers from 0 to $m+1$. The interior vertex corresponds to the number $m+1$. The directed edges are denoted by $e_1, ..., e_{m+1}$, with j corresponding to the vertex of edge e_j. The length of the edge e_j is denoted by b_j.

On each edge e_j of the star graph Γ we study the initial boundary value problem (IBVP) for the wave equation with the individual continuous potential $q_j(x_j)$

$$\frac{\partial^2 \theta_j(x_j,t)}{\partial t^2} = \frac{\partial^2 \theta_j(x_j,t)}{\partial x_j^2} - q_j(x_j)\theta_j(x_j,t), \ t>0, \ x_j \in e_j \tag{1.1}$$

with the initial conditions

$$\theta_j(x_j,0) = a_j(x_j), \quad \frac{\partial \theta_j(x_j,0)}{\partial t} = b_j(x_j), \tag{1.2}$$

matching conditions

$$\theta_1(0,t) = \theta_2(0,t) = ... = \theta_m(0,t) = \theta_{m+1}(b_{m+1},t), \tag{1.3}$$

$$\frac{\partial \theta_{m+1}(b_{m+1},t)}{\partial x_{m+1}} = \sum_{j=1}^{m} \frac{\partial \theta_j(0,t)}{\partial x_j} \tag{1.4}$$

at the interior vertices of the star graph, and the Dirichlet boundary conditions

$$\theta_1(b_1,t) = \theta_2(b_2,t) = ... = \theta_m(b_m,t) = \theta_{m+1}(0,t) \tag{1.5}$$

at the boundary vertices. So, one IBVP (1.3) - (1.5) corresponds to the each edge e_j of the star graph Γ.

The goal of this work is to establish an analogue of the d'Alembert formula for the mixed problem (1.3) - (1.5). In the simplest case of a graph, representing a sequential connection of four intervals, such a formula is presented in [1].

We invoke some constructions from the monograph [1] for the statement of the main result. We extend the continuous function $q_j(x_j)$ from $(0, b_j)$ with $j = 1, ..., m+1$ continuously to the entire real axis. So, we consider the following Goursat problem

$$\frac{\partial^2 w_j(x_j,t,s)}{\partial t^2} = \frac{\partial^2 w_j(x_j,t,s)}{\partial s^2} - q_j(s) w_j(x_j,t,s), \quad t > 0, \quad s \in \mathbb{R},$$

$$w_j(x_j,t,t+x_j) = -\frac{1}{2}\int\limits_0^t q_j(\tau + x_j)d\tau,$$

$$w_j(x_j,t,x_j - t) = -\frac{1}{2}\int\limits_0^t q_j(x_j - \tau)d\tau$$

for $j = 1, ..., m+1$. The Goursat problem has a unique solution $w_j(x_j, t, s)$.

Now we exploit an assumption that ensures the simplicity of the spectrum for eigenvalue problem (2.1) - (2.4).

Assumption 1.1. *The lengths $b_1, ..., b_{m+1}$ of the edges $e_1, ..., e_{m+1}$ and the potentials $q_j(x_j)$, $j = 1, ..., m+1$ are chosen such that the spectra of the Sturm-Liouville operators L_j do not intersect pairwise. The operator L_j is defined by the differential expression:*
$\frac{d^2}{dx^2} - q_j(x_j)$ on the domain $D(L_j) = \{y(x_j) \in W_2^2[0, b_j] : y(b_j) = y(0) = 0\}$.

We note that such a choice is always possible.

We state the main result of the work. We establish an existence and uniqueness of the solution to problem (1.1) - (1.5) on the star graph in the following theorem. Moreover, we present an analogue of the d'Alembert formula for the solution to the initial boundary value problem for the wave equation on the star graph.

Theorem 1.1. *Suppose that the potentials $q_j(x_j)$ for $j = 1, ..., m+1$ represent a set of real continuous functions. Let Assumption 1.1 also be satisfied. We assume that the initial functions $a_j(x_j), b_j(x_j)$, $j = 1, ..., m+1$ are twice continuously differentiable on the corresponding edges and satisfy conditions (1.2).*

Then, the solution to problem (1.1) - (1.5) exists and is unique. Moreover, for the solution, an analogue of the d'Alembert formula holds and for $b_j(x_j) \equiv 0$, $j = 1, ..., m+1$ the solution can be represented as follows

$$\theta_j(x_j,t) = \frac{1}{2}\tilde{a}_j(x_j+t) + \frac{1}{2}\tilde{a}_j(x_j-t) + \frac{1}{2}\int\limits_{x_j-t}^{x_j+t} w_j(x_j,t,s)\tilde{a}_j(s)ds.$$

Here $\tilde{a}_j(s)$ is the special continuation of the function $\tilde{a}_j(x_j)$ from the interval $(0, b_j)$ to the entire real axis. The details of this continuation are given in the proof of Theorem 1.1.

We note that $q_j(x_j) \equiv 0$ implies that $w_j(x_j, t, s) \equiv 0$.

2 Proof of Theorem 1.1

It is sufficient to prove Theorem 1.1 for $b_j(x_j) \equiv 0$ with $j = 0, 1, ..., m+1$. To state and prove Theorem 1.1 for nontrivial $b_j(x_j)$, we invoke the standard procedure from [3]. Now we state the following eigenvalue problem

$$l_j(y_j) \equiv -y_j''(x_j) + q_j(x_j)y_j(x_j) = \lambda y_j(x_j),\ x_j \in e_j \tag{2.1}$$

with the following Dirichlet boundary conditions

$$y_1(b_1) = y_2(b_2) = ... = y_m(b_m) = y_{m+1}(0) = 0 \tag{2.2}$$

at the boundary vertices of the star graph and the following matching conditions

$$y_1(0) = y_2(0) = ... = y_m(0) = y_{m+1}(b_{m+1}) \tag{2.3}$$

and

$$y'_{m+1}(b_{m+1}) = \sum_{j=1}^{m} y'_j(0) \tag{2.4}$$

at the interior vertices of the star graph. The spectral properties of eigenvalue problem (2.1) - (2.4) are essential to prove Theorem 1.1.

We state some notations and auxiliary facts related to eigenvalue problem (2.1)-(2.4) in Appendix 1. In particular, Appendix 1 proves that if Assumption 1.1 holds, then all the eigenvalues of problem (2.1) - (2.4) are simple, even if the potentials $q_j(x_j)$ with $j = 1, ..., m+1$ are complex-valued continuous functions. If the potentials $q_j(x_j)$ are real continuous functions, then all the eigenvalues are real, and the corresponding system of eigenfunctions forms an orthogonal basis in the space $L_2(\Gamma)$.

We denote by λ_k, where $k \in \mathbb{N}$, the sequence of eigenvalues of problem (2.1) - (2.4). Hence, $\lambda_k \neq \lambda_s$ if $k \neq s$ by Lemma 3.1 from Appendix 1. The corresponding system of eigenfunctions can be written as follows

$$\vec{\Phi}(\vec{x}, \lambda_k) = \{(\varphi_1(x_1, \lambda_k), ..., \varphi_{m+1}(x_{m+1}, \lambda_k))^T,\ k \geq 1\}.$$

Now we expand the initial function

$$\vec{A}(\vec{x}) = (a_1(x_1), ..., a_{m+1}(x_{m+1}))^T$$

in terms of the system of eigenfunctions of problem (2.1) - (2.4) to the following series

$$a_j(x_j) = \sum_{k=1}^{\infty} c_{jk}\ \ \varphi_j(x_j, \lambda_k), \tag{2.5}$$

where

$$c_{jk} = \frac{\int\limits_0^{b_j} a_j(x_j)\overline{\varphi_j^+(x_j, \overline{\lambda_k})}dx_j}{\int\limits_0^{b_j} \varphi_j(x_j, \lambda_k)\overline{\varphi_j^+(x_j, \overline{\lambda_k})}dx_j}.$$

Since the smooth function $\vec{A}(\vec{x})$ satisfies all conditions (2.2), (2.3) and (2.4), series (2.5) converges absolutely and uniformly on the graph Γ.

We seek the solution of problem (1.1) - (1.5) in the following form

$$\theta_j(x_j) = \sum_{k=1}^{\infty} d_{jk}(t)\,\varphi_j(x_j,\lambda_k). \tag{2.6}$$

In a standard way, we find that

$$d_{jk}(t) = c_{jk}\cos\sqrt{\lambda_k}t.$$

In the next lemma we give the representation of the solution of problem (1.1) - (1.5) in terms of the special extension of the eigenfunctions.

Lemma 2.1. *The following formula*

$$\cos\sqrt{\lambda_k}t\,\varphi_j(x_j,\lambda_k) = \tilde{\varphi}_j(x_j+t,\lambda_k) + \tilde{\varphi}_j(x_j-t,\lambda_k) + \int\limits_{x_j-t}^{x_j+t} w_j(x_j,t,s)\tilde{\varphi}_j(s,\lambda_k)ds$$

holds for $t>0$ and $x_j \in e_j$, where $\tilde{\varphi}_j(x_j,\lambda_k)$ is the special extension of the function $\varphi_j(x_j,\lambda_k)$ from the interval e_j to the entire real axis.

Proof. In fact, the potential $q_j(x_j)$ is defined only on the edge e_j. We extend the potential $q_j(x_j)$ to the entire real axis, preserving its class but otherwise arbitrarily. The function $\varphi_j(x_j,\lambda_k)$ $(j = 1,...,m+1)$ is the solution of the homogeneous equation

$$-\varphi_j''(x_j) + q_j(x_j)\varphi_j(x_j) = \lambda_k\varphi_j(x_j),\ \ x_j \in e_j\ (j = 1,...,m+1)$$

with Cauchy conditions at $x_j = b_j$

$$\varphi_j(b_j,\lambda_k) = 0,\ \ \varphi_j'(b_j,\lambda_k) = 1. \tag{2.7}$$

Let $\vartheta_j(x,t) = 2\cos\sqrt{\lambda_k}t\varphi_j(x,\lambda_k)$ for $x \in e_j$, $t>0$. We note that $\vartheta_j(x,t)$ is the solution of the mixed problem

$$\frac{\partial^2\vartheta_j(x,t)}{\partial t^2} = \frac{\partial^2\vartheta_j(x,t)}{\partial x^2} - q_j(x_j)\vartheta_j(x,t), \tag{2.8}$$

$$\vartheta_j(x,0) = 2\varphi_j(x,\lambda_k),\ \ \frac{\partial\vartheta_j(x,0)}{\partial t} = 0, \tag{2.9}$$

$$\vartheta_j(0,t) = 2\cos\sqrt{\lambda_k}t\varphi_j(0,\lambda_k),\ \ \vartheta_j(b_j,t) = 0. \tag{2.10}$$

Invoking the results of the monograph [2], we write the solution to the mixed problem (2.8), (2.9), (2.10) as follows

$$\vartheta_j(x,t) = \tilde{\varphi}_j(x+t,\lambda_k) + \tilde{\varphi}_j(x-t,\lambda_k) + \int\limits_{x_j-t}^{x_j+t} w_j(x,t,s)\tilde{\varphi}_j(s,\lambda_k)ds. \tag{2.11}$$

Here, $w_j(x,t,s)$ can be uniquely constructed from the function $q_j(x)$, $x \in \mathbb{R}$, as the solution of the Goursat problem

$$\frac{\partial^2 w_j}{\partial t^2} = \frac{\partial^2 w_j}{\partial s^2} - q_j(s)w_j,$$

$$w_j(x,t,x+t) = -\frac{1}{2}\int\limits_0^t q_j(x+\tau)d\tau,$$

$$w_j(x,t,x-t) = -\frac{1}{2}\int_0^t q_j(x-\tau)d\tau.$$

Let us clarify how the continuation of the function $\tilde{\varphi}_j(x,\lambda_k)$ initially defined as $\varphi_j(x,\lambda_k)$ on $x \in e_j = (0,b_j)$ is extended to the entire real axis.

Substitution of representation (2.11) into the first of conditions (2.10) leads us to the following equality

$$2\cos\sqrt{\lambda_k}t\varphi_j(0,\lambda_k) = \varphi_j(t,\lambda_k) + \int_0^t w_j(0,t,s)\varphi_j(s,\lambda_k)ds$$
$$+\tilde{\varphi}_j(-t,\lambda_k) + \int_{-t}^0 w_j(0,t,s)\tilde{\varphi}_j(s,\lambda_k)ds.$$

For $0 \le t \le b_j$, this equality implies that

$$\tilde{\varphi}_j(-t,\lambda_k) + \int_{-t}^0 w_j(0,t,s)\tilde{\varphi}_j(s,\lambda_k)ds = F_j(t), \tag{2.12}$$

where

$$F_j(t) = 2\cos\sqrt{\lambda_k}t\varphi_j(0,\lambda_k) - \varphi_j(t,\lambda_k) - \int_0^t w_j(0,t,s)\varphi_j(s,\lambda_k)ds.$$

In [2] it is shown that such an extension of the function $\varphi_j(x_j,\lambda_k)$ belongs to the space $\mathbb{C}^2[-b_j,b_j]$.

Applying the second of boundary condition (2.9), we extend the function $\varphi_j(x_j,\lambda_k)$ to the interval $(b_j,2b_j)$. Therefore, the substitution of (2.10) into the second of conditions (2.9) gives the following integral equation

$$\tilde{\varphi}_j(b_j+t,\lambda_k) + \int_0^{b_j+t} w_j(b_j,t,s)\tilde{\varphi}_j(s,\lambda_k)ds = R_j(t) \tag{2.13}$$

for $0 \le t \le b_j$, where

$$R_j(t) = -\varphi_j(b_j-t,\lambda_k) + \int_{b_j-t}^0 w_j(b_j,t,s)\varphi_j(s,\lambda_k)ds\,(j=1,...,m+1).$$

The extension of $\varphi_j(x_j,\lambda_k)$ from the interval $(0,b_j)$ to the interval $(b_j,2b_j)$ belongs to $\mathbb{C}^2(0,2b_j)$. Integral equations (2.11) and (2.12) allow the function $\varphi_j(x_j,\lambda_k)$ to be extended to the entire real axis. □

Representation (2.6), relation (2.5) and Lemma 2.1 allow us to express the desired solution as follows

$$\theta_j(x_j,t) = \tilde{a}_j(x_j+t) + \tilde{a}_j(x_j-t) + \int_{x_j-t}^{x_j+t} w_j(x_j,t,s)\tilde{a}_j(s)ds$$

for $j=1,...,m+1$, where $\tilde{a}_j(\xi_j)$ is the extension of $a_j(x_j)$ from the interval $(0,b_j)$ to the entire real axis, realized by the formula

$$\tilde{a}_j(\xi_j) = \sum_{k=1}^{\infty} c_{jk}\tilde{\varphi}(\xi_k,\lambda_k), \quad \xi_j \in (-\infty,\infty).$$

3 Appendix 1. Spectral properties of problem (2.1) - (2.4)

In this appendix, we state the spectral properties of problem (2.1) - (2.4) that are necessary to prove Theorem 1.1. In the absence of potentials $q_1(x_1) \equiv 0, ..., q_{m+1}(x_{m+1}) \equiv 0$, the spectral properties can be found in the work of N.P. Bondarenko [4]. In the case of real potentials $q_1, ..., q_{m+1}$, problem (2.1) - (2.4) is self-adjoint in the function space $L_2(\Gamma)$. Therefore, its spectrum consists of real eigenvalues and the corresponding system of eigenfunctions forms an orthogonal basis in the space $L_2(\Gamma)$. We establish conditions under which problem (2.1) - (2.4) has only simple eigenvalues. In our considerations, the potentials $q_1(\cdot) \equiv 0, ..., q_{m+1}(\cdot) \equiv 0$ may represent complex-valued continuous functions.

We denote by $\varphi_j(x_j, \lambda)$ the solution of the homogeneous equation $l_j(\varphi_j) = \lambda\varphi_j$ with $x_j \in e_j = (0, b_j)$ that satisfy the Cauchy conditions

$$\varphi_j(b_j, \lambda) = 0,\ \varphi_j'(b_j, \lambda) = 1$$

at $x_j = b_j$ for $j = 1, ..., m$. Assumption 1.1 implies that the values $\varphi_1(0, \lambda), ..., \varphi_m(0, \lambda)$ do not vanish for all complex values of the spectral parameter λ.

We also introduce the system of functions $\psi_1(x_j, \lambda) = B_j\varphi_1(x_j, \lambda)$, $x_j \in e_j$. The numbers $B_1, ..., B_m$ are chosen so that condition (2.3) is satisfied

$$B_j = B_{m+1} \prod_{\substack{i=1 \\ i \neq j}}^{m} \varphi_j(0, \lambda),$$

where B_{m+1} is a common constant for all indices $j = 1, 2, ..., m$. The value of B_{m+1} may depend on the spectral parameter λ, but does not depend on $x_{m+1} \in e_{m+1}$.
Now we denote by $\varphi_{m+1}(x_{m+1}, \lambda)$ the solution of the homogeneous equation $l_{m+1}(\varphi_{m+1}) = \lambda\varphi_{m+1}(x_{m+1}, \lambda)$ for $x_{m+1} \in (0, b_{m+1})$ which at the point $x_{m+1} = b_{m+1}$ satisfies the following conditions

$$\varphi_{m+1}(b_{m+1}, \lambda) = \prod_{i=1}^{m} \varphi_j(0, \lambda),$$

$$\varphi_{m+1}'(b_{m+1}, \lambda) = \sum_{i=1}^{m} \prod_{\substack{s=1 \\ s \neq i}}^{m} \varphi_s(0, \lambda)\varphi_i'(0, x).$$

Therefore, we can write the following relation

$$\varphi_{m+1}(x_{m+1}, \lambda) = \varphi_{m+1}(b_{m+1}, \lambda)c_{m+1}(x_{m+1}, \lambda) + \varphi_{m+1}'(b_{m+1}, \lambda)s_{m+1}(x_{m+1}, \lambda)$$

for all $x_{m+1} \in (0, b_{m+1})$. Here, $c_{m+1}(x_{m+1}, \lambda)$ and $s_{m+1}(x_{m+1}, \lambda)$ form a fundamental system of solutions of the homogeneous equation $l_{m+1}(y_{m+1}) = \lambda y_{m+1}(x_{m+1}, \lambda)$, $x_{m+1} \in e_{m+1}$ with the Cauchy conditions

$$s_{m+1}(b_{m+1}, \lambda) = {c'}_{m+1}(b_{m+1}, \lambda) = 0$$

and

$${s'}_{m+1}(b_{m+1}, \lambda) = c_{m+1}(b_{m+1}, \lambda) = 1$$

at $x_{m+1} = b_{m+1}$.

We note that the function $\varphi_{m+1}(x_{m+1}, \lambda)$ is an entire function of the spectral parameter λ for all fixed $x_{m+1} \in [0, b_{m+1}]$. The characteristic determinant of the eigenvalue problem (2.1) - (2.4) has the following form

$$\Delta(\lambda) \equiv \varphi_{m+1}(0, \lambda),\ \ \lambda \in \mathbb{C}.$$

Lemma 3.1. *The characteristic determinant $\Delta(\lambda)$ has only simple zeros.*

Proof. Let $\lambda, \mu \in \mathbb{C}$. We denote by $l_j^+(y) = -y''(x_j) + \overline{q}_j(x_j)y_j(x_j)$, $x_j \in e_j$ the formally adjoint differential expression to $l_j(\cdot)$.

We consider the following difference

$$R = \sum_{j=1}^{m+1} \Big(\int_0^{b_j} l_j(\varphi_j(x_j,\lambda))\overline{\varphi_j^+(x_j,\overline{\mu})}dx_j - \int_0^{b_j} \varphi_j(x_j,\lambda)\overline{l_j^+(\varphi_j^+(x_j,\overline{\mu}))}dx_j \Big),$$

where the functions $\varphi_j^+(x_j,\overline{\mu})$, $j = 1,...,m+1$, are the solutions of the homogeneous equations $l_j^+(\varphi_j^+(x_j,\overline{\mu})) = \overline{\mu}\varphi_j^+(x_j,\overline{\mu})$, and their construction is analogous to the construction of $\varphi_j(x,\lambda)$, $j = 1,...,m+1$. The characteristic determinant of the adjoint problem is $\Delta^+(\overline{\mu}) = \varphi_{m+1}^+(0,\overline{\mu})$. Applying the Lagrange formula [5] and Cauchy conditions (2.7) to the difference R, we obtain

$$R = \sum_{j=1}^{m+1} \Big(\frac{d}{dx_j}\varphi_j(x_j,\lambda)\overline{\varphi_j^+(x_j,\overline{\mu})} - \varphi_j(x_j,\lambda)\overline{\frac{d\varphi_j^+(x_j,\overline{\mu})}{dx_j}} \Big)\Big|_{x_j=0}^{x_j=b_j-0}$$

$$= \frac{d\varphi_{m+1}(b_{m+1},\lambda)}{dx_{m+1}}\overline{\varphi_{m+1}^+(b_{m+1},\overline{\mu})} - \varphi_{m+1}(b_{m+1},\lambda)\overline{\frac{d\varphi_{m+1}^+(b_{m+1},\overline{\mu})}{dx_{m+1}}}$$

$$-\sum_{j=1}^{m} \frac{d\varphi_j(0,\lambda)}{dx_j}\overline{\varphi_{m+1}^+(b_{m+1},\overline{\mu})} + \varphi_{m+1}(b_{m+1},\lambda)\sum_{j=1}^{m}\overline{\frac{d\varphi_j^+(0,\overline{\mu})}{dx_j}}$$

$$-\frac{d\varphi_{m+1}(0,\lambda)}{dx_{m+1}}\overline{\varphi_{m+1}^+(0,\overline{\mu})} + \varphi_{m+1}(0,\lambda)\overline{\frac{d\varphi_{m+1}^+(0,\overline{\mu})}{dx_{m+1}}}$$

$$= \Delta(\lambda)\overline{\frac{d\varphi_{m+1}^+(0,\overline{\mu})}{dx_{m+1}}} - \frac{d\varphi_{m+1}(0,\lambda)}{dx_{m+1}}\overline{\Delta^+(\overline{\mu})}.$$

On the other hand, we have the following equality

$$R = (\lambda - \mu)\sum_{j=1}^{m+1}\int_0^{b_j} \varphi(x_j,\lambda)\overline{\varphi_j^+(x_j,\overline{\mu})}dx_j.$$

As a result, we obtain the identity

$$\sum_{j=1}^{m+1}\int_0^{b_j} \varphi(x_j,\lambda)\overline{\varphi_j^+(x_j,\overline{\mu})}dx_j =$$

$$= \frac{1}{\lambda-\mu}\Big(\Delta(\lambda)\overline{\frac{d\varphi_{m+1}^+(0,\overline{\mu})}{dx_{m+1}}} - \frac{d\varphi_{m+1}(0,\lambda)}{dx_{m+1}}\overline{\Delta^+(\overline{\mu})}\Big).$$

Let $\lambda = \lambda_0$ be an arbitrary eigenvalue of the problem (2.1) - (2.4). Then $\Delta(\lambda_0) = 0$. Thus, the following equality

$$\sum_{j=1}^{m+1}\int_0^{b_j} \varphi(x_j,\lambda)\overline{\varphi_j^+(x_j,\overline{\mu})}dx_j = -\frac{1}{\lambda_0-\mu}\overline{\Delta^+(\overline{\mu})}\frac{d\varphi_{m+1}(0,\lambda_0)}{dx_{m+1}}$$

holds. Now, let $\mu \to \lambda_0$. This equality implies the limiting relation

$$\sum_{j=1}^{m+1} \int_0^{b_j} \varphi(x_j, \lambda)\overline{\varphi_j^+(x_j, \overline{\mu})}dx_j = \frac{\overline{d\Delta^+(\overline{\lambda_0})}}{d\mu} \cdot \frac{d\varphi_{m+1}(0, \lambda_0)}{dx_{m+1}}.$$

The results of monograph [6] imply that $\overline{\lambda_0}$ is an eigenvalue of the adjoint problem, and $\varphi_j^+(x_j, \overline{\lambda_0})$ is the eigenfunction corresponding to the eigenvalue $\overline{\lambda_0}$. Moreover, $\sum\limits_{j=1}^{m+1} \int\limits_0^{b_j} \varphi(x_j, \lambda_0)\overline{\varphi_j^+(x_j, \overline{\mu})}dx_j \neq 0$. Hence, we get $\frac{\overline{d\Delta^+(\overline{\lambda_0})}}{d\mu} \cdot \frac{d\varphi_{m+1}(0,\lambda_0)}{dx_{m+1}} \neq 0$. We observe that $\frac{d\varphi_{m+1}(0,\lambda_0)}{dx_{m+1}} \neq 0$, since $\varphi_{m+1}(b_{m+1}, \lambda_0) = 0$. Otherwise, this would contradict Assumption 1.1. Therefore, it follows that

$$\frac{\overline{d\Delta^+(\overline{\lambda}_0)}}{d\mu} \neq 0.$$

Hence, the eigenvalue $\overline{\lambda}_0$ of the adjoint problem is simple. Consequently, the eigenvalue of original problem (2.1) - (2.4) must also be simple. □

Corollary 3.1. *If λ_0 is an eigenvalue of problem* (2.1) - (2.4)*, then the corresponding eigenfunction has the follwing form*

$$\vec{\Phi}(\lambda_0) = (\varphi_1(x_1, \lambda_0), \varphi_2(x_2, \lambda_0), ..., \varphi_{m+1}(x_{m+1}, \lambda_0))$$

and none of the components of $\vec{\Phi}(\lambda_0)$ can be identically zero.

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