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INFINITELY MANY PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS INVOLVING PIECEWISE ALTERNATELY ADVANCED AND RETARDED ARGUMENT

K.-S. Chiu, I. Berna Sepúlveda

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AMS Mathematics Subject Classification: 34A38, 34K13.

Abstract. The manuscript introduces an innovative framework for establishing the existence of infinitely many nontrivial periodic solutions within a class of differential equations characterized by a piecewise alternately advanced and retarded argument. It comprehensively delineates the essential criteria required for the existence of these solutions and provides detailed procedures for their determination. Additionally, the study incorporates illustrative examples, including cases with infinitely many solutions, to demonstrate the effectiveness and applicability of the proposed approach.

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1 Introduction

A differential equation with a piecewise constant argument which is alternately advanced and retarded (DEPCA) is expressed as:

$$x'(t) = f\left(t, x(t), x\left(p\left[\frac{t+l}{p}\right]\right)\right), \tag{1.1}$$

where $[\cdot]$ denotes the greatest integer function, f is a continuous function defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, and $p\left[\frac{t+l}{p}\right]$ is a piecewise constant function defined by

$$p\left[\tfrac{t+l}{p}\right] = kp \quad \text{for} \quad t \in [kp-l, (k+1)p-l), \quad k \in \mathbb{Z},$$

with p and l being positive constants satisfying $p > l$.

The deviation argument of DEPCA (1.1), defined as

$$\wp(t) := t - p\left[\tfrac{t+l}{p}\right]$$

is negative within the interval $[kp-l, kp)$ and positive in $[kp, (k+1)p-l)$. This alternating sign behavior classifies DEPCA (1.1) as a differential equation of alternately advanced and retarded type.

DEPCAs represent a hybrid class of equations that combine characteristics of both discrete dynamics and continuous systems. These equations are particularly relevant in modeling applications in biomedical sciences and physical processes, as discussed in [6, 24] and further elaborated in [16, 18, 19, 20, 27]. Moreover, extensive research has been conducted to investigate various properties of DEPCAs. See, for example, [2, 3, 4, 5].

Recent studies [1, 12, 14, 15, 17, 22, 26, 28, 29, 30, 31] have analyzed specific formulations of DEPCAs.

Additionally, the works in [7, 8, 9, 10, 11] simplified the problem of n-periodic solution solvability into a system of n linear equations. Utilizing foundational principles of linear algebra, these studies systematically identified the conditions required for the existence of n-periodic solutions and provided explicit methodologies for deriving these solutions.

In 2024, M.I. Muminov and T.A. Radjabov [11] investigated the conditions under which 2-periodic solutions exist for first-order differential equations with piecewise constant delay:

$$T'(t) = a(t)T(t) + b(t)T([t]) + f(t).$$

They introduced a systematic method to identify 2-periodic solutions, clearly defined the requisite existence criteria, and provided explicit solution formulas.

Subsequently, K.-S. Chiu and F. Cordova [23] explored the conditions for the existence of 4-periodic solutions for first-order DEPCA (1.1) with the parameters $p = 2$ and $l = 1$.

To the best of our knowledge, only the two studies [11] and [23] have investigated the existence of infinitely many periodic solutions for DEPCA. However, neither has formulated detailed criteria for identifying such solutions in differential equations with a general piecewise alternately advanced and retarded argument.

This paper investigates a non-homogeneous differential equation with piecewise alternately advanced and retarded argument, given by:

$$y'(t) = a(t)y(t) + b(t)y\left(p\left[\frac{t+l}{p}\right]\right) + g(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where the functions $a(t)$, $b(t)$, and $g(t)$ are continuous and non-zero on $\mathbb{R}$. The general framework of this problem was previously analyzed in [13, 21], where the authors derived conditions ensuring the existence of a solution and demonstrated a Gronwall-type integral inequality as a practical application.

In this paper, we focus on establishing the conditions necessary for the existence of $2p$-periodic solutions to the initial value problem. An illustrative example is provided to demonstrate a case where an infinite number of $2p$-periodic solutions exist, thereby offering new perspectives that complement earlier studies on uniqueness for homogeneous cases.

2 Alternately advanced and retarded differential equation

A solution of DEPCA (1.2) is defined as follows. A function y is considered to be a solution of DEPCA (1.2) on $\mathbb{R}$ if it satisfies the following criteria:

(i) y is continuous on $\mathbb{R}$,

(ii) the derivative $y'(t)$ exists at each $t \in \mathbb{R}$, except possibly at points $t = kp - l$ for $k \in \mathbb{Z}$, where one-sided derivatives are required to exist,

(iii) y satisfies DEPCA (1.2) within each interval $\big(kp - l, (k+1)p - l\big)$ for $k \in \mathbb{Z}$, and the equation is satisfied for the right-hand derivative at $t = pk - l$ for $k \in \mathbb{Z}$.

To determine a solution of DEPCA (1.2), we follow the integration methodology described in [25]. By integrating equation (1.2), the solution can be expressed as:

$$\begin{aligned} y(t) = e^{\int_p^t a(s)ds} y(p) &+ \int_p^t b(s)y\left(p\left[\frac{s+l}{p}\right]\right) e^{\int_s^t a(r)dr} ds \\ &+ \int_p^t g(s)e^{\int_s^t a(r)dr} ds, \quad t \in [p-l, 2p-l). \end{aligned}$$

We define

$$\lambda(t,s) := e^{\int_s^t a(r)dr} + \int_s^t e^{\int_u^t a(r)dr} b(u)du,$$

$$\Psi(t,s) = \frac{\lambda\left(t, p\left[\frac{t+l}{p}\right]\right)}{\lambda\left(s, p\left[\frac{s+l}{p}\right]\right)} \prod_{j=\left[\frac{s+l}{p}\right]+1}^{\left[\frac{t+l}{p}\right]} \frac{\lambda(pj-l, p(j-1))}{\lambda(pj-l, pj)}, \quad t \geq s,$$

$$G(t,s) = \int_s^t e^{\int_u^t a(\kappa)d\kappa} g(u)du,$$

where $t, s \in [p-l, \infty)$, $\lambda(pj-l, pj) \neq 0$, $j \in \mathbb{N}$.

The following theorem provides a representation formula for the solution to DEPCA (1.2) for $t > 0$. The proof is similar to proofs of Theorem 2.1 in [13] and Theorem 2.2 in [21].

Theorem 2.1. *If $\lambda(pj-l, pj) \neq 0$ for $j \in \mathbb{N}$, then $y(t)$ represents the unique solution to DEPCA (1.2) for $t \geq p-l$ if and only if $y(t)$ can be represented as*

$$\begin{aligned} y(t) =& \Psi(t,\tau) y(\tau) + \int_\tau^{p[(\tau+l)/p]} \Psi(t,\tau) G(\tau,s) ds \\ &+ \sum_{k=[(\tau+l)/p]}^{[(t+l)/p]-1} \int_{kp}^{(k+1)p} \Psi\Big(t, (k+1)p - l\Big) G\Big((k+1)p - l, s\Big) ds \\ &+ G\Big(t, p\left[(t+l)/p\right]\Big). \end{aligned} \tag{2.1}$$

Proof. First, we demonstrate that the function $y(t)$ defined in (2.1) satisfies DEPCA (1.2). This can be readily verified using the relation $\frac{d\lambda(t,s)}{dt} = a(t)\lambda(t,s) + b(t)$, where s is fixed. Using the notation introduced earlier, we proceed as follows:

$$\begin{aligned} \frac{d\Psi(t,s)}{dt} &= \frac{\lambda'\left(t, p\left[\frac{t+l}{p}\right]\right)}{\left(s, p\left[\frac{s+l}{p}\right]\right)} \prod_{j=\left[\frac{s+l}{p}\right]+1}^{\left[\frac{t+l}{p}\right]} \frac{\lambda(pj-l, p(j-1))}{\lambda(pj-l, pj)} \\ &= \frac{a(t)\lambda\left(t, p\left[\frac{t+l}{p}\right]\right) + b(t)}{\lambda\left(s, p\left[\frac{s+l}{p}\right]\right)} \prod_{j=\left[\frac{s+l}{p}\right]+1}^{\left[\frac{t+l}{p}\right]} \frac{\lambda(pj-l, p(j-1))}{\lambda(pj-l, pj)} \\ &= a(t)\Psi(t,s) + b(t) \left(\frac{\lambda\left(p\left[\frac{t+l}{p}\right], p\left[\frac{t+l}{p}\right]\right)}{\lambda\left(s, p\left[\frac{s+l}{p}\right]\right)} \right) \prod_{j=\left[\frac{s+l}{p}\right]+1}^{\left[\frac{t+l}{p}\right]} \frac{\lambda(pj-l, p(j-1))}{\lambda(pj-l, pj)} \\ &= a(t)\Psi(t,s) + b(t)\Psi\left(p\left[\frac{t+l}{p}\right], s\right), \quad s < t. \end{aligned}$$

Conversely, suppose that $y_i(t)$ is a solution to DEPCA (1.2) on the interval $ip-l \leq t < (i+1)p-l$. Then, it satisfies

$$y_i'(t) = a(t)y_i(t) + b(t)y_i(ip) + g(t).$$

By defining $G(t,u) = \int_u^t e^{\int_s^t a(\kappa)d\kappa} g(s)ds$, the solution of this equation on $I_i = [ip-l, (i+1)p-l)$ is expressed as:

$$\begin{aligned} y_i(t) &= \left(e^{\int_{ip}^t a(s)ds} + \int_{ip}^t e^{\int_s^t a(\kappa)d\kappa} b(s)ds\right) y_i(ip) + \int_{ip}^t e^{\int_s^t a(\kappa)d\kappa} g(s)ds \\ &= \lambda(t, ip)\, y_i(ip) + G(t, ip). \end{aligned} \tag{2.2}$$

From (2.2), substituting $t = ip - l$ and taking the limit as $t \to (i+1)p - l^-$, we obtain

$$y_i(ip) = \left(\frac{y_i(ip-l) - G(ip-l, ip)}{\lambda\,(ip-l, ip)}\right). \tag{2.3}$$

Thus, based on (2.3), we deduce:

$$y_i((i+1)p-l) = \left(\frac{\lambda((i+1)p-l, ip)}{\lambda(ip-l, ip)}\right)\Big(y_i(ip-l) - G(ip-l, ip)\Big) + G((i+1)p-l, ip).$$

Similarly,

$$y_{i-1}(ip-l) = \left(\frac{\lambda\,(ip-l,(i-1)p)}{\lambda\,(ip-l,(i-1)p)}\right)\Big(y_{i-1}((i-1)p-l) - G((i-1)p-l,(i-1)p)\Big) + G(ip-l,(i-1)p), \qquad i \geq \left[\frac{\tau+l}{p}\right] + 2,$$

and as $t \to p\left(\left[\frac{\tau+l}{p}\right]+1\right) - l^-$, we have

$$y\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right) = \left(\frac{\lambda\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l, p\left[\frac{\tau+l}{p}\right]\right)}{\lambda\left(\tau, p\left[\frac{\tau+l}{p}\right]\right)}\right)\left(y(\tau) - G\left(\tau, p\left[\frac{\tau+l}{p}\right]\right)\right) + G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l, p\left[\frac{\tau+l}{p}\right]\right).$$

Applying the two previous relations, we obtain

$$\begin{aligned}
y_i((i+1)p-l) &= \left(\frac{\lambda\,((i+1)p-l, ip)}{\lambda\left(\tau, p\left[\frac{\tau+l}{p}\right]\right)}\right)\left(\prod_{j=\left[\frac{\tau+l}{p}\right]+1}^{i} \frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l, jp)}\right) y(\tau) \\
&+ \sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i} \Bigg\{ \left(\frac{\lambda\,((i+1)p-l, ip)}{\lambda\,((k+1)p-l,(k+1)p)}\right)\left(\prod_{j=k+2}^{i} \frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l, jp)}\right) \times \\
&\qquad \left(\frac{\lambda\,((k+1)p-l, kp)}{\lambda\,(kp-l, kp)}\,(-G(kp-l, kp)) + G((k+1)p-l, kp)\right)\Bigg\} \\
&+ \left(\frac{\lambda\,((i+1)p-l, ip)}{\lambda\,((k+1)p-l,(k+1)p)}\right)\left(\prod_{j=\left[\frac{\tau+l}{p}\right]+2}^{i} \frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l, jp)}\right) \times \\
&\left(\frac{\lambda\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l, p\left[\frac{\tau+l}{p}\right]\right)}{\lambda\left(\tau, p\left[\frac{\tau+l}{p}\right]\right)}\left(-G\left(\tau, p\left[\frac{\tau+l}{p}\right]\right)\right) + G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l, p\left[\frac{\tau+l}{p}\right]\right)\right).
\end{aligned}$$

Using the notation Ψ, we have for all $i \geq \left[\frac{\tau+l}{p}\right]+1$:

$$y_i((i+1)p-l) = \Psi\left((i+1)p-l,\tau\right)y(\tau) + \sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i}\Big[\Psi\left((i+1)p-l,kp-l\right)\cdot(-G(kp-l,kp)) + \Psi\left((i+1)p-l,(k+1)p-l\right)\cdot G((k+1)p-l,kp)\Big] + \Psi\left((i+1)p-l,\tau\right)\cdot\left(-G\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right) + \Psi\left((i+1)p-l,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)\cdot G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right).$$

In particular,

$$y_i(ip-l) = \Psi\left(ip-l,\tau\right)y(\tau) + \sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i-1}\Big[\Psi\left(ip-l,kp-l\right)(-G(kp-l,kp)) + \Psi\left(ip-l,(k+1)p-l\right)\cdot G((k+1)p-l,kp)\Big] + \Psi\left(ip-l,\tau\right)\cdot\left(-\Psi\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right) + \Psi\left(ip-l,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right). \tag{2.4}$$

From equations (2.2), (2.3), and (2.4), it follows that:

$$y_i(t) = \lambda(t,ip)\,y_i(ip)+G(t,ip) = \frac{\lambda(t,ip)}{\lambda(ip-l,ip)}\cdot\Psi\left(ip-l,\tau\right)y(\tau) + \frac{\lambda(t,ip)}{\lambda(ip-l,ip)}\cdot\Big[\Psi\left(ip-l,\tau\right)\cdot\left(-G\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right) + \Psi\left(ip-l,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)\cdot G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right)\Big] + \frac{\lambda(t,ip)}{\lambda(ip-l,ip)}\cdot\sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i-1}\Big[\Psi\left(ip-l,kp-l\right)\cdot(-G(kp-l,kp)) + \Psi\left(ip-l,(k+1)p-l\right)\cdot G((k+1)p-l,kp)\Big] - \frac{\lambda(t,ip)}{\lambda(ip-l,ip)}G(ip-l,ip)+G(t,ip)$$

$$= \Psi\left(t,\tau\right)y(\tau) + \Psi\left(t,\tau\right)\left(-G\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right) + \Psi(t,(i(\tau)+1)p-l)G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right) + \sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i-1}\Big[\Psi\left(t,kp-l\right)(-G(kp-l,kp)) + \Psi\left(t,(k+1)p-l\right)G((k+1)p-l,kp)\Big]$$

$$
\begin{aligned}
&-\frac{\lambda(t,ip)}{\lambda(ip-l,ip)}G(ip-l,ip)+G(t,ip)\\
&=\Psi(t,\tau)\,y(\tau)+G(t,ip)+\Psi(t,\tau)\left(-G\left(\tau,p\left[\frac{\tau+l}{p}\right]\right)\right)\\
&+\Psi\left(t,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)G\left(p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l,p\left[\frac{\tau+l}{p}\right]\right)\\
&+\sum_{k=\left[\frac{\tau+l}{p}\right]+1}^{i-1}\Big[\Psi(t,kp-l)\left(-G(kp-l,kp)\right)+\Psi(t,(k+1)p-l)\,G((k+1)p-l,kp)\Big]\\
&-\frac{\lambda(t,ip)}{\lambda(ip-l,ip)}G(ip-l,ip)\\
&=\Psi(t,\tau)\,y(\tau)+\int_{\tau}^{i(\tau)p}\Psi(t,\tau)\,e^{\int_s^{\tau}a(\kappa)d\kappa}g(s)ds\\
&+\int_{p\left[\frac{\tau+l}{p}\right]}^{p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l}\Psi\left(t,p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l\right)e^{\int_s^{p\left(\left[\frac{\tau+l}{p}\right]+1\right)-l}a(\kappa)d\kappa}g(s)ds\\
&+\sum\nolimits_{k=i(\tau)+1}^{k=i}\left(\int_{kp-l}^{kp}\Psi(t,kp-l)\,e^{\int_s^{kp-l}a(\kappa)d\kappa}g(s)ds\right)\\
&+\sum\nolimits_{k=i(\tau)+1}^{k=i-1}\left(\int_{kp}^{(k+1)p-l}\Psi(t,(k+1)p-l)\,e^{\int_s^{(k+1)p-l}a(\kappa)d\kappa}g(s)ds\right)\\
&+\int_{ip}^{t}e^{\int_s^{t}a(\kappa)d\kappa}g(s)ds\\
&=\Psi(t,\tau)\,y(\tau)+\int_{\tau}^{p\left[\frac{\tau+l}{p}\right]}\Psi(t,\tau)\,e^{\int_s^{\tau}a(\kappa)d\kappa}g(s)ds\\
&+\sum\nolimits_{k=\left[\frac{\tau+l}{p}\right]}^{k=\left[\frac{t+l}{p}\right]-1}\left(\int_{kp}^{(k+1)\cdot p}\Psi(t,(k+1)\cdot p-l)\,e^{\int_s^{(k+1)\cdot p-l}a(\kappa)d\kappa}g(s)ds\right)\\
&+\int_{ip}^{t}e^{\int_s^{t}a(\kappa)d\kappa}g(s)ds.
\end{aligned}
$$

From this, equality (2.1) follows.

If $g(t)=0$ in (2.1), we obtain the solution to linear DEPCA (1.2), given by $y(t)=\Psi(t,\tau)\,y(\tau)$. □

3 Existence of infinitely many periodic solutions

In this section, we propose an approach to identify $2p$-periodic solutions to DEPCA (1.2), assuming that the functions $a(\cdot)$, $b(\cdot)$, and $g(\cdot)$ are continuous on the interval $[p-l,\infty)$ and exhibit $2p$-periodic characteristics.

Assuming that $y(t)$ satisfies DEPCA (1.2) within the interval $kp-l\le t<(k+1)p-l$, integrating DEPCA (1.2) yields the solution of the following form:

$$
\begin{aligned}
y(t)&=e^{\int_{kp}^{t}a(s)ds}y(kp)+\int_{kp}^{t}b(s)y\left(p\left[\frac{s+l}{p}\right]\right)e^{\int_s^t a(r)dr}ds+\int_{kp}^{t}g(s)e^{\int_s^t a(r)dr}ds\\
&=\left(e^{\int_{kp}^{t}a(s)ds}+\int_{kp}^{t}b(s)e^{\int_s^t a(r)dr}ds\right)y(kp)+\int_{kp}^{t}g(s)e^{\int_s^t a(r)dr}ds.
\end{aligned}
$$

Using the notations $\lambda(t,s)$, and $G(t,s)$, the solution $y(t)$ is expressed as:

$$y(t) = \lambda(t,kp)y(kp) + G(t,kp), \quad kp - l \le t < (k+1)p - l.$$

From the above, by setting $t = kp - l$, we derive:

$$y(kp) = \frac{y(kp-l) - G(kp-l,kp)}{\lambda(kp-l,kp)}.$$

Substituting this result back, we obtain:

$$y(t) = \frac{\lambda(t,kp)}{\lambda(kp-l,kp)}\Big(y(kp-l) - G(kp-l,kp)\Big) + G(t,kp).$$

Assuming that $y(t)$ is $2p$-periodic on the interval $[p-l,\infty)$, the function $y(t)$ on $[p-l,3p-l)$ can be represented as:

$$y(t) = \begin{cases} \frac{\lambda(t,p)}{\lambda(p-l,p)}\Big(y(p-l) - G(p-l,p)\Big) + G(t,p), & t \in [p-l,2p-l), \\ \frac{\lambda(t,2p)}{\lambda(2p-l,2p)}\Big(y(2p-l) - G(2p-l,2p)\Big) + G(t,2p), & t \in [2p-l,3p-l). \end{cases} \tag{3.1}$$

This formulation highlights that the expressions on the right-hand side of (3.1) rely exclusively on the values of the unknowns $y_{p-l} = y(p-l)$ and $y_{2p-l} = y(2p-l)$. By leveraging the continuity of $y(\cdot)$, these unknowns can be defined as follows:

(i) $y_{2p-l} = y(2p-l) = \lim_{t\to 2p-l^-} y(t)$, where $t \in [p-l,2p-l)$,

(ii) $y_{3p-l} = y(3p-l) = \lim_{t\to 3p-l^-} y(t)$, where $t \in [2p-l,3p-l)$.

Given the continuity and periodicity of $y(\cdot)$, it follows that $y(p-l) = y(3p-l)$. To determine $y_{p-l} = y_{3p-l}$ from (3.1), we obtain the following system of equations:

$$\begin{cases} \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} y(p-l) - y(2p-l) = \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} G(p-l,p) - G(2p-l,p), \\ y(p-l) - \frac{\lambda(3p-l,4)}{\lambda(2p-l,2p)} y(2p-l) = -\frac{\lambda(3p-l,2p)}{\lambda(2p-l,2p)} G(2p-l,2p) + G(3p-l,2p). \end{cases} \tag{3.2}$$

Let Δ denote the determinant of the matrix $\mathcal{M}$, where

$$\mathcal{M} = \begin{pmatrix} \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} & -1 \\ 1 & -\frac{\lambda(3p-l,2p)}{\lambda(2p-l,2p)} \end{pmatrix}.$$

Using this, we establish the following theorem regarding the existence of $2p$-periodic solutions to DEPCA (1.2).

Theorem 3.1. *Let $a(\cdot)$, $b(\cdot)$, and $g(\cdot)$ be $2p$-periodic continuous functions. The following results hold.*

(a) If $\Delta \neq 0$, DEPCA (1.2) has a unique $2p$-periodic solution, as given in (3.1), where (y_{p-l}, y_{2p-l}) represents the sole solution of the system (3.2).

(b) If $\Delta = 0$ and $G(p-l,p) = G(2p-l,p) = G(2p-l,2p) = G(3p-l,2p) = 0$, DEPCA (1.2) admits an infinite number of $2p$-periodic solutions, as described below:

$$y(t) = \begin{cases} \alpha \frac{\lambda(t,p)}{\lambda(p-l,p)} \Big(y(p-l) - G(p-l,p) \Big) + G(t,p), & t \in [p-l, 2p-l), \\ \alpha \frac{\lambda(t,2p)}{\lambda(2p-l,2p)} \Big(y(2p-l) - G(2p-l,2p) \Big) + G(t,2p), & t \in [2p-l, 3p-l). \end{cases}$$

Here, (y_{p-l}, y_{2p-l}) represents an eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0, and α denotes a real-valued scalar.

(c) If $\Delta = 0$ and the rank $\mathcal{M}$ is less than the rank of the augmented matrix $(\mathcal{M}|b)$, where

$$b = \begin{pmatrix} \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} G(p-l,p) - G(2p-l,p), \\ -\frac{\lambda(3p-l,2p)}{\lambda(2p-l,2p)} G(2p-l,2p) + G(3p-l,2p) \end{pmatrix},$$

then DEPCA (1.2) does not possess a $2p$-periodic solution.

Proof. (a) Assume $y(t)$ is a $2p$-periodic solution to DEPCA (1.2). This solution can be characterized by (3.1), where (y_{p-l}, y_{2p-l}) satisfies the system (3.2). The solvability of linear system (3.2) requires that $\Delta \neq 0$. Consequently, $\Delta \neq 0$ must hold. Conversely, when $\Delta \neq 0$, DEPCA (3.2) admits a unique solution (y_{p-l}, y_{2p-l}). Furthermore, it can be demonstrated that the function $y(\cdot)$, defined in (3.1), represents the periodic solution to DEPCA (1.2).

(b) The function G assumes a value of zero at the points $(p-l,p)$, $(2p-l,p)$, $(2p-l,2p)$, and $(3p-l,2p)$. Consequently, equation (3.2) simplifies into a homogeneous form. A non-trivial solution to this equation exists if and only if $\Delta = 0$.

The pair of non-zero solutions (y_{p-l}, y_{2p-l}) serves as an eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0. Thus, $(\alpha y_{p-l}, \alpha y_{2p-l})$ represents a non-trivial solution to equation (3.2), where α denotes an arbitrary non-zero scalar. Accordingly, the $2p$-periodic function is expressed as:

$$y(t) = \begin{cases} \alpha \frac{\lambda(t,p)}{\lambda(p-l,p)} \Big(y(p-l) - G(p-l,p) \Big) + G(t,p), & t \in [p-l, 2p-l), \\ \alpha \frac{\lambda(t,2p)}{\lambda(2p-l,2p)} \Big(y(2p-l) - G(2p-l,2p) \Big) + G(t,2p), & t \in [2p-l, 3p-l). \end{cases}$$

This function satisfies DEPCA (1.2), where α can take any value.

(c) If $\Delta = 0$ and the rank of $\mathcal{M}$ is strictly less than the rank of the augmented matrix $(\mathcal{M} \,|\, b)$, where

$$b = \begin{pmatrix} \frac{\lambda(2p-l,p)}{\lambda(p-l,p)} G(p-l,p) - G(2p-l,p) \\ -\frac{\lambda(3p-l,2p)}{\lambda(2p-l,2p)} G(2p-l,2p) + G(3p-l,2p) \end{pmatrix},$$

then equation (3.2) does not admit a solution. As a result, DEPCA (1.2) cannot have a $2p$-periodic solution.

□

4 Illustrative example

In this section, we provide a pertinent example to demonstrate the practical application of our theoretical framework. Specifically, we consider the following scalar differential equation involving a piecewise alternately advanced and retarded argument

$$y'(t) = \sin(2\pi t)\, y\left(3\left[\frac{t+1}{3}\right]\right) + g(t), \quad t \geq 2. \tag{4.1}$$

DEPCA (4.1) represents a particular case of DEPCA (1.2), specified by the parameters $p = 2$, $l = 1$, $a = 0$, $b(t) = \sin(2\pi t)$ with

$$g(t) = \begin{cases} \sin\left(\frac{t-2.5-3k}{\pi}\right), & t \in [2+3k, 3+3k), \\ -\left(\frac{\sin(0.5\cdot\pi^{-1})}{\sin(\pi^{-1})}\right)\sin\left(\frac{t-4-3k}{\pi}\right), & t \in [3+3k, 5+3k), \end{cases}$$

for $k \in \mathbb{N}_0$.

It can be readily verified that $G(2,3) = G(5,3) = G(5,6) = G(8,6) = 0$. The matrix associated with the linear system of equations involving the variables y_2 and y_5 is given as follows:

$$\mathcal{M} = \begin{pmatrix} \frac{\lambda(5,3)}{\lambda(2,3)} & -1 \\ 1 & -\frac{\lambda(8,6)}{\lambda(5,6)} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The determinant $\mathcal{M}$ is zero and vector $(1,1)$ serves as an eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 0. According to Theorem 3.1(b), the solution to DEPCA is given by

$$y_\alpha(t) = \begin{cases} \alpha\lambda(t,3)y(2) + \hat{g}_1(t), & t \in [2,3), \\ \alpha\lambda(t,3)y(2) - \hat{g}_2(t), & t \in [3,5), \\ \alpha\lambda(t,6)y(5) + \hat{g}_3(t), & t \in [5,6), \\ \alpha\lambda(t,6)y(5) - \hat{g}_4(t), & t \in [6,8), \end{cases}$$

where

$$\hat{g}_1(t) = \pi\left(-\cos\left(\frac{t-2.5}{\pi}\right) + \cos\left(\frac{-0.5}{\pi}\right)\right),$$

$$\hat{g}_2(t) = \left(\frac{\sin(0.5\cdot\pi^{-1})\pi}{\sin(\pi^{-1})}\right)\left(-\cos\left(\frac{t-4}{\pi}\right) + \cos\left(\frac{-1}{\pi}\right)\right),$$

$$\hat{g}_3(t) = \pi\left(-\cos\left(\frac{t-5.5}{\pi}\right) + \cos\left(\frac{-0.5}{\pi}\right)\right),$$

$$\hat{g}_4(t) = \left(\frac{\sin(0.5\cdot\pi^{-1})\pi}{\sin(\pi^{-1})}\right)\left(-\cos\left(\frac{t-7}{\pi}\right) + \cos\left(\frac{-1}{\pi}\right)\right).$$

This solution is 6-periodic for any non-zero value of α.

The graphs of $y_\alpha(t)$ for $\alpha = 0.7$ and $\alpha = -0.5$ are presented in Figure 1 and Figure 2, respectively.

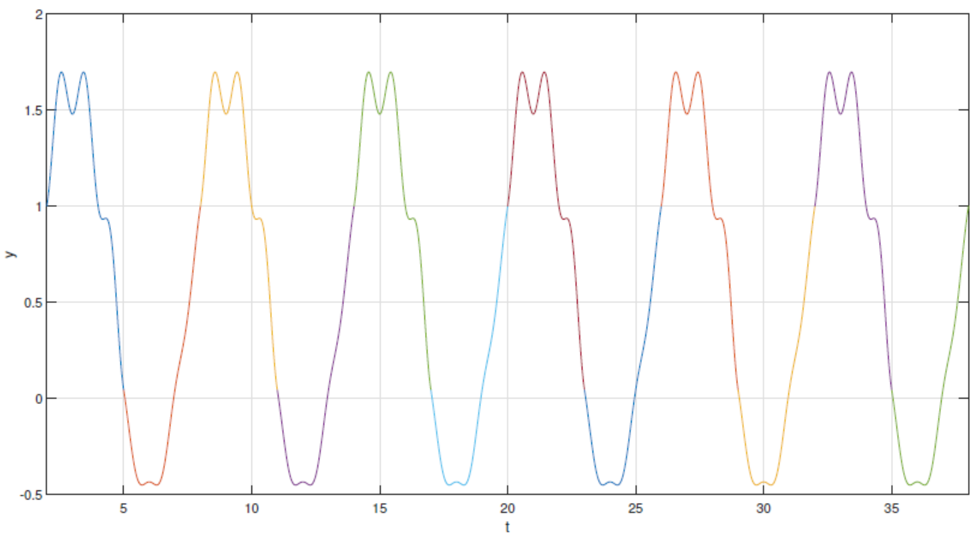

Fig. 1. 6-periodic solution to DEPCA (4.1) if $\alpha = 0.7$.

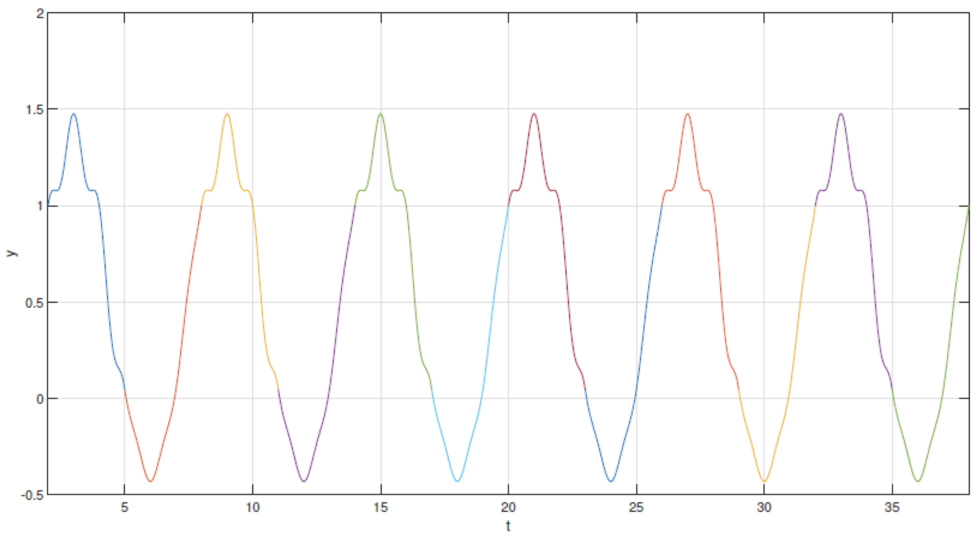

Fig. 2. 6-periodic solution to DEPCA (4.1) if $\alpha = -0.5$.

It is worth highlighting that the parameters of the equation in this example conform to the conditions outlined in the main results of [12]. Moreover, Example 4.1 extends and enhances the conclusions of Theorem 4.4 in [12], which establishes the uniqueness of the solution to the DEPCA (1.2).

5 Conclusion

This article examines the presence of infinitely many periodic solutions to first-order differential equations characterized by piecewise alternately advanced and retarded argument. Several theorems have been developed to establish both the existence and uniqueness of solutions to DEPCAs of this nature. Drawing inspiration from the methodologies in [11, 23], we have identified sufficient conditions ensuring the existence of infinitely many periodic solutions under the appropriate assumptions. Additionally, a range of numerical examples and simulations are provided to demonstrate the practical relevance and applicability of the theoretical results.

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