EURASIAN MATHEMATICAL JOURNAL
volume 16 , number 4, 2025

CONTENTS

EURASIAN NATIONAL UNIVERSITY

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

EURASIAN MATHEMATICAL JOURNAL

VOLUME 16, NUMBER 4 2025

ISSN 2077-9879
9 772077 987003

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2025, Volume 16, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Republic of Kazakhstan

The Moscow Editorial Office
The Patrice Lumumba Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russian Federation

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 4 (2025), 23 – 32

BOUNDEDNESS OF THE GENERALIZED RIEMANN-LIOUVILLE OPERATOR IN LOCAL MORREY-TYPE SPACES WITH MIXED QUASI-NORMS

V.I. Burenkov, M.A. Senouci

Communicated by N.A. Bokayev

Key words: generalized Riemann-Liouville operator, boundedness, local Morrey-type spaces with mixed quasi-norms.

AMS Mathematics Subject Classification: 34A55, 34B05, 58C40.

Abstract. The objective of this paper is to establish sufficient conditions for the boundedness of the generalized Riemann-Liouville operator in local Morrey-type spaces with mixed quasi-norms on a parallelepiped and to obtain sharp estimates of the norm of this operator with respect to the lengths of the edges of this parallelepiped.

DOI: https://doi.org/10.32523/2077-9879-2025-16-4-23-32

1 Introduction

First, we recall the definition of local Morrey-type spaces.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue-measurable set, $x_0 \in \overline{\Omega}$, $0 < p,\ \theta \le \infty$, $\lambda \ge 0$. Then $f \in LM^{\lambda}_{p\theta,x_0}(\Omega)$ if f is Lebesgue-measurable on Ω and

$$\|f\|_{LM^{\lambda}_{p\theta,x_0}(\Omega)} = \|r^{-\lambda-\frac{1}{\theta}}\|f\|_{L_p(\Omega\bigcap B(x_0,r))}\|_{L_\theta(0,\infty)} < \infty,$$

where $B(x_0,r)$ is the ball centered at x_0 of radius r.

If $\lambda = 0$, then, clearly, $LM^{0}_{p\infty,x_0}(\Omega) = L_p(\Omega)$. If $\lambda = 0$ and $\theta < \infty$, then $LM^{0}_{p\theta,x_0}(\Omega)$ consists only of functions f equivalent to 0 on Ω, because for any $\rho > 0$

$$\|f\|_{LM^{0}_{p\theta,x_0}(\Omega)} \ge \|\|f\|_{L_p(\Omega\bigcap B(x_0,r))}\|_{L_\theta(\rho,\infty)} \ge \|1\|_{L_\theta(\rho,\infty)}\|f\|_{L_p(\Omega\bigcap B(x_0,\rho))}.$$

For $\Omega = \mathbb{R}^n$, $x_0 = 0$ this definition was first introduced in [1], [2].

The boundedness of various operators acting from one local Morrey-type spaces $LM^{\lambda}_{p\theta,x_0}(\Omega)$ to $LM^{\mu}_{p\sigma,x_0}(\Omega)$ was investigated in a number of papers. See, for example, [1], [2].

In this paper, we consider the following two variants of the local Morrey-type spaces with mixed quasi-norms.

We shall use the following notation for vectors $x \in \mathbb{R}^n : x \equiv \overrightarrow{x} = (x_1, ..., x_n)$ and $\overleftarrow{x} = (x_n, ..., x_1)$.

Definition 2. Let $p = (p_1, ..., p_n)$, $\theta = (\theta_1, ..., \theta_n)$, $\lambda = (\lambda_1, ..., \lambda_n)$, $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ $-\infty < a_i < b_i \le \infty$, $0 < p_i$. $\theta_i \le \infty$, $0 \le \lambda_i < \infty$, $i = 1, ..., n$, $Q(a,b) = \{x \in \mathbb{R}^n, a_i < x_i < b_i, i = 1, ..., n\}$.

Let $LM^{\lambda}_{p\theta,a}(Q(a,b))$, $\overleftarrow{LM}^{\lambda}_{p\theta,a}(Q(a,b)) \equiv LM^{\overleftarrow{\lambda}}_{\overleftarrow{p}\,\overleftarrow{\theta},\overleftarrow{a}}(Q(a,b))$ be the spaces of all Lebesgue-measurable functions on $Q(a,b)$ for which the following quasi-norms are finite

$$\|f\|_{LM^{\lambda}_{p\theta,a}\left(Q(a,b)\right)}=\left\|...\|f(x_1,...,x_n)\|_{LM^{\lambda_1}_{p_1\theta_1,a_1,x_1}((a_1,b_1))}...\right\|_{LM^{\lambda_n}_{p_n\theta_n,a_n,x_n}((a_n,b_n))}$$

$$=\left\|r_n^{-\lambda_n-\frac{1}{\theta_n}}\right\|...\left\|r_1^{-\lambda_1-\frac{1}{\theta_1}}\left\|f(x_1,...,x_n)\right\|_{L_{p_1,x_1}((a_1,b_1)\bigcap(a_1-r_1,a_1+r_1))}\right\|_{L_{\theta_1}(0,\infty)}$$

$$...\Big\|_{L_{p_n,x_n}((a_n,b_n)\bigcap(a_n-r_n,a_n+r_n))}\Big\|_{L_{\theta_n}(0,\infty)} \tag{1.1}$$

and

$$\|f\|_{\overleftarrow{LM}_{p\theta,a}\left(Q(a,b)\right)}=\left\|...\|f(x_1,...,x_n)\|_{LM^{\lambda_n}_{p_n\theta_n,a_n,x_n}((a_n,b_n))}...\right\|_{LM^{\lambda_1}_{p_1\theta_1,a_1,x_1}((a_1,b_1))}$$

$$=\left\|r_1^{-\lambda_1-\frac{1}{\theta_1}}\right\|...\left\|r_n^{-\lambda_n-\frac{1}{\theta_n}}\|f(x_1,...,x_n)\|_{L_{p_n,x_n}((a_n,b_n)\bigcap(a_n-r_n,a_n+r_n))}\right\|_{L_{\theta_n}(0,\infty)}$$

$$...\Big\|_{L_{p_1,x_1}((a_1,b_1)\bigcap(a_1-r_1,a_1+r_1))}\Big\|_{L_{\theta_1}(0,\infty)}, \tag{1.2}$$

respectively.

If $a=0$, then, for brevity, we denote the corresponding spaces by $\|f\|_{LM^{\lambda}_{p\theta}(Q(0,b)}$ and $\|f\|_{\overleftarrow{LM}^{\lambda}_{p\theta}(Q(0,b))}$.

Definition 3. [5] Let $f\in L_1^{loc}(\mathbb{R}^n)$, $\alpha=(\alpha_1,...,\alpha_n)$, $0<\alpha_i<1$, $k=(k_1,...,k_n)$, $k_i\geq 0$, $a=(a_1,...,a_n)$, $x=(x_1,...,x_n)$, $0\leq a_i<x_i<\infty$, $i=1,...,n$. The generalized Riemann-Liouville fractional integral operator $I^{\alpha,k}_{a^+}$ of order α is defined by the following equality:

$$\left(I^{\alpha,k}_{a_+}f\right)(x)$$

$$=\prod_{i=1}^{n}\frac{(k_i+1)^{1-\alpha_i}}{\Gamma(\alpha_i)}\int_{a_n}^{x_n}...\int_{a_1}^{x_1}\prod_{i=1}^{n}\left[(x_i^{k_i+1}-t_i^{k_i+1})^{\alpha_i-1}t_i^{k_i}\right]f(t_1,...,t_n)dt_1...dt_n, \tag{1.3}$$

where Γ is the Euler Gamma-function.

2 One-dimensional case

Lemma 2.1. *Let $0<y<\infty$, $0<p,\theta\leq\infty$, $0<\lambda<\infty$ for $\theta<\infty$, $0\leq\lambda<\infty$ for $\theta=\infty$. Then* [1]

$$\|f\|_{L_p(0,y)}\leq(\lambda\theta)^{\frac{1}{\theta}}y^{\lambda}\|f\|_{LM^{\lambda}_{p\theta}(0,y)} \tag{2.1}$$

Proof. It suffices to note that for $\lambda>0$

$$\|f\|_{LM^{\lambda}_{p\theta}(0,y)}=\left(\int_0^{\infty}r^{-\lambda\theta-1}\|f\|^{\theta}_{L_p((0,y)\bigcap(-r,r))}dr\right)^{\frac{1}{\theta}}$$

$$\geq\left(\int_y^{\infty}r^{-\lambda\theta-1}\|f\|^{\theta}_{L_p((0,y)\bigcap(0,r))}dr\right)^{\frac{1}{\theta}}=\|f\|_{L_p(0,y)}\left(\int_y^{\infty}r^{-\lambda\theta-1}dr\right)^{\frac{1}{\theta}}=(\lambda\theta)^{-\frac{1}{\theta}}y^{-\lambda}\|f\|_{L_p(0,y)}$$

[1] If $\theta=\infty$, then here and in the sequel it is assumed that $(\lambda\theta)^{\frac{1}{\theta}}=1$ for all $0\leq\lambda<\infty$.

if $\theta < \infty$ and

$$\|f\|_{LM^{\lambda}_{p\infty}(0,y)} = \sup_{0<r<\infty} r^{-\lambda} \|f\|_{L_p((0,y)\bigcap(-r,r))} \le y^{-\lambda} \|f\|_{L_p((0,y))}$$

if $\theta = \infty$.

The case $\lambda = 0, \theta = \infty$ is trivial, since $LM^{\lambda}_{p\infty}(0,y) = L_p(0,y)$.

□

Lemma 2.2. *Let $0 \le a < b < \infty$, $1 < p \le \infty$, $0 < q \le \infty$, $\frac{1}{p} < \alpha < 1$, $k \ge 0$, $0 < \theta, \sigma \le \infty$, $0 < \lambda < \infty$ if $\theta < \infty$, $0 \le \lambda < \infty$ if $\theta = \infty$, $0 < \mu < \infty$ if $\sigma < \infty$, $0 \le \mu < \infty$ if $\sigma = \infty$.*

Then there exists $C_1 > 0$ such that

$$\|I^{\alpha,k}_{a+} f\|_{LM^{\mu}_{q\sigma,a}(a,b)} \le C_1 (b-a)^{\nu} \|f\|_{LM^{\lambda}_{p\theta,a}(a,b)} \tag{2.2}$$

for all finite intervals (a,b) and for all $f \in LM^{\lambda}_{p\theta,a}(a,b)$, where

$$\nu = \lambda + \frac{1}{q} - \frac{1}{p} + (k+1)\alpha - \mu \tag{2.3}$$

under the assumption $\nu > 0$ if $\sigma < \infty$ and $\nu \ge 0$ if $\sigma = \infty$.

Moreover, ν cannot be replaced by any other number.

Proof. It suffices to consider the case in which $a = 0$, $0 < b < \infty$.

Step 1. In [5] (see inequality (2.2)) it is proved that there exists $K_1 > 0$ such that

$$\left| \left(I^{\alpha,k}_{0+} f \right)(x) \right| \le K_1 x^{(k+1)\alpha - \frac{1}{p}} \|f\|_{L_p(0,x)}$$

for any $0 < x < b$ and for any $f \in L_p(0,x)$. By using (2.1), we obtain

$$\left| \left(I^{\alpha,k}_{0+} f \right)(x) \right| \le K_1 x^{(k+1)\alpha - \frac{1}{p} + \lambda} (\lambda\theta)^{\frac{1}{\theta}} \|f\|_{LM^{\lambda}_{p\theta}(0,x)}$$

and

$$\left\| I^{\alpha,k}_{0+} f \right\|_{L_q((-r,r)\cap(0,b))} \le K_1 (\lambda\theta)^{\frac{1}{\theta}} \left\| x^{(k+1)\alpha - \frac{1}{p} + \lambda} \right\|_{L_q((-r,r)\cap(0,b))} \|f\|_{LM^{\lambda}_{p\theta}(0,b)}.$$

Note that

$$\|I^{\alpha,k}_{0+} f\|_{LM^{\mu}_{q\sigma}(0,\infty)} = \left\| r^{-\mu - \frac{1}{\sigma}} \left\| I^{\alpha,k}_{0+} f \right\|_{L_q((-r,r)\cap(0,b))} \right\|_{L_\sigma(0,\infty)}$$

$$= \left(\left\| r^{-\mu - \frac{1}{\sigma}} \left\| I^{\alpha,k}_{0+} f \right\|_{L_q((-r,r)\cap(0,b))} \right\|^{\sigma}_{L_\sigma(0,b)} + \left\| r^{-\mu - \frac{1}{\sigma}} \left\| I^{\alpha,k}_{0+} f \right\|_{L_q((-r,r)\cap(0,b))} \right\|^{\sigma}_{L_\sigma(b,\infty)} \right)^{\frac{1}{\sigma}} \tag{2.4}$$

Moreover, since $(k+1)\alpha - \frac{1}{p} + \lambda > 0$, it follows that

$$r^{-\mu} \left\| x^{(k+1)\alpha - \frac{1}{p} + \lambda} \right\|_{L_q((-r,r)\cap(0,b))} = \begin{cases} \left(((k+1)\alpha - \frac{1}{p} + \lambda) q + 1 \right)^{-\frac{1}{q}} r^{\nu}, & r \le b, \\ \left(((k+1)\alpha - \frac{1}{p} + \lambda) q + 1 \right)^{-\frac{1}{q}} b^{\nu+\mu} r^{-\mu}, & r > b. \end{cases}$$

$$\le \begin{cases} r^{\nu}, & r \le b, \\ b^{\nu+\mu} r^{-\mu}, & r > b. \end{cases}$$

Therefore, if $r \le b$, then, since $\nu > 0$ for $\theta < \infty$ and $\nu \ge 0$ for $\theta = \infty$,

$$\left\|I_{0+}^{\alpha,k} f\right\|_{L_q((-r,r)\cap(0,b))} = \left\|r^{-\mu-\frac{1}{\theta}} \left\|I_{0+}^{\alpha,k} f\right\|_{L_q((-r,r)\cap(0,b))}\right\|_{L_\sigma(0,b)}$$

$$\leq K_1(\lambda\theta)^{\frac{1}{\theta}} \left\|r^{\nu-\frac{1}{\sigma}}\right\|_{L_\sigma(0,b)} \|f\|_{LM_{p\theta}^\lambda(0,b)}$$

$$= K_1(\lambda\theta)^{\frac{1}{\theta}}(\nu\sigma)^{-\frac{1}{\sigma}} b^\nu \|f\|_{LM_{p\theta}^\lambda(0,b)}. \tag{2.5}$$

If $r > b$, then, since $\mu > 0$ for $\sigma < \infty$ and $\mu \geq 0$ for $\sigma = \infty$,

$$\left\|r^{-\mu-\frac{1}{\sigma}} \left\|I_{0+}^{\alpha,k} f\right\|_{L_q((-r,r)\cap(0,b))}\right\|_{L_\sigma(b,\infty)}$$

$$\leq K_1(\lambda\sigma)^{\frac{1}{\sigma}} b^{(k+1)\alpha-\frac{1}{p}+\lambda+\frac{1}{q}} \left\|r^{-\mu-\frac{1}{\sigma}}\right\|_{L_\sigma(b,\infty)} \|f\|_{LM_{p\theta}^\lambda(0,b)}$$

$$= K_1(\lambda\sigma)^{\frac{1}{\sigma}}(\mu\sigma)^{-\frac{1}{\sigma}} b^\nu \|f\|_{LM_{p\theta}^\lambda(0,b)}. \tag{2.6}$$

So, by (2.4), (2.5), (2.6) it follows that

$$\|I_{0^+}^{\alpha,k} f\|_{LM_{q\sigma}^\mu(0,\infty)} \leq C_1 \|f\|_{LM_{p\theta}^\lambda(0,b)},$$

where

$$C_1 = K_1(\lambda\theta)^{\frac{1}{\theta}} \sigma^{-\frac{1}{\sigma}} \left(\frac{1}{\nu} + \frac{1}{\mu}\right)^{\frac{1}{\sigma}}.$$

Step 2. Suppose that for some $K_2(b) > 0$

$$\left\|I_{0_+}^{\alpha,k} f\right\|_{LM_{q\sigma}^\mu(0,b)} \leq K_2(b) \|f\|_{LM_{p\theta}^\lambda(0,b)}. \tag{2.7}$$

for all $f \in LM_{p\theta}^\lambda(0,b)$.

Let $f = \chi_{(\frac{b}{2},b)}$, then

$$\|\chi_{(\frac{b}{2},b)}\|_{L_p((\frac{b}{2},b)\bigcap(-r,r))} = 0, \text{ if } r \leq \frac{b}{2},$$

and

$$\|\chi_{(\frac{b}{2},b)}\|_{L_p((\frac{b}{2},b)\bigcap(-r,r))} \leq \|1\|_{L_p(\frac{b}{2},b)} = \left(\frac{b}{2}\right)^{\frac{1}{p}}, \text{ if } r > \frac{b}{2}.$$

Moreover,

$$\|f\|_{LM_{p\theta}^\lambda(0,b)} = \left\|\chi_{(\frac{b}{2},b)}\right\|_{LM_{p\theta}^\lambda(0,b)} = \left\|r^{-\lambda-\frac{1}{\theta}} \|\chi_{(\frac{b}{2},b)}\|_{L_p((\frac{b}{2},b)\bigcap(-r,r))}\right\|_{L_\theta(\frac{b}{2},\infty)}$$

$$\leq \left(\frac{b}{2}\right)^{\frac{1}{p}} \|r^{-\lambda-\frac{1}{\theta}}\|_{L_\theta(\frac{b}{2},\infty)} = K_3\, b^{\frac{1}{p}-\lambda},$$

where $K_3 = 2^{\lambda-\frac{1}{p}}(\lambda\theta)^{-\frac{1}{\theta}}$.

Next,

$$\left\|I_{0_+}^{\alpha,k} f\right\|_{LM_{q\sigma}^\mu(\frac{b}{2},b)} = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \left\|\int_0^x (x^{k+1} - t^{k+1})^{\alpha-1} t^k \chi_{(\frac{b}{2},b)}(t)dt\right\|_{LM_{q\sigma}^\mu(\frac{b}{2},b)}$$

$$(x^{k+1} - t^{k+1} = z) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \left\|\int_0^{x^{k+1}} z^{\alpha-1} \frac{dz}{k+1}\right\|_{LM_{p\theta}^\mu(\frac{b}{2},b)} = K_4 \left\|x^{\alpha(k+1)}\right\|_{LM_{q\sigma}^\mu(\frac{b}{2},b)},$$

where
$$K_4 = ((k+1)^\alpha \Gamma(\alpha+1))^{-1}.$$

Note that
$$\left\|x^{\alpha(k+1)}\right\|_{LM^{\mu}_{q\sigma}(\frac{b}{2},b)} \geq \left\| r^{-\frac{1}{\sigma}-\mu} \|x^{\alpha(k+1)}\|_{L_q((-r,r)\bigcap(\frac{b}{2},b))} \right\|_{L_\sigma(b,\infty)}$$
$$= \left\| r^{-\frac{1}{\sigma}-\mu} \|x^{\alpha(k+1)}\|_{L_q(\frac{b}{2},b)} \right\|_{L_\sigma(b,\infty)}$$
$$= \|x^{\alpha(k+1)}\|_{L_q(\frac{b}{2},b)} \|r^{-\frac{1}{\sigma}-\mu}\|_{L_\sigma(b,\infty)} = K_5\, b^{\alpha(k+1)+\frac{1}{q}-\mu},$$

where
$$K_5 = \left(\frac{1-2^{-\alpha(k+1)q-1}}{\alpha(k+1)q+1}\right)^{\frac{1}{q}} (\mu\sigma)^{-\frac{1}{\sigma}}.$$

Hence, we have
$$\left\| I^{\alpha,k}_{0+} f \right\|_{LM^{\mu}_{q\sigma}(\frac{b}{2},b)} \geq K_6\, b^{\alpha(k+1)+\frac{1}{q}-\mu}, \tag{2.8}$$

where $K_6 = K_5 K_4$.

By (2.7) and (2.8), we get
$$K_6\, b^{\alpha(k+1)+\frac{1}{q}-\mu} \leq K_2(b)\, K_3\, b^{\frac{1}{p}-\lambda},$$

so,
$$K_2(b) \geq \frac{K_6}{K_3} b^{\alpha(k+1)+\frac{1}{q}+\lambda-\frac{1}{p}-\mu} = \frac{K_6}{K_3} b^{\nu}$$

for all $b > 0$.

If (2.2) holds with $\tau \neq \nu$ replacing ν, then $\frac{K_6}{K_3} b^\nu \leq K_2(b) \leq C_1 b^\tau$ for all $b > 0$, which is impossible. □

Corollary 2.1. *If in Lemma 2.2 $\nu = 0$, then inequality (2.2) takes the form*
$$\left\| I^{\alpha,k}_{a+} f \right\|_{LM^{\mu}_{q\sigma,a}(a,b)} \leq C_1 \|f\|_{LM^{\lambda}_{p\theta,a}(a,b)}$$

for all $0 \leq a < b \leq \infty$ and for all $f \in LM^{\lambda}_{p\theta,a}(a,b)$, where
$$\mu = \lambda + (k+1)\alpha + \frac{1}{q} - \frac{1}{p}.$$

Remark 1. For $\sigma = \theta = \infty$ the statements of this section were proved in [6].

3 Multidimensional case

We start with proving a statement, in which we apply the generalized Minkowski's inequality for the Lebesgue spaces: let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ be Lebesgue-measurable sets $0 < p \leq q \leq \infty$, and $f : E \times F \to \mathbb{C}$ be a Lebesgue-measurable function. Then
$$\left\| \|f(x,y)\|_{L_{p,x}(E)} \right\|_{L_{q,y}(F)} \leq \left\| \|f(x,y)\|_{L_{q,y}(F)} \right\|_{L_{p,x}(E)}. \tag{3.1}$$

Lemma 3.1. *(Generalized Minkowski's inequality for local Morrey-type spaces.) Let* $-\infty \le a < b \le \infty$, $-\infty \le c \le d \le \infty$,

$$0 < \max\{p, \theta\} \le \min\{q, \sigma\} \le \infty,$$

$0 < \lambda$, $\mu < \infty$, *and* $f : (a, b) \times (c, d) \to \mathbb{C}$ *be a Lebesgue-measurable function.*

Then

$$\Big\| \|f(x,y)\|_{LM^{\lambda}_{p\theta,a,x}(a,b)} \Big\|_{LM^{\mu}_{q\sigma,c,y}(c,d)} \le \Big\| \|f(x,y)\|_{LM^{\mu}_{q\sigma,c,y}(c,d)} \Big\|_{LM^{\lambda}_{p\theta,a,x}(a,b)}. \tag{3.2}$$

Proof. By applying inequality (3.1) several times, we get

$$\Big\| \|f(x,y)\|_{LM^{\lambda}_{p\theta,a,x}(a,b)} \Big\|_{LM^{\mu}_{q\sigma,c,y}(c,d)} \le (\theta \le q)$$

$$\le \left\| \rho^{-\mu-\frac{1}{\sigma}} \left\| \left\| r^{-\lambda-\frac{1}{\theta}} \|f(x,y)\|_{L_{p,x}(a,b)\bigcap(a-r,a+r)} \right\|_{L_{q,y}((c,d)\bigcap(c-\rho,c+\rho))} \right\|_{L_\theta(0,\infty)} \right\|_{L_\sigma(0,\infty)}$$

$$\le (p \le q) \le \left\| \rho^{-\mu-\frac{1}{\sigma}} \left\| \left\| r^{-\lambda-\frac{1}{\theta}} \|f(x,y)\|_{L_{q,y}((c,d)\bigcap(c-\rho,c+\rho))} \right\|_{L_{p,x}(a,b)\bigcap(a-r,a+r)} \right\|_{L_\theta(0,\infty)} \right\|_{L_\sigma(0,\infty)}$$

$$\le (\theta \le \sigma) \le \left\| \rho^{-\mu-\frac{1}{\sigma}} \left\| \left\| r^{-\lambda-\frac{1}{\theta}} \|f(x,y)\|_{L_{q,y}((c,d)\bigcap(c-\rho,c+\rho))} \right\|_{L_{p,x}(a,b)\bigcap(a-r,a+r)} \right\|_{L_\sigma(0,\infty)} \right\|_{L_\theta(0,\infty)}$$

$$\le (p \le \sigma) \le \left\| r^{-\lambda-\frac{1}{\theta}} \left\| \left\| \rho^{-\mu-\frac{1}{\sigma}} \|f(x,y)\|_{L_{q,y}((c,d)\bigcap(c-\rho,c+\rho))} \right\|_{L_\sigma(0,\infty)} \right\|_{L_{p,x}(a,b)\bigcap(a-r,a+r)} \right\|_{L_\theta(0,\infty)}$$

$$= \Big\| \|f(x,y)\|_{LM^{\mu}_{q\sigma,c,y}(c,d)} \Big\|_{LM^{\lambda}_{p\theta,a,x}(a,b)}.$$

□

Theorem 3.1. *Let* $n \in \mathbb{N}, a, b, \alpha, k, p, q, \lambda, \mu, \theta, \sigma \in \mathbb{R}^n$,

$$0 \le a_i < b_i < \infty, 1 < p_i \le \infty, 0 < \theta_i, q_i, \sigma_i \le \infty, k_i \ge 0, \frac{1}{p_i} < \alpha_i < 1, i = 1, ..., n;$$

$$0 < \lambda_i < \infty \text{ if } \theta_i < \infty, \; 0 \le \lambda_i < \infty \text{ if } \theta_i = \infty, \; i = 1, ..., n.;$$

$$0 < \mu_i < \infty \text{ if } \sigma_i < \infty, \; 0 \le \mu_i < \infty \text{ if } \sigma_i = \infty, \; i = 1, ..., n.$$

Furthermore, let

$$\max\{p_i, \theta_i\} \le \min\{q_j, \sigma_j\}, i = 2, ..., n, \; j = 1, ..., i-1. \tag{3.3}$$

Then there exists $C_2 > 0$ *such that*

$$\|I_a^{\alpha,k} f\|_{\overleftarrow{LM}^{\mu}_{q\sigma,a}(Q(a,b))} \le C_2 \prod_{i=1}^{n} (b_i - a_i)^{\nu_i} \|f\|_{LM^{\lambda}_{p\theta,a}(Q(a,b))} \tag{3.4}$$

for all $f \in LM^{\lambda}_{p\theta,a}(Q(a,b)$, *where*

$$\nu_i = \lambda_i + \frac{1}{q_i} + \alpha_i(k_i + 1) - \frac{1}{p_i} - \mu_i > 0, \; i = 1, ..., n, \tag{3.5}$$

under the assumptions $\nu_i > 0$ *if* $\sigma_i < \infty$ *and* $\nu_i \ge 0$ *if* $\sigma_i = \infty$, $i = 1, ...n$.

Moreover, each ν_i, $i = 1, ...; n$, *cannot be replaced by any other number.*

Proof. Without loss of generality, we assure that $a = 0$.

Step 1. A typical case is $n = 3$, which we assume. In this case by Definition 2, (1.3) and (1.2) we have

$$A \equiv \left\| I_{0_+}^{\alpha,k} f \right\|_{\overleftarrow{LM}^{\mu}_{q\sigma}(Q(0,b))} = \left\| I_{0_+}^{\alpha,k} f \right\|_{LM^{\overleftarrow{\mu}}_{\overleftarrow{q}\,\overleftarrow{\sigma}}(Q(0,b))}$$

$$= \left\| I_{0_+}^{(\alpha_1,\alpha_2,\alpha_3),(k_1,k_2,k_3)} f \right\|_{LM^{(\mu_3,\mu_2,\mu_1)}_{(q_3,q_2 q_1)(\sigma_3,\sigma_2,\sigma_1)}(0,b_3)\times(0,b_2)\times(0,b_1)}$$

$$= \left\| \left\| \left\| I_{0+}^{\alpha_3,k_3} \left(I_{0+}^{\alpha_2,k_2} \left(I_{0+}^{\alpha_1,k_1} f \right) \right) \right\|_{LM^{\mu_3}_{q_3\sigma_3}(0,b_3)} \right\|_{LM^{\mu_2}_{q_2\sigma_2}(0,b_2)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)}.$$

By Lemma 2.2 with

$$\alpha = \alpha_3, k = k_3;\ \ \mu = \mu_3, q = q_3, \sigma = \sigma_3;\ \ \lambda = \lambda_3, p = p_3, \theta = \theta_3;\ \ a = 0, b = b_3$$

there exists $K_3 > 0$ such that

$$A \le K_3 b_3^{\nu_3} \left\| \left\| \left\| I_{0+}^{\alpha_2,k_2} \left(I_{0+}^{\alpha_1,k_1} f \right) \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)} \right\|_{LM^{\mu_2}_{q_2\sigma_2}(0,b_2)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)}.$$

By applying assumption (3.3) and inequality (2.2) first with

$$\max\{p_3, \theta_3\} \le \min\{q_2, \sigma_2\} \ \text{ and } \ \lambda = \lambda_3, p = p_3, \theta = \theta_3;\ \ \mu = \mu_2, q = q_2, \sigma = \sigma_2;$$

then with

$$\max\{p_3, \theta_3\} \le \min\{q_1, \sigma_1\} \ \text{ and } \ \lambda = \lambda_3, p = p_3, \theta = \theta_3;\ \ \mu = \mu_1, q = q_1, \sigma = \sigma_1;$$

we get

$$A \le K_3 b_3^{\nu_3} \left\| \left\| \left\| I_{0+}^{\alpha_2,k_2} \left(I_{0+}^{\alpha_1,k_1} f \right) \right\|_{LM^{\mu_2}_{q_2\sigma_2}(0,b_2)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)}$$

$$\le K_3 b_3^{\nu_3} \left\| \left\| \left\| I_{0+}^{\alpha_2,k_2} \left(I_{0+}^{\alpha_1,k_1} f \right) \right\|_{LM^{\mu_2}_{q_2\sigma_2}(0,b_2)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)}.$$

By Lemma 2.2 with

$$\alpha = \alpha_2, k = k_2;\ \ \mu = \mu_2, q = q_2, \sigma = \sigma_2;\ \ \lambda = \lambda_2, p = p_2, \theta = \theta_2;\ \ a = 0, b = b_2,$$

there exists $K_2 > 0$ such that

$$A \le K_2 K_3 b_2^{\nu_2} b_3^{\nu_3} \left\| \left\| \left\| I_{0+}^{\alpha_1,k_1} f \right\|_{LM^{\lambda_2}_{p_2\theta_2}(0,b_2)} \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)}.$$

By applying assumption (3.3) and inequality (3.1) with

$$\max\{p_2, \theta_2\} \le \min\{q_1, \sigma_1\} \ \text{ and } \ \lambda = \lambda_2, p = p_2, \theta = \theta_2;\ \ \mu = \mu_1, q = q_1, \sigma = \sigma_1;$$

we get

$$A \le K_2 K_3 b_2^{\nu_2} b_3^{\nu_3} = \left\| \left\| \left\| I_{0+}^{\alpha_1,k_1} f \right\|_{LM^{\mu_1}_{q_1\sigma_1}(0,b_1)} \right\|_{LM^{\lambda_2}_{p_2\theta_2}(0,b_2)} \right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)}.$$

Finally, by Lemma 2.2 with

$$\alpha=\alpha_1, k=k_1;\ \mu=\mu_1, q=q_1, \sigma=\sigma_1;\ \lambda=\lambda_1, p=p_1, \theta=\theta_1;\ a=0, b=b_1$$

there exists $K_1>0$ such that

$$A \le K_1K_2K_3 b_1^{\nu_1} b_2^{\nu_2} b_3^{\nu_3} = \left\|\left\|\|f\|_{LM^{\lambda_1}_{p_1\theta_1}(0,b_1)}\right\|_{LM^{\lambda_1}_{p_2\theta_2}(0,b_2)}\right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)}.$$

Also,

$$\left\|\left\|\|f\|_{LM^{\lambda_1}_{p_1\theta_1}(0,b_1)}\right\|_{LM^{\lambda_2}_{p_2\theta_2}(0,b_2)}\right\|_{LM^{\lambda_3}_{p_3\theta_3}(0,b_3)} = \|f\|_{LM^{\lambda}_{p\theta}(Q(0,b))}.$$

Therefore,

$$\left\|I^{\alpha,k}_{0+}f\right\|_{\overleftarrow{LM}^{\mu}_{q\sigma}(Q(0,b))} \le K_1K_2K_3 b_1^{\nu_1} b_2^{\nu_2} b_3^{\nu_3}\|f\|_{LM^{\lambda}_{p\theta}(Q(0,b))}.$$

Step 2. Suppose that the operator $I^{\alpha,k}_{0+}$ is bounded from $LM^{\lambda}_{p,\theta}(Q(0,b))$ to $\overleftarrow{LM}^{\mu}_{q\theta}(Q(0,b))$, so, there exists $K_4(b)>0$ such that

$$\left\|I^{\alpha,k}_{0+}f\right\|_{\overleftarrow{LM}^{\mu}_{q,\theta}(Q(0,b))} \le K_4(b)\,\|f\|_{LM^{\lambda}_{p\,\theta}(Q(0,b)}. \tag{3.6}$$

Let $f(x_1,x_2)=\chi_1(x_1)\chi_2(x_2)\chi_3(x_3)$, where $\chi_i(x_i)=\chi_{(\frac{b_i}{2},b_i)}(x_i), i=1,2,3$. Then

$$\|f\|_{LM^{\lambda}_{p\,\theta}(Q(0,b))} = \|\chi_1\|_{LM^{\lambda_1}_{p_1\theta_{1},x_1}(0,b_1)}\,\|\chi_2\|_{LM^{\lambda_2}_{p_2\theta_{2},x_2}(0,b_2)}\,\|\chi_3\|_{LM^{\lambda_3}_{p_3\theta_{3},x_3}(0,b_)},$$

where, as in Lemma 2.2,

$$\left\|\chi_{(\frac{b_i}{2},b_i)}\right\|_{LM^{\lambda_i}_{p_i\theta_{i},x_i}(\frac{b_i}{2},b_i)} \le 2^{\lambda_1-\frac{1}{p_1}}(\lambda_1\theta_1)^{-\frac{1}{\theta_1}} b_1^{\frac{1}{p_1}-\lambda_1},\ i=1,2,3.$$

Therefore, we obtain the following upper estimate:

$$\|f\|_{LM^{\lambda}_{p\theta}(Q(0,b))} \le K_5\prod_{i=1}^{3} b_i^{\frac{1}{p_i}-\lambda_i}, \tag{3.7}$$

where

$$K_5=\prod_{i=1}^{3}2^{\lambda_i-\frac{1}{p_i}}(\lambda_i\theta_i)^{-\frac{i}{\theta_i}}.$$

By applying inequality (2.8) to the equality

$$\left\|I^{\alpha,k}_{0+}f\right\|_{\overleftarrow{LM}^{\mu}_{q\,\sigma}(Q(0,b))} = \prod_{i=1}^{3}\left\|I^{\alpha_i,k_i}_{0+}\chi_i\right\|_{LM^{\mu_i}_{q_i\sigma_i}\left(\frac{b_i}{2},b_i\right)},$$

we also get the lower estimate:

$$\left\|I^{\alpha,k}_{0+}f\right\|_{\overleftarrow{LM}^{\mu}_{q\sigma}(Q(0,b))} \ge K_6\prod_{i=1}^{3} b_i^{\alpha_i(k_i+1)+\frac{1}{q_i}-\mu_i}, \tag{3.8}$$

where

$$K_6 = \prod_{i=1}^{3} (k_i+1)^{-\alpha_i} (\Gamma(\alpha_i+1)^{-1}(1-2^{-(\alpha_i(k_i+1)q_i+1)})^{\frac{1}{q_i}} (\alpha_i(k_i+1)q_i+1)^{-\frac{1}{q_i}} (\mu_i\sigma_i)^{-\frac{1}{\sigma_i}}.$$

By using inequalities (3.6), (3.7) and (3.8), we get

$$K_6 \prod_{i=1}^{3} b_1^{\alpha_i(k_i+1)+\frac{1}{q_i}-\mu_i} \leq K_4(b) K_5 \prod_{i=1}^{3} b_i^{\frac{1}{p_i}-\lambda_i}.$$

So,

$$K_4(b) \geq \frac{K_6}{K_5} b_1^{\nu_1} b_2^{\nu_2} b_3^{\nu_3}.$$

If for some $i \in \{1,2,3\}$, say for $i=1$, inequality (3.4) holds with $\tau \neq \nu_1$ replacing ν_1, then

$$\frac{K_6}{K_5} b_1^{\nu_1} \leq K_4(b,1,1) \leq C_1 b_1^{\tau_1}$$

for all $b_1 > 0$, which is impossible.

Thus, we obtain the required statements for $n=3$. The case $n>3$ is considered similarly. □

Remark 2. For $\sigma = \theta = \infty$ Theorem 2.1 is an anisotropic version of Theorem 2.1 of [6].

Acknowledgments

This work was carried out at the V.A. Steklov Mathematical Institute and supported by the Russian Science Foundation (project no. 24-11-00170, https://rscf.ru/project/24-11-00170/). Sections 1, 2 were written by M.A. Senouci, Section 3 by V.I. Burenkov.

References

[1] V.I. Burenkov, H.V. Guliev, *Necessary and sufficient conditions for the boundedness of the maximal operator in local spaces of Morrey type*, Dokl. Ross. Akad. Nauk, 391 (2003), no. 5, 591–594.

[2] V.I. Burenkov, H.V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces*, Studia Math. 163 (2004), no. 2, 157-176.

[3] V.I. Burenkov, P. Jain, T.V. Tararykova, *On boundedness of the Hardy operator in Morrey-type spaces*, Eurasian Math. J., 2 (2011), no. 1, 52-80.

[4] V.I. Burenkov, A. Gogatishvili, V.S. Guliyev, R.Ch. Mustafayev, *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Analysis 35 (2011), no. 1, 67-87.

[5] Z. Fu, J. Trujillo, Q. Wu, *Riemann-Liouville fractional calculus in Morrey spaces and applications* Comput Math Appl, 2016, http://doi.org/10.1016/j.camwa.2016.04.013

[6] M.A. Senouci, *Boundedness of the generalized Riemann-Liouville operator in local Morrey-type spaces*, Eurasian Math. J., 14 (2023), no. 4, 63-68.

[7] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1971.

Victor Ivanovich Burenkov, Mariam Abdelkaderovna Senouci
S.M. Nikolskii Mathematical Institute
RUDN University
6 Miklukho Maklay St
117198 Moscow, Russian Federation
and
V.A. Steklov Mathematical Institute
Russian Academy of Sciences
8 Gubkin St
117966 Moscow, Russian Federation
E-mails: burenkov_vi@pfur.ru, senoucim@yandex.ru

Received: 11.10.2024