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The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
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The Moscow Editorial Office
The Patrice Lumumba Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russian Federation

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APPROXIMATION BY T MEANS WITH RESPECT TO VILENKIN SYSTEM IN LEBESGUE SPACES

N. Anakidze, N. Areshidze, L.-E. Persson, G. Tephnadze

Communicated by N.A. Bokayev

Key words: Vilenkin group, Vilenkin system, T means, Nörlund means, Fejér means, approximation, Lebesgue spaces, inequalities.

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Abstract. In this paper we present and prove some new results concerning approximation properties of T means with respect to the Vilenkin system in Lebesgue spaces for any $1 \leq p < \infty$. As applications, we obtain extensions of some known approximation inequalities.

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1 Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers and $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m =: (m_0, m_1, \cdots)$ be a sequence of positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \cdots, m_k - 1\}$$

the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the group Z_{m_k} with the product of the discrete topologies of Z_{m_k}'s.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{k\in\mathbb{N}} m_k < +\infty$, then we call G_m a bounded Vilenkin group. If the sequence $\{m_k\}_{k\geq 0}$ is unbounded, then G_m is said to be an unbounded Vilenkin group. In this paper we consider only bounded Vilenkin groups.

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots) \qquad (\, x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m, namely

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \ (x \in G_m, \ n \in \mathbb{N}).$$

For brevity, we also define $I_n := I_n(0)$.

Next, we define a generalized number system based on m in the following way:

$$M_0 =: 1, \quad M_{k+1} =: m_k M_k \quad (k \in \mathbb{N})$$

Then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n=\sum_{j=0}^{\infty} n_j M_j, \quad \text{where} \quad n_j \in Z_{m_j} \quad (j \in \mathbb{N})$$

and only a finite number of n_j's differ from zero. Let

$$|n| =: \max\{j \in \mathbb{N}, n_j \neq 0\}.$$

Moreover, Vilenkin (see [33, 34, 35]) investigated the group G_m and introduced the Vilenkin system $\{\psi_j\}_{j=0}^{\infty}$ as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

where $r_k(x)$ are the generalized Rademacher functions defined by

$$r_k(x) := \exp(2\pi i x_k/m_k), \quad (k \in \mathbb{N}).$$

If $m_k = 2$ for any $k \in \mathbb{N}$, then the Vilenkin group coincides with the dyadic group, which will be denoted by G_2 and Vilenkin systems include as a special case the Walsh system.

The norms (or quasi-norms) $\|f\|_p$, $0 < p < \infty$, of the Lebesgue spaces $L^p(G_m)$ are defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu.$$

The Vilenkin system is orthonormal and complete in $L^2(G_m)$ (see e.g. [2] and [27]).

If $f \in L^1(G_m)$, we can define the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\begin{aligned}
\widehat{f}(k) &:= \int_{G_m} f \overline{\psi}_k d\mu, \quad (k \in \mathbb{N}), \\
S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad (n \in \mathbb{N}_+, \ S_0 f := 0), \\
\sigma_n f &:= \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad (n \in \mathbb{N}_+). \\
D_n &:= \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+). \\
K_n &:= \frac{1}{n} \sum_{k=1}^{n} D_k, \quad (n \in \mathbb{N}_+).
\end{aligned}$$

Recall that (see e.g. [2] and [25]),

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{1.1}$$

$$\begin{aligned}
D_{M_n - j}(x) &= D_{M_n}(x) - \overline{\psi}_{M_n-1}(-x) D_j(-x) \\
&= D_{M_n}(x) - \psi_{M_n-1}(x) \overline{D_j}(x), \quad 0 \leq j < M_n.
\end{aligned} \tag{1.2}$$

$$n\left|K_{n}\right| \leq 2R^{2}\sum_{l=0}^{|n|}M_{l}\left|K_{M_{l}}\right|, \tag{1.3}$$

and

$$\int_{G_m} K_n(x)d\mu(x)=1, \qquad \sup_{n\in\mathbb{N}}\int_{G_m}\left|K_n(x)\right|d\mu(x)\leq R^5. \tag{1.4}$$

where $R:=\sup_{k\in\mathbb{N}} m_k$. Moreover, if $n>t$, $t,n\in\mathbb{N}$, then

$$K_{M_n}(x)=\begin{cases}\frac{M_t}{1-r_t(x)}, & x\in I_t\backslash I_{t+1}, \quad x-x_te_t\in I_n,\\ \frac{M_n+1}{2}, & x\in I_n,\\ 0, & \text{otherwise.}\end{cases} \tag{1.5}$$

The n-th Nörlund mean t_n and T mean T_n of $f\in L^1(G_m)$ are defined by

$$t_nf:=\frac{1}{Q_n}\sum_{k=1}^{n}q_{n-k}S_kf$$

and

$$T_nf:=\frac{1}{Q_n}\sum_{k=0}^{n-1}q_kS_kf,$$

where

$$Q_n:=\sum_{k=0}^{n-1}q_k.$$

Here $\{q_k,\ k\geq 0\}$ is a sequence of nonnegative numbers, where $q_0>0$ and

$$\lim_{n\to\infty}Q_n=\infty. \tag{1.6}$$

Then, a T mean generated by $\{q_k,\ k\geq 0\}$ is regular if and only if condition (1.6) is satisfied (see [25]).

It is evident that

$$T_nf(x)=\int_{G_m}f(t)F_n(x-y)\,d\mu(y),$$

where

$$F_n:=\frac{1}{Q_n}\sum_{k=0}^{n-1}q_kD_k, \tag{1.7}$$

which are called the kernels of the T means.

By applying the Abel transformation, we get the following two useful identities:

$$Q_n:=\sum_{k=0}^{n-1}q_k\cdot 1=\sum_{k=0}^{n-2}(q_k-q_{k+1})k+q_{n-1}(n-1) \tag{1.8}$$

and

$$T_nf=\frac{1}{Q_n}\left(\sum_{k=0}^{n-2}(q_k-q_{k+1})k\sigma_kf+q_{n-1}(n-1)\sigma_{n-1}f\right). \tag{1.9}$$

2 Historical overview

It is well-known (see e.g. [15], [25] and [39]) that, for any $1 \leq p \leq \infty$ and $f \in L^p(G_m)$, there exists $C_p > 0$, depending only on p, such that

$$\left\|\sigma_n f\right\|_p \leq C_p \left\|f\right\|_p .$$

Moreover, Skvortsov [30] (see also [1]) proved that if $1 \leq p \leq \infty$, $M_N \leq n < M_{N+1}$, $f \in L^p(G_m)$ and $n \in \mathbb{N}$, then

$$\left\|\sigma_n f - f\right\|_p \leq 2R^5 \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right), \tag{2.1}$$

where $R := \sup_{k\in\mathbb{N}} m_k$ and $\omega_p(1/M_k, f)$ is the modulus of continuity of $L^p(G_m)$ functions, $1 \leq p < \infty$ functions defined by

$$\omega_p(1/M_k, f) = \sup_{|u|<1/M_k} \|f(\cdot - u) - f(\cdot)\|_p, \qquad k \in \mathbb{N},$$

where $-$ is the inverse operation of the sum $+$ defined on G_m and the modulus $|u|$ of $u \in G_m$ is defined by

$$|u| = \sum_{i=0}^{\infty} \frac{u_i}{M_{i+1}}.$$

It follows that if $f \in Lip\,(\alpha, p)$, i.e.,

$$Lip(\alpha, p) := \{f \in L^p(G_m) : \omega_p(1/M_k, f) = O(1/M_k^{\alpha}) \quad \text{as} \quad k \to 0\},$$

then

$$|\sigma_n f - f\|_p = \begin{cases} O(1/M_N), & \text{if} \quad \alpha > 1, \\ O(N/M_N), & \text{if} \quad \alpha = 1, \\ O(1/M_n^{\alpha}), & \text{if} \quad \alpha < 1. \end{cases}$$

Moreover, (see e.g. [25]) if $1 \leq p < \infty$, $f \in L^p(G_m)$ and

$$\left\|\sigma_{M_n} f - f\right\|_p = o\left(1/M_n\right), \text{ as } n \to \infty,$$

then f is a constant function.

For the maximal operators of Vilenkin-Fejér means σ^*, defined by

$$\sigma^* f = \sup_{n\in\mathbb{N}} |\sigma_n f|$$

the weak-$(1,1)$ type inequality

$$\left\|\sigma^* f\right\|_{weak-L_1} \leq C \left\|f\right\|_1, \qquad \left(f \in L^1(G_m)\right)$$

can be found in Schipp [26] for Walsh series and in Pál, Simon [24] and Weisz [36] for bounded Vilenkin series. The boundedness of the maximal operators of Vilenkin-Féjer means of the one- and two-dimensional cases can be found in Fridli [10], Gát [12], Goginava [14], Nagy and Tephnadze [22, 23], Simon [28, 29] and Weisz [37].

Convergence and summability of Nörlund means with respect to Vilenkin systems were studied by Areshidze and Tephnadze [3], Blahota and Nagy [4], Blahota, Persson and Tephnadze [7] (see also [5, 6]), Blyumin [8], Efimov [9], Fridli, Manchanda and Siddiqi [11], Goginava [13], Jastrebova [16], Nagy [20, 21], Memic [17], Tsutserova [31] and Zhantlesov [38].

Móricz and Siddiqi [19] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of $L^p(G_2)$ functions. In particular, they proved that if $f \in L^p(G_2)$, $1 \leq p \leq \infty$, $n = 2^j + k$, $1 \leq k \leq 2^j$ $(n \in \mathbb{N}_+)$ and $(q_k, k \in \mathbb{N})$ is a sequence of non-negative numbers, such that

$$\frac{n^{\gamma-1}}{Q_n^\gamma}\sum_{k=0}^{n-1} q_k^\gamma = O(1), \quad \text{for some} \quad 1 < \gamma \leq 2,$$

then there exists $C_p > 0$, depending only on p, such that

$$\|t_n f - f\|_p \leq \frac{C_p}{Q_n}\sum_{i=0}^{j-1} 2^i q_{n-2^i}\omega_p\left(\frac{1}{2^i}, f\right) + C_p\omega_p\left(\frac{1}{2^j}, f\right),$$

if the sequence $(q_k, k \in \mathbb{N})$ is non-decreasing, while

$$\|t_n f - f\|_p \leq \frac{C_p}{Q_n}\sum_{i=0}^{j-1}\left(Q_{n-2^i+1} - Q_{n-2^{i+1}+1}\right)\omega_p\left(\frac{1}{2^i}, f\right) + C_p\omega_p\left(\frac{1}{2^j}, f\right),$$

if the sequence $(q_k, k \in \mathbb{N})$ is non-increasing.

Tutberidze [32] (see also [25]) proved that if T_n are T means generated by either a non-increasing sequence $\{q_k, k \in \mathbb{N}\}$ or a non-decreasing sequence $\{q_k, k \in \mathbb{N}\}$ satisfying the condition

$$\frac{q_0}{Q_k} = O\left(\frac{1}{k}\right), \quad \text{as} \quad k \to \infty,$$

then there exists an absolute constant C, such that

$$\|T^* f\|_{weak-L_1} \leq C\,\|f\|_1, \qquad \left(f \in L^1(G_m)\right)$$

holds. From these results it follows that if $f \in L^p(G_m)$, where $1 \leq p < \infty$ and either the sequence $\{q_k, k \in \mathbb{N}\}$ is non-increasing, or $\{q_k, k \in \mathbb{N}\}$ is a sequence of non-decreasing numbers, such that the condition

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty, \tag{2.2}$$

is satisfied, then

$$\lim_{n\to\infty}\|T_n f - f\|_p = 0.$$

For the Walsh system in [18] Móricz and Rhoades proved that if $f \in L^p(G_2)$, where $1 \leq p < \infty$, and T_n are regular T means generated by a non-increasing sequence $\{q_k,\ k \in \mathbb{N}\}$, then, for any $2^N \leq n < 2^{N+1}$, we have the following approximation inequality:

$$\|T_n f - f\|_p \leq \frac{C_p}{Q_n}\sum_{s=0}^{N-1} 2^s q_{2^s}\omega_p\left(1/2^s, f\right) + C_p\omega_p\left(1/2^N, f\right). \tag{2.3}$$

In the case in which the sequence $\{q_k,\ k \in \mathbb{N}\}$ is non-decreasing and satisfying the condition

$$\frac{q_{k-1}}{Q_k} = O\left(\frac{1}{k}\right), \quad \text{as} \quad k \to \infty, \tag{2.4}$$

the following inequality holds:

$$\|T_nf - f\|_p \leq C_p \sum_{j=0}^{N-1} 2^{j-N} \omega_p \left(1/2^j, f\right) + C_p \omega_p \left(1/2^N, f\right). \tag{2.5}$$

In this paper we use a new approach and generalize inequalities in (2.3) and (2.5) for T means with respect to the Vilenkin system (see Theorems 1 and 2). We also prove a new inequality for the subsequences $\{T_{M_n}\}$ means if the sequence $\{q_k,\ k \in \mathbb{N}\}$ is non-decreasing (see Theorem 3).

3 The main results

Our first main result reads:

Theorem 3.1. *Let $f \in L^p(G_m)$, where $1 \leq p < \infty$ and T_n are T means generated by a non-increasing sequence $\{q_k,\ k \in \mathbb{N}\}$. Then, for any $n, N \in \mathbb{N}$, $M_N \leq n < M_{N+1}$, we have the following inequality:*

$$\|T_nf - f\|_p \leq \frac{6R^6}{Q_n} \sum_{j=0}^{N-1} M_j q_{M_j} \omega_p \left(1/M_j, f\right) + 4R^6 \omega_p \left(1/M_N, f\right). \tag{3.1}$$

Next we state and prove a similar inequality for non-decreasing sequences but under some restrictions.

Theorem 3.2. *Let $f \in L^p(G_m)$, where $1 \leq p < \infty$ and T_n are regular T means generated by a non-decreasing sequence $\{q_k,\ k \in \mathbb{N}\}$. Then, for any $n, N \in \mathbb{N}$, $M_N \leq n < M_{N+1}$, we have the following inequality:*

$$\|T_nf - f\|_p \leq \frac{6R^6 q_{n-1}}{Q_n} \sum_{j=0}^{N-1} M_j \omega_p \left(1/M_j, f\right) + \frac{4R^6 q_{n-1} M_N}{Q_n} \omega_p \left(1/M_N, f\right). \tag{3.2}$$

If, in addition, the sequence $\{q_k,\ k \in \mathbb{N}\}$ satisfies condition (2.2), then the inequality

$$\|T_nf - f\|_p \leq C_p \sum_{j=0}^{N} \frac{M_j}{M_N} \omega_p \left(1/M_j, f\right) \tag{3.3}$$

holds for $C_p > 0$, depending only on p.

Finally, we state and prove the third main result for non-decreasing sequences, in which we prove a more precise result than that in (3.3) and without restriction (2.2), but only for subsequences.

Theorem 3.3. *Let $f \in L^p(G_m)$, where $1 \leq p < \infty$ and T_k are regular T means generated by a non-decreasing sequence $\{q_k,\ k \in \mathbb{N}\}$. Then, for any $n \in \mathbb{N}$, the following inequality holds:*

$$\begin{aligned} \|T_{M_n}f - f\|_p \quad &\leq \quad R^2 \sum_{j=0}^{n-1} \frac{M_j}{M_n} \omega_p \left(1/M_j, f\right) \\ &+ \quad \frac{2R^4}{q_0} \sum_{j=0}^{n-1} \frac{(n-j) q_{M_{n-M_j}} M_j}{M_n} \omega_p \left(1/M_j, f\right) + \omega_p \left(1/M_n, f\right). \end{aligned} \tag{3.4}$$

We also point out the following generalizations of some results in [18] (in that paper, only the Walsh system was considered):

Corollary 3.1. *Let $\{q_k,\ k \geq 0\}$ be a sequence of non-negative and non-increasing numbers, while in case when the sequence is non-decreasing it is assumed that also condition (2.2) is satisfied. If $f \in Lip(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p < \infty$, then*

$$\|T_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad 0 < \alpha < 1, \\ O(n^{-1}\log n), & \text{if} \quad \alpha = 1, \\ O(n^{-1}), & \text{if} \quad \alpha > 1, \end{cases}$$

Corollary 3.2. *Let $\{q_k,\ k \geq 0\}$ be a sequence of non-negative and non-increasing numbers such that*

$$q_k \sim k^{-\beta} \quad \text{for some} \quad 0 < \beta \leq 1$$

is satisfied.

If $f \in Lip(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p < \infty$, then

$$\|T_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad \alpha + \beta < 1, \\ O(n^{-(1-\beta)}\log n + n^{-\alpha}), & \text{if} \quad \alpha + \beta = 1, \\ O(n^{-(1-\beta)}), & \text{if} \quad \alpha + \beta > 1,\ \beta > 1, \\ O((\log n)^{-1}), & \text{if} \quad \beta = 1. \end{cases}$$

Corollary 3.3. *Let $\{q_k,\ k \geq 0\}$ be a sequence of non-negative and non-increasing numbers such that the equivalence*

$$q_k \sim (\log k)^{-\beta} \quad \text{for some} \quad \beta > 0$$

is satisfied.

If $f \in Lip(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p < \infty$, then

$$\|T_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad 0 < \alpha < 1,\ \beta > 0, \\ O(n^{-1}\log n), & \text{if} \quad \alpha = 1,\ 0 < \beta < 1, \\ O(n^{-1}\log n \log\log n), & \text{if} \quad \alpha = \beta = 1, \\ O(n^{-1}(\log n)^{\beta}), & \text{if} \quad \alpha > 1,\ \beta > 0. \end{cases}$$

Corollary 3.4. *Let $f \in L^p(G_m)$, where $1 \leq p < \infty$ and $\{q_k,\ k \geq 0\}$ is a sequence of non-negative and non-increasing numbers, while in case when the sequence is non-decreasing it is also assumed that condition (2.2) is satisfied. Then,*

$$\lim_{n\to\infty} \|T_n f - f\|_p = 0.$$

4 Proofs

Proof of Theorem 1. Let $M_N \leq n < M_{N+1}$. Since T_n are regular T means generated by a sequence of non-increasing numbers $\{q_k : k \in \mathbb{N}\}$, we can combine (1.8) and (1.9) and conclude that

$$\begin{aligned} \|T_n f - f\|_p &\leq \frac{1}{Q_n}\left(\sum_{j=0}^{n-2}(q_j - q_{j+1})j\|\sigma_j f - f\|_p + q_{n-1}(n-1)\|\sigma_{n-1}f - f\|_p\right) \\ &:= I + II. \end{aligned} \tag{4.1}$$

Moreover,

$$\begin{aligned} I &= \frac{1}{Q_n}\sum_{j=1}^{M_N-1}(q_j - q_{j+1})\, j\|\sigma_j f - f\|_p + \frac{1}{Q_n}\sum_{j=M_N}^{n-1}(q_j - q_{j+1})\, j\|\sigma_j f - f\|_p \\ &:= I_1 + I_2. \end{aligned} \tag{4.2}$$

Now we estimate both terms separately. By applying estimate (2.1) for I_1 we obtain that

$$\begin{aligned} I_1 &\leq \frac{2R^5}{Q_n}\sum_{k=0}^{N-1}\sum_{j=M_k}^{M_{k+1}-1}(q_j - q_{j+1})\, j\sum_{s=0}^{k}\frac{M_s}{M_k}\omega_p\left(1/M_s, f\right) \\ &\leq \frac{2R^6}{Q_n}\sum_{k=0}^{N-1}M_k\sum_{j=M_k}^{M_{k+1}-1}(q_j - q_{j+1})\sum_{s=0}^{k}\frac{M_s}{M_k}\omega_p\left(1/M_s, f\right) \\ &\leq \frac{2R^6}{Q_n}\sum_{k=0}^{N-1}\left(q_{M_k} - q_{M_{k+1}}\right)\sum_{s=0}^{k}M_s\omega_p\left(1/M_s, f\right) \\ &\leq \frac{2R^6}{Q_n}\sum_{s=0}^{N-1}M_s\omega_p\left(1/M_s, f\right)\sum_{k=s}^{N-1}\left(q_{M_k} - q_{M_{k+1}}\right) \\ &\leq \frac{2R^6}{Q_n}\sum_{s=0}^{N-1}M_s q_{M_s}\omega_p\left(1/M_s, f\right). \end{aligned} \tag{4.3}$$

Moreover,

$$\begin{aligned} I_2 &\leq \frac{2R^5}{Q_n}\sum_{j=M_N}^{n-1}(q_j - q_{j+1})\, j\sum_{s=0}^{N}\frac{M_s}{M_N}\omega_p\left(1/M_s, f\right) \\ &\leq \frac{2R^6 M_N}{Q_n}\sum_{j=M_N}^{n-1}(q_j - q_{j+1})\sum_{s=0}^{N}\frac{M_s}{M_N}\omega_p\left(1/M_s, f\right) \\ &\leq \frac{2R^6 q_{M_N}}{Q_n}\sum_{s=0}^{N}M_s\omega_p\left(1/M_s, f\right) \\ &\leq \frac{2R^6}{Q_n}\sum_{s=0}^{N}M_s q_{M_s}\omega_p\left(1/M_s, f\right) \\ &\leq \frac{2R^6}{Q_n}\sum_{s=0}^{N-1}M_s q_{M_s}\omega_p\left(1/M_s, f\right) + 2R^6\omega_p\left(1/M_s, f\right). \end{aligned} \tag{4.4}$$

For II we have that

$$\begin{aligned} II &\leq \frac{2R^5 M_{N+1} q_{n-1}}{Q_n} \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right) \\ &\leq \frac{2R^6}{Q_n} \sum_{s=0}^{N-1} M_s q_{M_s} \omega_p \left(1/M_s, f\right) + 2R^6 \omega_p \left(1/M_N, f\right). \end{aligned} \tag{4.5}$$

The proof of (3.1) is complete by just combining (4.1)-(4.5). □

Proof of Theorem 2. Let $M_N \leq n < M_{N+1}$. Since T_n are regular T means, generated by a sequence of non-decreasing numbers $\{q_k : k \in \mathbb{N}\}$, by combining (1.8) and (1.9), we find that

$$\begin{aligned} \|T_n f - f\|_p &\leq \frac{1}{Q_n} \left(\sum_{j=1}^{n-1} \left(q_{j+1} - q_j\right) j \|\sigma_j f - f\|_p + q_{n-1}(n-1)\|\sigma_n f - f\|_p \right) \\ &:= I + II. \end{aligned} \tag{4.6}$$

Furthermore,

$$\begin{aligned} I &= \frac{1}{Q_n} \sum_{j=1}^{M_N - 1} \left(q_{j+1} - q_j\right) j \|\sigma_j f - f\|_p + \frac{1}{Q_n} \sum_{j=M_N}^{n-1} \left(q_{j+1} - q_j\right) j \|\sigma_j f - f\|_p \\ &:= I_1 + I_2. \end{aligned} \tag{4.7}$$

Analogously to (4.3) we get that

$$\begin{aligned} I_1 &\leq \frac{2R^6}{Q_n} \sum_{k=0}^{N-1} \left(q_{M_{k+1}} - q_{M_k}\right) \sum_{s=0}^{k} M_s \omega_p \left(1/M_s, f\right) \\ &\leq \frac{2R^6}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) \sum_{k=s}^{N-1} \left(q_{M_{k+1}} - q_{M_k}\right) \\ &= \frac{2R^6}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) \left(q_{M_N} - q_{M_s}\right) \\ &\leq \frac{2R^6 q_{M_N}}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) \\ &\leq \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right). \end{aligned} \tag{4.8}$$

In a similar way as in (4.4) we find that

$$I_2 \leq \frac{2R^5}{Q_n} \sum_{j=1}^{n-1} \left(q_{j+1} - q_j\right) j \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right)$$

$$= \frac{2R^5}{Q_n} \left((n-1)q_{n-1} - Q_n\right) \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right)$$

$$\leq \frac{2R^5 M_{N+1} q_{n-1}}{Q_n M_N} \sum_{s=0}^{N} M_s \omega_p \left(1/M_s, f\right)$$

$$\leq \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N} M_s \omega_p \left(1/M_s, f\right)$$

$$\leq \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) + \frac{2R^6 q_{n-1} M_N}{Q_n} \left(1/M_N, f\right). \tag{4.9}$$

For II we have that

$$\begin{aligned} II &\leq \frac{2R^5 q_{n-1} M_{N+1}}{Q_n} \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right) \\ &\leq \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N} M_s \omega_p \left(1/M_s, f\right) \\ &= \frac{2R^6 q_{n-1}}{Q_n} \sum_{s=0}^{N-1} M_s \omega_p \left(1/M_s, f\right) + \frac{2R^6 q_{n-1} M_N}{Q_n} \left(1/M_N, f\right). \end{aligned} \tag{4.10}$$

By combining (4.6)-(4.10) we find that (3.2) holds. Moreover, by using condition (2.2) we obtain estimate (3.3), so the proof is complete. □

Proof of Theorem 3. According to (1.2) we find that

$$T_{M_n} f = D_{M_n} * f - \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k \left(\left(\psi_{M_n-1} \overline{D_k} \right) * f \right).$$

Hence, by using the Abel transformation we get that

$$\begin{aligned} T_{M_n} f &= D_{M_n} * f \\ &- \frac{1}{Q_{M_n}} \sum_{j=0}^{M_n-2} \left(q_{M_n-j} - q_{M_n-j-1} \right) j((\psi_{M_n-1} \overline{K_j}) * f) \\ &- \frac{1}{Q_{M_n}} q_{M_n-1} (M_n - 1)(\psi_{M_n-1} \overline{K}_{M_n-1} * f) \\ &= D_{M_n} * f \\ &- \frac{1}{Q_{M_n}} \sum_{j=0}^{M_n-2} \left(q_{M_n-j} - q_{M_n-j-1} \right) j((\psi_{M_n-1} \overline{K_j}) * f) \\ &- \frac{1}{Q_{M_n}} q_{M_n-1} M_n (\psi_{M_n-1} \overline{K}_{M_n} * f) \\ &+ \frac{q_{M_n-1}}{Q_{M_n}} (\psi_{M_n-1} \overline{D}_{M_n} * f), \end{aligned}$$

so that

$$\begin{aligned} T_{M_n} f(x) - f(x) = &\int_{G_m} (f(x-t) - f(x)) D_{M_n}(t) dt \\ &- \frac{1}{Q_{M_n}} \sum_{j=0}^{M_n-2} \left(q_{M_n-j} - q_{M_n-j-1} \right) j \int_{G_m} \left(f(x-t) - f(x) \right) \psi_{M_n-1}(t) \overline{K}_j(t) dt \\ &- \frac{1}{Q_{M_n}} q_{M_n-1} M_n \int_{G_m} (f(x-t) - f(x)) \psi_{M_n-1}(t) \overline{K}_{M_n}(t) dt \end{aligned}$$

$$+\frac{q_{M_n-1}}{Q_{M_n}}\int_{G_m}(f(x-t)-f(x))\psi_{M_n-1}(t)\overline{D}_{M_n}(t)dt$$

$$=: I+II+III+IV. \tag{4.11}$$

By combining generalized Minkowski's inequality and (1.1) we find that

$$\|I\|_p \leq \int_{I_n}\|f(x-t)-f(x))\|_p D_{M_n}(t)dt \leq \omega_p\left(1/M_n,f\right) \tag{4.12}$$

and

$$\|IV\|_p \leq \int_{I_n}\|f(x-t)-f(x))\|_p D_{M_n}(t)dt \leq \omega_p\left(1/M_n,f\right). \tag{4.13}$$

Moreover, since

$$M_n q_{M_n-1} \leq Q_{M_n}, \quad \text{for any} \quad n\in\mathbb{N},$$

we can use (1.5) and generalized Minkowski's inequality to find that

$$\begin{aligned}
\|III\|_p &\leq \int_{G_m}\|f(x-t)-f(x)\|_p\left|\overline{K}_{M_n}(t)\right|d\mu(t)\\
&= \int_{I_n}\|f(x-t)-f(x)\|_p\left|\overline{K}_{M_n}(t)\right|d\mu(t)\\
&+ \sum_{s=0}^{n-1}\sum_{n_s=1}^{m_s-1}\int_{I_n(n_s e_s)}\|f(x-t)-f(x)\|_p\left|\overline{K}_{M_n}(t)\right|d\mu(t)\\
&\leq \int_{I_n}\|f(x-t)-f(x)\|_p\frac{M_n+1}{2}d\mu(t)\\
&+ \sum_{s=0}^{n-1}M_{s+1}\sum_{n_s=1}^{m_s-1}\int_{I_n(n_s e_s)}\|f(x-t)-f(x)\|_p\,d\mu(t)\\
&\leq \omega_p\left(1/M_n,f\right)\int_{I_n}\frac{M_n+1}{2}d\mu(t)\\
&+ \sum_{s=0}^{n-1}M_{s+1}\sum_{n_s=1}^{m_s-1}\int_{I_n(n_s e_s)}\omega_p\left(1/M_s,f\right)d\mu(t)\\
&\leq \omega_p\left(1/M_n,f\right)+R^2\sum_{s=0}^{n-1}\frac{M_s}{M_n}\omega_p\left(1/M_s,f\right).
\end{aligned} \tag{4.14}$$

From this inequality and the estimates in (4.14) it follows also that

$$M_n\int_{G_m}\|f(x-t)-f(x)\|_p\left|\overline{K}_{M_n}(t)\right|d\mu(t)\leq R^2\sum_{s=0}^{n}M_s\omega_p\left(1/M_s,f\right).$$

Let $M_k\leq j<M_{k+1}$. By applying (1.3) and the last estimate we find that

$$j\int_{G_m}\|f(x-t)-f(x)\|_p\left|\overline{K}_j(t)\right|d\mu(t)\leq 2R^4\sum_{l=0}^{k}\sum_{s=0}^{l}M_s\omega_p\left(1/M_s,f\right).$$

Hence, by also using (1.3) we obtain that

$$
\begin{aligned}
&\|II\|_p \\
\leq\ & \frac{1}{Q_{M_n}} \sum_{j=0}^{M_n-1} \left(q_{M_n-j} - q_{M_n-j-1}\right) j \int_{G_m} \|f(x-t) - f(x)\|_p |\overline{K}_j(t)| d\mu(t) \\
\leq\ & \frac{1}{Q_{M_n}} \sum_{k=0}^{n-1} \sum_{j=M_k}^{M_{k+1}-1} \left(q_{M_n-j} - q_{M_n-j-1}\right) j \int_{G_m} \|f(x-t) - f(x)\|_p |\overline{K}_j(t)| d\mu(t) \\
\leq\ & \frac{2R^4}{Q_{M_n}} \sum_{k=0}^{n-1} \sum_{j=M_k}^{M_{k+1}-1} \left(q_{M_n-j} - q_{M_n-j-1}\right) \sum_{l=0}^{k} \sum_{s=0}^{l} M_s \omega_p \left(1/M_s, f\right) \\
\leq\ & \frac{2R^4}{Q_{M_n}} \sum_{k=0}^{n-1} \left(q_{M_n-M_k} - q_{M_n-M_{k+1}}\right) \sum_{l=0}^{k} \sum_{s=0}^{l} M_s \omega_p \left(1/M_s, f\right) \\
\leq\ & \frac{2R^4}{Q_{M_n}} \sum_{l=0}^{n-1} \sum_{k=l}^{n-1} \left(q_{M_n-M_k} - q_{M_n-M_{k+1}}\right) \sum_{s=0}^{l} M_s \omega_p \left(1/M_s, f\right) \\
\leq\ & \frac{2R^4}{Q_{M_n}} \sum_{l=0}^{n-1} q_{M_n-M_l} \sum_{s=0}^{l} M_s \omega_p \left(1/M_s, f\right) \\
\leq\ & \frac{2R^4}{Q_{M_n}} \sum_{s=0}^{n-1} M_s \omega_p \left(1/M_s, f\right) \sum_{l=s}^{n-1} q_{M_n-M_l} \\
\leq\ & \frac{2R^4}{Q_{M_n}} \sum_{s=0}^{n-1} M_s \omega_p \left(1/M_s, f\right) q_{M_n-M_s} (n-s) \\
\leq\ & 2R^4 \sum_{s=0}^{n-1} \frac{(n-s)M_s}{M_n} \frac{q_{M_n-M_s}}{q_0} \omega_p \left(1/M_s, f\right). \qquad (4.15)
\end{aligned}
$$

Finally, by combining (4.11)-(4.15) and using Minkowski's inequality we obtain (3.4), so the proof is complete. □

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Nino Anakidze, George Tephnadze
School of Science and Technology
The University of Georgia
77a Merab Kostava St,
0171 Tbilisi, Georgia
E-mails: nino.anakidze@mail.ru, g.tephnadze@ug.edu.ge

Nika Areshidze
Department of Mathematics
University of California, Irvine
480 Rowland Hall,
92697 Irvine, CA, USA.
E-mail: nika.areshidze15@gmail.com

Lars-Erik Persson
Department of Mathematics and Computer Science
UiT The Arctic University of Norway
and
Department of Mathematics,
Uppsala University, Uppsala, Sweden.
E-mail: larserik6pers@gmail.com

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