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ON A THEOREM OF MUCHENHOUPT-WHEEDEN IN GENERALIZED MORREY SPACES

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Abstract. In this paper we find a condition on a function w which ensures the equivalence of norms of the Riesz potential and the fractional maximal function in generalized Morrey spaces $\mathcal{M}_{p,w}(\mathbb{R}^n)$.

Recall that $I_\alpha * \mu$ and $M_\alpha \mu$ denote the Riesz potential and the fractional maximal function associated with a non-negative measure μ on $\mathbb{R}^n$, respectively. That is,

$$I_\alpha * \mu(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n,$$

and

$$M_\alpha \mu(x) = \sup_{r>0} r^{\alpha-n} \mu(B(x,r)), \quad 0 \le \alpha < n,$$

where $B(x,r)$ denotes the open ball centered at x of radius r.

If $d\mu(x) = |f(x)|dx$, then $I_\alpha * \mu$ and $M_\alpha \mu$ will be denoted by $I_\alpha f$ and $M_\alpha f$, respectively.

Recall that, for $0 < \alpha < n$,

$$M_\alpha \mu(x) \lesssim I_\alpha * \mu(x) \tag{1}$$

for any $x \in \mathbb{R}^n$.

By $A \lesssim B$ we mean that $A \le cB$ with some positive constant c independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

Let us denote by $L_p^{\mathrm{loc},+}(\mathbb{R}^n)$ the set of all non-negative functions from $L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

The well-known Morrey spaces $\mathcal{M}_{p,\lambda}$ introduced by C. Morrey in 1938 in [6] in connection with the study of partial differential equations, were widely investigated during the last decades, including the study of classical operators of harmonic analysis - maximal, singular and potential operators - in generalizations of these spaces (the so-called Morrey-type spaces).

Definition 1. Let $1 \le p \le \infty$, w be a continuous weight function w defined on $(0,\infty)$. We say that $f \in \mathcal{M}_{p,w}$ if $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p,w}} \equiv \|f\|_{\mathcal{M}_{p,w}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, r>0} w(r) \|f\|_{L_p(B(x,r))} < \infty.$$

If $w(r) = r^{-\lambda/p}$, then $\mathcal{M}_{p,w}$ becomes the classical Morrey space $\mathcal{M}_{p,\lambda}$.

If $\|w(r)\|_{L_\infty(t,\infty)} = \infty$ for all $t > 0$ or $\|w(r)r^{\frac{n}{p}}\|_{L_\infty(0,t)} = \infty$ for all $t > 0$, then the space $\mathcal{M}_{p,w}$ coincides with the set of all functions equivalent to 0 on $\mathbb{R}^n$ (see [3, Lemma 1], for instance).

Definition 2 ([3]). Let $0 < p < \infty$. We denote by $\Omega_{p,\infty}$ the set of all non-negative measurable functions w on $(0,\infty)$ such that for some t_1, $t_2 > 0$,

$$\|w(r)\|_{L_\infty(t_1,\infty)} < \infty, \quad \|w(r)r^{\frac{n}{p}}\|_{L_\infty(0,t_2)} < \infty.$$

In what follows, we always assume that $w \in \Omega_{p,\infty}$.

The goal of the present work is to extend the theorem of B. Muckenhoupt and R.L. Wheeden to the generalized Morrey spaces (see [7]).

Our main result is given in the following theorem.

Theorem 1. *Let $1 < p < \infty$, $0 < \alpha < n$, w be a continuous weight function defined on $(0,\infty)$ such that $w \in \Omega_{p,\infty}$ and define*

$$\psi(x) := \sup_{x<s<\infty} s^{\alpha-n} \sup_{0<\tau<s} w(\tau)\tau^{\frac{n}{p}}. \tag{2}$$

If there exists a constant $c > 0$ such that for any $x > 0$

$$\int_x^\infty t^{\alpha-n-1} \left(\psi(t)\right)^{-1} dt \le c x^{\alpha-n} \left(\psi(x)\right)^{-1}, \tag{3}$$

then

$$\|I_\alpha f\|_{\mathcal{M}_{p,w}} \approx \|M_\alpha f\|_{\mathcal{M}_{p,w}} \tag{4}$$

for all $f \in L_1^{\mathrm{loc},+}(\mathbb{R}^n)$ and the constants in the equivalency do not depend on f.

Note that if the function $w(\tau)\tau^{\frac{n}{p}}$ is non-decreasing and the function $w(\tau)\tau^{\alpha-n+\frac{n}{p}}$ is non-increasing, then $\psi(x) = x^{\alpha-n+\frac{n}{p}} w(x)$.

In order to prove Theorem 1 we need the following statement.

Theorem 2. *Let $\beta > 0$, $\delta > 0$ and w be a continuous weight function defined on $(0,\infty)$. Let $\psi : (0,\infty) \to (0,\infty)$ be defined by*

$$\psi(x) := \sup_{x<s<\infty} s^{-\delta} \sup_{0<\tau<s} w(\tau)\tau^{\beta}.$$

Then the inequality

$$\sup_{r>0} w(r)r^\beta \int_r^\infty \frac{g(t)}{t^\delta} dt \lesssim \sup_{r>0} w(r)r^\beta \left(\sup_{t>r} t^{-\delta} \int_0^t g(s)ds \right). \tag{5}$$

holds for any non-negative measurable functions g on $(0,\infty)$ if and only if

$$\int_x^\infty t^{-\delta-1} \left(\psi(t)\right)^{-1} dt \lesssim x^{-\delta} \left(\psi(x)\right)^{-1}, \quad x > 0. \tag{6}$$

Inequalities of such type were studied in [4]. This theorem is of independent interest.

From Theorem 1, as a special case, follows the result of D.R. Adams and J. Xiao.

Theorem 3 ([2], Theorem 4.2). *Let $1 < p < \infty$, $0 < \alpha < n$, $0 \leq \lambda < n$. If $f \in L_1^{\mathrm{loc},+}(\mathbb{R}^n)$, then*

$$\|I_\alpha f\|_{\mathcal{M}_{p,\lambda}} \approx \|M_\alpha f\|_{\mathcal{M}_{p,\lambda}}. \tag{7}$$

To be more precise, in [2] Theorem 4.2 is formulated for non-negative measures.

Note that there is a gap in the proof of Theorem 3 of [2] as it depends on an incorrect estimate (see (4.8) of [2]).

The correct formulation of estimate (4.8) from [2] is given in the following lemma.

Lemma 1. *Let $1 < p < \infty$ and $0 < \alpha < n$. If μ is a non-negative measure on $\mathbb{R}^n$, then there exists $c > 0$ such that for any cube Q in $\mathbb{R}^n$*

$$\|I_\alpha * \mu\|_{L_p(Q)} \leq c\left(|Q|^{\frac{1}{p}}(I_\alpha * \mu)_Q + \|(I_\alpha * \mu)^{\#}\|_{L_p(Q)}\right). \tag{8}$$

Here $f^{\#}$ is the Fefferman-Stein sharp maximal function of $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy, \tag{9}$$

where the supremum is taken over all cubes Q containing x.

Estimate (4.8) in [2] is (8) but without the first summand in the right hand side.

The source of the error in [2] is in Lemma 4.1 (ii), in the proof of which the restriction $t > 2^{n+1}(I_\alpha * \mu)_Q$ was omitted from the local variant of the Calderón-Zygmund decomposition.

The correct formulation of Lemma 4.1 (ii) in [2] is the following.

Lemma 2. *Let $\alpha \in (0,n)$ and $I_\alpha * \mu \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ for a given non-negative measure μ on $\mathbb{R}^n$. Then, given a cube $Q \subset \mathbb{R}^n$ and numbers $t > 2^{n+1}(I_\alpha * \mu)_Q$, $\varepsilon > 0$,*

$$\begin{aligned} |\{x \in Q : I_\alpha * \mu(x) > t\}| \leq & |\{x \in Q : (I_\alpha * \mu)^{\#}(x) > 2^{-1}\varepsilon t\}| \\ & + \varepsilon|\{x \in Q : I_\alpha * \mu(x) > 2^{-n-1}t\}|. \end{aligned} \tag{10}$$

Note that the restriction $t > 2^{n+1}(I_\alpha * \mu)_Q$ is very important (see, for instance [5, Corollary 2.1.21]).

To avoid the restriction on t, we can formulate Lemma 2 in the following way.

Lemma 3. *Let $\alpha \in (0,n)$ and $I_\alpha * \mu \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ for a given non-negative measure μ on $\mathbb{R}^n$. Then, given a cube $Q \subset \mathbb{R}^n$ and numbers t, $\varepsilon > 0$,*

$$\begin{aligned} |\{x \in Q : |I_\alpha * \mu(x) - (I_\alpha * \mu)_Q| > t\}| \leq & |\{x \in Q : (I_\alpha * \mu)^{\#}(x) > 2^{-1}\varepsilon t\}| \\ & + \varepsilon|\{x \in Q : |I_\alpha * \mu(x) - (I_\alpha * \mu)_Q| > 2^{-n-1}t\}|. \end{aligned} \tag{11}$$

In fact, the main reason to prove Lemma 4.1 in [2] was to obtain the relation between L_p-norms of Riesz potential and fractional maximal function over cubes. Such an estimate is very useful in many applications. Our main two-sided estimate is formulated in the following theorem.

Theorem 4. *Let $0 < \alpha < n$, $1 < p < \infty$ and $f \in L_1^{\mathrm{loc},+}(\mathbb{R}^n)$. Then for any cube $Q = Q(x_0, r_0)$,*

$$\|I_\alpha f\|_{L_p(Q)} \approx \|M_\alpha f\|_{L_p(Q)} + |Q|^{\frac{1}{p}} \int_{\mathbb{R}^n \setminus Q} \frac{f(y)dy}{|y - x_0|^{n-\alpha}}, \tag{12}$$

where the constants in the equivalence do not depend on Q and f.

Remark 1. Since for any function $f \geq 0$ with compact support in $\mathbb{R}^n$ such that $I_\alpha f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$

$$(I_\alpha f)^{\#}(x) \approx M_\alpha f(x), \quad \text{for any} \quad x \in \mathbb{R}^n$$

(see [1, Proposition 3.3 and 3.4] or [2, Lemma 4.1 (i)]), then equivalence (12) could be written in the following form

$$\|I_\alpha f\|_{L_p(Q)} \approx \|(I_\alpha f)^{\#}\|_{L_p(Q)} + |Q|^{\frac{1}{p}} \int_{\mathbb{R}^n \setminus Q} \frac{f(y)dy}{|y - x_0|^{n-\alpha}}. \tag{13}$$

This inequality gave us a hint that inequality (4.8) in [2] is not likely to hold. The next example confirms our doubt: For $0 < r < R/4$, where R is a fixed real number, consider the function

$$f(y) = |y|^{-\alpha} \chi_{B(0,R)\setminus B(0,2r)}(y).$$

It is easy to calculate that

$$\|M_\alpha f\|_{L_p(B(0,r))} \approx r^{\frac{n}{p}}$$

and

$$\|I_\alpha f\|_{L_p(B(0,r))} \approx r^{\frac{n}{p}} \ln \frac{R}{r}.$$

Thus

$$\|I_\alpha f\|_{L_p(B(0,r))} \not\approx \|M_\alpha f\|_{L_p(B(0,r))} \approx \|(I_\alpha f)^{\#}\|_{L_p(B(0,r))}.$$

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