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BINARITY AND $\aleph_0$-CATEGORICITY FOR VARIANTS OF O-MINIMALITY[1]

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Abstract. The present work is a survey paper devoted to studying two variants of o-minimality: weak o-minimality and weak circular minimality (mostly in the $\aleph_0$-categorical case.

1 Introduction

In recent years there have been several approaches to generalizing the notion of *o-minimality*. Typically, for a structure, one imposes strong restrictions on the 1-variable definable sets. An o-minimal structure M can be viewed as an L-structure where $L \supset L_0 = \{<\}$, $<$ is a total order on M, and every definable subset of M is quantifier-free L_0-definable. This provides a template for other notions: replace L_0 by some other familiar language, consider L-structures such that the L_0-reduct is of stipulated type (e.g. a total order), and require that every definable subset of M is (quantifier-free) L_0-definable (one may require this for *all* models of the theory). This route was followed in [19], where notions such as *circularly minimal* and *C-minimal* were proposed and slightly explored. Other notions such as *P-minimal* [10] and *Boolean o-minimal* [24, 22] have since been developed.

In a slightly different direction, a totally ordered structure M is *weakly o-minimal* if every definable subset of M is a finite union of convex sets, and its theory is weakly o-minimal if this holds for all $N \equiv M$. Real closed fields with a proper convex valuation ring [8] provide an important example of weakly o-minimal (non-o-minimal) structures. The notion of weak o-minimality of a linearly ordered structure was introduced by M. Dickmann and originally deeply studied by D. Macpherson, D. Marker and C. Steinhorn in [20]. Some problems posed in [20] have been solved by logicians from Kazakhstan: B.S. Baizhanov [5] has obtained a classification of 1-types over a set of model of a weakly o-minimal theory and solved the problem of expanding a model of a weakly o-minimal theory by a unary convex predicate; R.D. Arefiev [3] has proved

[1]This survey contains results included in the DSc thesis by the author which was successfully defended on December 22, 2009 at the Institute of Mathematics of the Ministry of Education and Science of Republic of Kazakhstan (Scientific advisor: B.S. Baizhanov, opponents: K.Zh. Kudaibergenov, A.A. Stepanova, S.V. Sudoplatov and Novosibirsk State Tecnical University).

the monotonicity property for weakly o-minimal structures; V.V. Verbovskiy [23] has constructed an example of a weakly o-minimal ordered group not having a weakly o-minimal theory.

Here we continue studying the notion of weak o-minimality. The special accent is made on studying the $\aleph_0$-categorical case. As is known $\aleph_0$-categorical weakly o-minimal structures have been deeply studied in [11]: the 1-indiscernible case has been described up to binarity, the 2-indiscernible case has been described up to ternarity, and it has been proved that any 3-indiscernible structure is k-indiscernible for any natural $k \geq 3$. Here we consider questions of interacting types that have not been studied before.

Alongside with it we investigate the notion of weak circular minimality being a variant of o-minimality for circularly ordered sets. A *circular* (or *cyclic*) order relation is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x,y,z) \to K(y,z,x))$;
(co2) $\forall x \forall y \forall z (K(x,y,z) \wedge K(y,x,z) \Leftrightarrow x = y \vee y = z \vee z = x)$;
(co3) $\forall x \forall y \forall z (K(x,y,z) \to \forall t [K(x,y,t) \vee K(t,y,z)])$;
(co4) $\forall x \forall y \forall z (K(x,y,z) \vee K(y,x,z))$.

The following observation relates linear and circular orderings.

Fact 1.1 ([6], Theorem 11.9). *(i) If $\langle M, \leq \rangle$ is a linear ordering and K is the ternary relation derived from $\leq$ by the rule*

$$K(x,y,z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x)$$

then K is a circular order relation on M.

(ii) If $\langle N, K \rangle$ is a circular ordering and $a \in N$, then the relation $\leq_a$ defined on $M := N \setminus \{a\}$ by the rule

$$y \leq_a z :\Leftrightarrow K(a,y,z)$$

is a linear order. Furthermore, if we extend this linear order to the one, denoted by $\leq'$, on N by assuming that $a \leq' b$ for all $b \in M$, then the derived circular order relation is the original circular order K.

Any totally ordered structure carries an $\emptyset$-definable circularly ordered structure, and given a circularly ordered structure, over any parameter there is a definable linear order. There is a very tight connection between weak circular minimality and weak o-minimality. At the same time there are distinctions arising between these notions that involve definability over $\emptyset$, and thus determining independent interest for studying weak circular minimality.

2 Linear case

Let L be a countable first-order language. Everywhere in this section we consider L-structures and assume that L contains a binary relation symbol $<$ that is interpreted as a linear ordering in these structures.

Definition 2.1 ([12]). Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, and let $\phi(x)$ be an M-definable formula.
A rank of convexity for the formula $\phi(x)$ $(RC(\phi(x)))$ is defined as follows:

1) $RC(\phi(x)) \geq 0$ if $M \models \exists x \phi(x)$.
2) $RC(\phi(x)) \geq 1$ if $\phi(M)$ is infinite.
3) $RC(\phi(x)) \geq \alpha + 1$ if there is a parametrically definable equivalence relation $E(x, y)$ such that there are $b_i, i \in \omega$ which satisfy the following:

- For every $i, j \in \omega$, whenever $i \neq j$ then $M \models \neg E(b_i, b_j)$
- For every $i \in \omega$ $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of $\phi(M)$

4) $RC(\phi(x)) \geq \delta$ if $RC(\phi(x)) \geq \alpha$ for all $\alpha \leq \delta$ (δ is limit).

If $RC(\phi(x)) = \alpha$ for some α we say that $RC(\phi(x))$ is defined. Otherwise (i.e. if $RC(\phi(x)) \geq \alpha$ for all α) we put $RC(\phi(x)) = \infty$.

In particular, a theory has convexity rank 1 if there is no definable (with parameters) equivalence relation with infinitely many convex infinite classes. Obviously any o-minimal theory has convexity rank 1.

Example 2.1 ([11]). Let $M_n := \langle Q^n; =, <, E_1^2, E_2^2, \ldots, E_{n-1}^2 \rangle$, where Q^n is the set of n-tuples $x = (x_0, \ldots, x_{n-1})$ of rational numbers, ordered lexicographically by $<$, and for each $i = 1, \ldots, n-1$ let the equivalence relation E_i be given by $E_i(x, y) \Leftrightarrow$ for all $j < n - i$, $x_j = y_j$. Then for each i the equivalence classes of E_i are convex subsets of Q^n. Moreover, E_{i-1} refines E_i for each $2 \leq i \leq n-1$.

In [11] it is proved that $\aleph_0$-categorical 1-indiscernible weakly o-minimal structures are described up to a binary structure by this example. Obviously $Th(M_n)$ has convexity rank n.

Definition 2.2 (B.S. Baizhanov, [4]). Let M be a weakly o-minimal structure, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic.
(1) An A–definable formula $F(x, y)$ is said to be *p–stable* if there are $\alpha, \gamma_1, \gamma_2 \in p(M)$ such that $F(M, \alpha) \setminus \{\alpha\} \neq \emptyset$ and $\gamma_1 < F(M, \alpha) < \gamma_2$.
(2) A p–stable formula $F(x, y)$ is said to be *convex to the right (left)* if there is $\alpha \in p(M)$ such that $F(M, \alpha)$ is convex, α is a left (right) endpoint of $F(M, \alpha)$ and $\alpha \in F(M, \alpha)$.

In Example 2.1 $E_i(x, y)$ for each $1 \leq i \leq n-1$ is p–stable, where $p(x) := \{x = x\}$; $F_i(x, y) := E_i(x, y) \wedge y \leq x$ and $F_i'(x, y) := E_i(x, y) \wedge y \geq x$ are p–stable convex to the right and convex to the left formulas respectively.

Let $F(x, y)$ be a p–stable convex to the right (left) formula. We say $F(x, y)$ is *equivalence-generating* if for any $\alpha, \beta \in p(M)$ such that $M \models F(\beta, \alpha)$ the following holds:

$$M \models \forall x [x \geq \beta \rightarrow [F(x, \alpha) \leftrightarrow F(x, \beta)]]$$

$$(M \models \forall x [x \leq \beta \rightarrow [F(x, \alpha) \leftrightarrow F(x, \beta)]])$$

Obviously the above mentioned formulas $F_i(x, y)$ and $F_i'(x, y)$ are equivalence-generating.

Example 2.2. Let $M = \langle Q, =, <, R^2 \rangle$. M is a linearly ordered structure, Q is the ordering of rational numbers, for any $a, b \in M \quad M \models R(b,a) \Leftrightarrow a \leq b < a + \sqrt{2}$ and consequently $R(M,a) = \{b \in M | a \leq b < a + \sqrt{2}\}$ and $R(a,M) = \{b \in M | a - \sqrt{2} < b \leq a\}$.

For each $n < \omega$ consider the following formulas:

$$R^n(x,y) := \exists z_1, \ldots, z_n [R(z_1,y) \wedge \wedge_{1 \leq i < n} R(z_{i+1}, z_i) \wedge R(x, z_n)]$$

One can see that for any $a, b \in M$ $R^n(b,a) \Leftrightarrow a \leq b < (n+1)(a + \sqrt{2})$. Consequently, for any $a \in M$ we have $R(M,a) \subset R^1(M,a) \subset \ldots \subset R^n(M,a) \subset \ldots$ i.e. $Th(M)$ is not $\aleph_0$–categorical. The formulas $R^n(x,y)$ for each $n < \omega$ including in addition atomic formulas we declare to be basic. It can now be shown by standard arguments that $Th(M)$ admits elimination of quantifiers relative to these basic formulas, and consequently M is weakly o-minimal. Let $p(x) := \{x = x\}$. It is easy to see that $p(x) \in S_1(\emptyset)$, $R(x,y)$ is p-stable convex to the right and $R(x,y)$ is not equivalence-generating.

The following theorem is a characterization of behaviour of p-stable convex to the right (to the left) formulas ordered by type ω^* (the reverse ordering on the natural numbers). As is known, in the o-minimal case any such formula is the graph of a strictly increasing function and consequently the set of such 2-formulas cannot be ordered by ω^*. In the weakly o-minimal case any such formula generates an equivalence relation partitioning the set of realizations of 1-type into infinite convex classes.

Theorem 2.1. *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, M be $|A|^+$–saturated, $p \in S_1(A)$ be non-algebraic. Suppose that the set of all p-stable convex to the right formulas is ordered by ω^*. Then any p-stable convex to the right (left) formula is equivalence-generating.*

Definition 2.3 (B.S. Baizhanov, [5]). Let M be a weakly o-minimal structure, $A \subseteq M$, $p, q \in S_1(A)$ be non-algebraic. We say that p is not *weakly orthogonal* to q ($p \not\perp^w q$) if there are A-definable formula $H(x,y)$, $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M,\alpha)$ and $\beta_2 \notin H(M,\alpha)$.

Lemma 2.1 ([5], Corollary 34 (iii)). *A non-weak orthogonality relation is an equivalence relation on $S_1(A)$.*

We say that p is not *quite orthogonal* to q ($p \not\perp^q q$) if there is an A-definable bijection $f : p(M) \to q(M)$. We say that a weakly o-minimal theory is *quite o-minimal* if the notions of weak and quite orthogonality of 1-types coincide. Obviously, any o-minimal theory is quite o-minimal. An example of a quite o-minimal (non-o-minimal) theory is the field of algebraic numbers expanded by an unary predicate $(-\alpha, \alpha)$ where α is an arbitrary real transcendent number.

Fact 2.1. *A non-quite orthogonality relation is an equivalence relation on $S_1(A)$.*

Example 2.3 ([20]). Let M be the structure $\langle M, <, P^1, f^1 \rangle$. Here P is a unary predicate and f is a unary function with $Dom(f) = \neg P, Ran(f) = P$ (therefore, formally, M is 2-sorted). The universe of the structure M is a disjoint union of P and $\neg P$, where $x < y$ whenever $x \in P$ and $y \in \neg P$. To define f identify P with Q (where Q is the order of rational numbers) and $\neg P$ c $Q \times Q$ (which is lexicographically ordered),and for any $m, n \in Q$ let $f(m, n) = n$.

It is not difficult to prove that $Th(M)$ is a weakly o-minimal theory. Let $p(x) := \{\neg P\}$, $q(x) := \{P\}$. Obviously $p, q \in S_1(\emptyset)$, $p \not\perp^a q$, but $p \perp^q q$, i.e. $Th(M)$ is not quite o-minimal. Observe that the Exchange Principle for algebraic closure does not hold in M.

In the following theorem it is stated that the Exchange Principle for algebraic closure holds in quite o-minimal theories of finite convexity rank. A complete description of $\aleph_0$–categorical quite o-minimal theories will be later presented which implies their binarity (Theorem 2.6). Observe that the Exchange Principle for algebraic closure holds in any o-minimal theory, and $\aleph_0$–categorical o-minimal theories are binary. All these results testify that quite o-minimal theories "quite" inherit many properties of o-minimal theories.

Theorem 2.2. *Let T be a quite o-minimal theory of finite convexity rank. Then the Exchange Principle for algebraic closure holds in every model of T.*

We say an n-tuple $\bar{a} = \langle a_1, a_2, \ldots, a_n\rangle \in M^n$ is *increasing* if $a_1 < a_2 < \ldots < a_n$. Let $A \subseteq M$, $p \in S_1(A)$ be non-algebraic, $n \in \omega$. We say $p(M)$ is *n–indiscernible over A* if for any increasing n–tuples $\bar{a} = \langle a_1, a_2, \ldots, a_n\rangle, \bar{a}' = \langle a'_1, a'_2, \ldots, a'_n\rangle \in [p(M)]^n$ $tp(\bar{a}/A) = tp(\bar{a}'/A)$; also we say $p(M)$ is *indiscernible over A* if for every $n \in \omega$ $p(M)$ is n–indiscernible over A. Let $A \subseteq M$, A be finite, $p_1, p_2, \ldots, p_s \in S_1(A)$ be non-algebraic. We say that the family of 1-types $\{p_1, \ldots, p_s\}$ is *orthogonal over A* if for any sequence $(n_1, \ldots, n_s) \in \omega^s$ for any increasing tuples $\bar{a}_1, \bar{a}'_1 \in [p_1(M)]^{n_1}$, …, $\bar{a}_s, \bar{a}'_s \in [p_s(M)]^{n_s}$ such that $tp(\bar{a}_1/A) = tp(\bar{a}'_1/A)$, …, $tp(\bar{a}_s/A) = tp(\bar{a}'_s/A)$ we have $tp(\langle\bar{a}_1, \ldots, \bar{a}_s\rangle/A) = tp(\langle\bar{a}'_1, \ldots, \bar{a}'_s\rangle/A)$.

Orthogonality of families of pairwise weakly orthogonal non-algebraic 1-types for $\aleph_0$-categorical o-minimal theories was proved in [21]. However it isn't true for $\aleph_0$–categorical weakly o-minimal theories in general. We present an example of an $\aleph_0$–categorical weakly o-minimal theory of infinite convexity rank in which the condition of orthogonality of two weakly orthogonal non-algebraic 1-types fails (Example 2.4). The following theorem proves orthogonality for $\aleph_0$–categorical weakly o-minimal theories of finite convexity rank:

Theorem 2.3 ([13]). *Let T be an $\aleph_0$–categorical weakly o-minimal theory of finite convexity rank, $M \models T$, $p_1, p_2, \ldots, p_s \in S_1(\emptyset)$ be non-algebraic pairwise weakly orthogonal 1–types. Then $\{p_1, p_2, \ldots, p_s\}$ is orthogonal over $\emptyset$.*

Example 2.4. Let $M = \langle Q \cup W, <, E^3, P^1\rangle$ be a linearly ordered structure, where Q is the set of rational numbers, W is the set of all Q-sequences from $\{0, 1\}$ with finitely many non-zero coordinates ordered lexicographically, $P(M) = Q$, $\neg P(M) = W$ and $P(M) < \neg P(M)$. For any $a \in P(M)$ $E(a, y_1, y_2)$ is an equivalence relation on $\neg P(M)$ defined as follows: for any $a \in P(M)$, $b_1, b_2 \in \neg P(M)$ $E(a, b_1, b_2) \Leftrightarrow b_1(q) = b_2(q)$ for all $q \le a$, i.e. q–th coordinates of b_1 and b_2 coincide for all $q \le a$.

It can be proved M is an $\aleph_0$–categorical weakly o-minimal structure. Obviously if $a < a' \in P(M)$ we have $E(a', x_1, x_2)$ implies $E(a, x_1, x_2)$ and consequently $Th(M)$ has an infinite convexity rank. Let $p_1 := \{P(x)\}$, $p_2 := \{\neg P(x)\}$. It isn't difficult to see $p_1 \perp^w p_2$. Consider arbitrary $a, a' \in p_1(M)$, $b_1 < b_2$, $b'_1 < b'_2 \in p_2(M)$ with $a < a'$, $E(a, b_1, b_2)$ and $\neg E(a', b'_1, b'_2)$. Then $tp(\langle a, b_1, b_2\rangle/\emptyset) \neq tp(\langle a', b'_1, b'_2\rangle/\emptyset)$, although $tp(\langle b_1, b_2\rangle/\emptyset) = tp(\langle b'_1, b'_2\rangle/\emptyset)$.

Recall that a complete theory is *binary* if any formula is equivalent to a Boolean combination of formulas in at most two free variables. A. Pillay and C. Steinhorn have described all $\aleph_0$–categorical o-minimal theories [21]. Their description implies binarity for these theories. However $\aleph_0$–categorical weakly o-minimal theories aren't binary in general (Example 2.4). The following theorem is a criterion for binarity of $\aleph_0$–categorical weakly o-minimal theories:

Theorem 2.4 ([13]). *Let T be an $\aleph_0$–categorical weakly o-minimal theory. Then T is binary if and only if T has finite convexity rank.*

This criterion allows us in particular to describe the following two subclasses of the class of $\aleph_0$–categorical weakly o-minimal theories. These theorems generalize the result of A. Pillay and C. Steinhorn in the o-minimal case.

Theorem 2.5 ([14]). *Let T be an $\aleph_0$–categorical weakly o-minimal theory of convexity rank 1, $M \models T$, $|M| = \aleph_0$. Then there exist*
(i) a finite $C = \{c_0, \ldots, c_n\} \subseteq M$ ($M \cup \{-\infty, +\infty\}$, if M does not have a first or last element), consisting of all of the $\emptyset$–definable elements in M (with the possible exceptions of $-\infty, +\infty$), such that $M \models c_i < c_j$ for all $i < j \leq n$ and for each $j \in \{1, \ldots, n\}$ either $M \models \neg(\exists x) c_{j-1} < x < c_j$ or $I_j = \{\ x \in M : M \models c_{j-1} < x < c_j\ \}$ is a dense linear order without endpoints and there are $k_j \in \omega$ and $p^j_1, \ldots, p^j_{k_j} \in S_1(\emptyset)$ so that $I_j = \bigcup_{s=1}^{k_j} p^j_s(M)$;
(ii) equivalence relations $E_1,\ E_2 \subseteq (\{s : 1 \leq s \leq k\})^2$, where $\{p_s \mid s \leq k < \omega\}$ is an arbitrary enumeration of all non-algebraic 1-types over $\emptyset$, such that

- *for each $(i, j) \in E_1$ there is a unique $\emptyset$–definable monotonic bijection $f_{i,j} : p_i(M) \to p_j(M)$ so that $f_{i,i} = id_{p_i(M)}$ and $f_{j,k} \circ f_{i,j} = f_{i,k}$ for all (i, j), $(j, k) \in E_1$;*
- *for each $(i, j) \in E_2$ there is a unique $\emptyset$–definable formula $R_{i,j}(x, y)$ such that for any $a \in p_i(M)$ $R_{i,j}(a, M) \subset p_j(M)$, $R_{i,j}(a, M)^- = p_j(M)^-$, $R_{i,j}(a, M)$ is convex and open and $g_{i,j}(x) := sup\ R_{i,j}(x, M)$ is strictly monotonic on $p_i(M)$*
- *for each $(i, j) \in E_1$ we have $(i, j) \in E_2$ and $R_{i,j}(x, y) \equiv y < f_{i,j}(x)$*

so that T admits elimination of quantifiers down to the language $\{=, <\} \bigcup \{c_i : i \leq n\} \bigcup \{U_s : s \leq k\} \bigcup \{f_{i,j} : (i, j) \in E_1\} \bigcup \{R_{i,j} : (i, j) \in E_2 \setminus E_1\}$, where U_s isolates p_s for each $s \leq k$.
Moreover to any ordering with distinguished elements as in (i) and any suitable equivalence relations E_1, E_2 as in (ii), there corresponds an $\aleph_0$–categorical weakly o-minimal theory of convexity rank 1.

Recall that *convexity rank for an one-type* p ($RC(p)$) is an infimum of the set $\{RC(\phi(x))|\ \phi(x) \in p\}$. The following theorem completely describes $\aleph_0$–categorical quite o-minimal theories:

Theorem 2.6. *Let T be an $\aleph_0$–categorical quite o-minimal theory, $M \models T$, $|M| = \aleph_0$. Then there exist*
(i) a finite $C = \{c_0, \dots, c_n\} \subseteq M$ ($M \cup \{-\infty, +\infty\}$, if M does not have a first or last element), consisting of all of the $\emptyset$–definable elements in M (with the possible exceptions of $-\infty, +\infty$), such that $M \models c_i < c_j$ for all $i < j \leq n$ and for each $j \in \{1, \dots, n\}$ either $M \models \neg(\exists x)c_{j-1} < x < c_j$ or $I_j = \{\ x \in M : M \models c_{j-1} < x < c_j\ \}$ is a dense linear order without endpoints and there are $k_j \in \omega$ and $p_1^j, \dots, p_{k_j}^j \in S_1(\emptyset)$ so that $I_j = \bigcup_{s=1}^{k_j} p_s^j(M)$;
(ii) for every non-algebraic type $p \in S_1(\emptyset)$ there is $n_p \in \omega$ such that $RC(p) = n_p$, i.e. there exist $\emptyset$-definable equivalence relations $E_1^p(x, y)$, ..., $E_{n_p-1}^p(x, y)$ such that

- *$E_{n_p-1}^p$ partitions $p(M)$ into infinitely many $E_{n_p-1}^p$-classes, every $E_{n_p-1}^p$-class is convex and open so that the induced order on classes is a dense linear order without endpoints*
- *for each $i \in \{1, \dots, n_p - 2\}$ E_i^p partitions every E_{i+1}^p-class into infinitely many E_i^p-classes, every E_i^p-class is convex and open so that E_i^p-subclasses of each E_{i+1}^p-class are densely ordered without endpoints*

(iii) there exists an equivalence relation $\varepsilon \subseteq (\{s : 1 \leq s \leq k\})^2$ where $\{p_s \mid s \leq k < \omega\}$ is an arbitrary enumeration of all non-algebraic 1-types over $\emptyset$ such that for every $(i, j) \in \varepsilon$ there is a unique $\emptyset$–definable locally monotonic bijection $f_{i,j} : p_i(M) \to p_j(M)$ so that $RC(p_i) = RC(p_j)$, $f_{i,i} = id_{p_i(M)}$ and $f_{j,l} \circ f_{i,j} = f_{i,l}$ for all (i, j), $(j, l) \in \varepsilon$

so that T admits elimination of quantifiers down to the language $\{=, <\} \bigcup \{c_i : i \leq n\} \bigcup \{U_s : s \leq k\} \bigcup \{f_{i,j} : (i, j) \in \varepsilon\}$, where U_s isolates p_s for each $s \leq k$.
Moreover to any ordering with distinguished elements as in (i)-(ii) and any suitable equivalence relations ε as in (iii), there corresponds an $\aleph_0$–categorical quite o-minimal theory.

3 Circular case

Let L be a countable first-order language. Everywhere in this section we consider L-structures and assume that L contains a ternary relation symbol K that is interpreted as a circular ordering in these structures.

A set $A \subseteq M$ is said to be *convex* if for any $a, b \in A$ the following holds: for any $c \in M$ with $K(a, c, b)$ we have $c \in A$ or for any $c \in M$ with $K(b, c, a)$ we have $c \in A$. If $M = \langle M, \leq, \dots \rangle$ be a linearly ordered structure, we denote by $c(M)$ the structure $\langle M, K, \dots \rangle$ where we replace the linear order $\leq$ by a ternary relation K, which is derived from $\leq$.

Definition 3.1 ([19], [15]). A circularly ordered structure M is said to be *circularly minimal (weakly circularly minimal)* if any definable (with parameters) subset of M is a finite union of intervals and points (convex sets) in M.

The next two examples indicate some issues, concerning parameters, on which minimality in the linear and circular cases have a different behaviour.

Example 3.1. Let $M = c(\omega + \omega^* + Q + \omega + \omega^* + Q)$ where ω is the ordering of the natural numbers, ω^* the reverse ordering on the natural numbers and Q the ordering of the rational numbers. Then M is a circularly minimal structure. Denote the first elements of the copies of $\omega + \omega^*$ by a and c, and the last elements by b and d. Consider the following formulas:

$$K_0(x, y, z) := K(x, y, z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$$

$$\phi(x) := \forall y \forall z [K_0(y, x, z) \rightarrow \exists t_1 \exists t_2 (K_0(y, t_1, x) \wedge K_0(x, t_2, z))]$$

We have $\phi(M) = \{x \in M | M \models K_0(b, x, c) \vee K_0(d, x, a)\}$, i.e. $\phi(M)$ is a union of two open intervals, but we see the endpoints of these intervals are not definable over $\emptyset$: $a, b, c, d \in \mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)$. This is in contrast with the o-minimal case where $\mathrm{acl}(A) = \mathrm{dcl}(A)$ for all $A \subseteq M$. Also, in this example, $\phi(M)$ is the set of realisations of a complete type, but is not convex; this is not possible in weakly o-minimal structures.

Example 3.2. Let $M = \langle c(Q_1 + Q_2 + Q_3 + Q_4), P\rangle$ where Q_i is a copy of the ordering of rational numbers for each $i \leq 4$, and $P(M) = Q_1 \cup Q_3$. Then M is a weakly circularly minimal structure. It is easy to see $P(M)$ is a union of two convex sets which are not intervals and points. Therefore, M is not circularly minimal. Also, $P(M)$ is the union of of two convex sets neither of which is $\emptyset$-definable. This is in contrast with the weakly o-minimal case where every parametrically definable set is a finite union of convex sets definable over the same parameters.

A *cut* in a circularly ordered structure M is a maximal consistent set of formulas over M of the form $K(a, x, b)$, where $a, b \in M$. We will say a cut is *algebraic* if there is $c \in M$ which realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for any $b \in M$, $K(a, x, b) \in C(x)$ or for any $b \in M$, $K(b, x, a) \in C(x)$ then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*.

In [21], a criterion is given for o-minimality of a linearly ordered structure in terms of cuts and one-types, and in [12] there is given a criterion for weak o-minimality of a linearly ordered structure in terms of realisations of one-types. As noted in Example 3.1, in a circularly minimal structure the set of realisations of a complete type over $\emptyset$ may not be convex. This is in contrast with the weakly *o-minimal* case where the set of realisations of any $p \in S_1(A)$ with $A \subseteq M$ is convex in any elementary extension of M. Nevertheless, the following is an analogue of Theorem 3.1 of [12].

Theorem 3.1 ([15]). *Let M be a circularly ordered structure. Then the following conditions are equivalent:*
(1) M is weakly circularly minimal;
(2) for any non-empty $A \subseteq M$ the set of realizations of any complete 1-type over A is a convex set in any elementary extension of M;
(3) the set of realizations of any complete 1-type over M is a convex set in any elementary extension of M.

Definition 3.2. Let M be a circularly ordered structure.

(i) Let $p \in S_1(\emptyset)$. We say p is *n-convex* if for any elementary extension N of M, $p(N)$ is the disjoint union of n maximal convex sets (which are called the *convex components* of $p(N)$). We say p is *convex* if p is 1-convex. Otherwise, we say p is *non-convex*.

(ii) We say M is *n-convex* if every type $p \in S_1(\emptyset)$ is n-convex, and we say $Th(M)$ is *n-convex* if this holds for all $N \equiv M$.

Theorem 3.2 ([15]). *Let M be a weakly circularly minimal structure. Then there is $n < \omega$ such that M is n-convex.*

In particular, if M is weakly circularly minimal and $p \in S_1(\emptyset)$, then $p(M)$ is a finite union of convex sets.

In [21] there is a characterisation of all o-minimal linear orderings in the signature $\{<\}$, and in [12] there is a characterisation of all weakly o-minimal orderings in the same signature. Here we present our characterisation of weakly circularly minimal orderings in the signature $\{K\}$.

Let F be the set of all finite linear orderings, and

$$G := F \cup \{\omega, \omega^*, \omega + \omega^*, \omega^* + \omega, Q\}.$$

Here, as usual, ω represents the ordering of the natural numbers, ω^* its reverse, and Q is the ordering of the rationals. Also, let WCO be the collection of all circularly ordered sums of the form $c(M_1 + \ldots + M_m)$, where each M_i is elementarily equivalent to some member of G.

Theorem 3.3 ([15]). *Any weakly circularly minimal structure M restricted to the signature $\{K\}$ is a member of WCO, and conversely, the first-order theory of any member of WCO is a weakly circularly minimal theory.*

Observe that if $\langle M, < \rangle$ is an o-minimal ordering then $\langle M, K \rangle$ is not a circularly minimal ordering in general. Nevertheless, we can present a slightly different characterization of circularly minimal orderings. Let CO be the collection of all circularly ordered sums of the form $c(M_1 + \ldots + M_m)$, where M_i is elementarily equivalent to some member of G for each $i \leq m$, and for each $i \leq m-1$ if M_i does not have a last element then M_{i+1} has a first element and if M_m does not have a last element then M_1 has a first element.

Theorem 3.4 ([15]). *Any circularly minimal structure M restricted to the signature $\{K\}$ is a member of CO, and conversely, the first-order theory of any member of CO is a circularly minimal theory of circular order.*

We further investigate unary definable functions from a weakly circularly minimal structure M not just to M, but to a definable completion of M (as done in [20], Section 3, in the weakly o-minimal setting).

The formalism is as follows. Let M be weakly circularly minimal. A *definable cut* in M is a cut $C(x)$ with the following property: there are $a, b \in M$ such that $K(a, x, b) \in C(x)$ and $\{y \in M : K(a, y, b) \text{ and } K(a, x, y) \in C(x)\}$ is definable. The *definable completion* $\overline{M}$ of M consists of M together with all definable cuts of M which are *irrational*. There is a natural way to extend the circular ordering K to $\overline{M}$ which we do not give explicitly. Observe that $\overline{M}$ is essentially a union of certain sorts of M^{eq}. We shall consider definable (strictly speaking, *interpretable*) partial functions $M \to \overline{M}$.

Let f be a unary function to $\overline{M}$ with $\mathrm{Dom}(f) = I \subseteq M$ where I is an open convex set. We say f is *monotonic* on I if it either preserves or reverses the relation K_0, i.e. either for any $a, b, c \in I$ such that $K_0(a, b, c)$ we have $K_0(f(a), f(b), f(c))$ or for any $a, b, c \in I$ such that $K_0(a, b, c)$ we have $K_0(f(c), f(b), f(a))$. In particular, we say f is *monotonic-to-right (left)* on I if f preserves (reverses) the relation K_0.

Finally, if $\langle M, K\rangle$ is circularly ordered, then the set of open intervals is the basis of a topology on M. If I is convex, we write $\mathrm{Int}(I)$ for the interior of I with respect to this topology.

Let f be a unary function to $\overline{M}$ with $\mathrm{Dom}(f) = I \subseteq M$ where I is an open convex set. We say f is *locally monotonic-to-right (locally monotonic-to-left, locally constant)* on I if for all $x \in I$ there is an open convex set $J \subseteq I$ such that $x \in \mathrm{Int}(J)$ and f is monotonic-to-right (monotonic-to-left, constant) on J.

We say that a weakly circularly minimal structure M *has monotonicity* if whenever $A \subseteq M$ and f is an A-definable unary function to $\overline{M}$, there are some $m < \omega$ and a partition of $\mathrm{Dom}(f)$ into sets $X, I_1, \ldots, I_m$ such that X is finite, each I_i is open and convex, and on each set I_i the function f is locally monotonic or locally constant.

Theorem 3.5 ([15]). *Any weakly circularly minimal structure has monotonicity.*

We also have investigated weakly circularly minimal (ordered) groups and proved their dense ordering. As it has been earlier proved that both weakly o-minimal groups and circularly minimal ones are divisible. Here we present an example of weakly circularly minimal group which is not divisible. By Lemma 5.2 [20] any definable subgroup of a weakly o-minimal group is convex. However there are non-convex subgroups in the circular case. For example, in the weakly circularly minimal group $G = \langle \{z \in C | |z| = 1\}, =, K, *, 1\rangle$ the set $H := \{1, -1\}$ is a finite non-convex $\emptyset$–definable subgroup.

Proposition 3.1. *There is a weakly circularly minimal group that is abelian and non-divisible.*

Proof of Proposition 3.1. Consider the multiplication group $G' := \langle R', =, *, 1\rangle$ where $R' := R \setminus \{0\}$, R is the set of real numbers. It is not difficult to see that G' is abelian and -1 is the only non-unity element of finite order (all the rest non-unity elements have an infinite order). Let's order group circularly as follows. Let R_+ be the set of positive real numbers, i.e. $R_+ = \{a \in R \mid a > 0\}$, R_- be the set of negative real numbers, i.e. $R_- = \{a \in R \mid a < 0\}$. Let R_-^* be the reverse order on the set of negative real numbers. Then let $G := \langle c(R_+ + R_-^*), =, K, *, 1\rangle$. We assert that G is the required group. □

It has been proved earlier that any circularly minimal group doesn't contain proper infinite definable subgroups (Claim 5.1.1, [19]). However for a weakly circularly minimal group it is not true in general:

Proposition 3.2. *There is a weakly circluarly minimal group having an infinite non-convex $\emptyset$–definable subgroup.*

Proof of Proposition 3.2. Consider the set $R' := R \setminus \{0\}$ where R is the set of real numbers. Let $R_+ := \{a \in R' \mid a > 0\}$, $R_- := \{a \in R' \mid a < 0\}$, and i be the imaginary unit of the field of complex numbers, i.e. $i^2 = -1$. Consider $iR_+ := \{ir \mid r \in R_+\}$, $iR_- := \{ir \mid r \in R_-\}$, and for this iR_+ is ordered as follows: for any $r_1, r_2 \in R_+$ $ir_1 < ir_2 \Leftrightarrow r_1 < r_2$. iR_- is ordered similarly. Let R^*_- and iR^*_- denote the reverse orderings on R_- and iR_- respectively. Let $G := \langle c(R_+ + iR^*_- + R^*_- + iR_+), =, K, *, 1\rangle$. We assert that G is the required group. □

Let M be a circularly ordered structure and $G := \mathrm{Aut}(M)$. We say M is *k-homogeneous*, where $k \in \omega$, if for any two k-element sets $A, B \subseteq M$ there is $g \in G$ with $g(A) = B$; also, M is called *highly homogeneous* if it is k-homogeneous for all $k \in N$. We say M is *k-transitive* if for distinct $a_1, a_2, \ldots, a_k$ and distinct $b_1, b_2, \ldots, b_k$ there is $g \in G$ with $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. By a *congruence* on M we mean a G-invariant equivalence relation on M. We say M is *primitive* if it is 1-transitive and there are no non-trivial proper congruences on M.

Obviously the notions of 1-homogeneity and 1-transitivity coincide. These notions also coincide with the notion of 1-indiscernibility (ordered indiscernibility) introduced for linearly ordered structures. Also obviously that if M is n-transitive circularly ordered structure then $n \leq 2$.

In [11] it is proved that $\aleph_0$-categorical 1-indiscernible weakly o-minimal structures are described up to binary structure by Example 2.1. However, in the circular case we have a greater richness of examples, even under a primitivity assumption. For the $\aleph_0$-categorical weakly circularly minimal case we present descriptions of 1-transitive theories up to binarity with partition into the following classes: 2-homogeneous non-2-transitive (Theorem 3.6), primitive non-2-homogeneous (Theorem 3.7) and non-primitive (Theorems 3.12–3.15) and 2-transitive theories up to quaternarity (Theorems 3.8 and 3.9).

Example 3.3 ([9], [7]). Let n be a positive integer with $n \geq 2$, and let $L = \{\sigma_0, \ldots, \sigma_{n-1}\}$ where $\sigma_0, \ldots, \sigma_{n-1}$ are binary relation symbols.
Let Q^*_n be a structure $\langle Q_n, K, L\rangle$ such that
i) its domain Q_n is a countable dense subset of the unit circle, no two points making an angle of $2\pi k/n$ at the centre, where k ranges over integers, and
(ii) for distinct $x, y \in Q_n$, $(x, y) \in \sigma_i \Leftrightarrow 2\pi i/n < \arg(x/y) < 2\pi(i + 1)/n$.

It can be proved that Q^*_n is an $\aleph_0$-categorical primitive weakly circularly minimal structure. The structure Q^*_2 is essentially the *countable homogeneous local order*, or *circular tournament*, discussed for example in [7] and [18].

Let M, N be circularly ordered structures. By *2–reduct* of M we mean a circularly ordered structure with the same domain as M, and having a relation symbol for each $\emptyset$–definable relation of M of arity at most 2 and also a ternary relation symbol K for circular ordering, but no other relation symbols of higher arity. We say M is isomorphic to N *up to binarity* if the 2–reduct of M is isomorphic to N.

The following two theorems describe $\aleph_0$–categorical primitive weakly circularly minimal structures up to binarity:

Theorem 3.6 ([15]). *Let M be an $\aleph_0$-categorical weakly circularly minimal structure such that $\mathrm{Aut}(M)$ is 2-homogeneous but not 2-transitive. Then either there is an $\emptyset$-definable linear order on M so that M is 2-indiscernible weakly o-minimal under this order, or M is isomorphic to Q_2^* up to binarity.*

Theorem 3.7 ([15]). *Let M be an $\aleph_0$-categorical weakly circularly minimal structure such that $\mathrm{Aut}(M)$ is primitive but not 2-homogeneous. Then there is some natural $n \geq 3$ such that M is isomorphic to Q_n^* up to binarity.*

Further we consider the 2-transitive case by the notions of C-relation and D-relation investigated in detail in [2], [1].

Definition 3.3. (1) A ternary relation $C(x; y, z)$ on a set X is a *C-relation* if it satisfies the following:

(C1) $\forall x \forall y \forall z[C(x; y, z) \to C(x; z, y)]$;

(C2) $\forall x \forall y \forall z[C(x; y, z) \to \neg C(y; x, z)]$;

(C3) $\forall x \forall y \forall z \forall w[(C(x; y, z) \wedge \neg C(w; y, z)) \to C(x; w, z)]$;

(C4) $\forall x \forall y[x \neq y \to \exists z(z \neq y \wedge C(x; y, z))]$;

(C5) $\forall y \forall z \exists x C(x; y, z)$.

(2) A quaternary relation $D(x, y; z, w)$ on X is a *D-relation* if we have:

(D1) $\forall x \forall y \forall z \forall w[D(x, y; z, w) \to (D(y, x; z, w) \wedge D(z, w; y, x))]$;

(D2) the restriction of D to its last three arguments is a C-relation; that is, if we fix x and define on $X \setminus \{x\}$ the relation $E(y; z, w)$ to hold if and only if $D(x, y; z, w)$, then E satisfies (C1)-(C5).

Theorem 3.8 ([15]). *Let M be $\aleph_0$-categorical 2-transitive weakly circularly minimal structure. Then M is 3-homogeneous.*

We now investigate the possible 4-ary relations in the 3-homogeneous 2-transitive case. We recall that in Section 4 of [11], a complete structure theory (up to ternary relations) is given for $\aleph_0$-categorical weakly o-minimal structures which are 2-indiscernible. Essentially, any example consists of a 'nested' family of C-relations, as described in Lemma 4.2 of [11]. Let M_n be an $\aleph_0$-categorical 2-indiscernible weakly o-minimal structure with n 3-types of strictly increasing elements where $n > 1$. The structure M_n can be parsed as having just the relation of linear ordering $<$ and finitely many C-relations $C_1, \ldots, C_m$. We define for each n an $\aleph_0$-categorical weakly circularly minimal structure P_n as follows: the structure P_n has domain $M_n \cup \{\alpha\}$ where $\alpha < x$ for all x in M_n. Replace the relation of linear ordering $<$ by the relation of circular ordering K derived from $<$ in the natural way. In addition, for each relation C_i of M_n, there is a quaternary relation D_i on P_n, which holds precisely when determined by one of the following clauses:

(I) If $x = y$ and $x \neq z$, $x \neq w$, or if $z = w$ and $z \neq x$, $z \neq y$, then $D_i(x, y; z, w)$;

(II) if $x, y, z \in M_n$ are distinct, then

$$C_i(x; y, z) \leftrightarrow (D_i(\alpha, x; y, z) \vee D_i(x, \alpha; y, z) \vee D_i(y, z; \alpha, x) \vee D_i(y, z; x, \alpha)).$$

(III) If $x, y, z, w \in M_n$ are distinct, then $D_i(x, y; z, w)$ holds if and only if

$$(C_i(x; z, w) \wedge C_i(y; z, w)) \vee (C_i(z; x, y) \wedge C_i(w; x, y)).$$

Lemma 3.1 ([15]). *(i) The structure P_n is $\aleph_0$-categorical and weakly circularly minimal.*

(ii) Aut(P_n) *is 2-transitive and 3-homogeneous, and if $n = 2$ then it is 5-homogeneous.*

(iii) Aut(P_2) *is not 6-homogeneous, and for $n \geq 3$,* Aut(P_n) *is not 4-homogeneous.*

By *k-reduct* of M, where $k \geq 3$, we mean a structure with the same domain as M, and having a relation symbol for each $\emptyset$-definable relation of M of arity at most k, but no relation symbols of higher arity.

Theorem 3.9 ([15]). *Let M be an $\aleph_0$-categorical weakly circularly minimal structure with 3-homogeneous and 2-transitive automorphism group, and let M^* be the 4-reduct of M. Assume that M is not highly homogeneous. Then M^* is isomorphic to P_n for some $n > 1$.*

It has been earlier proved that any 3-indiscernible $\aleph_0$-categorical weakly o-minimal structure is k-indiscernible for any $k \geq 3$. For the circular case there is an analogue for k-indiscernibility which is *k-homogeneity*. There is an example of 5-homogeneous structure which is not 6-homogeneous (Lemma 3.1). Nevertheless we prove that any 6-homogeneous structure is k-homogeneous for any $k \geq 6$.

Theorem 3.10 ([15]). *Let M be an $\aleph_0$-categorical 6-homogeneous weakly circularly minimal structure. Then M is highly homogeneous.*

Further we investigate the behaviour of unary definable functions in an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure and give their complete characterization. As against the weakly o-minimal case where each such function is locally constant, i.e. generates an equivalence relation with infinitely many infinite convex classes, in the weakly circularly minimal case a unary function has a series of different types of behaviour. Such a function can be constant, piecewise constant (i.e. generate an equivalence relation with finitely many infinite convex classes) and locally monotonic (including strict monotonicity).

Let $F(x, y)$ be an $\emptyset$-definable formula such that $F(M, b)$ is convex infinite co-infinite for each $b \in M$. Let $F^\ell(y)$ be the formula saying y is a left endpoint of $F(M, y)$:

$$\exists z_1 \exists z_2 [K_0(z_1, y, z_2) \wedge \forall t_1 (K(z_1, t_1, y) \wedge t_1 \neq y \rightarrow \neg F(t_1, y)) \wedge$$

$$\forall t_2 (K(y, t_2, z_2) \wedge t_2 \neq y \rightarrow F(t_2, y))].$$

We say that $F(x, y)$ is *convex-to-right* if

$$M \models \forall y \forall x [F(x, y) \rightarrow F^l(y) \wedge \forall z (K(y, z, x) \rightarrow F(z, y))].$$

Obviously if $F(x, y)$ is convex-to-right then $M \models \forall y F(y, y)$. Let $F_1(x, y)$, $F_2(x, y)$ be arbitrary convex-to-right formulas. We say F_2 is *bigger than* F_1 if there is $a \in M$

with $F_1(M,a) \subset F_2(M,a)$. If M is transitive and this holds for some a, it holds for all a. This gives a total ordering on the (finite) set of all convex-to-right formulas $F(x,y)$ (viewed up to equivalence modulo $\mathrm{Th}(M)$). If for some $a \in M$ we have $\mathrm{acl}(a) = \{a\}$, then for each convex-to-right formula $F(x,y)$ and each $a \in M$, $F(M,a)$ has no right endpoint in M (unless this is a). We will write $f(y) := \mathrm{rend}\, F(M,y)$, meaning that $f(y)$ is a right endpoint of $F(M,y)$ which lies in the definable completion $\overline{M}$ of M. Then f is a function which maps M into $\overline{M}$.

Let f be an $\emptyset$-definable function in $\overline{M}$ such that f is locally monotonic on M, $E(x,y)$ be an $\emptyset$-definable equivalence relation partitioning M into infinitely many convex classes. We say that f is *piecewise monotonic-to-right (left) on* M/E if there is an $\emptyset$-definable non-trivial equivalence relation $E'(x,y)$ partitioning M into finitely many infinite convex classes so that f is monotonic-to-right (left) on $E'(M,a)/E$ for each $a \in M$ and f is not monotonic-to-right (left) on M/E'. We also say that f is *monotonic-to-right (left) on* E'/E if f is monotonic-to-right (left) on $E'(M,a)/E$ for each $a \in M$.

Definition 3.4. Let f be an $\emptyset$-definable function in $\overline{M}$ that is locally monotonic-to-right (left) on M, $n,m \in \omega$. We say that f *has rank* $\langle n,m\rangle$ if there are $\emptyset$-definable equivalence relations $E_1^f(x,y),\ldots,E_n^f(x,y)$ such that E_i^f partitions M into infinitely many infinite convex classes for each $i \le n-1$, E_n^f partitions M into m infinite convex classes so that

- $E_1^f(M,a) \subset E_2^f(M,a) \subset \ldots \subset E_{n-1}^f(M,a) \subset E_n^f(M,a)$ for any $a \in M$
- f is monotonic-to-right (left) on every E_1^f-class
- f is monotonic-to-left (right) on E_j^f/E_{j-1}^f for every even $2 \le j \le n$
- f is monotonic-to-right (left) on E_j^f/E_{j-1}^f for every odd $2 \le j \le n$
- If $m = 1$ and n is odd (even) then f is monotonic-to-right (left) on M/E_{n-1}^f
- If $m \ne 1$ and n is odd then f is monotonic-to-left (right) on M/E_n^f
- If $m \ne 1$ and n is even then f is monotonic right (left) on M/E_n^f

Example 3.4. Let $M = \langle M, =, K, E^2, f^1\rangle$ be a circularly ordered structure, $Q^2 = \langle \{(x_0,x_1) | x_i \in Q\}, <_{lex}\rangle$ be the set of all possible pairs of the rational numbers ordered lexicographically, and M is a disjoint union of Q_1^2 and Q_2^2 where for each $i = \overline{1,2}$ Q_i^2 is copy of Q^2. The equivalence relation E is defined as follows: for any elements $x = (x_0,x_1)$, $y = (y_0,y_1) \in M$ $E(x,y) \Leftrightarrow x_0 = y_0$, i.e. their first coordinates coincide. Define the function f: $f(Q_1^2) = Q_2^2$, $f(Q_2^2) = Q_1^2$ and for any $x = (x_0,x_1) \in M$ $f((x_0,x_1)) = (-x_0,x_1)$.

It can be proved that M is $\aleph_0$-categorical 1-transitive weakly circularly minimal and f is locally monotonic-to-right of rank $\langle 2,1\rangle$.

Theorem 3.11 ([16]). *Let M be an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, $F(x,y)$ be a convex-to-right formula. Then $f(y) := \operatorname{rend} F(M, y)$ has only one of the following behaviours on M:*
(1) f is locally constant;
(2) f is monotonic-to-right and $f^n(a) = a$ for some $n \in \omega$;
(3) f is monotonic-to-left and $f^2(a) = a$;
(4) f is locally monotonic-to-right (left) of rank $\langle n, 1\rangle$, $n > 1$ and $f^m(a) = a$ for some even $m \in \omega$; moreover if n is even (odd) then $f^2(a) = a$;
(5) f is locally monotonic-to-right (left) of rank $\langle n, m\rangle$, where $m > 2$, n is even (odd) and $f^k(a) = a$ for some even $k \in \omega$ and k divides m.

We finish our investigations by considering the 1-transitive non-primitive case. The following two theorems completely characterize all $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures of convexity rank 1 up to binarity:

Theorem 3.12 ([17]). *Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank 1 with $dcl(a) \neq \{a\}$ for some $a \in M$. Then M is isomorphic to one of the following structures up to binarity:*

- $M_m := \langle M, =, K, f^1\rangle$ *is a circularly ordered structure, M is dense, f is a monotonic-to-right bijection on M so that $f^m(a) = a$ for all $a \in M$ ($m \geq 2$).*
- $M_* := \langle M, =, K, f^1\rangle$ *is a circularly ordered structure, M is dense, f is a monotonic-to-left bijection on M so that $f^2(a) = a$ for all $a \in M$.*
- $M^1_{n,m} := \langle M, =, K, E^2, f^1\rangle$ *is a circularly ordered structure, M is dense, E is an equivalence relation partitioning M into n infinite convex classes without endpoints, f is a monotonic-to-right bijection on M so that $f^m(a) = a$, $\neg E(a, f(a))$ and $f(E(M, a)) = E(M, f(a))$ for all $a \in M$, m divides n ($m \geq 2$).*
- $M^2_{n,m} := \langle M, =, K, E^2, f^1\rangle$ *is a circularly ordered structure, M is dense, E is an equivalence relation partitioning M into n infinite convex classes without endpoints, f is a bijection so that f is monotonic-to-left on each E-class and f is monotonic-to-right on M/E, $f^m(a) = a$, $\neg E(a, f(a))$ and $f(E(M, a)) = E(M, f(a))$ for all $a \in M$, m is even, m divides n ($n \geq 4$).*

Let $E(x,y)$ be an $\emptyset$–definable equivalence relation partitioning M into infinite convex classes. Suppose that y lies in $\overline{M}$ (possibly not in M). Then

$$E^*(x,y) := \exists y_1 \exists y_2 [y_1 \neq y_2 \wedge \forall t (K(y_1, t, y_2) \to E(t, x)) \wedge K_0(y_1, y, y_2)]$$

If $E(x,y)$ partitions M into finitely many infinite convex classes, $F(x,y)$ is a convex-to-right formula, $f(y) := \operatorname{rend} F(M, y)$ and $k \in \omega$, then

$$\Phi_k^{f,E}(x) := \neg E^*(x, f(x)) \wedge \exists u_1 \ldots \exists u_k [\wedge_{i \neq j} \neg E(u_i, u_j) \wedge \wedge_{i=1}^k \{\neg E(u_i, x) \wedge$$
$$\wedge \neg E^*(u_i, f(x))\} \wedge K_0(x, u_1, \ldots, u_k, f(x)) \wedge \forall t [K(x, t, f(x)) \to$$
$$\to \vee_{i=1}^k E(t, u_i) \vee E(t, x) \vee E^*(t, f(x))]]$$

Theorem 3.13 ([17]). *Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank 1 with $dcl(a) = \{a\}$ for some $a \in M$. Then M is isomorphic to one of the following structures up to binarity:*

- $M'_n := \langle M, =, K, E^2 \rangle$ *is a circularly ordered structure, M is dense, E is an equivalence relation partitioning M into n infinite convex classes without endpoints ($n \geq 2$).*
- $M'_* := \langle M, =, K, R^2 \rangle$ *is a circularly ordered structure, M is dense, $R(x, y)$ is convex-to-right so that $R(M, a)$ has no right endpoint in M for all $a \in M$ and $r(y) := \mathrm{rend}\, R(M, y)$ is monotonic-to-left on M.*
- $M^3_{n,k} := \langle M, =, K, E^2, R^2 \rangle$ *is a circularly ordered structure, M is dense, E is an equivalence relation partitioning M into n infinite convex classes without endpoints, $R(x, y)$ is convex-to-right so that $R(M, a)$ has no right endpoint in M for all $a \in M$ and $r(y) := \mathrm{rend}\, R(M, y)$ is monotonic-to-right on M, $\neg E^*(a, r(a))$ and there is $k \geq 0$ with $\Phi^{r,E}_k(a)$ for all $a \in M$, $k + 1$ divides n ($n \geq 2$).*
- $M^4_{n,k} := \langle M, =, K, E^2, R^2 \rangle$ *is a circularly ordered structure, M is dense, E is an equivalence relation partitioning M into n infinite convex classes without endpoints, $R(x, y)$ is convex-to-right so that $R(M, a)$ has no right endpoint in M for all $a \in M$, $r(y) := \mathrm{rend}\, R(M, y)$ is monotonic-to-left on each E–class, r is monotonic-to-right on M/E, $\neg E^*(a, r(a))$ and there is $k \geq 0$ with $\Phi^{r,E}_k(a)$ for all $a \in M$, $k + 1$ divides n, n is even ($n \geq 4$).*

The following two theorems completely characterize all $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures of convexity rank greater than 1 up to binarity:

Theorem 3.14 ([16]). *Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater that 1 so that $dcl(a) \neq \{a\}$ for some $a \in M$. Then M is isomorphic to $M_{s,m,k} := \langle M, =, K, f^1, E^2_1, \ldots, E^2_s, E^2_{s+1} \rangle$ up to binarity, where M is a circularly ordered structure, M is dense, $s \geq 1$, $k \geq 2$, $m = 1$ or k divides m; E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints, for each $1 \leq i \leq s$ E_i is an equivalence relation partitioning every E_{i+1}–class into infinitely many infinite convex E_i–subclasses so that the induced order on E_i–subclasses is dense without endpoints; f is a bijection on M so that $f^k(a) = a$ for any $a \in M$, for each $1 \leq i \leq s+1$ $f(E_i(M, a)) = E_i(M, f(a))$ and $\neg E_i(a, f(a))$, and f has only one of the following behaviours on M:*

- *f is monotonic-to-right*
- *f is monotonic-to-left, $k = m = 2$*
- *f is piecewise monotonic-to-left, k is even, $m \geq 4$, f is monotonic-to-left on every E_{s+1}–class and f is monotonic-to right on M/E_{s+1}*
- *f is locally monotonic-to-right (left) of rank $\langle n + 1, 1 \rangle$ for some $1 \leq n \leq s$, and there are $1 \leq i_1 < i_2 < \ldots < i_n \leq s$ such that $E^f_j \equiv E_{i_j}$ for each $1 \leq j \leq n$, k is even, moreover if n is odd then $k = 2$*

- *f is locally monotonic-to-right (left) of rank $\langle n+1, m\rangle$ for some $1 \le n \le s$, and there are $1 \le i_1 < i_2 < \ldots < i_n < i_{n+1} = s+1$ such that $E^f_j \equiv E_{i_j}$ for each $1 \le j \le n+1$, k is even, $m > 2$, n is odd (even).*

Theorem 3.15 ([16]). *Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1 so that $dcl(a) = \{a\}$ for some $a \in M$. Then M is isomorphic to one of the following structures up to binarity:*

- *$M_{s,m} := \langle M, =, K, E^2_1, \ldots, E^2_s, E^2_{s+1}\rangle$, where M is a circularly ordered structure, M is dense, $s, m \ge 1$; E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints, for each $1 \le i \le s$ E_i is an equivalence relation partitioning every E_{i+1}–class into infinitely many infinite convex E_i–subclasses without endpoints so that the induced order on E_i–subclasses is dense without endpoints*

- *$M'_{s,m,k} := \langle M, =, K, E^2_1, \ldots, E^2_s, E^2_{s+1}, R^2\rangle$, where M is a circularly ordered structure, M is dense, $s, m \ge 1$; E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints, for each $1 \le i \le s$ E_i is an equivalence relation partitioning every E_{i+1}–class into infinitely many infinite convex E_i–subclasses without endpoints so that the induced order on E_i–subclasses is dense without endpoints; $R(x, y)$ is a convex-to-right formula such that $R(M, a)$ doesn't have right endpoint in M for all $a \in M$ and $r(y) := \mathrm{rend}\, R(M, y)$ is non-identity locally monotonic function on M so that for some $k \ge 2$ $r^k(a) = a$ for all $a \in M$, where $r^k(y) := r(r^{k-1}(y))$; for each $1 \le i \le s+1$ and any $a \in M$*

  $$M'_{s,m,k} \models \neg E^*_i(a, r(a)) \wedge \forall y(E_i(y, a) \rightarrow \exists u[E^*_i(u, r(a)) \wedge E^*_i(u, r(y))])$$

  $m = 1$ or k divides m, and r has only one of the following behaviours on M:

  1. *r is monotonic-to-right*
  2. *r is monotonic-to-left, $k = m = 2$*
  3. *r is piecewise monotonic-to-left, k is even, $m \ge 4$, and r is monotonic-to-left on every E_{s+1}–class and r is monotonic-to-right on M/E_{s+1}*
  4. *r is locally monotonic-to-right (left) of rank $\langle n+1, 1\rangle$ for some $1 \le n \le s$, and there are $1 \le i_1 < i_2 < \ldots < i_n \le s$ such that $E^r_j \equiv E_{i_j}$ for each $1 \le j \le n$, k is even, moreover if n is odd then $k = 2$*
  5. *r is locally monotonic-to-right (left) of rank $\langle n+1, m\rangle$ for some $1 \le n \le s$, and there are $1 \le i_1 < i_2 < \ldots < i_n < i_{n+1} = s+1$ such that $E^r_j \equiv E_{i_j}$ for each $1 \le j \le n+1$, k is even, $m > 2$, n is odd (even).*

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