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CONTENTS

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MIKHAIL L'VOVICH GOLDMAN

Doctor of physical and mathematical sciences, Professor Mikhail L'vovich Goldman passed away on July 5, 2025, at the age of 80 years.

Mikhail L'vovich was an internationally known expert in scienceand education. His fundamental scientific articles and text books in variousfields of the theory of functions of several variablesand functional analysis, the theory of approximation of functions, embedding theorems and harmonic analysis are a significant contribution to the development of mathematics.

Mikhail L'vovich was born on Aprill 13, 1945 in Moscow. In 1963, he graduated from School No. 128 in Moscow with a gold medal and entered the Physics Faculty of the Lomonosov Moscow State University. He graduated in 1969 and became a postgraduate student in the Mathematics Department. In 1972, he defended his PhD thesis "On integral representations and Fourier series of differentiable functions of several variables" under the supervision of Professor Ilyin Vladimir Aleksandrovich, and in 1988, his doctoral thesis "Study of spaces of differentiable functions of several variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor of Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an Assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he was a Professor of the Department of Theory of Functions and Differential Equations, then of the S.M. Nikol'skii Mathematical Institute at the Patrice Lumumba Peoples' Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman were: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators.Among the most important scientific achievements of M.L. Goldman, we note his research related to the optimal embedding of spaces with generalized smoothness, exact conditions for the convergence of spectral decompositions, descriptions of the integral and differential properties of generalized potentials of the Bessel and Riesz types, exact estimates for operators on cones, descriptions of optimal spaces for cones of functions with monotonicity properties.

M.L. Goldman has published more than 150 scientific articles in central mathematical journals. He is a laureate of the Moscow government competition, a laureate of the RUDN University Prize in Science and Innovation, and a laureate of the RUDN University Prize for supervision of postgraduate students. Under the supervision of Mikhail L'vovich 11 PhD theses were defended. His pupilss are actively involved in professional work at leading universities and research institutes in Russia, Kazakhstan, Ethiopia, Rwanda, Colombia, and Mongolia.

Mikhail L'vovich has repeatedly been a guest lecturer and guest professor at universities in Russia, Germany, Sweden, Great Britain, etc., and an invited speaker at many international conferences. Mikhail L'vovich was not only an excellent mathematician and teacher (he always spoke about mathematics and its teaching with great passion), but also a man of the highest culture and erudition, with a deep knowledge of history, literature and art, a very bright, kind and responsive person. This is how he will remain in the hearts of his family, friends, colleagues and students.

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Mikhail L'vovich Goldman.

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ASYMPTOTICS OF SOLUTIONS OF THE STURM-LIOUVILLE EQUATION IN VECTOR-FUNCTION SPACE

Ya.T. Sultanaev, N.F. Valeev, A. Yeskermessuly

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Key words: asymptotic methods, rapidly oscillating coefficients, vector-function, L-diagonal system, resonance case.

AMS Mathematics Subject Classification: 34L05, 34E05.

Abstract. In this paper, we study the asymptotic behaviour of fundamental systems of solutions to the Sturm-Liouville equation with rapidly oscillating potentials in a two-dimensional vector-function space. We consider different cases in which the coefficients do not satisfy the regularity conditions. Additionally, we investigate the asymptotic behaviour of solutions in resonance cases.

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1 Introduction

Numerous studies have focused on the asymptotic properties of solutions to singular Sturm-Liouville equations and differential equations of arbitrary orders, as discussed in papers [1, 2, 12] and papers cited there. These studies predominantly assumed that the equation's coefficients exhibit regular growth to infinity. In contrast, works [3, 4, 5, 7, 8, 10, 11] explored the asymptotic properties of solutions to ordinary differential equations with coefficients from broader classes, particularly those that do not meet the Titchmarsh-Levitan conditions.

In [11], a method was proposed to study the asymptotic behaviour of solutions of the Sturm-Liouville equation

$$y'' + (1 + q(x))y = 0, \quad x_0 < x < \infty \tag{1.1}$$

for the case in which $q(x)$ is a rapidly oscillating function belonging to the class σ as defined in [11]. This method enables the construction of asymptotic formulas for solutions whether $q(x)$ influences the leading term of the asymptotic expansion or not. However, this method does not address the classes in which $q(x)$ oscillates but does not belong to the class described in [11]. An example of such a function is $\sin(x)/x^\alpha$, where $\alpha > 0$.

In [9], this approach was modified to construct the asymptotics of perturbations of the form $p(x)/x^\alpha$, where $\alpha > 0$ and $p(x)$ is a quasi-periodic function.

Note that for $\alpha > 1$ the condition $\int |p(x)/x^\alpha| dx < \infty$ is satisfied, hence, due to Theorem 1 in [1] (p. 133), all solutions of equation (1.1) are bounded. Therefore, further study of this case is not of interest.

In this paper, we extend the methods of [8, 9] to construct asymptotic formulas for the solution of the Sturm-Liouville equation in the two-dimensional vector-function space:

$$\vec{y}'' + \left(A_0 + \frac{p(x)}{x^\alpha} A_1\right)\vec{y} = 0, \quad x_0 < x < \infty,$$

where

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A_j = \left(a_{lk}^j\right), \ A_j = const, \ j = \overline{0,1}, \ A_0^* = A_0 > 0,$$

and $p(x)$ is a quasi-periodic function.

2 Construction of asymptotic formulas

We consider the Sturm-Liouville equation in the two-dimensional vector-function space:

$$\vec{\varphi}'' + \left(A_0 + \frac{p(x)}{x^\alpha} A_1\right) \vec{\varphi} = 0, \quad A_0^* = A_0 > 0, \quad \alpha > 0, \tag{2.1}$$

$$p(x) = \sum_{k=1}^{m} s_k e^{ip_k x}, \quad s_k \in \mathbb{C}, \, p_k \in \mathbb{R} \setminus \{0\}. \tag{2.2}$$

The substitution

$$\vec{\varphi} = T\vec{y}, \tag{2.3}$$

transforms equation (2.1) to the equation

$$\vec{y}'' + \begin{pmatrix} \mu_1^2 & 0 \\ 0 & \mu_2^2 \end{pmatrix} \vec{y} + \frac{p(x)}{x^\alpha} B\vec{y} = 0, \ \ x_0 \le x < \infty, \ \ \alpha > 0, \tag{2.4}$$

where

$$T^{-1}A_0T = \begin{pmatrix} \mu_1^2 & 0 \\ 0 & \mu_2^2 \end{pmatrix}, \quad B = T^{-1}A_1T = \left(b_{jk}\right), \ j,k = \overline{1,2}.$$

We present the main result of this paper.

Theorem 2.1. *Let $\alpha > 1/3$, and let a function $p(x)$ have form (2.2). Moreover, suppose that the following conditions hold.*

1. *For any set of numbers $\{c_1, ..., c_m\}$, where $c_j \in 0 \cup \mathbb{N}$, $\sum_{j=1}^{m} c_j \neq 0$, the following condition is satisfied:*

$$\sum_{k=1}^{m} c_k p_k \neq 0. \tag{2.5}$$

2. *For any p_k, $k = 1, \dots, m$, it is true that*

$$p_k \notin \{\pm 2\mu_1, \, \pm 2\mu_2, \, \pm\mu_1 \pm \mu_2\}. \tag{2.6}$$

Then, for the fundamental system of solutions of equation (2.4), as $x \to +\infty$, the following asymptotic relation holds:

$$\vec{y} \sim \begin{pmatrix} c_{11}e^{i\mu_1 x} & c_{12}e^{-i\mu_1 x} \\ c_{21}e^{i\mu_2 x} & c_{22}e^{-i\mu_2 x} \end{pmatrix} (I + o(1))\vec{y}_0, \ c_{jk} = const, \ j,k = \overline{1,2}, \ \vec{y}_0 = const.$$

Proof. We reduce equation (2.4) to an equivalent first-order system of equations.

Let us introduce the following vector-function:

$$\vec{z}(x,\mu) = col(z_1, z_2, z_3, z_4): \ z_1 = y_1, \ z_2 = y_2, \ z_3 = y_1', \ z_4 = y_2'.$$

Then, equation (2.4) transforms into the following form:

$$\vec{z}' = (L + \frac{p(x)}{x^\alpha} A)\vec{z}, \tag{2.7}$$

where

$$L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\mu_1^2 & 0 & 0 & 0 \\ 0 & -\mu_2^2 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -b_{11} & -b_{12} & 0 & 0 \\ -b_{21} & -b_{22} & 0 & 0 \end{pmatrix}.$$

The substitution

$$\vec{z}(x) = T_1\vec{u}, \quad T_1 = \begin{pmatrix} -\frac{i}{\mu_1} & \frac{i}{\mu_1} & 0 & 0 \\ 0 & 0 & -\frac{i}{\mu_2} & \frac{i}{\mu_2} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{2.8}$$

transforms system (2.7) into the system

$$\vec{u}' = i\Lambda_0\vec{u} + \frac{1}{x^\alpha}\tilde{B}(x)\vec{u}, \tag{2.9}$$

$$\Lambda_0 = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & -\mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & -\mu_2 \end{pmatrix}, \quad \tilde{B}(x) = \frac{ip(x)}{2} \begin{pmatrix} \frac{b_{11}}{\mu_1} & \frac{-b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2} & \frac{-b_{12}}{\mu_2} \\ \frac{b_{11}}{\mu_1} & \frac{-b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2} & \frac{-b_{12}}{\mu_2} \\ \frac{b_{21}}{\mu_1} & \frac{-b_{21}}{\mu_1} & \frac{b_{22}}{\mu_2} & \frac{-b_{22}}{\mu_2} \\ \frac{b_{21}}{\mu_1} & \frac{-b_{21}}{\mu_1} & \frac{b_{22}}{\mu_2} & \frac{-b_{22}}{\mu_2} \end{pmatrix}.$$

We apply the following substitution:

$$\vec{u} = C(x)\vec{v}, \quad C(x) = C_0(x) + \frac{1}{x^\alpha}C_1(x), \tag{2.10}$$

which leads us to the following system:

$$C'(x)\vec{v} + C(x)\vec{v}' = i\Lambda_0 C(x)\vec{v} + \frac{1}{x^\alpha}\tilde{B}(x)C(x)\vec{v}. \tag{2.11}$$

We seek the matrices $C_0(x)$ and $C_1(x)$ from the following system of matrix equations:

$$\begin{cases} C_0'(x) = i\Lambda_0 C_0(x), \\ C_1'(x) = i\Lambda_0 C_1(x) + \tilde{B}(x)C_0(x). \end{cases} \tag{2.12}$$

From (2.12), we obtain

$$C_0(x) = e^{i\Lambda_0 x} = \begin{pmatrix} e^{i\mu_1 x} & 0 & 0 & 0 \\ 0 & e^{-i\mu_1 x} & 0 & 0 \\ 0 & 0 & e^{i\mu_2 x} & 0 \\ 0 & 0 & 0 & e^{-i\mu_2 x} \end{pmatrix}.$$

Also, from (2.12), we have

$$C_1(x) = C_0(x) - C_0(x)D(x), \quad D'(x) = D_1(x) = -C_0^{-1}(x)\tilde{B}(x)C_0(x), \tag{2.13}$$

$$D_1(x) = \frac{ip(x)}{2} \begin{pmatrix} \frac{b_{11}}{\mu_1} & -\frac{b_{11}}{\mu_1}e^{-2i\mu_1 x} & \frac{b_{12}}{\mu_2}e^{i(-\mu_1+\mu_2)x} & -\frac{b_{12}}{\mu_2}e^{i(-\mu_1-\mu_2)x} \\ \frac{b_{11}}{\mu_1}e^{2i\mu_1 x} & -\frac{b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2}e^{i(\mu_1+\mu_2)x} & -\frac{b_{12}}{\mu_2}e^{i(\mu_1-\mu_2)x} \\ \frac{b_{21}}{\mu_1}e^{i(\mu_1-\mu_2)x} & -\frac{b_{21}}{\mu_1}e^{i(-\mu_1-\mu_2)x} & \frac{b_{22}}{\mu_2} & -\frac{b_{22}}{\mu_2}e^{-2i\mu_2 x} \\ \frac{b_{21}}{\mu_1}e^{i(\mu_1+\mu_2)x} & -\frac{b_{21}}{\mu_1}e^{i(-\mu_1+\mu_2)x} & \frac{b_{22}}{\mu_2}e^{2i\mu_2} & -\frac{b_{22}}{\mu_2} \end{pmatrix}.$$

We define the matrix $D(x)$ as the antiderivative of the matrix function $D_1(x)$

$$D(x) = \begin{pmatrix} p_{11}(x,0) & -p_{11}(x,-2\mu_1) & p_{12}(x,-\mu_1+\mu_2) & -p_{12}(x,-\mu_1-\mu_2) \\ p_{11}(x,2\mu_1) & -p_{11}(x,0) & p_{12}(x,\mu_1+\mu_2) & -p_{12}(x,\mu_1-\mu_2) \\ p_{21}(x,\mu_1-\mu_2) & -p_{21}(x,-\mu_1-\mu_2) & p_{22}(x,0) & -p_{22}(x,-2\mu_2) \\ p_{21}(x,\mu_1+\mu_2) & -p_{21}(x,-\mu_1+\mu_2) & p_{22}(x,2\mu_2) & -p_{22}(x,0) \end{pmatrix},$$

where

$$p_{jk}(x,\sigma_{ml}) = \int \frac{ib_{jk}p(x)}{2\mu_k} e^{i\sigma_{ml}x}dx,\ j,k=\overline{1,2},\ m,l=\overline{1,4},$$
$$\sigma_{ml} \in \{0, \pm 2\mu_1, \pm 2\mu_2, \pm\mu_1 \pm \mu_2\}.$$

Thus, the solution $C_1(x)$ of system (2.12) has the form

$$C_1(x) = C_0(x)\cdot(I - D(x)). \tag{2.14}$$

It is easy to prove that due to conditions (2.5) and (2.6) of Theorem 2.1, all elements of the matrix $D(x)$ are bounded. Hence, the matrices $C_0(x)$ and $C_1(x)$ are bounded. Taking into account the last expressions, for the matrix $C(x)$ we obtain

$$C(x) = C_0(x)\left(I + \frac{1}{x^\alpha}(I - D(x))\right). \tag{2.15}$$

Since $C_0(x)$ is a diagonal matrix, $D(x)$ is bounded, and $x^{-\alpha} \to 0$ as $x \to \infty$, the matrix $C(x)$ admits a bounded inverse.

Considering (2.12) and (2.15), we can rewrite system (2.11) in the following form:

$$\begin{aligned}(\vec{v})' = {} & \frac{1}{x^{2\alpha}}\left(C_0(x) + \frac{1}{x^\alpha}C_1(x)\right)^{-1}\tilde{B}(x)C_1(x)\vec{v} \\ & + \frac{\alpha}{x^{\alpha+1}}\left(C_0(x) + \frac{1}{x^\alpha}C_1(x)\right)^{-1}C_1(x)\vec{v}.\end{aligned} \tag{2.16}$$

From the boundedness of the matrices $C_0(x)$ and $C_1(x)$, it follows that the matrices $C^{-1}\tilde{B}C_1$, $C^{-1}C_1$ are also bounded.

Let us consider the case $\alpha > 1/2$. We rewrite system (2.16) as follows:

$$(\vec{v})' = \tilde{C}(x)\vec{v}, \tag{2.17}$$

where

$$\tilde{C}(x) = \frac{1}{x^{2\alpha}}C^{-1}(x)\tilde{B}(x)C_1(x) + \frac{\alpha}{x^{\alpha+1}}C^{-1}(x)C_1(x).$$

If $\alpha > 1/2$, then the boundedness of $C^{-1}\tilde{B}C_1$ and $C^{-1}C_1$ obviously implies that all elements of the matrix $\tilde{C}(x)$ are summable, i.e., $\|\tilde{C}(x)\| \in L^1(x_0,\infty)$. Therefore, using successive approximations for system (2.17), we obtain

$$\vec{v} = \vec{v}_0 + \int_x^\infty \tilde{C}(\xi)\vec{v}(\xi)d\xi, \quad \vec{v}_0 = const,$$

which implies

$$\vec{v} = (I + o(1))\vec{v}_0.$$

Taking into account substitutions (2.10), (2.8), we obtain the solution to equation (2.7) as follows:

$$\vec{z}(x) = T_1 \cdot \left(C_0(x) + \frac{1}{x^\alpha} C_1(x) \right) \cdot (I + o(1)) \cdot \vec{v}_0, \quad v_0 = const. \tag{2.18}$$

Now, let us consider the case $1/3 < \alpha < 1/2$. In this case, the elements of the matrix $\frac{\alpha}{x^{\alpha+1}} C^{-1} C_1$ are summable. Given that $2\alpha < 1$, the asymptotic relation $x^{\alpha+1} = o(x^{3\alpha})$ as $x \to \infty$ holds. Therefore, we can rewrite system (2.16) by using the Neumann series for the inverse matrix

$$C^{-1}(x) = \left(I + \frac{1}{x^\alpha}(I - D(x)) \right)^{-1} C_0^{-1}(x) = C_0^{-1}(x) + \mathcal{O}(x^{-\alpha}),$$

in the following form:

$$(\vec{v})' = \frac{1}{x^{2\alpha}} C_0^{-1}(x) \tilde{B}(x) C_1(x) \vec{v} + \frac{1}{x^{3\alpha}} F(x, \mu_1, \mu_2) \vec{v}, \tag{2.19}$$

where

$$\frac{1}{x^{3\alpha}} F(x, \mu_1, \mu_2) = \frac{1}{x^{2\alpha}} \mathcal{O}(x^{-\alpha}) + \frac{\alpha}{x^{\alpha+1}} C^{-1}(x) C_1(x),$$

and $\left\| \frac{1}{x^{3\alpha}} F(x, \mu) \right\| \in L^1(x_0, \infty)$. Taking into account (2.13) and (2.14), for the matrix $C_0^{-1}(x) \tilde{B}(x) C_1(x)$ we have

$$C_0^{-1}(x) \tilde{B}(x) C_1(x) = C_0^{-1}(x) \tilde{B}(x) C_0(x) D(x) = D'(x) D(x).$$

Hence, the elements of the matrix $C_0^{-1} \tilde{B} C_1$ are oscillating functions and can be represented as

$$G(x, \mu_1, \mu_2) = \sum G_k e^{i\sigma_k x}, \quad \sigma_k \in \{p_k, \pm 2\mu_1, \pm 2\mu_2, \pm \mu_1 \pm \mu_2\}, \quad G_k = const.$$

Thus, system (2.19) takes the form

$$(\vec{v})' = \frac{1}{x^{2\alpha}} G(x, \mu_1, \mu_2) \vec{v} + \frac{1}{x^{3\alpha}} F(x, \mu_1, \mu_2) \vec{v}. \tag{2.20}$$

The substitution

$$\xi = \frac{x^{1-2\alpha}}{1 - 2\alpha}, \quad x = ((1 - 2\alpha)\xi)^{\frac{1}{1-2\alpha}}, \quad \vec{v}(x) = \vec{w}(\xi), \tag{2.21}$$

$$\beta = \frac{1}{1 - 2\alpha}, \quad \gamma = \frac{\alpha}{1 - 2\alpha},$$

transforms system (2.20) to the system

$$(\vec{w})'_\xi = G\left(a\xi^\beta, \mu_1, \mu_2\right) \vec{w} + \frac{\beta^\gamma}{\xi^\gamma} F\left(a\xi^\beta, \mu_1, \mu_2\right) \vec{w}, \tag{2.22}$$

where $a = \beta^{-\beta}$ is a constant, which does not affect the asymptotic behaviour of the solutions.

The condition $1/3 < \alpha < 1/2$ implies $3 < \beta < \infty$, hence $\gamma = \frac{\alpha}{1-2\alpha} = \beta\alpha > 1$. Therefore, the second term of system (2.22) is summable.

By by integration, from (2.22), we obtain

$$\vec{w}(\xi) = \vec{w}(\xi_0) + \int_\xi^\infty G\left(\tau^\beta, \mu_1, \mu_2\right) \vec{w}(\tau) d\tau + \beta^\gamma \int_\xi^\infty \tau^{-\gamma} F\left(\tau^\beta, \mu_1, \mu_2\right) \vec{w}(\tau) d\tau. \tag{2.23}$$

Integrating by part the second term of expression (2.23), we have

$$\int_{\xi}^{\infty} G\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)d\tau = \hat{G}\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)\Big|_{\xi}^{\infty} - \int_{\xi}^{\infty}\hat{G}\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}'(\tau)d\tau, \tag{2.24}$$

where

$$\left|\hat{G}\left(\xi^{\beta},\mu_1,\mu_2\right)\right| = \left|\int_{\xi}^{\infty} G\left(\tau^{\beta},\mu_1,\mu_2\right)d\tau\right| = \left|\frac{1}{\beta}\int_{\xi}^{\infty} G\left(\tau^{\beta},\mu_1,\mu_2\right)\frac{d\tau^{\beta}}{\tau^{\beta-1}}\right| = O\left(\frac{1}{\tau^{\beta-1}}\right).$$

Hence, $\|\hat{G}\left(\xi,\mu_1,\mu_2\right)\| \in L^1(\xi_0,\infty)$. By using (2.22) for (2.24), we get

$$J := \int_{\xi}^{\infty} G\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)d\tau = \hat{G}\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)\Big|_{\xi}^{\infty}$$
$$-\int_{\xi}^{\infty}\hat{G}\left(\tau^{\beta},\mu_1,\mu_2\right)\left(G\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau) + \frac{\beta^{\gamma}}{\tau^{\gamma}}F\left(\tau^{\beta},\mu_1,\mu_2\right)\vec{w}(\tau)\right)d\tau.$$

Taking into account the last expressions and (2.23), we obtain the following estimate:

$$\|\vec{w}-\vec{w}_0\|_{C(\xi_0,\infty)} \le K\|\vec{w}\|_{C(\xi_0,\infty)}, \quad K = const.$$

Hence, it follows that

$$\vec{w}(\xi) = \vec{w}(\xi_0) + o(1), \tag{2.25}$$

where $\vec{w}(\xi) = \vec{w}_0$. Returning from system (2.25) to system (2.7), taking into account substitutions (2.21), (2.10) and (2.8), we obtain (2.18).

Let us consider the case $\alpha = 1/2$. In this case, system (2.24) takes the form

$$(\vec{v})' = \frac{1}{x}G(x,\mu_1,\mu_2)\vec{v} + \frac{1}{x^{3/2}}F(x,\mu_1,\mu_2)\vec{v}. \tag{2.26}$$

Substituting

$$\xi = \ln x, \quad x = e^{\xi}, \quad \vec{v}(x) = \vec{w}(\xi),$$

converts system (2.26) to the system

$$(\vec{w})'_{\xi} = G(e^{\xi},\mu_1,\mu_2)\vec{w} + e^{-\frac{\xi}{2}}F(e^{\xi},\mu_1,\mu_2)\vec{w}.$$

Similarly to (2.22), by using successive approximations, we obtain

$$\vec{w}(\xi) = \vec{w}(\xi_0) + o(1).$$

Taking into account (2.21), (2.10) and (2.8), we obtain (2.18).

Finally, we obtain the asymptotics of the solutions to system (2.4) in the following form

$$\vec{y} \sim \begin{pmatrix} c_{11}e^{i\mu_1 x} & c_{12}e^{-i\mu_1 x} \\ c_{21}e^{i\mu_2 x} & c_{22}e^{-i\mu_2 x}\end{pmatrix}(I + o(1))\vec{y}_0,$$

where $c_{jk} = const$, $j,k = \overline{1,2}$, $\vec{y}_0 = const$. □

Remark 1. From the proven theorem, it follows that the perturbation $p(x)/x^\alpha$ does not affect the dominant part of the asymptotics of the solution to equation (2.4) provided that conditions (2.5) and (2.6) are satisfied.

Remark 2. The condition $\alpha > 1/3$ arises from the selection of the substitution:

$$C(x) = C_0(x) + \frac{1}{x^\alpha} C_1(x).$$

For the general case, the substitution takes the form:

$$C(x) = \sum_{k=0}^{m} \frac{1}{x^{k\alpha}} C_k(x),$$

as provided in [9] for the scalar case.

Remark 3. If condition (2.5) of Theorem 2.1 is not satisfied, a resonance case occurs, and the asymptotics will significantly differ from those obtained above.

3 Resonance case

In this section, we will show that the conditions of Theorem 2.1 are essential. To do this, let us consider the case in which $p_k \in \{\pm 2\mu_1, \pm 2\mu_2, \pm\mu_1 \pm \mu_2\}$, i.e., condition (2.5) is not satisfied. For the matrix $C(x)$ in (2.15), we obtain

$$C(x) = C_0(x) \left(I - \frac{1}{x^\alpha} \tilde{D}(x) - x^{1-\alpha} D_2(x) \right),$$

where

$$\tilde{D}(x) = \sum \tilde{D}_k e^{i\sigma_k}, \quad D_2(x) = \sum (D_2)_k e^{i\sigma_k}, \quad \tilde{D}_k,\ (D_2)_k = const,$$
$$\sigma_k \in \{\pm 2\mu_1, \pm 2\mu_2, \pm\mu_1 \pm \mu_2, p_k.\}$$

For the case $\alpha < 1$, the matrix $C(x)$ becomes unbounded as $x \to \infty$. This generates the resonance case and the method described in the previous section is no longer applicable. Therefore, we apply a different approach to study the asymptotic behaviour of solutions.

Let $p(x) = \cos(\mu_1 + \mu_2)x$, $\mu_1, \mu_2 \in \mathbb{R} \setminus 0$. Then, system (2.9) takes the following form:

$$\vec{u}' = \frac{i}{2} \Lambda_0 \vec{u} + \frac{i\cos(\mu_1 + \mu_2)x}{2x^\alpha} B\vec{u}, \tag{3.1}$$

where

$$\Lambda_0 = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & -\mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & -\mu_2 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{b_{11}}{\mu_1} & \frac{-b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2} & \frac{-b_{12}}{\mu_2} \\ \frac{b_{11}}{\mu_1} & \frac{-b_{11}}{\mu_1} & \frac{b_{12}}{\mu_2} & \frac{-b_{12}}{\mu_2} \\ \frac{b_{21}}{\mu_1} & \frac{-b_{21}}{\mu_1} & \frac{b_{22}}{\mu_2} & \frac{-b_{22}}{\mu_2} \\ \frac{b_{21}}{\mu_1} & \frac{-b_{21}}{\mu_1} & \frac{b_{22}}{\mu_2} & \frac{-b_{22}}{\mu_2} \end{pmatrix}.$$

In the case $\mu_1 = \mu_2$, system (3.1) becomes equivalent to two second-order scalar linear differential equations. Therefore, we consider the case $\mu_1 \neq \mu_2$.

The substitution

$$\vec{u} = e^{\frac{i}{2}\Lambda_0 x} \vec{v} \tag{3.2}$$

transforms system (3.1) to

$$\vec{v}' = \frac{i\cos(\mu_1 + \mu_2)x}{2x^\alpha} e^{-\frac{i}{2}\Lambda_0 x} B e^{\frac{i}{2}\Lambda_0 x} \vec{v}.$$

After some modifications, we get

$$\vec{v}' = \frac{1}{x^\alpha}\left(B_0 + \sum_{k=1}^{m} e^{i\sigma_k x} B_k\right)\vec{v}, \tag{3.3}$$

where

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & -\frac{ib_{12}}{4\mu_2} \\ 0 & 0 & \frac{ib_{12}}{4\mu_2} & 0 \\ 0 & -\frac{ib_{21}}{4\mu_1} & 0 & 0 \\ \frac{ib_{21}}{4\mu_1} & 0 & 0 & 0 \end{pmatrix}, \quad B_k = const, \quad k = 1, \dots, m,$$

$$\sigma_k \in \{\pm 2\mu_1;\ \pm 2\mu_2;\ \pm 2(\mu_1+\mu_2);\ \pm(\mu_1 \pm \mu_2);\ \pm(3\mu_1+\mu_2);\ \pm(\mu_1+3\mu_2)\}.$$

Replacing independent variable x by ξ as in (2.21):

$$\xi = \frac{x^{1-\alpha}}{1-\alpha}, \quad x = ((1-\alpha)\xi)^{\frac{1}{1-\alpha}}, \quad \vec{v}(x) = \vec{w}(\xi), \tag{3.4}$$

$$\beta = \frac{1}{1-\alpha}, \quad a = (1-\alpha)^\beta,$$

we obtain

$$\vec{w}'_\xi = B_0\vec{w} + \sum_{k=1}^{m} e^{ia\sigma_k \xi^\beta} B_k \vec{w}. \tag{3.5}$$

Denote

$$\phi_k(\xi) = \int_\xi^\infty e^{ia\sigma_k \tau^\beta} d\tau, \quad k = 1, \dots, m.$$

Assume that $1/2 < \alpha < 1$, which implies $\beta > 2$. Then

$$\phi_k(\xi) = \int_\xi^\infty e^{ia\sigma_k \tau^\beta} d\tau = \frac{1}{\beta}\int_\xi^\infty \frac{e^{ia\sigma_k \tau^\beta}}{\tau^{\beta-1}} d\tau^\beta \in L^1(\xi_0, \infty). \tag{3.6}$$

Applying the substitution

$$\vec{w} = e^{-\phi_1(\xi)B_1}\vec{w}_1, \tag{3.7}$$

we get

$$\vec{w}_1'(\xi) = e^{\phi_1(\xi)B_1} B_0 e^{-\phi_1(\xi)C_1}\vec{w}_1 + e^{\phi_1(\xi)B_1} \cdot \left(\sum_{k=2}^{m} e^{ia\sigma_k \xi^\beta} B_k\right) \cdot e^{-\phi_1(\xi)B_1}\vec{w}_1. \tag{3.8}$$

Using the properties of the matrix exponent, from (3.6) we obtain

$$e^{\phi_1(\xi)B_1} = I + F_1(\xi), \quad \|F_1(\xi)\| \in L^1(\xi_0, \infty),$$

and

$$e^{-\phi_1(\xi)B_1} = I + F_2(\xi), \quad \|F_2(\xi)\| \in L^1(\xi_0, \infty).$$

Therefore,

$$e^{\phi_1(\xi)B_1} B_0 e^{-\phi_1(\xi)B_1} = B_0 + F_3(\xi), \quad \|F_3(\xi)\| \in L^1(\xi_0, \infty),$$

where

$$F_3(\xi) = B_0F_2(\xi) + F_1(\xi)B_0 + F_1(\xi)B_0F_2(\xi).$$

For the remaining terms of system (3.8), we have

$$e^{\phi_1(\xi)B_1}\cdot\left(e^{ia\sigma_k\xi^\beta}B_k\right)\cdot e^{-\phi_1(\xi)B_1}=(I+F_1(\xi))e^{ia\sigma_k\xi^\beta}B_k(I+F_2(\xi))=e^{ia\sigma_k\xi^\beta}B_k+G_k(\xi),$$

$$G_k(\xi)=e^{ia\sigma_k\xi^\beta}\left(B_kF_2(\xi)+F_1(\xi)B_k+F_1(\xi)B_kF_2\right),\quad k=2,\dots,m.$$

The matrices $e^{ia\sigma_k\xi^\beta}B_k$ are bounded, and the matrices $F_1(\xi)$, $F_2(\xi)$ are summable. Hence, the matrices $G_k(\xi)$ are summable. On the base of these relations, system (3.8) takes the form

$$\vec{w}_1'(\xi)=B_0\vec{w}_1+\sum_{k=2}^{m}e^{ia\sigma_k\xi^\beta}B_k\vec{w}_1+G(\xi)\vec{w}_1, \tag{3.9}$$

where

$$G(\xi)=\sum_{k=2}^{m}G_k(\xi),\quad \|G(\xi)\|\in L^1(\xi_0,\infty).$$

Using the substitutions

$$\vec{w}_{k-1}=e^{ia\sigma_k\xi^\beta B_k}\vec{w}_k,\quad k=2,\dots,m, \tag{3.10}$$

one by one and conducting similar calculations, we finally obtain

$$\vec{w}_m'(\xi)=B_0\vec{w}_m+P(\xi)\vec{w}_m,\quad \|P(\xi)\|\in L^1(\xi_0,\infty). \tag{3.11}$$

Applying Levinson's Theorem to (3.11) (see [6], p. 292), we obtain the solution

$$\vec{w}_m(\xi)=e^{\xi B_0}\cdot(I+M\cdot o(1)),\quad M=const.$$

Using substitutions (3.9), (3.7), (3.4), (3.2) and (2.8), we obtain the solution for system (2.7) in the following form:

$$\vec{z}=T_1\cdot e^{\frac{i}{2}\Lambda_0\xi^\beta}\cdot\prod_{k=1}^{m}e^{-\phi_k(\xi)B_k}e^{\xi B_0}(I+M\cdot o(1))\vec{w}_m(\xi_0).$$

The dominant part of asymptotics of solutions is

$$\vec{z}\sim e^{\frac{i}{2}\Lambda_0 x}\cdot\exp\left\{\frac{x^{1-\alpha}}{1-\alpha}B_0\right\}(I+M\cdot o(1))z_0,\quad z_0=const.$$

Let us consider the case $\alpha=1$. Then, system (3.3) takes the form

$$\vec{v}'=\frac{1}{x}\left(B_0+\sum_{k=1}^{m}e^{i\sigma_k x}B_k\right)\vec{v}.$$

By using the substitution

$$\xi=\ln x,\quad x=e^\xi,\quad \vec{v}(x)=\vec{w}(\xi) \tag{3.12}$$

we obtain the system

$$\vec{w}_\xi'=\left(B_0+\sum_{k=1}^{m}e^{i\sigma_k e^\xi}B_k\right)\vec{w}.$$

Denote

$$\psi_k(\xi)=\int_\xi^\infty e^{i\sigma_k e^\tau}d\tau,\quad k=1,\dots,m.$$

As $\psi_k(\xi) \in L^1(\xi_0, \infty)$, $j = 1, \ldots, m$, using the substitutions

$$\vec{w} = e^{-\psi_1(\xi)B_1}\vec{w}_1, \quad \vec{w}_{k-1} = e^{-\psi_k(\xi)B_k}\vec{w}_k, \quad k = 2, \ldots, m, \tag{3.13}$$

and conducting similar calculations as in the previous case, we obtain

$$\vec{w}_m'(\xi) = B_0\vec{w}_m + G(\xi)\vec{w}_m, \quad \|G(\xi)\| \in L^1(\xi_0, \infty).$$

Applying Levinson's Theorem (see [6], p. 292) and using substitutions (3.13), (3.12), (3.2) and (2.8), we obtain the following expression for the solution to system (2.7):

$$\vec{z} = T_1 \cdot e^{\frac{i}{2}\Lambda_0 e^{\xi}} \cdot \prod_{k=1}^{m} e^{-\psi_k(\xi)B_k} \cdot e^{\xi B_0} \cdot (I + M \cdot o(1)) \cdot \vec{w}_m(\xi_0).$$

The dominant part of the asymptotics of the solution is given by

$$\vec{z} \sim e^{\frac{i}{2}\Lambda_0 x} \cdot e^{\ln x B_0} \cdot (I + M \cdot o(1))\vec{z}_0, \quad \vec{z}_0 = const.$$

Remark 4. In both cases $\alpha < 1$ and $\alpha = 1$, the asymptotics of solutions to system (2.7), as described by equation (2.4), will depend on the elements of the constant matrix B.

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References

[1] R. Bellman, *Stability theory of differential equations*, McGraw-Hill Book Company, New York, New York, 1953.

[2] M.V. Fedoryuk, *Asymptotic methods for linear ordinary differential equations* (in Russian). USSR. Moscow, 2009.

[3] N.N. Konechnaya, K.A. Mirzoev, Ya.T. Sultanaev, *On the asymptotic of solutions of some classes of linear differential equations*, Azerbaijan Journal of Mathematics 10 (2020), no. 1, 162–171.

[4] K. Magnus, *Vibrations: an introduction to the study of vibrational systems.* Moscow, Mir, 1982 (in Russian).

[5] O.V. Myakinova, Ya.T. Sultanaev, N.F. Valeev, *On the asymptotic of solutions of a singular nth-order differential equation with nonregular coefficients.* 104 (2018), no. 4, 626–631 (in Russian). English translation in Math. Notes 104 (2018), no. 4, 606–611. https://doi.org/10.1134/S0001434618090262

[6] M.A. Naimark, *Linear differential operators.* Moscow, Nauka, 1969 (in Russian).

[7] E.A. Nazirova, Ya.T. Sultanaev, N.F. Valeev, *Distribution of the eigenvalues of singular differential operators in space of vector-functions.* Tr. Mosk. Mat. Obs. 75 (2014), no. 2, MCCME, M., 107–123 (in Russian). English translation in Trans. Moscow Math. Soc. 75 (2014), 89–102. https://doi.org/10.1090/S0077-1554-2014-00238-7

[8] E.A. Nazirova, Ya.T. Sultanaev, L.N. Valeeva, *On a method for studying the asymptotic of solutions of Sturm-Liouville differential equations with rapidly oscillating coefficients.* Mat. Zametki 112 (2022), no. 6, 935–940 (in Russian). English translation in Math. Notes 112 (2022), no. 6, 1059–1064. https://doi.org/10.1134/S0001434622110372

[9] E.A. Nazirova, Ya.T. Sultanaev, N.F. Valeev, *Construction of asymptotics for solutions of Sturm-Liouville differential equations in classes of oscillating coefficients.* Vestnik Moskov. Univ. Ser. 1. Mat. Mekh. (2023), no. 5, 61–65 (in Russian). English translation in Moscow Univ. Math. Bull. 78 (2023), no. 5, 253–257 . https://doi.org/10.3103/S0027132223050066

[10] P.N. Nesterov, *Construction of the asymptotics of the solutions of the one-dimensional Schrodinger equation with rapidly oscillating potential.* Mat. Zametki 80 (2006), no. 2, 240–250 (in Russian). English translation in Math. Notes 80 (2006), no. 2, 233–243. https://doi.org/10.1007/s11006-006-0132-5

[11] N.F. Valeev, A. Eskermesuly, Ya.T. Sultanaev, *On the deficiency index of a differential operator with fast oscillating coefficients.* Mat. Zametki 100 (2016), no.3, 465–468 (in Russian). English translation in Math Notes 100 (2016), no. 3, 486-490. https://doi.org/10.1134/S0001434616090170

[12] A. Zettl, *Sturm-Liouville Theory*, Math. Surv. and Mon., Amer. Math. Soc., Providence, RI 121 (2005). https://api.semanticscholar.org/CorpusID:117287500

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