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CONTENTS

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- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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MIKHAIL L'VOVICH GOLDMAN

Doctor of physical and mathematical sciences, Professor Mikhail L'vovich Goldman passed away on July 5, 2025, at the age of 80 years.

Mikhail L'vovich was an internationally known expert in scienceand education. His fundamental scientific articles and text books in variousfields of the theory of functions of several variablesand functional analysis, the theory of approximation of functions, embedding theorems and harmonic analysis are a significant contribution to the development of mathematics.

Mikhail L'vovich was born on Aprill 13, 1945 in Moscow. In 1963, he graduated from School No. 128 in Moscow with a gold medal and entered the Physics Faculty of the Lomonosov Moscow State University. He graduated in 1969 and became a postgraduate student in the Mathematics Department. In 1972, he defended his PhD thesis "On integral representations and Fourier series of differentiable functions of several variables" under the supervision of Professor Ilyin Vladimir Aleksandrovich, and in 1988, his doctoral thesis "Study of spaces of differentiable functions of several variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor of Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an Assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he was a Professor of the Department of Theory of Functions and Differential Equations, then of the S.M. Nikol'skii Mathematical Institute at the Patrice Lumumba Peoples' Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman were: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators.Among the most important scientific achievements of M.L. Goldman, we note his research related to the optimal embedding of spaces with generalized smoothness, exact conditions for the convergence of spectral decompositions, descriptions of the integral and differential properties of generalized potentials of the Bessel and Riesz types, exact estimates for operators on cones, descriptions of optimal spaces for cones of functions with monotonicity properties.

M.L. Goldman has published more than 150 scientific articles in central mathematical journals. He is a laureate of the Moscow government competition, a laureate of the RUDN University Prize in Science and Innovation, and a laureate of the RUDN University Prize for supervision of postgraduate students. Under the supervision of Mikhail L'vovich 11 PhD theses were defended. His pupilss are actively involved in professional work at leading universities and research institutes in Russia, Kazakhstan, Ethiopia, Rwanda, Colombia, and Mongolia.

Mikhail L'vovich has repeatedly been a guest lecturer and guest professor at universities in Russia, Germany, Sweden, Great Britain, etc., and an invited speaker at many international conferences. Mikhail L'vovich was not only an excellent mathematician and teacher (he always spoke about mathematics and its teaching with great passion), but also a man of the highest culture and erudition, with a deep knowledge of history, literature and art, a very bright, kind and responsive person. This is how he will remain in the hearts of his family, friends, colleagues and students.

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Mikhail L'vovich Goldman.

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EXACT SOLUTION TO A STEFAN-TYPE PROBLEM FOR A GENERALIZED HEAT EQUATION WITH THE THOMSON EFFECT

T.A. Nauryz, S.N. Kharin, A.C. Briozzo, J. Bollati

Communicated by Ya.T. Sultanaev

Key words: Stefan problem, generalized heat equation, Thomson effect, similarity solution, nonlinear integral equations, nonlinear thermal coefficient, fixed point theorem.

AMS Mathematics Subject Classification: 80A22, 80A05, 35C11.

Abstract. We study a one-dimensional Stefan type problem which models the behavior of electromagnetic fields and heat transfer in closed electrical contacts that arises, when an instantaneous explosion of the micro-asperity occurs. This model involves vaporization, liquid and solid zones, in which the temperature satisfies a generalized heat equation with the Thomson effect. Accounting for the nonlinear thermal coefficient, the model also incorporates temperature-dependent electrical conductivity. By employing a similarity transformation, the Stefan-type problem is reduced to a system of coupled nonlinear integral equations. The existence of a solution is established using the fixed point theory in Banach spaces.

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1 Introduction

Stefan problems are fundamental in understanding the phase transition phenomena, particularly in situations involving heat transfer and solidification processes. They were first introduced by J. Stefan in his seminal work in [24]. These problems concern the determination of the moving boundary between phases during the process of solidification or melting.

The classical Stefan problem arises in scenarios, in which a material undergoes a phase change, such as freezing or melting, subject to certain boundary conditions and physical constraints. One of the key aspects of Stefan problems is the existence of a sharp interface, known as the Stefan interface, which separates the regions of different phases.

Significant theoretical contributions to Stefan problems have been done in [21], [1]. Further, the study of free and moving boundary problems, including Stefan problems, has garnered considerable attention. In [6] J. Crank provides a comprehensive treatment of such problems, offering valuable insights into their mathematical formulation and solution techniques.

Stefan problems, which traditionally deal with phase-change phenomena under classical heat conduction assumptions, have seen extensions to encompass more complex physical scenarios. These extensions, often referred to as non-classical Stefan problems, involve variations in thermal coefficients, boundary conditions, or latent heat dependencies, among other factors. The investigation of non-classical Stefan problems has significant implications in various fields, including materials science, engineering, and mathematical physics.

One avenue of research in non-classical Stefan problems involves the consideration of thermal coefficients that vary with temperature or position. In [4], [2] were explored Stefan problems for diffusion-convection equations with temperature-dependent thermal coefficients, providing insights into the behavior of phase-change processes under such conditions. Similarly, in [18], [17] A. Kumar et al. investigated Stefan problems with variable thermal coefficients, highlighting the impact of these variations on the phase-change dynamics. Furthermore, exact and approximate solutions to the Stefan problem in ellipsoidal coordinates were obtained in [8]

Another aspect of non-classical Stefan problems involves incorporating convective boundary conditions or heat flux conditions on fixed faces. In paper [4] there is examined the existence of exact solutions for one-phase Stefan problems with nonlinear thermal coefficients, incorporating Tirskii's method to handle such complexities. Additionally, paper [5] is devoted to the one-phase Stefan problem for a non-classical heat equation with a heat flux condition on the fixed face, contributing to the understanding of phase-change phenomena under non-standard boundary conditions.

Non-linear Stefan problems offer a valuable mathematical framework to model and analyze complex phenomena, providing insights, for example, into heat transfer processes during phase transitions within electrical contacts [3], [12]-[20].

Thermal phenomena in electrical apparatus, such as welding, arcing, and bridging, contribute to their failure and are highly complex. These phenomena depend on various factors including current, voltage, contact force, contact material properties, and arc duration [23], [7]. Experimental investigations usually focus on cumulative probability representations of resulting values as direct experimental observation of these processes is often challenging or even impossible due to their extremely short duration.

Hence, mathematical modelling plays a crucial role in understanding the dynamics of such processes, improving the endurance and reliability of contact systems, and predicting and preventing failures in electrical apparatus.

Efforts have been made in [22], [9]-[11] to address these aspects and the study of electrical contacts involves intricate thermal dynamics influenced by non-linearities in material properties and heat generation mechanisms.

This paper aims to further develop the existing models to models, also incorporating the Thomson effect.

The Thomson effect refers to the phenomenon, in which a temperature difference is created across an electrical conductor when an electric current flows through it. This effect occurs due to the interaction between the current-carrying electrons and the lattice structure of the conductor.

In the context of a closure of electrical contact after the instantaneous explosion of a micro-asperity, it is important to take into account that micro-asperities are tiny protrusions or irregularities on the surface of a material. An explosion or sudden release of energy can cause these micro-asperities to rupture or deform.

After such an explosion, the closure of electrical contact can manifest itself in several ways. The intense energy release can lead to the melting or vaporization of micro-asperities, altering the surface characteristics of the contact. This can potentially disrupt the normal flow of electric current and create temperature variations due to the Thomson effect.

The Thomson effect in this scenario could result in localized heating or cooling at the contact points, depending on the direction of the current flow. This temperature difference might affect the electrical conductivity and overall performance of the closure of electrical contact.

In the initial phase of a closed electrical contact, when a micro-asperity undergoes sudden ignition, the contact region comprises both a metallic vaporization zone and a liquid domain, see Figure 1. Modelling the metallic vapour zone, denoted as Z_0 with a height range of $0 < z < s(t)$, is a complex undertaking. We propose that the temperature within this region decreases linearly from the ionization temperature of the metallic vapour, denoted as T_{ion}, which occurs after the explosion at the fixed face $z = 0$, to the boiling temperature T_b at the free boundary that separates the vapour and liquid phases. The temperature field within the vapour zone Z_0 exhibits a gradual and linear

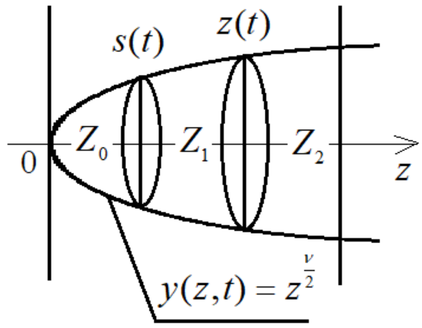

Figure 1: Contact zones: $Z_0 : (0 < z < s(t))$-vaporization zone, $Z_1 : (s(t) < z < r(t))$-liquid zone, $Z_2 : (r(t) < z)$-solid zone.

decrease

$$T_V(z,t) = \frac{z}{s(t)}(T_b - T_{ion}) + T_{ion}, \quad 0 \le z \le s(t), \tag{1.1}$$

where the following boundary conditions hold

$$T_V(0,t) = T_{ion}, \tag{1.2}$$

$$T_V(s(t),t) = T_b. \tag{1.3}$$

Temperature distribution and electrical potential field of the zones Z_1 and Z_2 are defined by the following relations:

$$c(T_1)\gamma(T_1)\frac{\partial T_1}{\partial t} = \frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\lambda(T_1)z^\nu\frac{\partial T_1}{\partial z}\right] + \sigma_{T_1}\frac{\partial T_1}{\partial z}\frac{\partial \varphi_1}{\partial z} + \frac{1}{\rho(T_1)}\left(\frac{\partial \varphi_1}{\partial z}\right)^2, \tag{1.4}$$

$$\frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\frac{1}{\rho(T_1)}z^\nu\frac{\partial \varphi_1}{\partial z}\right] = 0, \quad s(t) < z < r(t), \quad t > 0, \quad 0 < \nu < 1, \tag{1.5}$$

$$c(T_2)\gamma(T_2)\frac{\partial T_2}{\partial t} = \frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\lambda(T_2)z^\nu\frac{\partial T_2}{\partial z}\right] + \sigma_{T_2}\frac{\partial T_2}{\partial z}\frac{\partial \varphi_2}{\partial z} + \frac{1}{\rho(T_2)}\left(\frac{\partial \varphi_2}{\partial z}\right)^2, \tag{1.6}$$

$$\frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\frac{1}{\rho(T_2)}z^\nu\frac{\partial \varphi_2}{\partial z}\right] = 0, \quad r(t) < z, \quad t > 0, \quad 0 < \nu < 1, \tag{1.7}$$

$$T_1(s(t),t) = T_b, \quad t > 0, \tag{1.8}$$

$$-\lambda\left(T_1(s(t),t)\right)\frac{\partial T_1}{\partial z}\bigg|_{z=s(t)} = \frac{Q_0 e^{-s_0^2}}{2a\sqrt{\pi t}}, \quad t > 0, \tag{1.9}$$

$$\varphi_1(s(t),t) = 0, \quad t > 0, \tag{1.10}$$

$$T_1(r(t),t) = T_2(r(t),t) = T_m > 0, \quad t > 0, \tag{1.11}$$

$$\varphi_1(r(t),t) = \varphi_2(r(t),t), \quad t > 0, \tag{1.12}$$

$$-\lambda\left(T_1(r(t),t)\right)\frac{\partial T_1}{\partial z}\bigg|_{z=r(t)} + \lambda\left(T_2(r(t),t)\right)\frac{\partial T_2}{\partial z}\bigg|_{z=r(t)} = l_m\gamma_m\frac{dr}{dt}, \quad t > 0, \tag{1.13}$$

$$\frac{1}{\rho(T_1(r(t),t))}\frac{\partial \varphi_1}{\partial z}\bigg|_{z=r(t)} = \frac{1}{\rho(T_2(r(t),t))}\frac{\partial \varphi_2}{\partial z}\bigg|_{z=r(t)}, \quad t > 0, \tag{1.14}$$

$$T_2(+\infty,t) = 0, \quad t > 0, \tag{1.15}$$

$$\varphi_2(+\infty,t) = \frac{U_c}{2}, \quad t>0, \tag{1.16}$$

$$T_2(z,0) = \varphi_2(z,0) = 0, \quad z>0, \qquad s(0) = r(0) = 0, \tag{1.17}$$

where T_1, T_2 and φ_1, φ_2 are temperatures and electrical potential fields for liquid and solid zones, $c(T_i)$,$\gamma(T_i)$ and $\lambda(T_i)$ are specific heat, density and thermal conductivity which depend on the temperature, σ_{T_i} is the Thomson coefficient, $\rho(T_i)$ is the electrical resistivity, $Q_0>0$ is the power of the heat flux, T_m is the melting temperature, U_c is the contact voltage, $s(t)$ and $r(t)$ are locations of the boiling and melting interfaces.

This paper is structured as follows. In Section 2, we use the similarity transformation to obtain an equivalent system of coupled integral equations for problem (1.4)-(1.17). In Section 3, we define proper spaces in order to apply the fixed point Banach theorem to prove the existence of a solution to the system of coupled integral equations.

The contribution of the problem addressed in our paper has significant implications for electrical engineering. By developing a mathematical model that captures the behavior of electromagnetic fields and heat transfer in closed electrical contacts, particularly during instantaneous micro-asperity explosions, we offer valuable insights into the complex dynamics of these systems.

Our model accounts for the non-linear nature of thermal coefficients and temperature-dependent electrical conductivity, factors that are crucial in accurately representing real-world scenarios. By considering vaporization, liquid, and solid zones within the contact, we provide a comprehensive framework for analyzing the thermal and electromagnetic effects associated with such phenomena.

Furthermore, our approach, which utilizes similarity transformations to reduce the Stefan-type problem to a system of nonlinear integral equations, offers practical methodologies for analysing and predicting the closure of electrical contacts under extreme conditions. The rigorous establishment of the validity of this approach through discussions and proofs supported by the fixed point theory in Banach spaces enhances the reliability and applicability of our proposed solutions.

2 Integral formulation

In this section, taking into account that problem (1.4)-(1.17) can be thought as a Stefan-type problem, we look for similarity type solutions that depend on the similarity variable

$$\eta = \frac{z}{2a\sqrt{t}},$$

with $a=\sqrt{\frac{\lambda_0}{\rho_0 c_0}}$ where λ_0, ρ_0 and c_0 are reference thermal coefficients.

We propose the following transformation

$$f_i(\eta) = \frac{T_i(z,t)-T_m}{T_m}, \qquad \phi_i(\eta) = \varphi_i(z,t), \quad , \quad i=1,2. \tag{2.1}$$

According to this transformation, the location of the boiling and melting fronts are given by

$$s(t) = 2as_0\sqrt{t}, \qquad r(t) = 2ar_0\sqrt{t}, \tag{2.2}$$

where s_0 and r_0 must be determined as a part of the solution.

Therefore, problem (1.4)-(1.17) can be rewritten in the following form:

$$[L(f_i)\eta^\nu f_i']' + 2a\eta^{\nu+1}N(f_i)f_i' + \frac{\sigma_{f_i}}{c_0\gamma_0 a}\eta^\nu f_i'\phi_i' + \frac{\eta^\nu}{c_0\gamma_0 T_m a K(f_i)}(\phi_i')^2 = 0, \tag{2.3}$$

$$\left[\frac{1}{K(f_i)}\eta^\nu \phi_i'\right]' = 0, \tag{2.4}$$

$$i = 1: \quad s_0 < \eta < r_0, \qquad i = 2: \quad \eta > r_0,$$

$$f_1(s_0) = B, \tag{2.5}$$

$$L(f_1(s_0))f_1'(s_0) = -Qe^{-s_0^2}, \tag{2.6}$$

$$\phi_1(s_0) = 0, \tag{2.7}$$

$$f_1(r_0) = f_2(r_0) = 0, \tag{2.8}$$

$$\phi_1(r_0) = \phi_2(r_0), \tag{2.9}$$

$$-L(f_1(r_0))f_1'(r_0) = -L(f_2(r_0))f_2'(r_0) + Mr_0, \tag{2.10}$$

$$\frac{1}{K(f_1(r_0))}\phi_1'(r_0) = \frac{1}{K(f_2(r_0))}\phi_2'(r_0), \tag{2.11}$$

$$f_2(+\infty) = -1, \tag{2.12}$$

$$\phi_2(+\infty) = \frac{U_c}{2}, \tag{2.13}$$

where

$$B = \frac{T_b - T_m}{T_m}, \qquad Q = \frac{Q_0}{\lambda_0 T_m \sqrt{\pi}} > 0, \quad M = \frac{2l_m\gamma_m a^2}{\lambda_0 T_m} > 0 \tag{2.14}$$

and for $i = 1, 2$:

$$N(f_i) = \frac{c(f_iT_m + T_m)\gamma(f_iT_m + T_m)}{c_0\gamma_0}, \tag{2.15}$$

$$L(f_i) = \frac{\lambda(f_iT_m + T_m)}{\lambda_0}, \tag{2.16}$$

$$K(f_i) = \rho(f_iT_m + T_m), \tag{2.17}$$

$$\sigma_{f_i} = \sigma_{T_i}, \tag{2.18}$$

From (2.4), (2.7), (2.9), (2.11) and (2.13), we obtain the solution for electrical potential field for liquid and solid zones explicitly depending on f_1, f_2, s_0 and r_0 as

$$\phi_1(\eta, s_0, r_0, f_1, f_2) = \frac{U_cF_1(\eta, s_0, f_1)}{2H(r_0, s_0, f_1, f_2)}, \quad s_0 \leq \eta \leq r_0, \tag{2.19}$$

$$\phi_2(\eta, s_0, r_0, f_1, f_2) = \frac{U_c\left(F_1(r_0, s_0, f_1) + F_2(\eta, r_0, f_2)\right)}{2H(r_0, s_0, f_1, f_2)}, \quad \eta \geq r_0, \tag{2.20}$$

where

$$F_1(\eta, s_0, f_1) = \int_{s_0}^{\eta} \frac{K(f_1(v))}{v^\nu}dv, \quad s_0 \leq \eta \leq r_0, \tag{2.21}$$

$$F_2(\eta, r_0, f_2) = \int_{r_0}^{\eta} \frac{K(f_2(v))}{v^\nu}dv, \quad \eta \geq r_0, \tag{2.22}$$

and

$$H(r_0, s_0, f_1, f_2) = F_1(r_0, s_0, f_1) + F_2(+\infty, r_0, f_2). \tag{2.23}$$

In addition, from (2.3), (2.6) and (2.8), we get

$$f_1(\eta) = s_0^\nu Q \exp(-s_0^2)\left[\Phi_1(r_0, s_0, f_1, f_2) - \Phi_1(\eta, s_0, f_1, f_2)\right]$$

$$+\frac{D_1^*}{H^2(r_0,s_0,f_1,f_2)}\left[G_1(r_0,s_0,f_1,f_2)-G_1(\eta,s_0,f_1,f_2)\right],\quad s_0\leq\eta\leq r_0, \tag{2.24}$$

and from (2.3), (2.8) and (2.12) we get

$$f_2(\eta)=\left[\frac{D_2^*}{H^2(r_0,s_0,f_1,f_2)}G_2(+\infty,r_0,f_1,f_2)-1\right]\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}$$
$$-\frac{D_2^*}{H^2(r_0,s_0,f_1,f_2)}G_2(\eta,r_0,f_1,f_2),\quad \eta\geq r_0. \tag{2.25}$$

Moreover, from conditions (2.5) and (2.10) we obtain the following equalities:

$$s_0^\nu Q\exp(-s_0^2)\Phi_1(r_0,s_0,f_1,f_2)+\frac{D_1^*}{H^2(r_0,s_0,f_1,f_2)}G_1(r_0,s_0,f_1,f_2)=B, \tag{2.26}$$

and

$$E_1(r_0,s_0,f_1,f_2)\left[Q\exp(-s_0^2)s_0^\nu+\frac{D_1^*}{H^2(r_0,s_0,f_1,f_2)}H_1(r_0,s_0,f_1,f_2)\right]$$
$$-\frac{1}{\Phi_2(+\infty,r_0,f_1,f_2)}\left[1-\frac{D_2^*}{H^2(r_0,s_0,f_1,f_2)}G_2(+\infty,r_0,f_1,f_2)\right]=Mr_0^{\nu+1}, \tag{2.27}$$

where

$$\Phi_1(\eta,s_0,f_1,f_2)=\int_{s_0}^{\eta}\frac{E_1(v,s_0,f_1,f_2)}{L(f_1(v))v^\nu}dv,\quad s_0\leq\eta\leq r_0, \tag{2.28}$$

$$\Phi_2(\eta,r_0,f_1,f_2)=\int_{r_0}^{\eta}\frac{E_2(v,r_0,f_1,f_2)}{L(f_2(v))v^\nu}dv,\quad \eta\geq r_0, \tag{2.29}$$

$$G_1(\eta,s_0,f_1,f_2)=\int_{s_0}^{\eta}\frac{E_1(v,s_0,f_1,f_2)}{L(f_1(v))v^\nu}H_1(v,r_0,f_1,f_2)dv,\quad s_0\leq\eta\leq r_0, \tag{2.30}$$

$$G_2(\eta,r_0,f_1,f_2)=\int_{r_0}^{\eta}\frac{E_2(v,s_0,f_1,f_2)}{L(f_2(v))v^\nu}H_2(v,r_0,f_1,f_2)dv\quad \eta\geq r_0 \tag{2.31}$$

$$H_1(\eta,s_0,f_1,f_2)=\int_{s_0}^{\eta}\frac{K(f_1(v))}{v^\nu E_1(v,s_0,f_1,f_2)}dv,\quad s_0\leq\eta\leq r_0, \tag{2.32}$$

$$H_2(\eta,r_0,f_1,f_2)=\int_{r_0}^{\eta}\frac{K(f_2(v))}{v^\nu E_2(v,r_0,f_1,f_2)}dv\quad \eta\geq r_0 \tag{2.33}$$

$$E_1(\eta,s_0,f_1,f_2)=\exp\left(-\int_{s_0}^{\eta}\left[2av\tfrac{N(f_1(v))}{L(f_1(v))}+\tfrac{D_1}{H(r_0,s_0,f_1,f_2)}\tfrac{K(f_1(v))}{L(f_1(v))v^\nu}\right]dv\right),\quad s_0\leq\eta\leq r_0, \tag{2.34}$$

$$E_2(\eta,r_0,f_1,f_2)=\exp\left(-\int_{r_0}^{\eta}\left[2av\tfrac{N(f_2(v))}{L(f_2(v))}+\tfrac{D_2}{H(r_0,s_0,f_1,f_2)}\tfrac{K(f_2(v))}{L(f_2(v))v^\nu}\right]dv\right),\quad \eta\geq r_0, \tag{2.35}$$

and the coefficients D_i and D_i^* for $i = 1, 2$ are given by

$$D_i = \frac{\sigma_{f_i} U_c}{2c_0\gamma_0 a}, \qquad D_i^* = \frac{U_c D_i}{2}. \tag{2.36}$$

In conclusion, to find a similarity solution to problem (1.4)-(1.17) is equivalent to obtain f_1, f_2, s_0 and r_0 such that (2.24), (2.25), (2.26) and (2.27) hold. Notice that the electric potential fields ϕ_1 and ϕ_2 are explicitly given by (2.19) and (2.20) as functions of f_1, f_2, s_0 and r_0.

In the next section, to address the existence and uniqueness of solutions, we employ a rigorous analytical approach. We leverage similarity transformations to reduce the problem to a set of ordinary differential equations, facilitating a more tractable analysis. Additionally, we draw upon the fixed point theory in Banach spaces to establish the validity of our proposed solutions.

3 Existence of solution

In order to prove the existence and uniqueness of solution f_1, f_2 to equations (2.24) and (2.25), we fix positive constants $0 < s_0 < r_0$ and consider the Banach space

$$\mathcal{C} = C[s_0, r_0] \times C_b[r_0, +\infty) \tag{3.1}$$

endowed with the norm

$$||\vec{f}|| = \|(f_1, f_2)\| = \max\left\{||f_1||_{C[s_0,r_0]}, ||f_2||_{C_b[r_0,+\infty)}\right\},$$

where $C[s_0, r_0]$ denotes the space of all continuous functions defined on the interval $[s_0, r_0]$ and $C_b[r_0, +\infty)$ represents the space of all continuous and bounded functions on the interval $[r_0, +\infty)$. We define the closed subset $\mathcal{M}$ of $C_b[r_0, +\infty)$ by

$$\mathcal{M} = \{f_2 \in C_b[r_0, +\infty) : f_2(r_0) = 0, f_2(+\infty) = -1\}.$$

We consider the operator Ψ on $\mathcal{K} = C[s_0, r_0] \times \mathcal{M}$ given by

$$\Psi(\vec{f}) = (V_1(\vec{f}), V_2(\vec{f})), \tag{3.2}$$

where $V_1(\vec{f})$, $V_2(\vec{f})$ are defined by

$$\begin{aligned} V_1(\vec{f})(\eta) &= s_0^{\nu} Q \exp(-s_0^2)\left[\Phi_1(r_0, s_0, f_1, f_2) - \Phi_1(\eta, s_0, f_1, f_2)\right] \\ &+ \tfrac{D_1^*}{H^2(r_0,s_0,f_1,f_2)}\Big[G_1(r_0, s_0, f_1, f_2) - G_1(\eta, s_0, f_1, f_2)\Big], \quad s_0 \le \eta \le r_0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} V_2(\vec{f})(\eta) &= \Big[\tfrac{D_2^*}{H^2(r_0,s_0,f_1,f_2)} G_2(+\infty, r_0, f_1, f_2) - 1\Big] \tfrac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)} \\ &- \tfrac{D_2^*}{H^2(r_0,s_0,f_1,f_2)} G_2(\eta, r_0, f_1, f_2), \quad \eta \ge r_0. \end{aligned} \tag{3.4}$$

Notice that solving the system of equations (2.24) and (2.25) is equivalent to obtaining a fixed point to the operator Ψ.

Taking into account that $\mathcal{K}$ is a closed subset of $\mathcal{C}$ we will prove that $\Psi(\mathcal{K}) \subset \mathcal{K}$ and Ψ is a contraction mapping in order to apply the fixed point Banach theorem.

For this purpose we will assume that there exists positive coefficients μ, L_{im}, L_{iM}, N_{im} and N_{iM}, $\widetilde{L}_i$, $\widetilde{N}_i$ and $\widetilde{K}_i$ for $i = 1, 2$ such that

(A1) for each $f_1 \in C[s_0, r_0] : s_0 \leq v \leq r_0$

$$L_{1m}\eta^{\mu} \leq L(f_1)(\eta) \leq L_{1M}\eta^{\mu}, \tag{3.5}$$

$$N_{1m}\eta^{-\mu} \leq N(f_1)(\eta) \leq N_{1M}\eta^{-\mu}, \tag{3.6}$$

$$K_{1m}\eta^{-\mu} \leq K(f_1)(\eta) \leq K_{1M}\eta^{-\mu}, \tag{3.7}$$

(A2) for each $f_2 \in \mathcal{M},\ \eta \geq r_0$:

$$L_{2m}\eta^{\mu} \leq L(f_2)(\eta) \leq L_{2M}\eta^{\mu}, \tag{3.8}$$

$$N_{2m}\eta^{-\mu} \leq N(f_2)(\eta) \leq N_{2M}\eta^{-\mu}, \tag{3.9}$$

$$K_{2m}\eta^{-\mu} \leq K(f_2)(\eta) \leq K_{2M}\eta^{-\mu}, \tag{3.10}$$

(A3) for each $f_1, g_1 \in C[s_0, r_0], s_0 \leq \eta \leq r_0$:

$$|L(f_1(\eta)) - L(g_1(\eta))| \leq \widetilde{L}_1||f_1 - g_1||, \tag{3.11}$$

$$|N(f_1(\eta)) - N(g_1(\eta))| \leq \widetilde{N}_1||f_1 - g_1||, \tag{3.12}$$

$$|K(f_1(\eta)) - K(g_1(\eta))| \leq \widetilde{K}_1\eta^{-\mu}||f_1 - g_1||, \tag{3.13}$$

(A4) for each $f_2, g_2 \in \mathcal{M}, \eta \geq r_0$:

$$|L(f_2(\eta)) - L(g_2(\eta))| \leq \widetilde{L}_2||f_2 - g_2||, \tag{3.14}$$

$$|N(f_2(\eta)) - N(g_2(\eta))| \leq \widetilde{N}_2||f_2 - g_2||, \tag{3.15}$$

$$|K(f_2(\eta)) - K(g_2(\eta))| \leq \widetilde{K}_2\eta^{-\mu}||f_2 - g_2||, \tag{3.16}$$

(A5) $\mu > 2$.

From now on, hypothesis (A1)-(A5) will be assumed to hold throughout the paper.

We will present preliminary results that will be useful to prove the existence and uniqueness of a fixed point of the operator Ψ.

Lemma 3.1. *For every $\vec{f} = (f_1, f_2)$, $\vec{g} = (g_1, g_2) \in \mathcal{K}$, the following inequalities hold:*

$$H(r_0, s_0, f_1, f_2) \geq H_{inf}(r_0, s_0), \tag{3.17}$$

$$H(r_0, s_0, f_1, f_2) \leq H_{sup}(r_0, s_0), \tag{3.18}$$

$$|H(r_0, s_0, f_1, f_2) - H(r_0, s_0, g_1, g_2)| \leq \widetilde{H}(r_0, s_0)||\vec{f} - \vec{g}||, \tag{3.19}$$

where

$$H_{inf}(r_0, s_0) := \tfrac{K_{1m}}{\mu+\nu-1}\left(\tfrac{1}{s_0^{\mu+\nu-1}} - \tfrac{1}{r_0^{\mu+\nu-1}}\right), \tag{3.20}$$

$$H_{sup}(r_0, s_0) := \tfrac{1}{\mu+\nu-1}\left(\tfrac{K_{1M}}{s_0^{\mu+\nu-1}} + \tfrac{K_{2M}}{r_0^{\mu+\nu-1}}\right), \tag{3.21}$$

$$\widetilde{H}(r_0, s_0) := \tfrac{1}{\mu+\nu-1}\left(\tfrac{\widetilde{K}_1}{s_0^{\mu+\nu-1}} + \tfrac{\widetilde{K}_2}{r_0^{\mu+\nu-1}}\right). \tag{3.22}$$

Proof. Taking into account the definition of H given by (2.23) and assumptions (3.7) and (3.10), we have

$$H(r_0,s_0,f_1,f_2) \geq \frac{K_{1m}}{\mu+\nu-1}\left(\frac{1}{s_0^{\mu+\nu-1}}-\frac{1}{r_0^{\mu+\nu-1}}\right)+\frac{K_{2m}}{\mu+\nu-1}\frac{1}{r_0^{\mu+\nu-1}}$$

$$\geq \frac{K_{1m}}{\mu+\nu-1}\left(\frac{1}{s_0^{\mu+\nu-1}}-\frac{1}{r_0^{\mu+\nu-1}}\right),$$

and then we get (3.17). Inequality (3.18) follows analogously. In addition, taking into account assumptions (3.13) and (3.16), we get

$$|H(r_0,s_0,f_1,f_2)-H(r_0,s_0,g_1,g_2)|$$

$$\leq \left(\widetilde{K}_1\int_{s_0}^{r_0}\frac{1}{v^{\mu+\nu}}dv+\widetilde{K}_2\int_{r_0}^{+\infty}\frac{1}{v^{\mu+\nu}}dv\right)||\vec{f}-\vec{g}||$$

$$\leq \frac{1}{\mu+\nu-1}\left(\widetilde{K}_1\left(\frac{1}{s_0^{\mu+\nu-1}}-\frac{1}{r_0^{\mu+\nu-1}}\right)+\frac{\widetilde{K}_2}{r_0^{\mu+\nu-1}}\right)||\vec{f}-\vec{g}||$$

$$\leq \frac{1}{\mu+\nu-1}\left(\frac{\widetilde{K}_1}{s_0^{\mu+\nu-1}}+\frac{\widetilde{K}_2}{r_0^{\mu+\nu-1}}\right)||\vec{f}-\vec{g}||,$$

and, as a corollary, inequality (3.19) holds. □

Lemma 3.2. *For every $\vec{f}=(f_1,f_2)$, $\vec{g}=(g_1,g_2)\in\mathcal{K}$, the following inequalities hold:*

1)

$$E_1(\eta,s_0,f_1,f_2)\geq E_{1inf}(r_0,s_0), \tag{3.23}$$

$$E_1(\eta,s_0,f_1,f_2)\leq 1, \tag{3.24}$$

$$|E_1(\eta,s_0,f_1,f_2)-E_1(\eta,s_0,g_1,g_2)|\leq \widetilde{E}_1(r_0,s_0)||\vec{f}-\vec{g}||, \tag{3.25}$$

where

$$E_{1inf}(r_0,s_0):=\exp\left(-\left[a\frac{N_{1M}}{L_{1m}(\mu-1)}\frac{1}{s_0^{2\mu-2}}+\frac{D_1K_{1M}}{H_{inf}(r_0,s_0)L_{1m}(2\mu+\nu-1)}\frac{1}{s_0^{2\mu+\nu-1}}\right]\right), \tag{3.26}$$

$$\begin{aligned}\widetilde{E}_1(r_0,s_0):=&\,2a\left[\frac{\widetilde{N}_1}{L_{1m}(\mu-2)}\frac{1}{s_0^{\mu-2}}+\frac{N_{1M}\widetilde{L}_1}{L_{1m}^2(3\mu-2)}\frac{1}{s_0^{3\mu-2}}\right]\\&+D_1\left(\frac{\widetilde{K}_1}{H_{inf}(r_0,s_0)L_{1m}(2\mu+\nu-1)}\frac{1}{s_0^{2\mu+\nu-1}}\right.\\&\left.+\frac{K_{1M}}{H_{inf}(r_0,s_0)L_{1m}}\left(\frac{\widetilde{H}(r_0,s_0)}{H_{inf}(r_0,s_0)(2\mu+\nu-1)}\frac{1}{s_0^{2\mu+\nu-1}}+\frac{\widetilde{L}_1}{L_{1m}(3\mu+\nu-1)}\frac{1}{s_0^{3\mu+\nu-1}}\right)\right);\end{aligned} \tag{3.27}$$

2)

$$|\Phi_1(\eta,s_0,f_1,f_2)-\Phi_1(\eta,s_0,g_1,g_2)|\leq \widetilde{\Phi}_1(r_0,s_0)||\vec{f}-\vec{g}||, \tag{3.28}$$

where

$$\widetilde{\Phi}_1(r_0,s_0):=\frac{\widetilde{E}_1(r_0,s_0)}{L_{1m}(\mu+\nu-1)}\frac{1}{s_0^{\nu+\mu-1}}+\frac{\widetilde{L}_1}{L_{1m}^2(2\mu+\nu-1)}\frac{1}{s_0^{\nu+2\mu-1}}; \tag{3.29}$$

3)

$$H_1(\eta,s_0,f_1,f_2)\geq H_{1inf}(\eta,s_0), \tag{3.30}$$

$$H_1(\eta,s_0,f_1,f_2)\leq H_{1sup}(r_0,s_0), \tag{3.31}$$

$$|H_1(\eta,s_0,f_1,f_2)-H_1(\eta,s_0,g_1,g_2)|\leq \widetilde{H}_1(r_0,s_0)||\vec{f}-\vec{g}||, \tag{3.32}$$

where

$$H_{1inf}(\eta, s_0) := \frac{K_{1m}}{(\mu+\nu-1)} \left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}} \right), \tag{3.33}$$

$$H_{1sup}(r_0, s_0) := \frac{K_{1M}}{E_{1inf}(r_0,s_0)} \frac{1}{(\mu+\nu-1)} \frac{1}{s_0^{\mu+\nu-1}}, \tag{3.34}$$

$$\widetilde{H}_1(r_0, s_0) := \left(\widetilde{K}_1 + \frac{K_{1M}\widetilde{E}_1(r_0,s_0)}{E_{1inf}(r_0,s_0)} \right) \frac{1}{E_{1inf}(r_0,s_0)(\mu+\nu-1)} \frac{1}{s_0^{\mu+\nu-1}}; \tag{3.35}$$

4)

$$G_1(\eta, s_0, f_1, f_2) \geq G_{1inf}(\eta, r_0, s_0) \tag{3.36}$$

$$G_1(\eta, s_0, f_1, f_2) \leq G_{1sup}(r_0, s_0) \tag{3.37}$$

$$|G_1(\eta, s_0, f_1, f_2) - G_1(\eta, s_0, g_1, g_2)| \leq \widetilde{G}_1(r_0, s_0)||\vec{f} - \vec{g}|| \tag{3.38}$$

where

$$G_{1inf}(\eta, r_0, s_0) := \frac{K_{1m}E_{1inf}(r_0,s_0)}{2L_{1M}(\mu+\nu-1)^2} \left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}} \right)^2, \tag{3.39}$$

$$G_{1sup}(r_0, s_0) := \frac{H_{1sup}(r_0,s_0)}{L_{1m}} \frac{1}{(\mu+\nu-1)} \frac{1}{s_0^{\mu+\nu-1}}, \tag{3.40}$$

$$\widetilde{G}_1(r_0, s_0) := H_{1sup}(r_0, s_0)\widetilde{\Phi}_1(r_0, s_0) + \frac{\widetilde{H}_1(r_0,s_0)}{L_{1m}} \frac{1}{(\mu+\nu-1)} \frac{1}{s_0^{\mu+\nu-1}}. \tag{3.41}$$

Proof. From the definition of E_1 given by (2.34), assumptions (3.5)-(3.7) and inequality (3.17) we obtain that

$$\int_{s_0}^{\eta} 2av\frac{N(f_1(v))}{L(f_1(v)} + \frac{D_1}{H(r_0,s_0,f_1,f_2)} \frac{K(f_1(v))}{L(f_1(v))v^\nu} dv$$

$$\leq \int_{s_0}^{\eta} 2a\frac{N_{1M}}{L_{1m}} \frac{1}{v^{2\mu-1}} + \frac{D_1K_{1M}}{H_{inf}(r_0,s_0)L_{1m}} \frac{1}{v^{2\mu+\nu}} dv$$

$$\leq 2a\frac{N_{1M}}{L_{1m}(2\mu-2)} \frac{1}{s_0^{2\mu-2}} + \frac{D_1K_{1M}}{H_{inf}(r_0,s_0)L_{1m}(2\mu+\nu-1)} \frac{1}{s_0^{2\mu+\nu-1}}.$$

As a corollary it follows that $E_1(\eta, s_0, f_1, f_2) \geq E_{1inf}(r_0, s_0)$ with $E_{1inf}(r_0, s_0)$ given by (3.26). In addition, as E_1 is a negative exponential function, it follows that $E_1(\eta, s_0, f_1, f_2) \leq 1$.

Let us analyse the difference of E_1. From the inequality $|\exp(-x) - \exp(-y)| \leq |x - y|$ we have

$$\begin{aligned} &|E_1(\eta, s_0, f_1, f_2) - E_1(\eta, s_0, g_1, g_2)| \leq \int_{s_0}^{\eta} 2av \left| \frac{N(f_1(v))}{L(f_1(v))} - \frac{N(g_1(v))}{L(g_1(v))} \right| dv \\ &+D_1 \int_{s_0}^{\eta} \left| \frac{K(f_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^\nu} - \frac{K(g_1(v))}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^\nu} \right| dv. \end{aligned} \tag{3.42}$$

On one hand,

$$\begin{aligned} &\int_{s_0}^{\eta} 2av \left| \frac{N(f_1(v))}{L(f_1(v))} - \frac{N(g_1(v))}{L(g_1(v))} \right| dv \\ &\leq 2a \int_{s_0}^{\eta} \left| v\frac{N(f_1(v))L(g_1(v)) - N(g_1(v))L(g_1(v)) + N(f_1(v))L(g_1(v)) - N(g_1(v))L(f_1(v))}{L(f_1(v))L(g_1(v))} \right| dv \\ &\leq 2a \left[\int_{s_0}^{\eta} \frac{v\widetilde{N}_1||\vec{f}-\vec{g}||}{L_{1m}v^\mu} dv + \int_{s_0}^{\eta} \frac{N_{1M}\widetilde{L}_1 v^{1-\mu}||\vec{f}-\vec{g}||}{L_{1m}^2 v^{2\mu}} dv \right] \\ &\leq 2a \left[\frac{\widetilde{N}_1}{L_{1m}(\mu-2)} \frac{1}{s_0^{\mu-2}} + \frac{N_{1M}\widetilde{L}_1}{L_{1m}^2(3\mu-2)} \frac{1}{s_0^{3\mu-2}} \right] ||\vec{f} - \vec{g}||. \end{aligned} \tag{3.43}$$

On the other hand,

$$
\begin{aligned}
& D_1 \int_{s_0}^{\eta} \left| \frac{K(f_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^\nu} - \frac{K(g_1(v))}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^\nu} \right| dv \\
& \leq D_1 \Bigg(\int_{s_0}^{\eta} \left| \frac{K(f_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^\nu} - \frac{K(g_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^\nu} \right| dv \\
& + \int_{s_0}^{\eta} \left| \frac{K(g_1(v))}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^\nu} - \frac{K(g_1(v))}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^\nu} \right| dv \Bigg) \\
& \leq D_1 \Bigg(\int_{s_0}^{\eta} \frac{|K(f_1(v)) - K(g_1(v))|}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^\nu} dv \\
& + \int_{s_0}^{\eta} |K(g_1(v))| \left| \frac{1}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^\nu} - \frac{1}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^\nu} \right| dv \Bigg).
\end{aligned}
\tag{3.44}
$$

From assumptions (3.5), (3.7), (3.13) and inequalities (3.17), (3.19) we get that

$$
\int_{s_0}^{\eta} \frac{|K(f_1(v)) - K(g_1(v))|}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^\nu} dv \leq \frac{\widetilde{K}_1}{H_{inf}(r_0,s_0)L_{1m}(2\mu+\nu-1)} \frac{1}{s_0^{2\mu+\nu-1}} ||\vec{f} - \vec{g}|| \tag{3.45}
$$

and

$$
\begin{aligned}
& \int_{s_0}^{\eta} |K(g_1(v))| \left| \frac{1}{H(r_0,s_0,f_1,f_2)L(f_1(v))v^\nu} - \frac{1}{H(r_0,s_0,g_1,g_2)L(g_1(v))v^\nu} \right| dv \\
& \leq K_{1M} \Bigg(\int_{s_0}^{\eta} \frac{|H(r_0,s_0,g_1,g_2) - H(r_0,s_0,f_1,f_2)|}{H(r_0,s_0,f_1,f_2)L(f_1(v))H(r_0,s_0,g_1,g_2)} \frac{dv}{v^{\mu+\nu}} \\
& + \int_{s_0}^{\eta} \frac{|L(g_1(v)) - L(f_1(v))|}{L(f_1(v))H(r_0,s_0,g_1,g_2)L(g_1(v))} \frac{dv}{v^{\mu+\nu}} \Bigg) \\
& \leq \frac{K_{1M}}{H_{inf}(r_0,s_0)L_{1m}} \left(\frac{\widetilde{H}(r_0,s_0)}{H_{inf}(r_0,s_0)(2\mu+\nu-1)} \frac{1}{s_0^{2\mu+\nu-1}} + \frac{\widetilde{L}_1}{L_{1m}(3\mu+\nu-1)} \frac{1}{s_0^{3\mu+\nu-1}} \right) ||\vec{f} - \vec{g}||.
\end{aligned}
\tag{3.46}
$$

Then inequalities (3.42)-(3.46) imply that

$$
|E_1(\eta, s_0, f_1, f_2) - E_1(\eta, s_0, g_1, g_2)| \leq \widetilde{E}_1(r_0, s_0) ||\vec{f} - \vec{g}||
$$

with $\widetilde{E}_1$ given by (3.27).

From the definition of Φ_1 given by (2.19) we have that

$$
\begin{aligned}
& |\Phi_1(\eta, s_0, f_1, f_2) - \Phi_1(\eta, s_0, g_1, g_2)| \leq \int_{s_0}^{\eta} \left| \frac{E_1(v,s_0,f_1,f_2)}{L(f_1(v))} - \frac{E_1(v,s_0,g_1,g_2)}{L(g_1(v))} \right| \frac{dv}{v^\nu} \\
& \leq \int_{s_0}^{\eta} \frac{|E_1(v,s_0,f_1,f_2) - E_1(v,s_0,g_1,g_2)|}{L(f_1(v))} \frac{dv}{v^\nu} + \int_{s_0}^{\eta} \frac{E_1(v,s_0,g_1,g_2)|L(f_1(v)) - L(g_1(v))|}{L(f_1(v))L(g_1(v))} \frac{dv}{v^\nu} \\
& \leq \left(\frac{\widetilde{E}_1(r_0,s_0)}{L_{1m}} \int_{s_0}^{\eta} \frac{1}{v^{\mu+\nu}} dv + \frac{\widetilde{L}_1}{L_{1m}^2} \int_{s_0}^{\eta} \frac{1}{v^{2\mu+\nu}} dv \right) ||\vec{f} - \vec{g}|| \leq \widetilde{\Phi}_1(r_0, s_0) ||\vec{f} - \vec{g}||,
\end{aligned}
$$

where $\widetilde{\Phi}_1(r_0, s_0)$ is given by (3.29).

Taking into account the definition of H_1 given by (2.32), we easily obtain that

$$H_1(\eta, s_0, f_1, f_2) \geq K_{1m} \int_{s_0}^{\eta} \frac{1}{v^{\mu+\nu}} dv \geq \frac{K_{1m}}{(\mu+\nu-1)} \left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}} \right),$$

$$|H_1(\eta, s_0, f_1, f_2)| \leq \frac{K_{1M}}{E_{1inf}(r_0,s_0)} \int_{s_0}^{\eta} \frac{1}{v^{\mu+\nu}} dv \leq \frac{K_{1M}}{E_{1inf}(r_0,s_0)} \frac{1}{(\mu+\nu-1)} \frac{1}{s_0^{\mu+\nu-1}},$$

then (3.30) and (3.31) hold. In addition,

$$|H_1(\eta, s_0, f_1, f_2) - H_1(\eta, s_0, g_1, g_2)| \leq \int_{s_0}^{\eta} \left| \frac{K(f_1(v))}{E_1(v,s_0,f_1,f_2)} - \frac{K(g_1(v))}{E_1(v,s_0,g_1,g_2)} \right| \frac{dv}{v^{\nu}}$$

$$\leq \int_{s_0}^{\eta} \frac{|K(f_1(v))-K(g_1(v))|}{E_1(v,s_0,f_1,f_2)} \frac{dv}{v^{\nu}} + \int_{s_0}^{\eta} \frac{K(g_1(v))|E_1(v,s_0,f_1,f_2)-E_1(v,s_0,g_1,g_2)|}{E_1(v,s_0,f_1,f_2)E_1(v,s_0,g_1,g_2)} \frac{dv}{v^{\nu}}$$

$$\leq \frac{1}{E_{1inf}(r_0,s_0)} \left(\widetilde{K}_1 + \frac{K_{1M}\widetilde{E}_1(r_0,s_0)}{E_{1inf}(r_0,s_0)} \right) \int_{s_0}^{\eta} \frac{1}{v^{\mu+\nu}} dv \quad ||\vec{f} - \vec{g}|| \leq \widetilde{H}_1(r_0, s_0)||\vec{f} - \vec{g}||,$$

where $\widetilde{H}_1$ is given by (3.35).

From the definition of G_1 it follows that

$$\begin{aligned} G_1(\eta, s_0, f_1, f_2)| &\geq \frac{E_{1inf}(r_0,s_0)}{L_{1M}} \int_{s_0}^{\eta} \frac{H_{1inf}(v,s_0)}{v^{\mu+\nu}} dv \\ &\geq \frac{K_{1m}E_{1inf}(r_0,s_0)}{L_{1M}(\mu+\nu-1)} \int_{s_0}^{\eta} \frac{1}{v^{\mu+\nu}} \left(\frac{1}{s_0^{\mu+\nu-1}} - \frac{1}{v^{\mu+\nu-1}} \right) dv \geq G_{1inf}(\eta, r_0, s_0), \end{aligned}$$

where G_{1inf} is given by (3.39) and

$$|G_1(\eta, s_0, f_1, f_2)| \leq \int_{s_0}^{\eta} \left| \frac{E_1(v,s_0,f_1,f_2)H_1(v,s_0,f_1,f_2)}{L(f_1(v))} \right| \frac{dv}{v^{\nu}}$$

$$\leq \frac{H_{1sup}(r_0,s_0)}{L_{1m}} \int_{s_0}^{\eta} \frac{dv}{v^{\mu+\nu}} \leq G_{1sup}(r_0, s_0),$$

where G_{1sup} is given by (3.40). Moreover,

$$|G_1(\eta, s_0, f_1, f_2) - G_1(\eta, s_0, g_1, g_2)|$$

$$\leq \int_{s_0}^{\eta} |H_1(v, s_0, f_1, f_2)| \left| \frac{E_1(v,s_0,g_1,g_2)}{L(g_1(v))} - \frac{E_1(v,s_0,f_1,f_2)}{L(f_1(v))} \right| \frac{dv}{v^{\nu}}$$

$$+ \int_{s_0}^{\eta} \frac{E_1(v,s_0,g_1,g_2)|H_1(v,s_0,f_1,f_2)-H_1(v,s_0,g_1,g_2)|}{L(g_1(v))} \frac{dv}{v^{\nu}}$$

$$\leq \widetilde{G}_1(r_0, s_0)||\vec{f} - \vec{g}||,$$

with $\widetilde{G}_1$ defined by (3.41).

□

Lemma 3.3. *For every $\vec{f} = (f_1, f_2)$, $\vec{g} = (g_1, g_2) \in \mathcal{K}$, the following inequalities hold:*

1)

$$E_2(\eta, r_0, f_1, f_2) \geq E_{2inf}(r_0, s_0), \tag{3.47}$$

$$E_2(\eta, r_0, f_1, f_2) \leq 1, \tag{3.48}$$

$$|E_2(\eta, r_0, f_1, f_2) - E_2(\eta, r_0, g_1, g_2)| \leq \widetilde{E}_2(r_0, s_0)||\vec{f} - \vec{g}||, \tag{3.49}$$

where

$$E_{2inf}(r_0,s_0) := \exp\left(-\left[a\frac{N_{2M}}{L_{2m}(\mu-1)}\frac{1}{r_0^{2\mu-2}} + \frac{D_2K_{2M}}{H_{inf}(r_0,s_0)L_{2m}(2\mu+\nu-1)}\frac{1}{r_0^{2\mu+\nu-1}}\right]\right), \tag{3.50}$$

$$\widetilde{E}_2(r_0,s_0) := 2a\left[\frac{\widetilde{N}_2}{L_{2m}(\mu-2)}\frac{1}{r_0^{\mu-2}} + \frac{N_{2M}\widetilde{L}_2}{L_{2m}^2(3\mu-2)}\frac{1}{r_0^{3\mu-2}}\right]$$

$$+D_2\Bigg(\frac{\widetilde{K}_2}{H_{inf}(r_0,s_0)L_{2m}(2\mu+\nu-1)}\frac{1}{r_0^{2\mu+\nu-1}} \tag{3.51}$$

$$+\frac{K_{2M}}{H_{inf}(r_0,s_0)L_{2m}}\left(\frac{\widetilde{H}(r_0)}{H_{inf}(r_0,s_0)(2\mu+\nu-1)}\frac{1}{r_0^{2\mu+\nu-1}} + \frac{\widetilde{L}_2}{L_{2m}(3\mu+\nu-1)}\frac{1}{r_0^{3\mu+\nu-1}}\right)\Bigg);$$

2)

$$\Phi_2(\eta,r_0,f_1,f_2) \geq \Phi_{2inf}(\eta,r_0,s_0), \tag{3.52}$$

$$\Phi_2(\eta,r_0,f_1,f_2) \leq \Phi_{2sup}(r_0), \tag{3.53}$$

$$|\Phi_2(\eta,r_0,f_1,f_2) - \Phi_2(\eta,r_0,g_1,g_2)| \leq \widetilde{\Phi}_2(r_0,s_0)||\vec{f}-\vec{g}||, \tag{3.54}$$

where

$$\Phi_{2inf}(\eta,r_0,s_0) := \frac{E_{2inf}(r_0,s_0)}{L_{2M}}\frac{1}{(\mu+\nu-1)}\left(\frac{1}{r_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}}\right), \tag{3.55}$$

$$\Phi_{2sup}(r_0) := \frac{1}{L_{2m}}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}, \tag{3.56}$$

$$\widetilde{\Phi}_2(r_0,s_0) := \frac{\widetilde{E}_2(r_0,s_0)}{L_{2m}}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}} + \frac{\widetilde{L}_2}{L_{2m}^2}\frac{1}{(2\mu+\nu-1)}\frac{1}{r_0^{2\mu+\nu-1}}; \tag{3.57}$$

3)

$$H_2(\eta,r_0,f_1,f_2) \leq H_{2inf}(\eta,r_0,s_0), \tag{3.58}$$

$$H_2(\eta,r_0,f_1,f_2) \leq H_{2sup}(r_0,s_0), \tag{3.59}$$

$$|H_2(\eta,r_0,f_1,f_2) - H_2(\eta,r_0,g_1,g_2)| \leq \widetilde{H}_2(r_0,s_0)||\vec{f}-\vec{g}||, \tag{3.60}$$

where

$$H_{2inf}(\eta,r_0) := \frac{K_{2m}}{(\mu+\nu-1)}\left(\frac{1}{r_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}}\right), \tag{3.61}$$

$$H_{2sup}(r_0,s_0) := \frac{K_{2M}}{E_{2inf}(r_0,s_0)}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}, \tag{3.62}$$

$$\widetilde{H}_2(r_0,s_0) := \left(\widetilde{K}_2 + \frac{K_{2M}\widetilde{E}_2(r_0,s_0)}{E_{2inf}(r_0,s_0)}\right)\frac{1}{E_{2inf}(r_0,s_0)(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}; \tag{3.63}$$

4)

$$G_2(\eta,r_0,f_1,f_2) \leq G_{2inf}(\eta,r_0,s_0), \tag{3.64}$$

$$G_2(\eta,r_0,f_1,f_2) \leq G_{2sup}(r_0,s_0), \tag{3.65}$$

$$|G_2(\eta,r_0,f_1,f_2) - G_2(\eta,r_0,g_1,g_2)| \leq \widetilde{G}_2(r_0,s_0)||\vec{f}-\vec{g}||, \tag{3.66}$$

where

$$G_{2inf}(\eta,r_0,s_0) := \frac{K_{2m}E_{2inf}(r_0,s_0)}{2L_{2M}(\mu+\nu-1)^2}\left(\frac{1}{r_0^{\mu+\nu-1}} - \frac{1}{\eta^{\mu+\nu-1}}\right)^2, \tag{3.67}$$

$$G_{2sup}(r_0,s_0) := \frac{H_{2sup}(r_0,s_0)}{L_{2m}}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}, \tag{3.68}$$

$$\widetilde{G}_2(r_0,s_0) := H_{2sup}(r_0,s_0)\widetilde{\Phi}_2(r_0,s_0) + \frac{\widetilde{H}_2(r_0,s_0)}{L_{2m}}\frac{1}{(\mu+\nu-1)}\frac{1}{r_0^{\mu+\nu-1}}. \tag{3.69}$$

Proof. The proof follows analogously to the previous lemma. □

Lemma 3.4. *For every* $\vec{f}=(f_1,f_2),\vec{g}=(g_1,g_2)\in\mathcal{K}$ *it follows that*

$$||V_1(\vec{f})-V_1(\vec{g})||_{C[s_o,r0]}\leq \varepsilon_1(r_0,s_0)||\vec{f}-\vec{g}||,$$

where

$$\begin{aligned}\varepsilon_1(r_0,s_0)&=2s_0^{\nu}Q\exp(-s_0^2)\widetilde{\Phi}_1(r_0,s_0)\\ &+2D_1^*\left(\frac{G_{1sup}(r_0,s_0)2H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)}+\frac{\widetilde{G}_1(r_0,s_0)}{H^2_{inf}(r_0,s_0)}\right).\end{aligned}\tag{3.70}$$

Proof. Taking into account that

$$\begin{aligned}&\left|\frac{G_1(\eta,s_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}-\frac{G_1(\eta,s_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\right|\leq\frac{|G_1(\eta,s_0,f_1,f_2)||H^2(r_0,s_0,f_1,f_2)-H^2(r_0,s_0,g_1,g_2)|}{H^2(r_0,s_0,f_1,f_2)H^2(g_1,g_2)}\\ &+\frac{|G_1(\eta,s_0,f_1,f_2)-G_1(\eta,s_0,g_1,g_2)|}{H^2(r_0,s_0,g_1,g_2)}\leq\left[\frac{G_{1sup}(r_0,s_0)2H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)}+\frac{\widetilde{G}_1(r_0,s_0)}{H^2_{inf}(r_0,s_0}\right]||\vec{f}-\vec{g}||,\end{aligned}\tag{3.71}$$

for each $\eta\in[s_0,r_0]$ it follows that

$$\begin{aligned}&|V_1(\vec{f})(\eta)-V_1(\vec{g})(\eta)|\leq\\ &\leq s_0^{\nu}Q\exp(-s_0^2)\big[|\Phi_1(r_0,s_0,f_1,f_2)-\Phi_1(r_0,s_0,g_1,g_2)|\\ &+|\Phi_1(\eta,s_0,f_1,f_2)-\Phi_1(\eta,s_0,g_1,g_2)|\big]\\ &+\left|\frac{D_1^*G_1(r_0,s_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}-\frac{D_1^*G_1(r_0,s_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\right|+\left|\frac{D_1^*G_1(\eta,s_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}-\frac{D_1^*G_1(\eta,s_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\right|\\ &\leq\Big[2s_0^{\nu}Q\exp(-s_0^2)\widetilde{\Phi}_1(r_0,s_0)\\ &+2D_1^*\Big(\frac{G_{1sup}(r_0,s_0)2H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)}+\frac{\widetilde{G}_1(r_0,s_0)}{H^2_{inf}(r_0,s_0)}\Big)\Big]||\vec{f}-\vec{g}||=\varepsilon_1(r_0,s_0)||\vec{f}-\vec{g}||.\end{aligned}\tag{3.72}$$

□

Lemma 3.5. *For every* $\vec{f}=(f_1,f_2),\vec{g}=(g_1,g_2)\in\mathcal{K}$ *it follows that*

$$||V_2(\vec{f})-V_2(\vec{g})||_{C_b[r_0,+\infty)}\leq\varepsilon_2(r_0,s_0)||\vec{f}-\vec{g}||,$$

where

$$\varepsilon_2(r_0,s_0)=\varepsilon_{21}(r_0,s_0)+\varepsilon_{22}(r_0,s_0)+\varepsilon_{23}(r_0,s_0)\tag{3.73}$$

with

$$\varepsilon_{21}(r_0,s_0)=\frac{2\widetilde{\Phi}_2(r_0,s_0)}{\Phi_{2inf}(+\infty,r_0,s_0)},$$

$$\varepsilon_{22}(r_0,s_0)=\frac{\widetilde{G}_2(r_0,s_0)}{H^2_{inf}(r_0,s_0)}+\frac{2G_{2sup}(r_0,s_0)H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)},$$

$$\varepsilon_{23}(r_0,s_0)=\frac{\Phi_{2sup}(r_0,s_0)}{\Phi_{2inf}(+\infty,r_0,s_0)}\varepsilon_{22}(r_0,s_0)+\frac{G_{2sup}(r_0,s_0)}{H^2_{inf}(r_0,s_0)}\varepsilon_{21}(r_0,s_0).$$

Proof. On one hand, we have that

$$\begin{aligned}&\left|\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right|\leq\frac{|\Phi_2(\eta,r_0,f_1,f_2)-\Phi_2(\eta,r_0,g_1,g_2)|}{\Phi_2(+\infty,r_0,f_1,f_2)}\\ &+\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\frac{|\Phi_2(+\infty,r_0,f_1,f_2)-\Phi_2(+\infty,r_0,g_1,g_2)|}{\Phi_2(+\infty,r_0,f_1,f_2)}\\ &\leq\frac{|\Phi_2(\eta,r_0,f_1,f_2)-\Phi_2(\eta,r_0,g_1,g_2)|}{\Phi_2(+\infty,r_0,f_1,f_2)}+\frac{|\Phi_2(+\infty,r_0,f_1,f_2)-\Phi_2(+\infty,r_0,g_1,g_2)|}{\Phi_2(+\infty,r_0,f_1,f_2)}\\ &\leq\frac{2\widetilde{\Phi}_2(r_0,s_0)}{\Phi_{2inf}(+\infty,r_0,s_0)}||\vec{f}-\vec{g}||=\varepsilon_{21}(r_0,s_0)||\vec{f}-\vec{g}||.\end{aligned}\tag{3.74}$$

On the other hand, we obtain that

$$\begin{aligned}
&\left|\frac{G_2(\eta,r_0,f_1,f_2)}{H^2(s_0,r_0,f_1,f_2)}-\frac{G_2(\eta,r_0,g_1,g_2)}{H^2(s_0,r_0,g_1,g_2)}\right| \\
&\leq \frac{|G_2(\eta,r_0,f_1,f_2)-G_2(\eta,r_0,g_1,g_2)|}{H^2(s_0,r_0,f_1,f_2)}+\frac{|G_2(\eta,r_0,g_1,g_2)||H^2(s_0,r_0,g_1,g_2)-H^2(s_0,r_0,f_1,f_2)|}{H^2(s_0,r_0,f_1,f_2)H^2(s_0,r_0,g_1,g_2)} \\
&\leq \left(\frac{\widetilde{G}_2(r_0,s_0)}{H^2_{inf}(r_0,s_0)}+\frac{2G_{2sup}(r_0,s_0)H_{sup}(r_0,s_0)\widetilde{H}(r_0,s_0)}{H^4_{inf}(r_0,s_0)}\right)||\vec{f}-\vec{g}|| \\
&= \varepsilon_{22}(r_0,s_0)||\vec{f}-\vec{g}||.
\end{aligned} \tag{3.75}$$

In addition,

$$\begin{aligned}
&\left|\frac{G_2(+\infty,r_0,f_1,f_2)}{H^2(s_0,r_0,f_1,f_2)}\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\frac{G_2(+\infty,r_0,g_1,g_2)}{H^2(s_0,r_0,g_1,g_2)}\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right| \\
&\leq \frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}\left|\frac{G_2(+\infty,r_0,f_1,f_2)}{H^2(s_0,r_0,f_1,f_2)}-\frac{G_2(+\infty,r_0,g_1,g_2)}{H^2(s_0,r_0,g_1,g_2)}\right| \\
&+\frac{G_2(+\infty,r_0,g_1,g_2)}{H^2(s_0,r_0,g_1,g_2)}\left|\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right| \\
&\leq \left(\frac{\Phi_{2sup}(r_0,s_0)}{\Phi_{2inf}(+\infty,r_0,s_0)}\varepsilon_{22}(r_0,s_0)+\frac{G_{2sup}(r_0,s_0)}{H^2_{inf}(r_0,s_0)}\varepsilon_{21}(r_0,s_0)\right)||\vec{f}-\vec{g}|| \\
&= \varepsilon_{23}(r_0,s_0)||\vec{f}-\vec{g}||.
\end{aligned} \tag{3.76}$$

From the previous inequalities, for each $\eta \geq r_0$, it follows that

$$\begin{aligned}
&|V_2(\vec{f})(\eta)-V_2(\vec{g})(\eta)| \\
&\leq \left|\frac{D_2^*G_2(+\infty,r_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\frac{D_2^*G_2(+\infty,r_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right| \\
&+\left|\frac{\Phi_2(\eta,r_0,f_1,f_2)}{\Phi_2(+\infty,r_0,f_1,f_2)}-\frac{\Phi_2(\eta,r_0,g_1,g_2)}{\Phi_2(+\infty,r_0,g_1,g_2)}\right|+\left|\frac{D_2^*G_2(\eta,r_0,f_1,f_2)}{H^2(r_0,s_0,f_1,f_2)}-\frac{D_2^*G_2(\eta,r_0,g_1,g_2)}{H^2(r_0,s_0,g_1,g_2)}\right| \\
&\leq \varepsilon_2(r_0,s_0)||\vec{f}-\vec{g}||.
\end{aligned} \tag{3.77}$$

□

Theorem 3.1. *For every $\vec{f}=(f_1,f_2)$, $\vec{g}=(g_1,g_2)\in\mathcal{K}$ it follows that*

$$||\Psi(\vec{f})-\Psi(\vec{g})||\leq\varepsilon(r_0,s_0)||\vec{f}-\vec{g}||$$

with

$$\varepsilon(r_0,s_0)=\max\left\{\varepsilon_1(r_0,s_0),\varepsilon_2(r_0,s_0)\right\}, \tag{3.78}$$

where $\varepsilon_1(r_0,s_0)$ and $\varepsilon_2(r_0,s_0)$ are given by (3.70) *and* (3.73)*, respectively.*

Proof. From the previous lemmas we have that

$$\begin{aligned}
||\Psi(\vec{f})-\Psi(\vec{g})|| &= \max\left\{||V_1(\vec{f})-V_1(\vec{g})||_{C[s_0,r_0]},||V_2(\vec{f})-V_2(\vec{g})||_{C_b[r_0,+\infty)}\right\} \\
&= \max\left\{\varepsilon_1(r_0,s_0)||\vec{f}-\vec{g}||,\varepsilon_2(r_0,s_0)||\vec{f}-\vec{g}||\right\}=\varepsilon(r_0,s_0)||\vec{f}-\vec{g}||.
\end{aligned}$$

□

Now we will look for conditions that guarantee that Ψ is a contraction mapping.

For each $s_0 > 0$ fixed, we define the functions

$$\varepsilon_{1,s_0}(r_0) = \varepsilon_1(r_0, s_0) \text{ and } \varepsilon_{2,s_0}(r_0) = \varepsilon_2(r_0, s_0), \text{ for all } r_0 > s_0,$$

where ε_1, ε_2 are given by (3.70) and (3.73), respectively. The following results hold.

Lemma 3.6. a) *The function ε_{1,s_0} is a decreasing function that satisfies $\varepsilon_{1,s_0}(s_0) = +\infty$ and $\varepsilon_{1,s_0}(+\infty) = j_1(s_0)$, where*

$$\begin{aligned} j_1(s_0) &= 2s_0^\nu Q \exp(-s_0^2)\widetilde{\Phi}_1(+\infty, s_0) \\ &+2D_1^*\left(\tfrac{G_{1sup}(+\infty,s_0)2H_{sup}(+\infty,s_0)\widetilde{H}(+\infty,s_0)}{H_{inf}^4(+\infty,s_0)} + \tfrac{\widetilde{G}_1(+\infty,s_0)}{H_{inf}^2+\infty,s_0)}\right). \end{aligned} \tag{3.79}$$

b) *If*

$$\tfrac{2D_1^*\widetilde{K}_1}{L_{1m}K_{1m}^2}\left(\tfrac{2K_{1M}}{K_{1m}^2} + 1\right) < 1, \tag{3.80}$$

then there exists a unique $s_1 > 0$ such that $j_1(s_0) < 1$ for all $s_0 > s_1$.

Moreover, for each $s_0 > s_1$ there exists $r_1 = r_1(s_0) > s_0$ such that $\varepsilon_{1,s_0}(r_1) = 1$ and $\varepsilon_{1,s_0}(r_0) < 1$ for all $r_0 > r_1$.

Proof. a) According to the definition of ε_1 given by (3.70), the proof follows straightforwardly from Lemmas 3.1 and 3.2.

b) From the definition of j_1 given by (3.79), we have that it is a decreasing function that satisfies $j_1(0) = +\infty$ and $j_1(+\infty) = \tfrac{2D_1^*\widetilde{K}_1}{L_{1m}K_{1m}^2}\left(\tfrac{2K_{1M}}{K_{1m}^2} + 1\right)$. Then, assuming (3.80), it follows that there exists a unique $s_1 > s_0$ such that $j_1(s_1) = 1$ and $j_1(s_0) < 1$ for all $s_0 > s_1$. Moreover, from item a), for each $s_0 > s_1$ there exists $r_1 = r_1(s_0) > s_0$ such that $\varepsilon_{1,s_0}(r_1) = 1$ and $\varepsilon_{1,s_0}(r_0) < 1$ for all $r_0 > r_1$.

□

Lemma 3.7. a) *The function ε_{2,s_0} is a decreasing function that satisfies the equalities $\varepsilon_{2,s_0}(s_0) = +\infty$ and $\varepsilon_{2,s_0}(+\infty) = 0$.*

b) *For each $s_0 > 0$ there exists $r_2 = r_2(s_0) > s_0$ such that $\varepsilon_{2,s_0}(r_2) = 1$ and $\varepsilon_{2,s_0}(r_0) < 1$ for all $r_0 > r_2$.*

Proof. a) It follows from Lemmas 3.1 and 3.3, by taking into account that ε_2 is defined by (3.73).

b) It clearly follows from item a).

□

Theorem 3.2. *If inequality (3.80) holds, then for each $(r_0, s_0) \in \Sigma$ with*

$$\Sigma = \{(r_0, s_0) : \ s_0 > s_1, \ r_0 > \overline{r}_0(s_0)\} \tag{3.81}$$

we have that $\varepsilon(r_0, s_0) < 1$, where ε is given by (3.78) and

$$\overline{r}_0(s_0) = \max\{r_1(s_0), r_2(s_0)\} \tag{3.82}$$

with s_1, r_1 and r_2 defined in Lemmas 3.6 and 3.7, respectively.

Proof. The proof follows immediately by Lemmas 3.6 and 3.7. □

Corollary 3.1. *Under assumption (3.80), for each $(r_0, s_0) \in \Sigma$, the operator Ψ defined by (3.2) is a contraction mapping.*

Theorem 3.3. *Under assumption (3.80), for each $(r_0, s_0) \in \Sigma$, there exists a unique fixed point $(f_1^*, f_2^*) \in \mathcal{K}$ of the operator Ψ.*

Proof. First, notice that $\mathcal{K}$ is a closed subset of the Banach space $\mathcal{C}$ given by (3.1). In addition, it is easy to see that $\Psi(\vec{f}) \in \mathcal{K}$ given that $V_1(\vec{f}) \in C[s_0, r_0]$, $V_2(\vec{f}) \in C_b[r_0, +\infty)$, $V_2(\vec{f})(r_0) = 0$ and $V_2(\vec{f})(+\infty) = 0$. Finally, according to Corollary 3.1, under assumption (3.80), for each $(r_0, s_0) \in \Sigma$ it follows that Ψ is a contraction mapping. As a corollary, applying the fixed point Banach theorem, we get that there exists a unique fixed point $(f_1^*, f_2^*) \in \mathcal{K}$ of the operator Ψ for each $(r_0, s_0) \in \Sigma$. $\square$

Corollary 3.2. *If (3.80) holds, for each $(r_0, s_0) \in \Sigma$, then there exists a unique solution (f_1^*, f_2^*) to the system of equations (2.24)-(2.25).*

It remains to prove the existence of solution $(r_0, s_0) \in \Sigma$ to the system of equations given by (2.26) and (2.27), where $f_1 = f_1^*$ and $f_2 = f_2^*$ are the unique solutions to equations (2.24)-(2.25). For that purpose we will need some preliminary results.

Let us notice that equation (2.26) can be rewritten as

$$X(r_0, s_0) = Y(r_0, s_0), \tag{3.83}$$

where

$$X(r_0, s_0) = Z(r_0, s_0) - B, \quad Z(r_0, s_0) = \frac{D_1^* G_1(r_0, s_0, f_1^*, f_2^*)}{H^2(r_0, s_0, f_1^*, f_2^*)}, \tag{3.84}$$

and

$$Y(r_0, s_0) = -Q s_0^{\nu} \exp(-s_0^2) \Phi_1(r_0, s_0, f_1^*, f_2^*). \tag{3.85}$$

Lemma 3.8. *The following properties hold:*

a) *$Y(r_0, s_0) < 0$ for each $(r_0, s_0) \in \Sigma$,*

b) *$Z(r_0, s_0) > Z_{inf}(r_0, s_0)$ for each $(r_0, s_0) \in \Sigma$, where*

$$Z_{inf}(r_0, s_0) = \frac{D_1^* E_{1inf}(r_0, s_0) K_{1m}}{2L_{1M}} \left(\frac{r_0^{\mu+\nu-1} - s_0^{\mu+\nu-1}}{K_{1M} r_0^{\mu+\nu-1} + K_{2M} s_0^{\mu+\nu-1}} \right)^2,$$

c) *for a fixed $s_0 > s_1$, if we assume that*

$$X(\overline{r}_0(s_0), s_0) < Y(\overline{r}_0(s_0), s_0), \tag{3.86}$$

then $Z_{inf}(\cdot, s_0)$ is an increasing function that satisfies the conditions

$$Z_{inf}(\overline{r_0}(s_0), s_0) < B, \qquad Z_{inf}(+\infty, s_0) = j_2(s_0), \tag{3.87}$$

where

$$j_2(s_0) = \frac{D_1^* E_{1inf}(+\infty, s_0) K_{1m}}{L_{1M} K_{1M}^2} \tag{3.88}$$

is an increasing function that satisfies the equality

$$j_2(+\infty) = \frac{D_1^* K_{1m}}{2L_{1M} K_{1M}^2},$$

d) *if we assume that*

$$\frac{D_1^* K_{1m}}{2L_{1M} K_{1M}^2} > B, \tag{3.89}$$

then there exists a unique $s_2 = \min\{s_0 \geq s_1 : j_2(s_0) \geq B\}$. Moreover, for each $s_0 > s_2$, we have that $j_2(s_0) > B$,

e) *if* (3.86) *and* (3.89) *hold for each* $s_0 > s_2$, *then there exists a unique* $r_B(s_0) > s_0$ *such that* $Z_{inf}(r_0, s_0) > B$ *for all* $r_0 > r_B(s_0)$.

Proof.

a) It is clear from the definition of the function Y given by (3.85).

b) It follows from the inequalities obtained in Lemmas 3.1 and 3.2.

c) From the definition of Z_{inf} it it easy to see that $Z_{inf}(\cdot, s_0)$ is an increasing function for each fixed $s_0 > s_1$. In addition, assumption (3.86) and item a) lead to the inequalities

$$Z_{inf}(\overline{r}_0(s_0), s_0) - B < Z(\overline{r}_0(s_0), s_0) - B < Y(\overline{r}_0(s_0), s_0) < 0.$$

Hence, it follows that $Z_{inf}(\overline{r}_0(s_0), s_0) < B$. Finally, taking a limit gives that $Z_{inf}(+\infty, s_0) = j_2(s_0)$ for each $s_0 > s_1$.

d) First, notice that hypothesis (3.89) can be rewritten as $j_2(+\infty) > B$. From the fact that j_2 is an increasing function, we can conclude that there exists a unique $s_2 = \min\{s_0 \geq s_1 : j_2(s_0) \geq B\}$. Notice that $s_2 = s_1$ in the case $j_2(s_1) > B$. As a corollary, for each $s_0 > s_2$, we get that $j_2(s_0) > B$.

e) For each fixed $s_0 > s_2$, we have that $Z_{inf}(\overline{r}_0(s_0), s_0) < B$ from item c) and $Z_{inf}(+\infty, s_0) > B$ from item d). Then, there exists a unique $r_B = r_B(s_0) > \overline{r}_0(s_0)$ such that $Z_{inf}(r_B(s_0), s_0) = B$ and $Z_{inf}(r_0, s_0) > B$ for all $r_0 > r_B(s_0)$. □

Lemma 3.9. *For each* $s_0 > s_2$, *if we assume that inequalities* (3.86) *and* (3.89) *hold, then there exists at least one solution* $r_0^* = r_0^*(s_0, f_1^*, f_2^*) \in (\overline{r}_0(s_0), r_B(s_0))$ *to equation* (2.26).

Proof. For each $s_0 > s_2$, taking into account assumption (3.86) and the fact that from item e) of Lemma 3.8 the following inequality holds

$$X(r_B(s_0), s_0) \geq Z_{inf}(r_B(s_0), s_0) - B = 0 > Y(r_B(s_0), s_0),$$

we obtain that there exists at least one solution $r_0^* \in (\overline{r}_0(s_0), r_B(s_0))$ to equation (2.26). □

Now we will analyze equation (2.27). If we replace r_0 by $r_0^*(s_0)$ and (f_1, f_2) by (f_1^*, f_2^*), the resulting equation is equivalent to the equation

$$W(r_0^*(s_0), s_0) = M, \tag{3.90}$$

where

$$\begin{aligned} W(r_0^*(s_0), s_0) = {} & \frac{E_1(r_0^*(s_0), s_0, f_1^*, f_2^*)}{r_0^{\nu+1}} \Bigg[Q \exp(-s_0^2) s_0^{\nu} \\ & + \frac{D_1^*}{H^2(r_0^*(s_0), s_0, f_1^*, f_2^*)} H_1(r_0^*(s_0), s_0, f_1^*, f_2^*) \Bigg] \\ & - \frac{1}{r_0^*(s_0)^{\nu+1} \Phi_2(+\infty, r_0^*(s_0), f_1^*, f_2^*)} \left[1 - \frac{D_2^*}{H^2(r_0^*(s_0), s_0, f_1^*, f_2^*)} G_2(+\infty, r_0^*(s_0), f_1^*, f_2^*) \right]. \end{aligned} \tag{3.91}$$

Lemma 3.10. *If any of the following two systems of inequalities hold*

$$\begin{cases} W_{inf}(s_2) > M \\ W_{sup}(+\infty) < M \end{cases} \quad \text{or} \quad \begin{cases} W_{sup}(s_2) < M \\ W_{inf}(+\infty) > M, \end{cases} \tag{3.92}$$

then there exists at least one solution $\widehat{s_0} > s_2$ *to equation* (3.90)*, where*

$$W_{inf}(s_0) = \tfrac{E_{1inf}(r_0^*(s_0),s_0)}{r_B^{\nu+1}(s_0)}\left[Q\exp(-s_0^2)s_0^\nu + \tfrac{D_1^*}{H_{sup}^2(r_0^*(s_0),s_0)}H_{1inf}(r_0^*(s_0),s_0)\right]$$
$$-\tfrac{1}{\overline{r_0}^{\nu+1}(s_0)\Phi_{2inf}(+\infty,r_0^*(s_0),s_0)} + \tfrac{1}{r_B^{\nu+1}(s_0)\Phi_{2sup}(r_0^*(s_0))}\cdot\tfrac{D_2^*}{H_{sup}^2(r_0^*(s_0),s_0)}G_{2inf}(+\infty,r_0^*(s_0),s_0), \tag{3.93}$$

$$W_{sup}(s_0) = \tfrac{1}{\overline{r_0}^{\nu+1}(s_0)}\left[Q\exp(-s_0^2)s_0^\nu + \tfrac{D_1^*}{H_{inf}^2(r_0^*(s_0),s_0)}H_{1sup}(r_0^*(s_0),s_0)\right.$$
$$\left. + \tfrac{1}{\Phi_{2inf}(+\infty,r_0^*(s_0),s_0)}\tfrac{D_2^*}{H_{inf}^2(r_0^*(s_0),s_0)}G_{2sup}(r_0^*(s_0),s_0).\right]. \tag{3.94}$$

The above analysis allows to establish the following existence theorem.

Theorem 3.4. *If hypotheses* $(A1)-(A5)$ *and inequalities* (3.80)*,* (3.86)*,* (3.89) *and* (3.92) *hold, then there exists at least one solution* $(\widehat{s_0}, r_0^*(\widehat{s_0}), f_1^*, f_2^*)$ *to the system of equations* (2.24)*-*(2.27)*, where* (f_1^*, f_2^*) *is the unique fixed point of the operator* Ψ *corresponding to* $(\widehat{s_0}, r_0^*(\widehat{s_0})) \in \Sigma$*.*

Corollary 3.3. *If hypotheses* $(A1)-(A5)$ *and inequalities* (3.80)*,* (3.86)*,* (3.89) *and* (3.92) *hold, then there exists at least one solution to problem* (1.4)*-*(1.17)*, where*

$$\begin{cases} T_1(z,t) = T_m f_1^*\left(\frac{z}{2a\sqrt{t}}\right) + T_m, & s(t) \le z \le r(t),\ t > 0, \\ T_2(z,t) = T_m f_2^*\left(\frac{z}{2a\sqrt{t}}\right) + T_m, & z \ge r(t),\ t > 0, \\ \varphi_1(z,t) = \dfrac{U_c}{2}\cdot\dfrac{F_1\left(\frac{z}{2a\sqrt{t}}, \widehat{s_0}, f_1^*\right)}{H(r_0^*(\widehat{s_0}), \widehat{s_0}, f_1^*, f_2^*)}, & s(t) \le z \le r(t),\ t > 0, \\ \varphi_2(z,t) = \dfrac{U_c}{2}\cdot\dfrac{F_1(r_0^*(\widehat{s_0}), \widehat{s_0}, f_1^*) + F_2\left(\frac{z}{2a\sqrt{t}}, r_0^*(\widehat{s_0}), f_2^*\right)}{H(r_0^*(\widehat{s_0}), \widehat{s_0}, f_1^*, f_2^*)}, & z \ge r(t),\ t > 0, \end{cases}$$

with $s(t) = 2a\widehat{s_0}\sqrt{t}$ *and* $r(t) = 2ar_0^*(\widehat{s_0})\sqrt{t}$*.*

Conclusion

We have considered a two-phase Stefan type problem governed by the generalized heat equation with the Thomson effect and nonlinear thermal coefficients, that models the dynamics of electromagnetic fields and heat transfer within closed electrical contacts, particularly focusing on the instantaneous explosion of micro-asperities.

By employing similarity transformations, we have effectively reduced the problem to a set of coupled ordinary differential equations, thereby facilitating tractable analysis and solution.

The validity and utility of our approach have been rigorously demonstrated through discussions and proofs grounded on the fixed point theory within the framework of Banach spaces. This theoretical underpinning not only enhances our confidence in the proposed solutions, but also provides a solid foundation for future research endeavors in related domains.

Furthermore, the insights gained from this study hold significant implications for various practical applications involving electrical contacts, such as in the design and optimization of electronic devices, electrical connectors, and power transmission systems. By elucidating the intricate interplay between electromagnetic fields and heat transfer phenomena, our work contributes to advancing the understanding and engineering of such systems in both industrial and academic contexts.

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