EURASIAN MATHEMATICAL JOURNAL
volume 16 , number 3, 2025

CONTENTS

Events

EURASIAN NATIONAL UNIVERSITY

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

EURASIAN MATHEMATICAL JOURNAL

ISSN 2077-9879
9 772077 987003

VOLUME 16, NUMBER 3 2025

ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2025, Volume 16, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;
- compliance of the title of the paper to its content;
- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);
- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);
- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);
- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Republic of Kazakhstan

The Moscow Editorial Office
The Patrice Lumumba Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russian Federation

MIKHAIL L'VOVICH GOLDMAN

Doctor of physical and mathematical sciences, Professor Mikhail L'vovich Goldman passed away on July 5, 2025, at the age of 80 years.

Mikhail L'vovich was an internationally known expert in scienceand education. His fundamental scientific articles and text books in variousfields of the theory of functions of several variablesand functional analysis, the theory of approximation of functions, embedding theorems and harmonic analysis are a significant contribution to the development of mathematics.

Mikhail L'vovich was born on Aprill 13, 1945 in Moscow. In 1963, he graduated from School No. 128 in Moscow with a gold medal and entered the Physics Faculty of the Lomonosov Moscow State University. He graduated in 1969 and became a postgraduate student in the Mathematics Department. In 1972, he defended his PhD thesis "On integral representations and Fourier series of differentiable functions of several variables" under the supervision of Professor Ilyin Vladimir Aleksandrovich, and in 1988, his doctoral thesis "Study of spaces of differentiable functions of several variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor of Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an Assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he was a Professor of the Department of Theory of Functions and Differential Equations, then of the S.M. Nikol'skii Mathematical Institute at the Patrice Lumumba Peoples' Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman were: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators.Among the most important scientific achievements of M.L. Goldman, we note his research related to the optimal embedding of spaces with generalized smoothness, exact conditions for the convergence of spectral decompositions, descriptions of the integral and differential properties of generalized potentials of the Bessel and Riesz types, exact estimates for operators on cones, descriptions of optimal spaces for cones of functions with monotonicity properties.

M.L. Goldman has published more than 150 scientific articles in central mathematical journals. He is a laureate of the Moscow government competition, a laureate of the RUDN University Prize in Science and Innovation, and a laureate of the RUDN University Prize for supervision of postgraduate students. Under the supervision of Mikhail L'vovich 11 PhD theses were defended. His pupilss are actively involved in professional work at leading universities and research institutes in Russia, Kazakhstan, Ethiopia, Rwanda, Colombia, and Mongolia.

Mikhail L'vovich has repeatedly been a guest lecturer and guest professor at universities in Russia, Germany, Sweden, Great Britain, etc., and an invited speaker at many international conferences. Mikhail L'vovich was not only an excellent mathematician and teacher (he always spoke about mathematics and its teaching with great passion), but also a man of the highest culture and erudition, with a deep knowledge of history, literature and art, a very bright, kind and responsive person. This is how he will remain in the hearts of his family, friends, colleagues and students.

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Mikhail L'vovich Goldman.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 16, Number 3 (2025), 57 – 67

RECONSTRUCTION OF THE WEIGHTED DIFFERENTIAL OPERATOR WITH POINT δ-INTERACTION

M.Dzh. Manafov, A. Kablan

Communicated by B.E. Kangyzhin

Key words: differential operator, uniqueness results, point δ-interactions.

AMS Mathematics Subject Classification: 34A55, 34B24, 47E05.

Abstract. In this article, we provide various uniqueness results for inverse problems of weighted differential operator with point $\delta-$interaction. In terms of the method of spectral mappings, we also offer step by step strategies for finding their potential and boundary conditions basing either on the Weyl function, on spectral data, or on two spectra.

DOI: https://doi.org/10.32523/2077-9879-2025-16-3-57-67

1 Introduction

In this work, we study inverse problems for the boundary value problems generated by the differential equation

$$ly := -y'' + q(x)\,y = \lambda y, \quad x \in (0,a) \cup (a,T) \tag{1.1}$$

with the Robin boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(T) + Hy(T) = 0, \tag{1.2}$$

and the transmission conditions at the point $x = a$

$$I(y) := \begin{cases} y(a+0) = y(a-0) \equiv y(a) \\ y'(a+0) - y'(a-0) = -\alpha\lambda y(a), \end{cases} \tag{1.3}$$

where $q(x)$ is a real function belonging to the space $L_2[0,T]$, λ is a spectral parameter and h, H, and α are real numbers with $\alpha > 0$. Denote the boundary value problems, defined above, by $L(q(x), h, H)$.

It is important to note that, we can interpret problem (1.1) and (1.3) as analyzing the equation

$$-y'' + q(x)\,y = \lambda\rho(x)\,y, \quad x \in (0,T), \tag{1.4}$$

when $\rho(x) = 1 + \alpha\delta(x)$ where $\delta(x)$ is the Dirac Delta-function (see [1]).

One type of problems, the direct problem, consists of examining the spectral properties of an operator. But some problems in mathematical physics require the investigation of inverse problems of spectral analysis for various differential operators, which require the recovery of operators from some of their given spectral data. Such problems are often considered in mathematics and various branches of natural science and technical science. Direct and inverse problems for the classical Sturm-Liouville operators have been comprehensively investigated in [6, 10, 15] and references therein. Some classes of direct and inverse problems for discontinuous boundary value problems in various statements have been considered in [2, 7, 8, 12, 13, 16, 17, 18]. Notice that, spectral characteristics for weighted Sturm-Liouville operator with point $\delta-$interactions have been investigated in [9, 14]. Here, we provide procedures for finding the potential of a problem and its boundary conditions basing either on the Weyl function, on spectral data, or on two spectra in terms of the method of spectral mappings.

2 Constructing the Hilbert space relevant to the problem and some of its spectral properties

We will start this section by defining the Hilbert space $\mathbb{H} := L_2[0,T] \oplus \mathbb{C}$ of the two component vectors, equipped with the inner product

$$\langle f, g\rangle_{\mathbb{H}} := \int_0^T f_1(x)\overline{g}_1(x)dx + \frac{1}{\alpha} f_2\overline{g}_2$$

for

$$f = \begin{pmatrix} f_1(x) \\ f_2 \end{pmatrix}, g = \begin{pmatrix} g_1(x) \\ g_2 \end{pmatrix},$$

where $f_1(x)$, $g_1(x) \in L_2(0,T)$ and $f_2, g_2 \in \mathbb{C}$. In the space $\mathbb{H}$, we define the operator L

$$L : \mathbb{H} \to \mathbb{H}$$

with the domain

$$D(L) = \{f \in \mathbb{H} | f_1, f_1' \in AC((0,a) \cup (a,T)), lf_1 \in L_2[(0,T) \setminus \{a\}], f_2 = \alpha f_1(a), U(f_1) = V(f_1) = 0\}$$

and the operator rule

$$L(f) = \begin{pmatrix} lf_1 \\ f_1'(a-0) - f'(a+0) \end{pmatrix}.$$

Here, $AC(\cdot)$ stands for the set of all functions that are absolutely continuous on a related interval.

Theorem 2.1. *The operator L is symmetric.*

Proof. We obtain the equality $\langle Lf, g\rangle_{\mathbb{H}} = \langle f, Lg\rangle_{\mathbb{H}}$ for $f, g \in D(L)$ immediately from the conditions at the point $x = a$ and the fact that f and $\overline{g}$ satisfy the same boundary conditions (1.2). So, L is symmetric.

□

Corollary 2.1. *The function W defined by $W\{f,g;x\} = f(x)g'(x) - f'(x)g(x)$ is continuous on $(0,T)$.*

Lemma 2.1. *If $y(x,\lambda)$ and $z(x,\mu)$ are solutions to the equations $ly = \lambda y$ and $lz = \mu z$, respectively, then*

$$\frac{d}{dx} W\{y, z; x\} = (\lambda - \mu) yz.$$

Let $C(x,\lambda)$, $S(x,\lambda)$, $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be solutions to equation (1.1) under the following initial conditions:

$$C(0,\lambda) = 1,\ C'(0,\lambda) = 0,\ S(0,\lambda) = 0,\ S'(0,\lambda) = 1,$$

$$\varphi(0,\lambda) = 1,\ \varphi'(0,\lambda) = h,\ \psi(T,\lambda) = 1,\ \psi'(T,\lambda) = -H$$

and under transmission conditions (1.3). Then,

$$U(\varphi) = V(\psi) = 0.$$

Let us denote
$$\Delta(\lambda) = W\{\varphi, \psi; x\}. \tag{2.1}$$
Due to Corollary 1 and the Ostrogradskii-Liouville theorem (see [4, p. 83]) $W\{\varphi, \psi; x\}$ does not depend on x. Here, the function $\Delta(\lambda)$ is called the *characteristic function* of L. It is easly seen that
$$\Delta(\lambda) = -V(\varphi) = U(\psi), \tag{2.2}$$
and $\Delta(\lambda)$ is an entire function of λ, so it has at most countable set of zeros $\{\lambda_n\}_{n\geq 0}$.

Lemma 2.2. *The zeros $\{\lambda_n\}_{n\geq 0}$ of the characteristic function are the eigenvalues of the boundary value problem L. Also the functions $\varphi(x, \lambda_n)$ and $\psi(x, \lambda_n)$ are the eigenfunctions, and there exists a sequence $\{\beta_n\}$ such that*
$$\psi(x, \lambda_n) = \beta_n.\varphi(x, \lambda_n), \;\; \beta_n \neq 0.$$

Denote
$$\alpha_n := \int_0^T \varphi^2(x, \lambda_n)dx + \alpha\varphi^2(a, \lambda_n). \tag{2.3}$$
The set $\Omega = \{\lambda_n, \alpha_n\}_{n\geq 0}$ is called *the spectral data* associated with problem (1.1)–(1.3).

Lemma 2.3. *The following relation holds*
$$\dot{\Delta}(\lambda_n) = \alpha_n\beta_n,$$
where $\dot{\Delta}(\lambda) = d\Delta(\lambda)/d\lambda$.

We omit the proofs of Lemma 2.2 and Lemma 2.3 since they are similiar to those for the classical Sturm-Liouville operators (see [11]).

Corollary 2.2. *The eigenvalues $\{\lambda_n\}$ and the eigenfunctions $\varphi(x, \lambda_n)$, $\psi(x, \lambda_n)$ are real. Also all zeros of $\Delta(\lambda)$ are simple, i.e. $\dot{\Delta}(\lambda_n) \neq 0$.*

Now, consider the solution $\varphi(x, \lambda)$. Let $C_0(x, \lambda)$ and $S_0(x, \lambda)$ be smooth solutions to equation (1.1) on the interval $[0, T]$ under the initial condition
$$C_0(x, \lambda) = S'(0, \lambda) = 1, S_0(x, \lambda) = C_0'(0, \lambda) = 0. \tag{2.4}$$
Then,
$$C(x, \lambda) = C_0(x, \lambda), \;\; S(x, \lambda) = S_0(x, \lambda), \;\; 0 < x < a \tag{2.5}$$
$$\begin{aligned} C(x, \lambda) &= A_1C_0(x, \lambda) + B_1S_0(x, \lambda), \\ S(x, \lambda) &= A_2C_0(x, \lambda) + B_2S_0(x, \lambda), \;\; a < x < T, \end{aligned} \tag{2.6}$$
where
$$\begin{aligned} A_1 &= 1 + \alpha\lambda C_0(a, \lambda)S_0(a, \lambda), \quad B_1 = -\alpha\lambda C_0^2(a, \lambda), \\ A_2 &= \alpha\lambda S_0^2(a, \lambda), \quad B_2 = 1 - \alpha\lambda C_0(a, \lambda)S_0(a, \lambda). \end{aligned} \tag{2.7}$$

Let $\lambda = \rho^2$, $\rho = \sigma + i\tau$. It is easy to show that, the function $C_0(x, \lambda)$ satisfies the following integral equation:
$$C_0(x, \lambda) = \cos\rho x + \frac{1}{\rho}\int_0^x \sin\rho(x-t)q(t)C_0(t, \lambda)dt. \tag{2.8}$$

Using the method of successive approximations to solve problem (2.8), we obtain

$$\begin{aligned} C_0(x,\lambda) &= \cos\rho x + \frac{\sin\rho x}{2\rho}\int_0^x q(t)dt + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)dt \\ &+ O\left(\frac{1}{\rho^2}\exp(|\tau|)x\right). \end{aligned} \tag{2.9}$$

Analogously,

$$\begin{aligned} S_0(x,\lambda) &= \frac{\sin\rho x}{\rho} - \frac{\cos\rho x}{2\rho^2}\int_0^x q(t)dt + \frac{1}{2\rho^2}\int_0^x q(t)\cos\rho(x-2t)dt \\ &+ O\left(\frac{1}{\rho^3}\exp(|\tau|)x\right). \end{aligned} \tag{2.10}$$

By virtue of (2.7) and (2.9)-(2.10),

$$A_1 = \frac{\alpha}{2}\rho\sin 2\rho a - 1 - \frac{\alpha}{2}\cos 2\rho a\int_0^a q(t)dt + O\left(\frac{1}{\rho}\right),$$

$$B_1 = -\frac{\alpha}{2}\rho^2\left(1+\cos 2\rho a\right) - \frac{\alpha}{2}\rho\sin\rho a\int_0^a q(t)dt + O\left(1\right),$$

$$A_2 = \frac{\alpha}{2}\left(1-\cos 2\rho a\right) + O\left(\frac{1}{\rho}\right), B_2 = -\frac{\alpha}{2}\rho\sin 2\rho a + O\left(1\right)$$

Since $\varphi(x,\lambda) = C(x,\lambda) + hS(x,\lambda)$, by using (2.5)-(2.10), we find

$$\varphi(x,\lambda) = \cos\rho x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin\rho x}{\rho} + O\left(\frac{1}{\rho}\exp\left(|\tau|x\right)\right), 0<x<a, \tag{2.11}$$

$$\varphi'(x,\lambda) = -\rho\sin\rho x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\cos\rho x + O\left(\exp\left(|\tau|x\right)\right), 0<x<a, \tag{2.12}$$

$$\begin{aligned} \varphi(x,\lambda) &= \frac{\alpha}{2}\rho\left(\sin\rho\left(2a-x\right) - \sin\rho x\right) + f_1(x)\cos\rho x + f_2(x)\cos\rho\left(2a-x\right) \\ &+ O\left(\exp\left(|\tau|x\right)\right),\ a<x<T, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \varphi'(x,\lambda) &= -\frac{\alpha}{2}\rho^2\left(\cos\rho x + \cos\rho\left(2a-x\right)\right) - \rho f_1(x)\sin\rho x + \rho f_2(x)\sin\rho\left(2a-x\right) \\ &+ O\left(\rho\exp\left(|\tau|x\right)\right),\ a<x<T, \end{aligned} \tag{2.14}$$

where

$$f_1(x) = 1 + \frac{\alpha}{2}h + \frac{\alpha}{4}\int_0^x q(t)dt, \quad f_2(x) = \frac{\alpha}{4}\left(-2h + \int_a^x q(t)dt - \int_0^a q(t)dt\right).$$

It follows from (2.2),(2.13), and (2.14) that

$$\begin{aligned} \Delta(\lambda) &= \frac{\alpha}{2}\rho^2\left(\cos\rho T + \cos\rho\left(2a-T\right)\right) + \omega_1\rho\sin\rho T + \omega_2\rho\sin\rho\left(2a-T\right) \\ &+ O\left(\rho\exp\left(|\tau|T\right)\right), \end{aligned} \tag{2.15}$$

where

$$\omega_1 = -\left(1 + \frac{\alpha}{2}h + \frac{\alpha}{2}H + \frac{\alpha}{4}\int_0^T q(t)dt\right)$$

$$\omega_2 = \frac{\alpha}{2}\left(h - H - \frac{1}{2}\int_a^T q(t)dt + \frac{1}{2}\int_0^a q(t)dt\right).$$

Let $\lambda_n^0 = (\rho_n^0)^2$ and $\lambda_n = (\rho_n)^2$ be the zeros of the functions $\Delta_0(\lambda) = \frac{\alpha}{2}\rho^2(\cos\rho T + \cos\rho\,(2a - T))$ and $\Delta(\lambda)$, respectively.

The following properties of the characteristic function $\Delta(\lambda)$ and the eigenvalues $\lambda_n = \rho_n^2$ of the boundary value problem of L can be discovered using (2.15) and the well-known methods (see, for example [3]).

(i) Denote $G_\delta = \{\rho : |\rho - \rho_n^0| \geq \delta, n \geq 0\}$. There is a constant $C_\delta > 0$ such that

$$|\Delta_0(\lambda)| \geq C_\delta |\lambda| \exp\left(|\tau|T|\right), \;\; \rho \in G_\delta.$$

(ii) For sufficiently large value of n, the following inequality is valid

$$|\Delta(\lambda) - \Delta_0(\lambda)| \leq \frac{1}{2} C_\delta \exp\left(|\tau|T\right), \;\; \rho \in \Gamma_n = \{\rho : |\rho| = |\rho_n^0| + \frac{1}{2}\inf_{n \neq m} |\rho_n^0 - \rho_m^0|\}.$$

Thus, for sufficiently large natural number n and $\rho \in \Gamma_n$,

$$|\Delta_0(\lambda)| \geq C_\delta |\lambda| \exp\left(|\tau|T\right) > \frac{1}{2} C_\delta |\lambda| \exp\left(|\tau|T\right) > |\Delta(\lambda) - \Delta_0(\lambda)|.$$

Then by Rouche's theorem, the number of zeros of $\Delta_0(\lambda)$, counting multiplicities, inside circuit G_n coincides with the number of zeros of $\Delta(\lambda)$. Analogously, applying Rouche's theorem, we say that for sufficiently large values of n, the function $\Delta(\lambda)$ has exactly one zero ρ_n inside each circle $G_\delta = \{\rho : |\rho - \rho_n^0| \leq \delta\}$. Since δ is arbitrary sufficiently small number, we have

$$\rho_n = \rho_n^0 + \epsilon_n, \;\; \epsilon_n = o\,(1)\,, \;\; n \to \infty \tag{2.16}$$

Since the function $\Delta_0(\lambda)$ is type of *sinus* (see [5, p. 119]), there exist the number $d_\delta > 0$ such that, for all n, $|\frac{d}{d\lambda}\Delta_0(\lambda)|_{\lambda=\lambda_n} \geq d_\delta > 0$. Since ρ_n are zeros of $\Delta(\lambda)$, from (2.15) we get

$$\epsilon_n = -\frac{2}{\alpha\rho_n^0}\left[\omega_1 \sin\rho_n^0 T + \omega_2 \sin\rho_n^0\,(2a - T)\right]\left[\frac{d}{d\lambda}\Delta_0(\lambda)|_{\lambda=\lambda_n^0}\right]. \tag{2.17}$$

Substituting (2.17) into (2.16) we get

$$\rho_n = \rho_n^0 + \frac{d_n}{\rho_n^0} + \frac{\gamma_n}{\rho_n^0}, \tag{2.18}$$

where

$$d_n = -\frac{2}{\alpha}\left[\omega_1 \sin\rho_n^0 T + \omega_2 \sin\rho_n^0\,(2a - T)\right]\left[\frac{d}{d\lambda}\Delta_0(\lambda)|_{\lambda=\lambda_n^0}\right]$$

and $\gamma_n = o\,(1)$.

Finally, using (2.11),(2.12) and (2.18) into (2.3) we obtain

$$\alpha_n = \alpha_n^0 + o\,(1)\,, \;\; n \to \infty, \tag{2.19}$$

where

$$\alpha_n^0 = \int_0^T \varphi^2(x, \lambda_n^0)dx + \alpha\varphi^2(a, \lambda_n^0).$$

3 Algorithm of solving the inverse problem

In this section, we first give the spectral characteristics of the boundary value problem L and then demonstrate relationships between their spectral characteristics. Moreover, we provide the formula to solve the inverse problem of the reconstruction of the problem L basing on the Weyl function, on the spectral data, and on two spectra.

We define the Weyl function by

$$M(\lambda) = \frac{\psi(0,\lambda)}{\Delta(\lambda)}. \tag{3.1}$$

Here the function $\psi(0,\lambda)$ is the characteristic function of the boundary value problem which consists of equation (1.1) along with the boundary conditions $y(0) = V(y) = 0$ and transmission conditions (1.3). Let $\{\mu_n\}_{n\geq 0}$ be the zeros of the entire function $\psi(0,\lambda)$. It is clear that, $\psi(0,\lambda)$ and $\Delta(\lambda)$ have no common zeros. Thus, the Weyl function $M(\lambda)$ is meromorphic which has poles at the points $\{\lambda_n\}_{n\geq 0}$ and zeros at the points $\{\mu_n\}_{n\geq 0}$.

The following lemma gives the relationships between the spectral characteristic of L : the spectral data Ω, the Weyl function $M(\lambda)$ and the two spectra $\{\lambda_n, \mu_n\}_{n\geq 0}$.

Lemma 3.1. *Let $M(\lambda)$, Ω and $\Delta(\lambda)$ be defined as above. Then the following representation holds:*

$$M(\lambda) = \sum_{h=0}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}. \tag{3.2}$$

Moreover, $\Delta(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta(\lambda) = T(\lambda_0 - \lambda) \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{\lambda_n^0}. \tag{3.3}$$

Since the arguments for proving this lemma are similar to those in [2], we skip the proof. Now, we will consider the following inverse problems of recovering L :

- *Inverse problem* 1*:* constructing $q(x)$, h, and H when the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$ is given.
- *Inverse problem* 2*:* constructing $q(x)$, h and H when the Weyl function $M(\lambda)$ is given.
- *Inverse problem* 3*:* constructing $q(x)$, h and H when the two spectra $\Omega = \{\lambda_n, \mu_n\}_{n\geq 0}$ are given.

Let us note that, according to (3.1), (3.2) and (3.3), the inverse problems of recovering L basing on the spectral data and on the two spectra are specifications of the inverse problem of recovering L from the Weyl function. Consequently, the inverse problems 1–3 are equivalent.

The inverse problems studied here can be seen as generalizations of the inverse problems for the classical Sturm-Liouville operators, see [6, Chapter 1]. In the next section, using results stated above, we provide a constructive procedure for solving these inverse problems.

4 Finding solutions to inverse problems

In this section, with the help of Cauchy's integral formula and the Residue theorem, we will solve the inverse problems of recovering the Sturm-Liouville problem $L(q(x), h, H)$ using the spectrum mappings approach. We first reduce an inverse problem to the so-called main equation which is a linear equation in a corresponding Banach space of sequences. Finally, we provide the algorithms for solving the inverse problems by using the solution of the main equation.

For this purpose we introduce a new problem with new notations: together with L we consider a boundary value problem $\tilde{L}$ of the same form but with different coefficients $\tilde{q}(x)$, $\tilde{h}$, $\tilde{H}$. Throughout next sections, if a certain symbol e denotes an object related to L, then the symbol $\tilde{e}$ with tilde denotes the analogous object related to $\tilde{L}$. Now we introduce the following notations for convenience of further discussions.

$$\lambda_{n0} = \lambda_n, \ \ \lambda_{n1} = \tilde{\lambda}_n, \ \ \alpha_{n0} = \alpha_n, \ \ \alpha_{n1} = \tilde{\alpha}_n,$$

$$\varphi_{ni}(x) = \varphi(x, \lambda_{ni}), \ \ \tilde{\varphi}_{ni}(x) = \tilde{\varphi}(x, \lambda_{ni}),$$

$$Q_{kj}(x, \lambda) = \frac{< \varphi(x, \lambda), \varphi_{kj}(x) >}{\alpha_{kj}(\lambda - \lambda_{kj})} = \frac{1}{\alpha_{kj}} \int_0^x \varphi(t, \lambda)\varphi_{kj}(t)dt,$$

$$Q_{ni,kj}(x) = Q_{kj}(x, \lambda_{ni}),$$

for $i, j = 0, 1$ and $n, k \geq 0$. Here $\tilde{\varphi}(x, \lambda)$ is the solution of (1.4) with the potential $\tilde{q}$ under the initial conditions $\tilde{\varphi}(0, \lambda) = 1$, $\tilde{\varphi}'(0, \lambda) = \tilde{h}$. Similarly, we can define $\tilde{Q}_{kj}(x, \lambda)$ by replacing φ by $\tilde{\varphi}$ in the above definition.

Using the Schwartz lemma, see [5, p. 130] and (2.11)-(2.14), (2.17) we obtain the following asymptotic estimates:

$$|\varphi_{ni}(x)| \leq C(|\rho_n^0| + 1), \ \ |\varphi_{n0}(x) - \varphi_{n1}(x)| \leq C(|\rho_n^0| + 1)^{\frac{1}{2}}, \tag{4.1}$$

$$|Q_{ni,kj}(x)| \leq \frac{C(|\rho_n^0| + 1)}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)},$$

$$|Q_{ni,k0}(x) - Q_{ni,k1}(x)| \leq \frac{C(|\rho_n| + 1)}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)^{\frac{3}{2}}}, \tag{4.2}$$

$$|Q_{n0,kj}(x) - Q_{n1,j1}(x)| \leq \frac{C(|\rho_n| + 1)^{\frac{1}{2}}}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)}$$

where $n, k \geq 0, 1$ and $C > 0$ is independent of n, k, i, j. Similar estimates are also valid for $\tilde{\varphi}_{ni}(x)$, $\tilde{Q}_{ni,kj}(x)$.

Lemma 4.1. *Let $\varphi_{ni}(x)$ and $Q_{ni,kj}(x)$ be defined as above. Then the following representations hold for $i, j = 0, 1$ and $n, k \geq 0$:*

$$\tilde{\varphi}_{ni}(x) = \varphi_{ni}(x) + \sum_{l=1}^{\infty} (\tilde{Q}_{ni,l0}(x)\varphi_{k0}(x) - \tilde{Q}_{ni,l1}(x)\varphi_{k1}(x)), \tag{4.3}$$

$$Q_{ni,kj}(x) - \tilde{Q}_{ni,kj}(x) + \sum_{l=0}^{\infty} (\tilde{Q}_{ni,l0}(x)Q_{l0,kj}(x) - \tilde{Q}_{ni,l1}(x)Q_{l1,kj}(x)) = 0, \tag{4.4}$$

Both series converge absolutely and uniformly with respect to $x \in [0, T] \setminus \{a\}$.

The proof of this lemma is similar to that of the lemma given in [13] and, hence, is omitted.

From all arguments mentioned above, it is seen that, for each fixed $x \in (0, T) \setminus \{a\}$, relation (4.3) can be thought as a system of linear equations with respect to $\varphi_{ni}(x)$ for $n \geq 0$ and $i = 0, 1$. But the series in (4.3) converges only with brackets. So, it is not convenient to use (4.3) as a main equation of the inverse problem. Below, we will transfer (4.3) to a linear equation in a corresponding Banach space of sequences.

Let V be a set of all indices $u = (n, i)$, $n \geq 0$, $i = 0, 1$. For each fixed $x \in (0, T) \setminus \{a\}$, we define the vector

$$\phi(x) = [\phi_u(x)] = \begin{bmatrix} \phi_{n0}(x) \\ \phi_{n1}(x) \end{bmatrix}_{n \geq 0}$$

by the formulas

$$\begin{bmatrix} \phi_{n0}(x) \\ \phi_{n1}(x) \end{bmatrix} = \begin{bmatrix} \frac{\rho_n^0+1}{\rho_n} & -\frac{\rho_n^0+1}{\rho_n} \\ 0 & \frac{1}{\rho_n^0} \end{bmatrix} \begin{bmatrix} \varphi_{n0}(x) \\ \varphi_{n1}(x) \end{bmatrix} = \begin{bmatrix} \frac{(\rho_n^0+1)(\varphi_{n0}(x)-\varphi_{n1}(x))}{\rho_n} \\ \frac{\varphi_{n1}(x)}{\rho_n} \end{bmatrix} \tag{4.5}$$

Further, we define the block matrix

$$H(x) = [H_{u,v}(x)]_{u,v \in V} = \begin{bmatrix} H_{n0,k0}(x) & H_{n0,k1}(x) \\ H_{n1,k0}(x) & H_{n1,k1}(x) \end{bmatrix}_{n,k \geq 0},$$

where $u = (n, i)$, $v = (k, j)$ and

$$\begin{bmatrix} H_{n0,k0}(x) & H_{n0,k1}(x) \\ H_{n1,k0}(x) & H_{n1,k1}(x) \end{bmatrix} = \begin{bmatrix} \frac{\rho_n^0+1}{\rho_n} & -\frac{\rho_n^0+1}{\rho_n} \\ 0 & \frac{1}{\rho_n^0} \end{bmatrix} \begin{bmatrix} Q_{n0,k0}(x) & Q_{n0,k1}(x) \\ Q_{n1,k0}(x) & Q_{n1,k1}(x) \end{bmatrix} \begin{bmatrix} \frac{\rho_k}{\rho_k^0+1} & \rho_k \\ 0 & -\rho_k \end{bmatrix}.$$

Analogously, we define $\tilde{\varphi}(x)$, $\tilde{H}(x)$ by replacing in the previous definitions $\varphi_{ni}(x)$, $Q_{ni,kj}(x)$ by $\tilde{\varphi}_{ni}(x)$, $\tilde{Q}_{ni,kj}(x)$, respectively. It follows from (2.11) - (2.14), (2.17), (2.18), (4.1), (4.2) and the Schwarz lemma that

$$|\phi_{nj}(x)|, |\tilde{\phi}_{nj}(x)| \leq C, \tag{4.6}$$

and

$$|H_{ni,kj}(x)|, |\tilde{H}_{ni,kj}(x)| \leq \frac{C}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)}, \tag{4.7}$$

where $C > 0$ is independent of n, k, i, j.

Let us consider the Banach space B of bounded sequences $\alpha = [\alpha_u]_{u \in V}$ with the norm $\|\alpha\|_B = sup_{u \in V} |\alpha_u|$ and consider the operator $E + \tilde{H}(x)$ acting from B to B. Here E is the identity operator. It follows from (4.6), (4.7) that for each fixed x, this operator is a linear bounded operator, and

$$\|H(x)\|, \|\tilde{H}(x)\| \leq C \sup_n \sum_{k=0}^{\infty} \frac{1}{(|\rho_n^0 - \rho_k^0| + 1)(|\rho_k^0| + 1)} < \infty.$$

Now we are ready to give the main result of this section.

Theorem 4.1. *For each fixed $x \in (0, T) \setminus \{a\}$, the vector $\varphi(x) \in B$ satisfies the equation*

$$\tilde{\varphi}(x) = (E + \tilde{H}(x))\varphi(x), \tag{4.8}$$

in the Banach space B. Moreover, the operator $E + \tilde{H}(x)$ has a bounded inverse operator, hence, equation (4.8) is uniquely solvable.

Proof. Using the notation $\tilde{\phi}(x)$, rewrite (4.3) as

$$\tilde{\phi}_{ni}(x) = \phi_{ni} + \sum_{n,j} \tilde{H}_{ni,kj}(x)\phi_{kj}(x), (n,i) \in V, (k,j) \in V,$$

which is equivalent to (4.3). Interchanging places for L and $\tilde{L}$, we obtain analogously the equalities

$$\phi(x) = (E - H(x))\tilde{\varphi}(x), (E - H(x))(E + \tilde{H}(x)) = E.$$

Hence, the operator $(E + \tilde{H}(x))^{-1}$ exist, and it is a linear bounded operator. □

Equation (4.8) is named a *basic equation* of the inverse problem. Solving (4.8) we find the vector $\phi(x)$, and hence, the functions $\varphi_{ni}(x)$. Thus, we get the following algorithms to find the solution of an inverse problem.

Algorithm 1. *When the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$ is given, to construct $q(x)$, h and H, we follow the steps:*

(i) first contruct $\tilde{L}$ and then calculate $\tilde{\phi}(x)$ and $\tilde{H}(x)$,

(ii) by solving equation (4.8) *find $\phi(x)$ and calculate $\varphi_{n0}(x)$ by using* (4.5)*,*

(iii) choose some n (for example, $n = 0$) and construct $q(x)$, h and H by using the following formulas:

$$q(x) = \frac{\varphi''_{n0}(x)}{\varphi_{n0}(x)} + \lambda_n, \;\; h = \varphi'_{n0}(0), \;\; H = -\frac{\varphi_{n0}(T)}{\varphi_{n0}(T)}.$$

Algorithm 2. *When the function $M(\lambda)$ is given, to construct $q(x)$, h and H, we follow the steps:*

(i) construct the spectral data Ω by using (3.2)*,*

(ii) construct $q(x)$, h and H by using Algorithm 1.

Algorithm 3. *When two spectra $\{\lambda_n, \mu_n\}_{n\geq 0}$ are given, to construct $q(x)$, h and H, we follow the steps:*

(i) calculate $M(\lambda)$ by using (3.1)*,*

(ii) construct $q(x)$, h and H by using Algorithm 2.

References

[1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden (with an appendix by P. Exner), *Solvable models in quantum mechanics (second edition).* AMS Chelsea Publishing, 2005.

[2] B. Bayram, M. Dzh. Manafov, A. Kablan, *Solution and construction of inverse problem for Sturm-Liouville equations with finitely many point δ-interactions.* J. Appl. Anal. Comput. 13 (2023), no. 1, 458–469.

[3] R. Bellman, K. Cooke, *Differential-difference equations.* Academic Press, New York, 1963.

[4] E.A. Coddington, N. Levinson, *Theory of ordinary differential equations.* McGraw-Hill, New York, 1955.

[5] J.B. Conway, *Functions of one complex variable (2nd edition).* Springer, New York, 1995.

[6] G. Freiling, V.A. Yurko, *Inverse Sturm-Liouville problems and their applications.* Nova Science Publishers, 2001.

[7] N.J. Guliyev, *Schrödinger operators with distributional potentials and boundary conditions dependent on the eigenvalue parameter.* J. Math. Phys. 60 (2019), no. 6, 063501.

[8] O.H. Hald, *Discontinuous inverse eigenvalue problems.* Comm. Pure Appl. Math. 37 (1984), no. 5, 53–72.

[9] A. Karakuş, M.Dzh. Manafov, *The regularized trace formula for "weighted" Sturm-Liouville equation with point δ-interaction.* Baku Math. J. 1 (2022), no. 2, 164–171.

[10] B.M. Levitan, *Inverse Sturm-Liouville problems.* VSP, Zeist, 1987.

[11] B.M. Levitan, I.S. Sargsyan, *Introduction to spectral theory.* AMS Transl. Math. Monogr. Vol. 39, Providence, 1975.

[12] Y. Liu, G. Shi, J. Yan, *An inverse problem for non-selfadjoint Sturm-Liouville operator with discontinuity conditions inside a finite interval.* Inverse Probl. Sci. Eng. 27 (2019), no. 3, 407–421.

[13] M.Dzh. Manafov, *Inverse spectral problems for energy-dependent Sturm-Liouville equations with finitely many point δ-interactions.* Electron. J. Differential Equations 2016 (2016), no. 11, 1–12.

[14] M.Dzh. Manafov, *A regularized trace formula for "weighted" Sturm-Liouville equation with point δ-interaction.* Turkish J. Math. 45 (2021), no. 4, 1767–1774.

[15] V.A. Marchenko, *Sturm-Liouville operators and their applications.* Operator Theory: Advances and Applications, Birkhäuser, Basel, 1986.

[16] X.-Ch. Xu, Ch.-F. Yang, *Inverse spectral problems for the Sturm-Liouville operator with discontinuity.* J. Differential Equations 262 (2017), 3093–3106.

[17] V.A. Yurko, *Boundary value problems with discontinuity conditions in an interior point of the interval.* Differ. Equ. 36 (2000), no. 8, 1266–1269. (Translation from: Differ. Uravneniya 36 (2000), no. 8, 1139–1140.)

[18] R. Zhang, N.P. Bondarenko, C.-F. Yang, *Solvability of an inverse problem for discontinuous Sturm-Liouville operators.* Math. Methods Appl. Sci. 44 (2021), no. 1, 124–139.

Manaf Dzh. Manafov
Faculty of Arts and Sciences, Department of Mathematics
Adıyaman University
02040 Adıyaman, Turkey
E-mail: mmanafov@adiyaman.edu.tr,

Institute of Mathematics and Mechanics
Ministry of Science and Education
AZ1141 Baku, Azerbaijan

and

Scientific and Innovation Center
Western Caspian University
AZ1001 Baku, Azerbaijan

Abdullah Kablan
Faculty of Arts and Sciences, Department of Mathematics
Gaziantep University
27310 Gaziantep, Turkey
E-mail: kablan@gantep.edu.tr

Received: 05.01.2024