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MIKHAIL L'VOVICH GOLDMAN

Doctor of physical and mathematical sciences, Professor Mikhail L'vovich Goldman passed away on July 5, 2025, at the age of 80 years.

Mikhail L'vovich was an internationally known expert in scienceand education. His fundamental scientific articles and text books in variousfields of the theory of functions of several variablesand functional analysis, the theory of approximation of functions, embedding theorems and harmonic analysis are a significant contribution to the development of mathematics.

Mikhail L'vovich was born on Aprill 13, 1945 in Moscow. In 1963, he graduated from School No. 128 in Moscow with a gold medal and entered the Physics Faculty of the Lomonosov Moscow State University. He graduated in 1969 and became a postgraduate student in the Mathematics Department. In 1972, he defended his PhD thesis "On integral representations and Fourier series of differentiable functions of several variables" under the supervision of Professor Ilyin Vladimir Aleksandrovich, and in 1988, his doctoral thesis "Study of spaces of differentiable functions of several variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor of Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an Assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he was a Professor of the Department of Theory of Functions and Differential Equations, then of the S.M. Nikol'skii Mathematical Institute at the Patrice Lumumba Peoples' Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman were: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators.Among the most important scientific achievements of M.L. Goldman, we note his research related to the optimal embedding of spaces with generalized smoothness, exact conditions for the convergence of spectral decompositions, descriptions of the integral and differential properties of generalized potentials of the Bessel and Riesz types, exact estimates for operators on cones, descriptions of optimal spaces for cones of functions with monotonicity properties.

M.L. Goldman has published more than 150 scientific articles in central mathematical journals. He is a laureate of the Moscow government competition, a laureate of the RUDN University Prize in Science and Innovation, and a laureate of the RUDN University Prize for supervision of postgraduate students. Under the supervision of Mikhail L'vovich 11 PhD theses were defended. His pupilss are actively involved in professional work at leading universities and research institutes in Russia, Kazakhstan, Ethiopia, Rwanda, Colombia, and Mongolia.

Mikhail L'vovich has repeatedly been a guest lecturer and guest professor at universities in Russia, Germany, Sweden, Great Britain, etc., and an invited speaker at many international conferences. Mikhail L'vovich was not only an excellent mathematician and teacher (he always spoke about mathematics and its teaching with great passion), but also a man of the highest culture and erudition, with a deep knowledge of history, literature and art, a very bright, kind and responsive person. This is how he will remain in the hearts of his family, friends, colleagues and students.

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Mikhail L'vovich Goldman.

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ALGEBRAS OF BINARY FORMULAS FOR WEAKLY CIRCULARLY MINIMAL THEORIES WITH EQUIVALENCE RELATIONS

B.Sh. Kulpeshov

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Abstract. Algebras of binary isolating formulas are described for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure, having only equivalence relations.

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1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of a one-type at the binary level with respect to the superposition of binary definable sets. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in papers [27, 28]. In recent years, algebras of binary formulas have been studied intensively and have been continued in works [1], [3], [7]–[14], [26], [29].

Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

Let $M = \langle M, \leq\rangle$ be a linearly ordered set. If we connect two endpoints of M (possibly, $-\infty$ and $+\infty$), then we obtain a circular order. More formally, the *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x\forall y\forall z(K(x, y, z) \to K(y, z, x))$;
(co2) $\forall x\forall y\forall z(K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;
(co3) $\forall x\forall y\forall z(K(x, y, z) \to \forall t[K(x, y, t) \vee K(t, y, z)])$;
(co4) $\forall x\forall y\forall z(K(x, y, z) \vee K(y, x, z))$.

The following observation relates linear and circular orders.

Fact 1.1. [4] (i) *If $\langle M, \leq\rangle$ is a linear ordering and K is the ternary relation derived from $\leq$ by the rule*

$$K(x, y, z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x),$$

then K is a circular order relation on M.

(ii) *If $\langle N, K\rangle$ is a circular ordering and $a \in N$, then the relation $\leq_a$ defined on $M := N \setminus \{a\}$ by the rule $y \leq_a z :\Leftrightarrow K(a, y, z)$ is a linear order.*

Thus, any linearly ordered structure is circularly ordered, since the relation of circular order is $\emptyset$-definable in an arbitrary linearly ordered structure. However, the opposite is not true. The following example shows that there are circularly ordered structures not being linearly ordered (in the sense that a linear ordering relation is not $\emptyset$-definable in an arbitrary circularly ordered structure).

Example 1. [5, 6] Let $\mathbb{Q}_2^* := \langle \mathbb{Q}_2, K, L\rangle$ be a circularly ordered structure, where $L = \{\sigma_0^2, \sigma_1^2\}$, for which the following conditions hold:

(i) its domain $\mathbb{Q}_2$ is a countable dense subset of the unit circle, no two points making the central angle π;

(ii) for distinct $a, b \in \mathbb{Q}_2$

$$(a, b) \in \sigma_0 \Leftrightarrow 0 < \arg(a/b) < \pi,$$

$$(a, b) \in \sigma_1 \Leftrightarrow \pi < \arg(a/b) < 2\pi,$$

where $\arg(a/b)$ means the value of the central angle between a and b clockwise.

Indeed, one can check that the linear order relation is not $\emptyset$-definable in this structure.

The notion of *weak circular minimality* was studied initially in [15]. Let $A \subseteq M$, where M is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a, c, b)$, $c \in A$ holds, or for any $c \in M$ with $K(b, c, a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $M = \langle M, K, \ldots\rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. The study of weakly circularly minimal structures was continued in papers [16]–[22].

Let M be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(M)$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on M is an arbitrary G-invariant equivalence relation on M. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on M.

Notation 1. (1) $K_0(x, y, z) := K(x, y, z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n\rangle$ having the length 3 (in ascending order) satisfy K; similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure M. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend naturally that notation, using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

Further, we need the notion of the definable completion of a circularly ordered structure, introduced in [15]. Its linear analogue was introduced in [25]. A *cut* $C(x)$ in a circularly ordered structure M is the maximal consistent set of formulas of the form $K(a, x, b)$, where $a, b \in M$. A cut is said to be *algebraic* if there exists $c \in M$ that realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for all $b \in M$ the formula $K(a, x, b) \in C(x)$, or for all $b \in M$ the formula $K(b, x, a) \in C(x)$, then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*. A *definable cut* in M is a cut $C(x)$ with the following property: there exist $a, b \in M$ such that $K(a, x, b) \in C(x)$ and the set $\{c \in M \mid K(a, c, b) \text{ and } K(a, x, c) \in C(x)\}$ is definable. The *definable completion* $\overline{M}$ of a structure M consists of M together with all definable cuts in M that are irrational (essentially $\overline{M}$ consists of endpoints of definable subsets of the structure M).

Notation 2. [15] Let $F(x, y)$ be an L-formula such that $F(M, b)$ is convex infinite co-infinite for each $b \in M$. Let $F^\ell(y)$ be the formula saying y is a left endpoint of $F(M, y)$:

$$\exists z_1 \exists z_2 [K_0(z_1, y, z_2) \wedge \forall t_1 (K(z_1, t_1, y) \wedge t_1 \neq y \rightarrow \neg F(t_1, y)) \wedge$$

$$\forall t_2 (K(y, t_2, z_2) \wedge t_2 \neq y \rightarrow F(t_2, y))].$$

We say that $F(x,y)$ is *convex-to-right* if

$$M \models \forall y \forall x[F(x,y) \to F^l(y) \land \forall z(K(y,z,x) \to F(z,y))].$$

If $F_1(x,y), F_2(x,y)$ are arbitrary convex-to-right formulas we say F_2 is *bigger than* F_1 if there is $a \in M$ with $F_1(M,a) \subset F_2(M,a)$. If M is 1-transitive and this holds for some a, it holds for all a. This gives a total ordering on the (finite) set of all convex-to-right formulas $F(x,y)$ (viewed up to equivalence modulo $Th(M)$).

Consider $F(M,a)$ for arbitrary $a \in M$. In general, $F(M,a)$ has no the right endpoint in M. For example, if $dcl(a) = \{a\}$ holds for some $a \in M$, then for any convex-to-right formula $F(x,y)$ and any $a \in M$ the formula $F(M,a)$ has no the right endpoint in M. We write $f(y) := \text{rend } F(M,y)$, assuming that $f(y)$ is the right endpoint of the set $F(M,y)$ that lies, in general, in the definable completion $\overline{M}$ of M. Then, f is a function mapping M in $\overline{M}$.

Let $F(x,y)$ be a convex-to-right formula. We say that $F(x,y)$ is *equivalence-generating* if for any $a,b \in M$ such that $M \models F(b,a)$ the following holds:

$$M \models \forall x(K(b,x,a) \land x \neq a \to [F(x,a) \leftrightarrow F(x,b)]).$$

Lemma 1.1. [22] *Let M be an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, $F(x,y)$ be a convex-to-right formula that is equivalence-generating. Then $E(x,y) := F(x,y) \lor F(y,x)$ is an equivalence relation partitioning M into infinite convex classes.*

Let M, N be circularly ordered structures. The 2-*reduct* of M is a circularly ordered structure with the same universe of M and consisting of predicates for each $\emptyset$-definable relation on M of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure M is *isomorphic* to N *up to binarity* or *binarily isomorphic to* N if the 2-reduct of M is isomorphic to the 2-reduct of N.

The following definition can be used in a circular ordered structure as well.

Definition 1. [23], [24] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set A* ($RC(A)$) is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if A is finite and non-empty.

3) $RC(A) \geq 1$ if A is infinite.

4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x,y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- for every $i,j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i,b_j)$;
- for every $i \in \omega$, $RC(E(x,b_i)) \geq \alpha$ and $E(M,b_i)$ is a convex subset of A.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x,\bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M,\bar{a})$, i.e. $RC(\phi(x,\bar{a})) := RC(\phi(M,\bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

The following theorem characterizes up to binarity $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures of convexity rank greater than 1 having both a trivial definable closure and the condition that any convex-to-right formula is equivalence-generating:

Theorem 1.1. [16] *Let M be an $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1 with $dcl(a) = \{a\}$ for some $a \in M$ such that any convex-to-right formula is equivalence-generating.*

Then, M is isomorphic up to binarity to $M_{s,m} := \langle M, K^3, E_1^2, E_2^2, \ldots, E_s^2, E_{s+1}^2 \rangle$, where M is a circularly ordered structure, M is densely ordered, $s, m \geq 1$; E_{s+1} is an equivalence relation, partitioning M into m infinite convex classes without endpoints; E_i for every $1 \leq i \leq s$ is an equivalence relation, partitioning each E_{i+1}-class into infinitely many infinite convex E_i-subclasses without endpoints so that the induced ordering on E_i-subclasses is dense without endpoints.

In [9] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. In [11] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [12]–[13] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [14] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure.

Here, we describe algebras of binary isolating formulas for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure.

2 Results

Definition 2. [28] Let $p \in S_1(\emptyset)$ be non-algebraic. The algebra $\mathcal{P}_{\nu(p)}$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_{\nu(p)}$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_{\nu(p)}$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_{\nu(p)}$. We also say that an m-deterministic algebra $\mathcal{P}_{\nu(p)}$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic.

We say that the algebra $\mathcal{P}_{\nu(p)}$ is *$\exists$-maximally absorbing* if there exist $u_1, u_2 \in \rho_{\nu(p)}$ such that $u_1 \cdot u_2$ consists of all the labels of $\mathcal{P}_{\nu(p)}$.

Example 2. Consider the structure $M_{1,1} := \langle M, K^3, E_1^2 \rangle$ from Theorem 1.1. We assert that $Th(M_{1,1})$ has four binary isolating formulas:

$$\theta_0(x, y) := x = y,$$

$$\theta_1(x, y) := E_1(x, y) \land x \neq y \land \forall t[K(x, t, y) \to E_1(x, t)],$$

$$\theta_2(x, y) := \neg E_1(x, y),$$

$$\theta_3(x, y) := E_1(x, y) \land x \neq y \land \forall t[K(y, t, x) \to E_1(x, t)].$$

Clearly,

$$K_0(\theta_0(a, M), \theta_1(a, M), \theta_2(a, M), \theta_3(a, M))$$

holds for every $a \in M$.

Define the labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x, y), \text{ where } 0 \leq k \leq 3.$$

It easy to check that for the algebra $\mathfrak{P}_{M_{1,1}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{0,1,3\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2,3\}$	$\{2\}$
3	$\{3\}$	$\{0,1,3\}$	$\{2\}$	$\{3\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,1}}$ is commutative and strictly 4-deterministic.

Example 3. Consider now the structure $M_{1,2} := \langle M, K^3, E_1^2, E_2^2\rangle$ from Theorem 1.1. We assert that $Th(M_{1,2})$ has six binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$
$$\theta_2(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(x,t,y) \to E_2(x,t)],$$
$$\theta_3(x,y) := \neg E_2(x,y),$$
$$\theta_4(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(y,t,x) \to E_2(x,t)],$$
$$\theta_5(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

Clearly,
$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M))$$
holds for every $a \in M$.

Define the labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \leq k \leq 5.$$

It easy to check that for the algebra $\mathfrak{P}_{M_{1,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{0,1,5\}$
2	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{0,1,2,4,5\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,1,2,4,5\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\{0,1,2,4,5\}$	$\{3\}$	$\{4\}$	$\{4\}$
5	$\{5\}$	$\{0,1,5\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,2}}$ is commutative and strictly 5-deterministic.

Proposition 2.1. *The algebra $\mathfrak{P}_{M_{1,m}}$ of binary isolating formulas has $m+4$ labels, is commutative and strictly 5-deterministic for every natural number $m \geq 2$.*

Proof. The universe M of the structure $M_{1,m}$ is partitioned by the equivalence relation E_2 into m infinite convex classes. Take an arbitrary element $a \in M$. It belongs to one of these convex classes. In this convex class five binary isolating formulas appear:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$
$$\theta_2(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(x,t,y) \to E_2(x,t)],$$
$$\theta_{m+2}(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(y,t,x) \to E_2(x,t)],$$

$$\theta_{m+3}(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

There remain $m-1$ convex classes, where there are no elements lying in the algebraic closure of the element a, defining additionally $m-1$ binary isolating formulas. These formulas are defined as follows:

$$\theta_i(x,y) := \neg E_2(x,y) \wedge \forall t[K(x,t,y) \wedge \neg E_1(x,t) \wedge \neg E_2(t,y) \to \vee_{s=2}^{i-1}\theta_s(x,t)], 3 \leq i \leq m+1.$$

Thus, there are $5+(m-1) = m+4$ binary isolating formulas, and we have defined the formulas so that for any $a \in M$ the following holds:

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \ldots, \theta_m(a,M), \theta_{m+1}(a,M), \theta_{m+2}(a,M), \theta_{m+3}(a,M)).$$

Prove now the commutativity. First, it is obvious, $0 \cdot k = k \cdot 0 = \{k\}$ for every $0 \leq k \leq m+3$. Suppose further that both $k_1 \neq 0$ and $k_2 \neq 0$.

Case 1. $k_1 + k_2 = m+4$.

If $k_1 = 1$, then $k_2 = m+3$. In this case each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains, as a conjunctive member, the formula $E_1(x,y)$, i.e. the formula $E_1(x,y)$ is compatible with

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)].$$

We have: for any t, satisfying the formula $\theta_{k_1}(x,t)$, it follows that $t \in E_1(x,M)$ and t is in this class to the right of the element x. Considering an arbitrary element y, satisfying the formula $\theta_{k_2}(t,y)$, we obtain that $y \in E_1(t,M)$ and y is in this class to the left of the element t, i.e. the formula

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with every formula from the list of formulas with labels $\{0,1,m+3\}$. Consequently, $k_1 \cdot k_2 = \{0,1,m+3\}$. We can show similarly that $k_2 \cdot k_1 = \{0,1,m+3\}$.

If $k_1 = 2$, then $k_2 = m+2$. In this case each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains as a conjunctive member the formula $E_2(x,y) \wedge \neg E_1(x,y)$, i.e. the formula $E_2(x,y) \wedge \neg E_1(x,y)$ is compatible with

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)].$$

We have: for any t, satisfying the formula $\theta_{k_1}(x,t)$, it follows that $t \in E_2(x,M)$, $t \notin E_1(x,M)$, and t is in this class to the right of the element x. Considering an arbitrary element y, satisfying the formula $\theta_{k_2}(t,y)$, we obtain that $y \in E_2(t,M)$, $y \notin E_1(t,M)$, and y is in this class to the left of the element t, i.e. the formula

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with every formula from the list of formulas with labels $\{0,1,2,m+2,m+3\}$. Consequently, $k_1 \cdot k_2 = \{0,1,2,m+2,m+3\}$. We can show similarly that

$$k_2 \cdot k_1 = \{0,1,2,m+2,m+3\}.$$

Let now $2 < k_1 < m+2$. Then, we also have that $2 < k_2 < m+2$. Consequently, each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains as a conjunctive member the formula $\neg E_2(x,y)$. We have: t lies in the (k_1-1)-th E_2-class from $E_2(x,M)$ (i.e. the E_2-class, containing x is the first E_2-class; the next clockwise E_2-class is the second, etc.); y lies in the (k_2-1)-th E_2-class from $E_2(t,M)$. Then, we obtain that y lies in the (k_1+k_2-2)-th E_2-class from $E_2(x,M)$. But $k_1+k_2-2 = m+2$, i.e. y falls into $E_2(x,M)$. Therefore, we get that

$$k_1 \cdot k_2 = k_2 \cdot k_1 = \{0,1,2,m+2,m+3\}.$$

Case 2. $k_1 + k_2 < m+4$.

Let us first assume that $k_1 = 1$. If $k_2 = 1$, then

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with the formula $E_1(x,y)$. We have: t lies in the same E_1-class with x and in this class to the right of it; y lies in the same E_1-class with t and also to the right of it in this class. Consequently, y lies in the same E_1-class with x and in this class to the right of it, i.e. $1 \cdot 1 = \{1\}$.

Suppose now that $k_1 = 2$. If $k_2 = 2$ then

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with the formula $E_2(x,y)$. We have: t lies in the same E_2-class with x and in this class to the right of it; y lies in the same E_2-class with t and also in this class to the right of it. Consequently, y lies in the same E_2-class with x and in this class to the right of it, i.e. $2 \cdot 2 = \{2\}$.

Let now $k_2 > 2$. Clearly, $k_2 < m+2$ (since $k_1 + k_2 < m+4$). We have: t lies in the same E_2-class with x and in this class to the right of it; y lies in the (k_2-1)-th E_2-class from $E_2(t,M)$. Consequently, y lies in the (k_2-1)-th E_2-class from $E_2(x,M)$, i.e. $2 \cdot k_2 = \{k_2\}$. We can show similarly that $k_2 \cdot 2 = \{k_2\}$.

Suppose now that $k_1 > 2$ and $k_2 > 2$. Clearly, $k_1 < m+2$ and $k_2 < m+2$. Then each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains as a conjunctive member the formula $\neg E_2(x,y)$. We have: t lies in the (k_1-1)-th E_2-class from $E_2(x,M)$; y lies in the (k_2-1)-th E_2-class from $E_2(t,M)$. Then, we obtain that y lies in the (k_1+k_2-2)-th E_2-class from $E_2(x,M)$, i.e. $k_1 \cdot k_2 = \{k_1+k_2-2\}$. We can show similarly that $k_2 \cdot k_1 = \{k_1+k_2-2\}$.

Case 3. $k_1 + k_2 > m+4$.

In this case $k_1 > 1$ and $k_2 > 1$ (since otherwise we would obtain that $k_1 + k_2 \le m+4$).

Suppose first that $k_1 = 2$. Then, we unambiguously obtain that $k_2 = m+3$. We have: t lies in $E_2(x,M)$ and t is in this class to the right of the element x; y lies in $E_1(t,M)$ and t is in this class to the left of the element t, whence we obtain that $k_1 \cdot k_2 = \{k_1\}$. We can show similarly that $k_2 \cdot k_1 = \{k_1\}$.

Let now $k_1 > 2$. We have: t lies in the (k_1-1)-th E_2-class from $E_2(x,M)$. In this case $k_2 \ge m+2$, i.e. k_2 can take only the following values: $m+2$ and $m+3$. Then, we obtain: y lies in $E_2(t,M) \setminus E_1(t,M)$ or $E_1(t,M)$ and t is in the corresponding class to the left of the element t, whence we obtain that $k_1 \cdot k_2 = \{k_1\}$. We can show similarly that $k_2 \cdot k_1 = \{k_1\}$.

Suppose now that $k_1 = m+2$. We have: t lies in $E_2(x,M)$ and t is in this class to left of the element x. In this case $k_2 > 2$. If $k_2 \ge m+2$, then again we get that $k_1 \cdot k_2 = \{k_1\}$. We can show similarly that $k_2 \cdot k_1 = \{k_1\}$.

Further, suppose that $2 < k_1 < m+2$ and $2 < k_2 < m+2$. We have: t lies in the (k_1-1)-th E_2-class from $E_2(x,M)$; y lies in the (k_2-1)-th E_2-class from $E_2(t,M)$, but at the same time y jumps over $E_2(x,M)$ that is consistent with five binary isolating formulas. Therefore, y lies in the $(k_1+k_2+2)[\text{mod } m+4]$-th E_2-class from $E_2(x,M)$. Consequently, the formula

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

uniquely determines the formula $\theta_{(k_1+k_2+2)[\text{mod } m+4]}(x,y)$. We can show similarly that

$$k_2 \cdot k_1 = (k_1+k_2+2)[\text{mod } m+4].$$

□

Example 4. Consider now the structure $M_{2,1} := \langle M, K^3, E_1^2, E_2^2 \rangle$ from Theorem 1.1. We assert that $Th(M_{2,1})$ has six binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$
$$\theta_2(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(x,t,y) \to E_2(x,t)],$$
$$\theta_3(x,y) := \neg E_2(x,y),$$
$$\theta_4(x,y) := E_2(x,y) \wedge \neg E_1(x,y) \wedge \forall t[K(y,t,x) \to E_2(x,t)],$$
$$\theta_5(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

Clearly, $K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M))$ holds for every $a \in M$. Define the labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \leq k \leq 5.$$

It easy to check that for the algebra $\mathfrak{P}_{M_{2,1}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{0,1,5\}$
2	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{0,1,2,4,5\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,1,2,3,4,5\}$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\{0,1,2,4,5\}$	$\{3\}$	$\{4\}$	$\{4\}$
5	$\{5\}$	$\{0,1,5\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{2,1}}$ is commutative and strictly 6-deterministic.

Proposition 2.2. *The algebra $\mathfrak{P}_{M_{s,1}}$ of binary isolating formulas has $2s+2$ labels, is commutative and strictly $(2s+2)$-deterministic for every natural number $s \geq 1$.*

Proof. The universe M of the structure $M_{s,1}$ is partitioned by the equivalence relation E_s into infinitely many infinite convex classes, so that the induced ordering on E_s-classes is dense without endpoints; in addition, for any $2 \leq i \leq s$, each E_i-class is partitioned into infinitely many convex E_{i-1}-subclasses, so that the induced order on E_{i-1}-subclasses is dense without endpoints.

We have the following binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$
$$\theta_i(x,y) := E_i(x,y) \wedge \neg E_{i-1}(x,y) \wedge \forall t[K(y,t,x) \to E_i(x,t)], 2 \leq i \leq s,$$
$$\theta_{s+1}(x,y) := \neg E_s(x,y),$$
$$\theta_j(x,y) := E_{2s+2-j}(x,y) \wedge \neg E_{2s+1-j}(x,y) \wedge \forall t[K(y,t,x) \to E_{2s+2-j}(x,t)], s+2 \leq j \leq 2s,$$
$$\theta_{2s+1}(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

Thus, there exist $2s+2$ binary isolating formulas, and we have defined the formulas so that

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \ldots, \theta_{2s}(a,M), \theta_{2s+1}(a,M))$$

holds for any $a \in M$.

Prove now the commutativity. First, it is obvious that $0 \cdot k = k \cdot 0 = \{k\}$ for any $0 \leq k \leq 2s+1$. Suppose further that $k_1 \neq 0$ and $k_2 \neq 0$.

Case 1. $k_1 + k_2 = 2s+2$.

If $k_1 = l$ for some $1 \leq l \leq s$, then $k_2 = 2s + 2 - l$. Then, each of the formulas $\theta_{k_1}(x, y)$ and $\theta_{k_2}(x, y)$ contains, as a conjunctive member, the formula $E_l(x, y)$, i.e. the formula $E_l(x, y)$ is compatible with

$$\exists t[\theta_{k_1}(x, t) \wedge \theta_{k_2}(t, y)].$$

We have: for any t, satisfying the formula $\theta_{k_1}(x, t)$, it follows that $t \in E_l(x, M) \setminus E_{l-1}(x, M)$ (if $l = 1$, then $t \in E_1(x, M)$) and t is in this class to the right of the element x. Considering an arbitrary element y, satisfying the formula $\theta_{k_2}(t, y)$, we obtain that $y \in E_l(t, M) \setminus E_{l-1}(t, M)$ (if $l = 1$, then $y \in E_1(t, M)$) and y is in this class to the left of the element t, i.e. the formula

$$\exists t[\theta_{k_1}(x, t) \wedge \theta_{k_2}(t, y)]$$

is compatible with every formula from the list of formulas with labels $\{0, 1, \ldots, l, 2s+2-l, \ldots, 2s+1\}$. Consequently, $k_1 \cdot k_2 = \{0, 1, \ldots, l, 2s + 2 - l, \ldots, 2s + 1\}$. We can show similarly that

$$k_2 \cdot k_1 = \{0, 1, \ldots, l, 2s + 2 - l, \ldots, 2s + 1\}.$$

Let now $k_1 = s + 1$. Then, we also have that $k_2 = s + 1$ and each of the formulas $\theta_{k_1}(x, y)$ and $\theta_{k_2}(x, y)$ contains as a conjunctive member the formula $\neg E_s(x, y)$.

We have: for any t satisfying the formula $\theta_{k_1}(x, t)$, $\neg E_s(x, t)$ holds. Considering an arbitrary element y, satisfying the formula $\theta_{k_2}(t, y)$, we obtain that $\neg E_s(t, y)$. Thus, both $\neg E_s(x, y)$ and $E_s(x, y)$ are possible. Consequently, $k_1 \cdot k_2 = \{0, 1, 2, \ldots, 2s, 2s + 1\}$. We can show similarly that

$$k_2 \cdot k_1 = \{0, 1, 2, \ldots, 2s, 2s + 1\}.$$

If $k_1 = l$ for some $s+2 \leq l \leq 2s+1$, then $k_2 = 2s+2-l$, i.e. $1 \leq k_2 \leq l$. We can show similarly that

$$k_1 \cdot k_2 = \{0, 1, \ldots, l, 2s + 2 - l, \ldots, 2s + 1\}$$

and

$$k_2 \cdot k_1 = \{0, 1, \ldots, l, 2s + 2 - l, \ldots, 2s + 1\}.$$

Thus, in the case $k_1 = k_2 = s + 1$ we obtain that the product of labels k_1 and k_2 contains all the labels of the algebra, whence we conclude that the algebra $\mathfrak{P}_{M_{s,1}}$ is strictly $(2s + 2)$-deterministic.

Case 2. $k_1 + k_2 < 2s + 2$.

Suppose first that $1 \leq k_1, k_2 \leq s$. If $k_1 = k_2$, then since each of the formulas $\theta_{k_1}(x, y)$ and $\theta_{k_2}(x, y)$ contains as a conjunctive member the formula $E_l(x, y)$ for some $1 \leq l \leq s$, we obtain that $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l\}$. If $k_1 < k_2$, then since $\theta_{k_1}(x, y)$ contains as a conjunctive member the formula $E_{l_1}(x, y)$, and $\theta_{k_2}(x, y)$ contains as a conjunctive member the formula $E_{l_2}(x, y)$ for some $1 \leq l_1 < l_2 \leq s$, we obtain that $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$. Similar reasoning is for the case $k_1 > k_2$.

Suppose now that $1 \leq k_1 \leq s$ and $k_2 > s$. If $k_2 = s + 1$, then for any t satisfying the formula $\theta_{k_1}(x, t)$, it follows that $t \in E_l(x, M)$ for some $1 \leq l \leq s$; while for any y, satisfying the formula $\theta_{k_2}(t, y)$, $\neg E_s(t, y)$ holds. Whence we conclude that $k_1 \cdot k_2 = k_2 \cdot k_1 = \{s + 1\}$. If $k_2 \neq s + 1$, then $s + 2 \leq k_2 < 2s + 1$ and for any y satisfying the formula $\theta_{k_2}(t, y)$, it follows that $y \in E_{l_2}(t, M)$ for some $1 \leq l_2 \leq s$ (here $l_2 = 2s + 2 - k_2$).

If $l > l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l\}$. If $l < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$. The case $l = l_2$ is impossible, since otherwise we obtain $l + l_2 = 2s + 2$.

The case in which $k_1 > s$ is considered similarly (in this case $1 \leq k_2 < s$).

Case 3. $k_1 + k_2 > 2s + 2$.

In this case $k_1 > 1$ and $k_2 > 1$ (indeed, if we suppose that $k_1 = 1$, then k_2 must be greater than $2s + 1$ that is impossible). If $2 \leq k_1 \leq s$, then $k_2 > s + 2$, i.e. $s + 3 \leq k_2 \leq 2s + 1$.

We have: $t \in E_{l_1}(x, M)$ for some $2 \leq l_1 \leq s$, $y \in E_{l_2}(t, M)$ for some $1 \leq l_2 \leq s - 1$.

If $l_1 > l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

The case $l_1 = l_2$ is also impossible, since otherwise we obtain $l_1 + l_2 = 2s + 2$.

Let now $k_1 > s$. In this case $s + 2 \leq k_2 \leq 2s + 1$. If $k_1 = s + 1$, then we obtain $\neg E_s(x, t)$. Consequently, $k_1 \cdot k_2 = k_2 \cdot k_1 = \{s + 1\}$.

If $k_1 \geq s + 2$, then $s + 1 \leq k_2 \leq 2s + 1$. If $k_2 = s + 1$, then we obtain $k_1 \cdot k_2 = k_2 \cdot k_1 = \{s + 1\}$. If $k_2 \geq s + 2$, then we have: $t \in E_{l_1}(x, M)$ for some $1 \leq l_1 \leq s$, $y \in E_{l_2}(t, M)$ for some $1 \leq l_2 \leq s$. If $l_1 \geq l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$. □

Corollary 2.1. *The algebra $\mathfrak{P}_{M_{s,1}}$ of binary isolating formulas is $\exists$-maximally absorbing for every natural number $s \geq 1$.*

Example 5. Consider now the structure $M_{2,2} := \langle M, K^3, E_1^2, E_2^2, E_3^2 \rangle$ from Theorem 1.1. Here $E_3(x, y)$ is an equivalence relation partitioning the universe of the structure into two infinite convex classes. We assert that $Th(M_{2,2})$ has eight binary isolating formulas:

$$\theta_0(x, y) := x = y,$$

$$\theta_1(x, y) := E_1(x, y) \land x \neq y \land \forall t[K(x, t, y) \to E_1(x, t)],$$

$$\theta_2(x, y) := E_2(x, y) \land \neg E_1(x, y) \land \forall t[K(x, t, y) \to E_2(x, t)],$$

$$\theta_3(x, y) := E_3(x, y) \land \neg E_2(x, y) \land \forall t[K(x, t, y) \to E_3(x, t)],$$

$$\theta_4(x, y) := \neg E_3(x, y),$$

$$\theta_5(x, y) := E_3(x, y) \land \neg E_2(x, y) \land \forall t[K(y, t, x) \to E_3(x, t)],$$

$$\theta_6(x, y) := E_2(x, y) \land \neg E_1(x, y) \land \forall t[K(y, t, x) \to E_2(x, t)],$$

$$\theta_7(x, y) := E_1(x, y) \land x \neq y \land \forall t[K(y, t, x) \to E_1(x, t)].$$

Clearly,

$$K_0(\theta_0(a, M), \theta_1(a, M), \theta_2(a, M), \theta_3(a, M), \theta_4(a, M), \theta_5(a, M), \theta_6(a, M), \theta_7(a, M))$$

holds for every $a \in M$.

Define the labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x, y), \text{ where } 0 \leq k \leq 7.$$

It easy to check that for the algebra $\mathfrak{P}_{M_{2,2}}$ the following equalities hold:
$0 \cdot k = k \cdot 0 = \{k\}$ for every $0 \leq k \leq 7$,
$1 \cdot k = k \cdot 1 = \{k\}$ for every $1 \leq k \leq 6$, and $1 \cdot 7 = \{0, 1, 7\}$,
$2 \cdot k = k \cdot 2 = \{k\}$ for every $2 \leq k \leq 5$, $2 \cdot 6 = \{0, 1, 2, 6, 7\}$, and $2 \cdot 7 = \{2\}$,
$3 \cdot k = k \cdot 3 = \{k\}$ for every $3 \leq k \leq 4$, $3 \cdot 5 = \{0, 1, 2, 3, 5, 6, 7\}$, and
$3 \cdot 6 = 6 \cdot 3 = \{3\}$, $3 \cdot 7 = 7 \cdot 3 = \{3\}$,
$4 \cdot k = k \cdot 4 = \{4\}$ for every $1 \leq k \leq 3$, $4 \cdot 4 = \{0, 1, 2, 3, 5, 6, 7\}$, and
$4 \cdot 5 = 5 \cdot 4 = \{4\}$, $4 \cdot 6 = 6 \cdot 4 = \{4\}$, $4 \cdot 7 = 7 \cdot 4 = \{4\}$,
$5 \cdot k = k \cdot 5 = \{5\}$ for every $5 \leq k \leq 7$, and $5 \cdot 3 = \{0, 1, 2, 3, 5, 6, 7\}$,
$6 \cdot 6 = \{6\}$, $6 \cdot 7 = 7 \cdot 6 = \{6\}$, and $6 \cdot 2 = \{0, 1, 2, 6, 7\}$,
$7 \cdot 7 = \{7\}$, and $7 \cdot 1 = \{0, 1, 7\}$.

According to these equalities, the algebra $\mathfrak{P}_{M_{2,2}}$ is commutative and strictly 7-deterministic.

Theorem 2.1. *The algebra $\mathfrak{P}_{M_{s,m}}$ of binary isolating formulas has $2s+m+2$ labels, is commutative and strictly $(2s + 3)$-deterministic for any natural numbers $s, m \geq 1$.*

Proof. The universe M of the structure $M_{s,m}$ is partitioned by the equivalence relation E_{s+1} into m infinite convex classes. Take an arbitrary element $a \in M$. It falls into one of these convex classes. In this convex class, $2s+3$ binary isolating formulas arise:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(x,t,y) \to E_1(x,t)],$$

$$\theta_i(x,y) := E_i(x,y) \wedge \neg E_{i-1}(x,y) \wedge \forall t[K(x,t,y) \to E_i(x,t)],\ 2 \leq i \leq s+1,$$

$$\theta_j(x,y) := E_{2s+m+2-j}(x,y) \wedge \neg E_{2s+m+1-j}(x,y) \wedge \forall t[K(x,t,y) \to E_{2s+m+2-j}(x,t)],$$

$$\text{where } s+m+1 \leq j \leq 2s+m,$$

$$\theta_{2s+m+1}(x,y) := E_1(x,y) \wedge x \neq y \wedge \forall t[K(y,t,x) \to E_1(x,t)].$$

There remain $m-1$ convex classes, where there are no elements lying in the algebraic closure of the element a, defining additionally $m-1$ binary isolating formulas. These formulas are defined as follows:

$$\theta_l(x,y) := \neg E_{s+1}(x,y) \wedge \forall t[K(x,t,y) \wedge \neg E_s(x,t) \wedge \neg E_{s+1}(t,y) \to \vee_{k=s+1}^{l-1} \theta_k(x,t)],$$

$$\text{where } s+2 \leq l \leq s+m.$$

Thus, we get $2s+3+(m-1) = 2s+m+2$ binary isolating formulas, and we have defined the formulas, so that

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \ldots, \theta_{2s+m}(a,M), \theta_{2s+m+1}(a,M)).$$

holds for any $a \in M$.

Prove now the commutativity. First, it is obvious that $0 \cdot k = k \cdot 0 = \{k\}$ for any $0 \leq k \leq 2s+m+1$. Suppose further that $k_1 \neq 0$ and $k_2 \neq 0$.

Case 1. $k_1 + k_2 = 2s+m+2$.

If $k_1 = 1$, then clearly $k_2 = 2s+m+1$ and each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains, as a conjunctive member, the formula $E_1(x,y)$, i.e. the formula $E_1(x,y)$ is compatible with

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)].$$

We have: for any t, satisfying the formula $\theta_{k_1}(x,t)$, it follows that $t \in E_1(x,M)$ and t is to the right of the element x. Considering an arbitrary element y satisfying the formula $\theta_{k_2}(t,y)$, we obtain that $y \in E_1(t,M)$ and y is to the left of the element t, i.e. we obtain that the formula

$$\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$$

is compatible with every formula of the list of formulas with labels $\{0, 1, 2s+m+1\}$. Consequently, $k_1 \cdot k_2 = \{0, 1, 2s+m+1\}$. We can show similarly that $k_2 \cdot k_1 = \{0, 1, 2s+m+1\}$.

If $k_1 = l$ for some $2 \leq l \leq s+1$, we have $k_2 = 2s+m+2-l$. Then, each of the formulas $\theta_{k_1}(x,y)$ and $\theta_{k_2}(x,y)$ contains as a conjunctive member the formula

$$E_l(x,y) \wedge \neg E_{l-1}(x,y).$$

We have the following: $t \in E_l(x,M) \setminus E_{l-1}(x,M)$ and t is in this class to the right of the element x; $y \in E_l(t,M) \setminus E_{l-1}(t,M)$ and y is in this class to the left of the element t. Whence we obtain that

$$k_1 \cdot k_2 = k_2 \cdot k_1 = \{0, 1, \ldots, l, 2s+m+2-l, \ldots, 2s+m+1\}.$$

We can show similarly that

$$k_2 \cdot k_1 = \{0, 1, \ldots, l, 2s + m + 2 - l, \ldots, 2s + m + 1\}.$$

Suppose now that $s + 2 \leq k_1 \leq s + m$. Then, $k_2 = 2s + m + 2 - k_1$, i.e. we also have $s + 2 \leq k_2 \leq s + m$ and each of the formulas $\theta_{k_1}(x, y)$ and $\theta_{k_2}(x, y)$ contains as a conjunctive member the formula $\neg E_{s+1}(x, y)$.

We have the following: t lies in the $(k_1 - s)$-th E_{s+1}-class from $E_{s+1}(x, M)$; y lies in the $(k_2 - s)$-th E_{s+1}-class from $E_{s+1}(t, M)$. Then, we obtain that y lies in the $(k_1 + k_2 - 2s - 1)$-th E_{s+1}-class from $E_{s+1}(x, M)$. But $k_1 + k_2 - 2s - 1 = m + 1$, i.e. y falls into $E_{s+1}(x, M)$, whence

$$k_1 \cdot k_2 = \{0, 1, \ldots, s + 1, s + m + 1, \ldots, 2s + m + 1\}.$$

We can show similarly that

$$k_2 \cdot k_1 = \{0, 1, \ldots, s + 1, s + m + 1, \ldots, 2s + m + 1\}.$$

Let now $s + m + 1 \leq k_1 \leq 2s + m + 1$. Then, obviously $1 \leq k_2 \leq s + 1$. If $k_1 = l$ for some $s + m + 1 \leq l \leq 2s + m + 1$, we can show similarly that

$$k_1 \cdot k_2 = k_2 \cdot k_1 = \{0, 1, \ldots, 2s + m + 2 - l, s + m + 1, \ldots, l\}.$$

Case 2. $k_1 + k_2 < 2s + m + 2$.

First, suppose that $1 \leq k_1 \leq s + 1$. If $1 \leq k_2 \leq s + 1$, then we have: $t \in E_{l_1}(x, M)$ for some $1 \leq l_1 \leq s + 1$ and t is in this class to the right of the element x; $y \in E_{l_2}(t, M)$ for some $1 \leq l_2 \leq s + 1$ and y is in this class to the right of the element t. Then, we obtain that if $l_1 \geq l_2$, $y \in E_{l_1}(x, M)$ and consequently $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $y \in E_{l_2}(x, M)$, and consequently $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

If $s + 2 \leq k_2 \leq s + m$, then we have: $t \in E_l(x, M)$ for some $1 \leq l \leq s + 1$, and $y \in \neg E_{s+1}(t, M)$, whence we obtain $\neg E_{s+1}(x, y)$, i.e. $k_1 \cdot k_2 = k_2 \cdot k_1 = \{k_2\}$.

Suppose now that $k_2 > s + m$. We have the following: $t \in E_{l_1}(x, M)$ for some $1 \leq l_1 \leq s + 1$ and t is in this class to the right of the element x; $y \in E_{l_2}(t, M)$ for some $1 \leq l_2 \leq s + 1$ and y is in this class to the left of the element t. And the case $l_1 = l_2$ is impossible, since $k_1 + k_2 < 2s + m + 2$. If $l_1 > l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

Other cases are considered similarly.

Case 3. $k_1 + k_2 > 2s + m + 2$.

In this case $k_1 > 1$ and $k_2 > 1$ (since otherwise we would obtain that $k_1 + k_2 \leq 2s + m + 2$).

If $2 \leq k_2 \leq s + 1$ then $k_2 > s + m + 1$. We have the following: $t \in E_{l_1}(x, M)$ for some $2 \leq l_1 \leq s + 1$ and t is in this class to the right of the element x; $y \in E_{l_2}(t, M)$ for some $2 \leq l_2 \leq s$ and y is in this class to the left of the element t. And the case $l_1 = l_2$ is impossible, since $k_1 + k_2 > 2s + m + 2$. If $l_1 > l_2$ then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$ then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

Suppose now that $s + 2 \leq k_1 \leq s + m$. Then, $k_2 > s + m$. We have the following: $t \in \neg E_{s+1}(x, M)$ and $y \in E_l(t, M)$ for some $2 \leq l \leq s + 1$, whence we obtain $k_1 \cdot k_2 = k_2 \cdot k_1 = \{k_1\}$.

Let now $s + m + 1 \leq k_1 \leq 2s + m + 1$. If $2 \leq k_2 \leq s + 1$, we have that $t \in E_{l_1}(x, M)$ for some $2 \leq l_1 \leq s + 1$ and t is in the this class to the left of the element x; $y \in E_{l_2}(t, M)$ for some $2 \leq l_2 \leq s$ and y is in this class to the right of the element t. And the case $l_1 = l_2$ is impossible, since $k_1 + k_2 > 2s + m + 2$. If $l_1 > l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$.

If $s + 2 \leq k_2 \leq s + m$, then we have: $t \in E_l(x, M)$ for some $1 \leq l \leq s + 1$, and $y \in \neg E_{s+1}(t, M)$, whence we obtain $\neg E_{s+1}(x, y)$, i.e. $k_1 \cdot k_2 = k_2 \cdot k_1 = \{k_2\}$.

Suppose now that $k_2 > s + m$. We have the following: $t \in E_{l_1}(x, M)$ for some $1 \leq l_1 \leq s + 1$ and t is in this class to the left of the element x; $y \in E_{l_2}(t, M)$ for some $1 \leq l_2 \leq s + 1$ and y is in this class to the left of the element t. If $l_1 \geq l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_1\}$. If $l_1 < l_2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{l_2\}$. □

Corollary 2.2. *The algebra $\mathfrak{P}_{M_{s,m}}$ is $\exists$-maximally absorbing if and only if $m = 1$.*

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