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MIKHAIL L'VOVICH GOLDMAN

Doctor of physical and mathematical sciences, Professor Mikhail L'vovich Goldman passed away on July 5, 2025, at the age of 80 years.

Mikhail L'vovich was an internationally known expert in scienceand education. His fundamental scientific articles and text books in variousfields of the theory of functions of several variablesand functional analysis, the theory of approximation of functions, embedding theorems and harmonic analysis are a significant contribution to the development of mathematics.

Mikhail L'vovich was born on Aprill 13, 1945 in Moscow. In 1963, he graduated from School No. 128 in Moscow with a gold medal and entered the Physics Faculty of the Lomonosov Moscow State University. He graduated in 1969 and became a postgraduate student in the Mathematics Department. In 1972, he defended his PhD thesis "On integral representations and Fourier series of differentiable functions of several variables" under the supervision of Professor Ilyin Vladimir Aleksandrovich, and in 1988, his doctoral thesis "Study of spaces of differentiable functions of several variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor of Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an Assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he was a Professor of the Department of Theory of Functions and Differential Equations, then of the S.M. Nikol'skii Mathematical Institute at the Patrice Lumumba Peoples' Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman were: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators.Among the most important scientific achievements of M.L. Goldman, we note his research related to the optimal embedding of spaces with generalized smoothness, exact conditions for the convergence of spectral decompositions, descriptions of the integral and differential properties of generalized potentials of the Bessel and Riesz types, exact estimates for operators on cones, descriptions of optimal spaces for cones of functions with monotonicity properties.

M.L. Goldman has published more than 150 scientific articles in central mathematical journals. He is a laureate of the Moscow government competition, a laureate of the RUDN University Prize in Science and Innovation, and a laureate of the RUDN University Prize for supervision of postgraduate students. Under the supervision of Mikhail L'vovich 11 PhD theses were defended. His pupilss are actively involved in professional work at leading universities and research institutes in Russia, Kazakhstan, Ethiopia, Rwanda, Colombia, and Mongolia.

Mikhail L'vovich has repeatedly been a guest lecturer and guest professor at universities in Russia, Germany, Sweden, Great Britain, etc., and an invited speaker at many international conferences. Mikhail L'vovich was not only an excellent mathematician and teacher (he always spoke about mathematics and its teaching with great passion), but also a man of the highest culture and erudition, with a deep knowledge of history, literature and art, a very bright, kind and responsive person. This is how he will remain in the hearts of his family, friends, colleagues and students.

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Mikhail L'vovich Goldman.

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OSCILLATORY AND SPECTRAL ANALYSIS OF HIGHER-ORDER DIFFERENTIAL OPERATORS

A. Kalybay, R. Oinarov

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Key words: higher-order differential equation, differential operator, oscillation, non-oscillation, spectrum, variational method, weighted inequality.

AMS Mathematics Subject Classification: 34C10, 34K08, 26D10.

Abstract. In the paper there are investigated the oscillatory properties of a $2n$th order differential equation and the spectral properties of a $2n$th order differential operator. These properties are established using the variational method, which relies on verifying a specific nth order differential inequality. Here, the coefficients of both the equation and the operator are the weights in this inequality. Furthermore, the characterization of the inequality occurs when the weights satisfy conditions, ensuring the existence of a certain combination of boundary values at infinity and at zero for the function involved in this inequality.

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1 Introduction

Let $I = (0, \infty)$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\lambda > 0$, and $n > 1$ be an integer. Let u be a positive function continuous on the interval I. Suppose that v is a positive function infinitely differentiable on I.

Let $W^n_{p,v} \equiv W^n_{p,v}(I)$ represent the space of functions $f : I \to \mathbb{R}$ possessing weak derivatives up to the nth order on the interval I, satisfying $\|f^{(n)}\|_{p,v} < \infty$, where $\|f\|_{p,v} = \left(\int\limits_0^\infty v(t)|f(t)|^p dt\right)^{\frac{1}{p}}$ denotes the norm of the weighted space $L_{p,v}(I)$. Under certain conditions on the function v, we observe that $C_0^\infty(I) \subset W^n_{p,v}(I)$, where $C_0^\infty(I)$ denotes the set of all functions infinitely differentiable and compactly supported on I. Let $\mathring{W}^n_{p,v} \equiv \mathring{W}^n_{p,v}(I)$ denote the closure of the set $C_0^\infty(I)$ with respect to the norm $\|f^{(n)}\|_{p,v}$.

In the paper there are discussed oscillatory properties of the following $2n$th order differential equation

$$(-1)^n(v(t)y^{(n)}(t))^{(n)} - \lambda u(t)y(t) = 0, \quad t \in I, \tag{1.1}$$

and spectral properties of the self-adjoint differential operator L generated by the differential expression

$$ly(t) = (-1)^n \frac{1}{u(t)}(v(t)y^{(n)}(t))^{(n)}, \tag{1.2}$$

in the space $L_{2,u}(I)$ equipped with the inner product $(f, g)_{2,u} = \int\limits_0^\infty f(t)g(t)u(t)dt$.

In the qualitative analysis of differential equations, there exist effective techniques for determining the oscillatory behavior of second-order equations of the form:

$$(v(t)y'(t))' - u(t)y(t) = 0,\ t \in I.$$

However, extending these methods to higher-order equations poses challenges. Recent studies have explored approaches suited for higher-order equations, often by selecting one of the coefficients to be a power function (see, e.g., [2], [3], [20], and [21]). In this paper, we employ the variational method. This method relies on establishing a connection between the oscillatory properties of equation (1.1) and characterizations of the following inequality:

$$\left(\int_0^\infty u(t)|f(t)|^q dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(t)|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}},\quad f \in \mathring{W}^n_{p,v}(I). \tag{1.3}$$

This approach allows us to relax the requirement that the weights in the equation must be power functions exclusively. Furthermore, we derive explicit conditions for oscillation and spectral properties in terms of the coefficients u and v of equation (1.1) and the operator L. Inequality (1.3) is a generalization of the famous Hardy inequality, which has a long-standing history (see, e.g., [9]). Its different extensions and applications have evolved into an independent area known as the "theory of Hardy-type inequalities" with numerous papers being published annually (see, e.g., the most recent works [12], [17], and [22]).

The investigation of inequality (1.3) hinges on the behavior of the function v at the endpoints of the interval I. According to [10] and [16], if $v^{1-p'} \notin L_1(1,\infty)$, then there exists $f \in W^n_{p,v}$ such that the limits $\lim\limits_{t\to\infty} f^{(i)}(t)$ do not exist for all $i = 0, 1, ..., n-1$; if $v^{1-p'} \in L_1(0,1)$, then for any $f \in W^n_{p,v}$ the limits $\lim\limits_{t\to 0^+} f^{(i)}(t) \equiv f^{(i)}(0)$ exist for all $i = 0, 1, ..., n-1$. The oscillation of equation (1.1) under the conditions $v^{1-p'} \notin L_1(1,\infty)$ and $v^{1-p'} \in L_1(0,1)$ was investigated in [14] using the variational method, as will be done here. This case can be termed the "standard case", since for the nth order inequality (1.3), there exist precisely n boundary conditions at the endpoints of the interval I, namely no conditions at infinity and n finite limits at zero. The spectral properties of the operator L in this "standard case" were examined in paper [18].

From [10] and [16] it also follows that if $v^{1-p'} \in L_1(1,\infty)$ and $t^{p'}v^{1-p'} \notin L_1(1,\infty)$, then for any $f \in W^n_{p,v}$ there exists exactly one limit $\lim\limits_{t\to\infty} f^{(n-1)}(t) \equiv f^{(n-1)}(\infty)$. Therefore, together with the above condition for v at zero $v^{1-p'} \in L_1(0,1)$, they entail $n+1$ conditions at the endpoints:

$$f^{(i)}(0) = 0,\ i = 0, 1, ..., n-1, \quad \text{and} \quad f^{(n-1)}(\infty) = 0.$$

This "overdetermined" case was studied in work [7].

In our study, we explore equation (1.1) and the operator L under the conditions:

$$t^{p'(n-1)}v^{1-p'} \in L_1(1,\infty),\ t^{p'(n-2)}v^{1-p'} \notin L_1(0,1),\ \text{and}\ t^{p'(n-1)}v^{1-p'} \in L_1(0,1), \tag{1.4}$$

which, according to [10] and [16], guaranty the existence of another $n+1$ values $\lim\limits_{t\to 0^+} f(t) \equiv f(0)$ and $\lim\limits_{t\to\infty} f^{(i)}(t) \equiv f^{(i)}(\infty)$, $i = 0, 1, ..., n-1$, at the endpoints of the interval I, so that

$$\mathring{W}^n_{p,v}(I) = \{f \in W^n_{p,v}(I) : f(0) = 0 \ \text{and} \ f^{(i)}(\infty) = 0,\ i = 0, 1, ..., n-1\}. \tag{1.5}$$

Note that the same problems as here, but specifically for $n = 2$, where the differential equation and operator are of fourth-order, were considered in the paper [15]. Consequently, this paper expands its scope to include the problems for any $n \ge 2$.

The paper is organized as follows. In Section 2, we present all the main results regarding the oscillatory properties of equation (1.1) and the spectral properties of the operator L. Additionally, Section 2 encompasses the characterizations of inequality (1.3). Section 3 offers a proof concerning inequality (1.3). In Section 4, we compile the proofs of the main results concerning equation (1.1) and the operator L. In Section 5, we improve some results obtained earlier.

Let us present notations used in the paper. Assume that $\overline{v}(t) = \frac{v(t)}{t^{p(n-1)}}$, $t \in I$. Since

$$t^{p'(n-1)}v^{1-p'} = t^{p'(n-1)}\overline{v}^{1-p'}t^{p(n-1)(1-p')} = \overline{v}^{1-p'}t^{(n-1)(p'+p-pp')} = \overline{v}^{1-p'},$$

from (1.4) we have that $\overline{v}^{1-p'} \in L_1(I)$. Therefore, for any $\tau \in I$ there exists k_τ such that

$$\int_0^\tau \overline{v}^{1-p'}(t)dt = k_\tau \int_\tau^\infty \overline{v}^{1-p'}(t)dt, \tag{1.6}$$

in addition, k_τ increases in τ, $\lim\limits_{\tau\to 0^+} k_\tau = 0$, and $\lim\limits_{\tau\to\infty} k_\tau = \infty$.

The symbol $A \ll B$ means $A \le CB$ with some constant C. Additionally, we define χ_M as the characteristic function of a set M.

2 Oscillatory properties of equation (1.1) and spectral properties of the operator L

Equation (1.1) is termed oscillatory at zero if, for any $T > 0$, there exists a (non-trivial) solution of this equation possessing more than one zero with multiplicity n to the left of T ([4, p. 69]). Otherwise, equation (1.1) is termed non-oscillatory at zero.

Equation (1.1) is termed strongly oscillatory or non-oscillatory at zero if it is oscillatory or non-oscillatory at zero for all values $\lambda > 0$, respectively.

The oscillatory properties of differential equation (1.1) can be established using the variational method, relying on the following well-known statement.

Lemma A. *Equation (1.1) is non-oscillatory at zero if and only if there exists $T > 0$ such that*

$$\int_0^T \left(v(t)|f^{(n)}(t)|^2 - \lambda u(t)|f(t)|^2\right) dt \ge 0, \quad f \in \mathring{W}^n_{2,v}(0,T).$$

It is obvious that Lemma A can be reformulated as follows.

Lemma 2.1. *(i) Equation (1.1) is non-oscillatory at zero if and only if there exists $T > 0$ and $C_T > 0$, depending only on T, such that the inequality*

$$\int_0^T \lambda u(t)|f(t)|^2dt \le \lambda C_T \int_0^T v(t)|f^{(n)}(t)|^2dt, \quad f \in \mathring{W}^n_{2,v}(0,T), \tag{2.1}$$

holds with the least constant λC_T such that $0 < \lambda C_T \le 1$;
(ii) Equation (1.1) is oscillatory at zero if and only if for any $T > 0$ the least constant in (2.1) is such that $\lambda C_T > 1$.

Inequality (2.1) is a particular case of inequality (1.3), the characterizations of which are provided in the following theorem.

Theorem 2.1. *Let $1 < p \leq q < \infty$ and (1.4) hold. For $\tau \in I$ suppose that*

$$B_1(\tau) = \sup_{z>\tau} \left(\int_\tau^z u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty (s-z)^{p'(n-1)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B_2(\tau) = \sup_{z>\tau} \left(\int_\tau^z (z-t)^{q(n-1)} u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B_3(\tau) = \frac{1}{\tau} \left(\int_0^\tau t^q u(t)dt \right)^{\frac{1}{q}} \left(\int_\tau^\infty (s-\tau)^{p'(n-1)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B_4(\tau) = \frac{1}{\tau} \left(\int_0^\tau t^q (\tau-t)^{q(n-2)} u(t)dt \right)^{\frac{1}{q}} \left(\int_\tau^\infty (s-\tau)^{p'} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$F_1(\tau) = \sup_{0<z<\tau} \frac{1}{\tau^{n-1}} \left(\int_0^z t^q (\tau-t)^{q(n-2)} u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\tau (\tau-s)^{p'} s^{p'(n-2)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$F_2(\tau) = \sup_{0<z<\tau} \frac{1}{\tau^{n-1}} \left(\int_z^\tau (\tau-t)^{q(n-1)} u(t)dt \right)^{\frac{1}{q}} \left(\int_0^z s^{p'(n-1)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B(\tau) = \max\{B_1(\tau), B_2(\tau), B_3(\tau), B_4(\tau)\}, \quad F(\tau) = \max\{F_1(\tau), F_2(\tau)\},$$

$$BF = \inf_{\tau \in I} \max\{B(\tau), F(\tau)\},$$

$$\varepsilon_l(n) = \frac{4^{-\frac{1}{p}}}{(n-1)!}, \quad \varepsilon_r(n) = \frac{1}{(n-1)!} \left((n-1)2^{n-2} + (n+8)p^{\frac{1}{q}}(p')^{\frac{1}{p'}} \right).$$

Then for the least constant C in (1.3) the estimates

$$\varepsilon_l(n)BF \leq C \leq \varepsilon_r(n)BF, \tag{2.2}$$

$$\frac{1}{(n-1)!} \sup_{\tau \in I} (1 + k_\tau^{p-1})^{-\frac{1}{p}} F(\tau) \leq C \leq \varepsilon_r(n) F(\tau_0) \tag{2.3}$$

hold, where

$$\tau_0 = \inf\{\tau > 0 : B(\tau) \leq F(\tau)\}. \tag{2.4}$$

By following the same steps as in the proof of Lemma 4.3 in [7], using Lemma 2.1, we can deduce the following statement.

Lemma 2.2. *Let C_T be the least constant in (2.1).*
(i) Equation (1.1) is strongly non-oscillatory at zero if and only if $\lim_{T\to 0^+} C_T = 0$.
(ii) Equation (1.1) is strongly oscillatory at zero if and only if $C_T = \infty$ for any $T > 0$.

Based on Lemma 2.2 and Theorem 2.1, we establish the criteria for strong oscillation and non-oscillation of equation (1.1) as follows:

Theorem 2.2. *Let* $t^{2(n-1)}v^{-1} \in L_1(I)$ *and* $t^{2(n-2)}v^{-1} \notin L_1(0,1)$.
(i) Equation (1.1) *is strongly non-oscillatory at zero if and only if*

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_0^z t^2 u(t)dt \int_z^\tau s^{2(n-2)}v^{-1}(s)ds = 0, \tag{2.5}$$

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_z^\tau u(t)dt \int_0^z s^{2(n-1)}v^{-1}(s)ds = 0. \tag{2.6}$$

(ii) Equation (1.1) *is strongly oscillatory at zero if and only if*

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_0^z t^2 u(t)dt \int_z^\tau s^{2(n-2)}v^{-1}(s)ds = \infty \tag{2.7}$$

or

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_z^\tau u(t)dt \int_0^z s^{2(n-1)}v^{-1}(s)ds = \infty. \tag{2.8}$$

Let the minimal differential operator $L_{\min}$ be generated by differential expression (1.2), i.e., $L_{\min}y = ly$ is an operator with the domain $D(L_{\min}) = C_0^\infty(I)$. It is known that all self-adjoint extensions of the minimal differential operator $L_{\min}$ have the same spectrum ([4]).

Now, we present conditions under which any self-adjoint extension L of the operator $L_{\min}$ has a spectrum which is discrete and bounded below. The significance of studying these spectral properties is fully elucidated in [5].

The relationship between the non-oscillation of equation (1.1) and the above spectral properties of the operator L is expounded in the following statement ([4]).

Lemma B. *The operator L is bounded below and has a discrete spectrum if and only if equation* (1.1) *is strongly non-oscillatory.*

On the basis of Lemma B and Theorem 2.2, we obtain the following statement.

Theorem 2.3. *Let the assumptions of Theorem* 2.2 *hold. Then the operator L has a spectrum discrete and bounded below if and only if both* (2.5) *and* (2.6) *hold.*

If the operator $L_{\min}$ is nonnegative, it possesses the Friedrichs extension L_F. According to Theorem 2.3, the operator L_F exhibits a discrete spectrum if and only if both conditions (2.5) and (2.6) are satisfied.

For $p = q = 2$ inequality (1.3) can be rewritten as $(f,f)_2 C^{-2} \le (L_F f, f)_{2,u}$. Then from Theorem 2.3 we have the following theorem, where the introduced above values BF, $\varepsilon_l(n)$, and $\varepsilon_r(n)$ are taken for $p = q = 2$.

Theorem 2.4. *Let the assumptions of Theorem* 2.2 *hold. Then the operator L_F is positive definite if and only if $BF < \infty$. Moreover, $\varepsilon_l(n)BF \le \lambda_1^{-\frac{1}{2}} \le \varepsilon_r(n)BF$ holds for the smallest eigenvalue λ_1 of the operator L_F.*

By Rellih's lemma ([11, p. 183]), the operator L_F^{-1} possesses a spectrum that is discrete and bounded below in $L_{2,u}$ if and only if the space equipped with the norm $(L_F f, f)_{2,u}^{\frac{1}{2}}$ is compactly embedded into the space $L_{2,u}$. Consequently, we derive another statement from Theorem 2.3.

Theorem 2.5. *Under the assumptions of Theorem 2.2, the embedding* $\overset{\circ}{W}{}^n_{2,v}(I) \hookrightarrow L_{2,u}$ *is compact, and the operator* L_F^{-1} *is uniformly continuous on* $L_{2,u}$ *if and only if both conditions (2.5) and (2.6) are satisfied.*

Let the operator L_F^{-1} be completely continuous on $L_{2,u}$. Suppose that $\{\lambda_k\}_{k=1}^{\infty}$ are the eigenvalues and $\{\varphi_k\}_{k=1}^{\infty}$ is the corresponding complete orthonormal system of eigenfunctions of the operator L_F^{-1}. Assume that

$$D(t) = \int_t^{\infty} \left(\int_0^t (s-x)^{n-2} dx \right)^2 v^{-1}(s) ds + \int_0^t \left(\int_0^s (s-x)^{n-2} dx \right)^2 v^{-1}(s) ds.$$

Theorem 2.6. *Let the assumptions of Theorem 2.2 hold. Let (2.5) and (2.6) hold.*
(i)

$$\frac{1}{((n-2)!)^2} D(t) \le \sum_{k=1}^{\infty} \frac{|\varphi_k(t)|^2}{\lambda_k} \le \frac{2}{((n-2)!)^2} D(t). \tag{2.9}$$

(ii) The operator L_F^{-1} *is nuclear if and only if* $\int_0^{\infty} u(t)D(t)dt < \infty$*, and for the nuclear norm* $\|L_F^{-1}\|_{\sigma_1}$ *of the operator* L_F^{-1} *the relation*

$$\frac{1}{((n-2)!)^2} \int_0^{\infty} u(t)D(t)dt \le \|L_F^{-1}\|_{\sigma_1} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \le \frac{2}{((n-2)!)^2} \int_0^{\infty} u(t)D(t)dt \tag{2.10}$$

holds.

3 Proof of Theorem 2.1

Let $-\infty \le a < b \le \infty$. To prove Theorem 2.1 we use characterizations of the standard weighted Hardy inequality provided in the following statement (see, e.g., [9]).

Theorem A. *Let* $1 < p \le q < \infty$.
(i) The inequality

$$\left(\int_a^b u(t) \left| \int_a^t f(s) ds \right|^q dt \right)^{\frac{1}{q}} \le C \left(\int_a^b v(t) \left| f(t) \right|^p dt \right)^{\frac{1}{p}} \tag{3.1}$$

holds if and only if

$$A^+ = \sup_{a<z<b} \left(\int_z^b u(t) dt \right)^{\frac{1}{q}} \left(\int_a^z v^{1-p'}(s) ds \right)^{\frac{1}{p'}} < \infty,$$

moreover,

$$A^+ \le C \le p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A^+,$$

where C *is the least constant in (3.1).*
(ii) The inequality

$$\left(\int_a^b u(t) \left| \int_t^b f(s) ds \right|^q dt \right)^{\frac{1}{q}} \le C \left(\int_c^b v(t) \left| f(t) \right|^p dt \right)^{\frac{1}{p}} \tag{3.2}$$

holds if and only if

$$A^{-} = \sup_{a<z<b} \left(\int_a^z u(t)dt \right)^{\frac{1}{q}} \left(\int_z^b v^{1-p'}(s)ds \right)^{\frac{1}{p'}} < \infty,$$

moreover,

$$A^{-} \le C \le p^{\frac{1}{q}} (p')^{\frac{1}{p}} A^{-},$$

where C is the least constant in (3.2).

We also need the statement, which follows from the results of the works [19] and [6]. Let

$$B_1 = \sup_{a<z<b} \left(\int_a^z (z-t)^{q(n-1)} u(t)dt \right)^{\frac{1}{q}} \left(\int_z^b v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$B_2 = \sup_{a<z<b} \left(\int_a^z u(t)dt \right)^{\frac{1}{q}} \left(\int_z^b (s-z)^{p'(n-1)} v^{1-p'}(s)ds \right)^{\frac{1}{p'}}.$$

Theorem B. *Let $1 < p \le q < \infty$. The inequality*

$$\left(\int_a^b u(t) \left| \int_t^b (s-t)^{n-1} f(s)ds \right|^q dt \right)^{\frac{1}{q}} \le C \left(\int_a^b v(t) \left| f(t) \right|^p dt \right)^{\frac{1}{p}} \tag{3.3}$$

holds if and only if $\max\{B_1, B_2\} < \infty$. *Moreover,*

$$\max\{B_1, B_2\} \le C \le 8p^{\frac{1}{q}} (p)^{\frac{1}{p'}} \max\{B_1, B_2\},$$

where C is the least constant in (3.3).

To establish Theorem 2.1, we adopt the approach outlined in the proof of Theorem 2.2 in [13].

Proof of Theorem 2.1. *Sufficiency.* By the conditions, we have (1.5). Let $\tau \in I$. We assume that $f(t) = \int\limits_0^t f'(x)dx$ for $0 < t < \tau$, $f(t) = -\int\limits_t^\infty f'(x)dx$ for $t > \tau$ and $f'(x) = \frac{(-1)^{n-1}}{(n-2)!} \int\limits_x^\infty (s-x)^{n-2} f^{(n)}(s)ds$ for $x \in I$. Then for $f \in \mathring{W}^n_{p,v}(I)$ we have

$$f(t) = \frac{(-1)^n}{(n-1)!} \int_t^\infty (s-t)^{n-1} f^{(n)}(s)ds \tag{3.4}$$

for $t > \tau$. Moreover, we have

$$f(t) = \frac{(-1)^{n-1}}{(n-2)!} \int_0^t \int_x^\infty (s-x)^{n-2} f^{(n)}(s)dsdx = \frac{(-1)^{n-1}}{(n-2)!} \left[\int_0^t \int_x^t (s-t)^{n-2} f^{(n)}(s)dsdx \right.$$

$$\left. + \int_0^t \int_t^\tau (s-x)^{n-2} f^{(n)}(s)dsdx + \int_0^t \int_\tau^\infty (s-x)^{n-2} f^{(n)}(s)dsdx \right]$$

$$= \frac{(-1)^{n-1}}{(n-2)!}\left[\int_0^t f^{(n)}(s)\int_0^s (s-x)^{n-2}dxds + \int_t^\tau f^{(n)}(s)\int_0^t (s-x)^{n-2}dxds \right.$$
$$\left. + \int_\tau^\infty f^{(n)}(s)\int_0^t (s-x)^{n-2}dxds\right] \quad (3.5)$$

$$= \frac{(-1)^{n-1}}{(n-2)!}\left[\int_0^t f^{(n)}(s)\frac{s^{n-1}}{n-1}ds + \int_t^\tau f^{(n)}(s)s^{n-1}\frac{\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)}{n-1}ds \right.$$
$$\left. + \int_\tau^\infty f^{(n)}(s)s^{n-1}\frac{\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)}{n-1}ds\right].$$

Assuming $g(s) = f^{(n)}(s)s^{n-1}$, the last equality gives that

$$f(t) = \frac{(-1)^n}{(n-1)!}\left[-\int_\tau^\infty g(s)\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)ds \right.$$
$$\left. - \int_t^\tau g(s)\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)ds - \int_0^t g(s)ds\right]. \quad (3.6)$$

Since $\int_0^\infty f'(x)dx = 0$, we get

$$f(t) = c_1\int_0^\infty\int_x^\infty (s-x)^{n-2}f^{(n)}(s)dsdx = c_2\int_0^\infty f^{(n)}(s)s^{n-1}ds = 0,$$

which gives that $\int_0^\infty g(s)ds = 0$. Therefore, for $f \in \mathring{W}^n_{p,v}(I)$ from (3.6) we get

$$f(t) = \frac{(-1)^n}{(n-1)!}\left[-\int_\tau^\infty g(s)\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)ds \right.$$
$$\left. - \int_t^\tau g(s)\left(1-\left(1-\frac{t}{s}\right)^{n-1}\right)ds - \int_0^t g(s)ds + \left(1-\left(1-\frac{t}{\tau}\right)^{n-1}\right)\int_0^\infty g(s)ds\right]$$
$$= \frac{(-1)^n}{(n-1)!}\left[\int_\tau^\infty g(s)\left(\left(1-\frac{t}{s}\right)^{n-1} - \left(1-\frac{t}{\tau}\right)^{n-1}\right)ds \right.$$
$$\left. - \int_t^\tau g(s)\left(\left(1-\frac{t}{\tau}\right)^{n-1} - \left(1-\frac{t}{s}\right)^{n-1}\right)ds - \left(1-\frac{t}{\tau}\right)^{n-1}\int_0^t g(s)ds\right] \quad (3.7)$$

for $0 < t < \tau$. Then, for $f \in \overset{\circ}{W}{}^n_{p,v}(I)$ from (3.4) and (3.7) we obtain

$$\frac{(n-1)!}{(-1)^n} f(t) = \chi_{(0,\tau)}(t) \left[\int_\tau^\infty g(s) \left(\left(1 - \frac{t}{s}\right)^{n-1} - \left(1 - \frac{t}{\tau}\right)^{n-1} \right) ds \right.$$
$$\left. - \int_t^\tau g(s) \left(\left(1 - \frac{t}{\tau}\right)^{n-1} - \left(1 - \frac{t}{s}\right)^{n-1} \right) ds - \left(1 - \frac{t}{\tau}\right)^{n-1} \int_0^t g(s) ds \right]$$
$$+ \chi_{(\tau,\infty)}(t) \int_t^\infty (s-t)^{n-1} \frac{g(s)}{s^{n-1}} ds. \quad (3.8)$$

Since

$$\int_0^\infty v(s)|f^{(n)}(s)|^p ds = \int_0^\infty \frac{v(s)}{s^{p(n-1)}} |f^{(n)}(s) s^{n-1}|^p ds = \int_0^\infty \overline{v}(s)|g(s)|^p,$$

the condition $f \in \overset{\circ}{W}{}^n_{p,v}(I)$ is equivalent to the condition $g \in \widetilde{L}_{p,\overline{v}}(I)$, where $\widetilde{L}_{p,\overline{v}}(I) = \big\{ g \in L_{p,\overline{v}}(I) : \int_0^\infty g(s) ds = 0 \big\}$. Taking into account that for $s > \tau$

$$\left(1 - \frac{t}{s}\right)^{n-1} - \left(1 - \frac{t}{\tau}\right)^{n-1} \le (n-1) \frac{(s-t)^{n-2}}{s^{n-2}} \left(\frac{t}{\tau} - \frac{t}{s} \right)$$
$$\le (n-1) 2^{n-3} \frac{\left[(s-\tau)^{n-2} + (\tau-t)^{n-2}\right] t(s-\tau)}{s^{n-1}\tau}$$
$$= (n-1) 2^{n-3} \left[\frac{(s-\tau)^{n-1} t}{s^{n-1}\tau} + \frac{(s-\tau)(\tau-t)^{n-2} t}{s^{n-1}\tau} \right]$$

and for $\tau > s$

$$\left(1 - \frac{t}{\tau}\right)^{n-1} - \left(1 - \frac{t}{s}\right)^{n-1} \le (n-1) \frac{(\tau-t)^{n-2}(\tau-s)t}{\tau^{n-1} s},$$

by (3.8) inequality (1.3) can be written in the form

$$\frac{1}{(n-1)!} \left[\left(\int_0^\tau u(t) \left| (n-1) 2^{n-3} \frac{t}{\tau} \int_\tau^\infty (s-\tau)^{n-1} \frac{g(s)}{s^{n-1}} ds \right. \right. \right.$$
$$+ (n-1) 2^{n-3} \frac{t}{\tau} (\tau - t)^{n-2} \int_\tau^\infty (s-\tau) \frac{g(s)}{s^{n-1}} ds$$
$$\left. - \frac{(\tau-t)^{n-1}}{\tau^{n-1}} \int_0^t g(s) ds - (n-1) \frac{t}{\tau^{n-1}} (\tau-t)^{n-2} \int_t^\tau (\tau - s) \frac{g(s)}{s} ds \right|^q dt$$
$$\left. \left. + \int_\tau^\infty u(t) \left| \int_t^\infty (s-t)^{n-1} \frac{g(s)}{s^{n-1}} ds \right|^q dt \right)^{\frac{1}{q}} \right] \le C \left(\int_0^\infty \overline{v}(s)|g(s)|^p ds \right)^{\frac{1}{p}}. \quad (3.9)$$

In the left-hand side of (3.9) applying the Minkowski inequality for sums, then the Hölder inequality, Theorem A, and Theorem B, we get

$$\left(\int_0^\infty u(t)|f(t)|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{(n-1)!}\left(p^{\frac{1}{q}}(p')^{\frac{1}{p'}}F_1(\tau)+(n-1)p^{\frac{1}{q}}(p')^{\frac{1}{p'}}F_2(\tau)\right)\left(\int_0^\tau v(s)|f^{(n)}(s)|^p ds\right)^{\frac{1}{p}}$$

$$+\frac{1}{(n-1)!}\left((n-1)2^{n-3}B_3(\tau)+(n-1)2^{n-3}B_4(\tau)\right.$$

$$\left.+8\,p^{\frac{1}{q}}(p')^{\frac{1}{p'}}\max\{B_1(\tau),B_2(\tau)\}\right)\left(\int_\tau^\infty v(s)|f^{(n)}(s)|^p ds\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{(n-1)!}\left(n\,p^{\frac{1}{q}}(p')^{\frac{1}{p'}}F(\tau)+\left((n-1)2^{n-2}+8p^{\frac{1}{q}}(p')^{\frac{1}{p'}}\right)B(\tau)\right)\left(\int_0^\infty v(s)|f^{(n)}(s)|^p ds\right)^{\frac{1}{p}}$$

$$\leq \varepsilon_r(n)\max\{B(\tau),F(\tau)\}\left(\int_0^\infty v(s)|f^{(n)}(s)|^p ds\right)^{\frac{1}{p}}. \quad (3.10)$$

Since the left-hand side of (3.10) is independent of $\tau\in I$, (3.10) implies the right estimate in (2.2).

Now, let us prove the right estimate in (2.3). Since

$$\lim_{\tau\to\infty}F_1(\tau)=\lim_{\tau\to\infty}\sup_{0<z<\tau}\left(\int_0^z t^q\left(1-\frac{t}{\tau}\right)^{q(n-2)}u(t)dt\right)^{\frac{1}{q}}$$

$$\times\left(\int_z^\tau\left(1-\frac{s}{\tau}\right)^{p'}s^{p'(n-2)}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}=\sup_{z>0}\left(\int_0^z t^q u(t)dt\right)^{\frac{1}{q}}\left(\int_z^\infty s^{p'(n-2)}v^{1-p'}(s)ds\right)^{\frac{1}{p'}},$$

we have that

$$B_4(\tau)=\left(\int_0^\tau t^q\left(1-\frac{t}{\tau}\right)^{q(n-2)}u(t)dt\right)^{\frac{1}{q}}\tau^{n-3}\left(\int_\tau^\infty(s-\tau)^{p'}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$

$$<\left(\int_0^\tau t^q u(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty s^{p'(n-2)}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}\leq\lim_{\tau\to\infty}F_1(\tau). \quad (3.11)$$

For $0 < N < \tau$ we obtain

$$B_3(\tau) < \left(\int_0^N \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} + \left(\int_N^\tau \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$
$$\le \left(\int_0^N \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} + \left(\int_N^\tau u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}.$$

Since

$$\lim_{\tau\to\infty} \left(\int_0^N \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} = 0,$$

then

$$B_3(\tau) \ll \left(\int_N^\tau u(t)dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$

for a sufficiently large $\tau > N$. If $\lim\limits_{\tau\to\infty} F_2(\tau) = \infty$, where

$$\lim_{\tau\to\infty} F_2(\tau) = \lim_{\tau\to\infty} \sup_{0<z<\tau} \left(\int_z^\tau \left(1-\frac{t}{\tau}\right)^{q(n-1)} u(t)dt\right)^{\frac{1}{q}} \left(\int_0^z s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} = \sup_{z>0} \left(\int_z^\infty u(t)dt\right)^{\frac{1}{q}} \left(\int_0^z s^{p'(n-1)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}},$$

then $\int\limits_z^\infty u(t)dt = \infty$ for any $z > 0$. Therefore, $B_3(\tau) < \lim\limits_{\tau\to\infty} F_2(\tau) = \infty$ for a sufficiently large $\tau > N$. If $\lim\limits_{\tau\to\infty} F_2(\tau) < \infty$, then $\int\limits_z^\infty u(t)dt < \infty$, which implies that $\lim\limits_{\tau\to\infty} B_3(\tau) = 0$, and we find that $B_3(\tau) < F(\tau)$ for a sufficiently large τ. It is also obvious that $B_i(\tau) < F(\tau)$, $i = 1, 2$. Combining these estimates with the obtained estimates $B_3(\tau) < F(\tau)$ and (3.11), we have that $B(\tau) \le F(\tau)$ in some neighborhood of infinity. Therefore, in relation (2.4) there exists $\tau_0 > 0$ such that $B(\tau_0) \le F(\tau_0)$. Consequently,

$$BF = \inf_{\tau\in I} \max\{B(\tau), F(\tau)\} \le F(\tau_0)$$

and the right estimate in (2.3) holds.

Necessity. Since $\overline{v}^{1-p'} \in L_1(I)$, then (1.6) holds. For $\tau \in I$ we consider two sets $\mathcal{L}_1 = \{g \in L_{p,\overline{v}}(0,\tau) : g \le 0\}$ and $\mathcal{L}_2 = \{g \in L_{p,\overline{v}}(\tau,\infty) : g \ge 0\}$. Repeating the proof of the necessary part of Theorem 1 in [8], for each $g_1 \in \mathcal{L}_1$ we construct a function $g_2 \in \mathcal{L}_2$ and for each $g_2 \in \mathcal{L}_2$ we construct a function $g_1 \in \mathcal{L}_1$ such that $g(t) = g_1(t)$ for $0 < t \le \tau$ and $g(t) = g_2(t)$ for $t > \tau$ belongs to the set $\widetilde{L}_{p,\overline{v}}(I)$. For the constructed function g we have (see [8, (26)])

$$\int_0^\infty \overline{v}(t)|g(t)|^p dt = (1 + k_\tau^{p-1}) \int_0^\tau \overline{v}(t)|g_1(t)|^p dt = (1 + k_\tau^{1-p}) \int_\tau^\infty \overline{v}(t)|g_2(t)|^p dt < \infty. \tag{3.12}$$

Taking into account (3.8) for the function $g \in \widetilde{L}_{p,\overline{v}}(I)$, we have

$$\frac{1}{(n-1)!}\left[\left(\int_0^\tau u(t)\left|\int_\tau^\infty g_2(s)\left(\left(1-\frac{t}{s}\right)^{n-1} - \left(1-\frac{t}{\tau}\right)^{n-1}\right)ds \right.\right.\right.$$
$$\left. + \int_t^\tau |g_1(s)|\left(\left(1-\frac{t}{\tau}\right)^{n-1} - \left(1-\frac{t}{s}\right)^{n-1}\right)ds + \left(1-\frac{t}{\tau}\right)^{n-1}\int_0^t |g_1(s)|ds\right|^q dt$$
$$\left.\left. + \int_\tau^\infty u(t)\left|\int_t^\infty (s-t)^{n-1}\frac{g_2(s)}{s^{n-1}}ds\right|^q dt\right)^{\frac{1}{q}}\right] \le C\left(\int_0^\infty \overline{v}(s)|g(s)|^p ds\right)^{\frac{1}{p}}. \tag{3.13}$$

In the left-hand side of (3.13), all terms are nonnegative. Using the estimate for $s > \tau$

$$\left(1-\frac{t}{s}\right)^{n-1} - \left(1-\frac{t}{\tau}\right)^{n-1} \ge \left(1-\frac{t}{s}\right)^{n-1} - \left(1-\frac{t}{s}\right)^{n-2}\left(1-\frac{t}{\tau}\right)$$
$$= \frac{(s-t)^{n-2}(s-\tau)t}{s^{n-1}\tau} \ge \max\left[\frac{(s-\tau)^{n-1}t}{s^{n-1}\tau}, \frac{(s-\tau)(\tau-t)^{n-2}t}{s^{n-1}\tau}\right],$$

assuming that the function $g \in \widetilde{L}_{p,\overline{v}}(I)$ is constructed by the function $g_2 \in \mathcal{L}_2$, from (3.12) and (3.13), we have

$$\frac{1}{(n-1)!}\left(\int_0^\tau u(t)\left(\frac{t}{\tau}\int_\tau^\infty (s-\tau)^{n-1}\frac{g_2(s)}{s^{n-1}}ds\right)^q dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{(n-1)!}\left(\int_0^\tau t^q u(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{\tau}\int_\tau^\infty (s-\tau)^{n-1}\frac{g_2(s)}{s^{n-1}}ds\right)$$
$$\le C(1+k_\tau^{1-p})^{\frac{1}{p}}\left(\int_\tau^\infty \overline{v}(t)|g_2(t)|^p dt\right)^{\frac{1}{p}},$$

$$\frac{1}{(n-1)!}\left(\int_0^\tau u(t)\left(\frac{t}{\tau}(\tau-t)^{n-2}\int_\tau^\infty (s-\tau)\frac{g_2(s)}{s^{n-1}}ds\right)^q dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{(n-1)!}\left(\int_0^\tau t^q(\tau-t)^{q(n-2)}u(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{\tau}\int_\tau^\infty (s-\tau)\frac{g_2(s)}{s^{n-1}}ds\right)$$
$$\le C(1+k_\tau^{1-p})^{\frac{1}{p}}\left(\int_\tau^\infty \overline{v}(t)|g_2(t)|^p dt\right)^{\frac{1}{p}},$$

$$\frac{1}{(n-1)!}\left(\int_\tau^\infty u(t)\left(\int_t^\infty (s-t)^{n-1}\frac{g_2(s)}{s^{n-1}}ds\right)^q dt\right)^{\frac{1}{q}} \le C(1+k_\tau^{1-p})^{\frac{1}{p}}\left(\int_\tau^\infty \overline{v}(t)|g_2(t)|^p dt\right)^{\frac{1}{p}}.$$

Due to the arbitrariness of $g_2 \in \mathcal{L}_2$, applying the reverse Hölder inequality to the first two inequalities and Theorem B to the last inequality, we obtain

$$\frac{1}{(n-1)!}\max\{B_1(\tau), B_2(\tau), B_3(\tau)B_4(\tau)\} = \frac{1}{(n-1)!}B(\tau) \le C(1+k_\tau^{1-p})^{\frac{1}{p}}. \tag{3.14}$$

Similarly, using the estimate for $\tau > s$

$$\left(1-\frac{t}{\tau}\right)^{n-1} - \left(1-\frac{t}{s}\right)^{n-1} \ge \frac{(\tau-t)^{n-2}(\tau-s)t}{\tau^{n-1}s},$$

for the function $g \in \widetilde{L}_{p,\overline{v}}(I)$ constructed by the function $g_1 \in \mathcal{L}_1$, from (3.12) and (3.13) we have

$$\frac{1}{(n-1)!}\left(\int_0^\tau u(t)\left(\frac{(\tau-t)^{n-1}}{\tau^{n-1}}\int_0^t |g_1(s)|ds\right)^q dt\right)^{\frac{1}{q}} \le C(1+k_\tau^{p-1})^{\frac{1}{p}}\left(\int_0^\tau \overline{v}(t)|g_1(t)|^p dt\right)^{\frac{1}{p}},$$

$$\frac{1}{(n-1)!}\left(\int_0^\tau u(t)\left(\frac{t}{\tau^{n-1}}(\tau-t)^{n-2}\int_t^\tau (\tau-s)\frac{|g_1(s)|}{s}ds\right)^q dt\right)^{\frac{1}{q}}$$

$$\le C(1+k_\tau^{p-1})^{\frac{1}{p}}\left(\int_0^\tau \overline{v}(t)|g_1(t)|^p dt\right)^{\frac{1}{p}}.$$

The latter, due to the arbitrariness of $g_1 \in \mathcal{L}_1$, by Theorem A, gives that

$$\frac{1}{(n-1)!}F(\tau) \le C(1+k_\tau^{p-1})^{\frac{1}{p}}. \tag{3.15}$$

From (3.14) and (3.15) we find that

$$\frac{1}{(n-1)!}BF \le C\inf_{\tau\in I}\left[\max\{(1+k_\tau^{p-1})(1+k_\tau^{1-p})\}\right]^{\frac{1}{p}} \le 4^{\frac{1}{p}}C,$$

which yields the left estimate in (2.2). From (3.15) we get the left estimate in (2.3). □

4 Proofs of Theorems 2.2 and 2.6

Theorems 2.3, 2.4, and 2.5 directly follow as corollaries from the combination of results presented in Section 2 and Theorem 2.1 proved above. Here we present the proofs of Theorems 2.2 and 2.6.

For clarity, let us write the squared values $F_1(\tau)$ and $F_2(\tau)$ for $p = q = 2$ in the form:

$$F_1^2(\tau) = \sup_{0<z<\tau}\int_0^z t^2\left(1-\frac{t}{\tau}\right)^{2(n-2)} u(t)dt\int_z^\tau \left(1-\frac{s}{\tau}\right)^2 s^{2(n-2)}v^{-1}(s)ds,$$

$$F_2^2(\tau) = \sup_{0<z<\tau}\int_z^\tau \left(1-\frac{t}{\tau}\right)^{2(n-1)} u(t)dt\int_0^z s^{2(n-1)}v^{-1}(s)ds,$$

$$F^2(\tau) = \max\{F_1^2(\tau), F_2^2(\tau)\}.$$

Proof of Theorem 2.2. (i) Suppose that equation (1.1) is strongly non-oscillatory at zero. Then by Lemma 2.2, we have that $\lim\limits_{T\to 0^+} C_T = 0$. From the left estimate in (2.3) we have

$$\frac{1}{(n-1)!}\sup_{0<\tau<T}(1+k_\tau)^{-1}F^2(\tau)\le C_T,$$

which gives that

$$\lim_{T\to 0^+}\sup_{0<\tau<T}(1+k_\tau)^{-1}F^2(\tau)=0.$$

Hence,

$$\lim_{\tau\to 0^+}(1+k_\tau)^{-1}F^2(\tau)=\lim_{\tau\to 0^+}F^2(\tau)=0,$$

i.e., $\lim\limits_{\tau\to 0^+}F_1^2(\tau)=\lim\limits_{\tau\to 0^+}F_2^2(\tau)=0$. Thus,

$$0=\lim_{\tau\to 0^+}F_1^2(\tau)\ge\lim_{\tau\to 0^+}\sup_{0<z<\frac{\tau}{2}}\int\limits_0^z t^2\left(1-\frac{t}{\tau}\right)^{2(n-2)}u(t)dt\int\limits_z^{\frac{\tau}{2}}\left(1-\frac{s}{\tau}\right)^2 s^{2(n-2)}\upsilon^{-1}(s)ds$$

$$\ge 4^{-(n-1)}\lim_{\tau\to 0^+}\sup_{0<z<\frac{\tau}{2}}\int\limits_0^z t^2u(t)dt\int\limits_z^{\frac{\tau}{2}}s^{2(n-2)}\upsilon^{-1}(s)ds,$$

i.e., (2.5) holds. Similarly, we prove that (2.6) also holds.

Inversely, let (2.5) and (2.6) hold. Since $1-\frac{t}{\tau}\le 1$ for $0<t<\tau$, we obtain

$$0=\lim_{\tau\to 0^+}\sup_{0<z<\tau}\int\limits_0^z t^2u(t)dt\int\limits_z^{\tau}s^{2(n-2)}\upsilon^{-1}(s)ds$$

$$\ge\lim_{\tau\to 0^+}\sup_{0<z<\tau}\int\limits_0^z t^2\left(1-\frac{t}{\tau}\right)^{2(n-2)}u(t)dt\int\limits_z^{\tau}\left(1-\frac{s}{\tau}\right)^2 s^{2(n-2)}\upsilon^{-1}(s)ds=\lim_{\tau\to 0^+}F_1^2(\tau).$$

Similarly, we find that $\lim\limits_{\tau\to 0^+}F_2^2(\tau)=0$, i.e., $\lim\limits_{\tau\to 0^+}F^2(\tau)=0$. From the right estimate in (2.3) we have

$$C_T\le\varepsilon_r(n)F^2(\tau_0),\ 0<\tau_0<T. \tag{4.1}$$

Therefore, we get

$$0=\varepsilon_r(n)\lim_{T\to 0^+}F^2(\tau_0)=\varepsilon_r(n)\lim_{\tau\to 0^+}F^2(\tau)\ge\lim_{T\to 0^+}C_T.$$

Thus, $\lim\limits_{T\to 0^+}C_T=0$ and, by Lemma 2.2, equation (1.1) is strongly non-oscillatory at zero.

(ii) Let equation (1.1) be strongly oscillatory at zero, then by Lemma 2.2, we have $C_T=\infty$ for any $T>0$. Consequently, from (4.1), we deduce $\lim\limits_{T\to 0^+}F(\tau_0)=\lim\limits_{\tau\to 0^+}F(\tau)=\infty$. This indicates that at least one of conditions (2.7) or (2.8) holds.

Inversely, let (2.7) hold. Then

$$\infty = \lim_{\frac{\tau}{2}\to 0^+} \sup_{0<z<\frac{\tau}{2}} \int_0^z t^2 u(t)dt \int_z^{\frac{\tau}{2}} s^{2(n-2)} v^{-1}(s)ds$$

$$= \lim_{\frac{\tau}{2}\to 0^+} \sup_{0<z<\frac{\tau}{2}} \int_0^z t^2 u(t) 4^{-(n-2)} dt \int_z^{\frac{\tau}{2}} 4^{-1} s^{2(n-2)} v^{-1}(s)ds$$

$$\leq \lim_{\frac{\tau}{2}\to 0^+} \sup_{0<z<\frac{\tau}{2}} \int_0^z t^2 \left(1-\frac{t}{\tau}\right)^{2(n-2)} u(t)dt \int_z^{\frac{\tau}{2}} \left(1-\frac{s}{\tau}\right)^2 s^{2(n-2)} v^{-1}(s)ds$$

$$= \lim_{\frac{\tau}{2}\to 0^+} F_1^2\left(\frac{\tau}{2}\right) = \lim_{\tau\to 0^+} F_1^2(\tau).$$

Thus, $\lim\limits_{\tau\to 0^+} F_1^2(\tau) = \infty$. Since $\frac{1}{(n-1)!} \sup\limits_{0<\tau<T} (1+k_\tau)^{-1} F_1^2(\tau) \leq C_T$ and

$$\frac{1}{(n-1)!} \lim_{T\to 0^+} \sup_{0<\tau<T} (1+k_\tau)^{-1} F_1^2(\tau) \geq \frac{1}{(n-1)!} \lim_{\tau\to 0^+} (1+k_\tau)^{-1} F_1^2(\tau) = \lim_{\tau\to 0^+} F_1^2(\tau),$$

from $\lim\limits_{\tau\to 0^+} F_1^2(\tau) = \infty$ we get that $C_T = \infty$ for any $T > 0$. Therefore, by Lemma 2.2, we conclude that equation (1.1) is strongly oscillatory at zero. Arguing similarly, we prove that if (2.8) holds, then equation (1.1) is strongly oscillatory at zero. □

To prove Theorem 2.6 we need the following lemma.

Lemma 4.1. *Let the assumptions of Theorem 2.2 hold. Then for* $t \in I$

$$\frac{1}{(n-1)!} \sup_{\tau\in I} D(t,\tau) \leq \sup_{f\in \mathring{W}_{2,v}^n} \frac{|f(t)|}{\|f^{(n)}\|_{2,v}} \leq \frac{\sqrt{2}}{(n-1)!} \inf_{\tau\in I} D(t,\tau), \tag{4.2}$$

where

$$D(t,\tau) = \Bigg\{ \chi_{(0,\tau)}(t)(n-1)^2 \int_\tau^\infty \left(\int_0^t (s-x)^{n-2} dx \right)^2 v^{-1}(s)ds$$
$$+ \chi_{(\tau,\infty)}(t) \int_t^\infty (s-t)^{2(n-1)} v^{-1}(s)ds + \chi_{(0,\tau)}(t)(n-1)^2 \int_t^\tau \left(\int_0^t (s-x)^{n-2} dx \right)^2 v^{-1}(s)ds$$
$$+ \chi_{(0,\tau)}(t)(n-1)^2 \int_0^t \left(\int_0^s (s-x)^{n-2} dx \right)^2 v^{-1}(s)ds \Bigg\}^{\frac{1}{2}}.$$

Proof of Lemma 4.1. From (3.4) and (3.5) for the function $f \in \overset{\circ}{W}{}^n_{2,v}$ we have

$$f(t) = \frac{(-1)^{n-1}}{(n-1)!}\Bigg\{(n-1)\chi_{(0,\tau)}\Bigg[\int_0^t f^{(n)}(s)\int_0^s (s-x)^{n-2}dxds + \int_t^\tau f^{(n)}(s)\int_0^t (s-x)^{n-2}dxds + \int_\tau^\infty f^{(n)}(s)\int_0^t (s-x)^{n-2}dxds\Bigg] - \chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)^{n-1}f^{(n)}(s)ds\Bigg\}. \quad (4.3)$$

Applying the Hölder inequality, we obtain

$$|f(t)| \le \frac{1}{(n-1)!}\Bigg\{\Bigg[(n-1)\chi_{(0,\tau)}(t)\left(\int_\tau^\infty\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac12} + \chi_{(\tau,\infty)}(t)\left(\int_t^\infty (s-t)^{2(n-1)}v^{-1}(s)ds\right)^{\frac12}\Bigg] \times \left(\int_\tau^\infty v(s)|f^{(n)}(s)|^2 ds\right)^{\frac12}$$

$$+ (n-1)\chi_{(0,\tau)}(t)\Bigg[\left(\int_t^\tau\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac12} + \left(\int_0^t\left(\int_0^s (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac12}\Bigg] \times \left(\int_0^\tau v(s)|f^{(n)}(s)|^2 ds\right)^{\frac12}\Bigg\}$$

$$\le \frac{1}{(n-1)!}\Bigg\{\Bigg[(n-1)\chi_{(0,\tau)}(t)\left(\int_\tau^\infty\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac12} + \chi_{(\tau,\infty)}(t)\left(\int_t^\infty (s-t)^{2(n-1)}v^{-1}(s)ds\right)^{\frac12}\Bigg]^2$$

$$+\chi_{(0,\tau)}(t)(n-1)^2\left[\left(\int_t^\tau\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac{1}{2}}\right.$$

$$\left.\left.+\left(\int_0^t\left(\int_0^s (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right)^{\frac{1}{2}}\right]^2\right\}^{\frac{1}{2}} \|f^{(n)}\|_{2,v}$$

$$\leq \frac{1}{(n-1)!}\left\{\chi_{(0,\tau)}(t)(n-1)^2\int_\tau^\infty\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right.$$

$$+\chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)^{2(n-1)}v^{-1}(s)ds+2\chi_{(0,\tau)}(t)(n-1)^2\int_t^\tau\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds$$

$$\left.+2\chi_{(0,\tau)}(t)(n-1)^2\int_0^t\left(\int_0^s (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right\}^{\frac{1}{2}} \|f^{(n)}\|_{2,v}$$

$$\leq \frac{\sqrt{2}}{(n-1)!}D(t,\tau)\|f^{(n)}\|_{2,v}.$$

Therefore, $|f(t)| \leq \frac{\sqrt{2}}{(n-1)!}\inf\limits_{\tau\in I} D(t,\tau)\|f^{(n)}\|_{2,v}$ and the right estimate in (4.2) holds.

Let us prove the left estimate in (4.2). We fix $t\in I$ in (4.3) and select a function $f^{(n)}$ depending on t as follows:

$$f_t^{(n)}(s)=\begin{cases}\chi_{(0,t)}(s)(n-1)\int\limits_0^s (s-x)^{n-2}dx\, v^{-1}(s) & \text{if } 0<t<\tau,\\ \chi_{(t,\tau)}(s)(n-1)\int\limits_0^t (s-x)^{n-2}dx\, v^{-1}(s) & \text{if } 0<t<\tau,\\ \chi_{(\tau,\infty)}(s)(n-1)\int\limits_0^t (s-x)^{n-2}dx\, v^{-1}(s) & \text{if } 0<t<\tau,\\ -\chi_{(t,\infty)}(s)(s-t)^{n-1}v^{-1}(s) & \text{if } t>\tau.\end{cases}$$

Placing this function in (4.3), we get

$$f_t(t)=\frac{(-1)^{n-1}}{(n-1)!}\left\{\chi_{(0,\tau)}(t)(n-1)^2\int_\tau^\infty\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right.$$

$$+(n-1)^2\int_t^\tau\left(\int_0^t (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds+(n-1)^2\int_0^t\left(\int_0^s (s-x)^{n-2}dx\right)^2 v^{-1}(s)ds$$

$$\left.+\chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)^{2(n-1)}v^{-1}(s)ds\right\}=\frac{(-1)^{n-1}}{(n-1)!}D^2(t,\tau). \quad (4.4)$$

Let us calculate $\|f_t^{(n)}\|_{2,v}$:

$$\left(\int_0^\infty v(s)|f_t^{(n)}(s)|^2ds\right)^{\frac{1}{2}} = \left(\int_0^\tau v(s)|f_t^{(n)}(s)|^2ds + \int_\tau^\infty v(s)|f_t^{(n)}(s)|^2ds\right)^{\frac{1}{2}}$$

$$= \left\{\chi_{(0,\tau)}(t)(n-1)^2\int_\tau^\infty\left(\int_0^t(s-x)^{n-2}dx\right)^2 v^{-1}(s)ds\right.$$

$$+\chi_{(0,\tau)}(t)(n-1)^2\int_t^\tau\left(\int_0^t(s-x)^{n-2}dx\right)^2 v^{-1}(s)ds$$

$$+\chi_{(0,\tau)}(t)(n-1)^2\int_0^t\left(\int_0^s(s-x)^{n-2}dx\right)^2 v^{-1}(s)ds$$

$$\left.+\chi_{(\tau,\infty)}(t)\int_t^\infty(s-t)^{2(n-1)}v^{-1}(s)ds\right\}^{\frac{1}{2}} = D(t,\tau). \quad (4.5)$$

From (4.4) and (4.5) we get

$$\sup_{f\in \mathring{W}^n_{2,v}}\frac{|f(t)|}{\|f^{(n)}\|_{2,v}} \geq \frac{|f_t(t)|}{\|f_t^{(n)}\|_{2,v}} = \frac{1}{(n-1)!}D(t,\tau)$$

for any $\tau\in I$. This relation proves the validity of the left estimate in (4.2). □

Together with Lemma 4.1 we need the following statement from work [1].

Lemma C. *Let $H=H(I)$ be a Hilbert function space, and $C[0,\infty)\bigcap H$ be dense in it. For any point $t\in I$, we define the operator $E_tf=f(t)$ on $C[0,\infty)\bigcap H$, which acts to the space of complex numbers. We assume that E_t is a closed operator. Then, the norm of this operator is equal to the value $\left(\sum_{k=1}^\infty|\varphi_k(t)|^2\right)^{\frac{1}{2}}$ (finite or infinite), where $\{\varphi_k(\cdot)\}_{k=1}^\infty$ is any complete orthonormal system of continuous functions in H.*

Proof of Theorem 2.6. By the condition, the operator L_F^{-1} is completely continuous on $L_{2,u}$. We assume that the space $\mathring{W}^n_{2,v}(I)$ with the norm $\|f^{(n)}\|_{2,v}$ is the space $H(I)$ of Lemma C. Since the system of functions $\{\lambda_k^{-\frac{1}{2}}\varphi_k\}_{k=1}^\infty$ is a complete orthonormal system in the space $\mathring{W}^n_{2,v}(I)$, then by Lemma C we have

$$\|E_t\|^2=\left(\sup_{f\in\mathring{W}^n_{2,v}(I)}\frac{|f(t)|}{\|f^{(n)}\|_{2,v}}\right)^2=\sum_{k=1}^\infty\frac{|\varphi_k(t)|^2}{\lambda_k},$$

where $E_tf=f(t)$. The latter and (4.2) give

$$\frac{1}{((n-1)!)^2}\sup_{\tau\in I}D^2(t,\tau)\leq\sum_{k=1}^\infty\frac{|\varphi_k(t)|^2}{\lambda_k}\leq\frac{2}{((n-1)!)^2}\inf_{\tau\in I}D^2(t,\tau). \quad (4.6)$$

Since

$$\inf_{\tau\in I}D^2(t,\tau)\leq\lim_{\tau\to\infty}D^2(t,\tau)=(n-1)^2D(t)\leq\sup_{\tau\in I}D^2(t,\tau),$$

from (4.6) we have (2.9). Multiplying both sides of (2.9) by u and integrating them from zero to infinity, we get (2.10). □

5 Remarks

As pointed out in Introduction, from [10] and [16] it follows that if

$$v^{-1} \in L_1(0,1), \quad v^{-1} \in L_1(1,\infty), \quad \text{and} \quad t^2 v^{-1} \notin L_1(1,\infty), \tag{5.1}$$

then for any $f \in W^n_{2,v}$ there exist the limits $\lim\limits_{t\to 0^+} f^{(i)}(t) \equiv f^{(i)}(0)$ for all $i = 0,1,...,n-1$, and $\lim\limits_{t\to\infty} f^{(n-1)}(t) \equiv f^{(n-1)}(\infty)$. In paper [7] there are investigated oscillatory properties of equation (1.1) and spectral properties of the operator L under conditions (5.1), which give that

$$\overset{\circ}{W}{}^n_{2,v}(I) = \{f \in W^n_{2,v}(I) : f^{(i)}(0) = 0, i = 0,1,...,n-1, f^{(n-1)}(\infty) = 0\}.$$

Item (i) of Theorem 4.2 in [7] can be equivalently rewritten in the form.

Theorem 5.1. *Let assumption (5.1) hold. Then the operator L has a spectrum discrete and bounded below if and only if*

$$\lim_{z\to\infty} \int_z^\infty t^{2(n-2)} u(t)dt \int_0^z s^2 v^{-1}(s)ds = 0, \tag{5.2}$$

$$\lim_{z\to\infty} \int_0^z t^{2(n-1)} u(t)dt \int_z^\infty v^{-1}(s)ds = 0. \tag{5.3}$$

Theorem 4.6 in [7] can be also rewritten in the following simpler form.

Theorem 5.2. *Let assumption (5.1) hold. Let (5.2) and (5.3) hold.*
(i)

$$\frac{1}{((n-2)!)^2}\mathcal{D}(t) \le \sum_{k=1}^\infty \frac{|\varphi_k(t)|^2}{\lambda_k} \le \frac{2}{((n-2)!)^2}\mathcal{D}(t), \tag{5.4}$$

where

$$\mathcal{D}(t) = \int_0^t \left(\int_0^s (t-x)^{n-2}dx\right)^2 v^{-1}(s)ds + \frac{1}{(n-1)^2} t^{2(n-1)} \int_t^\infty v^{-1}(s)ds.$$

(ii) The operator L_F^{-1} is nuclear if and only if $\int\limits_0^\infty u(t)\mathcal{D}(t)dt < \infty$ and for the nuclear norm $\|L_F^{-1}\|_{\sigma_1}$ of the operator L_F^{-1} the relation

$$\frac{1}{((n-2)!)^2}\int_0^\infty u(t)\mathcal{D}(t)dt \le \|L_F^{-1}\|_{\sigma_1} = \sum_{k=1}^\infty \frac{1}{\lambda_k} \le \frac{2}{((n-2)!)^2}\int_0^\infty u(t)\mathcal{D}(t)dt \tag{5.5}$$

holds.

This statement follows from the relation

$$\frac{1}{(n-1)!}\sup_{\tau\in I}\mathcal{D}(t,\tau) \le \sup_{f\in \overset{\circ}{W}{}^n_{2,v}} \frac{|f(t)|}{\|f^{(n)}\|_{2,v}} \le \frac{\sqrt{2}}{(n-1)!}\inf_{\tau\in I}\mathcal{D}(t,\tau), \tag{5.6}$$

where

$$\mathcal{D}(t,\tau) = \Bigg\{ \chi_{(0,\tau)}(t) \int_0^t (t-s)^{2(n-1)} v^{-1}(s)ds$$
$$+\chi_{(\tau,\infty)}(t)(n-1)^2 \int_0^\tau \left(\int_s^\tau (t-x)^{n-2}dx \right)^2 v^{-1}(s)ds$$
$$+\chi_{(\tau,\infty)}(t)(n-1)^2 \int_\tau^t \left(\int_\tau^s (t-x)^{n-2}dx \right)^2 v^{-1}(s)ds$$
$$+\chi_{(\tau,\infty)}(t)(t-\tau)^{2(n-1)} \int_t^\infty v^{-1}(s)ds \Bigg\}^{\frac{1}{2}},$$

found in [7, Lemma 4.5]. Arguing similarly as in the proof of Lemma 4.1 and taking into account that

$$\inf_{\tau\in I} \mathcal{D}^2(t,\tau) \le \lim_{\tau\to 0^+} \mathcal{D}^2(t,\tau) = (n-1)^2\mathcal{D}(t) \le \sup_{\tau\in I} \mathcal{D}^2(t,\tau),$$

from (5.6) we get (5.4). Multiplying both sides of (5.4) by u and integrating them from zero to infinity, we obtain (5.5).

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