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MIKHAIL L'VOVICH GOLDMAN

Doctor of physical and mathematical sciences, Professor Mikhail L'vovich Goldman passed away on July 5, 2025, at the age of 80 years.

Mikhail L'vovich was an internationally known expert in scienceand education. His fundamental scientific articles and text books in variousfields of the theory of functions of several variablesand functional analysis, the theory of approximation of functions, embedding theorems and harmonic analysis are a significant contribution to the development of mathematics.

Mikhail L'vovich was born on Aprill 13, 1945 in Moscow. In 1963, he graduated from School No. 128 in Moscow with a gold medal and entered the Physics Faculty of the Lomonosov Moscow State University. He graduated in 1969 and became a postgraduate student in the Mathematics Department. In 1972, he defended his PhD thesis "On integral representations and Fourier series of differentiable functions of several variables" under the supervision of Professor Ilyin Vladimir Aleksandrovich, and in 1988, his doctoral thesis "Study of spaces of differentiable functions of several variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor of Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an Assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he was a Professor of the Department of Theory of Functions and Differential Equations, then of the S.M. Nikol'skii Mathematical Institute at the Patrice Lumumba Peoples' Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman were: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators.Among the most important scientific achievements of M.L. Goldman, we note his research related to the optimal embedding of spaces with generalized smoothness, exact conditions for the convergence of spectral decompositions, descriptions of the integral and differential properties of generalized potentials of the Bessel and Riesz types, exact estimates for operators on cones, descriptions of optimal spaces for cones of functions with monotonicity properties.

M.L. Goldman has published more than 150 scientific articles in central mathematical journals. He is a laureate of the Moscow government competition, a laureate of the RUDN University Prize in Science and Innovation, and a laureate of the RUDN University Prize for supervision of postgraduate students. Under the supervision of Mikhail L'vovich 11 PhD theses were defended. His pupilss are actively involved in professional work at leading universities and research institutes in Russia, Kazakhstan, Ethiopia, Rwanda, Colombia, and Mongolia.

Mikhail L'vovich has repeatedly been a guest lecturer and guest professor at universities in Russia, Germany, Sweden, Great Britain, etc., and an invited speaker at many international conferences. Mikhail L'vovich was not only an excellent mathematician and teacher (he always spoke about mathematics and its teaching with great passion), but also a man of the highest culture and erudition, with a deep knowledge of history, literature and art, a very bright, kind and responsive person. This is how he will remain in the hearts of his family, friends, colleagues and students.

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Mikhail L'vovich Goldman.

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ON SOME GEOMETRIC ASPECTS OF EVOLUTION VARIATIONAL PROBLEMS

V.M. Filippov, V.M. Savchin

Communicated by V.I. Burenkov

Key words: Christoffel symbols, evolution equations, geodesics, dynamical systems.

AMS Mathematics Subject Classification: 34A55, 35A15, 58C40.

Abstract. The main objective of the work is to identify the relationship between evolution equations with potential operators and geometries of related configuration spaces of the given systems. Using the Hamilton principle, a wide class of such equations is derived. Their structural analysis is carried out, containing operator analogues of the Christoffel symbols of both the 1st and 2nd kind. It is shown that the study of the obtained evolution equations can be associated, in general, with an extended configuration space, the metric of which is determined by the kinetic energy of the given system.

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1 Introduction

One of the main properties of the metric tensor is that it completely defines the geometry of the space to which it belongs. The relationship of expressions for the metric tensor and the kinetic energy allows determining the components of the metric tensor of the configuration space of the system by the type of kinetic energy for the system and constructing its geometric model. The subject of the present paper lies between analytical mechanics, geometry and variational calculus. Tensor methods have long been applied in the dynamics of finite-dimensional systems [11]. They were initially aimed at using the ideas of Riemannian geometry in dynamics. In turn, the problems of mechanics contributed to the development of geometry. Significant results have been obtained over more than a hundred years (see, for example, [1, 4, 3, 7, 11, 12] and references therein). In particular, it was shown that the curvature of a manifold — an invariant distinguishing Riemannian metrics $a_{ij}(u^1, \ldots, u^n)$, $i, j = \overline{1, n}$, — significantly affects the form of geodesics on it, i.e. the motion in the corresponding dynamical system [2].

Geodesics are lines $u^i = u^i(t)$, $t \in [t_0, t_1]$, $i = \overline{1, n}$, which are solutions to the equations

$$\frac{d^2 u^j}{dt^2} + \Gamma^j_{ki} \frac{du^k}{dt} \frac{du^i}{dt} = 0, \ j = \overline{1, n},$$

where Γ^j_{ki} are the Christoffel symbols of the second kind.

Here and below, summation is implied by repeating indices of factors located at different levels.

If the metric $a_{ij}(u^1, \ldots, u^n)$ is non-degenerate (i.e. $\det(a_{ij}) \neq 0$), then

$$\Gamma^k_{ij} = \frac{1}{2} a^{kl} \left(\frac{\partial a_{lj}}{\partial u^i} + \frac{\partial a_{il}}{\partial u^j} - \frac{\partial a_{ij}}{\partial u^l} \right), \tag{1.1}$$

where $\left(a^{kl}\right)$ is the inverse matrix of the matrix (a_{lk}).

The Christoffel symbols of the first kind are found through the components of the metric tensor by the formulas

$$\Gamma_{k,ij} = \frac{1}{2}\left(\frac{\partial a_{kj}}{\partial u^i} + \frac{\partial a_{ik}}{\partial u^j} - \frac{\partial a_{ji}}{\partial u^k}\right). \tag{1.2}$$

As noted in work [6], in problems of mechanics it is natural to choose as a Riemannian metric the metric that is determined by the kinetic energy of the system.

2 Statement of the problem. Geodesic equations

Let $U = C^2\left([t_0, t_1], U_1\right)$, $V = C\left([t_0, t_1], V_1\right)$, where U_1, V_1 are normed linear spaces over the field of real numbers $\mathbb{R}$, $U_1 \subseteq V_1$.

Let the state of an infinite-dimensional dynamical system be determined by a function $u \in U$, satisfying the conditions $u|_{t=t_0} = u_0$, $u|_{t=t_1} = u_1$, where u_0, u_1 are given elements from U_1. A curve u in U_1 is a mapping $u : [t_0, t_1] \to U_1$.

We will follow the notation and terminology of [4, 5].

Let be given a symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle : V_1 \times V_1 \to \mathbb{R}$ and the kinetic energy of the system

$$T\left[t, u, u_t\right] = \frac{1}{2}\left\langle u_t, A_u u_t\right\rangle + \left\langle u_t, B\left(t, u\right)\right\rangle + \left\langle u, C\left(t, u\right)\right\rangle,$$

where A_u is a linear Gâteaux differential operator, in general, depending nonlinearly on t and u; $u_t = \frac{du}{dt} = \lim\limits_{\Delta t \to 0} \frac{u(t+\Delta t) - u(t)}{\Delta t} \in U_1$. Operators B, C are differentiable with respect to t, and u in the sense of Gâteaux.

$A'_u\left(h; g\right) = \left(\frac{d}{d\varepsilon} A_{u+\varepsilon g} h\right)\Big|_{\varepsilon=0}$; $F[u] = \int\limits_{t_0}^{t_1} T\left[t, u, u_t\right] dt$, $u \in D\left(F\right) = \{u \in U : u|_{t=t_0} = u_0, u|_{t=t_1} = u_1\}$; the Gâteaux differential $\delta F[u, h] = \frac{d}{d\varepsilon} F[u + \varepsilon h]\Big|_{\varepsilon=0}$. The construction of adjoint operators in the work is based on the Lagrange identity [8].

Definition 1. A function $u \in D\left(F\right)$ is called stationary for a functional F if $\delta F[u, h] = 0$ $\forall h \in D\left(F'_u\right)$.

Theorem 2.1. *The stationary function of the functional $F[u]$ is a solution to the operator equation*

$$\begin{aligned} N(u) \equiv \frac{1}{2}\left(A_u + A_u^*\right) u_{tt} + \frac{1}{2}\left[A'_u\left(u_t; u_t\right) + A_u^{*'}\left(u_t; u_t\right) - A_u^{'*}\left(u_t; u_t\right)\right] - \\ - \left(B_u^{'*} - B_u^{'}\right) u_t + \frac{1}{2}\left(\frac{\partial A_u}{\partial t} + \frac{\partial A_u^*}{\partial t}\right) u_t + \frac{\partial B}{\partial t} - C - C_u^{'*} u = 0, \end{aligned} \tag{2.1}$$

where $(\dots)^*$ is the operator adjoint to the operator $(\dots)$ with respect to the given bilinear form, $u_{tt} = \frac{d^2 u}{dt^2}$, $A_u^{'*}\left(u_t; u_t\right) = \left(A'_u\left(u_t; \cdot\right)\right)^* u_t$.

Proof. For further use, we note that if the Gâteaux derivative N'_u of N exists, then [9]

$$N(u + \varepsilon h) = N(u) + \varepsilon N'_u h + r(u, \varepsilon h), \ u \in D(N), \tag{2.2}$$

where

$$\lim_{\varepsilon \to 0} \frac{r(u, \varepsilon h)}{\varepsilon} = 0.$$

Let us denote

$$F_1[u] = \frac{1}{2}\int\limits_{t_0}^{t_1} \langle u_t, A_u u_t\rangle \, dt,$$

$$F_2[u] = \int\limits_{t_0}^{t_1} \left[\langle u_t, B(t,u)\rangle + \langle u, C(t,u)\rangle\right] dt, \;\; u \in D(F) = D(F_1) = D(F_2).$$

Using equality (2.2), we obtain

$$\begin{aligned} F_1[u+\varepsilon h] &= \frac{1}{2}\int\limits_{t_0}^{t_1} \langle u_t + \varepsilon h_t, A_{u+\varepsilon h}(u_t + \varepsilon h_t)\rangle \, dt \\ &= \frac{1}{2}\int\limits_{t_0}^{t_1} \langle u_t + \varepsilon h_t, A_{u+\varepsilon h} u_t + A_{u+\varepsilon h}\varepsilon h_t\rangle \, dt \\ &= \frac{1}{2}\int\limits_{t_0}^{t_1} \langle u_t + \varepsilon h_t, A_u u_t + A'_u(u_t;\varepsilon h) + A_u \varepsilon h_t + A'_u(\varepsilon h_t;\varepsilon h) + r(u,\varepsilon h)\rangle \, dt. \end{aligned}$$

From here we find

$$\begin{aligned} \delta F_1[u,h] &= \frac{1}{2}\int\limits_{t_0}^{t_1} \left[\langle h_t, A_u u_t\rangle + \langle u_t, A'_u(u_t;h) + A_u h_t\rangle\right] dt \\ &= \frac{1}{2}\int\limits_{t_0}^{t_1} \Big[D_t\langle h, A_u u_t\rangle - \langle h, D_t(A_u u_t)\rangle + \\ &+ \left\langle A'^{*}_u(u_t;\cdot)\, u_t, h\right\rangle + \langle A^*_u u_t, h_t\rangle\Big] \, dt \;\; \forall u \in D(F), \forall h \in D(F'_u), \end{aligned} \tag{2.3}$$

where D_t is a total derivative with respect to t.

Since

$$\begin{aligned} \langle A^*_u u_t, h_t\rangle &= D_t\langle A^*_u u_t, h\rangle - \langle D_t(A^*_u u_t), h\rangle \\ &= D_t\langle A^*_u u_t, h\rangle - \left\langle \frac{\partial A^*_u}{\partial t} u_t + A^{*'}_u(u_t;u_t) + A^*_u u_{tt}, h\right\rangle, \end{aligned}$$

then from (2.3) we get

$$\begin{aligned} \delta F_1[u,h] = \frac{1}{2}\langle (A_u + A^*_u)\, u_t, h\rangle \Big|_{t=t_0}^{t=t_1} + \frac{1}{2}\int\limits_{t_0}^{t_1} \Big[\Big\langle A'^{*}_u(u_t;\cdot)\, u_t - A^{*'}_u(u_t;u_t) - \\ - A'_u(u_t;u_t) - (A_u + A^*_u)\, u_{tt} - \left(\frac{\partial A_u}{\partial t} + \frac{\partial A^*_u}{\partial t}\right) u_t, h\Big\rangle\Big] \, dt. \end{aligned} \tag{2.4}$$

Taking into account that

$$h\big|_{t=t_0} = h\big|_{t=t_1} = 0,$$

from (2.4) we find

$$\delta F_1[u,h] = -\frac{1}{2}\int_{t_0}^{t_1}\left[\left\langle (A_u + A_u^*)\, u_{tt} + A_u'\,(u_t;u_t) + A_u^{*'}\,(u_t;u_t) - A_u'^{*}\,(u_t;u_t) + \right.\right.$$

$$\left.\left. + \left(\frac{\partial A_u}{\partial t} + \frac{\partial A_u^*}{\partial t}\right) u_t, h\right\rangle\right] dt \;\forall u \in D\,(F)\,, \forall h \in D\,(F_u')\,.$$

Using equality (2.2), in a similar way we get

$$F_2\,[u+\varepsilon h] = \int_{t_0}^{t_1} \left[\langle u_t + \varepsilon h_t, B\,(t, u+\varepsilon h)\rangle + \langle u + \varepsilon h, C\,(t, u+\varepsilon h)\rangle\right] dt,$$

$$\delta F_2\,[u,h] = \int_{t_0}^{t_1} \left[\langle h_t, B\,(t,u)\rangle + \langle u_t, B' h\rangle + \langle h, C\,(t,u)\rangle + \left\langle u, C_u' h\right\rangle\right] dt.$$

From here we obtain

$$\delta F_2\,[u,h] = \int_{t_0}^{t_1} \left[D_t\,\langle h, B\,(t,u)\rangle - \langle h, D_t B\,(t,u)\rangle + \left\langle h, B_u'^{*} u_t\right\rangle + \right.$$

$$\left. + \langle h, C\,(t,u)\rangle + \left\langle h, C_u'^{*}\right\rangle\right] dt = \tag{2.5}$$

$$= \langle h, B\,(t,u)\rangle\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left\langle h, \left(B_u'^{*} - B_u'\right) u_t - \frac{\partial B}{\partial t} + C\,(t,u) + C_u'^{*} u\right\rangle dt.$$

Since $h\big|_{t=t_0} = h\big|_{t=t_1} = 0$, from (2.5) we find

$$\delta F_2\,[u,h] = \int_{t_0}^{t_1} \left\langle h, \left(B_u'^{*} - B_u'\right) u_t - \frac{\partial B}{\partial t} + C\,(t,u) + C_u'^{*} u\right\rangle dt.$$

From the condition

$$\delta F[u,h] \equiv \delta F_1[u,h] + \delta F_2[u,h] = 0, u \in D\,(F)\,, \forall h \in D\,(F_u')$$

we obtain operator equation (2.1). □

Corollary 2.1. *If $A_u^* = A_u$, then equation (2.1) takes the form*

$$\begin{aligned} A_u u_{tt} + \frac{1}{2}\left[A_u'\,(u_t;u_t) + A_u^{*'}\,(u_t;u_t) - A_u'^{*}\,(u_t;u_t)\right] - \\ - \left(B_u'^{*} - B_u'\right) u_t + \frac{\partial A_u}{\partial t} u_t + \frac{\partial B}{\partial t} - C - C_u'^{*} = 0. \end{aligned} \tag{2.6}$$

Corollary 2.2. *If $A_u^* = A_u$, $B_u'^{*} = B_u'$, A_u and B are independent of t, $C = 0$ and there is an inverse operator A_u^{-1}, then equation (2.1) takes the form*

$$u_{tt} + \frac{1}{2} A_u^{-1}\left[A_u'\,(u_t;u_t) + A_u^{*'}\,(u_t;u_t) - A_u'^{*}\,(u_t;u_t)\right] = 0. \tag{2.7}$$

Consider a finite-dimensional system with coordinates $(u^1, \ldots, u^n)$, $u^i(t_0) = u_0^i$, $u^i(t_1) = u_1^i$, $t \in [t_0, t_1]$, $i = \overline{1,n}$, and the kinetic energy $T = \frac{1}{2}\dot{u}^i a_{ij}(u)\dot{u}^j$, where $(a_{ij})_{i,j=1}^n$ is a symmetric matrix, $\det(a_{ij})_{i,j=1}^n \neq 0$, $\dot{u}^i = \frac{du^i}{dt}$.

Theorem 2.2. *If $T = \frac{1}{2}\dot{u}^i a_{ij}(u)\dot{u}^j$, then equation (2.7) coincides with the geodesic equation*

$$\frac{d^2u^j}{dt^2} + \Gamma^j_{ik}\dot{u}^i\dot{u}^k = 0,\ \ j = \overline{1,n}, \tag{2.8}$$

where

$$\Gamma^j_{ik} = \frac{1}{2}a^{jl}\left(\frac{\partial a_{lk}}{\partial u^i} + \frac{\partial a_{il}}{\partial u^k} - \frac{\partial a_{ik}}{\partial u^l}\right)$$

are the Christoffel symbols.

Proof. In the case under consideration

$$\langle u_t, A_u u_t\rangle = \dot{u}^i a_{ij}(u)\dot{u}^j, F[u] = \frac{1}{2}\int_{t_0}^{t_1} \dot{u}^i a_{ij}(u)\dot{u}^j dt.$$

We have

$$F[u + \varepsilon h] = \frac{1}{2}\int_{t_0}^{t_1}\left(\dot{u}^i + \varepsilon\dot{h}^i\right)(a_{ij}(u + \varepsilon h))\left(\dot{u}^j + \varepsilon\dot{h}^j\right) dt.$$

From here we find

$$\delta F[u,h] = \frac{d}{d\varepsilon}F[u + \varepsilon h]\Big|_{\varepsilon=0} = \frac{1}{2}\int_{t_0}^{t_1}\left[\dot{h}^i\dot{u}^j a_{ij}(u) + \dot{u}^i\dot{h}^j a_{ij}(u) + \dot{u}^i\dot{u}^j\frac{\partial a_{ij}(u)}{\partial u^k}h^k\right] dt.$$

Integrating by parts, we obtain

$$\delta F[u,h] = \frac{1}{2}\left[h^i\dot{u}^j a_{ij} + h^j\dot{u}^i a_{ij}\right]\Big|_{t=t_0}^{t=t_1} +$$
$$+\frac{1}{2}\int_{t_0}^{t_1}\left[\dot{u}^i\dot{u}^j\frac{\partial a_{ij}}{\partial u^k}h^k - h^i\left(\ddot{u}^j a_{ij} + \dot{u}^j\frac{\partial a_{ij}}{\partial u^k}\dot{u}^k\right) - h^j\left(\ddot{u}^i a_{ij} + \dot{u}^i\frac{\partial a_{ij}}{\partial u^k}\dot{u}^k\right)\right] dt.$$

Since $h^i(t_0) = h^i(t_1) = 0$, $i = \overline{1,n}$, then changing the summation indices in the terms under the integral sign, we find

$$\delta F[u,h] = \frac{1}{2}\int_{t_0}^{t_1}\left[-h^k\ddot{u}^j(a_{kj} + a_{jk}) + h^k\dot{u}^i\dot{u}^j\left(\frac{\partial a_{ij}}{\partial u^k} - \frac{\partial a_{kj}}{\partial u^i} - \frac{\partial a_{ik}}{\partial u^j}\right)\right] dt.$$

Taking into account the symmetry of the matrix $(a_{ij})_{i,j=1}^n$, we arrive at the equality

$$\delta F[u,h] = -\int_{t_0}^{t_1} h^k\left[a_{kj}\ddot{u}^j + \frac{1}{2}\left(\frac{\partial a_{kj}}{\partial u^i} + \frac{\partial a_{ik}}{\partial u^j} - \frac{\partial a_{ij}}{\partial u^k}\right)\dot{u}^i\dot{u}^j\right] dt.$$

From the condition $\delta F[u,h] = 0$, $u \in D(F)$, $\forall h \in D(F'_u)$ we conclude that u is a solution to the system of equations

$$a_{kj}\ddot{u}^j + \Gamma_{k,ij}\dot{u}^i\dot{u}^j = 0, \tag{2.9}$$

where $\Gamma_{k,ij}$ are the Christoffel symbols of the first kind (1.2).

Since $\det\left(a_{ij}\right)_{i,j=1}^{n} \neq 0$, system of equations (2.9) can be solved with respect to $\ddot{u}^{j} (j = \overline{1,n})$. As a result, we arrive at system of equations (2.8).

Thus, equations of geodesics (2.8) are obtained. □

In the absence of forces, the motion of a system with kinetic energy $\frac{1}{2}\langle u_t, A_u u_t\rangle$ can be interpreted as motion in U by inertia with the metric

$$ds^2 = \langle u_t, A_u u_t\rangle \, dt^2.$$

Borrowing terminology from mechanics, for such a motion the trajectories are called geodesic lines with respect to indicated metric. Thus, the problem of inertial motion is reduced to finding geodesic lines. Operator equation (2.6) expresses a far-reaching generalization of this fact.

Corollary 2.3. [10] *Equation (2.7) is an operator analogue of geodesic equations (2.8), while the operator*

$$K_{1u}[\cdot] = \frac{1}{2}\left[A'_u(\cdot;\cdot) + A^{*'}_u(\cdot;\cdot) - A'^{*}_u(\cdot;\cdot)\right] \tag{2.10}$$

defines an analogue of the Christoffel symbols of the first kind $\Gamma_{k,ij}$, and

$$K_{2u}[\cdot] = A_u^{-1} K_{1u}[\cdot] \tag{2.11}$$

is an analogue of the Christoffel symbols of the second kind Γ^k_{ij}.

The operator $\frac{D}{dt}$, defined by the formula

$$\frac{Du_t}{dt} = u_{tt} + A_u^{-1} K_{1u}[u_t],$$

is an analogue of the covariant derivative of u_t with respect to t.

The above analogues are of particular interest in terms of their relationship with Riemannian geometry, as well as the geometry defined by the pseudo-Riemannian metric.

Using now operators (2.10), (2.11), we get the following.

Corollary 2.4. *If $A_u^* = A_u$ and there exists the inverse operator A_u^{-1}, then evolution equation (2.1) can be represented in the form*

$$N_1(u) \equiv u_{tt} + K_{2u}[u_t] + A_u^{-1}\left[\frac{\partial A_u}{\partial t}u_t - \left(B'^{*} - B'_u\right)u_t + \frac{\partial B}{\partial t} - C - C'^{*}_u u\right] = 0, \quad u \in D(N) = D(F). \tag{2.12}$$

It is an interesting problem to interpret this operator evolution equation in terms of rheonomic geometry with the metric

$$ds^2 = \frac{1}{2}\langle u_t, A_u u_t\rangle \, dt^2 + \langle u_t, B(t,u)\rangle \, dt^2 + \langle u, C(t,u)\rangle \, dt^2,$$

associated with the given kinetic energy $T[t, u, u_t]$.

3 Evolution equation and relative integral invariant

Let us establish the connection between evolution equation (2.1) and an relative integral invariant of the first order.

Let

$$u = u\left(\lambda; t\right), \ \ \lambda \in \Lambda = [0, 1] \tag{3.1}$$

be an arbitrary one-parameter set of elements from U continuously differentiable with respect to λ. It can be considered as a curve γ in U. We assume that $u\left(0; t\right) = u\left(1; t\right)$, i.e. γ is a closed curve.

Let us introduce the notation

$$\delta u = \frac{\partial u\left(\lambda; t\right)}{\partial \lambda} d\lambda.$$

Let us consider the functional

$$F\left[u\left(\lambda; t\right)\right] = \int\limits_{\tau_0}^{\tau_1} T\left[t, u\left(\lambda; t\right), u_t\left(\lambda; t\right)\right] dt,$$

where $[\tau_0, \tau_1]$ is an arbitrary segment from $[t_0, t_1]$.

We have

$$\begin{aligned} \delta F = \frac{\partial F\left[u\left(\lambda; t\right)\right]}{\partial \lambda} d\lambda = \int\limits_{\tau_0}^{\tau_1} \frac{\partial T}{\partial \lambda} d\lambda dt = \int\limits_{\tau_0}^{\tau_1} \delta T dt = \\ = \frac{1}{2} \int\limits_{\tau_0}^{\tau_1} \left[\left\langle \delta u_t, A_u u_t \right\rangle + \left\langle u_t, A'_u\left(u_t; \delta u\right) + A_u \delta u_t \right\rangle + \left\langle \delta u_t, B(t, u) \right\rangle + \right. \\ \left. + \left\langle u_t, B'_u \delta u \right\rangle + \left\langle \delta u, C\left(t, u\right) \right\rangle + \left\langle u, C'_u \delta u \right\rangle \right] dt. \end{aligned} \tag{3.2}$$

Since $\delta u_t = \frac{d}{dt} \delta u$, from (3.2) we get

$$\begin{aligned} \delta F = \int\limits_{\tau_0}^{\tau_1} \left\{ \frac{1}{2} \left[D_t \left\langle \delta u, A_u u_t \right\rangle - \left\langle \delta u, D_t\left(A_u u_t\right) \right\rangle + \left\langle A'^*_u\left(u_t; \cdot\right) u_t, \delta u \right\rangle + \right. \right. \\ \left. + \left\langle A^*_u u_t, \delta u_t \right\rangle \right] + D_t \left\langle \delta u, B(t, u) \right\rangle - \left\langle \delta u, D_t B(t, u) \right\rangle + \\ \left. + \left\langle B'^*_u u_t, \delta u \right\rangle + \left\langle \delta u, C(t, u) \right\rangle + \left\langle C'^*_u u, \delta u \right\rangle \right\} dt. \end{aligned} \tag{3.3}$$

Bearing in mind that

$$\begin{aligned} D_t\left(A_u u_t\right) &= \frac{\partial A_u}{\partial t} u_t + A'_u\left(u_t; u_t\right) + A_u u_{tt}, \\ \left\langle A^*_u u_t, \delta u_t \right\rangle &= D_t \left\langle A^*_u u_t, \delta u \right\rangle - \left\langle D_t\left(A^*_u u_t\right), \delta u \right\rangle \\ &= D_t \left\langle A^*_u u_t, \delta u \right\rangle - \left\langle \frac{\partial A^*_u}{\partial t} + A^{*'}_u\left(u_t; u_t\right) + A^*_u u_{tt}, \delta u \right\rangle, \\ D_t B(t, u) &= \frac{\partial B}{\partial t} + B'_u u_t, \end{aligned}$$

from (3.3) we obtain

$$\begin{aligned}\delta F &= \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B(t,u), \delta u \right\rangle \Big|_{t=\tau_0}^{t=\tau_1} - \\ &- \int_{\tau_0}^{\tau_1} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_{tt} + \frac{1}{2}\left[A_u'\left(u_t; u_t\right) + A_u^{*'}\left(u_t; u_t\right) - A_u'^*\left(u_t; \cdot\right) u_t\right] + \right. \\ &+ \left. \left(B_u' - B_u'^*\right) u_t + \frac{1}{2}\left(\frac{\partial A_u}{\partial t} + \frac{\partial A_u^*}{\partial t}\right) u_t + \frac{\partial B}{\partial t} - C - C_u'^* u, \delta u \right\rangle = \\ &= \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B(t,u), \delta u \right\rangle \Big|_{t=\tau_0}^{t=\tau_1} - \int_{\tau_0}^{\tau_1} \langle N(u), \delta u \rangle \, dt.\end{aligned} \tag{3.4}$$

Along the real trajectories, the solutions to evolution equation (2.1), we have

$$\delta F = \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B(t,u), \delta u \right\rangle \Big|_{t=\tau_0}^{t=\tau_1}.$$

Integrating this equality termwise with respect to λ from $\lambda = 0$ to $\lambda = 1$, we obtain

$$\begin{aligned} 0 = F\left[u(1;t)\right] - F\left[u(0;t)\right] &= \int_0^1 \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B(t,u), \delta u \right\rangle \Big|_{t=\tau_0}^{t=\tau_1} = \\ = \int_0^1 \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle \Big|_{t=\tau_1} &- \int_0^1 \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle \Big|_{t=\tau_0} = \\ = \oint_{\gamma_1} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle &- \oint_{\gamma_0} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle, \end{aligned}$$

i.e.

$$\oint_{\gamma_1} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle = \oint_{\gamma_0} \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle,$$

where γ_0, γ_1 are arbitrary closed curves, embracing the tube of trajectories.

Thus we proved the following.

Theorem 3.1. *Equation* (2.1) *has the relative integral invariant*

$$I = \oint \left\langle \frac{1}{2}\left(A_u + A_u^*\right) u_t + B, \delta u \right\rangle.$$

4 Example

Let us denote $U = C^2\left(\left[t_0, t_1\right], U_1\right)$, $V = C\left(\left[t_0, t_1\right], V_1\right)$. Let Ω be a bounded domain in R^3 with piecewise smooth boundary $\partial\Omega$, $U_1 = C^4\left(\overline{\Omega}\right)$, $V_1 = C\left(\overline{\Omega}\right)$, $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2}$ the Laplace operator, $x = (x^1, x^2, x^3)$. Let $A_u = \Delta^2 + \alpha u + \beta u^2$, $\Delta^2 = \Delta\Delta$, where $\alpha, \beta \in C^1\left[t_0, t_1\right]$. We will assume that the domain of definition $D(A_u)$ of the operator A_u consists of all those functions $u \in U$ that satisfy the conditions

$$u|_{t=t_0} = u_0, u|_{t=t_1} = u_1,$$
$$u|_\Gamma = \psi(t,x),$$
$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = \varphi(t,x),$$

where $\Gamma = [t_0, t_1] \times \partial\Omega$, $u_i \in C^4\left(\overline{\Omega}\right)$ $(i = 0, 1)$, $\psi, \varphi \in C(\Gamma)$.

Let us define the bilinear form

$$\langle v, g \rangle = \int_\Omega v(t,x) g(t,x) dx.$$

Let us define

$$T = \frac{1}{2} \langle u_t, A_u u_t \rangle,$$

which we will interpret as the kinetic energy of some system.

We will find the form of equation (2.1) for this case.

For this purpose we obtain

$$A_u v = \Delta^2 v + \alpha u v + \beta u^2 v,$$
$$A_{u+\varepsilon h} v = \Delta^2 v + \alpha v (u + \varepsilon h) + \beta v (u + \varepsilon h)^2,$$
$$A'_u(v; h) = \frac{d}{d\varepsilon} A_{u+\varepsilon h} v\Big|_{\varepsilon=0} = \alpha v h + 2\beta v u h = (\alpha v + 2\beta v u) h.$$

Let us find A^*_u.

We have

$$\int_{t_0}^{t_1}\int_\Omega h \cdot A_u g dx dt = \int_{t_0}^{t_1}\int_\Omega h\left(\Delta^2 g + \alpha u g + \beta u^2 g\right) dx dt =$$
$$= \int_{t_0}^{t_1}\int_\Omega g\left(\Delta^2 h + \alpha u h + \beta u^2 h\right) dx dt = \int_{t_0}^{t_1}\int_\Omega g \cdot A_u h dx dt \ \forall u \in D(A_u), \forall g, h \in D(A'_u).$$

Thus,

$$A^*_u = A_u \ \forall u \in D(A_u).$$

Next, we get

$$A'^*_u(v; \cdot)h = (\alpha v + 2\beta u v)h, \frac{\partial A_u}{\partial t} = \alpha_t u + \beta_t u^2.$$

According to formula (2.10), we find

$$K_{1u}[u_t] = \frac{1}{2}[\alpha u_t + 2\beta u u_t] u_t + \frac{1}{2}[\alpha u_t + 2\beta u u_t] u_t - \frac{1}{2}[\alpha u_t + 2\beta u u_t] u_t =$$
$$= \frac{1}{2}[\alpha + 2\beta u] u_t^2.$$

Thus, in the case under consideration, equation (2.1) takes the form

$$\left(\Delta^2 + \alpha u + \beta u^2\right) u_{tt} + \frac{1}{2}(\alpha + 2\beta u) u_t^2 + \alpha_t u + \beta_t u^2 = 0.$$

It has the following relative integral invariant

$$\oint \int_\Omega \left(\Delta^2 + \alpha u + \beta u^2\right) u_t \, dx \, \delta u.$$

5 Conclusion

In the work there is identified the relationship between evolution equations with potential operators and geometries of related configuration spaces of the given systems. Using the Hamilton principle, a wide class of such equations is derived. Their structural analysis is carried out, containing operator analogues of the Christoffel symbols of both the 1st and 2nd kind. It is shown that the study of the obtained evolution system can be associated, in general, with an extended configuration space, the metric of which is determined by the kinetic energy of the given system. It is shown that the obtained evolution operator equation has a relative integral invariant.

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