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ON SOLVABILITY OF A NONLINEAR PROBLEM IN THEORY OF INCOME DISTRIBUTION

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Abstract. We consider a nonlinear integro-differential equation with a Hammerstein type noncompact operator, arising in the theory of income distribution. We prove the existence of a positive solution of the nonlinear problem in Sobolev space $W_1^1(\mathbb{R}^+)$. We list some examples arising in applications. For one modeling problem a uniqueness theorem is proved. At the end of the paper the results of numerical calculations are given.

1 Introduction and statement of problem

Let consider the following equation

$$\frac{df}{dx} + \lambda(x)f(x) = \int_0^\infty K(x,t)G_0(f(t))dt, \quad x \in \mathbb{R}^+ \equiv [0, +\infty) \tag{1.1}$$

with the initial condition

$$f(0) = \eta_0 > 0 \tag{1.2}$$

for the unknown real valued measurable function f. Here λ- is a measurable function, defined on $\mathbb{R}^+$, moreover it is assumed that

$$\operatorname*{essinf}_{x\in\mathbb{R}^+} \lambda(x) \equiv \lambda_0 > 0, \quad \operatorname*{esssup}_{x\in\mathbb{R}^+} \lambda(x) \equiv \lambda_1 < +\infty. \tag{1.3}$$

The kernel $K(x,t) \geq 0$ is a measurable function on the set $\mathbb{R}^+ \times \mathbb{R}^+$. It is also assumed that there exist measurable functions μ and $\overset{\circ}{K}$ defined on sets $\mathbb{R}^+$ and $\mathbb{R}$ respectively such that

$$0 \leq \overset{\circ}{K} \in L_1(\mathbb{R}), \quad \int_{-\infty}^{+\infty} \overset{\circ}{K}(\tau)d\tau = \lambda_0, \quad \nu(\overset{\circ}{K}) \equiv \int_{-\infty}^{+\infty} x\,\overset{\circ}{K}(x)dx < -1, \tag{1.4}$$

$$m_j(\overset{\circ}{K}) \equiv \int\limits_{-\infty}^{+\infty} |x|^j \overset{\circ}{K}(x)dx < +\infty, \quad j = 1, 2. \tag{1.5}$$

$$0 \leq \mu(x) \leq 1, \quad x \in \mathbb{R}^+, \quad (1 - \mu(x))x^j \in L_1(\mathbb{R}^+), \quad j = 0, 1, \tag{1.6}$$

and

$$0 \leq K(x,t) \leq \mu(x) \overset{\circ}{K}(x-t), \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.7}$$

$$K(x,t) \not\equiv 0, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{1.8}$$

Moreover we assume that G_0 is a measurable function on $\mathbb{R}$, for which there exists a number $\eta \geq \eta_0$ such that

$$G_0 \in C[0,\eta), \quad G_0(\tau) \not\equiv 0, \ \tau \in [0,\eta), \ 0 \leq G_0(\tau) \leq \tau, \ \text{of } \tau \in [0,\eta), \tag{1.9}$$

We also assume that G_0 is non-increasing on $[0,\eta)$, briefly

$$G_0 \uparrow \text{ on } [0,\eta). \tag{1.10}$$

The problem (1.1)-(1.2) arises in a wide variety of applications, including particle systems, biology and mathematical finance. In particular the distribution of income in an economy can be defined by distribution function $f(x)$, where $f(x)dx$ is the number of separate income owners in the economy, whose wealth is between x and $x + dx$ (see [9]). K–is the redistribution function, conditioned by different economic causes, particularly by mean savings and capital gains, gifts (capital transfers) between different agencies, establishment of new organizations, disappearance of old enterprizes and assignment of their property (completely or partially) to other enterprizes as heritage. The function λ describes capital growth, mean savings as well as loss of wealth of enterprizes due to their bankruptcy. The basic problem is to find the quantity $\varkappa = \int\limits_0^\infty xf(x)dx$ which represents the mean income.

The problem (1.1)-(1.2) is also of great interest for its purely mathematical content. In particular when $G_0(x) \equiv x$ the problem (1.1)-(1.2) has been recently studied by the authors (see [6]). In the case, when $\lambda(x) = const$, $K(x,t) \equiv \mu(x) \overset{\circ}{K}(x-t)$, and function G_0 satisfies the conditions

$$G_0 \in C[0,\xi], \ G_0(x) \geq x, \ x \in [0,\xi]$$

$$G_0 \uparrow \text{ on } [0,\xi], \quad G_0(\xi) = \xi,$$

for some $\xi \geq \eta_0 > 0$, equation (1) was investigated in work [7].

A number of works (see for example [3-5,8]) are dedicated to investigation of nonlinear integral equations with compact operators. It should be noted that in this paper the Hammerstein type operator, standing on right hand of (1.1) is noncompact.

In the present work taking into consideration conditions (1.3)-(1.10) we prove the existence of a positive solution of nonlinear problem (1.1)-(1.2) in Sobolev space $W_1^1(\mathbb{R}^+)$ (the space of all functions summerable on $\mathbb{R}^+$ together with their first weak derivatives). We list some examples of the functions K and G_0 arising in applications. For one modeling problem a uniqueness theorem is proved. At the end of the work some numerical calculations are given.

2 Some auxiliary facts from linear theory of convolution type integral equations

Let E–be one of the following Banach spaces: $L_p(\mathbb{R}^+)$, $p \geq 1$, $M(\mathbb{R}^+)$, $C_M(\mathbb{R}^+)$, $C_0(\mathbb{R}^+)$, where $M(\mathbb{R}^+)$ is the space of all essentially bounded functions on $\mathbb{R}^+$, $C_M(\mathbb{R}^+) = M(\mathbb{R}^+) \cap C(\mathbb{R}^+)$ denotes the Banach space of all bounded, continuous functions on $\mathbb{R}^+$. Finally $C_0(\mathbb{R}^+)$ is the space of all continuous functions on $\mathbb{R}^+$, tending to zero limit at infinity.

We consider the following Wiener-Hopf operator

$$(\hat{T}f)(x) = \int_0^\infty T(x-t)f(t)dt, \quad f \in E, \tag{2.1}$$

with the kernel

$$T(x) = \int_0^\infty \overset{\circ}{K}(x-z)e^{-\lambda_1 z}dz + (\lambda_1 - \lambda_0)e^{-\lambda_1 x}\theta(x), \quad x \in \mathbb{R}, \tag{2.2}$$

where θ is the well known Heaviside function.

Taking into account (1.4) and Fubini's theorem by (2.2) we get

$$0 \leq T \in L_1(\mathbb{R}), \quad \int_{-\infty}^{+\infty} T(x)dx = 1, \quad \nu(T) = \int_{-\infty}^{+\infty} xT(x)dx < 0. \tag{2.3}$$

The following lemma holds.

Lemma 1. *Let conditions* (1.4), (1.5) *are fulfilled. Then* $m_j(T) < +\infty$, $j = 1, 2$.

Proof. Let δ_1, δ_2 be arbitrary real numbers and $m_j(\overset{\circ}{K}) < +\infty$, $\int_{-\infty}^{+\infty} \overset{\circ}{K}(\tau)d\tau = \lambda_0$. By (2.2) we have

$$J = \int_{\delta_1}^{\delta_2} |x|^j T(x)dx = \int_{\delta_1}^{\delta_2} |x|^j \int_0^\infty \overset{\circ}{K}(x-z)e^{-\lambda_1 z}dzdx + (\lambda_1 - \lambda_0)\int_{\delta_1}^{\delta_2} |x|^j e^{-\lambda_1 x}\theta(x)dx =$$

$$= \int_{\delta_1}^{\delta_2} |x|^j \int_0^\infty \overset{\circ}{K}(x-z)e^{-\lambda_1 z}dzdx + (\lambda_1 - \lambda_0)\int_{E_0} |x|^j e^{-\lambda_1 x}dx,$$

where $E_0 = (\delta_1, \delta_2) \cap \mathbb{R}^+$.

It is easy to check that

$$\int_{E_0} |x|^j e^{-\lambda_1 x}dx \leq \int_0^\infty x^j e^{-\lambda_1 x}dx < +\infty, \quad j = 1, 2.$$

Now we estimate (above) the following integral

$$J^* \equiv \int_{\delta_1}^{\delta_2} |x|^j \int_0^\infty \overset{\circ}{K}(x-z)e^{-\lambda_1 z}dzdx$$

Changing the order of integration we get

$$J^* = \int_0^\infty e^{-\lambda_1 z} \int_{\delta_1}^{\delta_2} |x|^j \overset{\circ}{K}(x-z)dxdz =$$

$$= \int_0^\infty e^{-\lambda_1 z} \int_{\delta_1 - z}^{\delta_2 + z} |t+z|^j \overset{\circ}{K}(t)dtdz \leq \int_0^\infty e^{-\lambda_1 z} \int_{-\infty}^{+\infty} |t+z|^j \overset{\circ}{K}(t)dtdz =$$

$$= \int_0^\infty e^{-\lambda_1 z} \left(\int_{-\infty}^{1-z} |t+z|^j \overset{\circ}{K}(t)dt + \int_{1-z}^{\infty} |t+z|^j \overset{\circ}{K}(t)dt \right) dz \leq \int_0^\infty e^{-\lambda_1 z} \int_{-\infty}^{1-z} |t+z|^j \overset{\circ}{K}(t)dtdz +$$

$$+ \int_0^\infty e^{-\lambda_1 z} \int_{1-z}^{\infty} (t+z)^2 \overset{\circ}{K}(t)dtdz \leq \int_0^\infty e^{-\lambda_1 z} \int_{-1-z}^{1-z} \overset{\circ}{K}(t)dtdz + 2\int_0^\infty e^{-\lambda_1 z} \int_{-\infty}^{+\infty} (t+z)^2 \overset{\circ}{K}(t)dtdz \leq$$

$$\leq \frac{\lambda_0}{\lambda_1} + 4\int_0^\infty e^{-\lambda_1 z} \int_{-\infty}^{+\infty} (t^2+z^2) \overset{\circ}{K}(t)dtdz = \frac{\lambda_0}{\lambda_1} + \frac{4}{\lambda_1} m_2(\overset{\circ}{K}) + \frac{8\lambda_0}{\lambda_1^3} < +\infty.$$

Since δ_1 and δ_2 are arbitrary numbers and

$$J \leq \int_0^\infty x^j e^{-\lambda_1 x}dx + \frac{\lambda_0}{\lambda_1} + \frac{4}{\lambda_1} m_2(\overset{\circ}{K}) + \frac{8\lambda_0}{\lambda_1^3},$$

it follows that $m_j(T) < +\infty,\ j = 1, 2.$ $\square$

From the results of work [1] it follows that the operator $I - \hat{T}$ admits the following factorization:

$$I - \hat{T} = (I - \hat{V}_-)(I - \hat{V}_+), \tag{2.4}$$

where I is the unite operator and $\hat{V}_\pm$ are the lower and upper Volterra type operators of the form

$$(\hat{V}_+ f)(x) = \int_0^x V_+(x-t)f(t)dt, \quad f \in E, \tag{2.5}$$

$$(\hat{V}_- f)(x) = \int_x^\infty V_-(t-x)f(t)dt, \quad f \in E, \tag{2.6}$$

where

$$0 \leq V_{\pm} \in L_1(\mathbb{R}^+), \quad \gamma_{\pm} = \int\limits_0^{\infty} V_{\pm}(\tau)d\tau, \tag{2.7}$$

$$\gamma_- = 1, \quad \gamma_+ < 1. \tag{2.8}$$

Let us consider the Wiener-Hopf homogeneous equation with the kernel T

$$S(x) = \int\limits_0^{\infty} T(x-t)S(t)dt, \quad x \in \mathbb{R}^+ \tag{2.9}$$

Using factorization (2.4) solving of equation (2.9) is reduced solving of the following couple of equations:

$$(I - \hat{V}_-)S^* = 0, \tag{2.10}$$

$$(I - \hat{V}_+)S = S^* \tag{2.11}$$

We rewrite the equation (2.10) in operator form

$$S^*(x) = \int\limits_x^{\infty} V_-(t-x)S^*(t)dt, \quad x \in \mathbb{R}^+. \tag{2.12}$$

Since $\gamma_- = 1$, the function $S^*(x) = const$ satisfies equation (2.10).

We choose $S^*(x) = \eta(1-\gamma_+) > 0$. Substituting this S^* in (2.11) we arrive at the following nonhomogeneous equation

$$S(x) = \eta(1-\gamma_+) + \int\limits_0^{x} V_+(x-t)S(t)dt, \quad x \in \mathbb{R}^+. \tag{2.13}$$

Taking into consideration the results of work [1] and since $\gamma_+ < 1$ we conclude that equation (2.13) has monotonically increasing, bounded solution S. Furthermore

$$\eta(1-\gamma_+) \leq S(x) \leq \eta, \quad S(x) \uparrow \eta, \text{ as } x \to +\infty. \tag{2.14}$$

Lemma 2. *Let* $m_j(\overset{\circ}{K}) < +\infty,\ j = 1,2$. *Then the solution* S *of equation* (2.13) *beside* (2.14) *possesses also the following property*

$$\eta - S \in L_1(\mathbb{R}^+). \tag{2.15}$$

Proof. We denote $\varphi(x) = \eta - S(x) \geq 0$. Then equation (2.13) with respect to the unknown function φ takes the form

$$\varphi(x) = \eta \int\limits_x^{\infty} V_+(\tau)d\tau + \int\limits_0^{x} V_+(x-t)\varphi(t)dt, \quad x \in \mathbb{R}^+. \tag{2.16}$$

It is easy to check that , if $\int\limits_0^{\infty} xV_+(x)dx < +\infty$, then $0 \leq g(x) \equiv \eta \int\limits_x^{\infty} V_+(\tau)d\tau \in L_1(\mathbb{R}^+)$.

Indeed, let $r>0$ an arbitrary number. Taking into consideration Fubini's theorem we get

$$I_r=\int\limits_0^r g(x)dx=\eta\int\limits_0^r\int\limits_x^\infty V_+(\tau)d\tau dx=\eta\int\limits_0^r\int\limits_x^r V_+(\tau)d\tau dx+\eta\int\limits_0^r\int\limits_r^\infty V_+(\tau)d\tau dx=$$

$$=\eta\left(\int\limits_0^r V_+(\tau)\tau d\tau+r\int\limits_r^\infty V_+(\tau)d\tau\right)\leq\eta\int\limits_0^\infty \tau V_+(\tau)d\tau<+\infty.$$

Since $r>0$ is arbitrary, it follows that $g\in L_1(\mathbb{R}^+)$. On the other hand, if $m_j(T)<+\infty,\ j=1,2$, then $\int\limits_0^\infty xV_+(x)dx<+\infty$ (see [1]).

Using Lemma 1 and the above mentioned facts we conclude that from $m_j(\overset{\circ}{K})<+\infty$, $j=1,2$, it follows that $\int\limits_0^\infty xV_+(x)dx<+\infty$. Since $\gamma_+<1$, the operator $\hat{V}_+$ will be contractive in each Banach spaces E, in particular in $L_1(\mathbb{R}^+)$. As $g\in L_1(\mathbb{R}^+)$ equation (2.16) has a unique positive solution in $L_1(\mathbb{R}^+)$. Thus $\varphi\in L_1(\mathbb{R}^+)$. □

Remark. *By Lemma 2 it follows that Wiener-Hopf homogeneous equation* (2.9) *possesses positive solution with properties* (2.14), (2.15).

3 Solvability of Basic Equation (1)

We denote by $W_1^1(\mathbb{R}^+)$ the space of all functions that belong to $L_1(\mathbb{R}^+)$ together with their first weak derivatives (Sobolev space).

The following theorem is true

Theorem 1. *We assume that conditions* (1.3)-(1.10) *are satisfied. Then problem* (1.1)-(1.2) *in Sobolev space $W_1^1(\mathbb{R}^+)$ has a positive solution.*

Proof. We consider the following auxiliary equation

$$B(x)=\mu(x)\int\limits_0^\infty T(x-t)B(t)dt,\quad x\in\mathbb{R}^+ \tag{3.1}$$

with respect to the unknown function B, where μ satisfies condition (1.6). It is known that equation (3.1) has a nontrivial, bounded and nonnegative solution B. Furthermore B may be represented in the form (see [2])

$$B(x)=S(x)-\psi(x),\quad x\in\mathbb{R}^+, \tag{3.2}$$

where S is the solution of equation (2.9) with properties (2.14), (2.15) and ψ satisfies the conditions $\psi(x)\not\equiv S(x)$, $(1-\mu(x))S(x)\leq\psi(x)\leq S(x)$.

The function ψ be determined from the following nonhomogeneous equation

$$\psi(x) = (1 - \mu(x))S(x) + \mu(x) \int_0^\infty T(x - t)\psi(t)dt \tag{3.3}$$

and belong to that space $L_1(\mathbb{R}^+) \cap M(\mathbb{R}^+)$

Taking into account Lemma 2 and $\psi \in L_1(\mathbb{R}^+) \cap M(\mathbb{R}^+)$ from (3.2) it follows

$$0 \leq \eta - B(x) \in L_1(\mathbb{R}^+). \tag{3.4}$$

Below we shall use this fact essentially.

We denote

$$F(x) = \frac{df}{dx} + \lambda_1 f(x), \quad x \in \mathbb{R}^+. \tag{3.5}$$

From (1.1), (1.2) we have

$$\begin{gathered} F(x) = (\lambda_1 - \lambda(x)) \left(\int_0^x e^{-\lambda_1(x-t)} F(t)dt + \eta_0 e^{-\lambda_1 x} \right) + \\ + \int_0^\infty K(x,t) G_0 \left(\int_0^t e^{-\lambda_1(t-\tau)} F(\tau)d\tau + \eta_0 e^{-\lambda_1 t} \right) dt, \quad x \in \mathbb{R}^+. \end{gathered} \tag{3.6}$$

We introduce in consideration the following iteration

$$\begin{gathered} F_{n+1}(x) = (\lambda_1 - \lambda(x)) \left(\int_0^x e^{-\lambda_1(x-t)} F_n(t)dt + \eta_0 e^{-\lambda_1 x} \right) + \\ + \int_0^\infty K(x,t) G_0 \left(\int_0^t e^{-\lambda_1(t-\tau)} F_n(\tau)d\tau + \eta_0 e^{-\lambda_1 t} \right) dt, \\ F_0(x) \equiv 0, \ n = 0, 1, 2, \ldots, \ x \in \mathbb{R}^+. \end{gathered} \tag{3.7}$$

First we prove that

$$a) \quad 0 \leq F_n(x) \leq \lambda_1(\eta - B(x)), \ n = 0, 1, 2, \ldots, \ x \in \mathbb{R}^+, \tag{3.8}$$

$$b) \quad F_n(x) \not\equiv 0, \ n = 1, 2, 3, \ldots, \ x \in \mathbb{R}^+, \tag{3.9}$$

In the case $n = 0$ inequality $a)$ is obvious. Let it be true for some $n \in \mathbb{N}$. We prove it for $n + 1$. Since

$$0 \leq F_n(x) \leq \lambda_1 \eta,$$

it follows that

$$0 \leq \int_0^x e^{-\lambda_1(x-t)} F_n(t)dt + \eta_0 e^{-\lambda_1 x} \leq \eta, \quad x \in \mathbb{R}^+. \tag{3.10}$$

Therefore from (3.7) we get

$$F_{n+1}(x) \geq \eta(\lambda_1-\lambda_0)e^{-\lambda_1 x}+\int\limits_0^\infty K(x,t)G_0(\eta e^{-\lambda_1 t})dt \geq 0, \tag{3.11}$$

$$F_{n+1}(x) \leq \lambda_1(\lambda_1-\lambda_0)\left(\int\limits_0^x e^{-\lambda_1(x-t)}(\eta-B(t))dt+\eta_0 e^{-\lambda_1 x}\right)+$$

$$+\int\limits_0^\infty K(x,t)G_0\left(\lambda_1\int\limits_0^t e^{-\lambda_1(t-\tau)}(\eta-B(\tau))d\tau+\eta_0 e^{-\lambda_1 t}\right)dt \leq$$

$$\leq (\lambda_1-\lambda_0)\left(\eta-\lambda_1\int\limits_0^x e^{-\lambda_1(x-t)}B(t)dt\right)+$$

$$+\mu(x)\int\limits_0^\infty \overset{\circ}{K}(x-t)\left(\eta-\lambda_1\int\limits_0^t e^{-\lambda_1(t-\tau)}B(\tau)d\tau\right)dt=(\lambda_1-\lambda_0)\eta+\lambda_0\eta-$$

$$-\lambda_1\left[(\lambda_1-\lambda_0)\int\limits_0^x e^{-\lambda_1(x-t)}B(t)dt+\mu(x)\int\limits_0^\infty \overset{\circ}{K}(x-t)\int\limits_0^t e^{-\lambda_1(t-\tau)}B(\tau)d\tau dt\right] \leq$$

$$\leq \eta\lambda_1-\lambda_1\left[(\lambda_1-\lambda_0)\mu(x)\int\limits_0^x e^{-\lambda_1(x-t)}B(t)dt+\mu(x)\int\limits_0^\infty \overset{\circ}{K}(x-t)\int\limits_0^t e^{-\lambda_1(t-\tau)}B(\tau)d\tau dt\right]=$$

$$=\eta\lambda_1-\lambda_1\mu(x)\int\limits_0^\infty T(x-\tau)B(\tau)d\tau=\lambda_1(\eta-B(x)).$$

Now we prove *b*). From (3.7) it immediately follows that $F_1 \not\equiv 0$, because of $G_0 \not\equiv 0$ when on $[0,\eta]$.

Assuming that $F_n(x) \neq 0$ for any $n \geq 2$, $n \in \mathbb{N}$, and taking into account the properties of the function G_0 on interval $[0,\eta)$, from (3.7) we have $F_{n+1} \neq 0$.

Below we prove that

$$F_n(x) \uparrow \text{ by } n. \tag{3.12}$$

Inequality $F_1(x) \geq F_0(x)$ immediately follows by statement *a*). Assuming that $F_n(x) \geq F_{n-1}(x)$ and taking into consideration both inequalities (3.10), from (3.7) we get

$$F_{n+1}(x) \geq (\lambda_1-\lambda_0)\left(\int\limits_0^x e^{-\lambda_1(x-t)}F_{n-1}(t)dt+\eta e^{-\lambda_1 x}\right)+$$

$$+\int\limits_0^\infty K(x,t)G_0\left(\int\limits_0^t e^{-\lambda_1(t-\tau)}F_{n-1}(\tau)d\tau+\eta e^{-\lambda_1 t}\right)dt=F_n(x).$$

Thus the sequence of functions $\{F_n(x)\}_0^\infty$ has pointwise limit

$$\lim_{n\to\infty} F_n(x) = F(x)$$

moreover in accordance with the B. Levi theorem the limit function satisfies equation (3.6) and the inequalities:

$$F(x) \neq 0, \quad 0 \leq F(x) \leq \lambda_1(\eta - B(x)), \quad x \in \mathbb{R}^+. \tag{3.13}$$

Since $\eta - B \in L_1(\mathbb{R}^+)$, we have $F \in L_1(\mathbb{R}^+)$. Solving equation (1.1) with condition (1.2) we obtain

$$f(x) = \eta e^{-\lambda_1 x} + \int_0^x e^{-\lambda_1(x-t)} F(t)dt, \quad x \in \mathbb{R}^+. \tag{3.14}$$

As $F \in L_1(\mathbb{R}^+)$ from last formula it follows that $f \in W_1^1(\mathbb{R}^+)$. □

Below we list some examples of the functions $K(x,t)$ and $G_0(x)$.

I) $K(x,t) = \mu(x) \overset{\circ}{K}(x-t)$,

II) $K(x,t) = \mu(x)(\overset{\circ}{K}(x-t) - \overset{\circ}{K}(x+t))$ with the additional conditions on $\overset{\circ}{K}$: $\overset{\circ}{K}(-x) > \overset{\circ}{K}(x)$, $x \in \mathbb{R}^+$, $\overset{\circ}{K} \downarrow$ in x on $\mathbb{R}$

III) $K(x,t) = R(x,t) \overset{\circ}{K}(x-t)$, where $0 \leq R(x,t) \leq \mu(x)$, $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$.

1) $G_0(x) = x^p$, $x \in \mathbb{R}^+$, $p \geq 1$, $p \in \mathbb{R}$, $\eta = 1$,

2) $G_0(x) = \sin x$, $x \in \mathbb{R}^+$, $\eta = \dfrac{\pi}{2}$

3) $G_0(x) = \dfrac{x}{x+1}$, $x \in \mathbb{R}^+$, $\forall \eta > 0$

4 Uniqueness theorem for one modeling problem

Let $G(x) = x^p$, $p \geq 2$, $p \in \mathbb{R}^+$, $x \in \mathbb{R}^+$. It is easy to see that as η we can take 1. Therefore by Theorem 1 it follows that if $G(x) = x^p$, $p \geq 2$, $x \in \mathbb{R}^+$, then problem (1.1)-(1.9) in $W_1^1(\mathbb{R}^+)$ has a positive solution, satisfying the inequality $f(x) \leq 1 - S(x) \leq \gamma_+$ (see formulae (2.14) and (3.14)).

Below we prove that if $0 \leq \gamma_+ < \frac{1}{2}$ then problem (1.1)-(1.2) has a unique solution in the following class of functions

$$\mathfrak{M} = \{f(x) : f(x) \in W_1^1(\mathbb{R}^+),\ 0 \leq f(x) \leq \gamma_+\}.$$

Theorem 2. *Let* $0 < \gamma_+ < \dfrac{1}{2}$, $p \geq 2$, $p \in \mathbb{R}^+$, *and let the functions* λ *and* K *possess properties* (1.3)-(1.5). *Then the problem*

$$\frac{dy}{dx} + \lambda(x)y = \int_0^\infty K(x,t)y^p(t)dt, \quad x \in \mathbb{R}^+ \tag{4.1}$$

$$y(0) = 1. \tag{4.2}$$

in the class of functions

$$\mathfrak{M} = \{f(x) : f \in W_1^1(\mathbb{R}^+),\ 0 \leq f(x) \leq \gamma_+\} \tag{4.3}$$

has a unique solution.

Proof. Let problem (1.1), (1.2) has two different solutions $y_1, y_2 \in \mathfrak{M}$. Then the function $\Delta y = y_1 - y_2$ satisfies the following relation

$$\frac{d\Delta y}{dx} + \lambda(x)\Delta y = \int\limits_0^\infty K(x,t)(y_1^p(t) - y_2^p(t))dt,\ x \in \mathbb{R}^+, \tag{4.4}$$

$$\Delta y(0) = 0. \tag{4.5}$$

We denote

$$F(x) = \frac{d\Delta y}{dx} + \lambda_1 \Delta y \tag{4.6}$$

Taking into account (4.5) from (4.6) we have

$$\Delta y(x) = \int\limits_0^x e^{-\lambda_1(x-t)} F(t)dt, \quad x \in \mathbb{R}^+. \tag{4.7}$$

Using (4.7) from (4.4) we get

$$F(x) = (\lambda_1 - \lambda(x)) \int\limits_0^x e^{-\lambda_1(x-t)} F(t)dt + \int\limits_0^\infty K(x,t)(y_1^p(t) - y_2^p(t))dt$$

By the Lagrange formulae and due to (1.3), (1.7) we get

$$|F(x)| \leq (\lambda_1 - \lambda_0) \int\limits_0^x e^{-\lambda_1(x-t)} |F(t)|dt + \int\limits_0^\infty K(x,t)|y_1^p(t) - y_2^p(t)|dt \leq$$

$$\leq (\lambda_1 - \lambda_0) \int\limits_0^x e^{-\lambda_1(x-t)} |F(t)|dt + \int\limits_0^\infty K(x,t)p|\theta|^{p-1}|y_1(t) - y_2(t)|dt \leq$$

$$\leq (\lambda_1 - \lambda_0) \int\limits_0^x e^{-\lambda_1(x-t)} |F(t)|dt + p\int\limits_0^\infty K(x,t)\gamma_+^{p-1}|\Delta y(t)|dt =$$

$$\leq (\lambda_1 - \lambda_0) \int\limits_0^x e^{-\lambda_1(x-t)} |F(t)|dt + p\int\limits_0^\infty K(x,t)\gamma_+^{p-1}|\int\limits_0^t e^{-\lambda_1(t-\tau)} F(\tau)d\tau|dt \leq$$

$$\leq (\lambda_1 - \lambda_0) \int\limits_0^x e^{-\lambda_1(x-t)} |F(t)|dt + p\gamma_+^{p-1}\int\limits_0^\infty K(x,t)\int\limits_0^t e^{-\lambda_1(t-\tau)} |F(\tau)|d\tau dt \leq$$

$$\leq (\lambda_1-\lambda_0)\int\limits_0^x e^{-\lambda_1(x-t)}|F(t)|dt + p\gamma_+^{p-1}\operatorname*{esssup}_{0\leq\tau\leq+\infty}|F(\tau)|\int\limits_0^\infty K(x,t)\int\limits_0^t e^{-\lambda_1(t-\tau)}d\tau dt \leq$$

$$\leq \frac{\lambda_1-\lambda_0}{\lambda_1}\operatorname*{esssup}_{0\leq\tau\leq+\infty}|F(\tau)| + \frac{p\gamma_+^{p-1}}{\lambda_1}\operatorname*{esssup}_{0\leq\tau\leq+\infty}|F(\tau)|\int\limits_0^\infty K(x,t)dt \leq$$

$$\leq \left(\frac{\lambda_1-\lambda_0}{\lambda_1} + \frac{\lambda_0}{\lambda_1}p\gamma_+^{p-1}\right)\operatorname*{esssup}_{0\leq\tau\leq+\infty}|F(\tau)|. \tag{4.8}$$

Below we prove, that if $0<\gamma_+<\frac{1}{2}$, then for arbitrary $p\geq 2,\ p\in\mathbb{R}^+$ the following inequality holds

$$\rho\equiv p\gamma_+^{p-1}<1. \tag{4.9}$$

We consider the function

$$\psi(p)=1-p\gamma_+^{p-1},\ p\in[2,+\infty).$$

We have

$$\psi(2)=1-2\gamma_+>0$$

and

$$\psi'(p)=p\gamma_+^{p-1}\ln\frac{1}{\gamma_+}-\gamma_+^{p-1}\geq\gamma_+^{p-1}(2\ln\frac{1}{\gamma_+}-1)\geq\gamma_+^{p-1}(2\ln 2-1)=\gamma_+^{p-1}\ln\frac{4}{e}>0$$

Therefore $\psi\uparrow$ on $[2,+\infty)$, from which it follows that

$$\psi(p)\geq\psi(2)=1-2\gamma_+$$

or

$$\rho\equiv p\gamma_+^{p-1}\leq 2\gamma_+<1.$$

From estimate (4.9) and taking into consideration (4.8) we obtain $\frac{\lambda_0}{\lambda_1}(1-\rho)\operatorname*{esssup}_{x\in\mathbb{R}^+}|F(x)|\leq 0$, i.e. $F(x)=0$ almost everywhere on $\mathbb{R}^+$. Solving simple Cauchy problem

$$\begin{cases}\frac{d\Delta y}{dx}+\lambda_1\Delta y=0,\\ \Delta y(0)=0\end{cases}$$

we that get $\Delta y=0$ almost everywhere in $\mathbb{R}^+$. The theorem is proved. $\square$

5 Numerical results

For demonstration of the developed method as an example $K(x)$ we take the function

$$\overset{\circ}{K}(x)=\begin{cases}\beta e^{-\alpha x}, & \text{if } x>0\\ (\alpha-\beta)e^{\alpha x}, & \text{if } x<0.\end{cases} \tag{5.1}$$

Then $T(x)$ in terms of $K(x)$ (see (2.2)) will be

$$T(x) = \begin{cases} \frac{\alpha^2-\alpha+2\beta}{\alpha^2+1} e^{-x} + \frac{\beta}{\alpha-1} e^{-\alpha x}, & \text{if } x > 0 \\ \frac{\alpha-\beta}{\alpha+1} e^{\alpha x}, & \text{if } x < 0. \end{cases} \tag{5.2}$$

It is easy to check that all conditions (1.4) and (2.3) are satisfied, if $\alpha + \frac{2\beta}{\alpha} < 1$ $(\alpha > \beta)$.

If we rewrite operator equality (2.4) in terms of its kernel we arrive at Yengibaryan's nonlinear factorization equations (see [1]).

$$V_\pm(x) = T(\pm x) + \int_0^\infty V_\mp(t) V_\pm(x+t) dt. \tag{5.3}$$

Simple computations show that

$$V_+(x) = \alpha e^{-x} + \frac{2\beta}{\alpha - 1}(e^{-x} - e^{-\alpha x}), \quad V_-(x) = \alpha e^{-\alpha x}. \tag{5.4}$$

It is obvious that

$$\gamma_- = \int_0^\infty V_-(x) dx = 1, \ \gamma_+ = \int_0^\infty V_+(x) dx = \alpha + \frac{2\beta}{\alpha} < 1.$$

We consider the following iteration (see (3.7)) in the case when $\lambda = \lambda_1 = \lambda_0 = 1$, $\eta_0 = \eta = 1$, $G_0(x) = x^p$, $\mu(x) \equiv 1$:

$$F_{n+1}(x) = \beta \int_0^x e^{-\alpha(x-t)} \left[\int_0^t e^{-(t-\tau)} F_n(\tau) d\tau + e^{-t} \right]^p dt +$$

$$+ (\alpha - \beta) \int_x^\infty e^{-\alpha(t-x)} \left[\int_0^t e^{-(t-\tau)} F_n(\tau) d\tau + e^{-t} \right]^p dt$$

$$F_0(x) \equiv 0, \ n = 0, 1, 2$$

The iteration process is stopped when $|F_{n+1}(x) - F_n(x)| < 10^{-6}$ for all x.

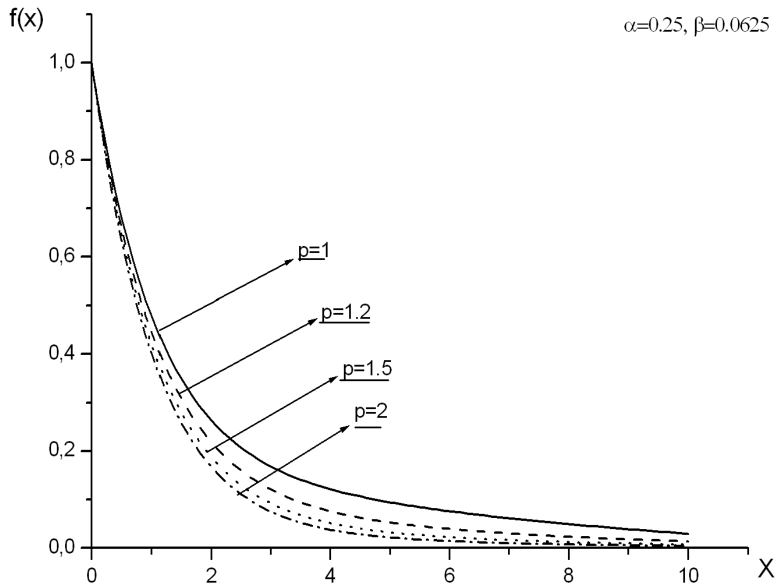

Figure. The graphs of function f(x) for different values p.

α/β	$p=1$	$p=1.2$	$p=1.5$	$p=2$
$\alpha=\frac{1}{3}, \beta=\frac{1}{12}$	$\varkappa=5.708$	$\varkappa=3.297$	$\varkappa=2.196$	$\varkappa=1.682$
$\alpha=\frac{1}{4}, \beta=\frac{1}{16}$	$\varkappa=4.677$	$\varkappa=2.926$	$\varkappa=2.084$	$\varkappa=1.656$
$\alpha=\frac{1}{6}, \beta=\frac{1}{24}$	$\varkappa=3.518$	$\varkappa=2.472$	$\varkappa=1.915$	$\varkappa=1.592$

Table. The table shows the values of the mean income $\varkappa=\int_0^\infty xf(x)dx$ for different powers of nonlinearity.

The graphs of the function

$$f(x)=e^{-x}\left[1+\int_0^x e^t F(t)dt\right]$$

are shown in the Figure. As we have already proved the solution of the linear problem is the largest of all the solutions of the nonlinear problem. Simple example show that nonlinearity may lead to large relative errors in calculation of the mean income value (see the Table).

It is hoped that the result stated in Theorem 1 will stimulate further work toward developing numerical methods at calculation of mean income for different economical models.

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