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ON INTUITIONISTIC FUZZY H-IDEALS IN H-HEMIREGULAR HEMIRINGS AND h^*-DUO-HEMIRINGS

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Abstract. In this paper, intuitionistic fuzzy h-bi-ideals and intuitionistic fuzzy h-quasi-ideals of hemirings are studied and some related properties are investigated. We characterize fully idempotent hemirings by the properties of their intuitionistic fuzzy h-ideals.We also characterize h-hemiregularity, h-intra-hemiregularity and h-quasi-hemiregularity in hemirings by the properties of their intuitionistic fuzzy h-bi-ideals and intuitionistic fuzzy h-quasi-ideals. The concept of intuitionistic fuzzy h^*-duo-hemirings is introduced and some of their characterizations are obtained.

1 Introduction

A semiring is a well-known universal algebra. This is a generalization of an associative ring $(R, +, \cdot)$. If $(R, +)$ is a semigroup instead of being a group then (R, +,$\cdot$) reduces to a semiring. Semiring has been found very useful for solving problem in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modeling and studying key factors in these applied areas. Ideals of semirings play a central role in the structure theory and are useful for many purposes. However they do not in general coincide with the usual ring ideals and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. In order to overcome this drawback Henriken [10] defined a more restricted class of ideals, which are called k-ideals. A still more restricted class of ideals in hemirings was introduced by Iizuka [12], which are called h-ideals. La Torre [14], investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of ring theorems for hemiring and to fill the gap between ring ideals and semiring ideals.

In 1965, Zadeh introduced the concept of a fuzzy set. Since then fuzzy sets have been applied to many branches of mathematics. The study of fuzzy algebraic structures has been started by Rosenfeld [17]. In [2], Ahsan et al. initiated the study of fuzzy semirings (see also [11, 3]). Fuzzy k-ideals in semirings are studied in [8], and fuzzy h-ideals are studied in a number of papers, in particular in [23, 16, 13]. Fuzzy algebraic structures play an important role in mathematics with wide applications in theoretical physics, computer science, control engineering, information science, coding theory and topological spaces [7, 20].

The concept of an intuitionistic fuzzy set was introduced by Atanassov [4], as a generalization of the notion of a fuzzy set. Fuzzy sets provide a degree of membership of an element in a given set while intuitionistic fuzzy sets provide both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership of an element is a real number between 0 and 1, this is also the case for the degree of non-membership, and the sum of these two degrees is not greater than 1.

In this paper, intuitionistic fuzzy h-bi-ideals and intuitionistic fuzzy h-quasi-ideals of a hemiring are studied and some related properties are investigated. We characterize fully idempotent hemirings by the properties of their intuitionistic fuzzy h-ideals.We also characterize h-hemiregularity, h-intra-hemiregularity and h-quasi-hemiregularity in hemirings by the properties of their intuitionistic fuzzy h-bi-ideals and intuitionistic fuzzy h-quasi-ideals. The concept of intuitionistic fuzzy h^*-duo-hemirings is introduced and some of their characterization are obtained.

2 Preliminaries

A set $R \neq \emptyset$ together with two binary operation addition "+" and multiplication "$\cdot$" is called a semiring and is denoted by $(R, +, \cdot)$ if $(R, +)$ and $(R, \cdot)$ are semigroups and multiplication is distributive from both sides over addition, that is for all $a, b, c \in R$,

$a\,(b + c) = ab + ac$ and$(a + b)\,c = ac + bc$.

An element $0 \in R$, satisfying the conditions $0 \cdot a = a \cdot 0 = 0$ and $0 + a = a + 0 = a$ for all $a \in R$, is called the zero of the semiring $(R, +, \cdot)$. An element $1 \in R$, satisfying the condition $1a = a1 = a$ for all $a \in R$, is called the identity of the semiring $(R, +, \cdot)$. A semiring with commutative multiplication is called commutative semiring. A semiring with commutative addition and the zero element is called a hemiring. A non-empty subset A of hemiring R is called a subhemiring of R if it contains zero and is closed with respect to the addition and multiplication of R. A non-empty subset I of hemiring R is called a left (right) ideal of R if I is closed under addition and $RI \subseteq I$ $(IR \subseteq I)$. Furthermore, I is called an ideal of R if it is both a left ideal and a right ideal of R. A non-empty subset Q of hemiring R is called a quasi-ideal of R if Q is closed under addition and $QR \cap RQ \subseteq Q$. A subhemiring B of a hemiring R is called bi-ideal of R if $BRB \subseteq B$. Every one sided ideal of a hemiring R is a quasi-ideal and every quasi-ideal is a bi-ideal but the converse is not true.

A left (right) ideal I of a hemiring R is called a left (right) h-ideal if for all $x, z \in R$ and for any $a, b \in I$, from $x + a + z = b + z$ it follows that $x \in I$. A bi-ideal B of a hemiring R is called an h-bi-ideal of R, if for all $x, z \in R$ and $a, b \in B$ from $x + a + z = b + z$ it follows that $x \in B$ [21].

The h-closure $\overline{A}$ of a non-empty subset A of a hemiring R is defined as

$\overline{A} = \{x \in R : x + a_1 + z = a_2 + z$ for some $a_1, a_2 \in A, z \in R\}$.

A quasi-ideal Q of a hemiring R is called an h-quasi-ideal of R if $\overline{QR} \cap \overline{RQ} \subseteq Q$ and $x + a + z = b + z$ implies $x \in Q$, for all $x, z \in R$ and $a, b \in Q$ [21]. Every left (right) h-ideal of a hemiring R is an h-quasi-ideal of R and every h-quasi-ideal is an h-bi-ideal of R. However, the converse in general is not true.

Lemma 2.1. [23] *For a hemiring R,*
(i) $A \subseteq \overline{A}$ *for all* $A \subseteq R$,
(ii) *if* $A \subseteq B \subseteq R$ *then* $\overline{A} \subseteq \overline{B}$,
(iii) $\overline{\overline{A}} = \overline{A}$ *for all* $A \subseteq R$,
(iv) $\overline{A} = A$ *for all left (right) h-ideal, h-bi-ideal or h-quasi-ideal* A *of* R.

A fuzzy set defined on a nonempty set X is an object having the form $A = \{x, \mu_A(x) \,|x \in X\}$ where $\mu_A : X \longrightarrow [0,1]$. $\mu_A(x)$ is called the degree of membership of $x \in X$.

Definition 1. [4] An intuitionistic fuzzy set A defined on a nonempty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ where $\mu_A : X \longrightarrow [0,1]$, $\gamma_A : X \longrightarrow [0,1]$ and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$. $\mu_A(x)$ is called the degree of membership of $x \in X$ and $\gamma_A(x)$ is called the degree of nonmembership. For the sake of brevity we shall write $A = (\mu_A, \gamma_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)), \mid x \in X\}$. $0_\sim$ and $1_\sim$ will denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set. They are defined by $\mu_{0_\sim}(x) = 0$ and $\gamma_{0_\sim}(x) = 1$ for all $x \in X$; $\mu_{1_\sim}(x) = 1$ and $\gamma_{1_\sim}(x) = 0$ for all $x \in X$. Briefly $0_\sim = (0,1)$, and $1_\sim = (1,0)$.

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy sets in X. Then by definition

(1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$.

(2) $A^c = \{(x, \gamma_A(x), \mu_A(x)), \mid x \in X\}$.

(3) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\gamma_A(x), \gamma_B(x)\}) \mid x \in X\}$.

(4) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\gamma_A(x), \gamma_B(x)\}) \mid x \in X\}$.

Let A be a non empty subset of a hemiring R, the intuitionistic characteristic function $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ is defined as

$$\mu_{\chi_A}(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A, \end{cases} \quad \text{and} \quad \gamma_{\chi_A}(x) = \begin{cases} 0 & \text{if} \quad x \in A, \\ 1 & \text{if} \quad x \notin A. \end{cases}$$

Definition 2. [6] Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy sets of a hemiring R. Then h-intrinsic product of A and B is defined by $A \odot_h B = (\mu_{A \odot_h B}, \gamma_{A \odot_h B})$ where

$$\mu_{A \odot_h B}(x) = \sup_{x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \left[\min\{\mu_A(a_i), \mu_A(a'_j), \mu_B(b_i), \mu_B(b'_j)\}\right],$$

$$\gamma_{A \odot_h B}(x) = \inf_{x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \left[\max\{\gamma_A(a_i), \gamma_A(a'_j), \gamma_B(b_i), \gamma_B(b'_j)\}\right],$$

and $\mu_{A \odot_h B}(x) = 0, \gamma_{A \odot_h B}(x) = 1$ if there do not exist $m, n, a_1, \ldots a_m, b_1, \ldots b_m, a'_1, \ldots a'_n, b'_1 \ldots b'_n$ and z such that $x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$.

Proposition 2.1. *Let R be a hemiring and A, B, C, D be intuitionistic fuzzy sets of R. If $A \subseteq B$ and $C \subseteq D$ then $A \odot_h C \subseteq B \odot_h D$.*

Proof. The proof is straightforward. □

Lemma 2.2. *Let R be a hemiring and A, B are subsets of R. Then we have*

$$(1)\ \ A \subseteq B \text{ if and only if } \chi_A \subseteq \chi_B,$$

$$(2)\ \ \chi_A \cap \chi_B = \chi_{A\cap B},$$

$$(3)\ \ \chi_A \odot_h \chi_B = \chi_{\overline{AB}}.$$

Proof. The proofs is straightforward. □

Definition 3. [6] An intuitionistic fuzzy subset $A = (\mu_{A,}\gamma_A)$ of a hemiring R is called an intuitionistic fuzzy left h-ideal of R if and only if for all $x, y, z, a, b \in R$

(i) $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(x+y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,
(ii) $\mu_A(xy) \geq \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(y)$,
(iii) If $x + a + z = b + z$, then $\mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\}$ and
$\gamma_A(x) \leq \max\{\gamma_A(a), \gamma_A(b)\}$.

Intuitionistic fuzzy right h-ideals are defined similarly. An intuitionistic fuzzy subset of a hemiring R is called an intuitionistic fuzzy h-ideal if it is both an intuitionistic fuzzy left and right h-ideal of R.

Definition 4. [6] An intuitionistic fuzzy subset $A = (\mu_A, \gamma_A)$ of a hemiring R is called an intuitionistic fuzzy h-bi-ideal if for all $x, y, z, a, b \in R$

(i) $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$ *and* $\gamma_A(x+y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

(ii) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ *and* $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

(iii) $\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\}$ *and* $\gamma_A(xyz) \leq \max\{\gamma_A(x), \gamma_A(z)\}$,

$$(iv)\ \ \begin{aligned} \text{If } x + a + z &= b + z, \text{ then } \mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\} \\ \text{and } \gamma_A(x) &\leq \max\{\gamma_A(a), \gamma_A(b)\}. \end{aligned}$$

Definition 5. An intuitionistic fuzzy subset $A = (\mu_A, \gamma_A)$ of a hemiring R is called an intuitionistic fuzzy h-quasi-ideal if for all $x, y, z, a, b \in R$

(i) $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$ *and* $\gamma_A(x+y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

(ii) $\mu_{A\odot_h\chi_R} \wedge \mu_{\chi_R\odot_h A} \leq \mu_A$ *and* $\gamma_{A\odot_h\chi_R} \vee \gamma_{\chi_R\odot_h A} \geq \gamma_A$,

$$(iii)\ \ \begin{aligned} \text{If } x + a + z &= b + z, \text{ then } \mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\}, \\ \text{and } \gamma_A(x) &\leq \max\{\gamma_A(a), \gamma_A(b)\}. \end{aligned}$$

Lemma 2.3. *An intutionistic fuzzy subset $A = (\mu_A, \gamma_A)$ of a hemiring R is an intutionistic fuzzy left h-ideal of R if and only if for all $x, y, z, a, b \in R$*

$$\begin{aligned} (1)\ \mu_A(x+y) &\geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \gamma_A(x+y) \leq \max\{\gamma_A(x), \gamma_A(y)\}, \\ (2)\quad \mu_{\chi_R \odot_h A} &\leq \mu_A \text{ and } \gamma_{\chi_R \odot_h A} \geq \gamma_A, \end{aligned}$$

$$\begin{aligned} (3)\ \text{If } x+a+z &= b+z, \text{ then } \mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\}, \\ \text{and } \gamma_A(x) &\leq \max\{\gamma_A(a), \gamma_A(b)\}. \end{aligned}$$

Definition 6. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy sets of a hemiring R. Then the h-sum of A and B is defined by

$$A +_h B = (\mu_{A+_hB},\ \gamma_{A+_hB}),$$

where

$$\begin{aligned} \mu_{A+_hB}(x) &= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\min\{\mu_A(a_1), \mu_A(a_2), \mu_B(b_1), \mu_B(b_2)\}], \\ \gamma_{A+_hB}(x) &= \inf_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\max\{\gamma_A(a_1), \gamma_A(a_2), \gamma_B(b_1), \gamma_B(b_2)\}] \end{aligned}$$

for all $x, a_1, a_2, b_1, b_2, z \in R$.

Theorem 2.1. *The h-sum of intuitionistic fuzzy h-ideals of a hemiring R is an intuitionistic fuzzy h-ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy h-ideals of R. Then for $x, y \in R$ we have

$$\begin{aligned} &\mu_{A+_hB}(x) \wedge \mu_{A+_hB}(y) \\ &\quad = \min\left(\begin{array}{c} \sup\limits_{x+(a_1+b_1)+z=(a_2+b_2)+z} \begin{bmatrix} \min\{\mu_A(a_1), \mu_A(a_2), \\ \mu_B(b_1), \mu_B(b_2)\} \end{bmatrix}, \\ \sup\limits_{y+(a_1'+b_1')+z'=(a_2'+b_2')+z'} \begin{bmatrix} \min\{\mu_A(a_1'), \mu_A(a_2'), \\ \mu_B(b_1'), \mu_B(b_2')\} \end{bmatrix} \end{array} \right) \\ &= \sup_{\substack{x+(a_1+b_1)+z=(a_2+b_2)+z \\ y+(a_1'+b_1')+z'=(a_2'+b_2')+z'}} \begin{bmatrix} \min\{\mu_A(a_1), \mu_A(a_2), \mu_B(b_1), \mu_B(b_2), \\ \mu_A(a_1'), \mu_A(a_2'), \mu_B(b_1'), \mu_B(b_2')\} \end{bmatrix} \\ &\leq \sup_{\substack{x+(a_1+b_1)+z=(a_2+b_2)+z \\ y+(a_1'+b_1')+z'=(a_2'+b_2')+z'}} \begin{bmatrix} \min\{\mu_A(a_1+a_1'), \mu_A(a_2+a_2'), \\ \mu_B(b_1+b_1'), \mu_B(b_2+b_2')\} \end{bmatrix} \\ &\leq \sup_{x+y+(c_1+d_1)+z''=(c_2+d_2)+z''} [\min\{\mu_A(c_1), \mu_A(c_2), \mu_B(d_1), \mu_B(d_2)] \\ &= \mu_{A+_hB}(x+y). \end{aligned}$$

Thus $\mu_{A+_hB}(x+y) \geq \mu_{A+_hB}(x) \wedge \mu_{A+_hB}(y)$.

Moreover,

$$\gamma_{A+_hB}(x) \vee \gamma_{A+_hB}(y)$$

$$= \max\left(\begin{array}{c} \inf\limits_{x+(a_1+b_1)+z=(a_2+b_2)+z} \left[\begin{array}{c} \max\{\gamma_A(a_1), \gamma_A(a_2), \\ \gamma_B(b_1), \gamma_B(b_2)\} \end{array} \right], \\ \inf\limits_{y+\left(a_1'+b_1'\right)+z'=\left(a_2'+b_2'\right)+z'} \left[\begin{array}{c} \max\{\gamma_A(a_1'), \gamma_A(a_2'), \\ \gamma_B(b_1'), \gamma_B(b_2')\} \end{array} \right] \end{array} \right)$$

$$= \inf_{\substack{x+(a_1+b_1)+z=(a_2+b_2)+z \\ y+(a_1'+b_1')+z'=(a_2'+b_2')+z'}} \left[\begin{array}{c} \max\{\gamma_A(a_1), \gamma_A(a_2), \gamma_B(b_1), \gamma_B(b_2), \\ \gamma_A(a_1'), \gamma_A(a_2'), \gamma_B(b_1'), \gamma_B(b_2')\} \end{array} \right]$$

$$\begin{aligned} &\geq \inf_{\substack{x+(a_1+b_1)+z=(a_2+b_2)+z \\ y+(a_1'+b_1')+z'=(a_2'+b_2')+z'}} \left[\begin{array}{c} \max\{\gamma_A(a_1+a_1'), \gamma_A(a_2+a_2'), \\ \gamma_B(b_1+b_1'), \gamma_B(b_2+b_2')\} \end{array} \right] \\ &\geq \inf_{x+y+(c_1+d_1)+z''=(c_2+d_2)+z''} [\max\{\gamma_A(c_1), \gamma_A(c_2), \gamma_B(d_1), \gamma_B(d_2)\}] \\ &= \gamma_{A+_hB}(x+y). \end{aligned}$$

Thus $\gamma_{A+_hB}(x+y) \leq \gamma_{A+_hB}(x) \vee \gamma_{A+_hB}(y)$.

Next,

$$\begin{aligned} \mu_{A+_hB}(x) &= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\min\{\mu_A(a_1), \mu_A(a_2), \mu_B(b_1), \mu_B(b_2)\}] \\ &\leq \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\min\{\mu_A(ra_1), \mu_A(ra_2), \mu_B(rb_1), \mu_B(rb_2)\}] \\ &\leq \sup_{rx+\left(a_1'+b_1'\right)+z'=\left(a_2'+b_2'\right)+z'} [\min\{\mu_A(a_1'), \mu_A(a_2'), \mu_B(b_1'), \mu_B(b_2')\}] \\ &= \mu_{A+_hB}(rx) \end{aligned}$$

Thus $\mu_{A+_hB}(rx) \geq \mu_{A+_hB}(x)$. Similarly $\mu_{A+_hB}(xr) \geq \mu_{A+_hB}(x)$.

Next

$$\begin{aligned} \gamma_{A+_hB}(x) &= \inf_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\max\{\gamma_A(a_1), \gamma_A(a_2), \gamma_B(b_1), \gamma_B(b_2)\}] \\ &\geq \inf_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\max\{\gamma_A(ra_1), \gamma_A(ra_2), \gamma_B(rb_1), \gamma_B(rb_2)\}] \\ &\geq \inf_{rx+\left(a_1'+b_1'\right)+z'=\left(a_2'+b_2'\right)+z'} [\max\{\gamma_A(a_1'), \gamma_A(a_2'), \gamma_B(b_1'), \gamma_B(b_2')\}] \\ &= \gamma_{A+_hB}(rx). \end{aligned}$$

Thus $\gamma_{A+_hB}(rx) \leq \gamma_{A+_hB}(x)$. Similarly $\gamma_{A+_hB}(xr) \leq \gamma_{A+_hB}(x)$.

Now we show that if $x+a+z=b+z$ then $\mu_{A+_hB}(x) \geq \mu_{A+_hB}(a) \wedge \mu_{A+_hB}(b)$ and $\gamma_{A+_hB}(x) \leq \gamma_{A+_hB}(a) \vee \gamma_{A+_hB}(b)$. To this end let $a+(a_1+b_1)+z_1=(a_2+b_2)+z_1$ and $b+(c_1+d_1)+z_2=(c_2+d_2)+z_2$. Then,

$$a+(c_2+d_2+z_2)+(a_1+b_1+z_1)=(a_2+b_2+z_1)+(b+c_1+d_1+z_2)$$

$$a+(a_1+c_2)+(b_1+d_2)+(z_1+z_2)=b+(a_2+c_1)+(b_2+d_1)+(z_1+z_2)$$

$$a+(a_1+c_2)+(b_1+d_2)+(z_1+z_2+z)=b+z+(a_2+c_1)+(b_2+d_1)+(z_1+z_2)$$

$$x+(a_2+c_1)+(b_2+d_1)+(z_1+z_2+z+a)=(a_1+c_2)+(b_1+d_2)+(z_1+z_2+z+a).$$

i.e $x+(a'+b')+z'=(a''+b'')+z'$ for some $a',b',a'',b'' \in R$.
Therefore,

$$\begin{aligned}
&\mu_{A+_hB}(a) \wedge \mu_{A+_hB}(b) \\
&= \min\left(\begin{array}{c} \sup\limits_{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1} \left[\begin{array}{c} \min\{\mu_A(a_1), \mu_A(a_2), \\ \mu_B(b_1), \mu_B(b_2)\} \end{array} \right], \\ \sup\limits_{b+(c_1+d_1)+z_2=(c_2+d_2)+z_2} \left[\begin{array}{c} \min\{\mu_A(c_1), \mu_A(c_2), \\ \mu_B(d_1), \mu_B(d_2) \end{array} \right] \end{array} \right) \\
&= \sup_{\substack{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1 \\ b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}} \left[\begin{array}{c} \min\{\mu_A(a_1), \mu_A(a_2), \mu_B(b_1), \mu_B(b_2), \\ \mu_A(c_1), \mu_A(c_2), \mu_B(d_1), \mu_B(d_2)\} \end{array} \right] \\
&\leq \sup_{\substack{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1 \\ b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}} \left[\begin{array}{c} \min\{\mu_A(a_1+c_2), \mu_A(a_2+c_1), \\ \mu_B(b_1+d_2), \mu_B(b_2+d_1)\} \end{array} \right] \\
&\leq \sup_{x+(a'+b')+z'=(a''+b'')+z'} [\min\{\mu_A(a'), \mu_A(a''), \mu_B(b'), \mu_B(b'')\}] \\
&= \mu_{A+_hB}(x).
\end{aligned}$$

Thus $\mu_{A+_hB}(x) \geq \mu_{A+_hB}(a) \wedge \mu_{A+_hB}(b)$.

Moreover,

$$\begin{aligned}
&\gamma_{A+_hB}(a) \vee \gamma_{A+_hB}(b) = \\
&\max\left(\begin{array}{c} \inf\limits_{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1} \left[\begin{array}{c} \max\{\gamma_A(a_1), \gamma_A(a_2), \\ \gamma_B(b_1), \gamma_B(b_2)\} \end{array} \right], \\ \inf\limits_{b+(c_1+d_1)+z_2=(c_2+d_2)+z_2} \left[\begin{array}{c} \max\{\gamma_A(c_1), \gamma_A(c_2), \\ \gamma_B(d_1), \gamma_B(d_2) \end{array} \right] \end{array} \right) \\
&= \inf_{\substack{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1 \\ b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}} \left[\begin{array}{c} \max\{\gamma_A(a_1), \gamma_A(a_2), \gamma_B(b_1), \gamma_B(b_2), \\ \gamma_A(c_1), \gamma_A(c_2), \gamma_B(d_1), \gamma_B(d_2)\} \end{array} \right]
\end{aligned}$$

$$\begin{aligned}
&\geq \inf_{\substack{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1 \\ b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}} \left[\begin{array}{c} \max\{\gamma_A(a_1+c_2),\gamma_A(a_2+c_1), \\ \gamma_B(b_1+d_2),\gamma_B(b_2+d_1)\} \end{array}\right] \\
&\geq \inf_{x+(a'+b')+z'=(a''+b'')+z'} [\max\{\mu_A(a'),\mu_A(a''),\mu_B(b'),\mu_B(b'')\}] \\
&= \gamma_{A+_hB}(x).
\end{aligned}$$

Hence $\gamma_{A+_hB}(x) \leq \gamma_{A+_hB}(a) \vee \gamma_{A+_hB}(b)$.

Thus $A +_h B$ is an intuitionistic fuzzy h-ideal of R. □

Lemma 2.4. *Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy h-right and h-left ideals of R, respectively. Then $A \cap B = (\mu_{A\cap B}, \gamma_{A\cup B})$ is an intuitionistic fuzzy h-quasi ideal of R.*

Proof. The proof is straightforward. □

Lemma 2.5. *Any intuitionistic fuzzy h-quasi ideal of a hemiring R is an intuitionistic fuzzy h-bi ideal of R.*

Proof. The proof is straightforward. □

Definition 7. [23] A hemiring R is said to be h-hemiregular if for each $x \in R$, there exist $a, a', z \in R$ such that $x + xax + z = xa'x + z$.

Lemma 2.6. [23] *A hemiring R is a h-hemiregular if and only if for any right h-ideal A and any left h-ideal B of R we have $\overline{AB} = A \cap B$.*

Proof. The proof is straightforward. □

Theorem 2.2. *A hemiring R is h-hemiregular if and only if for any intuitionistic fuzzy right h-ideal $A = (\mu_A, \gamma_A)$ and any intuitionistic fuzzy left h-ideal $B = (\mu_B, \gamma_B)$ of R we have $A \odot_h B = A \cap B$.*

Proof. Let R be a h-hemiregular hemiring, A is any intuitionistic fuzzy rihgt h-ideal and B any intuitionistic fuzzy left h-ideal of R. Then by Lemma 2.3 we have,

$$\begin{aligned}
\mu_{A\odot_hB}(x) &= (\mu_A \odot_h \mu_B)(x) \\
&\leq (\mu_A \odot_h \mu_{\chi_R})(x) \\
&\leq \mu_A(x)
\end{aligned}$$

Thus $\mu_{A\odot_hB}(x) \leq \mu_A(x)$. Next,

$$\begin{aligned}
\gamma_{A\odot_hB}(x) &= (\gamma_A \odot_h \gamma_B)(x) \\
&\geq (\gamma_A \odot_h \gamma_{\chi_R})(x) \\
&\geq \gamma_A(x)
\end{aligned}$$

Thus $\gamma_{A\odot_hB}(x) \geq \gamma_A(x)$. Hence $A \odot_h B \subseteq A$

$$\begin{aligned}
\mu_{A\odot_hB}(x) &= (\mu_A \odot_h \mu_B)(x) \\
&\leq (\mu_{\chi_R} \odot_h \mu_B)(x) \\
&\leq \mu_B(x)
\end{aligned}$$

Thus $\mu_{A\odot_h B}(x) \leq \mu_B(x)$. Next,

$$\begin{aligned}\gamma_{A\odot_h B}(x) &= (\gamma_A \odot_h \gamma_B)(x) \\ &\geq (\gamma_{\chi_R} \odot_h \gamma_B)(x) \\ &\geq \gamma_B(x)\end{aligned}$$

Thus $\gamma_{A\odot_h B}(x) \geq \gamma_B(x)$. Hence $A \odot_h B \subseteq B$ and $A \odot_h B \subseteq A \cap B$.

To show the reverse inclusion, let x be any element of R. Since R is h-hemiregular there exist $a, a', z \in R$ such that $x + xax + z = xa'x + z$. Then

$$\begin{aligned}\mu_{A\odot_h B}(x) &= \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left[\min\{\mu_A(a_i), \mu_A(a'_j), \mu_B(b_i), \mu_B(b'_j)\}\right] \\ &\geq \min\{\mu_A(xa), \mu_A(xa'), \mu_B(x), \mu_B(x)\} \\ &\geq \min\{\mu_A(x), \mu_A(x), \mu_B(x), \mu_B(x)\} \\ &= \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \wedge \mu_B)(x) = \mu_{A\cap B}(x).\end{aligned}$$

Thus $\mu_{A\odot_h B}(x) \geq \mu_{A\cap B}(x)$. Now,

$$\begin{aligned}\gamma_{A\odot_h B}(x) &= \inf_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left[\max\{\gamma_A(a_i), \gamma_A(a'_j), \gamma_B(b_i), \gamma_B(b'_j)\}\right] \\ &\leq \max\{\gamma_A(xa), \gamma_A(xa'), \gamma_B(x), \gamma_B(x)\} \\ &\leq \max\{\gamma_A(x), \gamma_A(x), \gamma_B(x), \gamma_B(x)\} \\ &= \max\{\gamma_A(x), \gamma_B(x)\} = (\gamma_A \vee \gamma_B)(x) = \gamma_{A\cup B}(x).\end{aligned}$$

Thus $\gamma_{A\odot_h B}(x) \leq \gamma_{A\cup B}(x)$. Thus $A \odot_h B \supseteq A \cap B$, which implies that $A \odot_h B = A \cap B$.

Conversely: Let A and B be any right h-ideal and left h ideal of R, respectively. Then the characteristic functions χ_A, and χ_B are intuitionistic fuzzy right h-ideal and intutionistic fuzzy left h-ideal of R, respectively. Now by the assumption and Lemma 2.2 we have,

$$\chi_{\overline{AB}} = \chi_A \odot_h \chi_B = \chi_A \cap \chi_B = \chi_{A\cap B}.$$

It follows by Lemma 2.2 that

$$\overline{AB} = A \cap B.$$

Therefore by Lemma 2.6, R is h-hemiregular hemiring. □

3 Characterizations of hemirings by the properties of their Intuitionistic Fuzzy h-ideals

Proposition 3.1. *The following statements are equivalent for a hemiring R.*

(1) *Each intuitionistic fuzzy h-ideals of R is idempotent.*

(2) *$A \odot_h B = A \cap B$ for all intuitionistic fuzzy h-ideals $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ of R.*

Proof. (1) $\Longrightarrow$ (2). Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy h-ideals of R. Then $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$ is also intuitionistic fuzzy h-ideal of R, and

$$\mu_A \wedge \mu_B \leq \mu_A \text{ and } \mu_A \wedge \mu_B \leq \mu_B.$$

Then by Proposition 2.1

$$(\mu_A \wedge \mu_B) \odot_h (\mu_A \wedge \mu_B) \leq \mu_A \odot_h \mu_B.$$

Since $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$ is an idempotent intuitionistic fuzzy h-ideal of R, we have

$$\mu_A \wedge \mu_B \leq \mu_A \odot_h \mu_B.$$

Also

$$\gamma_A \vee \gamma_B \geq \gamma_A \text{ and } \gamma_A \vee \gamma_B \geq \gamma_B,$$

$$\begin{aligned} (\gamma_A \vee \gamma_B) \odot_h (\gamma_A \vee \gamma_B) &\geq \gamma_A \odot_h \gamma_B, \\ \gamma_A \vee \gamma_B &\geq \gamma_A \odot_h \gamma_B. \end{aligned}$$

This shows that $A \cap B \subseteq A \odot_h B$.

Note that by Theorem 2.2 $A \cap B \supseteq A \odot_h B$, hence $A \cap B = A \odot_h B$.

(2) $\Longrightarrow$ (1). Assume that (2) holds. Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy h-ideal of R, then by the assumption we have $A \odot_h A = A \cap A = A$, which shows that each intuitionistic fuzzy h-ideal of R is idempotent. □

Theorem 3.1. *For a hemiring R with identity the following statements are equivalent.*

(1) *Each h-ideal of R is idempotent.*

(2) *$A \cap B = \overline{AB}$ for each pair of h-ideals of R.*

(3) *Each intuitionistic fuzzy h-ideal of R is idempotent.*

(4) *$A \odot_h B = A \cap B$ for all intuitionistic fuzzy h-ideals of R.*

Proof. (1) $\Longleftrightarrow$ (2). (Follows by Proposition 4.1 in [19]). (3) $\Longleftrightarrow$ (4). Follows by Proposition 3.1.

To prove that (1) and (3) are equivalent, first observe that the smallest h-ideal containing $x \in R$ has the form RxR. Its closure $\overline{RxR}$ is an h-ideal. Since by (1) all h-ideals of R are idempotent, we have

$$\overline{RxR} = \overline{\overline{RxR}.\overline{RxR}} = \overline{RxR.RxR} \qquad \text{(by Lemma 3.3 in [23])}.$$

Thus

$$x \in \overline{RxR} = \overline{RxR.RxR}$$

which implies that

$$x + \sum_{i=1}^{m} r_i x s_i u_i x t_i + z = \sum_{j=1}^{n} r'_j x s'_j u'_j x t'_j + z.$$

By Theorem 2.2 for every intuitionistic fuzzy h-ideal $A = (\mu_A, \gamma_A)$ of R, we have

$$\mu_A \odot_h \mu_A \leq \mu_A \qquad \text{and} \qquad \gamma_A \odot_h \gamma_A \geq \gamma_A.$$

Hence

$$\mu_A(x) = \mu_A(x) \wedge \mu_A(x) \leq \min_{1 \leq i \leq m} \{\mu_A(r_i x s_i), \mu_A(u_i x t_i)\}.$$

Also

$$\mu_A(x) = \mu_A(x) \wedge \mu_A(x) \leq \min_{1\leq j\leq n}\{\mu_A(r'_j x s'_j), \mu_A(u'_j x t'_j)\}.$$

Therefore

$$\begin{aligned} \mu_A(x) &\leq \min\{\mu_A(r_i x s_i), \mu_A(u_i x t_i), \mu_A(r'_j x s'_j), \mu_A(u'_j x t'_j) \,|_{1\leq j\leq n}^{1\leq i\leq m}\} \\ \leq & \sup_{x+\sum_{i=1}^{m} r_i x s_i u_i x t_i + z = \sum_{j=1}^{n} r'_j x s'_j u'_j x t'_j + z} \left[\min\{\mu_A(r_i x s_i), \mu_A(u_i x t_i), \mu_A(r'_j x s'_j), \mu_A(u'_j x t'_j)\}\right] \\ &= (\mu_A \odot_h \mu_A)(x) \end{aligned}$$

Thus $\mu_A \leq \mu_A \odot_h \mu_A$. Hence $\mu_A = \mu_A \odot_h \mu_A$. Next

$$\gamma_A(x) = \gamma_A(x) \vee \gamma_A(x) \geq \max_{1\leq i\leq m}\{\gamma_A(r_i x s_i), \gamma_A(u_i x t_i)\}$$

and

$$\gamma_A(x) = \gamma_A(x) \vee \gamma_A(x) \geq \max_{1\leq j\leq n}\{\gamma_A(r'_j x s'_j), \gamma_A(u'_j x t'_j)\}.$$

Therefore

$$\begin{aligned} \gamma_A(x) &\geq \max\{\gamma_A(r_i x s_i), \gamma_A(u_i x t_i), \gamma_A(r'_j x s'_j), \gamma_A(u'_j x t'_j) \,|_{1\leq j\leq n}^{1\leq i\leq m}\} \\ \geq & \inf_{x+\sum_{i=1}^{m} r_i x s_i u_i x t_i + z = \sum_{j=1}^{n} r'_j x s'_j u'_j x t'_j + z} \left[\max\{\gamma_A(r_i x s_i), \gamma_A(u_i x t_i), \gamma_A(r'_j x s'_j), \gamma_A(u'_j x t'_j)\}\right] \\ &= (\gamma_A \odot_h \gamma_A)(x). \end{aligned}$$

Thus $\gamma_A \geq \gamma_A \odot_h \gamma_A$. Hence $\gamma_A = \gamma_A \odot_h \gamma_A$, which shows that (3) holds.

(3) $\Longrightarrow$ (1). Let A be any h-ideal of R. Then the characteristic function $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ of R is an intuitionistic fuzzy h-ideal of R. Since by the assumption that $\chi_A = \chi_A \odot_h \chi_A = \chi_{\overline{AA}}$ by Lemma 2.2. Thus $A = \overline{AA}$. So (3) $\Longrightarrow$ (1). □

Lemma 3.1. [21] *Let R be a hemiring. Then the following conditions are equivalent.*

(1) *R is h-hemiregular.*

(2) *$B = \overline{BRB}$ for every h-bi ideal B of R.*

(3) *$Q = \overline{QRQ}$ for every h-quasi-ideal of R.*

Theorem 3.2. *Let R be a hemiring. Then the following conditions are equivalent.*

(1) *R is h-hemiregular.*

(2) *$A \subseteq A \odot_h \chi_R \odot_h A$ for every intuitionistic fuzzy h-bi ideal $A = (\mu_A, \gamma_A)$ of R.*

(3) *$A \subseteq A \odot_h \chi_R \odot_h A$ for every intuitionistic fuzzy h-quasi ideal $A = (\mu_A, \gamma_A)$ of R.*

Proof. (1) $\Longrightarrow$ (2). Assume that (1) holds. Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy h-bi ideal of R, and x any element of R. Since R is h-hemiregular there exist $a, a', z \in R$ such that $x + xax + z = xa'x + z$. Then we have

$$
\begin{aligned}
&(\mu_{A\odot_h\chi_R\odot_h\ A})(x)\\
&= \sup_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\left[\min\{\mu_{A\odot_h\ \chi_R}(a_i),\mu_{A\odot_h\ \chi_R}\left(a'_j\right),\mu_A(b_i),\mu_A\left(b'_j\right)\}\right]\\
&\geq \min\{\mu_{A\odot_h\ \chi_R}(xa),\mu_{A\odot_h\ \chi_R}(xa'),\mu_A(x),\mu_A(x)\}\\
&= \min\{\mu_{A\odot_h\ \chi_R}(xa),\mu_{A\odot_h\ \chi_R}(xa'),\mu_A(x)\}\\
&= \min\{\sup_{xa+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\left(\min\{\mu_A(a_i),\mu_A(a'_j)\}\right),\\
&\quad \sup_{xa'+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\left(\min\{\mu_A(a_i),\mu_A(a'_j)\}\right),\mu_A(x)\}\\
&\geq \min[\min\{\mu_A(xax),\mu_A(xa'x)\},\min\{\mu_A(xax),\mu_A(xa'x)\},\mu_A(x)]\\
&\geq \min\{\mu_A(x),\mu_A(x),\mu_A(x)\}=\mu_A(x),
\end{aligned}
$$

since

$$xa+xaxa+za=xa'xa+za \text{ and } xa'+xaxa'+za'=xa'xa'+za'.$$

Thus $(\mu_{A\odot_h\chi_R\odot_h\ A})(x)\geq\mu_A(x)$. Now,

$$
\begin{aligned}
&(\gamma_{A\odot_h\chi_R\odot_h\ A})(x)\\
&= \inf_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\left[\max\{\gamma_{A\odot_h\ \chi_R}(a_i),\gamma_{A\odot_h\ \chi_R}\left(a'_j\right),\gamma_A(b_i),\gamma_A\left(b'_j\right)\}\right]\\
&\leq \max\{\gamma_{A\odot_h\ \chi_R}(xa),\gamma_{A\odot_h\ \chi_R}(xa'),\gamma_A(x),\gamma_A(x)\}\\
&= \max\{\gamma_{A\odot_h\ \chi_R}(xa),\gamma_{A\odot_h\ \chi_R}(xa'),\gamma_A(x)\}\\
&= \max\left\{\begin{array}{c}\inf_{xa+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\left(\max\{\gamma_A(a_i),\gamma_A(a'_j)\}\right),\\ \inf_{xa'+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\left(\max\{\gamma_A(a_i),\gamma_A(a'_j)\right),\gamma_A(x)\end{array}\right\}\\
&\leq \max[\max\{\gamma_A(xax),\gamma_A(xa'x)\},\max\{\gamma_A(xax),\gamma_A(xa'x)\},\gamma_A(x)]\\
&\leq \max\{\gamma_A(x),\gamma_A(x),\gamma_A(x)\}=\gamma_A(x),
\end{aligned}
$$

since $xa+xaxa+za=xa'xa+za$ and $xa'+xaxa'+za'=xa'xa'+za'$. Thus $(\gamma_{A\odot_h\chi_R\odot_h\ A})(x)\leq\gamma_A(x)$. Hence $A\subseteq A\odot_h\chi_R\odot_h A$.

(2) $\Longrightarrow$ (1). This is straightforward by Lemma 2.5.

(3) $\Longrightarrow$ (1). Assume that (3) holds. Let Q be any h-quasi ideal of R. Then the charactersitic function χ_Q of Q is an intuitionistic fuzzy h-quasi ideal of R . By the assumption and Lemma 2.2 we have

$$\chi_Q\subseteq\chi_Q\odot_h\chi_R\odot_h\chi_Q=\chi_{\overline{QRQ}}$$

Then it follows by Lemma 2.2 that $Q\subseteq\overline{QRQ}$. On the other hand since Q is an h-quasi ideal of R. we have $\overline{QRQ}\subseteq\overline{RQ}\cap\overline{QR}\subseteq\overline{Q}=Q$ and so $\overline{QRQ}=Q$. Therefore R is h-hemiregular by Lemma 3.1. □

Theorem 3.3. *Let R be a hemiring. Then the following conditions are equivalent.*

(1) *R is a h-hemiregular.*

(2) $A\cap B\subseteq A\odot_h B\odot_h A$ *for every intuitionistic fuzzy h-bi ideal* $A=(\mu_A,\gamma_A)$ *and every intuitionistic fuzzy h-ideal* $B=(\mu_B,\gamma_B)$ *of* R.

(3) $A\cap B\subseteq A\odot_h B\odot_h A$ *for every intuitionistic fuzzy h-quasi ideal* $A=(\mu_A,\gamma_A)$ *and every intuitionistic fuzzy h-ideal* $B=(\mu_B,\gamma_B)$ *of* R.

Proof. (1) $\Longrightarrow$ (2). Assume that (1) holds. Let $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ be any intuitionistic fuzzy h-bi ideal and any intuitionistic fuzzy h-ideal of R, respectively. Next let x be any element of R. Since R is a h-hemiregular, there exist a, a' $z,\in R$ such that $x+xax+z=xa'x+z$. Then we have

$$
\begin{aligned}
&(\mu_{A\odot_h B\odot_h A})(x)\\
&\quad=\sup_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a_j'b_j'+z}\left[\min\{\mu_{A\odot_h B}(a_i),\mu_{A\odot_h B}\left(a_j'\right),\mu_A(b_i),\mu_A\left(b_j'\right)\}\right]\\
&\quad\geq \min\{\mu_{A\odot_h B}(xa),\mu_{A\odot_h B}(xa'),\mu_A(x),\mu_A(x)\}\\
&\quad= \min\{\mu_{A\odot_h B}(xa),\mu_{A\odot_h B}(xa'),\mu_A(x)\}\\
&= \min\{\sup_{xa+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a_j'b_j'+z}\left(\min\{\mu_A(a_i),\mu_A(a_j'),\mu_B(b_i),\mu_B(b_j')\}\right),\\
&\qquad\sup_{xa'+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a_j'b_j'+z}\left(\min\{\mu_A(a_i),\mu_A(a_j'),\mu_B(b_i),\mu_B(b_j')\}\right),\mu_A(x)\}\\
&\geq \min\{\min\{\mu_A(x),\mu_B(axa),\mu_B(xa'x)\},\min\{\mu_A(x),\mu_B(axa'),\mu_B(a'xa')\},\mu_A(x)\}\\
&\quad\geq \min\{\min\{\mu_A(x),\mu_B(x)\},\min\{\mu_A(x),\mu_B(x)\}\}\\
&\quad= \min\{\mu_A(x),\mu_B(x)\}=(\mu_A\wedge\mu_B)(x),
\end{aligned}
$$

since

$$xa+xaxa+za=xa'xa+za \text{ and } xa'+xaxa'+za'=xa'xa'+za'.$$

Thus $(\mu_{A\odot_h B\odot_h A})(x)\geq(\mu_A\wedge\mu_B)(x)$. Next,

$$
\begin{aligned}
&(\gamma_{A\odot_h B\odot_h A})(x)\\
&\quad=\inf_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a_j'b_j'+z}\left[\max\{\gamma_{A\odot_h B}(a_i),\gamma_{A\odot_h B}\left(a_j'\right),\gamma_A(b_i),\gamma_A\left(b_j'\right)\}\right]\\
&\quad\leq \max\{\gamma_{A\odot_h B}(xa),\gamma_{A\odot_h B}(xa'),\gamma_A(x),\gamma_A(x)\}\\
&\quad= \max\{\gamma_{A\odot_h B}(xa),\gamma_{A\odot_h B}(xa'),\gamma_A(x)\}\\
&= \max\{\inf_{xa+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a_j'b_j'+z}(\max\{\gamma_A(a_i),\gamma_A(a_j'),\gamma_B(b_i),\gamma_B(b_j')\})\\
&\qquad\inf_{xa'+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a_j'b_j'+z}(\max\{\gamma_A(a_i),\gamma_A(a_j'),\gamma_B(b_i),\gamma_B(b_j')\}),\gamma_A(x)\}\\
&\leq \max[\max\{\gamma_A(x),\gamma_B(axa),\gamma_B(a'xa)\},\max\{\gamma_A(x),\gamma_B(axa'),\gamma_B(a'xa')\},\gamma_A(x)]\\
&\quad\leq \max\left[\max\{\gamma_A(x),\gamma_B(x)\},\max\{\gamma_A(x),\gamma_B(x)\},\gamma_A(x)\right]\\
&\quad= \max\{\gamma_A(x),\gamma_B(x)\}=(\gamma_A\vee\gamma_B)(x).
\end{aligned}
$$

Thus $(\gamma_{A\odot_h B\odot_h A})(x)\leq(\gamma_A\vee\gamma_B)(x)$.
Hence $A\cap B\subseteq A\odot_h B\odot_h A$.

(2) $\Longrightarrow$ (3). This is straight forward by Lemma 2.5.

(3) $\Longrightarrow$ (1). Assume that (3) holds. Let A be any intuitionistic fuzzy h-quasi ideal of R. Then since χ_R is an intuitionistic fuzzy h-ideal of R, we have

$A = A \cap \chi_R \subseteq A \odot_h \chi_R \odot_h A, A \subseteq A \odot_h \chi_R \odot_h A$. Therefore R is a h-hemiregular by Theorem 3.2. □

Theorem 3.4. *Let R be a hemiring. Then the following conditions are equivalent.*

(1) R is h-hemiregular.

(2) $A \cap B \subseteq A \odot_h B$ for every intuitionistic fuzzy h-bi ideal A and every intuitionistic fuzzy left h-ideal B of R.

(3) $A \cap B \subseteq A \odot_h B$ for every intuitionistic fuzzy h-quasi ideal A and every intuitionistic fuzzy left h-ideal B of R.

(4) $A \cap B \subseteq A \odot_h B$ for every intuitionistic fuzzy right h-ideal A and every intuitionistic fuzzy h-bi-ideal B of R.

(5) $A \cap B \subseteq A \odot_h B$ for every intuitionistic fuzzy right h-ideal A and every intuitionistic fuzzy h-quasi-ideal B of R.

(6) $A \cap B \cap C \subseteq A \odot_h B \odot_h C$ for every intuitionistic fuzzy right h-ideal A, intuitionistic fuzzy h-bi ideal B and intuitonistic fuzzy left h-ideal C of R.

(7) $A \cap B \cap C \subseteq A \odot_h B \odot_h C$ for every intuitionistic fuzzy right h-ideal A, intuitionistic fuzzy h-quasi-ideal B and intuitonistic fuzzy left h-ideal C of R.

Proof. (1) $\Longrightarrow$ (2). Assume that (1) holds. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any intuitionistic fuzzy h-bi ideal and intuitionistic fuzzy left h-ideal R respectively. Now let x be any element of R. Since R is h-hemiregular, there exist $a, a', z \in R$ such that $x + xax + z = xa'x + z$. Hence we have

$$\begin{aligned}
\mu_{A \odot_h B}(x) &= \sup_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \left[\min\{\mu_A(a_i), \mu_A\left(a'_j\right), \mu_B(b_i), \mu_B\left(b'_j\right)\}\right] \\
&\geq \min\{\mu_A(x), \mu_A(x), \mu_B(ax), \mu_B(a'x)\} \text{ (since } x + xax + z = xa'x + z) \\
&= \min\{\mu_A(x), \mu_B(ax), \mu_B(a'x)\} \\
&\geq \min\{\mu_A(x), \mu_B(x), \mu_B(x)\} \\
&= \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \wedge \mu_B)(x).
\end{aligned}$$

Thus $\mu_{A \odot_h B}(x) \geq (\mu_A \wedge \mu_B)(x)$. Next

$$\begin{aligned}
\gamma_{A \odot_h B}(x) &= \inf_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \left[\max\{\gamma_A(a_i), \gamma_A\left(a'_j\right), \gamma_B(b_i), \gamma_B\left(b'_j\right)\}\right] \\
&\leq \max\{\gamma_A(x), \gamma_A(x), \gamma_B(ax), \gamma_B(a'x)\} \text{ (since } x + xax + z = xa'x + z) \\
&= \max\{\gamma_A(x), \gamma_B(ax), \gamma_B(a'x)\} \\
&\leq \max\{\gamma_A(x), \gamma_B(x), \gamma_B(x)\} \\
&= \max\{\gamma_A(x), \gamma_B(x)\} = (\gamma_A \vee \gamma_B)(x).
\end{aligned}$$

Thus $\gamma_{A \odot_h B}(x) \leq (\gamma_A \vee \gamma_B)(x)$. This implies that $A \cap B \subseteq A \odot_h B$.

(2) $\Longrightarrow$ (1). Assume that (2) holds. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any intuitionistic fuzzy h-right ideal and intuitionistic fuzzy left h-ideal R, respectively. Then it is easy to see that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy h-bi ideal of R. By the assumption, we have

$$A \cap B \subseteq A \odot_h B \subseteq A \odot_h \chi_R \cap \chi_R \odot_h B \subseteq A \cap B.$$

Hence $A \cap B = A \odot_h B$. So R is h-hemiregular by Theorem 2.2.

Similarly we can show that (1) $\Longleftrightarrow$ (3), (1) $\Longleftrightarrow$ (4), (1) $\Longleftrightarrow$ (5).

(1) $\Longrightarrow$ (6). Assume that (1) holds. Let A, B and C be any intuitionistic fuzzy right h-ideal, any intuitionistic fuzzy h-bi-ideal and any intuitionistic fuzzy left h-ideal of R, respectively. Next let x be any element of R, since R is h-hemiregular, there exist $a, a,' z \in R$, such that $x + xax + z = xa'x + z$. Then we have

$$\begin{aligned}
&\mu_{A\odot_h B\odot_h C}(x) \\
&= \sup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} \left[\min\{\mu_{A\odot_h B}(a_i), \mu_{A\odot_h B}\left(a'_j\right), \mu_C(b_i), \mu_C\left(b'_j\right)\}\right] \\
&\geq \min\{\mu_{A\odot_h B}(x), \mu_{A\odot_h B}(x), \mu_C(ax), \mu_C(a'x)\} \\
&= \min\{\mu_{A\odot_h B}(x), \mu_C(ax), \mu_C(a'x)\} \\
&= \min\left\{\begin{array}{c} \sup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} \left(\min\{\mu_A(a_i), \mu_A\left(a'_j\right), \mu_B(b_i), \mu_B\left(b'_j\right)\}\right) \\ , \mu_C(ax),\ \mu_C(a'x) \end{array}\right\} \\
&\geq \min\{\min\{\mu_A(xa),\ \mu_A(xa'),\ \mu_B(x)\},\ \mu_C(ax),\ \mu_C(a'x)\} \\
&\geq \min\{\mu_A(x), \mu_B(x), \mu_C(x)\} = (\mu_A \wedge \mu_B \wedge \mu_C)(x).
\end{aligned}$$

Thus $\mu_{A\odot_h B\odot_h C}(x) \geq (\mu_A \wedge \mu_B \wedge \mu_C)(x)$. Next

$$\begin{aligned}
&\gamma_{A\odot_h B\odot_h C}(x) \\
&= \inf_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} \left[\max\{\gamma_{A\odot_h B}(a_i), \gamma_{A\odot_h B}\left(a'_j\right), \gamma_C(b_i), \gamma_C\left(b'_j\right)\}\right] \\
&\leq \max\{\gamma_{A\odot_h B}(x), \gamma_{A\odot_h B}(x), \gamma_C(ax), \gamma_C(a'x)\} \\
&= \max\{\gamma_{A\odot_h B}(x), \gamma_C(ax), \gamma_C(a'x)\} \\
&= \max\left\{\begin{array}{c} \inf_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} (\max\{\gamma_A(a_i), \gamma_A\left(a'_j\right), \gamma_B(b_i), \gamma_B\left(b'_j\right)\}), \\ \gamma_C(ax),\ \gamma_C(a'x) \end{array}\right\} \\
&\leq \max\{\max\{\gamma_A(xa),\ \gamma_A(xa'),\ \gamma_B(x)\},\ \gamma_C(ax),\ \gamma_C(a'x)\} \\
&\leq \max\{\gamma_A(x), \gamma_B(x), \gamma_C(x)\} = (\gamma_A \vee \gamma_B \vee \gamma_C)(x).
\end{aligned}$$

Thus $\gamma_{A\odot_h B\odot_h C}(x) \leq (\gamma_A \vee \gamma_B \vee \gamma_C)(x)$, which shows that $A \cap B \cap C \subseteq A \odot_h B \odot_h C$.

(6) $\Longrightarrow$ (7). This is straight forward by Lemma 2.5.

(7) $\Longrightarrow$ (1). Assume that (7) holds. Let $A = (\mu_{A,}\gamma_A)$ and $B = (\mu_{B,}\gamma_B)$ be any intuitionistic fuzzy right h-ideal and any intuitionistic fuzzy left h-ideal of R, respectively. Since χ_R is a an intuitionistic fuzzy h-quasi ideal of R, by the assumption we have

$$\begin{aligned}
A \cap B &= A \cap \chi_R \cap B \subseteq A \odot_h \chi_R \odot_h B, \\
&\subseteq A \odot_h B \subseteq A \odot_h \chi_R \cap \chi_R \odot_h B. \\
&\subseteq A \cap B
\end{aligned}$$

Hence , $A \cap B = A \odot_h B$. So R is h-hemiregular by Theorem 2.2. □

Definition 8. [21] A hemiring R is said to be an h-intra-hemiregular hemiring if for each $x \in R$, there exist $a_i, a'_i, b_j, b'_j \in R$ such that

$$x+\sum_{i=1}^{m}a_ix^2a_i'+z=\sum_{j=1}^{n}b_j'x^2b_j'+z.$$

Equivalent definitions: (1) $x\in\overline{Rx^2R}\ \ \forall\ x\in R$, (2) $A\subseteq\overline{RA^2R}\ \ \forall\ A\subseteq R$.

Example 1. [21] *Let $R=\{0,a,b\}$ be a set with addition $(+)$, and multiplication $(\cdot)$ defined as follows:*

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

and

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	a	a

Then R is a hemiring which is both h-hemiregular and h-intra-hemiregular.

Lemma 3.2. [21] *Let R be hemirirng. Then the following conditions are equivalent.*
(1) *R is h-intra-hemiregular.*
(2) *$A\cap B\subseteq\overline{AB}$ for every left h-ideal A and every right h-ideal B of R.*

Theorem 3.5. [15] *Let R be a hemiring. Then the following conditions are equivalent.*
(1) *R is h-intra-hemiregular hemiring.*
(2) *$A\cap B\subseteq A\odot_h B$ for every intuitionistic fuzzy left h-ideal $A=(\mu_A,\gamma_A)$ and right h-ideal $B=(\mu_B,\gamma_B)$ of R.*

4 h-quasi-hemiregularity

Definition 9. A hemiring R is called (left, right) h-quasi hemiregular if every (left, right) h- ideal of R is idempotent.
It is easily seen that, a hemiring R is left (right) h-quasi-hemiregular if and only if $a\in\overline{RaRa}$ (respectively . $a\in\overline{aRaR}$).

Theorem 4.1. *A hemiring R is a left (right) h-quasi-hemiregular if and only if every intuitionistic fuzzy left (right) h-ideal is idempotent.*

Proof. Let R be a left h-quasi-hemiregular hemiring and $A=(\mu_A,\gamma_A)$ be any intuitionistic fuzzy left h-ideal of R. Let $a\in R$. Since R is left h-quasi-hemiregular $a\in\overline{RaRa}$, so there exist $z,x_i,y_i\in R$ i=1,2 such that $a+x_1ay_1a+z=x_2ay_2a+z$. Therefore, $\mu_{A\odot_h A}(a)=(\mu_A\odot_h\mu_A)(a)$

$$\begin{aligned}
&= \sup_{a+\sum_{i=1}^{n}a_ib_i+z=\sum_{j=1}^{m}c_jd_j+z}[\min\{\mu_A(a_i),\mu_A(c_j),\mu_A(b_i),\mu_A(d_j)\}]\\
&\geq \min\{\mu_A(x_1a),\mu_A(x_2a),\mu_A(y_1a),\mu_A(y_2a)\}\\
&\geq \min\{\mu_A(a),\mu_A(a),\mu_A(a),\mu_A(a)\}=\mu_A(a).
\end{aligned}$$

Hence $\mu_A\leq\mu_{A\odot_h A}$.

Since $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left h-ideal of R, it follows that $\mu_{A \odot_h A}(a) = (\mu_A \odot_h \mu_A)(a) \leq (\mu_{\chi_R} \odot_h \mu_A)(a) \leq \mu_A(a)$. Hence $\mu_A = \mu_{A \odot_h A}$.

Next, $\gamma_{A \odot_h A}(a) = (\gamma_A \odot_h \gamma_A)(a)$

$$\begin{aligned} &= \inf_{a+\sum_{i=1}^{n} a_i b_i + z = \sum_{j=1}^{m} c_j d_j + z} [\max\{\gamma_A(a_i), \gamma_A(c_j), \gamma_A(b_i), \gamma_A(d_j)\}] \\ &\leq \max\{\gamma_A(x_1 a), \gamma_A(x_2 a), \gamma_A(y_1 a), \gamma_A(y_2 a)\} \\ &\leq \max\{\gamma_A(a), \gamma_A(a), \gamma_A(a), \gamma_A(a)\} = \gamma_A(a). \text{ Hence } \gamma_{A \odot_h A} \leq \gamma_A. \end{aligned}$$

Since $\gamma_{A \odot_h A}(a) = (\gamma_A \odot_h \gamma_A)(a) \geq (\gamma_{\chi_A} \odot_h \gamma_A)(a) \geq \gamma_A(a)$. It follows that $\gamma_{A \odot_h A} = \gamma_A$, which show that every intuitionistic fuzzy left (right) h-ideal is idempotent.

Conversely, assume that every intuitionistic fuzzy left h-ideal is idempotent. Let $a \in R$, then $\overline{<a>}$ is a left h-ideal of R. So $\chi_{\overline{<a>}}$ is an intuitionistic fuzzy left h-ideal of R. Next $\mu_{\overline{<a><a>}}(a) = (\mu_{\overline{<a>}} \odot_h \mu_{\overline{<a>}})(a) = \mu_{\overline{<a>}}(a) = 1$ and $\gamma_{\overline{<a><a>}}(a) = \underline{(\gamma_{\overline{<a>}} \odot_h \gamma_{\overline{<a>}})(a)} = \gamma_{\overline{<a>}}(a) = 0$, so $a \in \overline{<a>} \cdot \overline{<a>} = \overline{Ra} \cdot \overline{Ra} \subseteq \overline{\overline{Ra} \cdot \overline{Ra}} = \overline{RaRa} \Longrightarrow a \in \overline{RaRa}$. Thus R is a left h-quasi-hemiregular. □

Theorem 4.2. *Let R be a hemiring. Then the following conditions are equivalent.*

(1) *R is h-quasi-hemiregular.*

(2) $A = (A \odot_h \chi_R)^2 \cap (\chi_R \odot_h A)^2$ *for every intuitionistic fuzzy h-quasi-ideal $A = (\mu_A, \gamma_A)$ of R.*

Proof. $(1) \Longrightarrow (2)$. Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy h-quasi-ideal of R. Since R is h-quasi-hemiregular, the intuitionistic fuzzy right h-ideal $A \odot_h \chi_R$ and the intuitionistic fuzzy left h-ideal $\chi_R \odot_h A$ are idempotent. Therefore,

$$(\mu_A \odot_h \mu_{\chi_R})^2 \wedge (\mu_{\chi_R} \odot_h \mu_A)^2 = (\mu_A \odot_h \mu_{\chi_R}) \wedge (\mu_{\chi_R} \odot_h \mu_A) \leq \mu_A$$

and

$$(\gamma_A \odot_h \gamma_{\chi_R})^2 \vee (\gamma_{\chi_R} \odot_h \gamma_A)^2 = (\gamma_A \odot_h \gamma_{\chi_R}) \vee (\gamma_{\chi_R} \odot_h \gamma_A) \geq \gamma_A.$$

To prove the reverse inclusion, let $a \in R$. Since R is a left h-quasi-hemiregular $a \in \overline{RaRa}$. So there exist $z, x_i, y_i \in R \ \ i = 1, 2$ such that $a + x_1 a y_1 a + z = x_2 a y_2 a + z$. Therefore

$$(\mu_{\chi_R} \odot_h \mu_A)^2(a) = ((\mu_{\chi_R} \odot_h \mu_A) \odot_h (\mu_{\chi_R} \odot_h \mu_A))(a)$$

$$= \sup_{a+\sum_{i=1}^{n} a_i b_i + z = \sum_{j=1}^{m} c_j d_j + z} \left[\begin{array}{c} \min\{(\mu_{\chi_R} \odot_h \mu_A)(a_i), (\mu_{\chi_R} \odot_h \mu_A)(c_j), \\ (\mu_{\chi_R} \odot_h \mu_A)(b_i), (\mu_{\chi_R} \odot_h \mu_A)(d_j)\} \end{array} \right]$$

$$\geq \min\{(\mu_{\chi_R} \odot_h \mu_A)(x_1 a), (\mu_{\chi_R} \odot_h \mu_A)(x_2 a), (\mu_{\chi_R} \odot_h \mu_A)(y_1 a), (\mu_{\chi_R} \odot_h \mu_A)(y_2 a)$$

So we have

$$x_1 a + (x_1 x_1 a y_1) a + x_1 z = (x_1 x_2 a y_2) a + x_1 z, x_2 a + (x_2 x_1 a y_1) a + x_2 z =$$

$$(x_2 x_2 a y_2) a + x_2 z, y_1 a + (y_1 x_1 a y_1) a + z =$$

$$(y_1 x_2 a y_2) a + z, y_2 a + (y_2 x_1 a y_1) a + y_2 z = (y_2 x_2 a y_2) a + y_2 z.$$

By the above inequalities it follows that

$$\begin{aligned}
&(\mu_{\chi_R} \odot_h \mu_A)^2 (a) \\
&\geq \min \Big\{ \mu_{\chi_R}(x_1x_1ay_1), \mu_{\chi_R}(x_1x_2ay_2), \mu_{\chi_R}(x_2x_1ay_1), \mu_{\chi_R}(x_2x_2ay_2) \\
&\qquad \mu_{\chi_R}(y_1x_1ay_1), \mu_{\chi_R}(y_1x_2ay_2), \mu_{\chi_R}(y_2x_1ay_1), \mu_{\chi_R}(y_2x_2ay_2), \\
&\qquad \mu_A(a), \mu_A(a), \mu_A(a), \mu_A(a), \mu_A(a), \mu_A(a), \mu_A(a), \mu_A(a) \Big\},
\end{aligned}$$

hence $(\mu_{\chi_R} \odot_h \mu_A)^2 (a) \geq \mu_A(a)$. Therefore $(\mu_{\chi_R} \odot_h \mu_A)^2 \geq \mu_A$.

Similarly we obtain that $(\mu_A \odot_h \mu_{\chi_R})^2 \geq \mu_A$. So $(\mu_A \odot_h \mu_{\chi_R})^2 \wedge (\mu_{\chi_R} \odot_h \mu_A)^2 \geq \mu_A$. Hence $(\mu_A \odot_h \mu_{\chi_R})^2 \wedge (\mu_{\chi_R} \odot_h \mu_A)^2 = \mu_A$.

Next

$$(\gamma_{\chi_R} \odot_h \gamma_A)^2 (a) = ((\gamma_{\chi_R} \odot_h \gamma_A) \odot_h (\gamma_{\chi_R} \odot_h \gamma_A))(a)$$

$$= \inf_{a+\sum\limits_{i=1}^{n} a_ib_i+z=\sum\limits_{j=1}^{m} c_jd_j+z} \left[\begin{array}{c} \max\{(\gamma_{\chi_R} \odot_h \gamma_A)(a_i), (\gamma_{\chi_R} \odot_h \gamma_A)(c_j), \\ (\gamma_{\chi_R} \odot_h \gamma_A)(b_i), (\gamma_{\chi_R} \odot_h \gamma_A)(d_j)\} \end{array} \right]$$

$$\begin{aligned}
&\leq \max\{(\gamma_{\chi_R} \odot_h \gamma_A)(x_1a), (\gamma_{\chi_R} \odot_h \gamma_A)(x_2a), (\gamma_{\chi_R} \odot_h \gamma_A)(y_1a), (\gamma_{\chi_R} \odot_h \gamma_A)(y_2a)\} \\
&\leq \max\{\gamma_{\chi_R}(x_1x_1ay_1), \gamma_{\chi_R}(x_1x_2ay_2), \gamma_{\chi_R}(x_2x_1ay_1), \gamma_{\chi_R}(x_2x_2ay_2), \\
&\qquad \gamma_{\chi_R}(y_1x_1ay_1), \gamma_{\chi_R}(y_1x_2ay_2), \gamma_{\chi_R}(y_2x_1ay_1), \gamma_{\chi_R}(y_2x_2ay_2), \\
&\qquad \gamma_A(a), \gamma_A(a), \gamma_A(a), \gamma_A(a), \gamma_A(a), \gamma_A(a), \gamma_A(a), \gamma_A(a)\}. = \gamma_A(a).
\end{aligned}$$

Therefore $(\gamma_{\chi_R} \odot_h \gamma_A)^2 \leq \gamma_A$.

Similarly we obtain $(\gamma_A \odot_h \gamma_{\chi_R})^2 \leq \gamma_A$. Hence $(\gamma_A \odot_h \gamma_{\chi_R})^2 \vee (\gamma_{\chi_R} \odot_h \gamma_A)^2 \leq \gamma_A$. Thus $(\gamma_A \odot_h \gamma_{\chi_R})^2 \vee (\gamma_{\chi_R} \odot_h \gamma_A)^2 = \gamma_A$, which shows that $A = (A \odot_h \chi_R)^2 \cap (\chi_R \odot_h A)^2$.

Conversely, assume that (2) holds. Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy right h-ideal of R. Then since $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy h-quasi-ideal of R, we have

$\mu_A = (\mu_A \odot_h \mu_{\chi_R})^2 \wedge (\mu_{\chi_R} \odot_h \mu_A)^2 \leq (\mu_A \odot_h \mu_{\chi_R})^2 \leq \mu_A \odot_h \mu_A \leq \mu_A \odot_h \mu_{\chi_R} \leq \mu_A$. So $\mu_A = \mu_A \odot_h \mu_A$.

Next, $\gamma_A = (\gamma_A \odot_h \gamma_{\chi_R})^2 \vee (\gamma_{\chi_R} \odot_h \gamma_A)^2 \geq (\gamma_A \odot_h \gamma_{\chi_R})^2 \geq \gamma_A \odot_h \gamma_A \geq \gamma_A \odot_h \gamma_{\chi_R} \geq \gamma_A$. So, $\gamma_A = \gamma_A \odot_h \gamma_A$. Therefore it follows by Theorem 4.1 that R is right h-quasi-hemiregular. Similarly, it can be proved that R is left h-quasi-hemiregular. □

Theorem 4.3. *Let R be a hemiring. Then the following conditions are equivalent:*

(1) *R is both h-intra -hemiregular and left h-quasi-hemiregular.*

(2) *$A \cap B \cap C \subseteq A \odot_h B \odot_h C$ for every intuitionistic fuzzy left h-ideal A, intuitionistic fuzzy right h-ideal B and intuitonistic fuzzy h-bi ideal C of R.*

(3) *$A \cap B \cap C \subseteq A \odot_h B \odot_h C$ for every intuitionistic fuzzy left h-ideal A, intuitionistic fuzzy right h-ideal B and intuitonistic fuzzy h-quasi-ideal C of R.*

Proof. (1) $\Longrightarrow$ (2). Let $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ be intuitionistic fuzzy left h-ideal, right h-ideal and h-bi-ideal respectively. Since R is h-intra-hemiregular and left h-quasi-hemiregular, any element $a \in R$ satisfy the equality

$a + x_1aaay_1a + z = x_2aaay_2a + z \quad (i)$
for some $z, x_i, y_i \in R$, $i = 1, 2$. Therefore

$$\begin{aligned}\mu_{A\odot_h B\odot_h C}(a) &= (\mu_A \odot_h \mu_B \odot_h \mu_C)(a) \\ &= \sup_{a+\sum_{i=1}^{n} a_ib_i+z=\sum_{j=1}^{m} c_jd_j+z} [\min\{\mu_A(a_i), \mu_A(c_j), (\mu_B \odot_h \mu_C)(b_i), (\mu_B \odot_h \mu_C)(d_j)\}] \\ &\geq \min\{\mu_A(x_1aa), \mu_A(x_2aa), (\mu_B \odot_h \mu_C)(ay_1a), (\mu_B \odot_h \mu_C)(ay_2a)\}. \\ &\geq \min\{\mu_A(a), \mu_A(a), (\mu_B \odot_h \mu_C)(ay_1a), (\mu_B \odot_h \mu_C)(ay_2a)\} \qquad (ii),\end{aligned}$$

because,

$$a + (x_1aa)\, ay_1a + z = (x_2aa)\, ay_2a + z$$

Next, from (i) we have

$$ay_1a + (ay_1x_1aa)\, ay_1a + ay_1z = (ay_1x_2aa)\, ay_2a + ay_1z,$$

and

$$ay_2a + (ay_2x_1aa)\, ay_1a + ay_2z = (ay_2x_2aa)\, ay_2a + ay_2z$$

so from (ii)

$$\begin{aligned}&\mu_{A\odot_h B\odot_h C}(a) \\ &\quad\geq \min\{\mu_A(a), \mu_A(a), \mu_B(ay_1x_1aa), \mu_B(ay_1x_2aa), \mu_C(ay_1a) \\ &\qquad, \mu_C(ay_2a), \mu_B(ay_2x_1aa), \mu_B(ay_2x_2aa)\} \\ &\quad\geq \min\{\mu_A(a), \mu_B(a), \mu_C(c)\} = (\mu_A \wedge \mu_B \wedge \mu_C)(a).\end{aligned}$$

Hence

$$\mu_A \wedge \mu_B \wedge \mu_C \leq \mu_A \odot_h \mu_B \odot_h \mu_C.$$

Now,

$$\begin{aligned}\gamma_{A\odot_h B\odot_h C}(a) &= (\gamma_A \odot_h \gamma_B \odot_h \gamma_C)(a) \\ &= \inf_{a+\sum_{i=1}^{n} a_ib_i+z=\sum_{j=1}^{m} c_jd_j+z} [\max\{\gamma_A(a_i), \gamma_A(c_j), (\gamma_B \odot_h \gamma_C)(b_i), (\gamma_B \odot_h \gamma_C)(d_j)\}] \\ &\leq \max\{\gamma_A(x_1aa), \gamma_A(x_2aa), (\gamma_B \odot_h \gamma_C)(ay_1a), (\gamma_B \odot_h \gamma_C)(ay_2a)\} \\ &\leq \max\{\gamma_A(a), \gamma_A(a), (\gamma_B \odot_h \gamma_C)(ay_1a), (\gamma_B \odot_h \gamma_C)(ay_2a)\} \\ &\leq \max\{\gamma_A(a), \gamma_A(a), \gamma_B(ay_1x_1aa), \gamma_B(ay_1x_2aa), \gamma_C(ay_1a), \gamma_C(ay_2a), \\ &\qquad \gamma_B(ay_2x_1aa), \gamma_B(ay_2x_2aa)\} \leq \max\{\gamma_A(a), \gamma_B(a), \gamma_C(a)\} \\ &\qquad = (\gamma_A \vee \gamma_B \vee \gamma_C)(a)\end{aligned}$$

Hence $\gamma_A \vee \gamma_B \vee \gamma_C \geq \gamma_A \odot_h \gamma_B \odot_h \gamma_C$. Thus $A \cap B \cap C \subseteq A \odot_h B \odot_h C$.

(2) $\Longrightarrow$ (3). Follows by Lemma 2.5.

(3) $\Longrightarrow$ (1). Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy left h-ideal of R. Then

$$\mu_A = \mu_A \wedge \mu_{\chi_R} \wedge \mu_A \leq \mu_A \odot_h \mu_{\chi_R} \odot_h \mu_A \leq \mu_A \odot_h \mu_A \leq \mu_A.$$

Hence $\mu_A = \mu_A \odot_h \mu_A$. Now,

$$\gamma_A = \gamma_A \vee \gamma_{\chi_R} \vee \gamma_A \geq \gamma_A \odot_h \gamma_{\chi_R} \odot_h \gamma_A \geq \gamma_A \odot_h \gamma_A \geq \gamma_A.$$

Hence $\gamma_A = \gamma_A \odot_h \gamma_A$. Therefore by Theorem 4.1, R is left h-quasi-hemiregular. Finally let $B = (\mu_B, \gamma_B)$ be an intuitionistic fuzzy right h-ideal of R. Then $A \cap B \subseteq A \odot_h B$. Hence by Theorm3.5, we deduce that R is h-intra-hemiregular. □

Definition 10. [11] A hemiring R is called h^*-duo if every one -sided h-ideal of R is a h-ideal of R. A hemiring R is called intuitionistic fuzzy h^*-duo if every one -sided intuitionistic fuzzy h-ideal of R is an intuitionistic fuzzy h-ideal of R. A hemiring R is called h-hemiregular intutiionistic fuzzy h^*-duo if it is both h-hemiregular and intuitionistic fuzzy h^*-duo .

For brevity in what follows we call h^*-duo hemiring a duo hemiring.

Lemma 4.1. *Let R be a hemiring. Then the following condition are equivalent.*
(1) *R is h-hemiregular duo-hemiring.*
(2) *$A \cap B = \overline{AB}$ for every left h-ideal A and every right h-ideal B of R.*

Proof. (1) $\Longrightarrow$ (2). Since R is h-hemiregular, then by Lemma 2.6, $A \cap B = \overline{AB}$ for every left h-ideal A and every right h-ideal B of R.

Conversely, assume that (ii) holds. Let A and B be any left and right h-ideal of R respectively. Since R itself is an h-ideal of R, we have $A = A \cap R = \overline{AR} \supseteq AR$ and $B = R \cap B = \overline{RB} \supseteq RB$. Hence A and B are h-ideals of R and so R is duo. Now for every right h-ideal A and left h-ideal B we have $A \cap B = \overline{AB}$, where R is h-hemiregular. □

Lemma 4.2. *Let R be a hemiring. Then R is h-hemiregular duo-hemiring if and only if $A \cap Q = \overline{AQA}$ for every h-ideal A and every quasi-ideal Q of R.*

Proof. Assume that R is h-hemiregular duo-hemiring. Let A and Q be any left h-ideal and any h-quasi-ideal of R, respectively. Then $\overline{AQA} \subseteq \overline{A} = A$ and $\overline{AQA} \subseteq \overline{AQ \cap QA} \subseteq \overline{AQ} \cap \overline{QA} \subseteq \overline{RQ} \cap \overline{QR} \subseteq Q$. Hence $\overline{AQA} \subseteq A \cap Q$. Now let $x \in A \cap Q$. Since R is h-hemiregular, there exist $a, b \in R$, such that $x + xax + z = xbx + z$. Then we have

$$xax + xaxax + zax = xbxax + zax, xbx + xaxbx + zbx = xbxbx + zbx,$$

and

$x + xax + z + xaxax + zax + xaxbx + zbx = xaxax + zax + xaxbx + zbx + xbx + z$
$x + xbxax + zax + z + xaxbx + zbx = xaxax + xbxbx + zbx + zax + z$
$x + xbxax + xaxbx + z + zax + zbx = xaxax + xbxbx + z + zax + zbx,$

so

$x + xbxax + xaxbx + z' = xaxax + xbxbx + z'$, where $z' = z + zax + zbx$, since A is an h-ideal of R.

We have $xa, xb, ax, bx \in A$, consequently
$(xbxax + xaxbx), (xaxax + xbxbx) \in AQA$ and so $x \in \overline{AQA}$.
Hence $\overline{AQA} = A \cap Q$.

Conversely, let B and C be any left, right h-ideal of R, respectively. Then both B and C are h-quasi ideals of R. Since R itself is an h-ideal of R, we have $\overline{RBR} = R \cap B = B$ and $\overline{RCR} = R \cap C = C$. Now, $BR = \overline{RBRR} \subseteq \overline{RBR} = B$ and $RC = \overline{RRCR} \subseteq \overline{RCR} = C$. Hence both B and C are h-ideals of R, so R is duo. Moreover by the assumption, we have $C \cap B = \overline{CBC} \subseteq \overline{CB} \subseteq \overline{C} \cap \overline{B} = C \cap B$. Hence R is h-hemiregular. □

Lemma 4.3. *Let R be h-hemiregular hemiring. Then R is duo if and only if it is an intuitionistic fuzzy duo.*

Proof. Assume that R is duo. Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy left h-ideal of R. Then since Rx is left h-ideal of R, it is an h-ideal by the assumption. Since R is h-hemiregular, we have $xy \in \overline{xRxR} \subseteq \overline{xRxR} \subseteq \overline{Rx}$. This implies that there exist $a, b, z \in R$ such that $xy + ax + z = bx + z$. Since $A = (\mu_A, \gamma_A)$ an intuitionistic fuzzy h-ideal of R, we have

$$\begin{aligned}
\mu_A(xy) &\geq \min\{\mu_A(ax), \mu_A(bx)\} \geq \mu_A(x), \\
\gamma_A(xy) &\leq \max\{\gamma_A(ax), \gamma_A(bx)\} \leq \gamma_A(x).
\end{aligned}$$

So $A = (\mu_A, \gamma_A)$ is an intuitonistic right h-ideal of R. It can be seen in a similar way that any intutionistic fuzzy right h-ideal of R is a intuitionistic fuzzy h-ideal of R. Thus R is an intuitionistic fuzzy duo.

Conversely , assume that R is an intuitionistic fuzzy duo. Let L be any left h-ideal of R, then the charateristic function χ_L of L is an intuitionistic fuzzy left h-ideal of R. By the assumption χ_L is an intuitonistic fuzzy ideal of R and so L is a h-ideal of R. Thus R is duo. □

Lemma 4.4. *Let R be a h-hemiregular intuitionistic fuzzy duo hemiring. Then every intuitionistic fuzzy h-bi-ideal of R is a h-ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy h-bi-ideal of R, and $x, y \in R$. Then since Rx is a left h-ideal of R, it is an h-ideal of R. Since R is h-hemiregular, $xy \in \overline{xRxy} \subseteq \overline{xRxR} \subseteq \overline{x\overline{Rx}R} \subseteq \overline{x\overline{Rx}} = \overline{xRx} \Longrightarrow xy + xax + z = xbx + z$ for some $a, b, z \in R$. Since $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy h-bi-ideal of R. we have

$$\begin{aligned}
\mu_A(xy) &\geq \min\{\mu_A(xax), \mu_A(xbx)\} \geq \mu_A(x), \\
\gamma_A(xy) &\leq \max\{\gamma_A(xax), \gamma_A(xbx)\} \leq \gamma_A(x).
\end{aligned}$$

So $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right h-ideal of R. In similar way it can be easily seen that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left h-ideal of R, and so A is an intuitionistic fuzzy h-ideal of R. □

Theorem 4.4. *Let R be hemiring. Then the following conditions are equivalent.*

(1) *R is h-hemiregular duo hemiring.*

(2) *$A \cap B = A \odot_h B \odot_h A$ for every intuitionistic fuzzy h-ideal $A = (\mu_A, \gamma_A)$ and intuitionistic fuzzy h-bi-ideal $B = (\mu_B, \gamma_B)$ of R.*

(3) *$A \cap B = A \odot_h B \odot_h A$ for every intuitionistic fuzzy h-ideal $A = (\mu_A, \gamma_A)$ and intuitionistic fuzzy h-quasi-ideal $B = (\mu_B, \gamma_B)$ of R.*

Proof. (1) $\Longrightarrow$ (2). Assume that (1) holds. Let $A = (\mu_A, \gamma_A)$ be any intuitonistic fuzzy h-ideal and $B = (\mu_B, \gamma_B)$ be any intuitionistic fuzzy h-bi-ideal of R. Then we have

$$\mu_A \odot_h \mu_B \odot_h \mu_A \leq (\mu_A \odot_h \mu_{\chi_R}) \odot_h \mu_{\chi_R} \leq \mu_A \odot_h \mu_{\chi_R} \leq \mu_A.$$

Also

$$\mu_A \odot_h \mu_B \odot_h \mu_A \leq \mu_{\chi_R} \odot_h \mu_B \odot_h \mu_{\chi_R} \leq \mu_B.$$

Hence $\mu_A \odot_h \mu_B \odot_h \mu_A \leq \mu_A \wedge \mu_B$. Next,

$$\gamma_A \odot_h \gamma_B \odot_h \gamma_A \geq (\gamma_A \odot_h \gamma_{\chi_R}) \odot_h \gamma_{\chi_R} \geq \gamma_A \odot_h \gamma_{\chi_R} \geq \gamma_A.$$

Also,

$$\gamma_A \odot_h \gamma_B \odot_h \gamma_A \geq \gamma_{\chi_R} \odot_h \gamma_B \odot_h \gamma_{\chi_R} \geq \gamma_B.$$

Hence $\gamma_A \odot_h \gamma_B \odot_h \gamma_A \geq \gamma_A \vee \gamma_B$.

Let x be any element R. Since R is h-hemiregular, there exist $a, b \in R$ such that $x + xax + z = xbx + z$. Then we have

$$xax + xaxax + zax = xbxax + zax, xbx + xaxbx + zbx = xbxbx + zbx,$$

and

$$x + xax + z + xaxax + zax + xaxbx + zbx = xaxax + zax + xaxbx + zbx + xbx + z,$$

$$x + xbxax + zax + z + xaxbx + zbx = xaxax + xbxbx + zbx + zax + z,$$

$$x + xbxax + xaxbx + z + zax + zbx = xaxax + xbxbx + z + zax + zbx,$$

so

$$x + xbxax + xaxbx + z' = xaxax + xbxbx + z',$$

where$z' = z + zax + zbx$, and

$$x + x(bxax) + x(axbx) + z' = x(axax) + x(bxbx) + z',$$

$$x + x(bxax + axbx) + z' = x(axax + bxbx) + z'.$$

Hence we have

$$(\mu_A \odot_h \mu_B \odot_h \mu_A)(x) = \sup_{x + \sum_{i=1}^{n} a_i b_i + z = \sum_{j=1}^{m} c_j d_j + z} \left[\begin{array}{c} \min\{(\mu_A \odot_h \mu_B)(a_i), (\mu_A \odot_h \mu_B)(c_j), \\ \mu_A(b_i), \mu_A(d_j)\} \end{array} \right]$$

$$\geq \min\{(\mu_A \odot_h \mu_B)(x), (\mu_A \odot_h \mu_B)(x), \mu_A(bxax + axbx), \mu_A(axax + bxbx)\}$$

$$\geq \min\{(\mu_A \odot_h \mu_B)(x), \mu_A(bxax + axbx), \mu_A(axax + bxbx)\}$$

$$= \min\{ \sup_{x + \sum_{i=1}^{n} a_i b_i + z = \sum_{j=1}^{m} c_j d_j + z} [\min\{\mu_A(a_i), \mu_A(c_j), \mu_B(b_i), \mu_B(d_j)\}],$$

and

$$\mu_A(bxax + axbx), \mu_A(axax + bxbx)\}$$

$$\geq \min\{\mu_A(xa), \mu_A(xb), \mu_B(x), \mu_B(x), \mu_A(bxax + axbx), \mu_A(axax + bxbx)\}$$
$$\geq \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \wedge \mu_B)(x).$$

Hence $\mu_A \wedge \mu_B \leq \mu_A \odot_h \mu_B \odot_h \mu_A$.

Next,

$$\begin{aligned}
&(\gamma_A \odot_h \gamma_B \odot_h \gamma_A)(x) \\
&= \inf_{x+\sum_{i=1}^{n} a_i b_i + z = \sum_{j=1}^{m} c_j d_j + z} \left[\begin{array}{c} \max\{(\gamma_A \odot_h \gamma_B)(a_i), (\gamma_A \odot_h \gamma_B)(c_j) \\ \gamma_A(b_i), \gamma_A(d_j)\} \end{array} \right] \\
&\leq \max\{(\gamma_A \odot_h \gamma_B)(x), \gamma_A(bxax + axbx), \gamma_A(axax + bxbx)\} \\
&= \max\{ \inf_{x+\sum_{i=1}^{n} a_i b_i + z = \sum_{j=1}^{m} c_j d_j + z} [\max\{\gamma_A(a_i), \gamma_A(c_j), \gamma_B(b_i), \gamma_B(d_j)\}],
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_A(bxax + axbx), \gamma_A(axax + bxbx)\} \\
&\leq \max\{\gamma_A(xa), \gamma_A(xb), \gamma_B(x), \gamma_B(x), \gamma_A(bxax + axbx), \gamma_A(axax + bxbx)\} \\
&\leq \max\{\gamma_A(x), \gamma_B(x)\} = (\gamma_A \vee \gamma_B)(x).
\end{aligned}$$

Hence $\gamma_A \vee \gamma_B \geq \gamma_A \odot_h \gamma_B \odot_h \gamma_A$. Thus $A \cap B = A \odot_h B \odot_h A$.

(2) $\Longrightarrow$ (3) This is straightforward by Lemma 2.5.

(3) $\Longrightarrow$ (1) Let A and Q be any h-ideal h-quasi-ideal of R, respectively. Then the charateristic functions χ_A, χ_Q of A , Q respectively are intuitionistic fuzzy h-ideal and intuitionistic fuzzy h-quasi-ideal of R. Now by the assumption we have

$$\chi_{A \cap Q} = \chi_A \cap \chi_Q = \chi_A \odot_h \chi_Q \odot_h \chi_A = \chi_{\overline{AQA}},$$

and so $A \cap Q = \overline{AQA}$. Hence by Lemma 4.2, we have that R is h-hemiregular duo hemiring. □

Theorem 4.5. *Let R be a hemiring. Then the following conditions are equivalent.*

(1) *R is h-hemiregular duo hemiring.*

(2) *R is h-hemiregular intuitionistic fuzzy duo hemiring.*

(3) *$A \cap B = A \odot_h B$ for every intuitionistic fuzzy h-bi-ideals $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ of R.*

(4) *$A \cap B = A \odot_h B$ for every intuitionistic fuzzy h-bi-ideal A and intuitionistic fuzzy quai ideal B of R.*

(5) *$A \cap B = A \odot_h B$ for every intuitionistic fuzzy h-bi-ideal A and every intuionistic fuzzy right h-ideal B of R.*

(6) *$A \cap B = A \odot_h B$ for every intuitionistic fuzzy h-quasi-ideal A and every intuionistic fuzzy h-bi-ideal B of R.*

(7) *$A \cap B = A \odot_h B$ for every intuitionistic fuzzy h-quasi-ideal A and B of R.*

(8) *$A \cap B = A \odot_h B$ for every intuitionistic fuzzy h-quasi-ideal A and every intuionistic fuzzy right h-ideal B of R.*

(9) *$A \cap B = A \odot_h B$ for every intuitionistic fuzzy left ideal A and every intuionistic fuzzy h-bi-ideal B of R.*

(10) *$A \cap B = A \odot_h B$ for every intuitionistic fuzzy left ideal A and every intuionistic fuzzy right h-ideal B of R.*

Proof. $(1) \Longleftrightarrow (2)$. (Follows by Lemma 4.3).
Now we prove that $(2) \Longrightarrow (3)$ Assume that (2) holds. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy h-bi-ideals of R. Now since $\mu_A \odot_h \mu_B \leq \mu_A \odot_h \mu_{\chi_R} \leq \mu_A$ and $\mu_A \odot_h \mu_B \leq \mu_{\chi_R} \odot_h \mu_B \leq \mu_B$. Thus $\mu_A \odot_h \mu_B \leq \mu_A \wedge \mu_B$. Now $\gamma_A \odot_h \gamma_B \geq \gamma_A \odot_h \gamma_{\chi_R} \geq \gamma_A$ and $\gamma_A \odot_h \gamma_B \geq \gamma_{\chi_R} \odot_h \gamma_B \geq \gamma_B$. Thus $\gamma_A \odot_h \gamma_B \geq \gamma_A \vee \gamma_B$. Now, $\mu_{A \odot_h B}(x) = (\mu_A \odot_h \mu_B)(x)$.
Therefore

$$
\begin{aligned}
&(\mu_A \odot_h \mu_B)(x) \\
&\qquad = \sup_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \left[\min\{\mu_A(a_i), \mu_A(a'_j), \mu_B(b_i), \mu_B(b'_j)\}\right] \\
&\geq \quad \min\{\mu_A(x), \mu_A(x), \mu_B(ax), \mu_B(a'x)\} \ \ (\text{since } x + xax + z = xa'x + z) \\
&\geq \quad \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \wedge \mu_B)(x). \Longrightarrow \mu_A \odot_h \mu_B \geq \mu_A \wedge \mu_B.
\end{aligned}
$$

Thus $\mu_A \odot_h \mu_B = \mu_A \wedge \mu_B$. Now, $\gamma_{A \odot_h B}(x) = (\gamma_A \odot_h \gamma_B)(x)$

$$
\begin{aligned}
&(\gamma_A \odot_h \gamma_B)(x) \\
&\qquad = \inf_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \left[\max\{\gamma_A(a_i), \gamma_A(a'_j), \gamma_B(b_i), \gamma_B(b'_j)\}\right] \\
&\leq \quad \max\{\gamma_A(x), \gamma_A(x), \gamma_B(ax), \gamma_B(a'x)\} \ \ (\text{since } x + xax + z = xa'x + z) \\
&\leq \quad \max\{\gamma_A(x), \gamma_B(x)\} = (\gamma_A \vee \gamma_B)(x). \Longrightarrow \gamma_A \odot_h \gamma_B \leq \gamma_A \vee \gamma_B.
\end{aligned}
$$

Thus $\gamma_A \odot_h \gamma_B = \gamma_A \vee \gamma_B$. Hence $A \cap B = A \odot_h B$.
$(3) \Longrightarrow (4) \Longrightarrow (5)$ (by Lemma 2.5), and $(5) \Longrightarrow (6)$ (by Lemma 4.4). Simlarly $(6) \Longrightarrow (7) \Longrightarrow (8)$. And $(8) \Longrightarrow (9)$ (by Lemma 4.4). $(9) \Longrightarrow (10)$ (by Lemma 4.4).
Finally we show that $(10) \Longrightarrow (1)$. Let A and B are right and left h-ideal of R. Then χ_A and χ_B are intuitionistic fuzzy right and left h-ideal of R. Then by assumption and Lemma 2.2,

$$\chi_{\overline{AB}} = \chi_A \odot_h \chi_B = \chi_A \wedge \chi_B = \chi_{A \cap B}.$$

Therefore it follows that $\overline{AB} = A \cap B$. Hence by Lemma 4.1, R is an h-hemiregular duo-hemiring. □

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