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ON THE DETERMINANTS OF PENTADIAGONAL MATRICES WITH THE CLASSICAL FIBONACCI, GENERALIZED FIBONACCI AND LUCAS NUMBERS

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Abstract. In this paper, we compute the determinants of several pentadiagonal matrices with the generalized Fibonacci, generalized Lucas numbers and the determinant of a pentadiagonal matrix with the classical Fibonacci numbers, and then we show how the classical Fibonacci numbers arise as determinants of some pentadiagonal matrices.

1 Introduction

Study of recurrence sequences is clearly of intrinsic interest and has been a central part of number theory for many years. Moreover, these sequences appear "almost everywhere" in mathematics and computer science. For example, in the theory of power series representing rational functions [48], pseudo-random number generators [43, 44, 45, 58], k-regular [1] and automatic sequences [36], and cellular automata [39]. Sequences of solutions of classes of interesting Diophantine equations form linear recurrence sequences, see e.g., [49, 50, 59, 60]. A great variety of power series, for example zeta-functions of algebraic varieties over finite fields [35], dynamical zeta functions of many dynamical systems [7, 31, 38], generating functions coming from group theory [15, 16], Hilbert series in commutative algebra [41], Poincare series [6, 13, 47] and the like are all known to be rational in many interesting cases. In such cases the coefficients of the series representing such functions are linear recurrence sequences, so many results from the present study may be applied. Linear recurrence sequences even enter the proof of Hilbert's Tenth Problem over $\mathbb{Z}$ [40, 61, 62]. In the proceedings [14], the problem is resolved for many other rings. The article [46] by Pheidas suggests using the arithmetic of bilinear recurrence sequences to deal with the still open rational case. Recurrence sequences also appear in many parts of the mathematical sciences in the wide sense (which includes applied mathematics and applied computer science). For example, many systems of orthogonal polynomials, including the Tchebychev polynomials and their finite field analogues, the Dickson polynomials, satisfy recurrence relations. Linear recurrence sequences are also of importance in approximation theory and cryptography and they have arisen in computer graphics [42] and time series analysis [8].

One of the simplest and most celebrated recurrence sequences is the Fibonacci sequence. The Fibonacci numbers are given by the sequence $0, 1, 1, 2, 3, 5, ...$ where each term is the sum of the previous two. This sequence can be defined via the recursive formulas: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, [32]. This recursive relation was introduced for the first time by the famous Italian mathematician Leonardo of Pisa (nicknamed Fibonacci). It is well known that the ratio of two consecutive classical Fibonacci numbers converges to the Golden Mean, or the Golden Section, $\tau = \frac{1+\sqrt{5}}{2}$, which appears in modern research in many fields from architecture [51, 52] to physics of high energy particles [17] – [19] or theoretical physics [20] – [26]. As is shown in [30], [53] – [56], the hyperbolic Fibonacci functions can lead to creation of the Lobachevsky–Fibonacci and Minkovsky–Fibonacci geometry which is of great importance for theoretical physics. In the 19th century the French mathematician Francois Edouard Anatole Lucas (1842 - 1891) introduced the so-called Lucas numbers given by the recursive relation $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$, with the seeds $L_0 = 2$ and $L_1 = 1$.

In [63], the relations have been studied between the Bell matrix and the Fibonacci matrix, which provide a unified approach to some lower triangular matrices, such as the Stirling matrices of both kinds, the Lah matrix, and the generalized Pascal matrix. To make the results more general, the discussion is also extended to the (s,t)-Fibonacci numbers and the corresponding matrix. Moreover, based on the matrix representations, various identities are derived.

For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the nth (s,t)-Fibonacci $\{F_n(s,t)\}_{n\in\mathbb{N}}$ and (s,t)-Lucas $\{L_n(s,t)\}_{n\in\mathbb{N}}$ sequences are defined recurrently by

$$F_{n+1}(s,t) = sF_n(s,t) + tF_{n-1}(s,t) \quad \text{for } n \geq 1, \tag{1.1}$$

and

$$L_{n+1}(s,t) = sL_n(s,t) + tL_{n-1}(s,t) \quad \text{for } n \geq 1, \tag{1.2}$$

with

$$F_0(s,t) = 0, \ F_1(s,t) = 1,$$

and

$$L_0(s,t) = 2, \ L_1(s,t) = s,$$

respectively.

It is well known that the (s,t)-Fibonacci and Lucas numbers are generalized Fibonacci and Lucas numbers. The following table summarizes special cases of $F_n(s,t)$ and $L_n(s,t)$:

(s,t)	F_n	L_n
$(1,1)$	Fibonacci numbers	Lucas numbers
$(2,1)$	Pell numbers	Pell-Lucas numbers
$(1,2)$	Jacobsthal numbers	Jacobsthal-Lucas numbers
$(3,-2)$	Mersenne numbers	Fermat numbers

In this paper we will write simply F_n, f_n, L_n, and l_n instead of $F_n(s,t)$, $F_n(1,1)$, $L_n(s,t)$, and $L_n(1,1)$ respectively.

Binet's formula is well known in theory of Fibonacci numbers [33]. Binet's formula allows us to express the generalized Fibonacci and Lucas numbers as functions of the roots $\alpha = \frac{s+\sqrt{s^2+4t}}{2}$ and $\beta = \frac{s-\sqrt{s^2+4t}}{2}$ of the characteristic equation $x^2 = sx + t$ associated with recurrence relations (1.1) and (1.2).

The following result is well known, and can be found, for example, in [33].

Theorem 1 (Binet's formula). *The n-th generalized Fibonacci and Lucas numbers are given by*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n.$$

There is a long tradition of using matrices and determinants to study the Fibonacci numbers. For example, Bicknell-Johnson and Spears [5] use elementary matrix operations and determinants to generate classes of identities for generalized Fibonacci numbers, and Cahill and Narayan [9] show how the Fibonacci and Lucas numbers arise as determinants of some tridiagonal matrices. The Hessenberg matrix [27]

$$\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & 0 \\ 1 & & & & 1 \\ 1 & \cdots & & 1 & 2 \end{bmatrix}$$

has as its determinant f_{n+2}. Several other Hessenberg matrices whose determinants are the Fibonacci numbers were introduced in [10] and [27], where cofactor expansions were used to compute these determinants. Combinatorial proofs were given for the determinant of Van-der-Monde's matrix [2] and of matrices whose entries are the Fibonacci [3] and Catalan [4] numbers. Strang [57] presents a family of tridiagonal matrices given by:

$$A(n) = \begin{pmatrix} 3 & 1 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{pmatrix},$$

where $A(n)$ is a $n \times n$ matrix. The determinants $|A(k)|$ are the Fibonacci numbers f_{2k+2}. Webb and Parberry [64] have showed the following complex factorization:

$$f_n = \prod_{k=1}^{n-1} \left(1 - 2i \cos \frac{\pi k}{n}\right), \quad n \geq 2,$$

where f_n is the nth Fibonacci number, by considering the roots of the Fibonacci polynomials. In [37] it is proposed to compute

$$\begin{vmatrix} l_{4n+8}+1 & l_{4n+6}-3 & 7-l_{4n} \\ l_{4n+4}+1 & l_{4n+2}-3 & 7-l_{4n-4} \\ l_{4n}+1 & l_{4n-2}-3 & 7-l_{4n-8} \end{vmatrix},$$

where l_n is the nth Lucas number. To study its generalization Kwong [34] first defined, for any real numbers a, b, c, d, e and f with $a, c, e \neq 0$, any integers $i, j, k \geq 1$, and any integer n,

$$\Delta(l) = \begin{vmatrix} al_{n+i+j+k+2} + b & cl_{n+i+j+k} + d & el_{n+i+j} + f \\ al_{n+i+k+2} + b & cl_{n+i+k} + d & el_{n+i} + f \\ al_{n+k+2} + b & cl_{n+k} + d & el_n + f \end{vmatrix},$$

and analogously

$$\Delta(f) = \begin{vmatrix} af_{n+i+j+k+2} + b & cf_{n+i+j+k} + d & ef_{n+i+j} + f \\ af_{n+i+k+2} + b & cf_{n+i+k} + d & ef_{n+i} + f \\ af_{n+k+2} + b & cf_{n+k} + d & ef_n + f \end{vmatrix},$$

where f_n is the nth Fibonacci number, and then he found that the values of these two determinants can be expressed in a rather neat manner, and that only differ by a constant. Civciv [12] studied the following determinant of a pentadiagonal matrix with Fibonacci numbers

$$E_k = \begin{bmatrix} 1 - f_k f_{k-1} & f_{k+1} & f_k f_{k-1} & & & \\ -f_{k+1} & 1 - 2f_k f_{k-1} & \ddots & \ddots & & \\ f_k f_{k-1} & -f_{k+1} & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & f_{k+1} & f_k f_{k-1} \\ & & \ddots & \ddots & 1 - 2f_k f_{k-1} & f_{k+1} \\ & & & f_k f_{k-1} & -f_{k+1} & 1 - f_k f_{k-1} \end{bmatrix}_{k \times k}.$$

In this note, we compute the determinants of several pentadiagonal matrices with the generalized Fibonacci, generalized Lucas numbers and the following determinant of pentadiagonal matrix with the classical Fibonacci numbers

$$G_k = \begin{bmatrix} 1 + f_k f_{k-1} & f_{k+1} & f_k f_{k-1} & & & \\ f_{k+1} & 1 + 2f_k f_{k-1} & \ddots & \ddots & & \\ f_k f_{k-1} & f_{k+1} & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & f_{k+1} & f_k f_{k-1} \\ & & \ddots & \ddots & 1 + 2f_k f_{k-1} & f_{k+1} \\ & & & f_k f_{k-1} & f_{k+1} & 1 + f_k f_{k-1} \end{bmatrix}_{k \times k},$$

and then show how the classical Fibonacci numbers arise as determinants of some pentadiagonal matrices.

2 Main results

In order to prove Theorems 2 – 5, we must first present the following lemma and its corollary.

Lemma 1. *For $n \geq 0$,*

$$L_n^2 + 4(-1)^{n+1} t^n = (\alpha - \beta)^2 F_n^2. \tag{2.1}$$

Proof. By Theorem 1 we get

$$\begin{aligned} L_n^2 - (\alpha-\beta)^2 F_n^2 &= (\alpha^n + \beta^n)^2 - (\alpha-\beta)^2 \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 \\ &= \alpha^{2n} + \beta^{2n} - 2(-1)^{n+1} t^n - \left(\alpha^{2n} + \beta^{2n} + 2(-1)^{n+1} t^n\right), \end{aligned}$$

from where the result follows. □

Corollary 1.

$$\alpha^n = \frac{\sqrt{(s^2+4t) F_n^2 + 4(-1)^n t^n} + \sqrt{s^2+4t} F_n}{2}, \tag{2.2}$$

or

$$\alpha^n = \frac{L_n + \sqrt{L_n^2 + 4(-1)^{n+1} t^n}}{2}, \tag{2.3}$$

and

$$\beta^n = \frac{\sqrt{(s^2+4t) F_n^2 + 4(-1)^n t^n} - \sqrt{s^2+4t} F_n}{2}, \tag{2.4}$$

or

$$\beta^n = \frac{L_n - \sqrt{L_n^2 + 4(-1)^{n+1} t^n}}{2}. \tag{2.5}$$

Theorem 2. *Let A_k be the following $k \times k$ $(k \geq 3)$ pentadiagonal matrix*

$$A_k = \begin{bmatrix} 1-(-t)^k & L_k & (-t)^k & & & \\ -L_k & 1-2(-t)^k & L_k & (-t)^k & & \\ (-t)^k & -L_k & \ddots & \ddots & \ddots & \\ & (-t)^k & \ddots & \ddots & L_k & (-t)^k \\ & & \ddots & -L_k & 1-2(-t)^k & L_k \\ & & & (-t)^k & -L_k & 1-(-t)^k \end{bmatrix}_{k\times k}.$$

Then

$$\det A_k = \prod_{j=1}^{k} \left[1 - 2iL_k \cos\frac{\pi j}{k+1} - 4(-t)^k \cos^2\frac{\pi j}{k+1}\right],\ k \geq 3.$$

Proof. In order to derive the value of the determinant of the matrix A_k, we introduce the real sequences $\left\{S_k^{(A)}\right\}_{k=1}^{\infty}$ and $\left\{T_k^{(A)}\right\}_{k=1}^{\infty}$ such that

$$\begin{aligned} S_1^{(A)} &= 1, \\ S_2^{(A)} &= 1 + \alpha^4, \\ S_k^{(A)} &= S_{k-1}^{(A)} + a^{2k} S_{k-2},\ k \geq 3, \end{aligned}$$

and

$$\begin{aligned} T_1^{(A)} &= 1, \\ T_2^{(A)} &= 1+\beta^4, \\ T_k^{(A)} &= T_{k-1}^{(A)} + \beta^{2k} T_{k-2}^{(A)}, \ k \geq 3. \end{aligned}$$

Then, by identities (2.3) and (2.5) we obtain

$$\det A_k = S_k^{(A)} T_k^{(A)}, \ k \geq 3. \tag{2.6}$$

In order to compute $\left\{S_k^{(A)}, k = 1, 2, ...\right\}$, we define the $k \times k$ tridiagonal matrix of the form

$$M_k^{(A)} = i\alpha^k N_k, \text{with } i = \sqrt{-1}, \tag{2.7}$$

where

$$N_k = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}_{k\times k}.$$

Note that $S_k^{(A)} = \det\left(I + M_k^{(A)}\right)$, $k \geq 1$. Here I is the $k \times k$ identity matrix. We know that the determinant of a square matrix can be found by taking the product of its eigenvalues. Therefore, we will compute the spectrum of $M_k^{(A)}$ in order to find an alternative expression for $S_k^{(A)}$. Let λ_j, $j = 1, 2, ..., k$, be the eigenvalues of $I + M_k^{(A)}$ and let μ_j, $j = 1, 2, ..., k$, be the eigenvalues of $M_k^{(A)}$(with the associated eigenvectors x_k). Thus, since, for each j,

$$\left(I + M_k^{(A)}\right) x_j = [1 + \mu_j]\, x_j,$$

we write $\lambda_j = 1 + \mu_j$, $j = 1, 2, ..., k$. Therefore,

$$S_k^{(A)} = \prod_{j=1}^{k} (1 + \mu_j)\,, k \geq 1. \tag{2.8}$$

Since [11] the eigenvalues of the matrix N_k are

$$\theta_j = -2\cos\frac{\pi j}{n+1}, \quad j = 1, 2, ..., n, \tag{2.9}$$

from (2.7) we have

$$\mu_j = -2i\alpha^k \cos\frac{\pi j}{k+1}, \ j = 1, 2, ..., k. \tag{2.10}$$

Combining (2.8) and (2.10), we get

$$S_k^{(A)} = \prod_{j=1}^{k} \left(1 - 2i\alpha^k \cos\frac{\pi j}{k+1}\right), \ k \geq 1. \tag{2.11}$$

Similarly, for $\left\{T_k^{(A)}\right\}_{k=1}^{\infty}$ we obtain

$$T_k^{(A)} = \prod_{j=1}^{k}\left(1 - 2i\beta^k \cos\frac{\pi j}{k+1}\right),\ k \geq 1. \tag{2.12}$$

Taking into account (2.6), (2.11) and (2.12) we compute

$$\det A_k = \prod_{j=1}^{k}\left[1 - 2iL_k \cos\frac{\pi j}{k+1} - 4\left(-t\right)^k \cos^2\frac{\pi j}{k+1}\right],\ k \geq 3,$$

and the proof is completed. □

Theorem 3. *Let B_k be the following $k \times k$ $(k \geq 3)$ pentadiagonal matrix*

$$B_k = \begin{bmatrix} 1+(-t)^k & L_k & (-t)^k & & & \\ L_k & 1+2(-t)^k & L_k & (-t)^k & & \\ (-t)^k & L_k & \ddots & \ddots & \ddots & \\ & (-t)^k & \ddots & \ddots & L_k & (-t)^k \\ & & \ddots & L_k & 1+2(-t)^k & L_k \\ & & & (-t)^k & L_k & 1+(-t)^k \end{bmatrix}_{k\times k}.$$

Then

$$\det B_k = \prod_{j=1}^{k}\left[1 - 2L_k \cos\frac{\pi j}{k+1} + 4\left(-t\right)^k \cos^2\frac{\pi j}{k+1}\right],\ k \geq 3.$$

Proof. The proof is similar to the proof of Theorem 2, and we only show an outline of it. In order to compute the determinant of the matrix B_k, we introduce the real sequences $\left\{S_k^{(B)}\right\}_{k=1}^{\infty}$ and $\left\{T_k^{(B)}\right\}_{k=1}^{\infty}$ such that

$$\begin{aligned} S_1^{(B)} &= 1, \\ S_2^{(B)} &= 1 - \alpha^4, \\ S_k^{(B)} &= S_{k-1}^{(B)} - a^{2k} S_{k-2}^{(B)},\ k \geq 3, \end{aligned}$$

and

$$\begin{aligned} T_1^{(B)} &= 1, \\ T_2^{(B)} &= 1 - \beta^4, \\ T_k^{(B)} &= T_{k-1}^{(B)} - \beta^{2k} T_{k-2}^{(B)},\ k \geq 3. \end{aligned}$$

Then, by identities (2.3) and (2.5) we get

$$\det B_k = S_k^{(B)} T_k^{(B)},\ k \geq 3. \tag{2.13}$$

In order to compute $\left\{S_k^{(B)}, k = 1, 2, ...\right\}$, we define the $k \times k$ tridiagonal matrix of the form

$$M_k^{(B)} = \alpha^k N_k. \tag{2.14}$$

Therefore, since $S_k^{(B)} = \det\left(I + \alpha^k N_k\right)$, $k \geq 1$, and $T_k^{(B)} = \det\left(I + \beta^k N_k\right)$, $k \geq 1$, taking into account (2.9) and (2.13) we compute

$$\det B_k = \prod_{j=1}^{k} \left[1 - 2L_k \cos\frac{\pi j}{k+1} + 4\left(-t\right)^k \cos^2\frac{\pi j}{k+1}\right], \ k \geq 3,$$

and the proof is completed. □

Theorem 4. *Let C_k be the following $k \times k$ $(k \geq 3)$ pentadiagonal matrix*

$$C_k = \begin{bmatrix} 1-\omega & F_k & \omega & & & \\ -F_k & 1-2\omega & F_k & \omega & & \\ \omega & -F_k & \ddots & \ddots & \ddots & \\ & \omega & \ddots & \ddots & F_k & \omega \\ & & \ddots & -F_k & 1-2\omega & F_k \\ & & & \omega & -F_k & 1-\omega \end{bmatrix}_{k\times k},$$

where $\omega = \frac{(-1)^{k+1}t^k}{s^2+4t}$. Then

$$\det C_k = \prod_{j=1}^{k} \left[1 - i\frac{2}{\sqrt{s^2+4t}}L_k \cos\frac{\pi j}{k+1} - 4\frac{(-t)^k}{s^2+4t}\cos^2\frac{\pi j}{k+1}\right], \ k \geq 3.$$

Proof. In order to compute the determinant of the matrix C_k, we introduce the real sequences $\left\{S_k^{(C)}\right\}_{k=1}^{\infty}$ and $\left\{T_k^{(C)}\right\}_{k=1}^{\infty}$ such that

$$\begin{aligned} S_1^{(C)} &= 1, \\ S_2^{(C)} &= 1 + \frac{\alpha^4}{s^2+4t}, \\ S_k^{(C)} &= S_{k-1}^{(C)} + \frac{\alpha^{2k}}{s^2+4t}S_{k-2}^{(C)}, \ k \geq 3, \end{aligned}$$

and

$$\begin{aligned} T_1^{(C)} &= 1, \\ T_2^{(C)} &= 1 + \frac{\beta^4}{s^2+4t}, \\ T_k^{(C)} &= T_{k-1}^{(C)} + \frac{\beta^{2k}}{s^2+4t}T_{k-2}^{(C)}, \ k \geq 3. \end{aligned}$$

Then, by identities (2.2) and (2.4) we have

$$\det C_k = S_k^{(C)}T_k^{(C)}, \ k \geq 3. \tag{2.15}$$

In order to compute $\left\{S_k^{(C)}, k=1,2,...\right\}$, we define the $k\times k$ tridiagonal matrix of the form

$$M_k^{(C)} = i\frac{\alpha^k}{\sqrt{s^2+4t}}N_k, \text{with } i=\sqrt{-1}. \tag{2.16}$$

Thus we get

$$S_k^{(C)} = \prod_{j=1}^{k}\left(1-2i\frac{\alpha^k}{\sqrt{s^2+4t}}\cos\frac{\pi j}{k+1}\right),\ \ k\geq 1. \tag{2.17}$$

Similarly, for $\left\{T_k^{(C)}\right\}_{k=1}^{\infty}$ we obtain

$$T_k^{(C)} = \prod_{j=1}^{k}\left(1-2i\frac{\beta^k}{\sqrt{s^2+4t}}\cos\frac{\pi j}{k+1}\right),\ \ k\geq 1. \tag{2.18}$$

Taking into account (2.15), (2.17) and (2.18) we compute

$$\det C_k = \prod_{j=1}^{k}\left[1-i\frac{2}{\sqrt{s^2+4t}}L_k\cos\frac{\pi j}{k+1}-4\frac{(-t)^k}{s^2+4t}\cos^2\frac{\pi j}{k+1}\right],\ \text{with } k\geq 3,$$

which completes the proof. □

Theorem 5. *Let D_k be the following $k\times k$ $(k\geq 3)$ pentadiagonal matrix*

$$D_k = \begin{bmatrix} 1+\omega & F_k & \omega & & & \\ F_k & 1+2\omega & F_k & \omega & & \\ \omega & F_k & \ddots & \ddots & \ddots & \\ & \omega & \ddots & \ddots & F_k & \omega \\ & & \ddots & F_k & 1+2\omega & F_k \\ & & & \omega & F_k & 1+\omega \end{bmatrix}_{k\times k},$$

where $\omega = \frac{(-1)^{k+1}t^k}{s^2+4t}$. Then

$$\det D_k = \prod_{j=1}^{k}\left[1-\frac{2}{\sqrt{s^2+4t}}L_k\cos\frac{\pi j}{k+1}+4\frac{(-t)^k}{s^2+4t}\cos^2\frac{\pi j}{k+1}\right],\ \textit{with } k\geq 3.$$

Proof. The proof is similar to the proof of Theorem 4, and we only show an outline of it. In order to compute the determinant of the matrix D_k, we introduce the real sequences $\left\{S_k^{(D)}\right\}_{k=1}^{\infty}$ and $\left\{T_k^{(D)}\right\}_{k=1}^{\infty}$ such that

$$\begin{aligned} S_1^{(D)} &= 1, \\ S_2^{(D)} &= 1-\frac{\alpha^4}{s^2+4t}, \\ S_k^{(D)} &= S_{k-1}^{(D)} - \frac{\alpha^{2k}}{s^2+4t}S_{k-2}^{(D)}, k\geq 3, \end{aligned}$$

and

$$\begin{aligned} T_1^{(D)} &= 1, \\ T_2^{(D)} &= 1 - \frac{\beta^4}{s^2+4t}, \\ T_k^{(D)} &= T_{k-1}^{(D)} - \frac{\beta^{2k}}{s^2+4t} T_{k-2}^{(D)}, k \geq 3. \end{aligned}$$

Then, by identities (2.2) and (2.4) we obtain

$$\det D_k = S_k^{(D)} T_k^{(D)}, \;\; k \geq 3. \tag{2.19}$$

In order to compute $\left\{ S_k^{(D)}, k = 1, 2, ... \right\}$, we define the $k \times k$ tridiagonal matrix of the form:

$$M_k^{(D)} = \frac{\alpha^k}{\sqrt{s^2+4t}} N_k. \tag{2.20}$$

Therefore, since $S_k^{(D)} = \det \left(I + \frac{\alpha^k}{\sqrt{s^2+4t}} N_k \right)$, $k \geq 1$, and $T_k^{(D)} = \det \left(I + \frac{\beta^k}{\sqrt{s^2+4t}} N_k \right)$, $k \geq 1$, taking into account (2.9) and (2.19) we compute

$$\det D_k = \prod_{j=1}^{k} \left[1 - \frac{2}{\sqrt{s^2+4t}} L_k \cos \frac{\pi j}{k+1} + 4 \frac{(-t)^k}{s^2+4t} \cos^2 \frac{\pi j}{k+1} \right], \text{ with } k \geq 3,$$

which completes the proof. □

Now using equations (2.2) and (2.4), similarly to the proof of Theorem 5 we can prove the following corollary.

Theorem 6. *Let G_k be the following $k \times k$ $(k \geq 3)$ pentadiagonal matrix*

$$G_k = \begin{bmatrix} 1+f_k f_{k-1} & f_{k+1} & f_k f_{k-1} & & & \\ f_{k+1} & 1+2f_k f_{k-1} & \ddots & \ddots & & \\ f_k f_{k-1} & f_{k+1} & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & f_{k+1} & f_k f_{k-1} \\ & & \ddots & \ddots & 1+2f_k f_{k-1} & f_{k+1} \\ & & & f_k f_{k-1} & f_{k+1} & 1+f_k f_{k-1} \end{bmatrix}_{k \times k},$$

where f_k is the kth classical Fibonacci number. Then

$$\det G_k = \prod_{j=1}^{k} \left[1 - 2 f_{k+1} \cos \frac{\pi j}{k+1} + 4 f_k f_{k-1} \cos^2 \frac{\pi j}{k+1} \right], \;\; k \geq 3.$$

Proof. The proof is similar to the proof of Theorem 5, and we only show an outline of it. From (2.19) we obtain

$$\det G_k = S_k^{(G)} T_k^{(G)}, \;\; k \geq 3, \tag{2.21}$$

where $\left\{S_k^{(G)}\right\}$ and $\left\{T_k^{(G)}\right\}$ are the real sequences such that

$$\begin{aligned} S_1^{(G)} &= 1, \\ S_2^{(G)} &= 1 - f_k^2, \\ S_k^{(G)} &= S_{k-1}^{(G)} - f_k^2 S_{k-2}^{(G)}, \ k \geq 3, \end{aligned}$$

and

$$\begin{aligned} T_1^{(G)} &= 1, \\ T_2^{(G)} &= 1 - f_{k-1}^2, \\ T_k^{(G)} &= T_{k-1}^{(G)} - f_{k-1}^2 T_{k-2}^{(G)}, \ k \geq 3. \end{aligned}$$

In order to compute $\left\{S_k^{(G)}, k = 1, 2, ...\right\}$, we define the $k \times k$ tridiagonal matrix of the form

$$M_k^{(G)} = f_k N_k. \tag{2.22}$$

Therefore, since $S_k^{(G)} = \det\left(I + f_k N_k\right), \ k \geq 1,$ and $T_k^{(G)} = \det\left(I + f_{k-1} N_k\right), \ k \geq 1,$ taking into account (2.9) and (2.21) we compute

$$\det G_k = \prod_{j=1}^{k} \left[1 - 2\left(f_k + f_{k-1}\right)\cos\frac{\pi j}{k+1} + 4 f_k f_{k-1} \cos^2\frac{\pi j}{k+1}\right], \ k \geq 3,$$

which completes the proof. □

Example 1.

$$\begin{vmatrix} 0 & 2 & 1 & & & \\ -2 & -1 & 2 & 1 & & \\ 1 & -2 & -1 & \ddots & \ddots & \\ & 1 & \ddots & \ddots & \ddots & 1 \\ & & \ddots & \ddots & -1 & 2 \\ & & & 1 & -2 & 0 \end{vmatrix}_{k \times k} = f_{k+1}^2, \quad k \geq 3.$$

Example 2.

$$\begin{vmatrix} 10/9 & 2/3 & 1/9 & & & \\ 2/3 & 11/9 & 2/3 & 1/9 & & \\ 1/9 & 2/3 & 11/9 & \ddots & \ddots & \\ & 1/9 & \ddots & \ddots & \ddots & 1/9 \\ & & \ddots & \ddots & 11/9 & 2/3 \\ & & & 1/9 & 2/3 & 0 \end{vmatrix}_{k \times k} = 3^{-2k} f_{2k+2}^2, \quad k \geq 3.$$

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