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GENERALIZED FRACTIONAL STEFFENSEN TYPE INEQUALITIES

J. Pečarić, I. Perić, K. Smoljak

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Abstract. In this paper, we state, prove and discuss new general Steffensen type inequality. As a special case of that general result we obtain fractional inequalities involving fractional integrals and derivatives of Riemann-Liouville, Canavati, Caputo, Hadamard and Erdelyi-Kóber types as well as fractional integrals of a function with respect to another function. Furthermore, we show that our main result covers much more general situations applying it to multidimensional settings. Finally we give mean value theorems for linear functionals related to obtained Steffensen type ineqalities.

1 Introduction

The well-known Steffensen inequality reads [9, p. 181]:

Theorem 1.1. *Suppose that f is decreasing and g is integrable on $[a, b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)dt.$$

The inequalities are reversed for increasing f.

We introduce the notation $x_+ = \max(x, 0)$. Also x_+^n denotes $(x_+)^n$ except that 0^0 will be interpreted as 0. Thus the characteristic function of $[t, \infty)$ is $(x - t)_+^0$.

Let M_k denote the class of functions f with the representation

$$f(x) = \int_0^1 (x - t)_+^k d\nu(t), \quad x \in [0, 1],$$

for some non-negative regular Borel measure ν.

The following generalizations of the Steffensen type inequality are given in [5]:

Theorem 1.2. *Let μ be a (signed) regular Borel measure such that $\int_0^1 |d\mu| < \infty$. Then*

$$\int_0^1 f d\mu \geq \int_0^a f dx \tag{1.1}$$

for all $f \in M_k$ *if and only if*

$$\int_0^1 (x-t)_+^k d\mu(x) \geq 0, \quad t \in [0,1] \tag{1.2}$$

and

$$a \leq \min_{0 \leq t \leq 1} \left\{ t + \left((k+1) \int_0^1 (x-t)_+^k d\mu(x) \right)^{\frac{1}{k+1}} \right\}. \tag{1.3}$$

Therefore the best possible choice for a *is when there is equality in* (1.3).

Theorem 1.3. *If* $\int_0^1 |d\mu| < \infty$, *then the inequality*

$$\int_0^1 f d\mu(x) \leq \int_a^1 f dx \tag{1.4}$$

holds for all $f \in M_k$ *if and only if*

$$\int_0^1 (x-t)_+^k d\mu(x) \leq \frac{(1-t)^{k+1}}{k+1}, \quad t \in [0,1] \tag{1.5}$$

and

$$a \leq \min_{0 \leq t \leq 1} \left\{ t + \left[(1-t)^{k+1} - (k+1) \int_0^1 (x-t)_+^k d\mu(x) \right]^{\frac{1}{k+1}} \right\}. \tag{1.6}$$

In particular, the best possible choice for a *is when there is equality in* (1.6).

The paper is organised in the following way. After this Introduction, in Section 2 we give new general inequalities which will be used for obtaining Steffensen type inequalities. In Section 3 we use our main results to obtain some Steffensen type inequalities given in [8] and some new Steffensen type ineqalities involving fractional integral of a function f with respect to a given function g, Hadamard fractional integral and Erdelyi-Kóber fractional integral. Furthermore, we apply our general result in multidimensional settings to obtain new results involving mixed Riemann-Liouville fractional integrals. In Section 4 we prove mean value theorems of the Lagrange and the Cauchy type.

First, let us recall some notions; log denotes the natural logarithm function, $\Gamma(\alpha)$ denotes the gamma function, ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function, an interval in $\mathbb{R}$ is any convex subset of $\mathbb{R}$ and by dx we denote the Lebesgue measure on $\mathbb{R}$.

2 Main results

Let $(\Omega_1, \Sigma_1, \mu_1)$ be a measure space with $\sigma-$finite (signed) regular Borel measure and Ω_2 be a set. Let $K : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative function and let U denote the class of all functions $f : \Omega_1 \to \mathbb{R}$ such that there exists a measure space $(\Omega_2, \Sigma_2, \mu_2)$ such that μ_2 is non-negative $\sigma-$finite regular Borel measure and

$$f(x) = \int_{\Omega_2} K(x,y) d\mu_2(y), \quad x \in \Omega_1. \tag{2.1}$$

Theorem 2.1. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *be a measure space with* $\sigma-$*finite (signed) regular Borel measure. Then for every* $f \in U$

$$\int_{\Omega_1} f(x)d\mu_1(x) \geq 0 \tag{2.2}$$

if and only if

$$\int_{\Omega_1} K(x,y)d\mu_1(x) \geq 0 \quad \textit{for } y \in \Omega_2. \tag{2.3}$$

Proof. Using the representation (2.1) in (2.2), and then using Fubini's theorem, (2.2) is equivalent to

$$\int_{\Omega_2}\int_{\Omega_1} K(x,y)d\mu_1(x)d\mu_2(y) \geq 0. \tag{2.4}$$

Since μ_2 is arbitrary non-negative regular Borel measure, (2.4) holds if and only if (2.3) holds. □

Theorem 2.2. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_1, \Sigma_3, \mu_3)$ *be measure spaces with* $\sigma-$*finite (signed) regular Borel measures. Then for every* $f \in U$

$$\int_{\Omega_1} f(x)d\mu_1(x) \geq \int_{\Omega_1} f(x)d\mu_3(x) \tag{2.5}$$

if and only if

$$\int_{\Omega_1} K(x,y)d\mu_1(x) \geq \int_{\Omega_1} K(x,y)d\mu_3(x) \quad \textit{for } y \in \Omega_2. \tag{2.6}$$

Proof. Apply Theorem 2.1 with measure μ_1 replaced by $\mu_1 - \mu_3$. □

Remark 1. Let $\Omega_1 = \Omega_2 = [0,1]$, $K(x,t) = (x-t)_+^k$, $d\mu_3(x) = \chi_{[0,a]}dx$ for $0 \leq a \leq 1$ and $d\mu_1(x) = d\mu(x)$ for some finite (signed) regular Borel measure μ. Then the class U reduces to M_k and (2.1) reduces to (1.1). Furthermore, the condition (2.6) reduces to

$$\int_0^1 (x-t)_+^k d\mu(x) \geq \int_0^a (x-t)_+^k dx. \tag{2.7}$$

Since the right hand side in (2.7) is non-negative, (1.2) is necessary. Moreover, from (2.7) we have (1.3) for $0 \leq t \leq a$. Since (1.2) holds, (1.3) is also true for $t \geq a$. Hence, considering the class of functions $f \in M_k$ and finite (signed) regular Borel measure μ, Theorem 2.2 reduces to the Steffensen type inequality given in Theorem 1.2.

Remark 2. Let $\Omega_1 = \Omega_2 = [0,1]$, $K(x,t) = (x-t)_+^k$, $d\mu_1(x) = \chi_{[a,1]}dx$ for $0 \leq a \leq 1$ and $d\mu_3(x) = d\mu(x)$ for some finite (signed) regular Borel measure μ. Then the class U reduces to M_k and (2.1) reduces to (1.4). Furthermore, the condition (2.6) reduces to

$$\int_0^1 (x-t)_+^k d\mu(x) \leq \int_a^1 (x-t)_+^k dx. \tag{2.8}$$

For $t > a$, from (2.8) we have

$$\int_0^1 (x-t)_+^k d\mu(x) \leq \frac{(1-t)^{k+1}}{k+1}. \tag{2.9}$$

Obviously, (2.9) also holds for $t \leq a$, so (1.5) is necessary. Moreover, from (2.8) we have (1.6) for $0 \leq t \leq a$. But since (1.5) holds, (1.6) is also true for $t \geq a$. Hence, considering the class of functions $f \in M_k$ and finite (signed) regular Borel measure μ, Theorem 2.2 reduces to the Steffensen type inequality given in Theorem 1.3.

3 New Steffensen type inequalities involving fractional integrals and derivatives

First, let us recall some facts about fractional derivatives needed in the sequel, for more details see [10] (or [1], [6], [7]). Let $0 < a < b \leq \infty$. By $C^m([a,b])$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order m, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m([a,b])$ we denote the space of all functions $g \in C^{m-1}([a,b])$ with $g^{(m-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$). By $L_1(a,b)$ we denote the space of all functions integrabile on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

Let us recall the definition of the Riemann-Liouville fractional integral, see [7]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integral $I_{a+}^\alpha f$ of order $\alpha > 0$ is defined by

$$(I_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1} dy, \quad (x > a).$$

This integral is called the left-sided fractional integral.

Remark 3. Applying Theorem 2.2 with $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_2(y) = f(y)dy$, $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$ (or $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$) for λ non-negative such that $a+\lambda \leq b$ (or $a \leq b-\lambda$), and

$$K(x,y) = \begin{cases} \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)}, & a \leq y \leq x; \\ 0, & x < y \leq b \end{cases}$$

we obtain Steffensen type inequalities for the left-sided fractional integral $I_{a+}^\alpha f$ given in [8, Theorems 2.1 and 2.2].

We define the generalized Riemann–Liouville fractional derivative of f of order $\alpha > 0$ by

$$(D_a^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\alpha-1} f(y) dy,$$

where $n = [\alpha] + 1$, $x \in [a,b]$.

In addition, we stipulate

$$D_a^0 f := f =: I_a^0 f, \quad I_a^{-\alpha} f := D_a^{\alpha} f \text{ if } \alpha > 0.$$

If $\alpha \in \mathbb{N}$ then $D_a^{\alpha} f = \frac{d^{\alpha} f}{dx^{\alpha}}$, the ordinary α-order derivative.

The space $I_a^{\alpha}(L(a,b))$ is defined as the set of all functions f on $[a,b]$ of the form $f = I_a^{\alpha}\varphi$ for some $\varphi \in L(a,b)$, (see [10, Chapter 1, Definition 2.3]). According to Theorem 2.3 in [10, p. 43], the latter characterization is equivalent to the condition

$$I_a^{n-\alpha} f \in AC^n[a,b],$$

$$\frac{d^j}{dx^j} I_a^{n-\alpha} f(a) = 0, \quad j = 0, 1, \ldots, n-1.$$

The following lemma summarizes conditions in composition identity for generalized Riemann-Liouville fractional derivatives. For details see [2].

Lemma 3.1. *Let $\beta > \alpha \geq 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Identity*

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(\beta - \alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^{\beta} f(y)\, dy, \quad x \in [a,b]$$

is valid if one of the following conditions holds:

(*i*) $f \in I_a^{\beta}(L(a,b))$.

(*ii*) $I_a^{n-\beta} f \in AC^n[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(*iii*) $D_a^{\beta-k} f \in C[a,b]$ *for* $k = 1, \ldots, n$, $D_a^{\beta-1} f \in AC[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(*iv*) $f \in AC^n[a,b]$, $D_a^{\beta} f \in L(a,b)$, $D_a^{\alpha} f \in L(a,b)$, $\beta - \alpha \notin \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, n$ *and* $D_a^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots, m$.

(*v*) $f \in AC^n[a,b]$, $D_a^{\beta} f \in L(a,b)$, $D_a^{\alpha} f \in L(a,b)$, $\beta - \alpha = l \in \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, l$.

(*vi*) $f \in AC^n[a,b]$, $D_a^{\beta} f \in L(a,b)$, $D_a^{\alpha} f \in L(a,b)$ *and* $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$.

(*vii*) $f \in AC^n[a,b]$, $D_a^{\beta} f \in L(a,b)$, $D_a^{\alpha} f \in L(a,b)$, $\beta \notin \mathbb{N}$ *and* $D_a^{\beta-1} f$ *is bounded in a neighbourhood of* $t = a$.

Remark 4. Let assumptions in Lemma 3.1 be satisfied. Then, applying Theorem 2.2 with $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_2(y) = (D_a^{\beta} f)(y)dy$, $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$ (or $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$) for λ non-negative such that $a + \lambda \leq b$ (or $a \leq b - \lambda$) and

$$K(x,y) = \begin{cases} \frac{(x-y)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a \leq y \leq x; \\ 0, & x < y \leq b, \end{cases} \tag{3.1}$$

we obtain Steffensen type inequalities for the generalized Riemann-Liouville fractional derivative $D_a^{\alpha} f$ given in [8, Theorems 2.3 and 2.4].

Let us recall the definition of the Caputo fractional derivative, for details see [1, p. 449].

Caputo fractional derivative $D^{\alpha}_{*a}g$ of order $\alpha > 0$ if defined by

$$(D^{\alpha}_{*a}g)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $g \in AC^n([a,b])$, $n = [\alpha]+1$, and $t \in [a,b]$.

Remark 5. Applying Theorem 2.2 with $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_2(y) = g^{(n)}(y)dy$, $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$ (or $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$) for λ non-negative such that $a+\lambda \leq b$ (or $a \leq b-\lambda$) and

$$K(x,y) = \begin{cases} \frac{(x-y)^{n-\alpha-1}}{\Gamma(n-\alpha)}, & a \leq y \leq x; \\ 0, & x < y \leq b \end{cases}$$

we obtain Steffensen type inequalities for the Caputo fractional derivative $D^{\alpha}_{*a}g$ given in [8, Theorems 2.5 and 2.6].

Next, we define generalized Canavati fractional derivative ($\alpha-$fractional derivative of f over $[a,b]$). The definition of generalized Canavati fractional derivative is given in [1] but we will use it with some new conditions given in [3]. We consider

$$C^{\alpha}_a([a,b]) = \{f \in C^n([a,b]) : I_a^{1-\bar{\alpha}} f^{(n)} \in C^1([a,b])\},$$

$\alpha > 0$, $n = [\alpha]$ and $\bar{\alpha} = \alpha - n, a \leq \bar{\alpha} < b$.
For $f \in C^{\alpha}_a([a,b])$ the generalized Canavati fractional derivative of f is defined by

$$D^{\alpha}_a f = D I_a^{1-\bar{\alpha}} f^{(n)},$$

where $D = d/dx$.

The following lemma gives conditions in composition rule for generalized Canavati fractional derivative (see [3]).

Lemma 3.2. *Let $\beta > \alpha > 0$, $n = [\beta]$, $m = [\alpha]$. Let $f \in C^{\beta}_a([a,b])$, be such that $f^{(i)}(a) = 0$, $i = m, m+1, ..., n-1$. Then*

(i) $f \in C^{\alpha}_a([a,b])$

(ii) $(D^{\alpha}_a f)(x) = \frac{1}{\Gamma(\beta-\alpha)} \int\limits_a^x (x-t)^{\beta-\alpha-1} (D^{\beta}_a f)(t) dt,$
for every $x \in [a,b]$.

Remark 6. Let assumptions in Lemma 3.2 be satisfied. Then, applying Theorem 2.2 with $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_2(y) = (D^{\beta}_a f)(y)dy$, $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$ (or $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$) for λ non-negative such that $a+\lambda \leq b$ (or $a \leq b-\lambda$) and K defined by (3.1) we obtain Steffensen type inequalities for generalized Canavati fractional derivative $D^{\alpha}_a f$ given in [8, Theorems 2.3 and 2.4].

We continue with definition and some properties of the fractional integral of a function f with respect to given function g. For details see e.g. [7, p. 99].

Let $(a,b)\,(-\infty \le a < b \le \infty)$ be a finite or infinite interval on the real line $\mathbb{R}$ and $\alpha > 0$. Let g be an increasing function on (a,b) such that g' is continuous on (a,b). The left-sided fractional integral of a function f with respect to another function g on $[a,b]$ is defined by

$$(I^{\alpha}_{a+;g}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(y)f(y)dy}{[g(x)-g(y)]^{1-\alpha}}, \quad x > a.$$

Theorem 3.1. *Let g be an increasing function on (a,b) such that g' is continuous on (a,b), let μ_1 be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every non-negative Borel measurable function f_1*

$$\int_a^b (I^{\alpha}_{a+;g}f_1)(x)d\mu_1(x) \ge \frac{1}{\Gamma(\alpha)}\int_a^{a+\lambda} g'(y)f_1(y)\int_y^{a+\lambda}(g(x)-g(y))^{\alpha-1}dx\,dy \tag{3.2}$$

if and only if

$$\int_y^b (g(x)-g(y))^{\alpha-1}d\mu_1(x) \ge 0, \quad y \in [a,b] \tag{3.3}$$

and

$$\int_y^{a+\lambda}(g(x)-g(y))^{\alpha-1}dx \le \int_y^b (g(x)-g(y))^{\alpha-1}d\mu_1(x). \tag{3.4}$$

Proof. Let (3.2) hold. Let $\Omega_1 = \Omega_2 = [a,b]$, λ be non-negative real number such that $a+\lambda \le b$,

$$K(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)}\frac{g'(y)}{(g(x)-g(y))^{1-\alpha}}, & a \le y \le x; \\ 0, & x < y \le b, \end{cases} \tag{3.5}$$

$d\mu_2(y) = f_1(y)dy$ and $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$. Notice that class U now reduces to class of functions $I^{\alpha}_{a+;g}f_1$ and that (2.1) reduces to (3.2). Now, from Theorem 2.2 it follows that (2.6) holds. Furthermore, (2.6) reduces to

$$\int_a^b K(x,y)d\mu_1(x) \ge \int_a^{a+\lambda} K(x,y)dx. \tag{3.6}$$

Since the right-hand side in (3.6) is non-negative, (3.3) is necessary. Now taking $a \le y \le b$, (3.6) is

$$\int_y^b (g(x)-g(y))^{\alpha-1}d\mu_1(x) \ge \int_y^{a+\lambda}(g(x)-g(y))^{\alpha-1}dx, \quad a \le y \le a+\lambda. \tag{3.7}$$

But since (3.3) holds, the inequality (3.7) is true for $y > a+\lambda$. Hence, (3.4) follows.

Conversely, let (3.3) hold and λ be such that (3.4) holds. As above, we see that (3.3) and (3.4) are obtained from (2.6). Now applying Theorem 2.2 it follows that (2.1) holds. Furthermore, from (2.1) we have

$$\int_a^b \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(y)f_1(y)}{(g(x)-g(y))^{1-\alpha}}dy\,d\mu_1(x) \ge \int_a^{a+\lambda}\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(y)f_1(y)}{(g(x)-g(y))^{1-\alpha}}dy\,dx. \tag{3.8}$$

Using Fubini's theorem, the right-hand side in (3.8) can be written as

$$\frac{1}{\Gamma(\alpha)}\int_a^{a+\lambda} g'(y)f_1(y)\int_y^{a+\lambda}(g(x)-g(y))^{\alpha-1}dx\,dy.$$

So we obtain (3.2). Hence, the proof is completed. □

Theorem 3.2. *Let g be an increasing function on (a,b) such that g' is continuous on (a,b), let μ_3 be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every non-negative Borel measurable function f_1*

$$\int_a^b (I^\alpha_{a+;g}f_1)(x)d\mu_3(x) \le \frac{1}{\Gamma(\alpha)}\left[\int_a^{b-\lambda} g'(y)f_1(y)\int_{b-\lambda}^b (g(x)-g(y))^{\alpha-1}dx\,dy + \int_{b-\lambda}^b g'(y)f_1(y)\int_y^b (g(x)-g(y))^{\alpha-1}dx\,dy\right]$$

if and only if

$$\int_y^b (g(x)-g(y))^{\alpha-1}d\mu_3(x) \le \int_{b-\lambda}^b \chi_{[y,b]}(g(x)-g(y))^{\alpha-1}dx, \quad y\in[a,b]. \tag{3.9}$$

Proof. Similar to the proof of Theorem 3.1, apply Theorem 2.2 with $\Omega_1=\Omega_2=[a,b]$, $d\mu_2(y)=f_1(y)dy$, $d\mu_1(x)=\chi_{[b-\lambda,b]}dx$ and K defined by (3.5). □

Remark 7. If $g(x)=x$, then $I^\alpha_{a+;x}f$ reduces to $I^\alpha_{a+}f$.

Now we continue with the definition of Hadamard type fractional integrals. Let (a,b) be finite or infinite interval on $\mathbb{R}_+$ and $\alpha>0$. The left-sided Hadamard type fractional integral of order $\alpha>0$ is defined by

$$(J^\alpha_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{y}\right)^{\alpha-1}\frac{f(y)dy}{y}, \quad x>a.$$

Notice that Hadamard fractional integral of order α is a special case of the fractional integral of a function f with respect to another function $g(x)=\log x$ on $[a,b]$ where $0\le a<b\le\infty$.

Corollary 3.1. *Let μ_1 be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every non-negative Borel measurable function f_1*

$$\int_a^b (J^\alpha_{a+}f_1)(x)d\mu_1(x) \ge \frac{1}{\Gamma(\alpha)}\int_a^{a+\lambda} f_1(y)\int_0^{\log\frac{a+\lambda}{y}} x^{\alpha-1}e^x dx\,dy$$

if and only if

$$\int_y^b \left(\log\frac{x}{y}\right)^{\alpha-1}d\mu_1(x)\ge 0, \quad y\in[a,b] \tag{3.10}$$

and

$$\int_y^b \left(\log\frac{x}{y}\right)^{\alpha-1}d\mu_1(x) \ge y\int_0^{\log\frac{a+\lambda}{y}} x^{\alpha-1}e^x dx. \tag{3.11}$$

Proof. Apply Theorem 3.1 for $g(x) = \log x$. □

Corollary 3.2. *Let μ_3 be $\sigma-$finite (signed) regular Borel measure on $[a, b]$. Then for every non-negative Borel measurable function f_1*

$$\int_a^b (J_{a+}^\alpha f_1)(x)d\mu_3(x) \leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{b-\lambda} f_1(y)\int_{\log\frac{b-\lambda}{y}}^{\log\frac{b}{y}} x^{\alpha-1}e^x dx\,dy + \int_{b-\lambda}^b f_1(y)\int_0^{\log\frac{b}{y}} x^{\alpha-1}e^x dx\,dy\right] \tag{3.12}$$

if and only if

$$\int_y^b \left(\log\frac{x}{y}\right)^{\alpha-1} d\mu_3(x) \leq y\int_{\log\frac{b-\lambda}{y}}^{\log\frac{b}{y}} \chi_{[0,\log\frac{b}{y}]}x^{\alpha-1}e^x dx, \quad y \in [a, b]. \tag{3.13}$$

Proof. Apply Theorem 3.2 for $g(x) = \log x$. □

Now will give the definition of Erdelyi-Kóber type fractional integrals. For details see [10] (see also [4, p, 154]).

Let (a, b) $(0 \leq a < b \leq \infty)$ be finite or infinite interval on $\mathbb{R}^+$. Let $\alpha > 0, \sigma > 0$, and $\eta \in \mathbb{R}$. The left-sided Erdelyi-Kóber type fractional integral of order $\alpha > 0$ is defined by

$$(I_{a+;\sigma;\eta}^\alpha f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x \frac{y^{\sigma\eta+\sigma-1}f(y)dy}{(x^\sigma - y^\sigma)^{1-\alpha}}, \quad x > a.$$

Theorem 3.3. *Let μ_1 be $\sigma-$finite (signed) regular Borel measure on $[a, b]$. Then for every non-negative Borel measurable function f_1*

$$\int_a^b (I_{a+;\sigma;\eta}^\alpha f_1)(x)d\mu_1(x) \geq \frac{1}{\Gamma(\alpha)}\int_a^{a+\lambda} y^{\sigma\eta+\sigma-1}f_1(y)\int_{y^\sigma}^{(a+\lambda)^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x - y^\sigma)^{1-\alpha}}dx\,dy \tag{3.14}$$

if and only if

$$\int_y^b \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}}d\mu_1(x) \geq 0, \quad y \in [a, b] \tag{3.15}$$

and

$$\int_y^b \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}}d\mu_1(x) \geq \frac{1}{\sigma}\int_{y^\sigma}^{(a+\lambda)^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x - y^\sigma)^{1-\alpha}}dx. \tag{3.16}$$

Proof. Let (3.14) hold. Let $\Omega_1 = \Omega_2 = [a, b]$, λ be non-negative real number such that $a + \lambda \leq b$,

$$K(x, y) = \begin{cases} \frac{1}{\Gamma(\alpha)}\frac{\sigma x^{-\sigma(\alpha+\eta)}y^{\sigma\eta+\sigma-1}}{(x^\sigma-y^\sigma)^{1-\alpha}}, & a \leq y \leq x; \\ 0, & x < y \leq b, \end{cases} \tag{3.17}$$

$d\mu_2(y) = f_1(y)dy$ and $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$. Notice that class U now reduces to class of functions $I^{\alpha}_{a+;g}f_1$ and that (2.1) reduces to (3.14). Now, from Theorem 2.2 it follows that (2.6) holds. Furthermore, (2.6) reduces to

$$\int_a^b K(x,y)d\mu_1(x) \geq \int_a^{a+\lambda} K(x,y)dx. \tag{3.18}$$

Since the right-hand side in (3.18) is non-negative, (3.15) is necessary. Now taking $a \leq y \leq b$, (3.18) is

$$\int_y^b \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} d\mu_1(x) \geq \int_y^{a+\lambda} \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} dx, \quad a \leq y \leq a+\lambda. \tag{3.19}$$

But since (3.15) holds, the inequality (3.19) is true for $y > a+\lambda$. Calculating integral on the right-hand side in (3.19), we obtain (3.16).

Conversely, let (3.15) hold and λ be such that (3.16) holds. As above, we see that (3.15) and (3.16) are obtained from (2.6). Now applying Theorem 2.2 it follows that (2.1) holds. Furthermore, from (2.1) we have

$$\begin{aligned} \int_a^b \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\sigma x^{-\sigma(\alpha+\eta)} y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} f_1(y)dy\, d\mu_1(x) \geq \\ \int_a^{a+\lambda} \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\sigma x^{-\sigma(\alpha+\eta)} y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} f_1(y)dy\, dx. \end{aligned} \tag{3.20}$$

Using Fubini's theorem, the right-hand side in (3.20) can be written as

$$\frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} \sigma y^{\sigma\eta+\sigma-1} f_1(y) \int_y^{a+\lambda} \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} dx\, dy. \tag{3.21}$$

Now using the definition of Erdelyi-Köber fractional integral and calculating the inner integral in (3.21) we obtain (3.14). Hence, the proof is completed. □

Theorem 3.4. *Let μ_3 be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every non-negative Borel measurable function f_1*

$$\begin{aligned} \int_a^b (I^{\alpha}_{a+;\sigma;\eta}f_1)(x)d\mu_3(x) \leq \frac{1}{\Gamma(\alpha)} \Bigg[\int_a^{b-\lambda} y^{\sigma\eta+\sigma-1} f_1(y) \int_{(b-\lambda)^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx\, dy \\ + \int_{b-\lambda}^b y^{\sigma\eta+\sigma-1} f_1(y) \int_{y^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx\, dy \Bigg] \end{aligned}$$

if and only if

$$\int_y^b \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} d\mu_3(x) \leq \int_{(b-\lambda)^\sigma}^{b^\sigma} \chi_{[y^\sigma,b^\sigma]} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx, \quad y \in [a,b]. \tag{3.22}$$

Proof. Similar to the proof of Theorem 3.3, apply Theorem 2.2 with $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_2(y) = f_1(y)dy$, $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$ and K defined by (3.17). □

In the previous theorems we derived only Steffensen type inequalities over some subsets of $\mathbb{R}$. Motivated by [5] we will show that Theorem 2.2 covers much more general situations. Let us observe multidimensional fractional integrals. Such type of fractional integrals are usually generalization of the corresponding one-dimensional fractional integral and fractional derivative. For details see [7].
For $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, we use the following notations:

$$\Gamma(\boldsymbol{\alpha}) = (\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)), \quad [\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n], \quad \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

and by $\mathbf{x} > \mathbf{a}$ we mean $x_1 > a_1, ..., x_n > a_n$.
We define the mixed Riemann-Liouville fractional integral of order $\alpha > 0$ as

$$(I_{\mathbf{a}+}^{\boldsymbol{\alpha}} f)(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} f(\mathbf{t})(\mathbf{x} - \mathbf{t})^{\boldsymbol{\alpha} - \mathbf{1}} d\mathbf{t}, \quad (\mathbf{x} > \mathbf{a}).$$

Let $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$, $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{t} = (t_1, \dots, t_n)$ and let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be non-negative such that $\mathbf{a} + \boldsymbol{\lambda} \le \mathbf{b}$ and $\mathbf{b} - \boldsymbol{\lambda} \ge \mathbf{a}$. Now we will give Steffensen type inequalities for mixed Riemann-Liouville fractional integrals.

Theorem 3.5. *Let μ_1 be $\sigma-$finite (signed) regular Borel measure on $[\mathbf{a}, \mathbf{b}]$. Then for every non-negative Borel measurable function f_1*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} (I_{\mathbf{a}+}^{\boldsymbol{\alpha}} f_1)(\mathbf{x}) d\mu_1(\mathbf{x}) \ge (I_{\mathbf{a}+}^{\boldsymbol{\alpha}+\mathbf{1}} f_1)(\mathbf{a} + \boldsymbol{\lambda}) \tag{3.23}$$

if and only if

$$\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha} - \mathbf{1}} d\mu_1(\mathbf{x}) \ge 0, \quad \mathbf{t} \in [\mathbf{a}, \mathbf{b}] \tag{3.24}$$

and

$$\prod_{i=1}^{n} \frac{(a_i + \lambda_i - y_i)_+^{\alpha_i}}{\alpha_i} \le \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha} - \mathbf{1}} d\mu_1(\mathbf{x}). \tag{3.25}$$

Proof. Let $\Omega_1 = \Omega_2 = [\mathbf{a}, \mathbf{b}]$,

$$K(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{(\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha} - \mathbf{1}}}{\Gamma(\boldsymbol{\alpha})}, & \mathbf{a} \le \mathbf{y} \le \mathbf{x}; \\ 0, & \text{otherwise} , \end{cases} \tag{3.26}$$

$d\mu_2(y) = f(\mathbf{y})d\mathbf{y}$ and $d\mu_3(x) = \chi_{[\mathbf{a}, \mathbf{a}+\boldsymbol{\lambda}]} d\mathbf{x}$. Notice that class U now reduces to class of functions $I_{\mathbf{a}+}^{\boldsymbol{\alpha}} f_1$. Applying Theorem 2.2, from (2.1) we obtain

$$\int_{\mathbf{a}}^{\mathbf{b}} \frac{1}{\Gamma(\boldsymbol{\alpha})} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \frac{f(\mathbf{y})}{(\mathbf{x} - \mathbf{y})^{\mathbf{1} - \boldsymbol{\alpha}}} d\mathbf{y}\, d\mu_1(\mathbf{x}) \ge \int_{\mathbf{a}}^{\mathbf{a}+\boldsymbol{\lambda}} \frac{1}{\Gamma(\boldsymbol{\alpha})} \int_{\mathbf{a}}^{\mathbf{x}} (\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha} - \mathbf{1}} f(\mathbf{y}) d\mathbf{y}\, d\mathbf{x}. \tag{3.27}$$

Using Fubini's theorem and then calculating the inner integral, the right-hand side in (3.27) can be written as

$$\frac{1}{\Gamma(\boldsymbol{\alpha} + \mathbf{1})} \int_{a_1}^{a_1 + \lambda_1} \cdots \int_{a_n}^{a_n + \lambda_n} f(\mathbf{y})(\mathbf{a} + \boldsymbol{\lambda} - \mathbf{y})^{\boldsymbol{\alpha}} d\mathbf{y}.$$

So we obtain (3.23). From (2.6) we obtain

$$\int_{\mathbf{a}}^{\mathbf{b}} K(\mathbf{x},\mathbf{y})d\mu_1(\mathbf{x}) \geq \int_{\mathbf{a}}^{\mathbf{a}+\boldsymbol{\lambda}} K(\mathbf{x},\mathbf{y})d\mathbf{x}. \tag{3.28}$$

Since the right-hand side in (3.28) is non-negative, (3.24) is necessary. Now taking $\mathbf{a} \leq \mathbf{y} \leq \mathbf{b}$, (3.28) is

$$\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}-\mathbf{1}} d\mu_1(\mathbf{x}) \geq \int_{y_1}^{a_1+\lambda_1} \cdots \int_{y_n}^{a_n+\lambda_n} (\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}-\mathbf{1}} d\mathbf{x}, \quad \mathbf{a} \leq \mathbf{y} \leq \mathbf{a}+\boldsymbol{\lambda}. \tag{3.29}$$

Calculating integral on the right-hand side in (3.29), we obtain

$$\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}-\mathbf{1}} d\mu_1(\mathbf{x}) \geq \prod_{i=1}^{n} \frac{(a_i+\lambda_i-y_i)^{\alpha_i}}{\alpha_i}, \quad a_i \leq y_i \leq a_i+\lambda_i, i=1,\ldots,n. \tag{3.30}$$

Hence, (3.25) follows and the proof is completed. □

Theorem 3.6. *Let μ_3 be $\sigma-$finite (signed) regular Borel measure on $[\mathbf{a},\mathbf{b}]$. Then for every non-negative Borel measurable function f_1*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} (I_{\mathbf{a}+}^{\boldsymbol{\alpha}} f_1)(\mathbf{x}) d\mu_3(\mathbf{x}) \leq (I_{\mathbf{a}+}^{\boldsymbol{\alpha}+\mathbf{1}} f_1)(\mathbf{b}) - (I_{\mathbf{a}+}^{\boldsymbol{\alpha}+\mathbf{1}} f_1)(\mathbf{b}-\boldsymbol{\lambda}) \tag{3.31}$$

if and only if

$$\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}-\mathbf{1}} d\mu_3(\mathbf{x}) \leq \int_{b_1-\lambda_1}^{b_1} \cdots \int_{b_n-\lambda_n}^{b_n} \chi_{[\mathbf{y},\mathbf{b}]} (\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}-\mathbf{1}} d\mathbf{x}. \tag{3.32}$$

Proof. Let $\Omega_1 = \Omega_2 = [\mathbf{a},\mathbf{b}]$, K be defined by (3.26), $d\mu_2(y) = f_1(\mathbf{y})d\mathbf{y}$ and $d\mu_1(x) = \chi_{[\mathbf{b}-\boldsymbol{\lambda},\mathbf{b}]} d\mathbf{x}$. Notice that class U now reduces to class of functions $I_{\mathbf{a}+}^{\boldsymbol{\alpha}} f_1$. Applying Theorem 2.2, from (2.1) we obtain

$$\int_{\mathbf{a}}^{\mathbf{b}} \frac{1}{\Gamma(\boldsymbol{\alpha})} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \frac{f_1(\mathbf{y})}{(\mathbf{x}-\mathbf{y})^{\mathbf{1}-\boldsymbol{\alpha}}} d\mathbf{y}\, d\mu_1(\mathbf{x}) \leq \int_{\mathbf{b}-\boldsymbol{\lambda}}^{\mathbf{b}} \frac{1}{\Gamma(\boldsymbol{\alpha})} \int_{\mathbf{a}}^{\mathbf{x}} (\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}-\mathbf{1}} f_1(\mathbf{y}) d\mathbf{y}\, d\mathbf{x}. \tag{3.33}$$

Using Fubini's theorem and then calculating the inner integral, the right-hand side in (3.33) can be written as

$$\frac{1}{\Gamma(\boldsymbol{\alpha}+\mathbf{1})} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f_1(\mathbf{y})(\mathbf{b}-\mathbf{y})^{\boldsymbol{\alpha}} d\mathbf{y} - \frac{1}{\Gamma(\boldsymbol{\alpha}+\mathbf{1})} \int_{a_1}^{b_1-\lambda_1} \cdots \int_{a_n}^{b_n-\lambda_n} f_1(\mathbf{y})(\mathbf{b}-\boldsymbol{\lambda}-\mathbf{y})^{\boldsymbol{\alpha}} d\mathbf{y}.$$

So we obtain (3.31). From (2.6) we obtain

$$\int_{\mathbf{a}}^{\mathbf{b}} K(\mathbf{x},\mathbf{y})d\mu_3(\mathbf{x}) \leq \int_{\mathbf{b}-\boldsymbol{\lambda}}^{\mathbf{b}} K(\mathbf{x},\mathbf{y})d\mathbf{x}.$$

i.e.

$$\int_{\mathbf{y}}^{\mathbf{b}} (\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}-\mathbf{1}} d\mu_3(\mathbf{x}) \leq \int_{\mathbf{b}-\boldsymbol{\lambda}}^{\mathbf{b}} \chi_{[\mathbf{y},\mathbf{b}]} (\mathbf{x}-\mathbf{y})^{\boldsymbol{\alpha}-\mathbf{1}} d\mathbf{x}.$$

Hence, (3.32) follows and the proof is completed. □

4 Mean value theorems

First, let us recall that U denotes the class of all functions $f : \Omega_1 \to \mathbb{R}$ such that there exists a measure space $(\Omega_2, \Sigma_2, \mu_2)$ such that μ_2 is non-negative $\sigma-$finite regular Borel measure and (2.1) holds. Now, we will define linear functionals which will be used in following theorems. For $f \in U$ let

$$A(f) = \int_{\Omega_1} f(x)d\mu_1(x) - \int_{\Omega_1} f(x)d\mu_3(x). \tag{4.1}$$

Now, we will give linear functionals related to fractional integrals and derivatives mentioned in Section 3. First, we define linear functionals related to fractional integral of a function f with respect to another function g.
Let

$$L_1(f_1) = \int_a^b (I^\alpha_{a+;g} f_1)(x)d\mu_1(x) - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} g'(y)f_1(y) \int_y^{a+\lambda} (g(x) - g(y))^{\alpha-1} dx\, dy, \tag{4.2}$$

and let

$$\begin{aligned} L_2(f_1) =& \frac{1}{\Gamma(\alpha)} \left[\int_a^{b-\lambda} g'(y)f_1(y) \int_{b-\lambda}^{b} (g(x) - g(y))^{\alpha-1} dx\, dy \right. \\ &+ \left. \int_{b-\lambda}^{b} g'(y)f_1(y) \int_y^b (g(x) - g(y))^{\alpha-1} dx\, dy \right] - \int_a^b (I^\alpha_{a+;g} f_1)(x)d\mu_3(x), \end{aligned} \tag{4.3}$$

where f_1 is non-negative Borel measurable function. Next, we define linear functionals related to Hadamard type fractional integral. Let

$$L_3(f_1) = \int_a^b (J^\alpha_{a+} f)(x)d\mu_1(x) - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} f_1(y) \int_0^{\log \frac{a+\lambda}{y}} x^{\alpha-1} e^x dx\, dy, \tag{4.4}$$

and let

$$\begin{aligned} L_4(f_1) = \frac{1}{\Gamma(\alpha)} &\left[\int_a^{b-\lambda} f_1(y) \int_{\log \frac{b-\lambda}{y}}^{\log \frac{b}{y}} x^{\alpha-1} e^x dx\, dy \right. \\ &+ \left. \int_{b-\lambda}^{b} f_1(y) \int_0^{\log \frac{b}{y}} x^{\alpha-1} e^x dx\, dy \right] - \int_a^b (J^\alpha_{a+} f_1)(x)d\mu_3(x), \end{aligned} \tag{4.5}$$

where f_1 is non-negative Borel measurable function.
Finally, we define linear functionals related to Erdelyi-Kóber type fractional integral.
Let

$$L_5(f_1) = \int_a^b (I^\alpha_{a+;\sigma;\eta} f_1)(x)d\mu_1(x) - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} y^{\sigma\eta+\sigma-1} f_1(y) \int_{y^\sigma}^{(a+\lambda)^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x - y^\sigma)^{1-\alpha}} dx\, dy, \tag{4.6}$$

and let

$$L_6(f_1) = \frac{1}{\Gamma(\alpha)}\left[\int_a^{b-\lambda} y^{\sigma\eta+\sigma-1} f_1(y) \int_{(b-\lambda)^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx\, dy \right.$$
$$\left. + \int_{b-\lambda}^{b} y^{\sigma\eta+\sigma-1} f_1(y) \int_{y^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx\, dy \right] - \int_a^b (I^\alpha_{a+;\sigma;\eta} f)(x) d\mu_3(x). \tag{4.7}$$

where f_1 is non-negative Borel measurable function.

Now we state and prove Lagrange-type mean value theorems.

Theorem 4.1. *Let Ω_1 be a compact set. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_1, \Sigma_3, \mu_3)$ be measure spaces with $\sigma-$finite regular Borel measures, let (2.6) hold and let $f \in C(\Omega_1)$. Then there exists $\xi \in \Omega_1$ such that*

$$A(f) = f(\xi)\left(\int_{\Omega_1} d\mu_1(x) - \int_{\Omega_1} d\mu_3(x)\right), \tag{4.8}$$

where A is defined by (4.1).

Proof. Notice that from Theorem 2.2 we have that if $f \geq 0$, then $A(f) \geq 0$, so A is positive linear functional.
Since f is continuous on Ω_1, there exists $m = \min_{x\in\Omega_1} f(x)$ and $M = \max_{x\in\Omega_1} f(x)$. Then $A(M-f) \geq 0$ and $A(f-m) \geq 0$. Therefore

$$m\left(\int_{\Omega_1} d\mu_1(x) - \int_{\Omega_1} d\mu_3(x)\right) \leq \int_{\Omega_1} f(x)d\mu_1(x) - \int_{\Omega_1} f(x)d\mu_3(x)$$
$$\leq M\left(\int_{\Omega_1} d\mu_1(x) - \int_{\Omega_1} d\mu_3(x)\right)$$

that is,

$$mA(1) \leq A(f) \leq MA(1).$$

If the function $A(1) = 0$, then $A(f) = 0$, so (4.8) holds for all $\xi \in \Omega_1$. Otherwise,

$$\min_{x\in\Omega_1} f(x) = m \leq \frac{A(f)}{A(1)} \leq M = \max_{x\in\Omega_1} f(x), \text{ so } \frac{A(f)}{A(1)} \in f(\Omega_1).$$

Since f is continuous, we have that $\frac{A(f)}{A(1)} = f(\xi)$ for some $\xi \in \Omega_1$. □

Theorem 4.2. *Let g be an increasing function on (a,b) such that g' is continuous on (a,b) and let f_1 be non-negative Borel measurable function such that $f_1 \in C([a,b])$. Let μ_1 be $\sigma-$finite (signed) regular Borel measure, let (3.3) and (3.4) hold. Then there exists $\xi \in [a,b]$ such that*

$$L_1(f_1) = \frac{f_1(\xi)}{\Gamma(\alpha)}\left(\int_a^b \frac{(g(x)-g(a))^\alpha}{\alpha} d\mu_1(x) - \int_a^{a+\lambda} g'(y) \int_y^{a+\lambda} (g(x)-g(y))^{\alpha-1} dx\, dy\right), \tag{4.9}$$

where L_1 is defined by (4.2).

Proof. Notice that from Theorem 3.1 we have that if $f_1 \geq 0$, then $L_1(f_1) \geq 0$, so L_1 is positive linear functional.
Set $m = \min_{x\in[a,b]} f_1(x)$, $M = \max_{x\in[a,b]} f_1(x)$. Then $L_1(M - f_1) \geq 0$ and $L_1(f_1 - m) \geq 0$. Using definition of the left-sided fractional integral of a function f_1 with respect to another function g, linear functional L_1 can be written as

$$L_1(f_1) = \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^x \frac{g'(y) f_1(y)}{(g(x) - g(y))^{1-\alpha}} dy\, d\mu_1(x) - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} g'(y) f_1(y) \int_y^{a+\lambda} (g(x) - g(y))^{\alpha-1} dx\, dy.$$

Therefore

$$\begin{aligned} &\frac{m}{\Gamma(\alpha)} \left(\int_a^b \frac{(g(x) - g(a))^\alpha}{\alpha} d\mu_1(x) - \int_a^{a+\lambda} g'(y) \int_y^{a+\lambda} (g(x) - g(y))^{\alpha-1} dx\, dy \right) \\ &\leq \int_a^b (I^\alpha_{a+;g} f_1)(x) d\mu_1(x) - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} g'(y) f_1(y) \int_y^{a+\lambda} (g(x) - g(y))^{\alpha-1} dx\, dy \\ &\leq \frac{M}{\Gamma(\alpha)} \left(\int_a^b \frac{(g(x) - g(a))^\alpha}{\alpha} d\mu_1(x) - \int_a^{a+\lambda} g'(y) \int_y^{a+\lambda} (g(x) - g(y))^{\alpha-1} dx\, dy \right), \end{aligned}$$

that is,

$$mL_1(1) \leq L_1(f_1) \leq ML_1(1).$$

Similar reasoning as in proof of Theorem 4.1 completes the proof. □

Theorem 4.3. *Let g be an increasing function on (a, b) such that g' is continuous on (a, b) and let f_1 be non-negative Borel measurable function such that $f_1 \in C([a, b])$. Let μ_3 be $\sigma-$finite (signed) regular Borel measure and let* (3.9) *hold. Then there exists $\xi \in [a, b]$ such that*

$$L_2(f_1) = \frac{f_1(\xi)}{\Gamma(\alpha)} \left(\int_a^{b-\lambda} g'(y) \int_{b-\lambda}^b (g(x) - g(y))^{\alpha-1} dx\, dy + \int_{b-\lambda}^b g'(y) \int_y^b (g(x) - g(y))^{\alpha-1} dx\, dy - \int_a^b \frac{(g(x) - g(a))^\alpha}{\alpha} d\mu_3(x) \right), \tag{4.10}$$

where L_2 is defined by (4.3)*.*

Proof. Similar to the proof of Theorem 4.2. □

Theorem 4.4. *Let μ_1 be $\sigma-$finite (signed) regular Borel measure, let* (3.10) *and* (3.11) *hold and let f_1 be non-negative Borel measurable function such that $f_1 \in C([a, b])$,*

$a > 0$. Then there exists $\xi \in [a,b]$ such that

$$L_3(f_1) = f_1(\xi)\left(\frac{1}{\Gamma(\alpha+1)}\int_a^b \left(\log\frac{x}{a}\right)^{\alpha} d\mu_1(x) - \frac{1}{\Gamma(\alpha)}\int_a^{a+\lambda}\int_0^{\log\frac{a+\lambda}{y}} x^{\alpha-1}e^{x}dx\,dy\right), \tag{4.11}$$

where L_3 is defined by (4.4).

Proof. Similar to the proof of Theorem 4.2. □

Theorem 4.5. *Let μ_3 be $\sigma-$finite (signed) regular Borel measure, let* (3.13) *hold and let f_1 be non-negative Borel measurable function such that $f_1 \in C([a,b])$, $a > 0$. Then there exists $\xi \in [a,b]$ such that*

$$\begin{aligned} L_4(f_1) =& f_1(\xi)\left(\frac{1}{\Gamma(\alpha)}\int_a^{b-\lambda}\int_{\log\frac{b-\lambda}{y}}^{\log\frac{b}{y}} x^{\alpha-1}e^{x}dx\,dy\right. \\ &\left.+\frac{1}{\Gamma(\alpha)}\int_{b-\lambda}^{b}\int_0^{\log\frac{a+\lambda}{y}} x^{\alpha-1}e^{x}dx\,dy - \frac{1}{\Gamma(\alpha+1)}\int_a^b\left(\log\frac{x}{a}\right)^{\alpha} d\mu_3(x)\right), \end{aligned} \tag{4.12}$$

where L_4 is defined by (4.5).

Proof. Similar to the proof of Theorem 4.2. □

Before we give Lagrange type mean value theorems for Erdelyi-Kóber type fractional integrals let us denote

$$HG(x) = {}_2F_1(1-\alpha, \eta+1; \eta+2, x^{\sigma}).$$

Theorem 4.6. *Let μ_1 be $\sigma-$finite (signed) regular Borel measure, let* (3.15) *and* (3.16) *hold and let f_1 be non-negative Borel measurable function such that $f_1 \in C([a,b])$. Then there exists $\xi \in [a,b]$ such that*

$$\begin{aligned} L_5(f_1) = f_1(\xi)&\left(\frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)}\int_a^b d\mu_1(x) - \frac{1}{(\eta+1)\Gamma(\alpha)}\int_a^b\left(\frac{a}{x}\right)^{\sigma\eta+\sigma} HG\left(\frac{a}{x}\right) d\mu_1(x)\right. \\ &\left.-\frac{1}{\Gamma(\alpha)}\int_a^{a+\lambda} y^{\sigma\eta+\sigma-1}\int_{y^{\sigma}}^{(a+\lambda)^{\sigma}} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^{\sigma})^{1-\alpha}}dx\,dy\right), \end{aligned} \tag{4.13}$$

where L_5 is defined by (4.6).

Proof. Similar to the proof of Theorem 4.2. □

Theorem 4.7. *Let μ_3 be $\sigma-$finite (signed) regular Borel measure, let* (3.22) *hold and let f_1 be non-negative Borel measurable function such that $f_1 \in C([a,b])$. Then there* *exists* $\xi \in [a,b]$ *such that*

$$L_6(f_1) = f_1(\xi)\Bigg(\frac{1}{\Gamma(\alpha)}\int_a^b y^{\sigma\eta+\sigma-1} f_1(y)\int_{(b-\lambda)^\sigma}^{b^\sigma}\frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}}dx\,dy$$
$$\frac{1}{\Gamma(\alpha)}\int_{b-\lambda}^b y^{\sigma\eta+\sigma-1} f_1(y)\int_{y^\sigma}^{b^\sigma}\frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}}dx\,dy \tag{4.14}$$
$$-\frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)}\int\limits_a^b d\mu_3(x) + \frac{1}{(\eta+1)\Gamma(\alpha)}\int_a^b\left(\frac{a}{x}\right)^{\sigma\eta+\sigma} HG\left(\frac{a}{x}\right)d\mu_3(x)\Bigg),$$

where L_6 *is defined by* (4.7).

Proof. Similar to the proof of Theorem 4.2. □

Following theorems are new analogues of the classical Cauchy mean value theorem.

Theorem 4.8. *Let conditions of Theorem 4.1 be satisfied and let* $f_1, f_2 \in C(\Omega_1)$ *be such that* $f_2(x) \neq 0$ *for every* $x \in \Omega_1$. *Then there exists* $\xi \in \Omega_1$ *such that*

$$\frac{f_1(\xi)}{f_2(\xi)} = \frac{A(f_1)}{A(f_2)}. \tag{4.15}$$

Proof. Set $\Phi(t) = f_1(t)A(f_2) - f_2(t)A(f_1)$. Obviously, $A(\Phi) = 0$. On the other hand, Theorem 4.1 yields that there exists $\xi \in [a,b]$ such that $A(\Phi) = \Phi(\xi)\cdot A(1)$. Since $A(1) \neq 0$, we have that

$$\Phi(\xi) = f_1(\xi)A(f_2) - f_2(\xi)A(f_1) = 0.$$

By the assumption $f_2(\xi) \neq 0$, so Theorem 2.2 assures that $A(f_2) \neq 0$. Thus, (4.15) follows. □

Theorem 4.9. *Let conditions of Theorems 4.2 – 4.7 be satisfied and let* $f_1, f_2 \in C([a,b])$ *be such that* $f_2(x) \neq 0$ *for every* $x \in [a,b]$. *Then there exists* $\xi_i \in [a,b]$ *such that*

$$\frac{f_1(\xi_i)}{f_2(\xi_i)} = \frac{L_i(f_1)}{L_i(f_2)}, \quad i = 1,2,3,4,5,6, \tag{4.16}$$

where L_i, $i = 1,2,3,4,5,6$ *are linear functionals defined by* (4.2)-(4.7).

Proof. Similar to the proof of Theorem 4.8. □

Remark 8. Theorem 4.9 enables us to define new types of means, because if f_1/f_2 has an inverse, from (4.16) we conclude that

$$\xi_i = \left(\frac{f_1}{f_2}\right)^{-1}\left(\frac{L_i(f_1)}{L_i(f_2)}\right), \quad i = 1,2,3,4,5,6. \tag{4.17}$$

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References

[1] G.A. Anastassiou, *Fractional Differentiation Inequalities.* Springer Science+Businness Media, LLC, 2009.

[2] M. Andrić, J. Pečarić, I. Perić, *An multiple Opial type inequality due to Fink for Riemann-Liouville fractional derivatives*, (accepted for publication in J. Math. Inequal.)

[3] M. Andrić, J. Pečarić, I. Perić, *Improvements of composition rule for Canavati fractional derivative and applications to Opial-type inequalities.* Dynam. Systems Appl. 20 (2011), 383–394.

[4] D.E. Edmunds, V. Kokilashvili, A. Meskhi, *Bounded and Compact Integral Operators.* Kluwer Academic Publishers, 101, Philip Drive, Norwell, MA02061, USA.

[5] A.M. Fink, *Steffensen type inequalities.* Rocky Mountain J. Math. 12, no. 4 (1982), 785–793.

[6] G.D. Handley, J.J. Koliha, J.E. Pečarić, *Hilbert-Pachpatte type integral inequalities for fractional derivatives.* Fract. Calc. Appl. Anal. 4 (2001), 37–46.

[7] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204 , Elsevier. New York-London, 2006.

[8] K. Krulić, J. Pečarić, K. Smoljak, *Fractional Steffensen type inequalities.* Complex Var. Elliptic Equ. 56 (10-11) (2011), 975–990.

[9] J.E. Pečarić, F. Proschan, Y.L. Tong, *Convex functions, partial orderings, and statistical applications.* Academic Press, San Diego, 1992.

[10] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications.* London-New-York: Gordon and Breach. Sci. Publ., 1993.

Josip Pečarić
Faculty of Textile Technology, University of Zagreb,
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
E-mail: pecaric@element.hr

Ivan Perić
Faculty of Food Technology and Biotechnology, University of Zagreb,
Pierottijeva 6, 10000 Zagreb, Croatia
E-mail: iperic@pbf.hr

Ksenija Smoljak
Faculty of Textile Technology, University of Zagreb,
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
E-mail: ksmoljak@ttf.hr

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