

**BRENNAN'S CONJECTURE FOR COMPOSITION OPERATORS
ON SOBOLEV SPACES**

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Communicated by V.I. Burenkov

Key words: Brennan's conjecture, conformal mappings, composition operators, Sobolev spaces.

AMS Mathematics Subject Classification: 30C35, 46E35.

Abstract. We show that Brennan's conjecture is equivalent to the boundedness of composition operators on homogeneous Sobolev spaces, that are generated by conformal homeomorphisms of simply connected plane domains to the unit disc. A geometrical interpretation of Brennan's conjecture in terms of integrability of p -distortion is given.

1 Introduction

The purpose of this paper is to point out a connection between Brennan's conjecture about integrability of derivatives of plane conformal mappings and boundedness of composition operators of homogeneous Sobolev spaces $L_p^1(\mathbb{D})$ to $L_q^1(\Omega)$ generated by conformal homeomorphisms $\varphi : \Omega \rightarrow \mathbb{D}$ of a simply connected plane domain with nonempty boundary $\Omega \subset \mathbb{R}^2$ to the unit disc $\mathbb{D} \subset \mathbb{R}^2$. Brennan's conjecture goes back to the approximation problem for analytic functions (see, for example, [2, 16, 19]) and is under intensive study at the present time.

We prove that *any conformal homeomorphism $\varphi : \Omega \rightarrow \mathbb{D}$ generates by the composition rule $\varphi^*(f) = f \circ \varphi$ a bounded composition operator*

$$\varphi^* : L_p^1(\mathbb{D}) \rightarrow L_{q(p,s)}^1(\Omega)$$

for any $p \in (2; +\infty)$ and $q(p, s) = ps/(p + s - 2)$. Here the number s is the summability exponent for Brennan's conjecture.

This connection of Brennan's conjecture with Sobolev spaces is new and possible applications to Sobolev type embedding theorems and elliptic boundary value problems are under investigation.

Brennan's conjecture. Brennan's conjecture concerns integrability of derivatives of plane conformal homeomorphisms $\varphi : \Omega \rightarrow \mathbb{D}$ of a simply connected plane domain with nonempty boundary $\Omega \subset \mathbb{R}^2$ to the unit disc $\mathbb{D} \subset \mathbb{R}^2$.

The conjecture [2] is that

$$\int_{\Omega} |\varphi'(z)|^s d\mu < +\infty, \quad \text{for all } \frac{4}{3} < s < 4. \quad (1.1)$$

We used the following notation: $z = x + iy$ is a complex number, $\varphi'(z)$ is complex derivative, $d\mu$ is the Lebesgue measure.

For $4/3 < s < 3$ it is a comparatively easy consequence of the Koebe distortion theorem (see, for example, [2]).

J. Brennan [2] (1973) extended this range to $4/3 < s < 3 + \delta$ where $\delta > 0$, and conjectured it to hold for $4/3 < s < 4$. The example $\Omega = \mathbb{R}^2 \setminus (-\infty, -1/4]$ shows that this range of s cannot be extended. The upper bound of those s for which (1.1) is known to hold has been increased by Ch. Pommerenke to $s \leq 3.399$, and then to $s \leq 3.421$ by D. Bertilsson. These results and additional information can be found in [1, 23].

2 Composition operators on homogeneous Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an Euclidean domain. The homogeneous Sobolev space $L_p^1(\Omega)$, $1 \leq p \leq \infty$, is the space of all locally summable, weakly differentiable functions $f : \Omega \rightarrow \mathbb{R}$ with the finite seminorm

$$\|f | L_p^1(\Omega)\| = \left(\int_{\Omega} |\nabla f(z)|^p d\mu \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \|f | L_{\infty}^1(\Omega)\| = \text{esssup}_{z \in \Omega} |\nabla f(z)|.$$

Let $\varphi : \Omega \rightarrow \Omega'$ be a homeomorphism of Euclidean domains $\Omega, \Omega' \subset \mathbb{R}^n$. We say that φ generates a bounded composition operator $\varphi^* : L_p^1(\Omega') \rightarrow L_q^1(\Omega)$, $1 \leq q \leq p \leq \infty$, by the composition rule $\varphi^*(f) = f \circ \varphi$, if for any $f \in L_p^1(\Omega')$ the function $\varphi^*(f) \in L_q^1(\Omega)$ and there exists a constant $K < \infty$ such that

$$\|\varphi^*(f) | L_q^1(\Omega)\| \leq K \|f | L_p^1(\Omega')\|.$$

Composition problem for homogeneous Sobolev spaces *is the problem about description of homeomorphisms $\varphi : \Omega \rightarrow \Omega'$ that generate bounded composition operators from $L_p^1(\Omega')$ to $L_q^1(\Omega)$, $1 \leq q \leq p \leq \infty$.*

The quasiconformal methods in the study of the composition problem arise in the Reshetnyak's problem stated in 1968. It is the problem about description of all isomorphisms of homogeneous Sobolev spaces $L_n^1(\Omega')$ and $L_n^1(\Omega)$ which are generated by quasiconformal mappings of Euclidean domains. This problem was solved in [28], where it was proved that a mapping $\varphi : \Omega \rightarrow \Omega'$ that maps a domain $\Omega \subset \mathbb{R}^n$ to a domain $\Omega' \subset \mathbb{R}^n$ generates by the composition rule $\varphi^*(f) = f \circ \varphi$ an isomorphism of $L_n^1(\Omega)$ and $L_n^1(\Omega')$ if and only if φ is a quasiconformal homeomorphism.

The homeomorphisms that generate bounded composition operators of Sobolev spaces $L_1^1(\Omega')$ and $L_1^1(\Omega)$ were introduced by V. G. Maz'ya in [18] as a class of sub-area mappings. This pioneering work established a connection between geometrical properties of homeomorphisms and corresponding Sobolev spaces. This study was extended in the book [20], where the theory of multipliers was applied to the change of variable problem in Sobolev spaces. In the framework of the quasiconformal approach to Composition problem it was proved in [29] that homeomorphisms that generate isomorphisms of Sobolev spaces $W_p^1(\Omega')$ and $W_p^1(\Omega)$, $p > n$, are quasiisometric. The composition

problem was studied for Sobolev spaces $W_p^1(\Omega')$ and $W_p^1(\Omega)$, $n - 1 < p < n$, in [5], and Sobolev spaces $W_p^1(\Omega')$ and $W_p^1(\Omega)$, $1 \leq q < n$, in [21]. Analytic characteristics of homeomorphisms that generate bounded composition operators $\varphi^* : L_p^1(\Omega') \rightarrow L_p^1(\Omega)$, $1 \leq p \leq \infty$, were studied in [27]. The geometric description of homeomorphisms that generate bounded composition operators $\varphi^* : L_p^1(\Omega') \rightarrow L_p^1(\Omega)$, $n - 1 < p < \infty$ was obtained in [4].

New problems arise when we study composition operators on Sobolev spaces with decreasing integrability of first weak derivatives. This problem was studied firstly in [24]. Note, that in this case a significant role in the description of the composition operators play (quasi)additive set functions, defined on open sets. This type of composition operators has applications in the Sobolev embedding theory [3, 6].

The main result of [24] is

Theorem A. *A homeomorphism $\varphi : \Omega \rightarrow \Omega'$, $\Omega, \Omega' \subset \mathbb{R}^n$, generates a bounded composition operator*

$$\varphi^* : L_p^1(\Omega') \rightarrow L_q^1(\Omega), \quad 1 \leq q < p < \infty,$$

if and only if $\varphi \in W_{1,\text{loc}}^1(\Omega)$, has finite distortion, and

$$K_{p,q}(\varphi; \Omega) = \left(\int_{\Omega} \left(\frac{|D\varphi(z)|^p}{|J(z, \varphi)|} \right)^{\frac{q}{p-q}} d\mu \right)^{\frac{p-q}{pq}} < \infty.$$

Here $D\varphi(z)$ is a formal Jacobi matrix of φ at $z \in \Omega$ and $J(z, \varphi) = \det D\varphi(z)$ its Jacobian. The norm $|D\varphi(z)|$ of the matrix is the norm of the linear operator defined by this matrix in the Euclidean space \mathbb{R}^n .

In [3] this class of mappings was studied in connection with the Sobolev type embedding theorems. A detailed study of composition operators on Sobolev spaces was given in [31]. The composition operators on Sobolev spaces in the limiting case $p = \infty$ were studied in [7, 8].

If $1 \leq q < p = n$ this integral condition of Theorem A can be expressed in terms of integrability of the conformal distortion of the mapping φ :

$$K(x, \varphi) = \frac{|D\varphi(z)|^n}{|J(z, \varphi)|} \quad \text{a. e. in } \Omega.$$

Existence of conformal distortion almost everywhere in Ω means that f is a mapping of finite distortion i. e. $|D\varphi(z)| = 0$ for almost all $z \in Z = \{z \in \Omega : J(z, \varphi) = 0\}$.

The theory of mappings of finite distortion is under intensive development at the last decades. In series of works geometrical and topological properties of these mappings were studied (see, e.g. [10, 12, 13, 14, 15, 22]).

The generalization of conformal distortion is the notion of so-called p -distortion

$$K_p(z, \varphi) = \frac{|D\varphi(z)|^p}{|J(z, \varphi)|} \quad \text{a. e. in } \Omega.$$

introduced in [4]. On this way mappings with a finite characteristics $K_{p,q}(\varphi; \Omega)$ are natural generalizations of quasiconformal mappings and we call them as (p, q) -quasiconformal homeomorphisms [30] or mappings with bounded (p, q) -distortion [25].

Composition operators on weighted Sobolev spaces were studied in [26]. These operators have applications in weighted Sobolev type embeddings and for degenerate elliptic equations [6, 9].

In the recent decade an interest to composition operators on Sobolev spaces increased [11, 17]. For example, in [17] was proved (by a different method than in [24]) that (p, q) -quasiconformal homeomorphisms generate a bounded composition operators on Sobolev spaces.

3 Composition theorem for conformal homeomorphisms

Let us rewrite Theorem A in the particular case of conformal homeomorphisms of plane domains Ω and Ω' . In this case $|D\varphi(z)|$ is equal to the absolute value of complex derivative $|\varphi'(z)|$ and $|\varphi'(z)|^2 = J(z, \varphi) > 0$ for any conformal homeomorphisms $w = \varphi(z) : \Omega \rightarrow \Omega'$. Note, that conformal homeomorphisms are smooth mappings, hence they belong to $W_{1,\text{loc}}^1(\Omega)$ and have finite distortion.

Composition Theorem. *A conformal homeomorphism $\varphi : \Omega \rightarrow \Omega'$ of plane domains $\Omega, \Omega' \subset \mathbb{R}^2$ generates a bounded composition operator*

$$\varphi^* : L_p^1(\Omega') \rightarrow L_q^1(\Omega), \quad 1 \leq q < p \leq \infty,$$

if and only if

$$\int_{\Omega} |\varphi'(z)|^{\frac{(p-2)q}{p-q}} d\mu < \infty \quad \left(\int_{\Omega} |\varphi'(z)|^q d\mu < +\infty \text{ for } p = \infty \right).$$

Remarks:

1. The theorem was proved for homeomorphisms in the case $1 \leq q < p < \infty$ in [24] and for the limiting case $p = \infty$ in [7, 8].

2. The case $p = q$ is not interesting for the present study because of absence of connections with Brennan's conjecture.

3. Recall, that a conformal homeomorphism $\varphi : \Omega \rightarrow \Omega'$ generates an isometry $\varphi^* : L_2^1(\Omega') \rightarrow L_2^1(\Omega)$ because $|\varphi'(z)|^2 = J(z, \varphi)$.

Indeed, in this case by the direct calculation for $f \in L_2^1(\Omega')$ we have:

$$\begin{aligned} \|\varphi^* f \mid L_2^1(\Omega)\| &= \left(\int_{\Omega} |\nabla(f \circ \varphi(z))|^2 d\mu \right)^{\frac{1}{2}} = \left(\int_{\Omega} |\nabla f|^2(\varphi(z)) \cdot |\varphi'(z)|^2 d\mu \right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega} |\nabla f|^2(\varphi(z)) \cdot J(z, \varphi) d\mu \right)^{\frac{1}{2}} = \left(\int_{\Omega'} |\nabla f|^2(w) d\mu \right)^{\frac{1}{2}} = \|f \mid L_2^1(\Omega')\|. \end{aligned}$$

Equivalence of Brennan's conjecture and Composition problem for conformal homeomorphisms. The Riemann Mapping Theorem states existence of a conformal homeomorphism φ of a simply connected plane domain Ω with nonempty

boundary onto the unit disc $\mathbb{D} \subset \mathbb{R}^2$. Therefore the Composition problem for conformal homeomorphisms $\varphi : \Omega \rightarrow \Omega'$ of simply connected plane domains with nonempty boundary can be reduced to case $\Omega' = \mathbb{D}$.

We prove that Brennan's conjecture is equivalent to the boundedness of composition operators generated by conformal homeomorphisms $\varphi : \Omega \rightarrow \mathbb{D}$.

Equivalence Theorem. *Brennan's conjecture (1.1) holds for a number $s \in (4/3; 4)$ if and only if any conformal homeomorphism $\varphi : \Omega \rightarrow \mathbb{D}$ generates a bounded composition operator*

$$\varphi^* : L_p^1(\mathbb{D}) \rightarrow L_{q(p,s)}^1(\Omega)$$

for any $p \in (2; +\infty)$ and $q(p, s) = ps/(p + s - 2)$.

Proof. By Composition Theorem

$$\int_{\Omega} |\varphi'(z)|^{\frac{(p-2)q}{p-q}} d\mu < \infty$$

if and only if $\varphi : \Omega \rightarrow \mathbb{D}$ generates a bounded composition operator

$$\varphi^* : L_p^1(\mathbb{D}) \rightarrow L_q^1(\Omega), \quad 1 \leq q < p \leq \infty.$$

If a number $s \in (4/3; 4)$ satisfies to Brennan's conjecture then

$$\int_{\Omega} |\varphi'(z)|^s d\mu < \infty.$$

On the other hand

$$\int_{\Omega} |\varphi'(z)|^s d\mu = \int_{\Omega} |\varphi'(z)|^{\frac{(p-2)q}{p-q}} d\mu$$

for $(p-2)q/(p-q) = s$ i. e. for $q := q(p, s) = ps/(p+s-2)$. Let us check the condition of Composition Theorem that $q(p, s) = ps/(p+s-2) < p$. Because $p > 2$ we have $p+s-2 > s > 0$. Hence $s/(p+s-2) < 1$ and $q(p, s) < p$. \square

Remarks:

1. The restriction $p \in (2; +\infty)$ is sharp. If $p < 2$ then $q(p, s) > p$ and we have no composition operators. If $p = 2$ then $q(p, s) = 2$ and $\varphi' \in L^2(\Omega)$ if and only if $|\Omega| < \infty$.
2. The result is correct also for $p = \infty$ and $q = s$.

4 Geometric versions Brennan's conjecture

Brennan's conjecture for p -distortion. The p -distortion for conformal homeomorphisms is

$$K_p(z, \varphi) = \frac{|\varphi'(z)|^p}{J(z, \varphi)} = |\varphi'(z)|^{p-2} \text{ for any } p \in [1, \infty).$$

Using the notion of p -distortion and motivated by the Composition Theorem and the Equivalence Theorem we shall reformulate the Brennan's conjecture in the following equivalent form:

Geometric Brennan's conjecture. The conjecture concerns integrability of p -dilatation of plane conformal homeomorphisms $\varphi : \Omega \rightarrow \mathbb{D}$ of a simply connected plane domain with nonempty boundary $\Omega \subset \mathbb{R}^2$ to the unit disc $\mathbb{D} \subset \mathbb{R}^2$.

The conjecture is that the p -distortion is integrable in power α for all $\frac{4}{3(p-2)} < \alpha < \frac{4}{p-2}$ if $2 < p < +\infty$ and $\frac{-4}{2-p} < \alpha < \frac{-4}{3(2-p)}$ if $1 \leq p < 2$.

In the context of Composition problem the following reformulation of Brennan's conjecture will be useful:

Inverse Brennan's conjecture Brennan's conjecture can be reformulated also for conformal homeomorphisms $\psi = \varphi^{-1} : \mathbb{D} \rightarrow \Omega$.

The conjecture becomes

$$\int_{\mathbb{D}} |\psi'(w)|^r d\mu < +\infty, \quad \text{for all } -2 < r < \frac{2}{3}.$$

Inverse Brennan's conjecture is proved for the range $-1.78 \leq r < 2/3$ (S. Shimorin (2005)).

The next theorem demonstrates connection between Inverse Brennan's conjecture and Composition Problem for conformal maps $\psi : D \rightarrow \Omega$.

Inverse Composition Theorem. A diffeomorphism $w = \varphi : \Omega \rightarrow \Omega'$ of plane domains $\Omega, \Omega' \subset \mathbb{R}^2$ generates a bounded composition operator

$$\varphi^* : L_p^1(\Omega') \rightarrow L_q^1(\Omega), \quad 1 < q < p < \infty,$$

if and only if $\varphi^{-1} : \Omega' \rightarrow \Omega$ generates a bounded composition operator

$$(\varphi^{-1})^* : L_{q'}^1(\Omega) \rightarrow L_{p'}^1(\Omega')$$

where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

Proof. Let φ generates the composition operator $\varphi^* : L_p^1(\Omega') \rightarrow L_q^1(\Omega)$. By Composition Theorem

$$\int_{\Omega} \left(\frac{|\varphi'(z)|^p}{|J(z, \varphi)|} \right)^{\frac{q}{p-q}} d\mu < \infty.$$

Using the equality $|(\varphi^{-1})'(\varphi(z))| = |\varphi'(z)|/|J(z, \varphi)|$ and the change of variable formula

we obtain

$$\begin{aligned}
 \int_{\Omega'} \left(\frac{|(\varphi^{-1})'(w)|^{q'}}{|J(w, \varphi^{-1})|} \right)^{\frac{p'}{q'-p'}} d\mu &= \int_{\Omega'} \left(\left(\frac{|\varphi'(\varphi^{-1}(w))|}{|J(\varphi^{-1}(w), \varphi)|} \right)^{q'} \frac{1}{|J(w, \varphi^{-1})|} \right)^{\frac{p'}{q'-p'}} d\mu \\
 &= \int_{\Omega'} \left(\frac{|\varphi'(\varphi^{-1}(w))|^{q'}}{|J(\varphi^{-1}(w), \varphi)|^{q'-1}} \right)^{\frac{p'}{q'-p'}} d\mu = \int_{\Omega'} \frac{|\varphi'(\varphi^{-1}(w))|^{\frac{pq}{p-q}}}{|J(\varphi^{-1}(w), \varphi)|^{\frac{p}{p-q}}} d\mu \\
 &= \int_{\Omega} \frac{|\varphi'(z)|^{\frac{pq}{p-q}}}{|J(z, \varphi)|^{\frac{p}{p-q}}} |J(z, \varphi)| d\mu = \int_{\Omega} \left(\frac{|\varphi'(z)|^p}{|J(z, \varphi)|} \right)^{\frac{q}{p-q}} d\mu < \infty.
 \end{aligned}$$

Hence, by Composition Theorem the composition operator

$$(\varphi^{-1})^* : L_{q'}^1(\Omega) \rightarrow L_{p'}^1(\Omega'), \quad 1/p + 1/p' = 1, \quad 1/q + 1/q' = 1.$$

is bounded. The inverse assertion can be proved in the same way. \square

Remarks:

1. The result is also correct for $p = \infty$ and $q = 1$.
2. By the Inverse Composition Theorem Inverse Brennan's conjecture (for conformal homeomorphisms $\varphi : \mathbb{D} \rightarrow \Omega$, see, for example, [1]) is equivalent to the Inverse Composition Problem, for $p \in (1, 2]$.

Brennan's conjecture for mappings with bounded (p, q) -distortion.

Brennan's conjecture means than conformal homeomorphisms are mappings with bounded (p, q) -distortion ((p, q) -quasiconformal homeomorphisms) for any $p \in (2; +\infty)$ and $q(p, s) = ps/(p + s - 2)$. This fact gives us examples of mappings with bounded (p, q) -distortion. The Brennan's conjecture in geometric terms of integrability of the p -distortion looks more natural for the geometric function theory and applications to PDE.

Let us recall finally that Equivalence theorem is proved only for $2 < p < \infty$ and for $1 < p < 2$ we have no information.

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