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GENERALIZED WEIGHTED MORREY SPACES AND HIGHER ORDER COMMUTATORS OF SUBLINEAR OPERATORS

V.S. Guliyev

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Abstract. In this paper, we study the boundedness of sublinear operators and their higher order commutators generated by Calderon-Zygmund operators and Riesz potentials on generalized weighted Morrey space.

1 Introduction

The classical Morrey spaces $L_{p,\lambda}$ were originally introduced by Morrey in [30] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [30, 32]. In [8], Chiarenza and Frasca showed the boundedness of the Hardy-Littlewood maximal operator, the Riesz potential and the Calderon-Zygmund singular integral operator these spaces. The boundedness of the Riesz potential was originally studied by Adams [1].

On the other hand, in harmonic analysis it is very important to study weighted estimates for these operators. On the weighted L_p spaces, the boundedness of operators above was obtained by Muckenhoupt [29], Mukenhoupt and Wheeden [30], and Coifman and Fefferman [9]. Recently, Komori and Shirai [22] defined the weighted Morrey spaces $L_{p,\kappa}(w)$ and studied the boundedness of the aforementioned classical operators these spaces. These results were extended to several other spaces. However, their boundedness in generalized weighted Morrey spaces $M_{p,\varphi}(w)$ have not yet been studied.

Therefore, in this paper, we shall investigate the boundedness of sublinear operators and their higher order commutators generated by Calderon-Zygmund operators and Riesz potentials in generalized weighted Morrey space, that is, the maximal operator, the fractional maximal operator, the Riesz potential, the Calderon-Zygmund operators, the Littlewood-Paley operator, the Marcinkiewicz operator, the Bochner-Riesz operator.

2 Definitions and notation

Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space of points $x=(x_1,...,x_n)$ with the norm $|x|=(\sum_{i=1}^n x_i^2)^{1/2}$. For $x\in\mathbb{R}^n$ and $r>0$, let $B(x,r)$ be the open ball centered at x of radius r, ${}^{\complement}B(x,r)$ denote its complement, and $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

The fractional maximal operator M_α and the Riesz potential I_α are defined by

$$M_\alpha f(x)=\sup_{t>0}|B(x,t)|^{-1+\frac{\alpha}{n}}\int_{B(x,t)}|f(y)|dy,\qquad 0\le\alpha<n,$$

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)\,dy}{|x-y|^{n-\alpha}},\qquad 0<\alpha<n.$$

If $\alpha=0$, then $M\equiv M_0$ is the Hardy-Littlewood maximal operator.

Let K be a Calderón-Zygmund singular integral operator, briefly a Calderón-Zygmund operator, i.e., a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to functions in $L_1^{\rm loc}(\mathbb{R}^n)$, represented for such functions by

$$Kf(x)=\int_{\mathbb{R}^n}k(x,y)f(y)\,dy\qquad \text{a.e. on } {}^{\complement}\text{supp}f.$$

Here $k(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $c_1>0$ and $0<\varepsilon\le 1$ such that

$$|k(x,y)|\le c_1|x-y|^{-n}$$

for all $x,y\in\mathbb{R}^n,\ x\ne y$, and

$$|k(x,y)-k(x',y)|+|k(y,x)-k(y,x')|\le c_1\left(\frac{|x-x'|}{|x-y|}\right)^{\varepsilon}|x-y|^{-n}$$

whenever $2|x-x'|\le|x-y|$. Such operators were introduced in [11].

It is well known that the fractional maximal operator, Riesz potential and Calderón-Zygmund operators play an important role in harmonic analysis (see [14, 28, 37, 39]).

Suppose that T_α, $\alpha\in[0,n)$ represents a linear or a sublinear operator, which satisfies, for any $f\in L_1(\mathbb{R}^n)$ with compact support and $x\notin suppf$, the inequality

$$|T_\alpha f(x)|\le c_1\int\limits_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy, \tag{2.1}$$

where c_1 is independent of f and x.

For a function b, suppose that the kth-order commutator operator $T_{b,\alpha,k}$, $\alpha\in[0,n)$ represents a linear or a sublinear operator, which satisfies, for any $f\in L_1(\mathbb{R}^n)$ with compact support and $x\notin suppf$, the inequality

$$|T_{b,\alpha,k}f(x)|\le c_2\int_{\mathbb{R}^n}|b(x)-b(y)|^k\,|x-y|^{-n+\alpha}|f(y)|dy, \tag{2.2}$$

where c_2 is independent of f and x.

We point out that the condition (2.1) with $\alpha = 0$ was first introduced by Soria and Weiss in [34] . Condition (2.1) is satisfied by many interesting operators in harmonic analysis, such as the Calderón–Zygmund operator, Carleson's maximal operator, Hardy-Littlewood maximal operators, fractional maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Riesz potentials, Ricci–Stein's oscillatory singular integrals, Bochner–Riesz means and so on (see [12], [27], [34] for details).

We define the generalized weighed Morrey spaces as follows.

Definition 1. *Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t\left(\int_{\{y\in B(x,r):\, |f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

Remark 1. (1) *If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.*

(2) *If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.*

(3) *If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space.*

(4) *If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.*

(5) *If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.*

In [20], we proved the boundedness of the sublinear operator T_0 satisfying condition (2.1) with $\alpha = 0$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ with $w \in A_p$, $1 < p < \infty$, and from $M_{1,\varphi_1}(w)$ to the weak space $WM_{1,\varphi_2}(w)$ with $w \in A_1$, where A_p is the Muckenhoupt class [29] (see the definition in Section 4).

In this work, we prove the boundedness of the sublinear operator T_α, $\alpha \in (0,n)$ satisfies the condition (2.1) generated by the Riesz potential operator from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$ with $w \in A_{p,q}$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/n$, and from $M_{1,\varphi_1}(w)$ to the weak space $WM_{q,\varphi_2}(w^q)$ with $w \in A_{1,q}$, $1 < q < \infty$, $1 - 1/q = \alpha/n$, where $A_{p,q}$ is the Muckenhoupt-Wheeden class [30] (see the definition in Section 4).

In the case $b \in BMO(\mathbb{R}^n)$ and $T_{b,0,k}$ a sublinear kth-order commutator operator, satisfying condition (2.2) with $\alpha = 0$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operator $T_{b,0,k}$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ with $w \in A_p$, $1 < p < \infty$. Also, in the case $b \in BMO(\mathbb{R}^n)$ and $T_{b,\alpha,k}$, $\alpha \in (0,n)$ a sublinear kth-order commutator operator, satisfying condition (2.2) with $\alpha \in (0,n)$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operator $T_{b,\alpha,k}$ from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$ with $w \in A_{p,q}$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/n$. Finally, as application, we apply this result to several particular operators such as Littlewood-Paley operator, Marcinkiewicz operator, Bochner-Riesz operator and fractional powers of some analytic semigroups.

By $A \lesssim B$ we mean that $A \le CB$ with some positive C is independent of insignificant quantities. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$ and say that A and B are equivalent.

3 Main results

The following statements, were proved in [20, 21].

Theorem 3.1. *Let $1 \le p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{3.1}$$

where C does not depend on x and r. Let T_0 be a sublinear operator satisfying condition (2.1) with $\alpha = 0$ bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{1,w}(\mathbb{R}^n)$. Then the operator T_0 is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.

Note that, in the case $w = 1$ Theorem 3.1 was proved in [17] and for the operators M and K in [3].

Theorem 3.2. *Let $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \ln\Big(e + \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{3.2}$$

where C does not depend on x and r. Let $T_{b,0,1}$ be a sublinear operator satisfying condition (2.2) with $\alpha = 0$, $k = 1$ and bounded on $L_{p,w}(\mathbb{R}^n)$. Then the operator $T_{b,0,1}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.

Note that for $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, from Theorems 3.1 and 3.2 we get the following new results.

Corollary 3.1. *Let $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Let also T_0 be a sublinear operator satisfying condition (2.1) with $\alpha = 0$ bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{1,w}(\mathbb{R}^n)$. Then the operator T_0 is bounded on $L_{p,\kappa}(w)$ for $p > 1$ and from $L_{1,\kappa}(w)$ to $WL_{1,\kappa}(w)$ (see [21]).*

Corollary 3.2. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ and let T_b be a sublinear operator satisfying condition (2.2) with $\alpha = 0$, $k = 1$. Let also $T_{b,0,1}$ be bounded on $L_{p,w}(\mathbb{R}^n)$. Then the operator $T_{b,0,1}$ is bounded on $L_{p,\kappa}(w)$ (see [21]).*

Note that from Corollaries 3.1 and 3.2 for the Hardy-Littlewood maximal operators M and the Calderón-Zygmund operators K we get results which were proved in [22].

From Theorem 3.1 we also get the following result.

Corollary 3.3. *Let $1 \le p < \infty$, $0 < \lambda < n$, $\lambda - n < \beta < n(p-1)$ ($\lambda - n < \beta \le 0$, if $p = 1$) and let T_0 be a sublinear operator satisfying condition (2.1) with $\alpha = 0$ bounded on $L_{p,|\cdot|^\beta}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,|\cdot|^\beta}(\mathbb{R}^n)$ to $WL_{1,|\cdot|^\beta}(\mathbb{R}^n)$. Then the operator T is bounded on $M_{p,\lambda}(|\cdot|^\beta)$ for $p > 1$ and from $M_{1,\lambda}(|\cdot|^\beta)$ to $WM_{1,\lambda}(|\cdot|^\beta)$.*

Corollary 3.4. *Let $1 \le p < \infty$, $0 < \lambda < n$, $\lambda - n < \beta < n(p-1)$ ($\lambda - n < \beta \le 0$, if $p = 1$). Then the operators M and K are bounded on $M_{p,\lambda}(|\cdot|^\beta)$ for $p > 1$ and from $M_{1,\lambda}(|\cdot|^\beta)$ to $WM_{1,\lambda}(|\cdot|^\beta)$ for $p = 1$.*

Next we state our main results. First we present some estimates which are the main tools for proving our theorems, on the boundedness of the operators T_α with $\alpha \in (0, n)$ on the generalized weighted Morrey spaces.

Theorem 3.3. *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $w \in A_{p,q}$. Let also T_α be a sublinear operator satisfying condition (2.1), bounded from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{q,w^q}(\mathbb{R}^n)$ for $p = 1$.*

Then, for $1 < p < \frac{n}{\alpha}$ the inequality

$$\|T_\alpha f\|_{L_{q,w^q}(B(x,r))} \lesssim \big(w^q(B(x,r))\big)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \big(w^q(B(x,t))\big)^{-\frac{1}{q}} \frac{dt}{t}$$

holds for any ball $B(x,r)$ and for all $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

Moreover, for $p = 1$ the inequality

$$\|T_\alpha f\|_{WL_{q,w^q}(B(x,r))} \lesssim \big(w^q(B(x,r))\big)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x,t))} \big(w^q(B(x,t))\big)^{-\frac{1}{q}} \frac{dt}{t}, \tag{3.3}$$

holds for any ball $B(x,r)$ and for all $f \in L^{\rm loc}_{1,w}(\mathbb{R}^n)$.

Theorem 3.4. *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w \in A_{p,q}$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) \big(w^p(B(x,s))\big)^{\frac{1}{p}}}{\big(w^q(B(x,t))\big)^{\frac{1}{q}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{3.4}$$

where C does not depend on x and r. Let T_α be a sublinear operator satisfying condition (2.1) *with $\alpha \in (0,n)$, bounded from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$ for $p>1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{q,w^q}(\mathbb{R}^n)$ for $p=1$. Then the operator T_α is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$ for $p>1$ and from $M_{1,\varphi_1}(w)$ to $WM_{q,\varphi_2}(w^q)$ for $p=1$. Moreover, for $p>1$*

$$\|T_\alpha f\|_{M_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M_{p,\varphi_1}(w^p)},$$

and for $p=1$

$$\|T_\alpha f\|_{WM_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M_{1,\varphi_1}(w)}.$$

Note that, in the case $w=1$ Theorem 3.4 was proved in [18].

Corollary 3.5. *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w \in A_{p,q}$ and (φ_1, φ_2) satisfy condition* (3.4). *Then the operators M_α and I_α are bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$ for $p>1$ and from $M_{1,\varphi_1}(w)$ to $WM_{q,\varphi_2}(w^q)$ for $p=1$.*

For $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, from Theorem 3.4 we get the following new result.

Corollary 3.6. *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 < \kappa < \frac{p}{q}$ and $w \in A_{p,q}$. Let also T_α be a sublinear operator satisfying condition* (2.1) *with $\alpha \in (0,n)$ bounded from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$ for $p>1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{q,w^q}(\mathbb{R}^n)$. Then the operator T_α is bounded from $L_{p,\kappa}(w^p, w^q)$ to $L_{q,\kappa q/p}(w^q)$ for $p>1$ and from $L_{1,\kappa}(w, w^q)$ to $WL_{q,\kappa q}(w^q)$ for $p=1$.*

Now we present some estimates which are the main tools for proving our theorems, on the boundedness of the operators $T_{b,\alpha,k}$, $\alpha \in [0,n)$ on the generalized weighted Morrey spaces.

Theorem 3.5. *Let $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$, and $T_{b,0,k}$ be a sublinear kth-order commutator operator satisfying condition* (2.2) *with $\alpha = 0$. Let also $T_{b,0,k}$ be bounded on $L_{p,w}(\mathbb{R}^n)$. Moreover, let*

$$\|T_{b,0,k} f\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|b\|_*^k \|f\|_{L_{p,w}(\mathbb{R}^n)},$$

where $\|b\|_*$ is the norm in $BMO(\mathbb{R}^n)$ (see Definition 2 below).

Then the inequality

$$\|T_{b,0,k} f\|_{L_{p,w}(B(x,r))} \lesssim \|b\|_*^k \, w(B(x,r))^{\frac{1}{p}} \int_{2r}^{\infty} \ln^k \left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x,t))} \, w(B(x,t))^{-1/p} \, \frac{dt}{t}$$

holds for any ball $B(x,r)$ and for all $f \in L_{p,w}^{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 3.6. *Let $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, and $w \in A_{p,q}$. Let also $T_{b,\alpha,k}$ be a sublinear kth-order commutator operator satisfying condition* (2.2), *and bounded from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$. Moreover, let*

$$\|T_{b,\alpha,k} f\|_{L_{q,w^q}(\mathbb{R}^n)} \lesssim \|b\|_*^k \|f\|_{L_{p,w^p}(\mathbb{R}^n)}.$$

Then the inequality

$$\|T_{b,\alpha,k} f\|_{L_{q,w^q}(B(x,r))} \lesssim \|b\|_*^k \left(w^q(B(x,r))\right)^{\frac{1}{q}} \int_{2r}^{\infty} \ln^k \left(e + \frac{t}{r}\right) \|f\|_{L_{p,w^p}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}$$

holds for any ball $B(x,r)$ and for all $f \in L_{p,w^p}^{\mathrm{loc}}(\mathbb{R}^n)$.

Now we give theorem about the boundedness of the operators $T_{b,\alpha,k}$ on the generalized weighted Morrey spaces.

Theorem 3.7. *Let $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \ln^k \left(e + \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{3.5}$$

where C does not depend on x and r. Let $T_{b,0,k}$ be a sublinear kth-order commutator operator satisfying condition (2.2) with $\alpha = 0$ and bounded on $L_{p,w}(\mathbb{R}^n)$. Moreover, let

$$\|T_{b,0,k} f\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|b\|_*^k \|f\|_{L_{p,w}(\mathbb{R}^n)}.$$

Then the operator $T_{b,0,k}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover,

$$\|T_{b,0,k} f\|_{M_{p,\varphi_2}(w)} \lesssim \|b\|_*^k \|f\|_{M_{p,\varphi_1}(w)}.$$

Note that from Theorem 3.7 we get new results in particular for the sublinear kth order commutator of the maximal operator $M_{b,k}$ and for the linear kth order commutator of the Calderón-Zygmund operator

$$[b^k, K] f(x) \equiv [\underbrace{b \ldots [b, K] \ldots}_{k}] f(x) = \int_{\mathbb{R}^n} \big(b(x) - b(y)\big)^k k(x,y) f(y) dy.$$

For $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, from Theorem 3.7 we also get the following new result.

Corollary 3.7. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ and let $T_{b,0,k}$ be a sublinear kth-order commutator operator satisfying condition (2.2) with $\alpha = 0$. Let also $T_{b,0,k}$ is bounded on $L_{p,w}(\mathbb{R}^n)$. Then the operator $T_{b,0,k}$ be bounded on $L_{p,\kappa}(w)$.*

Proof. Let $1 < p < \infty$, $w \in A_p$, $0 < \kappa < 1$, $b \in BMO(\mathbb{R}^n)$. Then the pair $(w(B(x,r))^{\frac{\kappa-1}{p}}, w(B(x,r))^{\frac{\kappa-1}{p}})$ satisfies condition (3.5). Indeed,

$$\begin{aligned}\int_r^\infty \ln^k \left(e + \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} w(B(x,s))^{\frac{\kappa}{p}}}{w(B(x,t))^{1/p}} \frac{dt}{t} &= \int_r^\infty \ln^k \left(e + \frac{t}{r}\right) w(B(x,t))^{\frac{\kappa-1}{p}} \frac{dt}{t} \\ &\le C\, w(B(x,r))^{\frac{\kappa-1}{p}},\end{aligned}$$

with where the last inequality with $C > 0$ independent of x and r follows from Lemma 13 in [4]. □

Note that, in the case $k = 1$, from Corollary 3.7 for the operator $[b^k, K]$ we get results which were proved in [22].

Theorem 3.8. *Let $1<p<q<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $w\in A_{p,q}$, $b\in BMO(\mathbb{R}^n)$ and (φ_1,φ_2) satisfy the condition*

$$\int_r^\infty \ln^k\left(e+\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)\big(w^p(B(x,s))\big)^{\frac{1}{p}}}{\big(w^q(B(x,t))\big)^{\frac{1}{q}}}\frac{dt}{t}\le C\,\varphi_2(x,r), \tag{3.6}$$

where C does not depend on x and r. Let $T_{b,\alpha,k}$ be a sublinear kth-order commutator operator satisfying condition (2.2), and bounded from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$. Moreover, let

$$\|T_{b,\alpha,k}f\|_{L_{q,w^q}(\mathbb{R}^n)}\lesssim\|b\|_*^k\,\|f\|_{L_{p,w^p}(\mathbb{R}^n)}.$$

Then the operator $T_{b,\alpha,k}$ is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$. Moreover,

$$\|T_{b,\alpha,k}f\|_{M_{q,\varphi_2}(w^q)}\lesssim\|b\|_*^k\,\|f\|_{M_{p,\varphi_1}(w^p)}.$$

For the sublinear kth order commutator of the fractional maximal operator

$$M_{b,\alpha,k}(f)(x)=\sup_{t>0}|B(x,t)|^{-1+\frac{\alpha}{n}}\int_{B(x,t)}|b(x)-b(y)|^k|f(y)|dy$$

and for the linear kth order commutator of the Riesz potential

$$[b^k,I_\alpha]f(x)\equiv[\underbrace{b\ldots[b,I_\alpha]\ldots}_{k}]f(x)=\int_{\mathbb{R}^n}\big(b(x)-b(y)\big)^k\frac{f(y)}{|x-y|^{n-\alpha}}dy$$

from Theorem 3.8 we get the following new result.

Corollary 3.8. *Let $1<p<q<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $w\in A_{p,q}$, $b\in BMO(\mathbb{R}^n)$ and (φ_1,φ_2) satisfy condition (3.6). Then, the operators $M_{b,\alpha,k}$ and $[b^k,I_\alpha]$ are bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$.*

In the case $\varphi_1(x,r)=\varphi_2(x,r)\equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, from Theorem 3.8 we get the following new results.

Corollary 3.9. *Let $1<p<q<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $0<\kappa<\frac{p}{q}$, $b\in BMO(\mathbb{R}^n)$, and $w\in A_{p,q}$. Let also $T_{b,\alpha,k}$ be a sublinear kth-order commutator operator satisfying condition (2.2) bounded from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$. Then the operator $T_{b,\alpha,k}$ is bounded from $L_{p,\kappa}(w^p,w^q)$ to $L_{q,\kappa q/p}(w^q)$.*

Corollary 3.10. *Let $1<p<q<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $0<\kappa<\frac{p}{q}$, $b\in BMO(\mathbb{R}^n)$, and $w\in A_{p,q}$. Then, the operators $M_{b,\alpha,k}$ and $[b^k,I_\alpha]$ are bounded from $L_{p,\kappa}(w^p,w^q)$ to $L_{q,\kappa q/p}(w^q)$.*

Note that in the case $k=1$ Corollary 3.10 was proved in [22].

4 Preliminaries and some lemmas

By a weight function, briefly weight, we mean a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E)=\int_E w(x)dx$, and denote the Lebesgue measure of E by $|E|$ and the characteristic function of E by $\chi_{_E}$. Given a weight w, we say that w satisfies the doubling condition if there exists a constant $D>0$ such that for any ball B, we have $w(2B)\leq Dw(B)$. When w satisfies this condition, we write brevity $w\in\Delta_2$.

If w is a weight function, we denote by $L_{p,w}(\mathbb{R}^n)\equiv L_p(\mathbb{R}^n,w)$ the weighted Lebesgue space defined by finiteness of the norm

$$\|f\|_{L_{p,w}}=\left(\int_{\mathbb{R}^n}|f(x)|^p w(x)dx\right)^{\frac{1}{p}}<\infty,\quad \text{if}\ \ 1\leq p<\infty$$

and by $\|f\|_{L_{\infty,w}}=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|f(x)|w(x)$ if $p=\infty$.

We recall that a weight function w is in the Muckenhoupt's class A_p [29], $1<p<\infty$, if

$$\begin{aligned}[w]_{A_p}:&=\sup_B[w]_{A_p(B)}\\ &=\sup_B\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1}<\infty,\end{aligned}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls B by Hölder's inequality

$$[w]^{1/p}_{A_p(B)}=|B|^{-1}\|w\|^{1/p}_{L_1(B)}\,\|w^{-1/p}\|_{L_{p'}(B)}\geq 1. \tag{4.1}$$

For $p=1$, the class A_1 is defined by the condition $Mw(x)\leq Cw(x)$ with $[w]_{A_1}=\sup\limits_{x\in\mathbb{R}^n}\frac{Mw(x)}{w(x)}$, and for $p=\infty$ $A_\infty=\bigcup_{1\leq p<\infty}A_p$ and $[w]_{A_\infty}=\inf\limits_{1\leq p<\infty}[w]_{A_p}$.

A weight function w belongs to the Muckenhoupt-Wheeden class $A_{p,q}$ [30] for $1<p,q<\infty$ if

$$\begin{aligned}[w]_{A_{p,q}}:&=\sup_B[w]_{A_{p,q}(B)}\\ &=\sup_B\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{1/q}\left(\frac{1}{|B|}\int_B w(x)^{-p'}dx\right)^{1/p'}<\infty,\end{aligned}$$

where the sup is taken with respect to all balls B. Note that, for all balls B by Hölder's inequality

$$[w]_{A_{p,q}(B)}=|B|^{\frac{1}{p}-\frac{1}{q}-1}\|w\|_{L_q(B)}\,\|w^{-1}\|_{L_{p'}(B)}\geq 1. \tag{4.2}$$

If $p=1$, w is in $A_{1,q}$ with $1<q<\infty$ if

$$\begin{aligned}[w]_{A_{1,q}}:&=\sup_B[w]_{A_{1,q}(B)}\\ &=\sup_B\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{1/q}\left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{w(x)}\right)<\infty,\end{aligned}$$

where the sup is taken with respect to all balls B.

Remark 2. *[14, 15] If $w \in A_{p,q}$ with $1 < p < q < \infty$, then the following statements are true:*

(a) $w^q \in A_t$ *with* $t = 1 + q/p'$.

(b) $w^{-p'} \in A_{t'}$ *with* $t' = 1 + p/q'$.

(c) $w \in A_{q,p}$.

(d) $w^p \in A_s$ *with* $s = 1 + p/q'$.

(e) $w^{-q'} \in A_{s'}$ *with* $s' = 1 + q'/p$.

Lemma 4.1. ([15]) (1) *If $w \in A_p$ for some $1 \le p < \infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$*

$$w(\lambda B) \le \lambda^{np}[w]_{A_p} w(B).$$

(2) *If $w \in A_\infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$*

$$w(\lambda B) \le 2^{\lambda^n}[w]_{A_\infty} w(B).$$

(3) *If $w \in A_p$ for some $1 \le p \le \infty$, then there exit $C > 0$ and $\delta > 0$ such that for any ball B and a measurable set $S \subset B$,*

$$\frac{w(S)}{w(B)} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta}.$$

We are going to use the following result on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)d\mu(r),\ 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0, \infty)$.

Theorem 4.1. ([7]) *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all functions g non-negative and non-increasing on $(0, \infty)$ if and only if

$$A := \sup_{t>0} \frac{w(t)}{t}\int_0^t \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

We also need the following statement on the boundedness of the Hardy type operator

$$(H_1 g)(t) := \frac{1}{t}\int_0^t \ln^k\Big(e + \frac{t}{r}\Big) g(r)d\mu(r),\ 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0, \infty)$.

Theorem 4.2. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)H_1g(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all functions g non-negative and non-increasing on $(0,\infty)$ if and only if

$$A_1 := \sup_{t>0} \frac{w(t)}{t} \int_0^t \ln^k \left(e + \frac{t}{r}\right) \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A_1$.

Note that, Theorem 4.2 can be proved analogously to Theorem 4.3 in [17].

Definition 2. *$BMO(\mathbb{R}^n)$ is the Banach space modulo constants with the norm $\|\cdot\|_*$ defined by*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|dy < \infty,$$

where $b \in L_1^{\rm loc}(\mathbb{R}^n)$ and

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)dy.$$

Lemma 4.2. ([30], Theorem 5, p. 236) *Let $w \in A_\infty$. Then the norm $\|\cdot\|_*$ is equivalent to the norm*

$$\|b\|_{*,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{w(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r),w}|w(y)dy,$$

where

$$b_{B(x,r),w} = \frac{1}{w(B(x,r))} \int_{B(x,r)} b(y)w(y)dy.$$

Remark 3. (1) *The John-Nirenberg inequality : there are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$*

$$|\{x \in B : |b(x) - b_B| > \beta\}| \le C_1|B|e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) *For $1 \le p < \infty$ the John-Nirenberg inequality implies that*

$$\|b\|_* \approx \sup_B \left(\frac{1}{|B|} \int_B |b(y) - b_B|^p dy\right)^{\frac{1}{p}} \tag{4.3}$$

and for $1 \le p < \infty$ and $w \in A_\infty$

$$\|b\|_* \approx \sup_B \left(\frac{1}{w(B)} \int_B |b(y) - b_B|^p w(y)dy\right)^{\frac{1}{p}}. \tag{4.4}$$

Note that, by the John-Nirenberg inequality and Lemma 4.1 (part 3) it follows that

$$w(\{x \in B : |b(x) - b_B| > \beta\}) \leq C_1^{\delta} w(B) e^{-C_2 \beta \delta / \|b\|_*}$$

for some $\delta > 0$. Hence

$$\begin{aligned}\int_B |b(y) - b_B|^p w(y) dy &= p \int_0^{\infty} \beta^{p-1} \, w(\{x \in B : |b(x) - b_B| > \beta\}) d\beta \\ &\leq p C_1^{\delta} \, w(B) \int_0^{\infty} \beta^{p-1} \, e^{-C_2 \beta \delta / \|b\|_*} \, d\beta = C_3 w(B) \|b\|_*^p,\end{aligned}$$

where $C_3 > 0$ depends only on C_1^{δ}, C_2, p, and δ, which implies (4.4).

Also (4.3) is a particular case of (4.4) with $w \equiv 1$.

The following lemma was proved in [23].

Lemma 4.3. *Let $b \in BMO(\mathbb{R}^n)$. Let also $1 \leq p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x, r_1)|} \int_{B(x, r_1)} |b(y) - b_{B(x, r_2)}|^p dy \right)^{\frac{1}{p}} \leq C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*,$$

where $C > 0$ is independent of f, x, r_1, and r_2.

The following lemma is valid.

Lemma 4.4. *i) Let $w \in A_{\infty}$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 \leq p < \infty$, $x \in \mathbb{R}^n$, $k > 0$ and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{w(B(x, r_1))} \int\limits_{B(x, r_1)} |b(y) - b_{B(x, r_2), w}|^{kp} w(y) dy \right)^{\frac{1}{p}} \leq C \, \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right)^k \|b\|_*^k,$$

where $C > 0$ is independent of f, w, x, r_1, and r_2.

ii) Let $w \in A_p$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$, $k > 0$ and $r_1, r_2 > 0$. Then

$$\left(\frac{1}{w^{1-p'}(B(x, r_1))} \int\limits_{B(x, r_1)} |b(y) - b_{B(x, r_2), w}|^{kp'} w(y)^{1-p'} dy \right)^{\frac{1}{p'}} \leq C \, \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right)^k \|b\|_*^k,$$

where $C > 0$ is independent of f, w, x, r_1, and r_2.

Proof. i) From (4.4) and Lemma 4.3 we have

$$\begin{aligned}
&\left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,r_2),w}|^{kp}w(y)dy\right)^{\frac{1}{p}}\\
&\le\left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,r_1)}|^{kp}w(y)dy\right)^{\frac{1}{p}}\\
&\quad+|b_{B(x,r_1)}-b_{B(x,r_2)}|^k+|b_{B(x,r_2)}-b_{B(x,r_2),w}|^k\\
&\le\left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,r_1)}|^{kp}w(y)dy\right)^{\frac{1}{p}}\\
&\quad+\Big(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}|b(y)-b_{B(x,r_2)}|dy\Big)^k\\
&\quad+\Big(\frac{1}{w(B(x,r_2))}\int_{B(x,r_2)}|b(y)-b_{B(x,r_2)}|w(y)dy\Big)^k\\
&\lesssim\left(1+\left|\ln\frac{r_1}{r_2}\right|\right)^k\|b\|_*^k.
\end{aligned}$$

This completes the proof of the first part of the lemma.

ii) From (4.4) and Lemma 4.3 we have

$$\begin{aligned}
&\left(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,r_2),w}|^{kp'}w(y)^{1-p'}dy\right)^{\frac{1}{p'}}\\
&\le\Big(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,r_1)}|^{kp'}w(y)^{1-p'}dy\Big)^{\frac{1}{p'}}\\
&+\left|b_{B(x,r_1)}-b_{B(x,r_2)}\right|^k+\left|b_{B(x,r_2)}-b_{B(x,r_2),w^{1-p'}}\right|^k\\
&\le\left(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,r_1),w^{1-p'}}|^{kp'}w(y)^{1-p'}dy\right)^{\frac{1}{p'}}\\
&\quad+\Big(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}|b(y)-b_{B(x,r_2)}|dy\Big)^k\\
&+\left(\frac{1}{w^{1-p'}(B(x,r_2))}\int_{B(x,r_2)}|b(y)-b_{B(x,r_2),w^{1-p'}}|w(y)^{1-p'}dy\right)^k\\
&\lesssim\left(1+\left|\ln\frac{r_1}{r_2}\right|\right)^k\|b\|_*^k.
\end{aligned}$$

This completes the proof of the second part of the lemma. □

The following lemma can be proved analogously.

Lemma 4.5. *i) Let $w\in A_\infty$ and $b\in BMO(\mathbb{R}^n)$. Let also $1\le q<\infty$, $x\in\mathbb{R}^n$, $k>0$, and $r_1,r_2>0$. Then*

$$\Big(\frac{1}{w^q(B(x,r_1))}\int\limits_{B(x,r_1)}|b(y)-b_{B(x,r_2),w^q}|^{kq}\,w^q(y)dy\Big)^{\frac{1}{q}}\le C\,\left(1+\left|\ln\frac{r_1}{r_2}\right|\right)^k\|b\|_*^k,$$

where $C>0$ is independent of f, w, x, r_1 and r_2.

ii) Let $1<p<q<\infty$, $w\in A_{p,q}$ and $b\in BMO(\mathbb{R}^n)$. Let also $x\in\mathbb{R}^n$, $k>0$, and $r_1,r_2>0$. Then

$$\Big(\frac{1}{w^{-p'}(B(x,r_1))}\int\limits_{B(x,r_1)}|b(y)-b_{B(x,r_2),w^{-p'}}|^{kp'}w(y)^{-p'}dy\Big)^{\frac{1}{p'}}\le C\left(1+\left|\ln\frac{r_1}{r_2}\right|\right)^k\|b\|_*^k,$$

where $C>0$ is independent of f, w, x, r_1 and r_2.

Note that, Lemma 4.3 is a particular case of Lemma 4.4 (statement i) with $w\equiv 1$ and $k=1$).

5 Proofs of Theorems 3.3, 3.4, 3.5, 3.6, 3.7, and 3.8

In this section we shall prove Theorems 3.3, 3.4, 3.5, 3.6, 3.7, and 3.8. First we shall prove Theorem 3.3.

Proof of Theorem 3.3. Let $1<p<q<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, and $w\in A_{p,q}$. For arbitrary $x\in\mathbb{R}^n$, set $B=B(x,r)$, $2B\equiv B(x,2r)$. We represent f as

$$f=f_1+f_2,\quad f_1(y)=f(y)\chi_{2B}(y),\quad f_2(y)=f(y)\chi_{\complement(2B)}(y),\quad r>0,\tag{5.1}$$

and have

$$\|T_\alpha f\|_{L_{q,w^q}(B)}\le\|T_\alpha f_1\|_{L_{q,w^q}(B)}+\|T_\alpha f_2\|_{L_{q,w^q}(B)}.$$

Since $f_1\in L_{p,w^p}(\mathbb{R}^n)$, $T_\alpha f_1\in L_{q,w^q}(\mathbb{R}^n)$ and from the boundedness of T_α from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$ it follows that:

$$\|T_\alpha f_1\|_{L_{q,w^q}(B)}\le\|T_\alpha f_1\|_{L_{q,w^q}}\le C\|f_1\|_{L_{p,w^p}}=C\|f\|_{L_{p,w^p}(2B)},$$

where constant $C>0$ is independent of f.

It is clear that $z\in B$, $y\in\complement(2B)$ implies $\frac{1}{2}|x-y|\le|z-y|\le\frac{3}{2}|x-y|$. We get

$$|T_\alpha f_2(z)|\le 2^{n-\alpha}c_0\int_{\complement(2B)}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\complement(2B)}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy&\approx\int_{\complement(2B)}|f(y)|\Big(\int_{|x-y|}^{\infty}\frac{dt}{t^{n+1-\alpha}}\Big)\,dy\\&\approx\int_{2r}^{\infty}\Big(\int_{2r\le|x-y|<t}|f(y)|dy\Big)\,\frac{dt}{t^{n+1-\alpha}}\\&\le\int_{2r}^{\infty}\Big(\int_{B(x,t)}|f(y)|dy\Big)\frac{dt}{t^{n+1-\alpha}}.\end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned}\int_{\complement(2B)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \|w^{-1}\|_{L_{p'}(B(x,t))} \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}.\end{aligned} \tag{5.2}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|T_\alpha f_2\|_{L_{q,w^q}(B)} \lesssim w^q(B)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} w^q(B(x,t)^{-\frac{1}{q}} \frac{dt}{t} \tag{5.3}$$

is valid. Thus

$$\|T_\alpha f\|_{L_{q,w^q}(B)} \lesssim \|f\|_{L_{p,w^p}(2B)} + w^q(B)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \|w^{-1}\|_{L_{p'}(B(x,t))} \frac{dt}{t^{n+1-\alpha}}.$$

On the other hand,

$$\begin{aligned}\|f\|_{L_{p,w^p}(2B)} &\approx |B|^{1-\frac{\alpha}{n}} \|f\|_{L_{p,w^p}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1-\alpha}} \\ &\le |B|^{1-\frac{\alpha}{n}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim w^q(B)^{\frac{1}{q}} \|w^{-1}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim w^q(B)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \|w^{-1}\|_{L_{p'}(B(x,t))} \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim [w]_{A_{p,q}} w^q(B)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}.\end{aligned} \tag{5.4}$$

Thus

$$\|T_\alpha f\|_{L_{q,w^q}(B)} \lesssim w^q(B)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}.$$

Let $p = 1$. From the weak $(1,q)$ boundedness of T_α and (5.4) it follows that:

$$\begin{aligned}\|T_\alpha f_1\|_{WL_{q,w^q}(B)} &\le \|Tf_1\|_{WL_{q,w^q}} \\ &\lesssim \|f_1\|_{L_{1,w}} = \|f\|_{L_{1,w}(2B)} \\ &\approx |B|^{1-\frac{\alpha}{n}} \|f\|_{L_{1,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1-\alpha}} \\ &\le |B|^{1-\frac{\alpha}{n}} \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x,t))} \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim w^q(B)^{\frac{1}{q}} \|w^{-1}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x,t))} \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim w^q(B)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x,t))} \|w^{-1}\|_{L_\infty(B(x,t))} \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim w^q(B)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}.\end{aligned} \tag{5.5}$$

By (5.3) and (5.5) we get the inequality (3.3).

□

Proof of Theorem 3.4. By Lemma 3.3 and Theorem 4.1 we have for $p>1$

$$\begin{aligned}
\|T_\alpha f\|_{M_{q,\varphi_2}(w^q)} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_{p,w^p}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t} \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-1}} \|f\|_{L_{p,w^p}(B(x,t^{-1}))} \left(w^q(B(x,t^{-1}))\right)^{-\frac{1}{q}} \frac{dt}{t} \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-1})^{-1}\, r\, \frac{1}{r} \int_0^{r} \|f\|_{L_{p,w^p}(B(x,t^{-1}))} \left(w^q(B(x,t^{-1}))\right)^{-\frac{1}{q}} \frac{1}{t}\, dt \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r^{-1})^{-1} \left(w^p(B(x,r^{-1}))\right)^{-\frac{1}{p}} \|f\|_{L_{p,w^p}(B(x,r^{-1}))} \\
&= \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \left(w^p(B(x,r))\right)^{-\frac{1}{p}} \|f\|_{L_{p,w^p}(B(x,r))} = \|f\|_{M_{p,\varphi_1}(w^p)}
\end{aligned}$$

and for $p=1$

$$\begin{aligned}
\|T_\alpha f\|_{WM_{q,\varphi_2}(w^q)} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-1}} \|f\|_{L_{1,w}(B(x,t^{-1}))} \left(w^q(B(x,t^{-1}))\right)^{-\frac{1}{q}} \frac{dt}{t} \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-1})^{-1}\, r\, \frac{1}{r} \int_0^{r} \|f\|_{L_{1,w}(B(x,t^{-1}))} \left(w^q(B(x,t^{-1}))\right)^{-\frac{1}{q}} \frac{1}{t}\, dt \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r^{-1})^{-1} w(B(x,r^{-1}))^{-1} \|f\|_{L_{p,w^p}(B(x,r^{-1}))} \\
&= \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-1} \|f\|_{L_{1,w}(B(x,r))} = \|f\|_{M_{1,\varphi_1}(w)}.
\end{aligned}$$

□

Proof of Theorem 3.5. Let $p\in(1,\infty)$ and $b\in BMO(\mathbb{R}^n)$. For arbitrary $x\in\mathbb{R}^n$, set $B=B(x,r)$ for the ball centered at x and of radius r. Write $f=f_1+f_2$ with $f_1=f\chi_{2B}$ and $f_2=f\chi_{\complement(2B)}$. Hence

$$\|T_{b,0,k}f\|_{L_{p,w}(B)} \le \|T_{b,0,k}f_1\|_{L_{p,w}(B)} + \|T_{b,0,k}f_2\|_{L_{p,w}(B)}.$$

From the boundedness of $T_{b,0,k}$ in $L_{p,w}(\mathbb{R}^n)$ it follows that:

$$\begin{aligned}
\|T_{b,0,k}f_1\|_{L_{p,w}(B)} &\le \|T_{b,0,k}f_1\|_{L_{p,w}} \\
&\lesssim \|b\|_*^k \|f_1\|_{L_{p,w}} = \|b\|_*^k \|f\|_{L_{p,w}(2B)}.
\end{aligned}$$

For $z\in B$ we have

$$\begin{aligned}
|T_{b,0,k}f_2(z)| &\lesssim \int_{\mathbb{R}^n} \frac{|b(y)-b(z)|^k}{|z-y|^n} |f_2(y)|dy \\
&\approx \int_{\complement(2B)} \frac{|b(y)-b(z)|^k}{|x-y|^n} |f(y)|dy.
\end{aligned}$$

Then

$$\begin{aligned}\|T_{b,0,k}f_2\|_{L_{p,w}(B)} &\lesssim \left(\int_B \left(\int_{\complement(2B)} \frac{|b(y)-b(z)|^k}{|x-y|^n}|f(y)|dy\right)^p w(z)dz\right)^{\frac{1}{p}}\\ &\lesssim \left(\int_B \left(\int_{\complement(2B)} \frac{|b(y)-b_{B,w}|^k}{|x-y|^n}|f(y)|dy\right)^p w(z)dz\right)^{\frac{1}{p}}\\ &\quad + \left(\int_B \left(\int_{\complement(2B)} \frac{|b(x)-b_{B,w}|^k}{|x-y|^n}|f(y)|dy\right)^p w(z)dz\right)^{\frac{1}{p}}\\ &= I_1 + I_2.\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned}I_1 &= w(B)^{\frac{1}{p}} \int_{\complement(2B)} \frac{|b(y)-b_{B,w}|^k}{|x-y|^n}|f(y)|dy\\ &\approx w(B)^{\frac{1}{p}} \int_{\complement(2B)} |b(y)-b_{B,w}|^k|f(y)| \int_{|x-y|}^{\infty} \frac{dt}{t^{n+1}}dy\\ &\approx w(B)^{\frac{1}{p}} \int_{2r}^{\infty}\int_{2r\le|x-y|\le t} |b(y)-b_{B,w}|^k|f(y)|dy\frac{dt}{t^{n+1}}\\ &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty}\int_{B(x,t)} |b(y)-b_{B,w}|^k|f(y)|dy\frac{dt}{t^{n+1}}.\end{aligned}$$

Applying Hölder's inequality and by Lemma 4.4, we get

$$\begin{aligned}I_1 &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \left(\int_{B(x,t)} |b(y)-b_{B(x,r),w}|^{kp'}w(y)^{1-p'}dy\right)^{\frac{1}{p'}} \|f\|_{L_{p,w}(B(x,t))}\frac{dt}{t^{n+1}}\\ &\lesssim [w]_{A_p}^{\frac{1}{p}}\|b\|_*^k\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \left(1+\ln^k\frac{t}{r}\right)\|w^{-1/p}\|_{L_{p'}(B(x,t))}\,\|f\|_{L_{p,w}(B(x,t))}\frac{dt}{t^{n+1}}\\ &\lesssim [w]_{A_p}^{\frac{1}{p}}\|b\|_*^k\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \ln^k\left(e+\frac{t}{r}\right)\|f\|_{L_{p,w}(B(x,t))}\,w(B(x,t))^{-1/p}\,\frac{dt}{t}.\end{aligned}$$

In order to estimate I_2 note that

$$I_2 = \left(\int_B |b(z)-b_{B,w}|^{kp}w(z)dz\right)^{\frac{1}{p}} \int_{\complement(2B)} \frac{|f(y)|}{|x-y|^n}dy.$$

By Lemma 4.4, we get

$$I_2 \lesssim \|b\|_*^k\, w(B)^{\frac{1}{p}} \int_{\complement(2B)} \frac{|f(y)|}{|x-y|^n}dy.$$

Applying Hölder's inequality, we get

$$\begin{aligned}\int_{\complement(2B)} \frac{|f(y)|}{|x-y|^n}dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\,\|w^{-1/p}\|_{L_{p'}(B(x,t))}\,\frac{dt}{t^{n+1}}\\ &\le [w]_{A_p}^{1/p}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x,t))}\,w(B(x,t))^{-1/p}\,\frac{dt}{t}.\end{aligned} \tag{5.6}$$

Thus, by (5.9)

$$I_2 \lesssim \|b\|_*^k\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-1/p}\, \frac{dt}{t}.$$

Summing up I_1 and I_2, for all $p \in [1,\infty)$ we get

$$\|T_{b,0,k} f_2\|_{L_{p,w}(B)} \lesssim \|b\|_*^k\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \ln^k \left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-1/p}\, \frac{dt}{t}. \quad (5.7)$$

On the other hand,

$$\begin{aligned}
\|f\|_{L_{p,w}(2B)} &\approx |B| \|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\
&\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))} \frac{dt}{t^{n+1}} \qquad (5.8) \\
&\le w(B)^{\frac{1}{p}} \|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))} \frac{dt}{t^{n+1}} \\
&\le w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\, \|w^{-1/p}\|_{L_{p'}(B(x,t))}\, \frac{dt}{t^{n+1}} \\
&\le [w]_{A_p}^{1/p}\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-1/p}\, \frac{dt}{t}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|T_{b,0,k} f\|_{L_{p,w}(B)} &\lesssim \|b\|_*^k\, \|f\|_{L_{p,w}(2B)} \\
&\quad + \|b\|_*^k\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \ln^k \left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-1/p}\, \frac{dt}{t},
\end{aligned}$$

and the statement of Theorem 3.5 follows by (5.8).

□

Now we shall get to the proof of Theorem 3.6.

Proof of Theorem 3.6.

Let $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ and $w \in A_{p,q}$. For arbitrary $x \in \mathbb{R}^n$, set $B = B(x,r)$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Hence

$$\|T_{b,\alpha,k} f\|_{L_{q,w^q}(B)} \le \|T_{b,\alpha,k} f_1\|_{L_{q,w^q}(B)} + \|T_{b,\alpha,k} f_2\|_{L_{q,w^q}(B)}.$$

From the boundedness of $T_{b,\alpha,k}$ from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$ it follows that:

$$\begin{aligned}
\|T_{b,\alpha,k} f_1\|_{L_{q,w^q}(\mathbb{R}^n)(B)} &\le \|T_{b,\alpha,k} f_1\|_{L_{q,w^q}(\mathbb{R}^n)} \\
&\lesssim \|b\|_*^k\, \|f_1\|_{L_{p,w^p}(\mathbb{R}^n)} = \|b\|_*^k\, \|f\|_{L_{p,w^p}(\mathbb{R}^n)(2B)}.
\end{aligned}$$

For $z \in B$ we have

$$\begin{aligned}
|T_{b,\alpha,k} f_2(z)| &\lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b(z)|^k}{|z-y|^{n-\alpha}} |f_2(y)|\, dy \\
&\approx \int_{\complement(2B)} \frac{|b(y) - b(z)|^k}{|x-y|^{n-\alpha}} |f(y)|\, dy.
\end{aligned}$$

Then

$$\begin{aligned}\|T_{b,\alpha,k}f_2\|_{L_{q,w^q}(\mathbb{R}^n)(B)} &\lesssim \left(\int_B \left(\int_{\complement(2B)} \frac{|b(y)-b(z)|^k}{|x-y|^{n-\alpha}}|f(y)|dy\right)^q w^q(z)dz\right)^{\frac{1}{q}}\\ &\lesssim \left(\int_B \left(\int_{\complement(2B)} \frac{|b(y)-b_{B,w}|^k}{|x-y|^{n-\alpha}}|f(y)|dy\right)^q w^q(z)dz\right)^{\frac{1}{q}}\\ &\quad + \left(\int_B \left(\int_{\complement(2B)} \frac{|b(z)-b_{B,w}|^k}{|x-y|^{n-\alpha}}|f(y)|dy\right)^q w^q(z)dz\right)^{\frac{1}{q}}\\ &= I_1 + I_2.\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned}I_1 &= \left(w^q(B)\right)^{\frac{1}{q}} \int_{\complement(2B)} \frac{|b(y)-b_{B,w}|^k}{|x-y|^{n-\alpha}}|f(y)|dy\\ &\approx \left(w^q(B)\right)^{\frac{1}{q}} \int_{\complement(2B)} |b(y)-b_{B,w}|^k |f(y)| \int_{|x-y|}^{\infty} \frac{dt}{t^{n-\alpha+1}}dy\\ &\approx \left(w^q(B)\right)^{\frac{1}{q}} \int_{2r}^{\infty}\int_{2r\le|x-y|\le t} |b(y)-b_{B,w}|^k |f(y)|dy\frac{dt}{t^{n-\alpha+1}}\\ &\lesssim \left(w^q(B)\right)^{\frac{1}{q}} \int_{2r}^{\infty}\int_{B(x,t)} |b(y)-b_{B,w}|^k |f(y)|dy\frac{dt}{t^{n-\alpha+1}}.\end{aligned}$$

Applying Hölder's inequality and by Lemma 4.4, we get

$$\begin{aligned}I_1 &\lesssim \left(w^q(B)\right)^{\frac{1}{q}} \int_{2r}^{\infty} \left(\int_{B(x,t)} |b(y)-b_{B(x,r),w}|^{kp'} w(y)^{-p'}dy\right)^{\frac{1}{p'}} \|f\|_{L_{p,w^p}(B(x,t))}\frac{dt}{t^{n-\alpha+1}}\\ &\lesssim \|b\|_*^k \left(w^q(B)\right)^{\frac{1}{q}} \int_{2r}^{\infty} \left(1+\ln^k\frac{t}{r}\right)\|w^{-1}\|_{L_{p'}(B(x,t))}\|f\|_{L_{p,w^p}(B(x,t))}\frac{dt}{t^{n-\alpha+1}}\\ &\lesssim [w]_{A_{p,q}}\|b\|_*^k \left(w^q(B)\right)^{\frac{1}{q}} \int_{2r}^{\infty} \ln^k\left(e+\frac{t}{r}\right)\|f\|_{L_{p,w^p}(B(x,t))}\left(w^q(B(x,t))\right)^{-\frac{1}{q}}\frac{dt}{t}.\end{aligned}$$

In order to estimate I_2 note that

$$I_2 = \left(\int_B |b(z)-b_{B,w}|^{kq}w^q(z)dz\right)^{\frac{1}{q}} \int_{\complement(2B)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy.$$

By Lemma 4.4, we get

$$I_2 \lesssim \|b\|_*^k \left(w^q(B)\right)^{\frac{1}{q}} \int_{\complement(2B)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy.$$

Applying Hölder's inequality, we get

$$\begin{aligned}\int_{\complement(2B)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))}\|w^{-1}\|_{L_{p'}(B(x,t))}\frac{dt}{t^{n-\alpha+1}}\\ &\le [w]_{A_{p,q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))}\left(w^q(B(x,t))\right)^{-\frac{1}{q}}\frac{dt}{t}.\end{aligned} \tag{5.9}$$

Thus, by (5.9)

$$I_2 \lesssim \|b\|_*^k \left(w^q(B)\right)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}.$$

Summing up I_1 and I_2, for all $p \in [1,\infty)$ we get

$$\|T_{b,\alpha,k} f_2\|_{L_{p,w^p}(B)} \lesssim \|b\|_*^k \left(w^q(B)\right)^{\frac{1}{q}} \int_{2r}^{\infty} \ln^k \left(e + \frac{t}{r}\right) \|f\|_{L_{p,w^p}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}. \tag{5.10}$$

Finally,

$$\|T_{b,\alpha,k} f\|_{L_{p,w^p}(B)} \lesssim \|b\|_*^k \|f\|_{L_{p,w^p}(2B)} + \|b\|_*^k \left(w^q(B)\right)^{\frac{1}{q}} \int_{2r}^{\infty} \ln^k \left(e + \frac{t}{r}\right) \|f\|_{L_{p,w^p}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t},$$

and the statement of Theorem 3.6 follows by (5.4).

□

Now we shall get to the proof of Theorem 3.7

Proof of Theorem 3.7. By Theorem 3.5 and Theorem 4.1 we have

$$\|T_{b,0,k} f\|_{M_{p,\varphi_2}(w)} \lesssim \|b\|_*^k \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \ln^k \left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t}$$

$$= \|b\|_*^k \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-1}} \ln^k \left(e + \frac{1}{tr}\right) \|f\|_{L_{p,w}(B(x,t^{-1}))} w(B(x,t^{-1}))^{-\frac{1}{p}} \frac{dt}{t}$$

$$= \|b\|_*^k \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-1})^{-1}\, r\, \frac{1}{r} \int_0^{r} \ln^k \left(e + \frac{r}{t}\right) \|f\|_{L_{p,w}(B(x,t^{-1}))} w(B(x,t^{-1}))^{-\frac{1}{p}} \frac{dt}{t}$$

$$\lesssim \|b\|_*^k \sup_{x\in\mathbb{R}^n,r>0} \varphi_1(x,r^{-1})^{-1} w(B(x,r^{-1}))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r^{-1}))}$$

$$= \|b\|_*^k \sup_{x\in\mathbb{R}^n,r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))}$$

$$= \|b\|_*^k \|f\|_{M_{p,\varphi_1}(w)}.$$

□

Now we shall get to the proof of Theorem 3.8.

Proof of Theorem 3.8. By Theorem 3.6 and Theorem 4.2 we have

$$\|T_{b,\alpha,k}f\|_{M_{p,\varphi_2}(w^q)} \lesssim \|b\|_*^k \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \ln^k\left(e+\frac{t}{r}\right) \|f\|_{L_{p,w^p}(B(x,t))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}$$

$$= \|b\|_*^k \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-1}} \ln^k\left(e+\frac{1}{tr}\right) \|f\|_{L_{p,w^p}(B(x,t^{-1}))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}$$

$$= \|b\|_*^k \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-1})^{-1}\; r\; \frac{1}{r} \int_0^{r} \ln^k\left(e+\frac{r}{t}\right) \|f\|_{L_{p,w^p}(B(x,t^{-1}))} \left(w^q(B(x,t))\right)^{-\frac{1}{q}} \frac{dt}{t}$$

$$\lesssim \|b\|_*^k \sup_{x\in\mathbb{R}^n,r>0} \varphi_1(x,r^{-1})^{-1}\left(w^p(B(x,r^{-1}))\right)^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r^{-1}))}$$

$$= \|b\|_*^k \sup_{x\in\mathbb{R}^n,r>0} \varphi_1(x,r)^{-1}\left(w^p(B(x,r))\right)^{-\frac{1}{p}} \|f\|_{L_{p,w^p}(B(x,r))}$$

$$= \|b\|_*^k \|f\|_{M_{p,\varphi_1}(w^p)}.$$

□

6 Some applications

In this section, we shall apply Theorems 3.7 and 3.8 to several particular operators such as the Littlewood-Paley operator, the Marcinkiewicz operator, the Bochner-Riesz operator and the fractional powers of some analytic semigroups.

6.1 Littlewood-Paley operator

The Littlewood-Paley functions play an important role in classical harmonic analysis, for example in the study of non-tangential convergence of Fatou type and boundedness of Riesz transforms and multipliers [35, 36, 37, 39]. The Littlewood-Paley operator (see [24, 39]) is defined as follows.

Definition 3. *Suppose that $\psi \in L_1(\mathbb{R}^n)$ satisfies*

$$\int_{\mathbb{R}^n} \psi(x)dx = 0. \tag{6.1}$$

Then the generalized Littlewood-Paley g function g_ψ is defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$ and $F_t(f) = \psi_t * f$.

The sublinear kth order commutator of the operator g_ψ is defined by

$$[b^k, g_\psi](f)(x) = \left(\int_0^\infty |F_t^{b,k}(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^{b,k}(f)(x)=\int_{\mathbb{R}^n}[b(x)-b(y)]^k\psi_t(x-y)f(y)dy.$$

The following theorem is valid (see [28], Theorem 5.2.2).

Theorem 6.1. *Suppose that $\psi\in L_1(\mathbb{R}^n)$ satisfies* (6.1) *and the following properties:*

$$|\psi(x)|\leq\frac{C}{(1+|x|)^{n+1}}, \tag{6.2}$$

$$|\nabla\psi(x)|\leq\frac{C}{(1+|x|)^{n+2}}, \tag{6.3}$$

where $C>0$ is independent of x. Then g_ψ is bounded on $L_{p,w}(\mathbb{R}^n)$ for $1<p<\infty$ and $w\in A_p$.

Let H be the space $H=\{h:\|h\|=(\int_0^\infty|h(t)|^2dt/t)^{1/2}<\infty\}$, then, for each fixed $x\in\mathbb{R}^n$, $F_t(f)(x)$ may be viewed as a mapping from $[0,\infty)$ to H, and it is clear that $g_\psi(f)(x)=\|F_t(f)(x)\|$.

In fact, by Minkowski inequality and the conditions on ψ, we get

$$\begin{aligned}\big|[b^k,g_\psi]\big|&\leq\int_{\mathbb{R}^n}|b(x)-b(y)|^k|f(y)|\left(\int_0^\infty|\psi_t(x-y)|^2\frac{dt}{t}\right)^{1/2}dy\\&\lesssim\int_{\mathbb{R}^n}|b(x)-b(y)|^k|f(y)|\left(\int_0^\infty\frac{t^{-2n}}{(1+|x-y|/t)^{2(n+1)}}\frac{dt}{t}\right)^{1/2}dy\\&=\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|^k}{|x-y|^n}|f(y)|dy.\end{aligned}$$

Thus we get

Corollary 6.1. *Let $1<p<\infty$, and $w\in A_p$. Suppose that (φ_1,φ_2) satisfy condition* (3.5)*, $b\in BMO(\mathbb{R}^n)$ and $\psi\in L_1(\mathbb{R}^n)$ satisfies* (6.1)-(6.3)*. Then the kth order commutator of Littlewood-Paley operator $[b^k,g_\psi]$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

From Corollary 3.7 we get the following

Corollary 6.2. *Let $1<p<\infty$, $0<\kappa<1$, $w\in A_p$, and $b\in BMO(\mathbb{R}^n)$. Suppose that $\psi\in L_1(\mathbb{R}^n)$ satisfies* (6.1)-(6.3)*. Then the operator $[b^k,g_\psi]$ is bounded on $L_{p,\kappa}(w)$.*

6.2 Marcinkiewicz operator

Let $S^{n-1}=\{x\in\mathbb{R}^n:|x|=1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n\setminus\{0\}$, that is,

$$\Omega(\mu x)=\Omega(x),\quad\text{for any}\quad\mu>0,x\in\mathbb{R}^n\setminus\{0\}.$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \leq 1$, that is there exists a constant $M > 0$ such that,

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, Stein [36] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [28, 35, 37, 39].

The Marcinkiewicz operator is defined by (see [40])

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy.$$

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

The sublinear kth-order commutator of the operator $\mu_{\Omega,\alpha}$ is defined by

$$\mu_{b,\Omega,\alpha,k}(f)(x) = \left(\int_0^\infty |F^b_{\Omega,\alpha,k,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F^b_{\Omega,\alpha,k,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} [b(x) - b(y)]^k f(y)dy.$$

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then, it is clear that $\mu_\Omega(f)(x) = \|F_{\Omega,t}(x)\|$.

By Minkowski inequality and the above conditions on Ω, we get

$$\mu_{\Omega,\alpha}(f)(x) \leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3}\right)^{1/2} dy \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy$$

and

$$\begin{aligned}\mu_{b,\Omega,\alpha,k}(f)(x) &\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|\,|b(x)-b(y)|^k}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3}\right)^{1/2} dy \\ &\leq C \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|^k}{|x-y|^{n-\alpha}} |f(y)|dy.\end{aligned}$$

Thus, $\mu_{\Omega,\alpha}$ and $\mu_{b,\Omega,\alpha,k}$ satisfies conditions (2.1) and (2.2) respectively. It is known that for $w\in A_{p,q}$ $\mu_{\Omega,\alpha}$ is bounded from $L_{p,w^p}(\mathbb{R}^n)$ on $L_{q,w^q}(\mathbb{R}^n)$ for $p>1$, and from $L_{1,w}(\mathbb{R}^n)$ to $WL_{q,w^q}(\mathbb{R}^n)$ (see [40]), then from Theorems 3.7 and 3.8 we get the following new results.

Corollary 6.3. *Let $1\le p<q<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $w\in A_{p,q}$, (φ_1,φ_2) satisfy condition (3.4), and Ω satisfies conditions $(a)-(c)$. Then $\mu_{\Omega,\alpha}$ is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$ for $p>1$ and bounded from $M_{1,\varphi_1}(w)$ to $WM_{q,\varphi_2}(w^q)$.*

Corollary 6.4. *Let $1<p<q<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $w\in A_{p,q}$, $b\in BMO(\mathbb{R}^n)$, (φ_1,φ_2) satisfy condition (3.6), and Ω satisfies conditions $(a)-(c)$. Then $\mu_{b,\Omega,\alpha,k}$ is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$.*

Note that, in the case $w=1$, $\alpha=0$ and $k=1$ Corollaries 6.3, 6.4 was proved in [19].

Corollary 6.5. *Let $1\le p<q<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $0<\kappa<\frac{p}{q}$, $w\in A_{p,q}$, and Ω satisfies conditions $(a)-(c)$. Then $\mu_{\Omega,\alpha}$ is bounded from $L_{p,\kappa}(w^p,w^q)$ to $L_{q,\kappa q/p}(w^q)$ for $p>1$ and from $L_{1,\kappa}(w,w^q)$ to $WL_{q,\kappa q}(w^q)$ for $p=1$.*

Corollary 6.6. *Let $1<p<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $0<\kappa<\frac{p}{q}$, $w\in A_{p,q}$, $b\in BMO(\mathbb{R}^n)$, and Ω satisfies conditions $(a)-(c)$. Then $\mu_{b,\Omega,\alpha,k}$ is bounded from $L_{p,\kappa}(w^p,w^q)$ to $L_{q,\kappa q/p}(w^q)$.*

6.3 Bochner-Riesz operator

Let $\delta>(n-1)/2$, $\widehat{B_t^\delta(f)}(\xi)=(1-t^2|\xi|^2)_+^\delta\hat{f}(\xi)$ and $B_t^\delta(x)=t^{-n}B^\delta(x/t)$ for $t>0$. The maximal Bochner-Riesz operator is defined by (see [25, 26])

$$B_{\delta,*}(f)(x)=\sup_{t>0}|B_t^\delta(f)(x)|.$$

Let H be the space $H=\{h:\|h\|=\sup_{t>0}|h(t)|<\infty\}$, then it is clear that $B_{\delta,*}(f)(x)=\|B_t^\delta(f)(x)\|$.

By [14]

$$B_{\delta,*}(f)(x)\lesssim\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}dy.$$

Thus, $B_{\delta,*}$ satisfies condition (2.1) with $\alpha=0$. It is known that $B_{\delta,*}$ is bounded on $L_{p,w}(\mathbb{R}^n)$ for $1<p<\infty$ and $w\in A_p$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{1,w}(\mathbb{R}^n)$ for $w\in A_1$ (see [33, 38]), then from Theorem 3.7 we get

Corollary 6.7. *Let $1<p<\infty$, and $w\in A_p$. Suppose that (φ_1,φ_2) satisfy condition (3.5), $\delta>(n-1)/2$ and $b\in BMO(\mathbb{R}^n)$. Then the operator $[b^k,B_{\delta,*}]$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

From Corollary 6.7 we get the following

Corollary 6.8. *Let $1<p<\infty$, $0<\kappa<1$, $w\in A_p$, $b\in BMO(\mathbb{R}^n)$, and $\delta>(n-1)/2$. Then the operator $[b^k,B_{\delta,*}]$ is bounded on $L_{p,\kappa}(w)$.*

6.4 Fractional powers of some analytic semigroups

The theorems of the previous sections can be applied to various operators which are estimated from above by the Riesz potentials. We give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x,y)| \leq \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{6.4}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where c_1, $c_2 > 0$ are independent of x, y and t.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α. See, for example, Chapter 5 in [35].

Property (6.4) is satisfied for large classes of differential operators (see, for example [4]). In [4] also other examples of operators which are estimates from above by the Riesz potentials are given. In these cases Theorems 3.4 and 3.8 are also applicable for proving boundedness of those operators and commutators from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$.

Theorem 6.2. *Let condition* (6.4) *be satisfied. Moreover, let* $1 \leq p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w \in A_{p,q}$, *and* (φ_1, φ_2) *satisfy condition* (3.4). *Then* $L^{-\alpha/2}$ *is bounded from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$ *for* $p > 1$ *and from* $M_{1,\varphi_1}(w)$ *to* $WM_{q,\varphi_2}(w^q)$ *for* $p = 1$.

Proof. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (6.4), it follows that

$$|L^{-\alpha/2} f(x)| \lesssim I_\alpha(|f|)(x)$$

(see [13]). Hence by the aforementioned theorems we have

$$\|L^{-\alpha/2} f\|_{M_{q,\varphi_2}(w^q)} \lesssim \|I_\alpha(|f|)\|_{M_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M_{p,\varphi_1}(w^p)}.$$

□

Corollary 6.9. *Let condition* (6.4) *be satisfied. Moreover, let* $1 \leq p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 < \kappa < \frac{p}{q}$, *and* $w \in A_{p,q}$. *Then* $L^{-\alpha/2}$ *is bounded from* $L_{p,\kappa}(w^p, w^q)$ *to* $L_{q,\kappa q/p}(w^q)$ *for* $p > 1$ *and from* $L_{1,\kappa}(w, w^q)$ *to* $WL_{q,\kappa q}(w^q)$ *for* $p = 1$.

Let b be a locally integrable function on $\mathbb{R}^n$, the $k$the order commutator of b and $L^{-\alpha/2}$ is defined as follows

$$[b^k, L^{-\alpha/2}] f(x) = L^{-\alpha/2}\big((b(x) - b(\cdot))^k f\big)(x).$$

In [13] extended the result of [6] from $(-\Delta)$ to the more general operator L defined above. More precisely, they showed that when $b \in BMO(\mathbb{R}^n)$, then the $k$the order commutator operator $[b^k, L^{-\alpha/2}]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then from Theorem 3.8 we get

Theorem 6.3. *Let condition* (6.4) *be satisfied. Moreover, let* $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, $w \in A_{p,q}$, *and* (φ_1, φ_2) *satisfy condition* (3.6). *Then* $[b^k, L^{-\alpha/2}]$ *is bounded from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$.

Proof. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (6.4), it follows that

$$|[b^k, L^{-\alpha/2}]f(x)| \lesssim [b^k, I_\alpha](|f|)(x)$$

(see [13]). Hence by the aforementioned theorems we have

$$\|[b^k, L^{-\alpha/2}]f\|_{M_{q,\varphi_2}(w^q)} \lesssim \|[b^k, I_\alpha](|f|)\|_{M_{q,\varphi_2}(w^q)} \lesssim \|b\|_*^k \|f\|_{M_{p,\varphi_1}(w^p)}.$$

□

Corollary 6.10. *Let condition* (6.4) *be satisfied. Moreover, let* $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, $0 < \kappa < \frac{p}{q}$, *and* $w \in A_{p,q}$. *Then* $[b, L^{-\alpha/2}]$ *is bounded from* $L_{p,\kappa}(w^p, w^q)$ *to* $L_{q,\kappa q/p}(w^q)$.

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Vagif Guliyev
Department of Mathematics
Ahi Evran University
40100 Kirsehir, Turkey
Institute of Mathematics and Mechanics
Academy of Sciences of Azerbaijan
AZ 1141 Baku, Azerbaijan
E-mail: vagif@guliyev.com

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