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LOCAL SMOOTHNESS OF THE CONJUGATE FUNCTIONS

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Abstract. In the paper the author deals with the following problem: what can we say about the behaviour of the conjugate function at a fixed point, if the global smoothness as well as the behaviour at this point of the original function are known? Sharp results on this and related problems are obtained.

1 Introduction

Let $T = [-\pi, \pi]$ and $f \in L(T)$ be a 2π-periodic function. The conjugate function is defined by

$$\tilde{f}(x) = -\frac{1}{\pi} \int\limits_0^\pi \frac{f(x+t) - f(x-t)}{2tg\frac{t}{2}} dt \tag{1}$$

for $x \in T$ if the integral exists (at 0 this integral is understood in the improper sense). N.N. Luzin [9] (for $f \in L_2(T)$) and I.I. Privalov [12] (for $f \in L(T)$) proved the existence of integral (1) almost everywhere on T (see also [1]). In the sequel the function f will be assumed to satisfy conditions which ensure that integral (1) exists as the usual Lebesgue integral.

Works of well-known mathematicians were devoted to the following very important problem: *If a function f has certain smoothness in some space, then what can be said about the smoothness of the conjugate function in the same space?*

We start with recalling the following well-known Riesz's theorem [13].

Theorem A. *For any $1 < p < \infty$ the conjugation operator is a continuous linear operator on the $L_p(T)$ space.*

In $L(T)$ and $C(T)$ spaces the situation is more complicated.

The study of the problem stated above in $C(T)$ was started in the work of I.I. Privalov [12], who obtained the following result.

Theorem B. *If $\alpha \in (0,1)$, and a 2π-periodic function $f \in \operatorname{Lip}\alpha$, then the conjugate function $\tilde{f} \in \operatorname{Lip}\alpha$.*

Later on this theorem has been repeatedly generalized by various mathematicians, until N.K. Bari and S.B. Stechkin [2] obtained a criterion for the Nikol'skii space

$$H^{\omega}(T) = \{f \in C(T) : \omega(f,t)_{C(T)} \leq \omega(t),\, 0 \leq t \leq \pi\},$$

where ω is a given modulus of continuity and

$$\omega(f,t)_{C(T)} = \max_{|h|\leq t} \max_{x\in T} |f(x+t) - f(x)|.$$

Theorem C. *Let ω be a modulus of continuity. Then the conjugate function $\tilde{f}$ of any function $f \in H^{\omega}(T)$ is also belongs to $H^{\omega}(T)$ if and only if the modulus of continuity satisfies the conditions*

$$\int_0^t \frac{\omega(u)}{u} du = O(\omega(t)) \tag{2}$$

as $t \to 0^+$ and

$$t \int_t^1 \frac{\omega(u)}{u^2} du = O(\omega(t)) \tag{3}$$

as $t \to 0^+$.

Note that conditions (2) and (3), as well as the ones equivalent to them (see [2]) play an important role in research on the theory of functions. They are special cases of more general Ul'yanov's conditions [15], [16]. In the sequel, we shall also use the following generalization of (3): for natural $k > 1$

$$t^k \int_t^1 \frac{\omega(u)}{u^{k+1}} du = O(\omega(t)) \tag{3_k}$$

as $t \to 0^+$.

Moreover, from the paper by N.K. Bari and S.B. Stechkin [2] we can conclude how the conjugation operator can deteriorate smoothness properties of functions in the space $H^{\omega}(T)$ in the case when conditions (2) and (3) are not satisfied. Thus the problem of preserving smoothness in the space $C(T)$ in its original formulation can be considered to be solved. However another question arose: is it possible to say something about the smoothness of the conjugate function at a certain point, if it is known that the original function, in addition to certain global smoothness on T, has better smoothness at this point? The first result in this direction was obtained by the Hungarian mathematician M. Salay [14], who established the following theorem.

Theorem D. *Let $0 < \beta < \alpha \leq 1$. If a 2π-periodic function $f \in \mathrm{Lip}\beta$ and for some $x_0 \in T : |f(x_0+t) - f(x_0)| \leq |t|^{\alpha}$ for $t \in T$, then*

$$|\tilde{f}(x_0+t) - \tilde{f}(x_0)| \leq C(\alpha,\beta)|t|^{\beta + \frac{\alpha-\beta}{1+\beta}\beta} \tag{4}$$

for $t \in T$, where $C(\alpha,\beta) > 0$ depends only on α and β.

P.L. Ul'yanov posed the problem of finding the best possible result in this direction and some related problems, solutions of which are given in this paper. These results were obtained by the author in his thesis [5] and have been deposited at VINITI [6]. However, the full text has not been published yet, although the results were later extended by other authors to the multi-dimensional case. One of the main aims of this article is the publication of these results with complete proofs.

2 Main results

First we introduce some notation.

Definition. If k is a natural number, then we say that a function α defined on $[0,\pi]$ belongs to B_k, if the following conditions hold:

1) the function α is continuous and nonnegative on $[0,\pi]$;
2) $\alpha(0)=0$ and $\alpha(t)\not\equiv 0$;
3) the function α is strictly increasing on $[0,\pi]$, briefly $\alpha\uparrow$ on $[0,\pi]$;
4) for any $n\in\mathbf{N}$ we have $\alpha(nt)\le n^k\alpha(t)$ when $t\in[0,\frac{\pi}{n}]$.

If ω is a strictly increasing on $[0,\pi]$ modulus of continuity which satisfies

$$\int_0^{\pi}\frac{\omega(u)}{u}du<\infty, \tag{5}$$

a function $\alpha\in B_k$ for some $k\ge 1$, $\alpha(t)=o(\omega(t))$ as $t\to 0^+$, and $\omega(\pi)>\alpha(\pi)$, then we can define the function

$$\varphi(\omega,\alpha,t)=\omega^{-1}(\alpha(t))$$

for $t\in[0,\pi]$, where ω^{-1} is the inverse function of ω, and the function

$$\eta(\omega,\alpha,t)=\int_0^{\varphi(\omega,\alpha,|t|)}\frac{\omega(u)}{u}du+\alpha(|t|)\left|\ln\frac{\varphi(\omega,\alpha,|t|)}{|t|}\right|$$

for $t\in T$. As will be shown in Section 3, depending on the functions α and ω either of two terms in the definition of $\eta(\omega,\alpha,t)$ can be main as $t\to 0^+$. Below, in the cases in which this cannot cause ambiguity, we write $\varphi(t)$ instead of $\varphi(\omega,\alpha,t)$ and $\eta(t)$ instead of $\eta(\omega,\alpha,t)$.

Let for any function $g\in C(T)$, for any $x,t\in T$ and for any natural k

$$\Delta_k(g,x,t)=\sum_{n=0}^{k}(-1)^n\binom{k}{n}g(x+nt).$$

Everywhere below, C denotes a positive constant (not necessarily the same in different cases), $C(k)$ denotes a positive quantity depending only on k, etc.

For functions f and g positive in some right half-neighbourhood of zero the relation $f(t)\sim g(t)$ as $t\to 0^+$ means that there exist $\delta>0$ and constants $0<C_1<C_2$ such that for $t\in(0,\delta)$ we have $C_1g(t)\le f(t)\le C_2g(t)$.

The following statements hold.

Theorem 1. *Let k be a natural number, ω be a strictly increasing on $[0,\pi]$ modulus of continuity satisfying (5), $\alpha \in B_k$, α satisfy (3_k), $\alpha(t) = o(\omega_0(t))$ as $t \to 0^+$, and $\omega(\pi) > \alpha(\pi)$.*

If $f \in H^\omega(T)$, a point $x_0 \in T$ and

$$|f(x_0 + t) - f(x_0)| \le \alpha(|t|)$$

for $t \in T$, then

$$|\Delta_k(\tilde{f}, x_0, t)| \le C(k, \omega, \alpha)\eta(\omega, \alpha, t)$$

for $t \in T$.

In Theorem 1 condition (3_k) on the function α is imposed in order to separate the local deterioration of smoothness from the global deterioration under the action of the conjugation operator.

Theorem 2. *Let k be a natural number, ω be a strictly increasing on $[0,\pi]$ modulus of continuity satisfying (5), $\alpha \in B_k$, α satisfy (3_k), $\alpha(t) = o(\omega(t))$ as $t \to 0^+$, and $\omega(\pi) > \alpha(\pi)$.*

Then, for any numerical sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \downarrow 0$ as $n \to \infty$, there exists a function $f \in H^\omega(T)$, a point $x_0 \in T$ and an increasing sequence of natural numbers $\{n_p\}_{p=1}^\infty$, for which

$$|f(x_0 + t) - f(x_0)| \le \alpha(|t|)$$

for $t \in T$ and

$$|\Delta_k(\tilde{f}, x_0, t_{n_p})| \ge \frac{1}{2\pi}\eta(\omega, \alpha, t_{n_p})$$

for all p.

Under the conditions of Theorem D function $\eta(\omega, \alpha, t) \sim t^\alpha \ln \frac{1}{t}$ as $t \to 0^+$. Hence Theorem 1, in particular, significantly improves Theorem D.

In the cases when the a priori known behaviour at the point of the difference of the second order is known, the situation is more complicated. The following result is true.

Theorem 3. *Let ω be a strictly increasing on $[0,\pi]$ modulus of continuity satisfying (5), $\alpha \in B_2$, α satisfy condition (3_2), $\alpha(t) = o(\omega(t))$ as $t \to 0^+$, $\frac{\alpha(t)}{t} \uparrow$ on $(0,\pi]$, and $\omega(\pi) > \alpha(\pi)$.*

If a function $f \in H^\omega(T)$, $x_0 \in T$,

$$|f(x_0) - 2f(x_0 + t) + f(x_0 + 2t)| \le \alpha(|t|) \tag{6}$$

and

$$|f(x_0 - t) - 2f(x_0) + f(x_0 + t)| \le \alpha(|t|) \tag{7}$$

for $t \in T$, then

$$|\Delta_2(\tilde{f}, x_0, t)| \le C(\omega, \alpha)\eta(\omega, \alpha, t) \tag{8}$$

for $t \in T$.

Note that Theorem 3 cannot be strengthened. This follows immediately by Theorem 2. In the case when only condition (6) holds the assertion of Theorem 3 is incorrect. This is a consequence of the following statements.

Theorem 4. *Let $0 < \beta \leq 1 < a < 2$, a function $f \in \operatorname{Lip}\beta$ and*

$$|f(0) - 2f(t) + f(2t)| \leq |t|^a$$

for $t \in T$.

Then the function γ defined by

$$\gamma(t) = \lim_{n\to\infty} 2^n (f(t \cdot 2^{-n}) - f(0))$$

exists for all $t \in T$ and

$$\left| \Delta_2(\tilde{f}, 0, t) - \frac{2}{\pi} t \int_{\frac{\pi}{2}}^{\pi} \frac{\gamma(u) + \gamma(-u)}{u^2} du \right| \leq C(a, \beta) |t|^a \ln \frac{1}{|t|}$$

for $t \in (-\frac{1}{2}, \frac{1}{2})$.

Corollary 1. *Let $0 < \beta \leq 1 < a < 2$, a function $f \in \operatorname{Lip}\beta$, $f(t) - 2f(0) + f(-t) \geq 0$ for $t \in (0, \pi]$ and*

$$|f(0) - 2f(t) + f(2t)| \leq |t|^a$$

for $t \in T$.

Then inequality (8) (here $\eta(t) \sim t^a \ln \frac{1}{t}$ as $t \to 0^+$) implies condition (7) with $\alpha(t) = t^a$ for $t \in [0, \pi]$ (possibly with a constant multiple in right-hand side).

Without loss of generality we will assume that $x_0 = 0$ and $f(x_0) = 0$ in the proofs of Theorems 1-4.

The developed methods allow us to give an answer to another question. Let a 2π-periodic function $f \in H^\omega(T)$, where the modulus of continuity ω satisfies conditions (2) and (3). Then, by Theorem C the conjugate function $\tilde{f} \in H^\omega(T)$. Let us denote

$$M(f) = \{x \in T : f(x + t) - f(x) = o(\omega(|t|)) \quad \text{as } t \to 0\}.$$

G. Freud [8] established the following result.

Theorem E. *Let $0 < a < 1$ and $f \in \operatorname{Lip} a$. Then the Lebesgue measure*

$$\mu(M(f) \Delta M(\tilde{f})) = 0,$$

where $M(f)\Delta M(\tilde{f})$ is the symmetric difference of $M(f)$ and $M(\tilde{f})$.

It turns out that a much stronger result holds.

Theorem 5. *Let a modulus of continuity ω satisfy conditions (2) and (3). Then for any function $f \in H^\omega(T)$ we have $M(\tilde{f}) = M(f)$.*

Another question to be discussed in the article is the problem of differentiability at a point of the conjugate function.

Let $x_0 \in T$, $r \uparrow$ on $[0,1]$ and $r(t) = o(t)$ as $t \to 0^+$. We say that a continuous on T 2π-periodic function $f \in P(x_0, r(t))$, if there exists $f'(x_0)$ and $|f(x_0+h) - f(x_0) - f'(x_0)h| \le r(|h|)$ for $|h| \le 1$.

Theorem 6. *Let ω be a strictly increasing on $[0,\pi]$ modulus of continuity satisfying (5), $x_0 \in T$, $r \uparrow$ on $[0,1]$, $r(t) = o(t)$ as $t \to 0^+$, and $\omega(\pi) > r(\pi)$.*

Then the derivative $\tilde{f}'(x_0)$ exists for any function $f \in H^\omega(T) \cap P(x_0, r(t))$ if and only if the following conditions hold:

$$\int_0^1 t^{-2} r(t)dt < \infty \qquad \text{and} \qquad \eta(\omega, r, t) = o(t) \quad as \quad t \to 0^+.$$

The settings of all problems mentioned above belong to the author's supervisor academician of the Russian Academy of Sciences P.L. Ul'yanov.

In connection with the results of this paper it would be interesting to investigate dependence on the global and the local smoothness of the original function of the smoothness at a point for other special integrals. For example, for Hilbert transform of non-periodic functions or for convolution operators, which were considered in the paper [11] by E.D. Nursultanov, S.Yu. Tikhonov and N.T. Tleukhanova. We note that similar problems can be solved in other spaces. For example, the author [7] studied them in the spaces $L_p(T)$, $1 < p < \infty$ (in contrast to global smoothness, where everything is determined by Riesz's theorem, for local smoothness these problems are not trivial). It would be interesting to investigate similar problems for isotropic or anisotropic Besov spaces, which were studied, in particular, by K.A. Bekmaganbetov and E.D. Nursultanov [3], or for spaces of Morrey type, which were studied by V.I. Burenkov and E.D. Nursultanov [4] or for Lorentz spaces (see, for example, the paper by E.D. Nursultanov [10]).

3 Auxiliary statements

Lemma 1. *Let $\delta \in (0,\pi)$, a modulus of continuity ω satisfy condition (5), $f \in H^\omega(T)$, and $f(x) = 0$ for $x \in [\pi - \delta, \pi + \delta]$. If*

$$\hat{\tilde{f}}(x) = -\frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_\varepsilon^\pi \frac{f(x+t) - f(x-t)}{t} dt,$$

then the function $R(x) = \tilde{f}(x) - \hat{\tilde{f}}(x)$ is infinitely differentiable for $x \in (-\delta, \delta)$.

Proof. Since the function $\psi(t) = \frac{1}{t} - \frac{1}{2tg\frac{t}{2}}$ is continuous at the point $t = 0$, and $f(x) = 0$ for $x \in [\pi - \delta, \pi + \delta]$, then for $x \in (-\delta, \delta)$ we have

$$R(x) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \int_\varepsilon^\pi (f(x+t) - f(x-t))\psi(t)dt =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\psi(t)dt = \frac{1}{\pi}\int_{-\pi+x}^{\pi+x} f(u)\psi(u-x)du =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\psi(u-x)du. \tag{9}$$

But for $\tau \in [-\pi-\delta, \pi+\delta]$ the function $\psi(\tau)$ is infinitely differentiable. Hence the function $R(x)$ is also infinitely differentiable for $x \in (-\delta, \delta)$. $\square$

Corollary 2. *If the assumptions of Lemma 1 hold, then for any natural number k and for any $|t| \le \frac{\delta}{2k}$*

$$|\Delta_k(R, 0, t)| \le |t|^k \max_{u\in[-\frac{\delta}{2},\frac{\delta}{2}]} |R^{(k)}(u)| \le C(k,\delta)|t|^k \max_{x\in T}|f(x)|.$$

This result immediately follows from (9).

Lemma 2. *Let $\gamma \in (0,\pi)$, $f \in C([-\pi,\pi])$ and $f(x) = 0$ for $x \in [-\gamma,\gamma]$. Then for $x \in (-\frac{\gamma}{2},\frac{\gamma}{2})$ the function $\tilde{f}(x)$ is infinitely differentiable.*

Proof. Let $x \in (-\frac{\gamma}{2},\frac{\gamma}{2})$. Then

$$\tilde{f}(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x+t)}{2tg\frac{t}{2}}dt =$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(u)}{2tg\frac{u-x}{2}}du = -\frac{1}{\pi}\int_{[-\pi,\pi]\setminus[-\gamma,\gamma]} \frac{f(u)}{2tg\frac{u-x}{2}}du,$$

which implies the statement of Lemma 2. $\square$

Corollary 3. *If the assumptions of Lemma 2 hold, then for any natural number k and for any $|t| \le \frac{\gamma}{2k}$*

$$|\Delta_k(\tilde{f}, 0, t)| \le |t|^k \max_{u\in[-\frac{\gamma}{2},\frac{\gamma}{2}]} |\tilde{f}^{(k)}(u)| \le$$

$$\le |t|^k \max_{x\in T}|f(x)| \frac{1}{\pi} \cdot \int_{\frac{\gamma}{2}}^{\pi} \left| \left(\text{ctg}\frac{u}{2}\right)^{(k)} \right| du.$$

Lemma 3. *For any natural number k and for any t and x such that $t \ne mx$, $m = 0, 1, ..., k$ the following equality holds*

$$\Delta_k\left(\frac{1}{u}, t, -x\right) = \sum_{l=0}^{k} (-1)^l \begin{pmatrix} k \\ l \end{pmatrix} \frac{1}{t-lx} = \frac{(-1)^k k! x^k}{t(t-x)...(t-kx)}.$$

Proof. Let us use the method of mathematical induction. For $k=1$ we have

$$\frac{1}{t}-\frac{1}{t-x}=\frac{-x}{t(t-x)}.$$

Let for $k=n$ Lemma holds. Then

$$\Delta_{n+1}\left(\frac{1}{u},t,-x\right)=\Delta_n\left(\frac{1}{u},t,-x\right)-\Delta_n\left(\frac{1}{u},t-x,-x\right)=$$

$$=(-1)^n n!x^n\cdot\left(\frac{1}{t(t-x)...(t-nx)}-\frac{1}{(t-x)(t-2x)...(t-(n+1)x)}\right)=$$

$$=\frac{(-1)^{n+1}(n+1)!x^{n+1}}{t(t-x)...(t-(n+1)x)}.$$

□

Lemma 4. *Let a 2π-periodic function f be such that $|f(x)|\le M<\infty$ for $x\in T$ and $|f(0)-2f(t)+f(2t)|\le\gamma(|t|)$ for $t\in T$, where $\gamma\uparrow$ on $[0,\pi]$. Then*

$$|f(t)-f(0)|\le 2|t|\int\limits_{|t|}^{2}\frac{\gamma(u)}{u^2}du+2M|t|$$

for $t\in[-1,1]$.

Proof. Let $t\in(0,1]$ and n be the natural number such that $2^{-n}<t\le 2^{1-n}$. Then we obtain

$$|f(t)-f(0)|\le\left|\frac{1}{2}f(0)-f(t)+\frac{1}{2}f(2t)\right|+\frac{1}{2}|f(0)-f(2t)|\le$$

$$\le\frac{1}{2}\gamma(t)+\frac{1}{2}|f(0)-f(2t)|.$$

Applying this inequality, we obtain

$$|f(t)-f(0)|\le\frac{1}{2}\gamma(t)+\frac{1}{2^2}\gamma(2t)+\frac{1}{2^2}|f(0)-f(2^2t)|\le...\le$$

$$\le\sum_{k=0}^{n-1}2^{-k-1}\gamma(2^kt)+\frac{1}{2^n}|f(0)-f(2^nt)|\le 2Mt+$$

$$+2^{-n}\sum_{k=0}^{n-1}2^{n-k-1}\gamma(2^{-n+k+1})\le 2Mt+t\sum_{m=0}^{n-1}2^m\gamma(2^{-m})\le$$

$$\le 2Mt+2t\sum_{k=0}^{n-1}\int\limits_{2^{-m}}^{2^{-m+1}}\frac{\gamma(u)}{u^2}du\le 2Mt+2t\int\limits_{t}^{2}\frac{\gamma(u)}{u^2}du.$$

The case $t\in[-1,0)$ is similar, and thus, Lemma 4 is proved. □

The following statement will not be used in the sequel, but is of particular interest in connection with Lemma 4.

Lemma 5. *Let a 2π-periodic function f be such that $|f(0) - 2f(t) + f(2t)| \le \lambda(|t|)$ for $t \in T$, where $\lambda \uparrow$ on $[0, \pi]$ and*

$$\int_0^1 \frac{\lambda(t)}{t^2} dt < \infty.$$

Moreover, let

$$|f(t) - f(0)| = o(|t|) \tag{10}$$

as $t \to 0$. Then

$$|f(t) - f(0)| \le 2|t| \int_0^{2|t|} \frac{\lambda(u)}{u^2} du$$

for $t \in T$.

Proof. Let $t \in (0, \pi]$. Note that

$$|f(t) - f(0)| \le \left| f(t) - 2f\left(\frac{t}{2}\right) + f(0) \right| + 2\left| f\left(\frac{t}{2}\right) - f(0) \right| \le$$

$$\le \lambda(t) + 2\left| f\left(\frac{t}{2}\right) - f(0) \right|.$$

Applying this inequality, we obtain that

$$|f(t) - f(0)| \le \lambda(t) + 2\lambda(2^{-1}t) + 2^2|f(2^{-2}t) - f(0)| \le ... \le$$

$$\le \sum_{k=0}^{m} 2^k \lambda(2^{-k}t) + 2^{m+1}|f(2^{-m-1}t) - f(0)|$$

for any integer $m \ge 1$.

Passing to the limit as $m \to \infty$ and using (10), we get

$$|f(t) - f(0)| \le \sum_{k=0}^{\infty} 2^k \lambda(2^{-k}t) \le 2t \sum_{k=0}^{\infty} \int_{2^{-k}}^{2^{-k+1}} \frac{\lambda(u)}{u^2} du \le 2t \int_0^{2t} \frac{\lambda(u)}{u^2} du.$$

For $t \in [-1, 0)$, the situation is similar, and Lemma 5 is proved. □

Note that the example $f(x) = |x|$ on T shows that condition (10) does not follow even from the condition $f(0) - 2f(t) + f(2t) = 0$ for $t \in [-1, 1]$.

Lemma 6. *Let ω be a strictly increasing on $[0, \pi]$ modulus of continuity satisfying (5), $\alpha \in B_2$, α satisfy condition (3_2), $\alpha(t) = o(\omega(t))$ as $t \to 0^+$, $\frac{\alpha(t)}{t} \uparrow$ on $(0, \pi]$, and $\omega(\pi) > \alpha(\pi)$.*

If $\delta \in (0, \frac{\pi}{4})$, an odd function $f \in H^\omega(T)$, $f(x) = 0$ for $x \in [\pi - \delta, \pi + \delta]$, and $|f(2t) - 2f(t)| \le \alpha(|t|)$ for $t \in T$, then for $t \in T$

$$|\hat{\tilde{f}}(0) - 2\hat{\tilde{f}}(t) + \hat{\tilde{f}}(2t)| \le C(\omega, \alpha, \delta)\eta(\omega, \alpha, t).$$

Proof. Let us denote $\psi(t) = \frac{f(t)}{t}$ for $t \in T \setminus \{0\}$. Then

$$|\psi(2t) - \psi(t)| = \left|\frac{f(2t) - 2f(t)}{2t}\right| \leq \frac{\alpha(|t|)}{|2t|}$$

for $t \in T \setminus \{0\}$. Since $\max_{x\in T}|f(x)| \leq \omega(\pi)$, then by Lemma 4

$$|f(t)| = |f(t) - f(0)| \leq 2|t| \int\limits_{|t|}^{2} \frac{\alpha(u)}{u^2} du + 2\omega(\pi)|t| \leq C(\omega, \alpha)|t| \ln \frac{1}{|t|}$$

for $|t| \leq \frac{1}{2}$, consequently $|\psi(t)| \leq C(\omega, \alpha) \ln \frac{1}{|t|}$. Thus $\psi \in L(T)$.

Let us consider for $t \in (0, \frac{\delta}{2})$ the following expression (integrals around the points $u = t$ and $u = 2t$ are understood in the sense of principal value)

$$-\pi(\hat{\tilde{f}}(0) - 2\hat{\tilde{f}}(t) + \hat{\tilde{f}}(2t)) =$$

$$= \int\limits_T f(u) \left(\frac{1}{u} - \frac{2}{u-t} + \frac{1}{u-2t}\right) du =$$

$$= \int\limits_T f(u) \frac{2t^2}{u(u-t)(u-2t)} du = 2t \int\limits_T \psi(u) \frac{t}{(u-t)(u-2t)} du =$$

$$= 2t \left(\int\limits_T \psi(u) \frac{du}{u-2t} - \int\limits_T \psi(u) \frac{du}{u-t} \right) =$$

$$= 2t \left(\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi(2v) \frac{dv}{v-t} - \int\limits_{-\pi}^{\pi} \psi(v) \frac{dv}{v-t} \right) =$$

$$= 2t \left(\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\psi(2v) - \psi(v)) \frac{dv}{v-t} - \int\limits_{[-\pi,\pi]\setminus[-\frac{\pi}{2},\frac{\pi}{2}]} \psi(v) \frac{dv}{v-t} \right) \equiv$$

$$\equiv 2t(I_1 + I_2). \tag{11}$$

Since ψ is an odd function,

$$|I_2| \leq \left| \int\limits_{\frac{\pi}{2}}^{\pi} \psi(v) \left(\frac{1}{v-t} + \frac{1}{-v-t}\right) dv \right| \leq$$

$$\leq 2t \int\limits_{\frac{\pi}{2}}^{\pi} \frac{|\psi(v)|}{v^2 - t^2} dv \leq 2t\pi\omega(\pi) = C(\omega)t. \tag{12}$$

Furthermore we consider $t \in (0, t_0) \cap (0, \frac{\delta}{2})$, where t_0 is such that $\varphi(t) < \frac{t}{2}$ for $t \in (0, t_0)$. Then

$$I_1 = \int\limits_{-\frac{\pi}{2}}^{-2t} (\psi(2v) - \psi(v)) \frac{dv}{v-t} + \int\limits_{-2t}^{t-\varphi(t)} (\psi(2v) - \psi(v)) \frac{dv}{v-t} +$$

$$+ \int\limits_{t-\varphi(t)}^{t+\varphi(t)} (\psi(2v) - \psi(v)) \frac{dv}{v-t} + \int\limits_{t+\varphi(t)}^{2t} (\psi(2v) - \psi(v)) \frac{dv}{v-t} +$$

$$+ \int\limits_{2t}^{\frac{\pi}{2}} (\psi(2v) - \psi(v)) \frac{dv}{v-t} \equiv I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_{1,5}, \tag{13}$$

where the integral $I_{1,3}$ ais understood in the sense of principal value around the point $v = t$. Since ψ is an even function, using condition (3_2), we find that

$$|I_{1,1} + I_{1,5}| = \left| \int\limits_{2t}^{\frac{\pi}{2}} (\psi(2v) - \psi(v)) \left(\frac{1}{v-t} - \frac{1}{v+t} \right) dv \right| \leq$$

$$\leq \int\limits_{2t}^{\frac{\pi}{2}} \frac{\alpha(v)}{2v} \frac{2t}{v^2 - t^2} dv \leq 4t \int\limits_{2t}^{\frac{\pi}{2}} \frac{\alpha(v)}{v^3} dv \leq C(\alpha) \frac{\alpha(t)}{t}. \tag{14}$$

Moreover, using the condition $\frac{\alpha(u)}{u} \uparrow$ on $(0, \pi]$, we get that

$$|I_{1,2}| \leq \frac{\alpha(2t)}{4t} \int\limits_{-2t}^{t-\varphi(t)} \frac{dv}{|v-t|} \leq \frac{\alpha(t)}{t} (|\ln \frac{\varphi(t)}{t}| + \ln 3) \tag{15}$$

and analogously

$$|I_{1,4}| \leq \frac{\alpha(t)}{t} \left| \ln \frac{\varphi(t)}{t} \right|. \tag{16}$$

Consequently

$$|I_{1,3}| = \left| \int\limits_0^{\varphi(t)} \left(\frac{f(2t+2u)}{2t+2u} - \frac{f(t+u)}{t+u} - \frac{f(2t-2u)}{2t-2u} + \frac{f(t-u)}{t-u} \right) \frac{d\,u}{u} \right| =$$

$$= \frac{1}{2} \cdot \left| \int\limits_0^{\varphi(t)} \frac{(f(2t+2u) - f(2t-2u)) - 2(f(t+u) - f(t-u))}{t-u} \frac{du}{u} + \right.$$

$$\left. + \int\limits_0^{\varphi(t)} (f(2t+2u) - 2f(t+u)) \left(\frac{1}{t+u} - \frac{1}{t-u} \right) \frac{du}{u} \right| \leq$$

$$\leq \frac{1}{2}\cdot\frac{2}{t}\int\limits_0^{\varphi(t)}\frac{8\omega(u)}{u}du+\frac{8\alpha(2t)}{t^2}\int\limits_0^{\varphi(t)}du\leq$$

$$\leq \frac{16}{t}\left(\int\limits_0^{\varphi(t)}\frac{\omega(u)}{u}du+\alpha(t)\right). \tag{17}$$

By formulas (11) - (17) we obtain that for $t\in(0,t_0)\cap(0,\frac{\delta}{2})$ the following estimate holds

$$|(\hat{\hat{f}}(0)-2\hat{\hat{f}}(t)+\hat{\hat{f}}(2t))|\leq$$

$$\leq C(\omega,\alpha)\left(\int\limits_0^{\varphi(t)}\frac{\omega(u)}{u}du+\alpha(t)|\ln\frac{\varphi(t)}{t}|+\alpha(t)+t^2\right)\leq$$

$$\leq C(\omega,\alpha,\delta)\eta(\omega,\alpha,t). \tag{18}$$

If we increase appropriately $C(\omega,\alpha,\delta)$, then inequality (18) holds for all $t\in(0,\pi]$. For $t\in[-1,0]$, the situation is similar, and so Lemma 6 is proved. □

Lemma 7. *Let $0<\beta\leq 1<a<2$, $f\in \operatorname{Lip}\beta$ and for $t\in T$ $|f(2t)-2f(t)|\leq|t|^a$. Then the function γ, which is defined in Theorem 4 exists and satisfies the inequality*

$$|\gamma(x)-\gamma(y)|\leq C(a,\beta)|x-y|^{\frac{a-1}{a-\beta}\beta}$$

for $x,y\in T$.

Proof. Note that since $f\in C(T)$ and $|f(2t)-2f(t)|\leq|t|^a$ for $t\in T$, we get that $f(0)=0$. By the assumptions of the lemma we have

$$\left|\frac{f(2u)}{2u}-\frac{f(u)}{u}\right|\leq\frac{|u|^{a-1}}{2} \tag{19}$$

for $u\in T\setminus\{0\}$. By (19) it follows that, if $m>n\geq n_0$, then

$$|2^nf(2^{-n}t)-2^mf(2^{-m}t)|\leq\sum_{k=n}^{m-1}|2^kf(2^{-k}t)-2^{k+1}f(2^{-k-1}t)|=$$

$$=|t|\sum_{k=n}^{m-1}\left|\frac{f(2^{-k}t)}{2^{-k}t}-\frac{f(2^{-k-1}t)}{2^{-k-1}t}\right|\leq$$

$$\leq|t|^a\sum_{k=n}^{m-1}\left(\frac{1}{2^{k+1}}\right)^{a-1}\leq C(a)|t|^a2^{-n_0(a-1)}. \tag{20}$$

Thus for the sequence $\{2^nf(2^{-n}t)\}_{n=0}^{\infty}$ the Cauchy criterion holds, and the existence of the function γ is established.

Note that for any $t \in T \setminus \{0\}$ and for any $m \geq 1$ we have that

$$\frac{\gamma(2^m t)}{2^m} = 2^{-m} \lim_{n\to\infty} 2^n f(2^{-n}2^m t) =$$

$$= \lim_{n\to\infty} 2^{n-m} f(2^{m-n}t) = \lim_{n\to\infty} 2^n f(2^n t) = \gamma(t).$$

Hence from inequality (20) it follows that

$$|\gamma(t) - 2^n f(2^{-n}t)| \leq C(a)|t|^a 2^{-n(a-1)} \tag{21}$$

for $t \in T$. Moreover, by the assumptions of the lemma we get that

$$|2^n f(2^{-n}x) - 2^n f(2^{-n}y)| \leq 2^n |x-y|^\beta 2^{-n\beta}. \tag{22}$$

Without loss of generality, we can assume that $x < y$. Given $0 < x < y \leq \pi$. Then we choose the integer $m \geq 0$ such that $2^{1-m} < y - x \leq 2^{2-m}$. Let us assume in inequalities (21) and (22) $n = [\frac{\beta}{a-\beta}m]$, where $[b]$ is the integer part of the number b. Then we get that

$$|\gamma(x) - \gamma(y)| \leq |\gamma(x) - 2^n\gamma(x2^{-n})| + |2^n\gamma(x2^{-n}) - 2^n\gamma(y2^{-n})| +$$

$$+ |2^n\gamma(y2^{-n}) - \gamma(y)| \leq 2C(a)\pi^a 2^{-[\frac{\beta}{a-\beta}m](a-1)} +$$

$$+ 4 \cdot 2^{-[\frac{\beta}{a-\beta}m](1-\beta) - m\beta} \leq 4(C(a)\pi^a + 1)2^{-m\frac{a-1}{a-\beta}\beta} \leq C(a,\beta)|x-y|^{\frac{a-1}{a-\beta}\beta}. \tag{23}$$

The continuity of $\gamma(t)$ at $t = 0$ follows directly by its definition. So, in inequality (23) we may assume that $0 \leq x < y \leq \pi$. It is clear, that for $-\pi \leq x < y \leq 0$, the situation is similar. If $-\pi \leq x < 0 < y \leq \pi$, then

$$|\gamma(x) - \gamma(y)| \leq |\gamma(x)| + |\gamma(y)| = |\gamma(x) - \gamma(0)| + |\gamma(y) - \gamma(0)| \leq$$

$$\leq C(a,\beta)\left(|x|^{\frac{a-1}{a-\beta}\beta} + y^{\frac{a-1}{a-\beta}\beta}\right) \leq C(a,\beta)|x-y|^{\frac{a-1}{a-\beta}\beta}.$$

The proof is complete. □

4 Local smoothness of the conjugate functions: the case of the first difference

Proof of Theorem 1. First, we prove Theorem 1 for the function $\hat{\tilde{f}}$. Since $\alpha(t) = o(\omega(t))$ as $t \to 0^+$, then $\varphi(t) < t$ for sufficiently small positive t. We establish the estimate of the theorem for t such that $0 < (k+1)t < \frac{\pi}{3}$ and $\varphi(t) < t$. For small in absolute value negative t, the situation is similar. Also by the appropriate increasing of the constant the estimate of the theorem can be extended to all $t \in [-\pi, \pi]$. Let us consider the case $f(x) = 0$ for $x \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$. Note that since condition (5) holds for $0 \leq m \leq k$, we have

$$\hat{\tilde{f}}(mt) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(mt+u) - f(mt)}{u} du =$$

$$= -\frac{1}{\pi}\int\limits_{-(k+m+1)t}^{(k-m+1)t}\frac{f(mt+u)-f(mt)}{u}du-\frac{1}{\pi}\int\limits_{-\pi}^{-(k+m+1)t}\frac{f(mt+u)}{u}du-$$

$$-\frac{1}{\pi}\int\limits_{(k-m+1)t}^{\pi}\frac{f(mt+u)}{u}du+\frac{f(mt)}{\pi}\left(\int\limits_{-\pi}^{-(k+m+1)t}\frac{du}{u}+\int\limits_{(k-m+1)t}^{\pi}\frac{du}{u}\right)=$$

$$= -\frac{1}{\pi}\int\limits_{-(k+m+1)t}^{(k-m+1)t}\frac{f(mt+u)-f(mt)}{u}du-\frac{1}{\pi}\int\limits_{-\pi+mt}^{-(k+1)t}\frac{f(v)}{v-mt}dv-$$

$$-\frac{1}{\pi}\int\limits_{(k+1)t}^{\pi+mt}\frac{f(v)}{v-mt}dv+\frac{f(mt)}{\pi}\ln\frac{k+m+1}{k-m+1}=$$

$$= -\frac{1}{\pi}\int\limits_{-(k+m+1)t}^{(k-m+1)t}\frac{f(mt+u)-f(mt)}{u}du-\frac{1}{\pi}\int\limits_{[-\pi,\pi]\setminus[-(k+1)t,(k+1)t]}\frac{f(v)}{v-mt}dv+$$

$$+\frac{f(mt)}{\pi}\ln\frac{k+m+1}{k-m+1}. \tag{24}$$

Next the following estimate is true

$$\left|\int\limits_{-(k+m+1)t}^{(k-m+1)t}\frac{f(mt+u)-f(mt)}{u}du\right|\le\left|\int\limits_{-(k+m+1)t}^{-\varphi(t)}\frac{f(mt+u)-f(mt)}{u}du\right|+$$

$$+\left|\int\limits_{-\varphi(t)}^{\varphi(t)}\frac{f(mt+u)-f(mt)}{u}du\right|+\left|\int\limits_{\varphi(t)}^{(k-m+1)t}\frac{f(mt+u)-f(mt)}{u}du\right|\le$$

$$\le 4\alpha(3kt)\left|\ln\frac{\varphi(t)}{3kt}\right|+2\int\limits_{0}^{\varphi(t)}\frac{\omega(u)}{u}du. \tag{25}$$

Now (see (24), (25), Lemma 3, (2) and (3_k)), we get

$$|\Delta_k(\hat{\tilde{f}},0,t)|=\left|\sum_{m=0}^{k}(-1)^m\begin{pmatrix}k\\m\end{pmatrix}\hat{\tilde{f}}(mt)\right|\le$$

$$\le\frac{1}{\pi}\sum_{m=0}^{k}\begin{pmatrix}k\\m\end{pmatrix}\left|\int\limits_{-(k+m+1)t}^{(k-m+1)t}\frac{f(mt+u)-f(mt)}{u}du\right|+$$

$$+\frac{1}{\pi}\int\limits_{[-\pi,\pi]\setminus[-(k+1)t,(k+1)t]}|f(v)|\left|\sum_{m=0}^{k}(-1)^m\left(\begin{array}{c}k\\m\end{array}\right)\frac{1}{v-mt}\right|dv+$$

$$+\frac{1}{\pi}\sum_{m=0}^{k}\left(\begin{array}{c}k\\m\end{array}\right)|f(mt)|\ln\frac{k+m+1}{k-m+1}\leq\frac{2^k}{\pi}\left(4\alpha(3kt)\left|\ln\frac{\varphi(t)}{3kt}\right|+\right.$$

$$\left.+2\int\limits_0^{\varphi(t)}\frac{\omega(u)}{u}du\right)+\frac{2(k+1)^k k!}{\pi}t^k\cdot\int\limits_t^{\pi}\frac{\alpha(u)}{u^{k+1}}du+$$

$$+\frac{2^k}{\pi}\alpha(kt)\ln(3k)\leq$$

$$\leq C(k,\omega,\alpha)\left(\alpha(t)\left|\ln\frac{\varphi(t)}{t}\right|+\int\limits_0^{\varphi(t)}\frac{\omega(u)}{u}du\right). \tag{26}$$

Next step is to get rid of restrictions on the function $f(x)$. Let μ be infinitely differentiable 2π-periodic even function, satisfying $0\leq\mu(x)\leq 1$ and

$$\mu(x)=\begin{cases}0 & \text{for } \frac{2\pi}{3}\leq x\leq\pi;\\ 1 & \text{for } 0\leq x\leq\frac{\pi}{2}.\end{cases}$$

Then

$$f(x)=f(x)\mu(x)+f(x)(1-\mu(x))\equiv f_1(x)+f_2(x).$$

Note that $|f_1(t)|\leq|f(t)|\leq\alpha(|t|)$ for $t\in T$ and $f_1(x)\in H^{C(\omega)\omega}(T)$. Then we have that (see (26), Corollary 2 and 3)

$$|\Delta_k(\tilde{f},0,t)|\leq|\Delta_k(\hat{\tilde{f}}_1,0,t)|+$$

$$+|\Delta_k(\tilde{f}_1-\hat{\tilde{f}}_1,0,t)|+|\Delta_k(\tilde{f}_2,0,t)|\leq$$

$$\leq C(k,\omega,\alpha)\left(\alpha(|t|)\left|\ln\frac{\varphi(|t|)}{|t|}\right|+\int\limits_0^{\varphi(|t|)}\frac{\omega(u)}{u}du\right)+$$

$$+C(k,\omega,\alpha)|t|^k\leq C(k,\omega,\alpha)\eta(|t|)$$

for $t\in T$. So Theorem 1 proved. □

Proof of Theorem 2. Let a sequence $t_n\downarrow 0$ as $n\to\infty$. Let us choose an increasing sequence $\{m_i\}_{i=1}^{\infty}$ such that:

1. m_1 is such that $t_{m_1}<\frac{\pi}{2k}$ and $\varphi(t)<t$ for $t<kt_{m_1}$,
2. m_{i+1} is such that $t_{m_{i+1}}<\frac{t_{m_i}}{6k}$ for $i=1,2,\ldots$.

Let $\tau_{m_i}=t_{m_i}k$ for $i=1,2,\ldots$. Let us define the function $f(x)$ for $x\in[\tau_{m_{i+1}},\tau_{m_i}]$, where $i=1,2,\ldots$:

$$f(x)=\begin{cases}\omega(x-\tau_{m_{i+1}}) & \text{for } x\in[\tau_{m_{i+1}},\tau_{m_{i+1}}+\varphi(t_{m_{i+1}})];\\ \omega(\varphi(t_{m_{i+1}}))=\alpha(t_{m_{i+1}}) & \text{for } x\in[\tau_{m_{i+1}}+\varphi(t_{m_{i+1}}),3\tau_{m_{i+1}}];\\ f(6\tau_{m_{i+1}}-x) & \text{for } x\in[3\tau_{m_{i+1}},5\tau_{m_{i+1}}];\\ 0 & \text{in other cases } [\tau_{m_{i+1}},\tau_{m_i}].\end{cases}$$

Let for $x \in [\tau_{m_1}, \pi]$ $f(x) = 0$ and $f(0) = 0$. Extending f as an even function to $[-\pi, 0)$, then 2π-periodically to the whole line, we obtain the desired function. Let us prove this.

Let $x \in [0, \pi]$. Then either $f(x) = 0$, or there exists a number $i \geq 1$ such that $x \in [\tau_{m_{i+1}}, \tau_{m_i}]$. Hence $|f(x)| \leq \alpha(t_{m_{i+1}}) \leq \alpha(x)$. Let us show that $f \in H^\omega(T)$. Let $x, y \in T$ and

$$f(x) \geq f(y). \tag{27}$$

We assume that $f(x) > 0$, otherwise there is nothing to prove. Since f is an even function it is enough to consider the case $x > 0$. Then there exists a number $i \geq 2$ such that $x \in [\tau_{m_i}, 5\tau_{m_i}]$. Three cases are possible.

1) The number $x \in [\tau_{m_i}, \tau_{m_i} + \varphi(\tau_{m_i})]$. Suppose that $y \in [\tau_{m_i}, x]$. Then taking into account (27), the properties of a modulus of continuity and the definition of the numbers τ_{m_i}, we have

$$0 \leq f(x) - f(y) = \omega(x - \tau_{m_i}) - \omega(y - \tau_{m_i}) \leq \omega(x-y)+$$

$$+\omega(y - \tau_{m_i}) - \omega(y - \tau_{m_i}) = \omega(x-y). \tag{28}$$

Let now $y < \tau_{m_i}$. Then (see (27))

$$0 \leq f(x) - f(y) \leq f(x) = \omega(x - \tau_{m_i}) \leq \omega(x-y). \tag{29}$$

If $y > x$, then taking into account (27), we get that $y > 3\tau_{m_i}$. At the same time

$$x < \tau_{m_i} + \varphi(\tau_{m_i}) \leq 2\tau_{m_i}.$$

Hence

$$0 \leq f(x) - f(y) \leq f(x) = \omega(x - \tau_{m_i}) \leq \omega(\tau_{m_i}) \leq \omega(y-x). \tag{30}$$

Combining estimations (28) – (30), we see that in case 1)

$$|f(x) - f(y)| \leq \omega(|x-y|). \tag{31}$$

2) The number $x \in [5\tau_{m_i} - \varphi(\tau_{m_i}), 5\tau_{m_i}]$. This case is almost not different from case 1) and again estimate (31) holds.

3) The number $x \in [\tau_{m_i} + \varphi(\tau_{m_i}), 5\tau_{m_i} - \varphi(\tau_{m_i})]$. If y belongs to the same segment, then $f(x) - f(y) = 0$. If $y < \tau_{m_i} + \varphi(\tau_{m_i})$, then (see (27), the definition of the numbers τ_{m_i} and (31))

$$0 \leq f(x) - f(y) = f(\tau_{m_i} + \varphi(\tau_{m_i})) - f(y) \leq$$
$$\leq \omega(\tau_{m_i} + \varphi(\tau_{m_i}) - y) \leq \omega(x-y).$$

If $y > 5\tau_{m_i} - \varphi(\tau_{m_i})$, then again by the same reasons

$$0 \leq f(x) - f(y) = f(5\tau_{m_i} - \varphi(\tau_{m_i})) - f(y) \leq$$

$$\leq \omega(y - 5\tau_{m_i} + \varphi(\tau_{m_i})) \leq \omega(y-x). \tag{32}$$

Since $f(x) = 0$ for $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, from estimates (31) and (32) it follows that the function $f \in H^\omega(T)$.

Let us consider now $\hat{\tilde{f}}(nt_{m_i})$, where $i > 1$ and $1 \le n \le k$. Then we have that (see the definition of the numbers t_{m_i} and of the function f)

$$\hat{\tilde{f}}(nt_{m_i}) = -\frac{1}{\pi}\int\limits_T \frac{f(nt_{m_i}+u)}{u}du = -\frac{1}{\pi}\int\limits_T \frac{f(v)}{v-nt_{m_i}}dv =$$

$$= -\frac{1}{\pi}\int\limits_{-3\tau_{m_i}}^{3\tau_{m_i}} \frac{f(v)}{v-nt_{m_i}}dv - \frac{1}{\pi}\int\limits_{-\pi}^{-3\tau_{m_i}} \frac{f(v)}{v-nt_{m_i}}dv-$$

$$-\frac{1}{\pi}\int\limits_{3\tau_{m_i}}^{\pi} \frac{f(v)}{v-nt_{m_i}}dv. \tag{33}$$

For $1 \le n < k$ by the definitions of the numbers t_{m_i} and of the function $f(x)$, we get that $f(v) = 0$ for $v \in [nt_{m_i} - \frac{1}{6}t_{m_i}, nt_{m_i} + \frac{1}{6}t_{m_i}]$. Hence

$$\left|\int\limits_{-3\tau_{m_i}}^{3\tau_{m_i}} \frac{f(v)}{v-nt_{m_i}}dv\right| \le 2\alpha(3\tau_{m_i})\int\limits_{\frac{t_{m_i}}{6}}^{6\tau_{m_i}} \frac{dv}{v} \le$$

$$\le C(k)\alpha(t_{m_i}). \tag{34}$$

Moreover,

$$\int\limits_{-3\tau_{m_i}}^{3\tau_{m_i}} \frac{f(v)}{v-kt_{m_i}}dv = \int\limits_{-3\tau_{m_i}}^{\tau_{m_i}} \frac{f(v)}{v-\tau_{m_i}}dv+$$

$$+\int\limits_{\tau_{m_i}}^{3\tau_{m_i}} \frac{f(v)}{v-\tau_{m_i}}dv \equiv J_1 + J_2. \tag{35}$$

It is clear that

$$|J_1| \le \alpha(3\tau_{m_i})\left|\int\limits_{-3\tau_{m_i}}^{\frac{5}{6}\tau_{m_i}} \frac{1}{v-\tau_{m_i}}dv\right| \le (3k)^k \ln 18 \cdot \alpha(t_{m_i}). \tag{36}$$

And by the definition of the function f it follows that

$$J_2 = \int\limits_{\tau_{m_i}}^{\tau_{m_i}+\varphi(t_{m_i})} \frac{\omega(v-\tau_{m_i})}{v-\tau_{m_i}}dv + \int\limits_{\tau_{m_i}+\varphi(t_{m_i})}^{3\tau_{m_i}} \frac{f(v)}{v-\tau_{m_i}}dv =$$

$$= \int\limits_0^{\varphi(t_{m_i})} \frac{\omega(v)}{v} + \alpha(t_{m_i})\int\limits_{\varphi(t_{m_i})}^{2\tau_{m_i}} \frac{dv}{v} =$$

$$= \eta(\omega, \alpha, t_{m_i}) + \alpha(t_{m_i}) \ln(2k). \tag{37}$$

Finally, using the parity of the function f, Lemma 3, (33) - (37) and the condition (3_k), we obtain

$$|\Delta_k(\hat{\tilde{f}}, 0, t_{m_i})| = \left| \sum_{n=0}^{k} (-1)^n C_k^n \hat{\tilde{f}}(nt_{m_i}) \right| \geq$$

$$\geq -\frac{1}{\pi} \sum_{n=0}^{k-1} C_k^n \cdot \left| \int_{-3\tau_{m_i}}^{3\tau_{m_i}} \frac{f(v)}{v - nt_{m_i}} dv \right| - \frac{1}{\pi}|J_1| + \frac{1}{\pi}|J_2| -$$

$$-\frac{1}{\pi} \int_{-\pi}^{-3\tau_{m_i}} \frac{|f(v)| k! t_{m_i}^k}{|v(v - t_{m_i})...(v - kt_{m_i})|} dv -$$

$$-\frac{1}{\pi} \int_{3\tau_{m_i}}^{\pi} \frac{|f(v)| k! t_{m_i}^k}{v(v - t_{m_i})...(v - kt_{m_i})} dv \geq$$

$$\geq -C(k)\alpha(t_{m_i}) - -C(k)\alpha(t_{m_i}) + \frac{1}{\pi}\eta(\omega, \alpha, t_{m_i}) - 2k! t_{m_i}^k \int_{t_{m_i}}^{\pi} \frac{\alpha(v)}{v^{k+1}} dv \geq$$

$$\geq \frac{2}{3\pi} \eta(\omega, \alpha, t_{m_i}) \tag{38}$$

for sufficiently large i. Using Corollary 2 and inequality (38) we find that for $i \geq i_0$ the following inequality holds

$$|\Delta_k(\tilde{f}, 0, t_{m_i})| \geq |\Delta_k(\hat{\tilde{f}}, 0, t_{m_i})| - |\Delta_k(\tilde{f} - \hat{\tilde{f}}, 0, t_{m_i})| \geq$$

$$\geq \frac{2}{3\pi}\eta(\omega, \alpha, t_{m_i}) - \max_{t \in T} |f(t)| \cdot C(k) \cdot t_{m_i}^k \geq \frac{1}{2\pi}\eta(\omega, \alpha, t_{m_i}).$$

Assuming that for all $i \geq 1$ the number $n_i = m_{i_0+i}$, we see that Theorem 2 is proved. $\square$

To conclude this section let us examine some examples of the behaviour of the two terms in the definition of the function $\eta(\omega, \alpha, t)$.

Case 1. Let $\omega(t) = t^\beta$, and $\alpha(t) = t^\gamma$, where $0 < \beta < \gamma < 1$. Then $\varphi(t) = t^{\frac{\gamma}{\beta}}$. Consequently

$$\eta(t) = \int_0^{t^{\frac{\gamma}{\beta}}} u^{\beta-1} du + t^\gamma |\ln t^{\frac{\gamma}{\beta}-1}| =$$

$$= \frac{1}{\beta} t^\gamma + (\frac{\gamma}{\beta} - 1) t^\gamma \ln \frac{1}{t} \sim t^\gamma \ln \frac{1}{t}$$

as $t \to 0^+$.

Note that in this case the main is the second term.

Case 2. Let $\omega(t) = \ln^{-p}\frac{1}{t}$ for $t \in (0, \frac{1}{2})$, where $p > 1$, and $\alpha(t) = t^{\gamma}$, for $0 < \gamma < 1$. Then $\varphi(t) = e^{-t^{-\frac{\gamma}{p}}}$. Hence

$$\eta(t) = \int\limits_0^{\varphi(t)} \frac{du}{u\ln^p\frac{1}{u}} + t^{\gamma}\left|\ln \frac{e^{-t^{-\frac{\gamma}{p}}}}{t}\right| \leq$$

$$\leq \frac{1}{p-1}\ln^{1-p}e^{t^{-\frac{\gamma}{p}}} + t^{\gamma-\frac{\gamma}{p}} + t^{\gamma}\ln\frac{1}{t} = \frac{p}{p-1}t^{\gamma\frac{p-1}{p}} + t^{\gamma}\ln\frac{1}{t} \sim$$

$$\sim \frac{p}{p-1}t^{\gamma\frac{p-1}{p}}$$

as $t \longrightarrow 0^+$. In that case both of terms in $\eta(t)$ have the same order as $t \rightarrow 0^+$.

Case 3. Let $\omega(t) = \ln^{-1}\frac{1}{t}(\ln\ln\frac{1}{t})^{-p}$ for $t \in (0, \frac{1}{10})$, where $p > 1$, and $\alpha(t) = t^{\gamma}$, for $0 < \gamma < 1$. Then by the definition of the function φ

$$\ln^{-1}\frac{1}{\varphi(t)}\left(\ln\ln\frac{1}{\varphi(t)}\right)^{-p} = t^{\gamma},$$

consequently it is clear that

$$\ln\ln\frac{1}{\varphi(t)} + p\ln\ln\ln\frac{1}{\varphi(t)} = \gamma\ln\frac{1}{t},$$

i.e.

$$\ln\ln\frac{1}{\varphi(t)} \sim \gamma\ln\frac{1}{t} \tag{39}$$

as $t \longrightarrow 0^+$, and

$$t^{\gamma}\ln\frac{1}{\varphi(t)} = \left(\ln\ln\frac{1}{\varphi(t)}\right)^{-p} \sim \gamma^{-p}\ln^{-p}\frac{1}{t} \tag{40}$$

as $t \longrightarrow 0^+$. By formulas (39) and (40) it follows that

$$\eta(t) = \int\limits_0^{\varphi(t)} \frac{du}{u\ln\frac{1}{u}(\ln\ln\frac{1}{u})^p} + t^{\gamma}|\ln\frac{\varphi(t)}{t}| =$$

$$= (p-1)\left(\ln\ln\frac{1}{\varphi(t)}\right)^{1-p} + t^{\gamma}\ln\frac{t}{\varphi(t)} \sim$$

$$\sim \gamma^{1-p}\ln^{1-p}\frac{1}{t} + \gamma^{-p}\ln^{-p}\frac{1}{t} \sim \gamma^{1-p}\ln^{1-p}\frac{1}{t}$$

as $t \longrightarrow 0^+$. Here the main contribution is of the term containing integral.

5 Generalization of G. Freud's theorem

Proof of Theorem 5. By Theorem C it suffices to prove that $M(f) \subseteq M(\tilde{f})$. Let $x \in M(f)$. As noted above, without loss of generality, we may assume that $x = 0$ and $f(x) = 0$. If $t \in (0, \frac{\pi}{4}]$, the following estimate holds (see the proof of Theorem 1)

$$\pi|\tilde{f}(t) - \tilde{f}(0)| = \left| \int_T \frac{f(t+u)}{2tg\frac{u}{2}} du - \int_T \frac{f(u)}{2tg\frac{u}{2}} du \right| =$$

$$= \left| \int_T \frac{f(u)}{2tg\frac{u-t}{2}} du - \int_T \frac{f(u)}{2tg\frac{u}{2}} du \right| \leq \left| \int_{-\pi}^{-2t} f(u) \left(\frac{1}{2tg\frac{u-t}{2}} - \frac{1}{2tg\frac{u}{2}} \right) du \right| +$$

$$+ \left| \int_{-2t}^{2t} \frac{f(u)}{2tg\frac{u}{2}} du \right| + \left| \int_{-2t}^{2t} \frac{f(u) - f(t)}{2tg\frac{u-t}{2}} du \right| +$$

$$+ \left| f(t) \int_{-2t}^{2t} \frac{1}{2tg\frac{u-t}{2}} du \right| + \left| \int_{2t}^{\pi} f(u) \left(\frac{1}{2tg\frac{u-t}{2}} - \frac{1}{2tg\frac{u}{2}} \right) du \right| \leq$$

$$\leq C \left(t \int_{-\pi}^{-t} \frac{|f(u)|}{u^2} du + \int_{-2t}^{2t} \frac{|f(u)|}{u} du + \right.$$

$$\left. + \int_{-2t}^{2t} \frac{|f(u) - f(t)|}{|u-t|} du + |f(t)| + t \int_t^{\pi} \frac{|f(u)|}{u^2} du \right) \equiv$$

$$\equiv C(J_1 + J_2 + J_3 + |f(t)| + J_4). \tag{41}$$

Since by the assumptions of Theorem 5 $|f(t)| = o(\omega(t))$ as $t \to 0^+$ and for ω the condition (3) holds, then by the well-known theorem (see [3, c. 38]) we get that

$$J_4 = t \int_t^{\pi} \frac{o(\omega(u))}{u^2} du = o\left(\int_t^{\pi} \frac{\omega(u)}{u^2} du \right) = o(\omega(t)) \tag{42}$$

as $t \to 0^+$, and analogously

$$J_1 = o(\omega(t)) \tag{43}$$

as $t \to 0^+$. Hence, since for ω condition (2) holds,

$$J_2 = \int_{-2t}^{2t} \frac{o(\omega(|u|))}{|u|} du = o\left(\int_{-2t}^{2t} \frac{\omega(|u|)}{|u|} du \right) = o(\omega(t)) \tag{44}$$

as $t \to 0^+$. Let us consider a function $\psi(t) \downarrow 0$ when $t \downarrow 0$ for $t \in (0, \pi]$ such that

$$|\ln \psi(t)| \max_{|u| \leq 2t} |f(u)| = o(\omega(t))$$

as $t \to 0^+$. In [5] it was established that the condition (2) is equivalent to

$$\liminf_{u \to +0} \frac{\omega(Au)}{\omega(u)} > 1$$

for any $A > 1$. But then $\omega(t\psi(t)) = o(\omega(t))$ as $t \to 0^+$. Therefore for sufficiently small positive t we have

$$J_3 \leq \int\limits_{[-2t,2t]\setminus[t-t\psi(t),t+t\psi(t)]} \frac{|f(u)-f(t)|}{|u-t|} du + \int\limits_{t-t\psi(t)}^{t+t\psi(t)} \frac{|f(u)-f(t)|}{|u-t|} du \leq$$

$$\leq 4|\ln \psi(t)| \max_{|u|\leq 2t} |f(u)| + 2 \int\limits_0^{t\psi(t)} \frac{\omega(u)}{u} du = o(\omega(t)) \tag{45}$$

as $t \to)^+$. Since for negative t everything is similar, from (41) - (45) it follows that $0 \in M(\tilde{f})$, i.e. $M(f) \subseteq M(\tilde{f})$. □

6 Local smoothness of the conjugate functions: the case of the second difference

Proof of Theorem 3. From condition (7) (taking into account the fact that $f(0) = 0$) it follows that if we define the odd function $f_1(x) = f(x)$ for $x \in [0, \pi]$, then the function $f_2(x) = f(x) - f_1(x)$ satisfies $|f_2(x)| \leq \alpha(x)$ for $x \in T$. Moreover, it is obvious that $f_1, f_2 \in H^{4\omega}(T)$. By Theorem 1 we have

$$|\Delta_2(\tilde{f}_2, 0, t)| \leq C(\omega, \alpha)\eta(\omega, \alpha, |t|) \tag{46}$$

for $t \in T$. Now if μ is a function defined in the proof of Theorem 1, we can define the decomposition

$$f_1(x) = f_1(x)\mu(x) + f_1(x)(1 - \mu(x)) \equiv f_3(x) + f_4(x) \tag{47}$$

for $x \in T$. Let us note that function f_3 satisfies the assumptions of Lemma 6. Hence

$$|\Delta_2(\hat{\tilde{f}}_3, 0, t)| \leq C(\omega, \alpha)\eta(\omega, \alpha, |t|) \tag{48}$$

for $x \in T$. Finally, by applying Corollary 2 and 3, we conclude that

$$|\Delta_2(\tilde{f}_3 - \hat{\tilde{f}}_3, 0, t)| \leq C(\omega)|t|^2 \tag{49}$$

and

$$|\Delta_2(\tilde{f}_4, 0, t)| \leq C(\omega)|t|^2 \tag{50}$$

for $x \in T$. From (46) – (50) it follows that

$$|\Delta_2(\tilde{f}, 0, t)| \leq C(\omega, \alpha)\eta(\omega, \alpha, |t|)$$

for $x \in T$. Theorem 3 is proved. □

Proof of Theorem 4. Let μ be a function defined in the proof of Theorem 1 and $\gamma_1(x) = \gamma(x)\mu(x)$ for $x \in T$. By Lemma 7 the function $\gamma_1 \in H^\omega(T)$, where $\omega(t) = C(a,\beta)t^{\frac{a-1}{a-\beta}\beta}$. If $0 < |t| < \frac{\pi}{4}$, then there exists an integer $k \geq 2$ such that $|t| \in (\frac{\pi}{2^{k+1}}, \frac{\pi}{2^k}]$. But then

$$\gamma_1(2t) - 2\gamma_1(t) = \gamma(2t) - 2\gamma(t) = \frac{\gamma(2^k t)}{2^{k-1}} - 2\frac{\gamma(2^k t)}{2^k} = 0. \tag{51}$$

Let $f_1(t) = f(t) - \gamma_1(t)$. Then

$$|f_1(x) - f_1(y)| \leq C(a,\beta)|x-y|^{\frac{a-1}{a-\beta}\beta}$$

for $x, y \in T$ and

$$|f_1(2t) - 2f_1(t)| \leq C(a)|t|^a$$

for $t \in T$. Moreover, assuming that in inequality (21) $n = 0$, we get that $|t| \leq \frac{\pi}{4}$, such that $|f_1(t)| = |f(t) - \gamma(t)| \leq C(a)|t|^a$ for $t \in T$. Using Theorem 1, we have that

$$|\Delta_2(\tilde{f}_1, 0, t)| \leq C(a,\beta)|t|^a \ln\frac{1}{|t|} \tag{52}$$

for $|t| \leq \frac{1}{2}$. Let us note that if $|t| \leq \frac{\pi}{8}$, then equality (11) for function γ_1 will have the form (see (51))

$$-\pi\Delta_2(\hat{\tilde{\gamma}}_1, 0, t) =$$

$$= 2t\left(\int\limits_{[-\frac{\pi}{2},\frac{\pi}{2}]\setminus[-\frac{\pi}{4},\frac{\pi}{4}]} \frac{\gamma_1(2v) - 2\gamma_1(v)}{2v(v-t)}dv - \int\limits_{[-\pi,\pi]\setminus[-\frac{\pi}{2},\frac{\pi}{2}]} \frac{\gamma_1(v)}{v(v-t)}dv\right). \tag{53}$$

Carrying out simple transformation we get that

$$\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\gamma_1(2v) - 2\gamma_1(v)}{2v(v-t)}dv - \int\limits_{\frac{\pi}{2}}^{\pi} \frac{\gamma_1(v)}{v(v-t)}dv =$$

$$= -\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\gamma(v)}{v(v-t)}dv + \int\limits_{\frac{\pi}{2}}^{\pi} \frac{\gamma(u)\mu(u)}{u(u-2t)}du -$$

$$-\int\limits_{\frac{\pi}{2}}^{\pi} \frac{\gamma(u)\mu(u)}{u(u-t)}du =$$

$$= -\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\gamma(v)}{v(v-t)}dv + t\int\limits_{\frac{\pi}{2}}^{\pi} \frac{\gamma(u)\mu(u)}{u(u-t)(u-2t)}du \tag{54}$$

and analogously

$$\int\limits_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \frac{\gamma_1(2v) - 2\gamma_1(v)}{2v(v-t)}dv - \int\limits_{-\pi}^{-\frac{\pi}{2}} \frac{\gamma_1(v)}{v(v-t)}dv =$$

$$= -\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \frac{\gamma(v)}{v(v-t)}dv + t\int_{\pi}^{\frac{\pi}{2}} \frac{\gamma(u)\mu(u)}{u(u-t)(u-2t)}du. \tag{55}$$

Note that

$$\left|\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\gamma(v)}{v(v-t)}dv - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\gamma(v)}{v^2}dv\right| \le 2|t|\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{|\gamma(v)|}{v^3}dv \tag{56}$$

and

$$\left|\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \frac{\gamma(v)}{v(v-t)}dv - \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \frac{\gamma(v)}{v^2}dv\right| \le 2|t|\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \frac{|\gamma(v)|}{v^3}dv. \tag{57}$$

And by Corollary 2, we get that

$$|\Delta_2(\tilde{\gamma}_1 - \hat{\tilde{\gamma}}_1, 0, t)| \le t^2 C(a,\beta)\max_{x\in T}|\gamma(x)|. \tag{58}$$

Taking into account formulas (52) – (58) and

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\gamma(v)+\gamma(-v)}{v^2}dv = \int_{\frac{\pi}{2}}^{\pi} \frac{\gamma(v)+\gamma(-v)}{v^2}dv,$$

we obtain

$$\left|\Delta_2(\tilde{f},0,t) - \frac{2}{\pi}t\int_{\frac{\pi}{2}}^{\pi} \frac{\gamma(u)+\gamma(-u)}{u^2}du\right| \le$$

$$\le |\Delta_2(\tilde{f}_1,0,t)| + |\Delta_2(\tilde{\gamma}_1 - \hat{\tilde{\gamma}}_1,0,t)| +$$

$$+\left|\Delta_2(\tilde{\gamma}_1,0,t) - \frac{2}{\pi}t\int_{\frac{\pi}{2}}^{\pi} \frac{\gamma(u)+\gamma(-u)}{u^2}du\right| \le C(a,\beta)|t|^a \ln\frac{1}{|t|} +$$

$$+C(a,\beta)t^2 \le C(a,\beta)|t|^a \ln\frac{1}{|t|}$$

for $t \in (-\frac{1}{2}, \frac{1}{2})$. The proof of Theorem 4 is complete. □

Proof of Corollary 1. By the assumptions of Corollary 1 it follows that $\gamma(v)+\gamma(-v) \ge 0$ for $v \in [0,\pi]$ and by Theorem 4 we have

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\gamma(v)+\gamma(-v)}{v^2}dv = 0.$$

Hence $\gamma(v)+\gamma(-v)=0$ for $v\in[\frac{\pi}{2},\pi]$, and according to the definition of the function γ, for every $v\in[0,\pi]$. In the proof of Theorem 4 it was noted that

$$|f_1(t)|=|f(t)-\gamma(t)|\le C(a)|t|^a$$

for $|t|<\frac{\pi}{4}$. So

$$|f(v)+f(-v)|=|f(v)+f(-v)-(\gamma(v)+\gamma(-v))|\le$$

$$\le|f_1(v)|+|f_1(-v)|\le C(a)v^a$$

for $v\in[0,\frac{\pi}{4}]$, which is equivalent to the statement. □

7 Differentiation of the conjugate functions

Proof of Theorem 6. We assume that $x_0=0$. Let us define the function $f_1(x)=f(x)\mu(x)$, where μ was defined in the proof of Theorem 1. Applying Lemma 2, we see that the existence of $\tilde{f}'(0)$ is equivalent to the existence of $\tilde{f_1}'(0)$, which is equivalent to the existence of $\hat{\tilde{f_1}}'(0)$ by Lemma 1. Let us note that for $f\in P(0,r(t))$ we have

$$|f_1(t)-At|\le C(f)r(|t|)$$

for $t\in T$. Let $\nu(t)=f_1(t)-At\mu(t)$ for $t\in T$. Then

$$|\nu(t)|=|f(t)-At|\cdot|\mu(t)|\le r(|t|)$$

for $t\in T$. Moreover, $\nu\in H^{C(f)\omega}(T)$. Moreover, it is clear that for some $t_0>0$ for all $t\in(0,t_0)$ the estimate $\varphi(t)\equiv\varphi(\omega,r,t)<t$ holds.

Let us establish the sufficiency in Theorem 6. Let $\varepsilon>0$. Let us consider for $0<t<\min(\frac{\pi}{6},t_0)$ the expression

$$\frac{1}{t}(\hat{\tilde{\nu}}(t)-\hat{\tilde{\nu}}(0))=-\frac{1}{t\pi}\left(\int\limits_{T\setminus[-2t,2t]}\nu(u)\left(\frac{1}{u-t}-\frac{1}{u}\right)du-\int\limits_{-2t}^{2t}\frac{\nu(u)}{u}du+\right.$$

$$\left.+\int\limits_{-2t}^{t-\varphi(2t)}\frac{\nu(u)}{u-t}du+\int\limits_{0}^{\varphi(2t)}\frac{\nu(t+u)-\nu(t-u)}{u}du+\int\limits_{t+\varphi(2t)}^{2t}\frac{\nu(u)}{u-t}du\right)\equiv$$

$$\equiv-\frac{1}{t\pi}(J_1+J_2+J_3+J_4+J_5). \tag{59}$$

Let us choose a number δ_0 such that (see the first condition of Theorem 6)

$$\int\limits_{-\delta_0}^{\delta_0}\frac{|\nu(u)|}{u^2}du<\frac{\varepsilon}{16}.$$

Then for $0 < t < \min(\frac{\pi}{6}, t_0, \delta_0)$ we get that

$$\left| -\frac{1}{t\pi} J_1 + \frac{1}{\pi} \int\limits_T \frac{\nu(u)}{u^2} du \right| \le$$

$$\le \frac{1}{\pi} \left| \int\limits_{T \setminus [-\delta_0, \delta_0]} \nu(u) \left(\frac{1}{u(u-t)} - \frac{1}{u^2} \right) du \right| +$$

$$+ \frac{1}{\pi} \int\limits_{-\delta_0}^{\delta_0} \frac{|\nu(u)|}{u^2} du + \frac{2}{\pi} \int\limits_{[-\delta_0, \delta_0] \setminus [-2t, 2t]} \frac{|\nu(u)|}{u^2} du \le$$

$$\le \frac{t}{\pi} \int\limits_{[-\pi, \pi] \setminus [-\delta_0, \delta_0]} \frac{|\nu(u)|}{u^2 |u - t|} du + \frac{3}{\pi} \int\limits_{-\delta_0}^{\delta_0} \frac{|\nu(u)|}{u^2} du < \frac{\varepsilon}{4}, \tag{60}$$

if t is sufficiency small. Let $\delta_1 > 0$ be such that for $0 < t < \delta_1$ the following inequality is true: $\nu(t) < \frac{\varepsilon}{16} t$. Then for the some t, we get

$$\left| -\frac{1}{t\pi} J_2 \right| \le \frac{\varepsilon}{16\pi t} \int\limits_{-2t}^{2t} du < \frac{\varepsilon}{4}. \tag{61}$$

And by the assumptions of Theorem 6

$$\left| -\frac{1}{t\pi} (J_3 + J_4 + J_5) \right| \le \frac{1}{t\pi} \left(r(2t) \left| \ln \frac{\varphi(2t)}{3t} \right| + r(2t) \left| \ln \frac{\varphi(2t)}{t} \right| + \right.$$

$$\left. + C \int\limits_0^{\varphi(2t)} \frac{\omega(u)}{u} du \right) < \frac{\varepsilon}{2}, \tag{62}$$

if t is sufficiency small. By formulas (59) – (62) it follows that there exists

$$\lim_{t \to +0} \frac{\hat{\tilde{\nu}}(t) - \hat{\tilde{\nu}}(0)}{t} = -\frac{1}{\pi} \int\limits_{-\pi}^{\pi} \frac{\nu(u)}{u^2} du.$$

Since for negative t everything is analogous, and the existence of $(t\hat{\tilde{\mu}}(t))'(0)$ is obvious, then the sufficiency in Theorem 6 is proved.

Now let us assume that

$$\int\limits_0^1 \frac{r(t)}{t^2} dt = \infty. \tag{63}$$

Let $t_i = 6^{-i}$ for $i = 1, 2, \ldots$. Let us consider the function f, which we have constructed in the proof of Theorem 2, for $k = 1$ and with the function r replacing α (function r does not satisfy the requirements imposed on α, but for the construction of an appropriate function f it is only necessary that $r \uparrow$ at $t \in [0,\pi]$). From the proof of Theorem 2 it is clear that we can take the sequence $\{t_{m_p}\}_{p=1}^{\infty}$ instead of $\{t_i\}_{i=i_0}^{\infty}$. Note that the function $f \in P(0, r(t))$ (here $f'(0) = 0$). Let us denote $\theta_i = \frac{11}{2}t_i$ when $i \geq i_0 + 2$. Let us consider the sequence $\{\frac{1}{\theta_i}\hat{\tilde{f}}(\theta_i)\}_{i=i_0+2}^{\infty}$. Since the function f is even, we have (see the proof of Theorem 2) that $\hat{\tilde{f}}(0) = 0$ and

$$-\pi\frac{1}{\theta_i}\hat{\tilde{f}}(\theta_i) = \frac{1}{\theta_i}\int\limits_{-\pi}^{\pi}\frac{f(u)}{u-\theta_i}du = \frac{1}{\theta_i}\int\limits_{-2\theta_i}^{2\theta_i}\frac{f(u)}{u-\theta_i}du+$$

$$+\frac{1}{\theta_i}\int\limits_{2\theta_i}^{\pi} f(u)\left(\frac{1}{u-\theta_i} - \frac{1}{u+\theta_i}\right)du \equiv I_1 + I_2. \tag{64}$$

Next, since $f(u) = 0$ for $u \in [\frac{10}{11}\theta_i, \frac{12}{11}\theta_i]$, we have that

$$|I_1| = \frac{1}{\theta_i}\left|\int\limits_{-2\theta_i}^{\frac{10}{11}\theta_i}\frac{f(u)}{u-\theta_i}du + \int\limits_{\frac{12}{11}\theta_i}^{2\theta_i}\frac{f(u)}{u-\theta_i}du\right| \leq$$

$$\leq \frac{r(2\theta_i)}{\theta_i}\cdot(\ln 33 + \ln 11) < 1, \tag{65}$$

if i is sufficiency small. At the same time since r is a monotone function on $[0,\pi]$, we obtain that (see the definition of the function f)

$$I_2 = \frac{1}{\theta_i}\int\limits_{2\theta_i}^{\pi} f(u)\frac{2\theta_i}{(u-\theta_i)(u+\theta_i)}du \geq 2\int\limits_{2\theta_i}^{\pi}\frac{f(u)}{u^2}du \geq$$

$$\geq 2\sum_{n=i_0+1}^{i-1}\int\limits_{2t_n}^{4t_n}\frac{f(u)}{u^2}du = 2\sum_{n=i_0+1}^{i-1} r(t_n)\int\limits_{2\cdot 6^{-n}}^{4\cdot 6^{-n}}\frac{du}{u^2} \geq$$

$$\geq C\sum_{n=i_0+1}^{i-1} r(t_n)\int\limits_{6^{-n}}^{6^{-n+1}}\frac{du}{u^2} = C\sum_{n=i_0+1}^{i-1} r(t_n)\int\limits_{t_n}^{t_{n-1}}\frac{du}{u^2} =$$

$$= C\int\limits_{t_{i-1}}^{t_{i_0}}\frac{r(u)du}{u^2}. \tag{66}$$

Taking into account formulas (63) – (66), we get that

$$\left|\frac{\hat{\tilde{f}}(\theta_i) - \hat{\tilde{f}}(0)}{\theta_i}\right| = \left|\frac{\hat{\tilde{f}}(\theta_i)}{\theta_i}\right| \to \infty$$

for $i \to \infty$. Thus, the necessity of the first condition of Theorem 6 is proved.

Now, assume that the first condition holds, then we prove the necessity of the second condition. Suppose that there exists a sequence of numbers $\{d_n\}_{n=1}^{\infty}$ and a constant $R > 0$ such that $d_n \downarrow 0$ for $n \to \infty$ and

$$\eta(\omega, r, d_n) \geq R d_n \tag{67}$$

for $n = 1, 2, ...$ As in the proof of Theorem 2, selecting from the sequence $\{d_n\}_{n=1}^{\infty}$ a subsequence $\{e_i = d_{n_i}\}_{i=1}^{\infty}$, let us construct the function $f(x)$ for $k = 1$ and with the functions $r(t)$ replacing α. Note that the function $f \in P(0, r(t))$ (here $f'(0) = 0$) and $\hat{\tilde{f}}(0) = 0$. Let us denote $s_i = \frac{11}{2} e_i$ for $i \geq 1$. Let us consider for $i \geq 2$ the following expression

$$\frac{1}{e_i} \int_{-\pi}^{\pi} \frac{f(u)du}{u - e_i} - \frac{1}{s_i} \int_{-\pi}^{\pi} \frac{f(u)du}{u - s_i} =$$

$$= \frac{1}{e_i} \int_{-2s_i}^{2s_i} \frac{f(u)du}{u - e_i} - \frac{1}{s_i} \int_{-2s_i}^{2s_i} \frac{f(u)du}{u - s_i} +$$

$$+2 \int_{2s_i}^{\pi} f(u) \left(\frac{1}{(u - e_i)(u + e_i)} - \frac{1}{(u - s_i)(u + s_i)} \right) du \equiv P_1 - P_2 + 2P_3. \tag{68}$$

As in the derivation of (65), we have

$$|P_2| = \frac{1}{s_i} \left| \int_{-2s_i}^{\frac{10}{11}s_i} \frac{f(u)}{u - s_i} du + \int_{\frac{12}{11}s_i}^{2s_i} \frac{f(u)}{u - s_i} du \right| \leq$$

$$\leq \frac{r(2s_i)}{s_i} \cdot (\ln 33 + \ln 11) < \frac{R}{4}, \tag{69}$$

if i is sufficiently large. Let us fix $\delta_0 > 0$ such that

$$\int_{0}^{\delta_0} \frac{r(u)}{u^2} du < \frac{R}{64}. \tag{70}$$

We assume that i is so large that $2s_i < \delta_0$. Then we get that (see (70))

$$|P_3| \leq 4 \int_{2s_i}^{\delta_0} \frac{|f(u)|}{u^2} du + \int_{\delta_0}^{\pi} f(u) \frac{s_i^2 - e_i^2}{(u^2 - e_i^2)(u^2 - s_i^2)} du \leq$$

$$\leq 4 \int_{0}^{\delta_0} \frac{r(u)}{u^2} du + 4s_i^2 \int_{\delta_0}^{\pi} \frac{r(u)}{u^4} du < \frac{R}{8}, \tag{71}$$

if i is sufficiently large. And (see the definition of the function f)

$$P_1 \geq \frac{1}{e_i}\int\limits_{e_i}^{2e_i}\frac{f(u)du}{u-e_i} - \frac{1}{e_i}\left|\int\limits_{-2s_i}^{\frac{5}{6}e_i}\frac{f(u)du}{u-e_i}\right| \geq$$

$$\geq \frac{1}{e_i}\left(\int\limits_{0}^{\varphi(e_i)}\frac{\omega(u)du}{u} + r(e_i)\int\limits_{\varphi(e_i)}^{e_i}\frac{du}{u}\right) -$$

$$-\frac{r(11e_i)}{e_i}\ln(66) \geq \frac{3}{4}R, \tag{72}$$

if i is sufficiently large. Thus we proved that (see (68), (69), (71) and (72))

$$\left|\frac{\hat{\tilde{f}}(e_i)-\hat{\tilde{f}}(0)}{e_i} - \frac{\hat{\tilde{f}}(s_i)-\hat{\tilde{f}}(0)}{s_i}\right| =$$

$$= \left|\frac{\hat{\tilde{f}}(e_i)}{e_i} - \frac{\hat{\tilde{f}}(s_i)}{s_i}\right| \geq \frac{R}{4\pi}$$

for sufficiently large i. So, the derivative $\hat{\tilde{f}}'(0)$ does not exist, hence the necessity of the second condition of Theorem 6 proved. □

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