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INDEX THEORY FOR REAL FACTORS

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Abstract. The notion of index for arbitrary real factors is introduced and investigated. The main tool in our approach is the reduction of real factors to involutive *-anti-automorphisms of their complex enveloping von Neumann algebras. Similarly to the complex case the values of the index for real factors are calculated.

1 Introduction

In [10], H. Kosaki extended the notion of index to an arbitrary (normal faithful) expectation from a factor onto a subfactor. While Jones' definition of the index is based on the coupling constant, Kosaki's definition of the index of an expectation relies on the notion of spatial derivatives due to A. Connes [6] as well as the theory of operator-valued weights due to U. Haagerup [8]. In [10, 11], it was shown that many fundamental properties of the Jones index of the type II_1 case can be extended to the general setting. At present, the theory of index is well developed thanks to works by V. Jones, P. Loi, R. Longo, H. Kosaki and other mathematicians and has many applications in the theory of operator algebras and physics (see also [12, 13]).

In parallel with the theory of index of complex subfactors the theory of index of real subfactors has also been intensively developed. In the papers [1, 2, 4, 15, 16, 17] real analogues of Jones' theory of index are considered. In particular, the notions of the real coupling constant and the index for finite real factors was introduced and investigated. In the present paper the notion of real index for arbitrary real factors is introduced and investigated. The main tool in our approach is the reduction of real factors to involutive *-anti-automorphisms of their complex enveloping von Neumann algebras. Similarly to the complex case the values of the index for real factors are calculated.

2 Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. A weakly closed *-subalgebra $\mathfrak{A}$ containing the identity operator $\mathbb{1}$ in $B(H)$ is called a W*-*algebra*. A real *-subalgebra $R \subset B(H)$ is called a *real* W*-*algebra* if it is closed in the weak operator topology and $R \cap iR = \{0\}$. A real W*-algebra R is called a *real factor* if its center $Z(R)$ consists of the elements $\{\lambda \mathbb{1}, \lambda \in \mathbb{R}\}$. We say that a real W*-algebra R is of the type I_{fin}, I_∞, II_1, II_∞, or III_λ, $(0 \leq \lambda \leq 1)$ if the enveloping W*-algebra $\mathfrak{A}(R)$ has the corresponding type in the ordinary classification of W*-algebras. A linear mapping α of an algebra into itself with $\alpha(x^*) = \alpha(x)^*$ is called an *-*automorphism* if $\alpha(xy) = \alpha(x)\alpha(y)$; it is called an *involutive* *-*antiautomorphism* if $\alpha(xy) = \alpha(y)\alpha(x)$ and $\alpha^2(x) = x$. If α is an involutive *-antiautomorphism of a W*-algebra M, we denote by (M, α) the real W*-algebra generated by α, i.e. $(M, \alpha) = \{x \in M : \alpha(x) = x^*\}$. Conversely, every real W*-algebra R is of the form (M, α), where M is the complex envelope of R and α is an involutive *-antiautomorphism of M (see [5, 7, 18]). Therefore we shall identify in the sequel the real von Neumann algebra R with the pair (M, α).

3 Extended positive part of a real W*-algebra

We recall the definitions of the extended positive part $\hat{N}^+$ of a W*-algebra N and an operator-valued trace on a W*-algebra M^+ with range in the extended positive part $\hat{N}^+$ of a W*-subalgebra $N \subset M$ [8]. Let N_*^+ be the set of all normal positive linear functionals on N. We consider the set $\hat{N}^+$ of all positively homogeneous additive lower semicontinuous functions $m : N_*^+ \to [0, +\infty]$ and embed the cone N^+ in $\hat{N}^+$ by identifying an element $x \in N^+$ with the function m_x defined by the relation $m_x(f) = f(x)$, for all $f \in N_*^+$. For an unbounded self-adjoint positive operator x affiliated with N we denote its support by e and define the corresponding function m_x by the formula

$$m_x(f) = \sum f(\overline{e}_n x) + (+\infty) f(\mathbb{1} - e),$$

where f is an arbitrary functional in N_*^+ and $\overline{e}_n = e_{[n-1,n]}$ are the spectral projections of x corresponding to the set $[n-1, n]$, $n \in \{1, 2, \ldots, \infty\}$. It was shown in [8] that for each $m \in \hat{N}^+$ there exists a positive self-adjoint (but not necessarily bounded) operator A affiliated with N such that $m = m_A$.

Let M be a W*-algebra, and let N be a W*-subalgebra of M. By an *operator-valued weight* on the W*-algebra M with range in $\hat{N}$, or an N-*valued weight*, we mean a linear map $E : M \to \hat{N}$ such that $E(yxy^*) = yE(x)y^*$, for $x \in M^+$ and $y \in N$ [8]. The properties of being normal, faithful, or semifinite are defined for E similarly to the case of linear functionals.

Let $M_E = \{x \in M : \|E(x^*x)\| < \infty\} = \{x \in M : E(x^*x) \in N^+\}$. As is known, M_E is a facial subalgebra of M and an N-bimodule (that is, $N \cdot M_E \cdot N \subset M_E$ and $E(yxz) = yE(x)z$, for all $x \in M_E$ and $y, z \in N$); moreover, E can be uniquely extended to a linear map $E : M_E \to N$, and $E(M_E)$ is a two-sided ideal of N. Hence, E can be uniquely extended to the map $\overline{E} : \hat{M}^+ \to \hat{N}^+$ (see [8]).

Now, we recall the definitions of the extended positive part $\hat{R}^+$ of a real W*-algebra $R = (M, \alpha)$ [19]. Let $(R_*^+)_r$ be the set of all normal positive linear functionals on R that vanish on skew-symmetric elements of R; each of these functionals has a unique α-invariant normal extension to M (see [3]). On the set $(R_*^+)_r$ we consider the family $\hat{R}^+$ of all positively homogeneous additive lower semicontinuous functions $m : (R_*^+)_r \to [0, +\infty]$; we can embed the cone R^+ in $\hat{R}^+$ by identifying an element

$x \in R^+$ with the function m_x such that $m_x(f) = f(x)$, for all $f \in (R_*^+)_r$. On the other hand, if x is an unbounded self-adjoint positive operator affiliated with R and with support e, then we define the corresponding function m_x by the formula

$$m_x(f) = \sum f(\overline{e}_n x) + (+\infty) f(\mathbb{1} - e),$$

where f is an arbitrary functional in $(R_*^+)_r$ and $\overline{e}_n = e_{[n-1,n]}$ are the spectral projections of x corresponding to the set $[n-1, n]$, $n \in \{1, 2, \ldots, \infty\}$.

Definition 1. The extended positive part of a real W*-algebra R is the set $\hat{R}^+$.

Proposition 3.1. [19, Proposition 3.1.] *For each $m \in \hat{R}^+$ there exist a projection $e \in K$ and a positive self-adjoint (not necessarily bounded) operator A on eH affiliated with R such that $m = m_A$.*

It follows from Proposition 1 that $\hat{R}^+ \subset \hat{M}^+$, where $M = \mathfrak{A}(R)$ is the enveloping W*-algebra of R.

By an *operator-valued weight* on a real W*-algebra R with range in the extended positive part $\hat{Q}$ of a real W*-subalgebra Q of R (or a *Q-valued weight*) we mean a linear map $T : R^+ \to \hat{Q}^+$ such that $T(yxy^*) = yT(x)y^*$, for $x \in R^+$ and $y \in Q$ [19]. The properties of being normal, faithful, or semifinite are defined for T similarly to the case of linear functionals. For completeness let us recall the definitions: an operator-valued weight T on R is said to be

- *normal* if the convergence $x_i \nearrow x$ implies $T(x_i) \nearrow T(x)$, for $x_i, x \in R^+$;
- *faithful* if the equality $T(x^*x) = 0$ implies $x = 0$;
- *semifinite* if the set $R_T = \{x \in R : \|T(x^*x)\| < \infty\}$ is ultraweakly dense in R.

The set R_T is also a facial subalgebra of R and R_T is a real Q-bimodule; moreover, T can be uniquely extended to a linear map $T : R_T \to Q$, the set $T(R_T)$ is a two-sided ideal of Q. Hence, the map T can be uniquely extended to the map $\overline{T} : \hat{R}^+ \to \hat{Q}^+$.

4 The existence of a normal operator-valued weights

Let M be a W*-algebra, and let N be a W*-subalgebra of M. The set of normal faithful semi-finite weights on M is denoted by $P(M)$; the set of normal faithful semi-finite operator-valued weights from M to N is denoted by $P(M, N)$.

As shown by U. Haagerup, the following results takes place

Theorem 4.1. [8, Theorem 5.1.] *Let $\psi \in P(M)$ and $\phi \in P(N)$. If $\sigma_t^{\psi}(x) = \sigma_t^{\phi}(x)$, for any $x \in N$ and $t \in \mathbb{R}$, then there exists a unique $T \in P(M, N)$, such that $\psi = \phi \circ T$.*

Theorem 4.2. [8, Theorem 5.9.] $P(M, N) \neq \emptyset \;\Leftrightarrow\; P(N', M') \neq \emptyset$

Here M' and N' are the commutant of M and N, respectively. Moreover, U. Haagerup constructed the canonical order-reversing bijection (denoted by $\Phi : E \to E^{-1}$) between $P(M, N)$ and $P(N', M')$. For a given $E \in P(M, N)$, the canonical bijection

$E^{-1} \in P(N', M')$ is characterized by $d(\phi \circ E)/d\psi = d\phi/d(\psi \circ E^{-1})$. Here $\psi \in P(M')$ and $\phi \in P(N)$, and E^{-1} depends only on E.

The result similar to Theorem 4.1 is also valid for real W*algebras. Namely the following theorem establishes necessary and sufficient conditions for the existence of a normal operator-valued weight on (M, α).

Theorem 4.3. [19, Theorem 4.1.] *The following conditions are equivalent:*

(1) *there exists a normal faithful semi-finite α-invariant weight φ on M^+ and a normal faithful semi-finite α-invariant weight ψ on N^+ such that $\sigma_t^{\varphi}(x) = \sigma_t^{\psi}(x)$, for any $x \in N$ and $t \in \mathbb{R}$;*

(2) *there exists a unique normal faithful semi-finite operator-valued weight T on (M, α) such that $\varphi = \psi \circ T$.*

It is easy to see that from Theorems 4.3 and 4.1 for real W*-algebras $R = (M, \alpha)$ and $Q = (N, \alpha)$ we have following corollary

Corollary 4.1. $P(R, Q) \neq \emptyset \;\Rightarrow\; P(M, N) \neq \emptyset.$

Let us prove the converse implication.

Theorem 4.4. $P(M, N) \neq \emptyset \;\Rightarrow\; P(R, Q) \neq \emptyset$

Proof. Let $T_1 : M^+ \to \hat{N}^+$ be a normal faithful semi-finite operator-valued weight, i.e. $T_1 \in P(M, N)$. We put

$$T(x) = \frac{1}{2}(T_1(x) + \overline{\alpha}T_1(x)),$$

where $\overline{\alpha}$ is the extention of α on $\hat{M}^+$.

Since $\overline{\alpha}T = T$, $R^+ \subset M^+$ and $\hat{Q}^+ \subset \hat{N}^+$, then for all $x \in R^+$, $y = T(x) \in \hat{N}^+$ we have $\overline{\alpha}(y) = \overline{\alpha}T(x) = T(x) = y = y^*$, i.e. $y \in \hat{Q}^+$. Therefore, $T : R^+ \to \hat{Q}^+$. The linearity of T_1 implies obviously the linearity of T. Let $x \in R^+$ and $y \in Q$. Then according to $T_1(yxy^*) = yT_1(x)y^*$ and $\overline{\alpha}(y^*) = y$ we have

$$T(yxy^*) = \frac{1}{2}(yT_1(x)y^* + \overline{\alpha}(y^*)T_1(x)\overline{\alpha}(y)) = y\Big(\frac{1}{2}(T_1(x) + \overline{\alpha}T_1(x))\Big)y^* = yT(x)y^*.$$

Thus, the map $T : R^+ \to \hat{Q}^+$ is an operator-valued weight.

If $x_i, x \in R^+ \subset M^+$ and $x_i \nearrow x$, then by normality of T_1 we obtain $T(x_i) \nearrow T(x)$, i.e. T is also normal. If $T(x^*x) = 0$, then $T_1(x^*x) = 0$, therefore from the faithfulness of T_1 the faithfulness of T follows. Finally, since $\|T(x^*x)\| < \infty \Leftrightarrow \|T_1(x^*x)\| < \infty$, semifiniteness of T_1 implies semifiniteness of T.

Thus the map $T : R^+ \to \hat{Q}^+$, defined as $T = \frac{1}{2}(T_1 + \overline{\alpha}T_1)$ is a normal faithful semi-finite operator-valued weight, i.e. $T \in P(R, Q)$. □

Now, by Theorems 4.1-4.4 and Corollary 4.1 we obtain the following corollaries

Corollary 4.2. $P(R,Q) \neq \emptyset \;\Leftrightarrow\; P(M,N) \neq \emptyset.$

Corollary 4.3. $P(R,Q) \neq \emptyset \;\Leftrightarrow\; P(Q',R') \neq \emptyset.$

Here $Q' = (N',\alpha')$, $R' = (M',\alpha')$ and α' is the involutive *-antiautomorphism of N' defined as $\alpha'(\cdot) = J\alpha(J \cdot J)J$, where $J : x \to x^*$ is the canonical conjugate linear isometry (see [5]).

5 Index for real finite factors

Let F ($\subset B(H)$) be a finite (complex or real) factor with the finite commutant F'. The *coupling constant* $\dim_F(H)$ of F is defined as $\mathrm{tr}_F(E_\xi^{F'})/\mathrm{tr}_{F'}(E_\xi^{F})$, where ξ is a non-zero vector in H, tr_A denotes the normalized trace and E_ξ^A is the projection of H onto the closure of the subspace $A\xi$. This definition, in the complex case, is due to F.J. Murray and J. von Neumann [14], and in the real case it is introduced in [1, 2]. In both cases it does not depend on ξ.

It is known [1, Theorem 6.4.], that if $M \subset B(H) = B(H_r) + iB(H_r)$ is a finite factor and $(M,\alpha) \subset B(H_r)$, where H_r is a real Hilbert space with $H_r + iH_r = H$, then

$$\dim_M(H) = \dim_{(M,\alpha)}(H_r) = \frac{1}{2}\dim_{(M,\alpha)}(H). \tag{5.1}$$

Consider a subfactor $N \subset M$ such that $\alpha(N) \subset N$. The *index* of N in M, denoted by $[M : N]$ is defined as $\dim_N(L^2(M))$ (see [9]), where $L^2(M)$ the completion of M with respect to the norm $\|x\|_2 = \tau(x^*x)^{1/2}$. Similarly, the *index* of (N,α) in (M,α), denoted by $[(M,\alpha) : (N,\alpha)]$, or by $[R : Q]$, is defined as $\dim_{(N,\alpha)}(L^2(M,\alpha))$ (see [1, 2]). Between real and complex indices there is the following relation

$$[(M,\alpha) : (N,\alpha)] = [M : N], \text{ i.e. } [R : Q] = [R + iR : Q + iQ].$$

(see [1, Theorem 7.2] and [2, Theorem 8]).

Considering a complex factor M as a real W*-algebra in view of (5.1) we may put $[M : (M,\alpha)] = 2[(M,\alpha) : (M,\alpha)] = 2$, i.e. $[M : R] = 2$.

For example, if M is a factor of type I_4, then up to isomorphisms it has seven real W*-subalgebras different from M, which are real or complex subfactors of M: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $M_2(\mathbb{R})$, $M_2(\mathbb{C})$, $M_2(\mathbb{H})$ and $M_4(\mathbb{R})$, where $\mathbb{H}$ is the quaternion algebra. The values of the indexes are respectively: $[M : M_4(\mathbb{R})] = [M : M_2(\mathbb{H})] = 2$, $[[M : M_2(\mathbb{C})] = [M_4(\mathbb{R}) : M_2(\mathbb{R})] = [M_2(\mathbb{H}) : \mathbb{H}] = 4$, $[M : M_2(\mathbb{R})] = 8$, $[M : \mathbb{C}] = [M_4(\mathbb{R}) : \mathbb{R}] = [M_2(\mathbb{H}) : \mathbb{R}] = 16$. $[M : \mathbb{R}] = 32$.

If M is of type I_5, then up to isomorphisms it has three real W*-subalgebras different from M, which are real or complex subfactors of M: $\mathbb{R}$, $\mathbb{C}$, $M_5(\mathbb{R})$. The corresponding indices are: $[M : M_5(\mathbb{R})] = 2$, $[M : \mathbb{C}] = [M_5(\mathbb{R}) : \mathbb{R}] = 25$, $[M : \mathbb{R}] = 50$.

We have calculated the value of the index in the above examples. It turns out that the index may be calculated also in the general case. V. Jones in [9] has proved a theorem on the values of the index for subfactors of finite factors. Let us recall this theorem

Theorem 5.1. [9, Theorem 4.3.1.] *Let M be a finite factor, and let N be a subfactor of M with $[M:N]<\infty$. Then one has either $[M:N]=4\cos^2\dfrac{\pi}{q}$ for some integer $q\geq 3$ or $[M:N]\geq 4$.*

From Theorem 5.1 we obtain the following real version of the above theorem.

Theorem 5.2. [1, Theorem 7.5.] *Let M be a finite factor and let N be a subfactor of M with $[M:N]<\infty$. Given be an involutive *-antiautomorphism α of M with $\alpha(N)\subset N$, put $R=(M,\alpha)$, $Q=(N,\alpha)$. Then one has either $[(M,\alpha):(N,\alpha)]=4\cos^2\dfrac{\pi}{q}$ for some integer $q\geq 3$ or $[(M,\alpha):(N,\alpha)]\geq 4$, i.e. $[R:Q]=4\cos^2\dfrac{\pi}{q}$ for some integer $q\geq 3$ or $[R:Q]\geq 4$.*

6 Extension of the index theory to arbitrary real factors

Let now M be a σ-finite factor and let N be a subfactor of M with $\alpha(N)\subset N$. We recall that a linear positive mapping $E:M\to N$ (or $E:(M,\alpha)\to(N,\alpha)$) is called *the conditional expectation* with respect to the W*-subalgebra N if the following conditions are satisfied:

(i) $E(\mathbb{1})=\mathbb{1}$;

(ii) $E(E(x)y)=E(x)E(y)=E(xE(y))$;

(iii) $E(x)^*E(x)\leq E(x^*x),\ \ \forall x,y\in M$.

We fix a normal conditional expectation E from (M,α) onto (N,α). The existence of E follows from [20, Theorem 1]. For this it suffices to take a normal faithful semi-finite α-invariant weight φ on M with $\sigma_t^\varphi(M)=M$ $(\forall t\in\mathbb{R})$, where σ_t^φ is the modular automorphism group of a weight φ. The extension of E on M will be denoted by $\overline{E}$. Since E is an operator-valued weight by Corollary 4.3 we get $E^{-1}\in P((N',\alpha'),(M',\alpha'))$. Similarly by Theorem 4.2 we have $\overline{E}\in P(M,N)$ and $\overline{E}^{-1}\in P(N',M')$. By the proof of Theorems 4.3 (i.e. of [19, Theorem 4.1.]), 4.4 and Theorem 1 of [20] we obtain $\overline{E}^{-1}|_{(N',\alpha')}=E^{-1}$, i.e. $\overline{E}^{-1}=\overline{E^{-1}}$.

For any unitary $u\in M'$, we have

$$u\overline{E}^{-1}(\mathbb{1})u^*=\overline{E}^{-1}(u\mathbb{1}u^*)=\overline{E}^{-1}(\mathbb{1}). \tag{6.1}$$

It is obvious that $\overline{E}(\mathbb{1})=E(\mathbb{1})=\mathbb{1}$, but in general we have $\overline{E}^{-1}(\mathbb{1})=E^{-1}(\mathbb{1})\neq\mathbb{1}$. Since M is a factor, by (6.1) $\overline{E}^{-1}(\mathbb{1})=E^{-1}(\mathbb{1})$ is a scalar (possibly $+\infty$).

H. Kosaki defined, in [10], the notion of the index as $[M:N]=\overline{E}^{-1}(\mathbb{1})$ and showed that when M is a finite factor, then his definition coincides with Jones' definition.

Moreover, in Theorem 5.4 [10] he proved that if M is a σ-finite factor and N its subfactor, then similarly to the finite case one has $\overline{E}^{-1}(\mathbb{1}) \in \{4\cos^2\pi/n(n \geq 3)\} \cup [4, +\infty]$.

Now following H. Kosaki we introduce the following

Definition 2. The index of $Q = (N, \alpha)$ in $R = (M, \alpha)$, denoted by $[R : Q]$ or by $[(M, \alpha) : (N, \alpha)]$, is defined as the scalar $E^{-1}(\mathbb{1})$.

By this definition between real and complex indices there is the following relation

$$[(M, \alpha) : (N, \alpha)] = [M : N], \text{ i.e. } [R : Q] = [R + iR : Q + iQ].$$

Thus, we obtain the following real version of the index theorem.

Theorem 6.1. *Let M be a σ-finite factor and let α be an involutive *-antiautomorphism of M. If N is a subfactor of M with $\alpha(N) \subset N$, then one has either $[(M, \alpha) : (N, \alpha)] = 4\cos^2\frac{\pi}{q}$ for some integer $q \geq 3$ or $[(M, \alpha) : (N, \alpha)] \geq 4$.*

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