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APPROXIMATION BY COMPOSITION OF SZĂSZ-MIRAKYAN AND DURRMEYER-CHLODOWSKY OPERATORS

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Abstract. We establish some approximation properties in weighted spaces and we give a Voronovskaya type asymptotic formula for the composition of the Szăsz-Mirakyan and Durrmeyer-Chlodowsky operators.

1 Introduction

For a function f defined on the interval $[0, \infty)$, O. Szăsz [9] defined the following operators, which are called the Szăsz-Mirakyan operators:

$$S_n(f\,;\,x) = \sum_{k=0}^{\infty} p_{n,k}\,(x)\; f\left(\frac{k}{n}\right),\;\; 0 \le x < \infty\,,$$

where $p_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$, in order to analyze the uniform approximation problem for functions defined on positive semi-axis. Recently Chlodowsky-Durremeyer type operators were defined in [5] (see also [8]) for a function f integrable on positive semi-axis as follows:

$$D_n(f\,;\,x) = \frac{n+1}{b_n}\sum_{k=0}^{n} \varphi_{n,k}\left(\frac{x}{b_n}\right) \int\limits_0^{b_n} \varphi_{n,k}\left(\frac{t}{b_n}\right)\, f\,(t)\,dt, \quad 0 \le x \le b_n$$

where $\varphi_{n,k}(u) = \begin{pmatrix} n \\ k \end{pmatrix} u^k\,(1-u)^{n-k}$ and (b_n) is a sequence of positive real numbers which satisfy $\lim\limits_{n\to\infty} b_n = \infty$, $\lim\limits_{n\to\infty} \dfrac{b_n}{n} = 0$.

In this paper we introduce composition of Szăsz-Mirakyan operators by taking the weight function of Chlodowsky-Durrmeyer operators on $C[0, \infty)$, as

$$F_n(f\,;\,x) = \frac{n+1}{b_n}\sum_{k=0}^{\infty} p_{n,k}(\frac{x}{b_n}) \int\limits_0^{b_n} \varphi_{n,k}(\frac{t}{b_n}) f(t)dt, \quad 0 \le x \le b_n \qquad (1.1)$$

The aim of this paper is to study approximation, the rate of approximation and to give a Voronovskaya type theorem for $F_n(f\,;x)$ in weighted spaces when the interval of approximation grows to infinity as $n \to \infty$. For our aim we will use the weighted Korovkin type theorems, proved by A.D. Gadzhiev [2], [3] and we will use the notation of [2].

Let $\rho(x) = 1 + x^2$, $x \in (-\infty, \infty)$ and B_ρ be the set of all functions f defined on the real axis satisfy the condition

$$|f(x)| \leq M_f \rho(x) \tag{1.2}$$

where M_f is a constant depending only on f. B_ρ is a normed space with the norm

$$\|f\|_\rho = \sup_{x\in(-\infty,\infty)} \frac{|f(x)|}{\rho(x)}, \; f \in B_\rho.$$

C_ρ denotes the subspace of all continuous functions in B_ρ and C_ρ^k denotes the subspace of all functions $f \in C_\rho$ with

$$\lim_{|x|\to\infty} \frac{|f(x)|}{\rho(x)} = K_f < \infty$$

where K_f is a constant depending only on f.

Theorem A ([2], [3]). *Let $\{T_n\}$ be a sequence of linear positive operators taking C_ρ into B_ρ and satisfying the conditions*

$$\lim_{n\to\infty} \|T_n(t^v;x) - x^v\|_\rho = 0, \; v = 0, 1, 2.$$

Then for any $f \in C_\rho^k$,

$$\lim_{n\to\infty} \|T_n f - f\|_\rho = 0$$

and there exist a function $g \in C_\rho \setminus C_\rho^k$ such that

$$\lim_{n\to\infty} \|T_n g - g\|_\rho \geq 1.$$

Applying Theorem A to the operators

$$T_n(f;x) = \begin{cases} V_n(f;x), & if \; x \in [0, a_n] \\ f(x), & if \; x > a_n \end{cases}$$

one then also has the following theorem.

Theorem B ([4]). *Let (a_n) be a sequence with $\lim_{n\to\infty} a_n = \infty$ and $\{V_n\}$ be a sequence of linear positive operators taking $C_\rho[0, a_n]$ into $B_\rho[0, a_n]$.*

If for $v = 0, 1, 2$

$$\lim_{n\to\infty} \|V_n(t^v;x) - x^v\|_{\rho,[0,a_n]} = 0,$$

then for any $f \in C_\rho^k[0, a_n]$

$$\lim_{n\to\infty} \|V_n f - f\|_{\rho,[0,a_n]} = 0,$$

where $B_\rho[0,a_n], C_\rho[0,a_n]$ and $C_\rho^k[0,a_n]$ denote the same as B_ρ, C_ρ and C_ρ^k respectively, but for functions defined on $[0,a_n]$ instead of $\mathbb{R}$ and the norm is taken as

$$\|f\|_{\rho,[0,a_n]} = \sup_{x\in[0,a_n]} \frac{|f(x)|}{\rho(x)}.$$

2 Auxiliary results

In this section we will study some properties of $F_n(f\,;x)$. Let (a_n) be any sequence of positive real numbers and $p = 0, 1, 2, ...$ Then we have

$$\frac{\partial^p}{\partial x^p}[(a_n x)^p e^{a_n x}] = \frac{\partial^p}{\partial x^p}\sum_{k=0}^{\infty}\frac{(a_n x)^{k+p}}{k!} = {a_n}^p \sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}\frac{(k+p)!}{k!}. \tag{2.1}$$

On the other hand, from formula of Leibnitz

$$\frac{\partial^p}{\partial x^p}[(a_n x)^p e^{a_n x}] = \sum_{i=0}^{p}\binom{p}{i}((a_n x)^p)^{(p-i)}(e^{a_n x})^{(i)}$$

$$= e^{a_n x} a_n^p \sum_{i=0}^{p}\binom{p}{i}\frac{p!}{i!}(a_n x)^i \tag{2.2}$$

Since left sides of (2.1)and (2.2) are equal, we get

$${a_n}^p \sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}\frac{(k+p)!}{k!} = e^{a_n x} a_n^p \sum_{i=0}^{p}\binom{p}{i}\frac{p!}{i!}(a_n x)^i$$

and

$$e^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}\frac{(k+p)!}{k!} = \sum_{i=0}^{p}\binom{p}{i}\frac{p!}{i!}(a_n x)^i.$$

Hence

$$\sum_{k=0}^{\infty} p_{n,k}\left(\frac{x}{b_n}\right)\frac{(k+p)!}{k!} = \sum_{i=0}^{p}\binom{p}{i}\frac{p!}{i!}(a_n x)^i. \tag{2.3}$$

Lemma 1. *For any $p \in \mathbb{N}$,*

$$F_n(t^p;\, x) = \frac{(n+1)!\, b_n^p}{(n+p+1)!}\sum_{i=0}^{p}\binom{p}{i}\frac{p!}{i!}\left(\frac{nx}{b_n}\right)^i. \tag{2.4}$$

Proof. It is easy to verify the following equality:

$$\binom{n}{k}\int_0^{b_n} t^p \left(\frac{t}{b_n}\right)^k\left(1-\frac{t}{b_n}\right)^{n-k} dt = \frac{n!\, b_n^{p+1}}{(n+p+1)!}\frac{(k+p)!}{k!}.$$

Thus we get

$$F_n(t^p;\, x) = \frac{(n+1)!\, b_n^p}{(n+p+1)!} \sum_{k=0}^{n} p_{n,k}(\frac{x}{b_n}) \frac{(k+p)!}{k!}.$$

By taking $a_n = \dfrac{n}{b_n}$ in (2.3), the proof will be completed. □

Now we will give some special cases of (2.4) for some p.

$$F_n(1;\, x) = 1, \tag{2.5}$$

$$F_n(t;\, x) = x + \frac{b_n - 2x}{n+2}, \tag{2.6}$$

$$F_n(t^2;\, x) = x^2 + \frac{x[4nb_n - (5n+6)x]}{(n+2)(n+3)} + \frac{2b_n^2}{(n+2)(n+3)}, \tag{2.7}$$

$$F_n(t^3;\, x) = x^3 + \frac{x[18nb_n^2 + 9n^2 b_n x - (9n^2+26n+24)x^2]}{(n+2)(n+3)(n+4)} + \frac{6b_n^3}{(n+2)(n+3)(n+4)}, \tag{2.8}$$

$$F_n(t^4;\, x) = x^4 + \frac{x[96nb_n^3 + 72n^2 b_n^2 x + 16n^3 b_n x^2 - (14n^3+71n^2+154n+120)x^3]}{(n+2)(n+3)(n+4)(n+5)} + \frac{24b_n^4}{(n+2)(n+3)(n+4)(n+5)}. \tag{2.9}$$

Lemma 2.

$$T_{n,m}(x) := F_n((t-x)^m;\, x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{(n+1)!\, b_n^m}{(n+m-j+1)!n^j} \sum_{i=0}^{m-j} \binom{m-j}{i} \frac{(m-j)!}{i!} \left(\frac{nx}{b_n}\right)^{i+j} \tag{2.10}$$

Proof. We have the equality

$$(t-x)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j x^j t^{m-j}$$

Because of linearity of the operator F_n and (2.4), we get desired result. □

In particular, for $p = 1, 2, 3, 4$ we have

$$F_n((t-x);\, x) = \frac{b_n - 2x}{n+2}, \tag{2.11}$$

$$T_{n,2}(x) = F_n((t-x)^2;\, x) = \frac{x[2(n-3)b_n - (n-6)x]}{(n+2)(n+3)} + \frac{2b_n^2}{(n+2)(n+3)}, \tag{2.12}$$

$$T_{n,2}(x) \le 2\frac{x[nb_n + 3x] + b_n^2}{(n+2)^2},$$

$$T_{n,4}(x) = F_n((t-x)^4;\, x) = \frac{x}{(n+2)(n+3)(n+4)(n+5)} \tag{2.13}$$

$$\times \left[24(3n-5)b_n^3 + 2(n^2-171n+20)b_n^2 x\right.$$

$$\left.-4(3n^2-73n+60)b_n x^2 + (3n^2-86n+120)x^3\right] + 24b_n^4,$$

$$T_{n,4}(x) \le 24\frac{x[3nb_n^3 + (n^2+20)b_n^2 x + 4nb_n x^2 + (n^2+40)x^3] + b_n^4}{(n+2)^4},$$

$$\sup_{x\in[0,b_n]} T_{n,2}(x) \le \begin{cases} \dfrac{b_n^2}{5} & , \quad n \le 3, \\ \dfrac{b_n^2}{n+3} & , \quad 3 \le n \le 6, \\ \dfrac{4b_n^2}{n+2} & , n \ge 7, \end{cases} \tag{2.14}$$

$$\sup_{x\in[0,b_n]} T_{n,4}(x) \le \frac{5(n+37)^2 b_n^4}{(n+2)^4} \le 845\frac{b_n^4}{(n+2)^2}. \tag{2.15}$$

3 Approximation of $F_n(f;x)$ in weighted spaces

Let (b_n) be a sequence of positive real numbers, increasing and such that

$$\lim_{n\to\infty} b_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{b_n^2}{n} = 0\ . \tag{3.1}$$

Lemma 3. *$\{F_n\}$ is a sequence of linear positive operators taking $C_\rho[0,b_n]$ into $B_\rho[0,b_n]$.*

Proof. In order to prove the lemma, it suffices to prove that $\lim_{n\to\infty} F_n(\rho(t);\, x) = \rho(x)$ uniformly on $[0,b_n]$ since $\rho(x) \in C_\rho[0,\infty]$. By the using (2.5) and (2.6), we have

$$F_n(\rho(t);x) = \rho(x) + \frac{x[4nb_n - (5n+6)x]}{(n+2)(n+3)} + \frac{2b_n^2}{(n+2)(n+3)}.$$

Therefore, $\|F_n(f;x)\|_{\rho,[0,b_n]}$ is uniformly bounded on $[0,b_n]$ because of

$$\lim_{n\to\infty} \sup_{x\in[0,b_n]} \frac{x[4nb_n - (5n+6)x]}{(n+2)(n+3)} + \frac{2b_n^2}{(n+2)(n+3)}$$

$$= \lim_{n\to\infty} \frac{2(2n^2+5n+6)b_n^2}{(n+2)(n+3)(5n+6)} = 0$$

with condition (3.1). □

Theorem 1. *Let $f \in C_\rho^k[0,\infty)$. Then*

$$\lim_{n\to\infty} \|F_n f - f\|_{\rho,[0,b_n]} = 0.$$

Proof. Using (2.5), (2.6) and (2.7) we get

$$\lim_{n\to\infty} \|F_n(1;x)-1\|_{\rho,(0,b_n]} = 0,$$

$$\lim_{n\to\infty} \|F_n(t;x)-x\|_{\rho,[0,b_n]} = \lim_{n\to\infty} \sup_{x\in[0,b_n]} \left|\frac{b_n-2x}{n+2}\frac{1}{1+x^2}\right| = \lim_{n\to\infty}\frac{b_n+1}{n+2} = 0,$$

$$\lim_{n\to\infty} \left\|F_n(t^2;x)-x^2\right\|_{\rho,[0,b_n]}$$

$$= \lim_{n\to\infty} \sup_{x\in[0,b_n]} \left|\left(\frac{x[4nb_n-(5n+6)x]}{(n+2)(n+3)} + \frac{2b_n^2}{(n+2)(n+3)}\right)\frac{1}{1+x^2}\right|$$

$$= \lim_{n\to\infty}\frac{2b_n^2+2nb_n+5n+6}{(n+2)(n+3)} = 0.$$

According to Theorem B, the proof is completed. □

4 Rate of approximation of $F_n(f;x)$ in weighted spaces

Now we want to find the rate of approximation of the sequence of linear positive operators $\{F_n\}$ for $f \in C_\rho^k[0,b_n]$. It is well known that the first modulus of continuity

$$\omega(f;\delta) = \sup\{|f(t)-f(x)| : t,x\in[a,b], |t-x|\le\delta\}$$

does not tend to zero, as $\delta\to 0$, on any infinite interval.

In [6] a weighted modulus of continuity $\Omega_n(f;\delta)$ was defined which tends to zero, as $\delta\to 0$, on infinite interval. A similar definition can be found in [1].

For each $f\in C_\rho^k[0,b_n]$ it is given by

$$\Omega_n(f;\delta) = \sup_{|h|\le\delta,\ x\in[0,b_n]} \frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}. \tag{4.1}$$

In [6] the following properties of $\Omega_n(f;\delta)$ are shown:

(i) $\lim_{\delta\to 0}\Omega_n(f;\delta) = 0$ for every $f\in C_\rho^k[0,b_n]$,

(ii) For every $f\in C_\rho^k[0,b_n]$ and $t,x\in[0,b_n]$,

$$|f(t)-f(x)| \le 2(1+\delta_n^2)(1+x^2)\Omega_n(f;\delta_n).S_n(t,x),$$

where

$$S_n(t,x) = \left(1+\frac{|t-x|}{\delta_n}\right)\left(1+(t-x)^2\right).$$

It is easy to see that

$$S_n(t,x) \le \begin{cases} 2(1+\delta_n^2), & \text{if } |t-x|\le\delta_n,\\ 2(1+\delta_n^2)\dfrac{(t-x)^4}{\delta_n^4}, & \text{if } |t-x|\ge\delta_n. \end{cases} \tag{4.2}$$

Theorem 2. *Let $f \in C_\rho^k[0,\infty)$. Then for all sufficiently large n*

$$\|F_n f - f\|_{\rho,[0,b_n]} \le C\Omega_n\left(f;\sqrt{\frac{b_n^2}{n+2}}\right).$$

where $C = 13536$.

Proof. If we use (2.3), it follows that

$$\begin{aligned}|F_n(f;x)-f(x)| &\le F_n(|f(t)-f(x)|;x)\\ &\le 2(1+\delta_n^2)(1+x^2)\Omega_n(f;\delta_n)F_n(S_n(t,x);x).\end{aligned}$$

By (4.2) we get

$$S_n(t,x) \le 2(1+\delta_n^2)[1+\frac{(t-x)^4}{\delta_n^4}]$$

for all $x \in [0,b_n]$, $t \in [0,\infty)$. Thus, for $x \in [0,b_n]$ and using (2.15) for $n \ge 7$, we get

$$\begin{aligned}|F_n(f;x)-f(x)| &\le 4(1+\delta_n^2)^2(1+x^2)\left[1+\frac{1}{\delta_n^4}T_{n,4}(x)\right]\Omega_n(f;\delta_n)\\ &\le 4(1+\delta_n^2)^2(1+x^2)\left[1+\frac{845}{\delta_n^4}\frac{b_n^4}{(n+2)^2}\right]\Omega_n(f;\delta_n).\end{aligned}$$

Put $\delta_n = \sqrt{\frac{b_n^2}{n+2}}$, then $\delta_n \le 1$ for sufficiently large n since $\lim_{n\to\infty}\frac{b_n^2}{n+2} = 0$ and the proof will be complete. □

Remark 1. This kind of theorems are studied for different operators (for instance, Szăsz-Mirakyan and Baskakov operators: see [6], [7]) in the norm $\|\cdot\|_{\rho^3}$. But in our Theorem we use the norm $\|\cdot\|_\rho$. Thus, Theorem 2 gives a better order of approximation compared with analogues theorems proved in [5], [6].

5 A Voronovskaya type theorem

In this section, we prove a Voronovskaya type theorem for the operators F_n.

Theorem 3. *For every $f \in C_\rho^k[0,b_n]$ such that f', $f'' \in C_\rho^k[0,b_n]$, we have*

$$\lim_{n\to\infty}\frac{n+2}{b_n}\{F_n(f;\,x)-f(x)\} = f'(x)+xf''(x)$$

for each fixed $x \in [0,b_n]$.

Proof. Let f, f', $f'' \in C_\rho^k[0,b_n]$. In order to prove the theorem, by Taylor's theorem we write

$$f(t) = \begin{cases} f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+(t-x)^2\eta(t-x), & if\ \ t\neq x\\ 0, & if\ \ t=x\end{cases}$$

where $\eta(h)$ tends to zero as h tends to zero.

Now from (2.5), (2.11) and (2.12)

$$\frac{n+2}{b_n}\left\{F_n(f;\,x)-f(x)\right\}=\frac{n+2}{b_n}\frac{b_n-2x}{n+2}f\,'(x)$$

$$+\frac{1}{2}\frac{n+2}{b_n}\left[\frac{x[2(n-3)b_n-(n-6)x]}{(n+2)(n+3)}+\frac{2b_n^2}{(n+2)(n+3)}\right]f''(x)$$

$$+\frac{n+2}{b_n}F_n((t-x)^2\eta(t-x);\,x).$$

If we apply the Cauchy-Schwarz -Bunyakovsky inequality to $F_n((t-x)^2\eta(t-x);\,x)$, we conclude that

$$\frac{n+2}{b_n}\left|F_n((t-x)^2\eta(t-x);\,x)\right|\leq\sqrt{\frac{n+2}{b_n^2}F_n((t-x)^4;\,x)}\sqrt{(n+2)F_n((\eta(t-x))^2;\,x)}.$$

If we consider (17) and condition (18) we get

$$\sqrt{\frac{n+2}{b_n^2}F_n((t-x)^4;\,x)}\leq\sqrt{\frac{n+2}{b_n^2}\left[845\frac{b_n^4}{(n+2)^2}\right]}$$

hence $\lim\limits_{n\to\infty}\sqrt{\dfrac{n+2}{b_n^2}\,F_n((t-x)^4;\,x)}=0$. On the other hand, by the assumption $\lim\limits_{t\to x}\eta(t-x)=0$. So, it follows that

$$\lim_{n\to\infty}\frac{n+2}{b_n}\left|F_n((t-x)^2\eta(t-x);\,x)\right|=0.$$

Then we have

$$\frac{n+2}{b_n}\left\{F_n(f;\,x)-f(x)\right\}=f\,'(x)-\frac{2x}{b_n}f'(x)+xf''(x)-\frac{6}{n+3}xf''(x)-\frac{9(n+2)x^2}{(n+3)b_n}f''(x).$$

If we take lim on both side with respect to $x\in[0,b_n]$, we will get the desired result. □

Example. For $f\,(x)=x^2$

$$\lim_{n\to\infty}\frac{n+2}{b_n}\left\{F_n(t^2;\,x)-x^2\right\}=4x.$$

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